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Abstract In this paper we derive the asymptotic distributions of residual and normalized residual empirical autocovariances and autocorrelations of (parsimonious) periodic autoregressive moving-average (PARMA) models under the assumption that the errors are uncorrelated but not necessarily independent. We then deduce the modified portmanteau statistics. We establish the asymptotic behavior of the proposed statistics. It is shown that the asymptotic distribution of the modified portmanteau tests is that of a weighted sum of independent chi-squared random variables, which can be different from the usual chi-squared approximation used under independent and identically distributed (iid) assumption on the noise. We also propose another test based on a self-normalization approach to check the adequacy of PARMA models. A set of Monte Carlo experiments and an application to financial data are presented.

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1. Introduction

Time series with periodic means and autocovariances frequently arise in various fields, such as meteorology, hydrology or economics, amongst others (see for instance Jones and Brelsford (1967), (Hipel and McLeod, 1994, Chapter 14) and Holan et al. (2010)). Modeling methods for periodic series focus on the important class of periodic autoregressive moving-average (PARMA) models as its second moment structure is periodic. PARMA models are an extension of autoregressive moving average (ARMA) models allowing parameters to vary with respect to time. A second order stationary process $X = (X_t)_{t\in\mathbb{Z}}$ with mean such that $\mu_t = \mu_{t+T}$ is said to be a PARMA model with period T (T is assumed to be known) if it is a solution to the periodic linear difference equation

$$(X_t - \mu_t) - \sum_{i=1}^{p} \phi_i(t)(X_{t-i} - \mu_{t-i}) = \epsilon_t - \sum_{j=1}^{q} \theta_j(t)\epsilon_{t-j}, \tag{1}$$

where the coefficients satisfy $\phi_i(t+T) = \phi_i(t)$ for $1 \le i \le p$ and $\theta_j(t+T) = \theta_j(t)$ for $1 \le j \le q$.

A notation that emphasizes seasonality uses $X_{nT+\nu}$ as the data point during the ν-th season of the n-th cycle of data. Here ν is a seasonal suffix that satisfies $1 \le \nu \le T$; we allow a 0-th cycle of data so that X_1 denotes the first observation (season 1 of cycle 0). One can also allow p and q to vary periodically if needed (see Lund and Basawa (2000); Francq et al. (2011)). Formally the PARMA model in (1) during a season ν with p and q varying periodically is equivalently written as

$$(X_{nT+\nu} - \mu_\nu) - \sum_{k=1}^{p_\nu} \phi_k(\nu)(X_{nT+\nu-k} - \mu_{\nu-k}) = \epsilon_{nT+\nu} - \sum_{l=1}^{q_\nu} \theta_l(\nu)\epsilon_{nT+\nu-l} \tag{2}$$

and is denoted by $\mathrm{PARMA}_T(p_1, \dots, p_T; q_1, \dots, q_T)$. Process $(\epsilon_t) = (\epsilon_{nT+\nu})_{n\in\mathbb{Z}}$ can be interpreted, as in Francq et al. (2011), as the linear innovation of $(X_t) = (X_{nT+\nu})_{n\in\mathbb{Z}}$, *i.e.* $\epsilon_t = X_t - \mathbb{E}[X_t|\mathcal{H}_X(t-1)]$, where $\mathcal{H}_X(t-1)$ is the Hilbert space generated by $(X_s, s < t)$. This linear periodic innovation process is assumed to be a stationary sequence satisfying

(A0): $\quad \mathbb{E}[\epsilon_t] = 0,\ \mathrm{Var}(\epsilon_t) = \sigma_\nu^2 > 0$ and $\mathbb{E}(\epsilon_t\epsilon_{t'}) = 0, \quad \forall\, t \ne t'.$

Under the above assumptions the process $(\epsilon_{nT+\nu})_{n\in\mathbb{Z}}$ is called a weak periodic white noise. An example of weak periodic white noise is the periodic generalized autoregressive conditional heteroscedastic (PGARCH) model (see for instance Bollerslev and Ghysels (1996)). Many other examples can also be found in Francq et al. (2011). It is customary to say that $(X_{nT+\nu})_{n\in\mathbb{Z}}$ is a strong $\mathrm{PARMA}_T(p_1,\dots,p_T;q_1,\dots,q_T)$ representation and we will do so if in (1) or (2) $(\epsilon_{nT+\nu}/\sigma_\nu)_{n\in\mathbb{Z}}$ is a strong periodic white noise, namely an independent and identically distributed (iid for short) sequence of random variables with mean 0 and variance 1. A strong white noise is obviously a weak white noise because independence entails uncorrelatedness. Of course the converse is not true. In contrast with this previous definition, the representation (1) or (2) is said to be a weak $\mathrm{PARMA}_T(p_1,\dots,p_T;q_1,\dots,q_T)$ if no additional assumption is made on $(\epsilon_{nT+\nu})_{n\in\mathbb{Z}}$, that is if $(\epsilon_{nT+\nu})_{n\in\mathbb{Z}}$ is only a weak periodic white noise (not necessarily iid). As mentioned by Francq et al. (2011) a weak PARMA processes can be viewed as an approximation of the Wold decomposition of periodically stationary processes of the form:

$$X_{nT+\nu}=\sum_{i\geq 0}\zeta_i(\nu)\epsilon_{nT+\nu-i}\qquad\text{and}\qquad\sum_{i\geq 0}\zeta_i^2(\nu)<\infty,\tag{3}$$

where $(\epsilon_{nT+\nu})_{n\in\mathbb{Z}}$ is the linear periodic innovation of $(X_{nT+\nu})_{n\in\mathbb{Z}}$. The linear model (3), which consists of the PARMA models and their limits is very general under the noise uncorrelatedness, but can be quite restrictive if the assumption of strong noise is made. Note that many time series encountered in practice can't be described by strong PARMA models (see for instance Wang et al. (2006), Francq et al. (2011)).

When the order of both the autoregressive and moving average components are not allowed to vary with season, *i.e.*, when $p_1=\cdots=p_T=p$ and $q_1=\cdots=q_T=q$ we simply write $\mathrm{PARMA}_T(p;q)$ instead of $\mathrm{PARMA}_T(p,\dots,p;q,\dots,q)$. In the sequel we suppose that the process $(X_{nT+\nu})_{n\in\mathbb{Z}}$ is centered, that is $\mu_1=\cdots=\mu_T=0$. Thus (2) becomes

$$X_{nT+\nu}-\sum_{k=1}^{p_\nu}\phi_k(\nu)X_{nT+\nu-k}=\epsilon_{nT+\nu}-\sum_{l=1}^{q_\nu}\theta_l(\nu)\epsilon_{nT+\nu-l}.\tag{4}$$

After identification and estimation of the PARMA processes, the next important step in the modeling consists in checking if the estimated model fits the data satisfactorily. Thus, under the null hypothesis that the model has been correctly identified, the residuals $\hat\epsilon_{nT+\nu}$ are approximately a periodic white noise. This adequacy checking step validates or invalidates the choice of the orders $(p_1,\dots,p_T;q_1,\dots,q_T)$.

It is important to check the validity of a $\mathrm{PARMA}_T(p_1,\dots,p_T;q_1,\dots,q_T)$ model, for given orders $(p_1,\dots,p_T;q_1,\dots,q_T)$ because the number of parameters quickly increases with the orders $(p_1, \dots, p_T; q_1, \dots, q_T)$ which entails statistical difficulties. Based on the residual empirical autocorrelations, Box and Pierce (1970) derived a goodness-of-fit test, the portmanteau test, for univariate strong ARMA models (i.e. under the assumption that the error term is iid). Ljung and Box (1978) proposed a modified portmanteau test which is nowadays one of the most popular diagnostic checking tool in ARMA modeling of time series. Since the article by Ljung and Box (1978), portmanteau tests have been important tools in time series analysis, in particular for testing the adequacy of an estimated ARMA model. See also Li (2004) for a reference book on the portmanteau tests.

This paper is devoted to the problem of the validation step of the PARMA representations (4) processes. The former works on the portmanteau statistic of PARMA models are generally performed under the assumption that the errors $\epsilon_{nT+\nu}$ are independent (see McLeod (1994, 1995) or Ursu and Duchesne (2009); Duchesne and Lafaye de Micheaux (2013)). This independence assumption is often considered too restrictive by practitioners. It precludes PGARCH and/or other forms of periodic nonlinearity (see Francq et al. (2011)). All these above results have been obtained under the iid assumption on the periodic noise and they may be invalid when the periodic series is uncorrelated but dependent. In any case, the standard portmanteau test needs to be adapted to take into account the possible lack of independence of the errors terms. In this framework we relax the standard independence assumption on the periodic error terms in order to be able to cover PARMA representations of general nonlinear models and to extend the range of application of the PARMA models. For such models we show that the asymptotic distributions of the proposed statistics are no longer chi-squared distributions but a mixture of chi-squared distributions, weighted by eigenvalues of the asymptotic covariance matrix of the vector of autocorrelations as in Francq et al. (2005). We also proposed another modified statistics based on a self-normalization approach which are asymptotically distribution-free under the null hypothesis (see Boubacar Maïnassara and Saussereau (2018) for a reference in a weak ARMA case). Contrary to the standard tests the proposed tests can be used safely for m small, where m is the number of autocorrelations used in the portmanteau test statistic. Another contribution is the improvement the results concerning the statistical analysis of weak PARMA models by considering the adequacy problem.

The article is organized as follows. In Section 2 we recall the results on the least squares estimator (LSE) asymptotic distribution of a weak PARMA model obtained by Francq et al. (2011) under ergodic and mixing assumptions. We also propose an extension of these results to weak parsimonious PARMA models. Section 3 presents our mains results. Then we study the asymptotic behaviour of the residuals autocovariances and autocorrelations of a weak PARMA model in Section 3.1.1. In Section 3.1.2 we derive the asymptotic distribution of residuals autocovariances and autocorrelations using self-normalization approach and we establish the asymptotic behaviour of the proposed statistics. Section 3.2 presents the goodness-of-fit portmanteau tests in PARMA models under the standard assumption that the innovations are iid. Examples are also proposed in order to illustrate our results in the online supplementary materials. Section 4 proposes numerical illustrations. The empirical power are also investigated. Section 5 illustrates the portmanteau test for PARMA models applied to stock market indices. The proofs of the main results are available in the online supplementary materials.

2. Preliminaries

We recall in this section some technical results which we are need on weak PARMA and are contained in Francq et al. (2011).

2.1. Causality and invertibility of PARMA models

There is no loss of generality in taking that $p_1 = \cdots = p_T = p$ and $q_1 = \cdots = q_T = q$ in (4) by adding coefficients equal to zero (see Lund and Basawa (2000) for more details). Thus the $\mathrm{PARMA}_T(p;q)$ process $(X_{nT+\nu})_{n\in\mathbb{Z}}$ satisfies the following difference equations

$$X_{nT+\nu} - \sum_{k=1}^{p} \phi_k(\nu) X_{nT+\nu-k} = \epsilon_{nT+\nu} - \sum_{l=1}^{q} \theta_l(\nu)\epsilon_{nT+\nu-l}. \tag{5}$$

We denote by $\lfloor a \rfloor$ the integer part of the real a. Let $p^* = \lfloor (p-1)/T \rfloor + 1$ and $q^* = \lfloor (q-1)/T \rfloor + 1$. As in Vecchia (1985) the difference equations (5) can be written in the following T-dimensional vector ARMA (VARMA) form:

$$\Phi_0 \boldsymbol{X}_n - \sum_{k=1}^{p^*} \Phi_k \boldsymbol{X}_{n-k} = \Theta_0 \boldsymbol{\epsilon}_n - \sum_{l=1}^{q^*} \Theta_l \boldsymbol{\epsilon}_{n-l} \tag{6}$$

where $\boldsymbol{X}_n = (X_{nT+1}, \dots, X_{nT+T})'$ and $\boldsymbol{\epsilon}_n = (\epsilon_{nT+1}, \dots, \epsilon_{nT+T})'$. The $(T \times T)-$matrices coefficients Φ_k, $k = 0, \dots, p^*$ and Θ_l, $l = 0, \dots, q^*$, are defined by

$$(\Phi_0)_{i,j} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i < j \\ -\phi_{i-j}(i) & \text{if} \quad i > j \end{cases}, \qquad (\Theta_0)_{i,j} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i < j \\ -\theta_{i-j}(i) & \text{if} \quad i > j \end{cases},$$

and $(\Phi_k)_{i,j} = \phi_{kT+i-j}(i)$, $k = 1, \dots, p^*$ and $(\Theta_l)_{i,j} = \theta_{lT+i-j}(i)$, $l = 1, \dots, q^*$. Here it is implicit that $\phi_h(\nu) = 0$, for $h \notin \{1, \dots, p\}$ and $\theta_h(\nu) = 0$, for $h \notin \{1, \dots, q\}$. The covariance matrix of the T-dimensional white noise $\boldsymbol{\epsilon}_n$ is $\Sigma_{\boldsymbol{\epsilon}} = \mathrm{Diag}(\sigma_1^2, \dots, \sigma_T^2) > 0$. We denote by L the back-shift operator such that $L^k \boldsymbol{\epsilon}_n = \boldsymbol{\epsilon}_{n-k}$. It is important to note that the lag operator L operates on the cycle index n. Equation (6) can be written more compactly as

$$\Phi(L)\boldsymbol{X}_n = \Theta(L)\boldsymbol{\epsilon}_n \tag{7}$$

where $\Phi(z) = \Phi_0 - \sum_{i=1}^{p^*} \Phi_i z^i$ and $\Theta(z) = \Theta_0 - \sum_{i=1}^{q^*} \Theta_i z^i$. From (6) we can in principle deduce the properties of weak PARMA parameters estimation, identification and validation from existing results on parameters estimation, identification and validation of the weak VARMA models (see for instance Boubacar Mainassara and Francq (2011); Boubacar Maïnassara (2012); Boubacar Maïnassara and Kokonendji (2016); Boubacar Mainassara (2011); Boubacar Maïnassara and Saussereau (2018); Katayama (2012)). Therefore we have preferred to work in the univariate PARMA setting for various reasons. In particular the results obtained directly in terms of the univariate PARMA representation are more directly usable because fewer parameters are involved and their estimation is easier (see Francq et al. (2011) for more details).

We denote by $\det(A)$ the determinant of a matrix A. We assume that the $\mathrm{PARMA}_T(p;q)$ process $(X_{nT+\nu})_{n\in\mathbb{Z}}$ corresponds to stable and invertible representations, namely

(A1): we have $\det \Phi(z) \det \Theta(z) \neq 0$ for all $|z| \leq 1$. Furthermore as in Boubacar Mainassara and Francq (2011) we assume that the weak VARMA model (7) is identifiable (see for instance Brockwell and Davis (1991); Lütkepohl (2005); Reinsel (1997)).

2.2. Estimating weak PARMA models

In this section we recall the results which we need on the weighted least squares (WLS) asymptotic distribution obtained by Francq et al. (2011) when $(\epsilon_{nT+\nu})$ satisfies α−mixing assumptions. For notation, let $\phi(\nu) = (\phi_1(\nu), \dots, \phi_p(\nu))'$ and $\theta(\nu) = (\theta_1(\nu), \dots, \theta_q(\nu))'$ respectively denote the vectors of autoregressive and moving average parameters for a specified season ν. The $T(p+q)$-dimensional collection of all PARMA parameters is denoted by

$$\alpha = \left(\phi(1)', \dots, \phi(T)', \theta(1)', \dots, \theta(T)'\right)' \in \mathbb{R}^{(p+q)T}.$$

The periodic white noise variances $\sigma^2 = \left(\sigma_1^2, \dots, \sigma_T^2\right)'$ will be treated as a nuisance parameter.

Let $X_1, \dots, X_{NT}$ be a data sample from the causal and invertible PARMA model (5) with the true parameter value $\alpha = \alpha_0$ and $\sigma^2 = \sigma_0^2$. The sample contains N full periods of data which are indexed from 0 to $N-1$. Indeed when $0 \le n \le N-1$ and $1 \le \nu \le T$, $nT+\nu$ goes from 1 to NT. The unknown parameter of interest α_0 is supposed to belong to the parameter space

$$\Delta = \left\{ \alpha = \left(\phi(1)', \dots, \phi(T)', \theta(1)', \dots, \theta(T)'\right)' \text{such that } \textbf{(A1)} \text{ is verified } \right\}.$$

For $\alpha \in \Delta$, let $\epsilon_{nT+\nu}(\alpha)$ be the periodically second-order stationary solution of

$$\epsilon_{nT+\nu}(\alpha) = X_{nT+\nu} - \sum_{k=1}^{p} \phi_k(\nu) X_{nT+\nu-k} + \sum_{l=1}^{q} \theta_l(\nu) \epsilon_{nT+\nu-l}(\alpha). \tag{8}$$

The variable $\epsilon_{nT+\nu}(\alpha)$ can be approximated, for $0 < nT+\nu \le NT$, by $e_{nT+\nu}(\alpha)$ defined recursively by

$$e_{nT+\nu}(\alpha) = X_{nT+\nu} - \sum_{k=1}^{p} \phi_k(\nu) X_{nT+\nu-k} + \sum_{l=1}^{q} \theta_l(\nu) e_{nT+\nu-l}(\alpha), \tag{9}$$

where the unknown initial values are set to zero: $e_0(\alpha) = \cdots = e_{1-q}(\alpha) = X_0 = \cdots = X_{1-p} = 0$. As showed by (Francq et al., 2011, Lemma 7) these initial values are asymptotically negligible uniformly in α.

Let δ be a strictly positive constant chosen so that the true parameter α_0 belongs to the compact set

$$\Delta_\delta = \left\{ \alpha \in \mathbb{R}^{(p+q)T};\ \text{the zeros of } \det \Phi(z) = 0 \text{ and } \det \Theta(Z) = 0 \text{ have modulus } \ge 1+\delta \right\}.$$

The random variable $\hat{\alpha}_{\mathrm{OLS}}$ is called the ordinary least squares (OLS) estimator of α if it satisfies, almost surely,

$$S_N(\hat{\alpha}_{\mathrm{OLS}}) = \min_{\alpha\in\Delta_\delta} S_N(\alpha) \quad \text{where} \quad S_N(\alpha) = \frac{1}{2N}\sum_{n=0}^{N-1}\sum_{\nu=1}^{T} e^2_{nT+\nu}(\alpha).$$

Because of the presence of heteroscedastic innovations, the OLS estimator might be inefficient. Given some vectors of weights $\omega^2 = (\omega_1^2, \dots, \omega_T^2)'$, Francq et al. (2011) and Basawa and Lund (2001) showed that the OLS estimator is asymptotically outperformed by the weighted least squares (WLS) estimator $\hat{\alpha}_{\mathrm{WLS}} = \hat{\alpha}^{\omega^2}_{\mathrm{WLS}}$ defined by

$$Q_N^{\omega^2}(\hat{\alpha}_{\mathrm{WLS}}) = \min_{\alpha\in\Delta_\delta} Q_N^{\omega^2}(\alpha) \quad \text{where} \quad Q_N^{\omega^2}(\alpha) = \frac{1}{2N}\sum_{n=0}^{N-1}\sum_{\nu=1}^{T} \omega_\nu^{-2} e^2_{nT+\nu}(\alpha).$$

Francq et al. (2011) and Basawa and Lund (2001) showed that an optimal WLS estimator is the generalized least squares (GLS) estimator defined by

$$\hat{\alpha}_{\mathrm{GLS}} = \hat{\alpha}^{\sigma^2}_{\mathrm{WLS}}, \text{ where } \sigma^2 = (\sigma_1^2, \dots, \sigma_T^2)'. \tag{10}$$

As mentioned by Francq et al. (2011) the GLS estimator assumes that σ^2 is known. In practice this parameter has also to be estimated. Given any consistent estimator $\hat{\sigma}^2 = (\hat{\sigma}_1^2, \dots, \hat{\sigma}_T^2)'$ of σ^2, a quasi-generalized least squares (QLS) estimator of α_0 is defined by

$$\hat{\alpha}_{\mathrm{QLS}} = \hat{\alpha}^{\hat{\sigma}^2}_{\mathrm{WLS}}. \tag{11}$$

One possible consistent estimator of σ_ν^2 can be obtained by

$$\hat{\sigma}_\nu^2 = \frac{1}{N}\sum_{n=0}^{N-1} e^2_{nT+\nu}(\hat{\alpha}_{\mathrm{OLS}}).$$

To establish the consistency of the least squares estimators (LSE), an additional assumption is needed.

(A2): The T-dimensional white noise $\{\boldsymbol{\epsilon}_n = (\epsilon_{nT+1}, \dots, \epsilon_{nT+T})', n \in \mathbb{Z}\}$ is ergodic and strictly stationary.

For the asymptotic normality of the LSE, additional assumptions are also required. It is necessary to assume that α_0 is not on the boundary of the parameter space Δ_δ.

(A3): We have $\alpha_0 \in \mathring{\Delta}_\delta$, where $\mathring{\Delta}_\delta$ denotes the interior of Δ_δ.

To control the serial dependence of the stationary process $(\boldsymbol{\epsilon}_n)$, we introduce the strong mixing coefficients $\alpha_{\boldsymbol{\epsilon}}(h)$ defined by

$$\alpha_{\boldsymbol{\epsilon}}(h) = \sup_{A \in \mathcal{F}_{-\infty}^{n}, B \in \mathcal{F}_{n+h}^{+\infty}} \left| \mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B) \right|,$$

where $\mathcal{F}_{-\infty}^{n} = \sigma(\boldsymbol{\epsilon}_u, u \leq n)$ and $\mathcal{F}_{n+h}^{+\infty} = \sigma(\boldsymbol{\epsilon}_u, u \geq n+h)$. We use $\|\cdot\|$ to denote the Euclidian norm of a vector. We will make an integrability assumption on the moment of the noise and a summability condition on the strong mixing coefficients $(\alpha_{\boldsymbol{\epsilon}}(k))_{k\geq 0}$.

(A4): We have $\mathbb{E}\|\boldsymbol{\epsilon}_n\|^{4+2\kappa} < \infty$ and $\sum_{k=0}^{\infty} \{\alpha_{\boldsymbol{\epsilon}}(k)\}^{\frac{\kappa}{2+\kappa}} < \infty$ for some $\kappa > 0$.

In all the sequel we denote by $\xrightarrow{\mathrm{d}}$ the convergence in distribution. The symbol $\mathrm{o}_{\mathbb{P}}(1)$ is used for a sequence of random variables that converges to zero in probability. Under the above assumptions, Francq et al. (2011) showed that $\hat{\alpha}_{\mathrm{OLS}} \to \alpha_0$, $\hat{\alpha}_{\mathrm{WLS}} \to \alpha_0$, $\hat{\alpha}_{\mathrm{GLS}} \to \alpha_0$ and $\hat{\alpha}_{\mathrm{QLS}} \to \alpha_0$ *a.s.* as $N \to \infty$ and

$$\sqrt{N}\left(\hat{\alpha}_{\mathrm{LS}} - \alpha_0\right) \xrightarrow[N\to\infty]{\mathrm{d}} \mathcal{N}\left(0, \Omega_{\alpha_0}^{\mathrm{LS}}\right), \tag{12}$$

where the exponent LS stands for OLS, WLS, GLS and QLS with

$$\Omega_{\alpha_0}^{\mathrm{OLS}} = \Omega(\alpha_0, (1,\dots,1)'), \quad \Omega_{\alpha_0}^{\mathrm{WLS}} = \Omega(\alpha_0, \omega^2), \quad \Omega_{\alpha_0}^{\mathrm{GLS}} = \Omega(\alpha_0, \sigma^2), \quad \Omega_{\alpha_0}^{\mathrm{QLS}} = \Omega(\alpha_0, \hat{\sigma}^2)$$

and

$$\Omega(\alpha_0, \omega^2) = \left(J(\alpha_0, \omega^2)\right)^{-1} I(\alpha_0, \omega^2) \left(J(\alpha_0, \omega^2)\right)^{-1}.$$

The matrices $I(\alpha_0, \omega^2)$ and $J(\alpha_0, \omega^2)$ are defined as

$$J(\alpha_0, \omega^2) = \sum_{\nu=1}^{T} \omega_\nu^{-2} \mathbb{E}\left[\left(\frac{\partial \epsilon_{nT+\nu}(\alpha)}{\partial \alpha}\right)\left(\frac{\partial \epsilon_{nT+\nu}(\alpha)}{\partial \alpha}\right)'\right]_{\alpha=\alpha_0}, \tag{13}$$

$$I(\alpha_0, \omega^2) = \sum_{\nu=1}^{T}\sum_{\nu'=1}^{T} \omega_\nu^{-2}\omega_{\nu'}^{-2} \sum_{k=-\infty}^{\infty} \mathbb{E}\left[\epsilon_\nu(\alpha)\epsilon_{kT+\nu'}(\alpha)\left(\frac{\partial \epsilon_\nu(\alpha)}{\partial \alpha}\right)\left(\frac{\partial \epsilon_{kT+\nu'}(\alpha)}{\partial \alpha}\right)'\right]_{\alpha=\alpha_0} \tag{14}$$

2.3. Estimating weak parsimonious PARMA models

The PARMA model in (5) has $(p+q)T$ autoregressive and moving average parameters. This parameter total can be large for even moderate T, making some PARMA inference matters unwieldy. Consequently, many authors have investigated parsimonious versions of (5) (see for instance Lund et al. (2006), Dudek et al. (2016)). Thus we suppose that: α_0 is function of a parameter vector β_0 of lower dimension $s_0 \leq (p+q)T$. Note that the dimension s_0 can be considerably smaller than $(p+q)T$. The notations $\phi_k(\nu;\beta_0)$ and $\theta_l(\nu;\beta_0)$ will explicitly emphasize dependence of the autoregressive and moving average parameters on β_0, we then write $\alpha_0 = \alpha(\beta_0)$ when $s_0 = (p+q)T$. Therefore we can rewrite, respectively, (8) and (9) as:

$$\epsilon_{nT+\nu}(\beta) = X_{nT+\nu} - \sum_{k=1}^{p} \phi_k(\nu;\beta) X_{nT+\nu-k} + \sum_{l=1}^{q} \theta_l(\nu;\beta)\epsilon_{nT+\nu-l}(\beta), \tag{15}$$

$$e_{nT+\nu}(\beta) = X_{nT+\nu} - \sum_{k=1}^{p} \phi_k(\nu;\beta) X_{nT+\nu-k} + \sum_{l=1}^{q} \theta_l(\nu;\beta) e_{nT+\nu-l}(\beta). \tag{16}$$

Let $\hat{\beta}_{\mathrm{LS}}$ be the LS estimator of β_0 and $\hat{\alpha}_{\mathrm{LS}} := \alpha(\hat{\beta}_{\mathrm{LS}})$ the estimator of α_0 when $s_0 = (p+q)T$. From a practical point of view, it may be worth studying the asymptotic distribution of $\hat{\beta}_{\mathrm{LS}}$.

To ensure the asymptotic properties of the LS estimator of β_0, we need assumptions similar to those we assumed in the previous case that α_0 is not parsimonious. We will assume Assumptions **(A0)**, **(A1)**, **(A2)**, **(A3)** and **(A4)** with parameter α_0 replaced by β_0 and the space parameter Δ_δ replaced by $\mathfrak{B}_\delta$ (the parameter space of β).

Proposition 1. *Suppose that $(X_{nT+\nu})_{n\in\mathbb{Z}}$ is a PARMA$_T(p;q)$ process satisfying (15). Under the assumptions* **(A0)**, **(A1)**, **(A2)**, **(A3)** *and* **(A4)**, *we have:* $\hat{\beta}_{\mathrm{LS}} \xrightarrow[N\to\infty]{\text{a.s}} \beta_0$ *and*

$$\sqrt{N}\left(\hat{\beta}_{\mathrm{LS}} - \beta_0\right) \xrightarrow[N\to\infty]{\mathrm{d}} \mathcal{N}\left(0, \Omega^{\mathrm{LS}}_{\beta_0}\right), \tag{17}$$

where the exponent LS stands for OLS, WLS, GLS and QLS with

$$\Omega^{\mathrm{OLS}}_{\beta_0} = \Omega(\beta_0, (1,\dots,1)'), \quad \Omega^{\mathrm{WLS}}_{\beta_0} = \Omega(\beta_0, \omega^2), \quad \Omega^{\mathrm{GLS}}_{\beta_0} = \Omega(\beta_0, \sigma^2), \quad \Omega^{\mathrm{QLS}}_{\beta_0} = \Omega(\beta_0, \hat{\sigma}^2)$$

and

$$\Omega(\beta_0, \omega^2) = \left(J(\beta_0,\omega^2)\right)^{-1} I(\beta_0,\omega^2) \left(J(\beta_0,\omega^2)\right)^{-1}.$$

The matrices $I(\beta_0,\omega^2)$ and $J(\beta_0,\omega^2)$ are defined as

$$J(\beta_0,\omega^2) = \sum_{\nu=1}^{T} \omega_\nu^{-2} \mathbb{E}\left[\left(\frac{\partial \epsilon_{nT+\nu}(\beta)}{\partial \beta}\right)\left(\frac{\partial \epsilon_{nT+\nu}(\beta)}{\partial \beta}\right)'\right]_{\beta=\beta_0}, \tag{18}$$

$$I(\beta_0,\omega^2) = \sum_{\nu=1}^{T}\sum_{\nu'=1}^{T} \omega_\nu^{-2}\omega_{\nu'}^{-2} \sum_{k=-\infty}^{\infty} \mathbb{E}\left[\epsilon_\nu(\beta)\epsilon_{kT+\nu'}(\beta)\left(\frac{\partial \epsilon_\nu(\beta)}{\partial \beta}\right)\left(\frac{\partial \epsilon_{kT+\nu'}(\beta)}{\partial \beta}\right)'\right]_{\beta=\beta_0} \tag{19}$$

The proof of this proposition is similar to that of Theorems 1 and 2 of Francq et al. (2011).

Remark 1. *Using the multivariate chain rule as in Lund et al. (2006), we obtain:*

$$\frac{\partial \epsilon_{nT+\nu}}{\partial \beta} = V' \frac{\partial \epsilon_{nT+\nu}}{\partial \alpha} \quad \textit{where} \quad V = \frac{\partial \alpha}{\partial \beta'}. \tag{20}$$

Now using (20) in (18) and (19), we have:

$$J(\beta_0, \omega^2) = V' J(\alpha_0, \omega^2) V \quad \textit{and} \quad I(\beta_0, \omega^2) = V' I(\alpha_0, \omega^2) V.$$

Remark 2. *In the strong SPARMA case,* i.e. *when* **(A2)** *is replaced by the assumption that $(\epsilon_{nT+\nu})_{n\in\mathbb{Z}}$ is iid, the independence of the $\epsilon_{nT+\nu}{}'s$ implies that only the terms for $k = 0$ and $\nu = \nu'$ are non-zero. Then we have $I(\beta_0, \omega^2) = J(\beta_0, \omega^2)$, so that the covariance matrix in the strong case is $\Omega_{\mathrm{s}}(\beta_0, \omega^2) := (J(\beta_0, \omega^2))^{-1}$. Therefore in view of Remark 1, we retrieve the well-known result obtained by Lund et al. (2006).*

3. Diagnostic checking in PARMA models

After the estimation phase, the next important step consists in checking if the estimated model fits satisfactorily the data. In this section we derive the limiting distribution of the residual autocorrelations and that of the portmanteau statistics based on the standard and the self-normalized approaches in the framework of weak PARMA models (4). Note that the results stated in this section extend directly for models (2) with constants. In order to check the validity of the PARMA models, it is a common in practice to examine the least squares periodic residuals $\hat{\epsilon}_{nT+\nu} = \hat{e}_{nT+\nu} = e_{nT+\nu}(\hat{\beta}_{\mathrm{LS}})$ when $p + q > 0$. We use (16) to notice that we have $\hat{e}_t = 0$ for $t := nT + \nu \leq 0$ and $t > NT$. By (15), for $n = 0, 1, \ldots, N-1$, it holds that

$$\hat{e}_{nT+\nu} \quad = X_{nT+\nu} - \textstyle\sum_{k=1}^{p} \hat{\phi}_k(\nu) X_{nT+\nu-k} + \sum_{l=1}^{q} \hat{\theta}_l(\nu) \hat{e}_{nT+\nu-l},$$

for $1 \leq t \leq NT$, with $\hat{X}_t = 0$ for $t \leq 0$ and $\hat{X}_t = X_t$ for $t \geq 1$ where $\hat{\phi}_k(\nu) = \phi_k(\nu; \hat{\beta}_{\mathrm{LS}})$ and $\hat{\theta}_l(\nu) = \theta_l(\nu; \hat{\beta}_{\mathrm{LS}})$. We denote by

$$\gamma(\nu, l) = \frac{1}{N} \sum_{n=l+1}^{N} \epsilon_{nT+\nu} \epsilon_{nT+\nu-l} \text{ and } \hat{\gamma}(\nu, l) = \frac{1}{N} \sum_{n=l+1}^{N} \hat{e}_{nT+\nu} \hat{e}_{nT+\nu-l} \tag{21}$$

the periodic white noise "empirical" autocovariances and residuals autocovariances. It should be noted that $\gamma(\nu, l)$ is not a computable statistic because it depends on the unobserved innovations $\epsilon_{nT+\nu} = \epsilon_{nT+\nu}(\alpha_0)$. For a fixed integer $m \geq 1$ we let

$$\gamma_m(\nu) := (\gamma(\nu,1),\dots,\gamma(\nu,m))' \text{ and } \hat{\gamma}_m(\nu) := (\hat{\gamma}(\nu,1),\dots,\hat{\gamma}(\nu,m))',$$

be the vectors of "empirical" autocovariances and residuals autocovariances for a specified season ν. Similarly the global "empirical" autocovariances and residuals autocovariances across all the seasons are denoted by

$$\gamma_{mT} := (\gamma_m(1),\dots,\gamma_m(T))' \text{ and } \hat{\gamma}_{mT} := (\hat{\gamma}_m(1),\dots,\hat{\gamma}_m(T))'.$$

Note that the mixing assumption **(A4)** will entail the asymptotic normality of $\gamma_m(\nu)$ and γ_{mT}. The theoretical and sample autocorrelations at lag l are respectively defined by

$$\rho(\nu,l) = \frac{\Gamma(\nu,l)}{\sqrt{\Gamma(\nu,0)\Gamma(\nu-l,0)}} \text{ and } \hat{\rho}(\nu,l) = \frac{\hat{\gamma}(\nu,l)}{\sqrt{\hat{\gamma}(\nu,0)\hat{\gamma}(\nu-l,0)}} \text{ with } \hat{\gamma}(\nu-l,0) = \frac{1}{N}\sum_{n=1}^{N}\hat{e}^2_{nT+\nu-l},$$

and where $\Gamma(\nu,l) := \mathrm{Cov}(\epsilon_{nT+\nu},\epsilon_{nT+\nu-l})$ with $\Gamma(\nu,0) := \sigma^2_\nu$. See for instance Hipel and McLeod (1994); Shao and Lund (2004) for some properties of $\Gamma(\nu,l)$. In the sequel we will also need these expressions:

$$\hat{\rho}_m(\nu) = (\hat{\rho}(\nu,1),\dots,\hat{\rho}(\nu,m))' \quad \text{and} \quad \hat{\rho}_{mT} = (\hat{\rho}_m(1),\dots,\hat{\rho}_m(T))',$$

where $\hat{\rho}_m(\nu)$ denotes the vector of the first m sample autocorrelations for a specified season ν and $\hat{\rho}_{mT}$ is the vector of the global sample autocorrelations across all the seasons.

Based on the residual empirical autocorrelations, Box and Pierce (1970) have proposed a goodness-of-fit test, the so-called portmanteau test (BP), for strong ARMA models. A modification of their test has been proposed by Ljung and Box (1978) which is nowadays one of the most popular diagnostic checking tools in ARMA modeling of time series. To test simultaneously whether all residual autocorrelations at lags $1,\dots,m$ of a PARMA model are equal to zero for a specified period ν, the portmanteau of Box and Pierce (1970); Ljung and Box (1978) can be adapted as proposed by McLeod (1994, 1995) (see also Hipel and McLeod (1994)). Based on these results McLeod (1994, 1995) suggested the following portmanteau statistics defined by

$$Q_m^{\text{BP}}(\nu) = N\sum_{h=1}^{m}\hat{\rho}^2(\nu,h) \text{ and } Q_m^{\text{LBM}}(\nu) = N\sum_{h=1}^{m}\varrho(h,\nu,N,T)\hat{\rho}^2(\nu,h), \tag{22}$$

where $\varrho(h,\nu,N,T)$ is called the Ljung-Box-McLeod (LBM) correction and is defined by

$$\varrho(h,\nu,N,T)=\begin{cases}(N+2)/(N-h/T) & \text{if} \quad h\equiv 0 \mod T\\ N/\left(N-\lfloor(h-\nu+T)/T\rfloor\right) & \text{otherwise.}\end{cases}$$

Note that the lag m used in (22) could be chosen to be different across the seasons but in most applications it is reasonable to use the same value of m for all seasons (see for instance Hipel and McLeod (1994)).

Across seasons the portmanteau test statistics are asymptotically independent for $\nu = 1,\dots,T$. Consequently for the case of the portmanteau test statistics in (22) an overall check to test if the residuals across all the seasons are periodic white noise is given by

$$Q_m^{\text{BP}}=\sum_{\nu=1}^{T}Q_m^{\text{BP}}(\nu) \text{ and } Q_m^{\text{LBM}}=\sum_{\nu=1}^{T}Q_m^{\text{LBM}}(\nu). \tag{23}$$

The statistics (23) (resp. (22)) are usually used to test the following null hypothesis

(H0) : $(X_{nT+\nu})_{n\in\mathbb{Z}}$ satisfies a $\text{PARMA}_T(p;q)$ representation (resp. for a specified period $\nu\in\{1,2,\dots,T\}$);

against the alternative

(H1) : $(X_{nT+\nu})_{n\in\mathbb{Z}}$ does not admit a $\text{PARMA}_T(p;q)$ representation (resp. for a specified period ν) or $(X_{nT+\nu})_{n\in\mathbb{Z}}$ satisfies a $\text{PARMA}_T(p';q')$ representation (resp. for a specified period ν) with $p'>p$ or $q'>q$.

3.1. Diagnostic checking in weak PARMA models

For weak $\text{PARMA}_T(p;q)$ models we will show that the asymptotic distributions of the statistics defined in (22) and (23) are a mixture of chi-squared distributions weighted by eigenvalues of the asymptotic covariance matrix of the vector of autocorrelations.

3.1.1. Asymptotic distribution of the residual autocorrelations

By a Taylor expansion of $\sqrt{N}\hat{\gamma}(\nu,h)$ we have

$$\sqrt{N}\hat{\gamma}(\nu,h)=\sqrt{N}\gamma(\nu,h)+\left(\mathbb{E}\left[\epsilon_{nT+\nu-h}\frac{\partial}{\partial\beta'}\epsilon_{nT+\nu}(\beta_0)\right]\right)\sqrt{N}\left(\hat{\beta}_{\text{LS}}-\beta_0\right)+\mathrm{o}_{\mathbb{P}}(1). \tag{24}$$

We now remark that in Equation (24), $\mathbb{E}[\epsilon_{nT+\nu-h}(\partial\epsilon_{nT+\nu}(\beta_0)/\partial\beta^{'})]$ is the row h of the following matrix $\Psi_m(\nu) \in \mathbb{R}^{m\times s_0}$ defined by

$$\Psi_m(\nu) = \mathbb{E}\left\{\begin{pmatrix}\epsilon_{nT+\nu-1}\\ \vdots \\ \epsilon_{nT+\nu-m}\end{pmatrix}\frac{\partial\epsilon_{nT+\nu}}{\partial\beta'}\right\} = \mathbb{E}\left\{\begin{pmatrix}\epsilon_{nT+\nu-1}\\ \vdots \\ \epsilon_{nT+\nu-m}\end{pmatrix}\frac{\partial\epsilon_{nT+\nu}}{\partial\alpha'}V\right\}, \tag{25}$$

using (20). So for $h = 1, \dots, m$, Equation (24) becomes

$$\sqrt{N}\hat{\gamma}_m(\nu) = \sqrt{N}\left(\hat{\gamma}(\nu,1),\dots,\hat{\gamma}(\nu,m)\right)^{'} = \sqrt{N}\gamma_m(\nu) + \Psi_m(\nu)\sqrt{N}\left(\hat{\beta}_{\mathrm{LS}} - \beta_0\right) + \mathrm{o}_{\mathbb{P}}(1). \tag{26}$$

From (26) it is clear that the asymptotic distribution of the residual autocovariances $\sqrt{N}\hat{\gamma}_m(\nu)$ is related to the asymptotic behavior of $\sqrt{N}(\hat{\beta}_{\mathrm{LS}}' - \beta_0^{'}, \gamma_m^{'}(\nu))^{'}$.

In view of Proposition 1 and **(A3)** we have almost surely $\hat{\beta}_{\mathrm{LS}} \to \beta_0 \in \overset{\circ}{\mathfrak{B}}_\delta$. Thus $\partial Q_N^{\omega^2}(\hat{\beta}_{\mathrm{LS}})/\partial\beta = 0$ for sufficiently large N and a Taylor expansion gives

$$\sqrt{N}\frac{\partial}{\partial\beta}O_N^{\omega^2}(\beta_0) + J(\beta_0,\omega^2)\sqrt{N}(\hat{\beta}_{\mathrm{LS}} - \beta_0) = \mathrm{o}_{\mathbb{P}}(1), \text{ where } O_N^{\omega^2}(\alpha) = \frac{1}{2N}\sum_{n=0}^{N-1}\sum_{\nu=1}^{T}\omega_\nu^{-2}\epsilon_{nT+\nu}^2(\alpha). \tag{27}$$

Consequently under the assumption that $J(\beta_0,\omega^2)$ is invertible and from (27) we deduce that

$$\sqrt{N}(\hat{\beta}_{\mathrm{LS}} - \beta_0) = -J(\beta_0,\omega^2)^{-1}\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\sum_{\nu=1}^{T}\omega_\nu^{-2}\epsilon_{nT+\nu}\frac{\partial\epsilon_{nT+\nu}(\beta_0)}{\partial\beta} + \mathrm{o}_{\mathbb{P}}(1). \tag{28}$$

Proposition 2. *Assume that $p + q > 0$. Under the assumptions* **(A0)**, **(A1)**, **(A2)**, **(A3)** *and* **(A4)**, *the random vector $\sqrt{N}\left(\left(\hat{\beta}_{\mathrm{LS}} - \beta_0\right)^{'}, \gamma_m^{'}(\nu)\right)^{'}$ has a limiting centered normal distribution with covariance matrix*

$$\Xi(\nu) = \begin{pmatrix}\Omega_{\beta_0}^{\mathrm{LS}} & \Sigma_{\hat{\beta},\gamma_m(\nu)}\\ \Sigma_{\hat{\beta},\gamma_m(\nu)}^{'} & \Sigma_{\gamma_m(\nu)}\end{pmatrix} = \sum_{h=-\infty}^{\infty}\mathbb{E}\left[W_{nT+\nu}W_{(n-h)T+\nu}^{'}\right],$$

where from (21) and (28) we have

$$W_{nT+\nu} = \begin{pmatrix}W_{1,nT+\nu}\\ W_{2,nT+\nu}\end{pmatrix} = \begin{pmatrix}-J(\beta_0,\omega^2)^{-1}\sum_{\nu=1}^{T}\omega_\nu^{-2}\epsilon_{nT+\nu}\frac{\partial\epsilon_{nT+\nu}(\beta_0)}{\partial\beta}\\ \left(\epsilon_{nT+\nu-1},\dots,\epsilon_{nT+\nu-m}\right)^{'}\epsilon_{nT+\nu}\end{pmatrix}. \tag{29}$$

The proof of this result is available in the extended online version of this article.

The following theorem which is an extension of the results given in Francq et al. (2005) provides the limit distribution of the residual autocovariances and autocorrelations of weak PARMA models for a specified season ν. We also provide the limit distribution of the global residual autocovariances and autocorrelations of weak PARMA models across all the seasons. Let $D_m(\nu) = \mathrm{Diag}(\sigma_\nu\sigma_{\nu-1}, \dots, \sigma_\nu\sigma_{\nu-m})$ and

$$D_{mT} = \mathrm{Diag}(\sigma_1\sigma_{1-1}, \dots, \sigma_1\sigma_{1-m}, \dots, \sigma_T\sigma_{T-1}, \dots, \sigma_T\sigma_{T-m}).$$

Theorem 3.1. *Under the assumptions of Proposition 2 we have*

$$\sqrt{N}\hat{\gamma}_m(\nu) \xrightarrow[N\to\infty]{d} \mathcal{N}\left(0, \nabla_{\hat{\gamma}_m}(\nu,\nu)\right), \quad \text{where} \tag{30}$$

$$\begin{aligned} \nabla_{\hat{\gamma}_m}(\nu,\nu) &= \Sigma_{\gamma_m(\nu)} + \Psi_m(\nu)\Omega_{\beta_0}^{\mathrm{LS}}\Psi'_m(\nu) + \Psi_m(\nu)\Sigma_{\hat{\beta},\gamma_m(\nu)} + \Sigma'_{\hat{\beta},\gamma_m(\nu)}\Psi'_m(\nu) \text{ when } p+q>0 \\ \text{and } \nabla_{\hat{\gamma}_m}(\nu,\nu) &= \Sigma_{\gamma_m(\nu)} \quad \text{when} \quad p=q=0. \end{aligned}$$

Then:

$$\sqrt{N}\hat{\gamma}_{mT} \xrightarrow[N\to\infty]{d} \mathcal{N}\left(0, \nabla_{\hat{\gamma}_m}\right) \tag{31}$$

where the asymptotic covariance matrix $\nabla_{\hat{\gamma}_m}$ is a block matrix, with the asymptotic variances given by $\nabla_{\hat{\gamma}_m}(\nu,\nu)$, for $\nu = 1, \dots, T$ and the asymptotic covariances given by

$$\nabla_{\hat{\gamma}_m}(\nu,\nu') = \Sigma_{\gamma_m}(\nu,\nu') + \Psi_m(\nu)\Omega_{\beta_0}^{\mathrm{LS}}\Psi'_m(\nu') + \Psi_m(\nu)\Sigma_{\hat{\beta},\gamma_m}(\nu,\nu') + \Sigma'_{\hat{\beta},\gamma_m(\nu)}\Psi'_m(\nu'), \quad \nu \neq \nu'.$$

We also have

$$\sqrt{N}\hat{\rho}_m(\nu) \xrightarrow[N\to\infty]{d} \mathcal{N}\left(0, \nabla_{\hat{\rho}_m}(\nu,\nu)\right) \quad \text{where} \quad \nabla_{\hat{\rho}_m}(\nu,\nu) = D_m^{-1}(\nu)\nabla_{\hat{\gamma}_m}(\nu,\nu)D_m^{-1}(\nu) \tag{32}$$

$$\text{and} \quad \sqrt{N}\hat{\rho}_{mT} \xrightarrow[N\to\infty]{d} \mathcal{N}\left(0, \nabla_{\hat{\rho}_m}\right) \quad \text{where} \quad \nabla_{\hat{\rho}_m} = D_{mT}^{-1}\nabla_{\hat{\gamma}_m}D_{mT}^{-1}. \tag{33}$$

The proof of this result is available in the extended online version of this article.

The asymptotic variance matrices $\nabla_{\hat{\gamma}_m}(\nu,\nu)$ and $\nabla_{\hat{\rho}_m}(\nu,\nu)$ depend on the unknown matrices $\Xi(\nu)$, $\Psi_m(\nu)$ and the scalar $\sigma_{\nu-h}$ for $h = 1, \dots, m$. Matrix $\Psi_m(\nu)$ and $\sigma_{\nu-h}$ can be estimated by its empirical counterpart, respectively by

$$\hat{\Psi}_m(\nu) = \frac{1}{N}\sum_{n=0}^{N-1}\left\{(\hat{\epsilon}_{nT+\nu-1}, \dots, \hat{\epsilon}_{nT+\nu-m})'\frac{\partial\hat{\epsilon}_{nT+\nu}}{\partial\alpha'}V\right\} \text{ and } \hat{\sigma}_{\nu-h} = \sqrt{\frac{1}{N}\sum_{n=0}^{N-1}\hat{\epsilon}^2_{nT+\nu-h}}.$$

Similarly the matrix $J(\alpha_0,\omega^2)$ can be estimated empirically by the square matrix $J_N(\hat{\alpha}_{\rm LS},\omega^2)$ of order $(p+q)T$ defined by:

$$J_N(\hat{\alpha}_{\rm LS},\omega^2)=\sum_{\nu=1}^{T}\omega_\nu^2\frac{1}{N}\sum_{n=0}^{N-1}\left\{\frac{\partial}{\partial\alpha}e_{nT+\nu}\left(\hat{\alpha}_{\rm LS}\right)\right\}\left\{\frac{\partial}{\partial\alpha}e_{nT+\nu}\left(\hat{\alpha}_{\rm LS}\right)\right\}'. \tag{34}$$

Thus in view of Remark 1, the matrix $J(\beta_0,\omega^2)$ can be estimated by $J_N(\hat{\beta}_{\rm LS},\omega^2):=V'J_N(\hat{\alpha}_{\rm LS},\omega^2)V$. The estimation of the long-run variance $\Xi(\nu)$ is more complicated. In the econometric literature the nonparametric kernel estimator, also called heteroscedastic autocorrelation consistent (HAC) estimator (see Newey and West (1987), or Andrews (1991)), is widely used to estimate covariance matrices of the form $\Xi(\nu)$. Interpreting $(2\pi)^{-1}\Xi(\nu)$ as the spectral density of the multivariate stationary periodic process $(W_{nT+\nu})_{n\in\mathbb{Z}}=(W'_{1,nT+\nu},W'_{2,nT+\nu})'_{n\in\mathbb{Z}}$ evaluated at frequency 0 (see Brockwell and Davis (1991), p. 459 for more details), an alternative method consists in using a parametric vector AR (VAR) estimate of the spectral density of $(W_{nT+\nu})_{n\in\mathbb{Z}}$. The stationary process $(W_{nT+\nu})_{n\in\mathbb{Z}}$ admits the Wold decomposition

$$W_{nT+\nu}=u_{nT+\nu}+\sum_{i=1}^{\infty}\varpi_i(\nu)u_{nT+\nu-i},$$

where $u_{nT+\nu}$ is a $(s_0+m)-$variate weak periodic white noise. Assume that the covariance matrix $\Sigma_u(\nu):=\mathrm{Var}(u_{nT+\nu})$ is non-singular, that $\sum_{i=1}^{\infty}\|\varpi_i(\nu)\|<\infty$, where $\|\cdot\|$ denotes any norm on the space of the real $(s_0+m)\times(s_0+m)$ matrices, and that $\det\left\{I_{s_0+m}+\sum_{i=1}^{\infty}\varpi_i(\nu)z^i\right\}\neq 0$ if $|z|\leq 1$. Then $(W_{nT+\nu})_{n\in\mathbb{Z}}$ admits an AR(∞) representation (see Akutowicz (1957)) of the form

$$\mathbf{\Phi}_\nu(L)W_{nT+\nu}=W_{nT+\nu}+\sum_{i=1}^{\infty}\Phi_i(\nu)W_{nT+\nu-i}=u_{nT+\nu}, \tag{35}$$

such that $\sum_{i=1}^{\infty}\|\Phi_i(\nu)\|<\infty$ and $\det\left\{\mathbf{\Phi}_\nu(z)\right\}\neq 0$ if $|z|\leq 1$.

This parametric approach has been studied by Berk (1974) (see also den Haan and Levin (1997)) is based on the expression

$$\Xi(\nu)=\mathbf{\Phi}_\nu^{-1}(1)\Sigma_u(\nu)\mathbf{\Phi}_\nu^{-1'}(1). \tag{36}$$

Since $W_{nT+\nu}$ is unobserved we introduce $\hat{W}_{nT+\nu}\in\mathbb{R}^{s_0+m}$ obtained by replacing $\epsilon_{nT+\nu}(\beta_0)$ by $e_{nT+\nu}(\hat{\beta}_{\rm LS})$ and $J(\beta_0,\omega^2)$ by its empirical or observable counterpart $J_N(\hat{\alpha}_{\rm LS},\omega^2)$ in (29). Let $\hat{\mathbf{\Phi}}_{\nu,r}(z)=\mathrm{I}_{s_0+m}+\sum_{i=1}^{r}\hat{\Phi}_{r,i}(\nu)z^i$, where $\hat{\Phi}_{r,1}(\nu),\dots,\hat{\Phi}_{r,r}(\nu)$ denote the coefficients of the least squares regression of $\hat{W}_{nT+\nu}$ on $\hat{W}_{nT+\nu-1},\dots,\hat{W}_{nT+\nu-r}$ for a specified season ν. Let $\hat{u}_{r,nT+\nu}$ be the residuals of this regression, and let $\hat{\Sigma}_{\hat{u}_r}(\nu)$ be the empirical variance of $\hat{u}_{r,1},\dots,\hat{u}_{r,N}$.

In the case of linear processes with independent innovations, Berk (1974) has shown that the spectral density can be consistently estimated by fitting autoregressive models of order $r=r(N)$, whenever r tends to infinity and r^3/N tends to 0 as N tends to infinity. There are differences with Berk (1974): $(W_{nT+\nu})_{n\in\mathbb{Z}}$ is multivariate, is not directly observed and is replaced by $(\hat{W}_{nT+\nu})_{n\in\mathbb{Z}}$. It is shown that this result remains valid for the multivariate linear process $(W_{nT+\nu})_{n\in\mathbb{Z}}$ with non-independent innovations (see Francq et al. (2005); Boubacar Mainassara et al. (2012); Boubacar Mainassara and Francq (2011) for references in weak (multivariate) ARMA models). We will extend these results to weak PARMA models. We are now able to state the following theorem.

Theorem 3.2. *In addition to the assumptions of Theorem 2, assume that the process $(W_{nT+\nu})_{n\in\mathbb{Z}}$ defined in (29) admits a periodic VAR(∞) representation (35) in which the roots of* $\det\mathbf{\Phi}(z)=0$ *are outside the unit disk, $\|\Phi_i\|=\mathrm{o}(i^{-2})$, and $\Sigma_u(\nu)=Var(u_{nT+\nu})$ is non-singular. Moreover we assume that $\mathbb{E}\,\|\boldsymbol{\epsilon}_n\|^{8+4\kappa}<\infty$ and $\sum_{k=0}^{\infty}\{\alpha_{\boldsymbol{\epsilon}}(k)\}^{\kappa/(2+\kappa)}<\infty$ for some $\kappa>0$. Then the spectral estimator of $\Xi(\nu)$:*

$$\hat{\Xi}^{\mathrm{SP}}(\nu):=\hat{\mathbf{\Phi}}_{\nu,r}^{-1}(1)\hat{\Sigma}_{\hat{u}_r}(\nu)\hat{\mathbf{\Phi}}_{\nu,r}^{-1\prime}(1)\to\Xi(\nu)$$

in probability when $r=r(N)\to\infty$ and $r^3/N\to 0$ as $N\to\infty$.

The proof of this theorem is similar to the proof of Theorem 5.2 in Francq et al. (2005) and it is omitted.

From Theorem 3.1 we can deduce the following result which gives the exact limiting distribution of the standard portmanteau statistics (22) and (23) under general assumptions on the innovation process of the fitted PARMA$(p;q)$ model.

Theorem 3.3. *Under Assumptions of Theorem 3.1 and* (**H0**)*, the statistics $Q_m^{\mathrm{LB}}(\nu)$ and $Q_m^{\mathrm{LBM}}(\nu)$ defined in (22) converge in distribution, as $N\to\infty$, to*

$$Z_m^{\nu}(\xi_m^{\nu})=\sum_{i=1}^{m}\xi_{i,m}^{\nu}Z_i^2$$

where $\xi_m^{\nu}=(\xi_{1,m}^{\nu},\dots,\xi_{m,m}^{\nu})'$ is the vector of the eigenvalues of the matrix $\nabla_{\hat{\rho}_m}(\nu,\nu)$ given in (32) and $Z_1,\dots,Z_m$ are independent $\mathcal{N}(0,1)$ variables.

The asymptotic distribution of the global portmanteau test statistics (that takes into account all the seasons) Q_m^{LB} and Q_m^{LBM} defined in (23) is also a weighted sum of chi-square random variables:

$$Z_m(\xi_m) = \sum_{i=1}^{mT} \xi_{i,mT} Z_i^2$$

where $\xi_{mT} = (\xi_{1,mT}, \ldots, \xi_{mT,mT})'$ denotes the vector of the eigenvalues of the matrix $\nabla_{\hat{\rho}_m}$ given in (33).

The true asymptotic distributions depend on the periodic nuisance parameters involving $\sigma_{\nu-h}$ for $h = 1, \ldots, m$ and the elements of $\Xi(\nu)$. Consequently in order to obtain the asymptotic distribution of the portmanteau statistics (22) (resp. (23)) under weak assumptions on the periodic noise, one needs a consistent estimator of the asymptotic covariance matrix $\nabla_{\hat{\rho}_m}(\nu,\nu)$ (resp. $\nabla_{\hat{\rho}_m}$). We let $\hat{\nabla}_{\hat{\rho}_m}(\nu,\nu)$ (resp. $\hat{\nabla}_{\hat{\rho}_m}$) the matrix obtained by replacing $\Xi(\nu)$ by $\hat{\Xi}(\nu)$, $\Psi_m(\nu)$ by $\hat{\Psi}_m(\nu)$ and $\sigma_{\nu-h}$ by $\hat{\sigma}_{\nu-h}$ in $\nabla_{\hat{\rho}_m}(\nu,\nu)$ (resp. $\nabla_{\hat{\rho}_m}$). Denote by $\hat{\xi}_m^{\nu} = (\hat{\xi}_{1,m}^{\nu}, \ldots, \hat{\xi}_{m,m}^{\nu})'$ (resp. $\hat{\xi}_m = (\hat{\xi}_{1,mT}, \ldots, \hat{\xi}_{mT,mT})'$) the vector of the eigenvalues of $\hat{\nabla}_{\hat{\rho}_m}(\nu,\nu)$ (resp. $\hat{\nabla}_{\hat{\rho}_m}$). At the asymptotic level α and for a specified season ν, the LBM test (resp. the BP test) consists in rejecting the adequacy of the weak $\text{PARMA}_T(p;q)$ model when

$$\lim_{N\to\infty} \mathbb{P}\left(Q_m^{\text{BP}}(\nu) > S_m^{\nu}(1-\alpha)\right) = \lim_{N\to\infty} \mathbb{P}\left(Q_m^{\text{LBM}}(\nu) > S_m^{\nu}(1-\alpha)\right) = \alpha,$$

where $S_m^{\nu}(1-\alpha)$ is such that $\mathbb{P}\left\{Z_m^{\nu}(\hat{\xi}_m^{\nu}) > S_m^{\nu}(1-\alpha)\right\} = \alpha$. We emphasize the fact that the proposed modified versions of the BP and LBM statistics are more difficult to implement because their critical values have to be computed from the data while the critical values of the standard method are simply deduced from a χ^2-table. We shall evaluate the p-values

$$\mathbb{P}\left\{Z_m(\hat{\xi}_m^{\nu}) > Q_m^{\text{BP}}(\nu)\right\} \quad \text{and} \quad \mathbb{P}\left\{Z_m(\hat{\xi}_m^{\nu}) > Q_m^{\text{LBM}}(\nu)\right\}, \ \text{with} \ \ Z_m(\hat{\xi}_m^{\nu}) = \sum_{i=1}^{m} \hat{\xi}_{i,m}^{\nu} Z_i^2,$$

by means of the Imhof algorithm (see Imhof (1961)) or other exact methods.

The test procedures for Q_m^{BP} and Q_m^{LBM} defined in (23) are similar but they are based on the mT empirical eigenvalues of $\hat{\nabla}_{\hat{\rho}_m}$.

3.1.2. Self-normalized asymptotic distribution of the residual autocorrelations

The nonparametric kernel estimator (see Andrews (1991); Newey and West (1987)) used to estimate the matrix $\Xi(\nu)$ causes serious difficulties regarding the choice of the sequence of weights. The parametric approach based on an autoregressive estimate of the spectral density of $(W_{nT+\nu})_{n\in\mathbb{Z}}$ studied in Berk (1974); den Haan and Levin (1997) is also facing the problem of choosing the truncation parameter. So the choice of the order of truncation is often crucial and difficult.

In this section, we propose an alternative method where we do not estimate an asymptotic covariance matrix which is an extension to the results obtained by Boubacar Maïnassara and Saussereau (2018). It is based on a self-normalization approach to construct a test-statistic which is asymptotically distribution-free under the null hypothesis. This approach has been studied by Boubacar Maïnassara and Saussereau (2018) in the weak ARMA case by proposing new portmanteau statistics. In this case the critical values are not computed from the data since they are tabulated by Lobato (2001). In some sense this method is finally closer to the standard method in which the critical values are simply deduced from a $\mathcal{X}^2$-table. The idea comes from Lobato (2001) and has been already extended by Boubacar Maïnassara and Saussereau (2018), Kuan and Lee (2006), Shao (010a), Shao (010b) and Shao (2012) to name a few in more general frameworks. See also Shao (2015) for a review on some recent developments on the inference of time series data using the self-normalized approach.

Let $(B_K(r))_{r\geq 0}$ be a K-dimensional Brownian motion starting from 0. For $K\geq 1$, we denote $\mathcal{U}_K$ the random variable defined by

$$\mathcal{U}_K = B_K'(1)V_K^{-1}B_K(1) \text{ where } V_K = \int_0^1 \left(B_K(r)-rB_K(1)\right)\left(B_K(r)-rB_K(1)\right)' dr.$$

We denote $\Lambda(\nu)$ the matrix in $\mathbb{R}^{m\times(s_0+m)}$ defined in block formed by $\Lambda(\nu)=(\Psi_m(\nu)|I_m)$. In view of (26) and (28) we deduce that $\sqrt{N}\,\hat{\gamma}_m(\nu) = N^{-1/2}\sum_{n=1}^{N-1}\Lambda(\nu)W_{nT+\nu} + \mathrm{o}_{\mathbb{P}}(1)$.

Contrary to Subsection 3.1.1 we do not rely on the classical method that would consist in estimating the asymptotic covariance matrix of $(\Lambda(\nu)W_{nT+\nu})_{n\in\mathbb{Z}}$. We rather try to apply Lemma 1 in Lobato (2001). So we need to check that a functional central limit theorem holds for the process $(W_{nT+\nu})_{n\in\mathbb{Z}}$.

Finally we define the normalization matrix $C_m(\nu,\nu)\in\mathbb{R}^{m\times m}$ for a specified season ν by

$$C_m(\nu,\nu) = \frac{1}{N^2}\sum_{n=0}^{N-1} S_n(\nu)S_n'(\nu) \text{ where } S_n(\nu) = \sum_{j=0}^{n}\left(\Lambda(\nu)W_{jT+\nu}-\gamma_m(\nu)\right). \qquad (37)$$

Similarly we denote by $C_{mT} \in \mathbb{R}^{mT\times mT}$ the global normalization block matrix across all the seasons with diagonal block given by (37) for $\nu = 1,\dots,T$ and

$$C_m(\nu,\nu') = \frac{1}{N^2}\sum_{n=0}^{N-1} S_n(\nu)S_n'(\nu') \text{ where } S_n(\nu') = \sum_{j=0}^{n}\left(\Lambda(\nu')W_{jT+\nu'} - \gamma_m(\nu')\right) \quad \text{for} \quad \nu \neq \nu'.$$

To ensure the invertibility of the normalization matrix $C_m(\nu,\nu)$ we need the following technical assumption on the distribution of $\boldsymbol{\epsilon}_n$.

(A5): The process $(\boldsymbol{\epsilon}_n)_{n\in\mathbb{Z}}$ has a positive density on some neighborhood of zero.

The following proposition gives the invertibility of the matrices $C_m(\nu,\nu)$ and C_{mT}.

Proposition 3. *Under the assumptions of Theorem 3.1 and* **(A5)***, the matrices $C_m(\nu,\nu)$ and C_{mT} are almost surely non singular.*

The proof of this result is available in the extended online version of this article.

The following theorem states the asymptotic distributions of the residual autocovariances and autocorrelations of weak PARMA models for a specified season ν. We also provide the limit distribution of the global residual autocovariances and autocorrelations of weak PARMA models across all the seasons.

Theorem 3.4. *We assume that $p+q>0$. Under Assumptions of Theorem 3.1,* **(A5)** *and under the null hypothesis* **(H0)** *we have for a specified season ν*

$$N\hat{\gamma}_m'(\nu)C_m^{-1}(\nu,\nu)\hat{\gamma}_m(\nu) \xrightarrow[N\to\infty]{\mathrm{d}} \mathcal{U}_m \text{ and } N\hat{\rho}_m'(\nu)D_m(\nu)C_m^{-1}(\nu,\nu)D_m(\nu)\hat{\rho}_m(\nu) \xrightarrow[N\to\infty]{\mathrm{d}} \mathcal{U}_m.$$

For the global residual autocovariances and autocorrelations across all the seasons we also have

$$N\hat{\gamma}_{mT}'C_{mT}^{-1}\hat{\gamma}_{mT} \xrightarrow[N\to\infty]{\mathrm{d}} \mathcal{U}_{mT} \text{ and } N\hat{\rho}_{mT}'D_{mT}C_{mT}^{-1}D_{mT}\hat{\rho}_{mT} \xrightarrow[N\to\infty]{\mathrm{d}} \mathcal{U}_{mT}.$$

The proof of this result is available in the extended online version of this article.

Remark 3. *When $p=q=0$ we don't need to estimate the unknown parameter α_0. Thus a careful reading of the proofs shows that the vector $W_{nT+\nu}$ is replaced by*

$$\tilde{W}_{nT+\nu} = W_{2,nT+\nu} = \left(\epsilon_{nT+\nu}\epsilon_{nT+\nu-1},\dots,\epsilon_{nT+\nu}\epsilon_{nT+\nu-m}\right)'.$$

Then we generalized for weak periodic noise the result obtained by Boubacar Maïnassara and Saussereau (2018).

Of course the above theorem is useless for practical purpose because the normalization matrix $C_m(\nu,\nu)$, C_{mT} and the parameter $\sigma_{\nu-h}$ are not observable. This gap will be fixed below (see Theorem 3.5) when one replaces the matrix $C_m(\nu,\nu)$, C_{mT} and the scalar $\sigma_{\nu-h}$ by their empirical or observable counterparts. Then we denote

$$\hat{C}_m(\nu,\nu) = \frac{1}{N^2}\sum_{n=0}^{N-1}\hat{S}_n(\nu)\hat{S}_n'(\nu) \text{ where } \hat{S}_n(\nu) = \sum_{j=0}^{n}\left(\hat{\Lambda}(\nu)\hat{W}_{jT+\nu} - \hat{\gamma}_m(\nu)\right),$$

with $\hat{\Lambda}(\nu) = (\hat{\Psi}_m(\nu)|I_m)$ and where $\hat{W}_{nT+\nu}$ and $\hat{\sigma}_{\nu-h}$ are defined in Subsection 3.1.1.

We are able to state the following result which is the applicable counterpart of Theorem 3.4.

Theorem 3.5. *Under the assumptions of Theorem 3.4, for a specified season ν we have*

$$N\hat{\gamma}_m'(\nu)\hat{C}_m^{-1}(\nu,\nu)\hat{\gamma}_m(\nu) \xrightarrow[N\to\infty]{\mathrm{d}} \mathcal{U}_m \text{ and } Q_m^{\mathrm{SN}}(\nu) =: N\hat{\rho}_m'(\nu)\hat{D}_m(\nu)\hat{C}_m^{-1}(\nu,\nu)\hat{D}_m(\nu)\hat{\rho}_m(\nu) \xrightarrow[N\to\infty]{\mathrm{d}} \mathcal{U}_m.$$

We also have for the global seasons

$$N\hat{\gamma}_{mT}'\hat{C}_{mT}^{-1}\hat{\gamma}_{mT} \xrightarrow[N\to\infty]{\mathrm{d}} \mathcal{U}_{mT} \text{ and } Q_m^{\mathrm{SN}} =: N\hat{\rho}_{mT}'\hat{D}_{mT}\hat{C}_{mT}^{-1}\hat{D}_{mT}\hat{\rho}_{mT} \xrightarrow[N\to\infty]{\mathrm{d}} \mathcal{U}_{mT}.$$

The proof of this result is available in the extended online version of this article.

Based on the above result we propose a modified version of the LBM statistic for a specified season ν when one uses the statistic

$$\tilde{Q}_m^{\mathrm{SN}}(\nu) = N\hat{\rho}_m'(\nu)M_m^{1/2}(\nu,N,T)\hat{D}_m(\nu)\hat{C}_m^{-1}(\nu,\nu)\hat{D}_m(\nu)M_m^{1/2}(\nu,N,T)\hat{\rho}_m(\nu), \tag{38}$$

where the matrix $M_m(\nu,N,T) \in \mathbb{R}^{m\times m}$ is diagonal with $(\varrho(1,\nu,N,T),\dots,\varrho(m,\nu,N,T))$ as diagonal terms.

Similarly we propose a modified version of the LBM statistic for the global seasons when one uses the following statistic

$$\tilde{Q}_m^{\mathrm{SN}} = N\hat{\rho}_{mT}'M_{mT}^{1/2}\hat{D}_{mT}\hat{C}_{mT}^{-1}\hat{D}_{mT}M_{mT}^{1/2}\hat{\rho}_{mT}, \tag{39}$$

where the matrix $M_{mT} \in \mathbb{R}^{mT\times mT}$ is diagonal with

$$(\varrho(1,1,N,T),\dots,\varrho(m,1,N,T),\dots,\varrho(1,T,N,T),\dots,\varrho(m,T,N,T))$$

as diagonal terms.

At the asymptotic level α and for a specified season ν, the self-normalized LBM test (resp. the BP test) consists in rejecting the adequacy of the weak $\mathrm{PARMA}_T(p;q)$ model when

$$\tilde{Q}_m^{\text{SN}}(\nu) > \mathcal{U}_m(1-\alpha) \quad (\text{resp. } Q_m^{\text{SN}}(\nu) > \mathcal{U}_m(1-\alpha)),$$

where the critical values $\mathcal{U}_m(1-\alpha)$ are tabulated in (Lobato, 2001, Table 1). Similarly for the global test we have

$$\tilde{Q}_m^{\text{SN}} > \mathcal{U}_{mT}(1-\alpha) \quad (\text{resp. } Q_m^{\text{SN}} > \mathcal{U}_{mT}(1-\alpha)).$$

3.2. Diagnostic checking in strong PARMA models

Under the assumption that the data generating process (DGP) follows a strong $\mathrm{PARMA}_T(p;q)$ model, the statistic $Q_m^{\text{LBM}}(\nu)$ has the same asymptotic chi-squared distribution as $Q_m^{\text{BP}}(\nu)$ and has the reputation of doing better for small or medium sized sample (see McLeod (1994, 1995)). If the strong $\mathrm{PARMA}_T(p;q)$ model is adequate for a fixed period ν, McLeod (1994, 1995) shown that $Q_m^{\text{BP}}(\nu)$ or $Q_m^{\text{LBM}}(\nu)$ is asymptotically distributed as a chi-squared variable with $m-(p+q)$ (or $m-(p_\nu+q_\nu)$ for strong PARMA (4)) degrees of freedom. Thus if the strong $\mathrm{PARMA}_T(p;q)$ model is adequate the global statistics (23) will also be asymptotically distributed as a chi-squared variable with $mT - T(p+q)$ (or $mT - \sum_{\nu=1}^{T}(p_\nu+q_\nu)$ for strong PARMA (4)) degrees of freedom (see Hipel and McLeod (1994)).

Remark 4. *For weak $\mathrm{PARMA}_T(p;q)$ models, Theorem 3.3 shows that: the asymptotic distributions of the statistics defined in (22) and (23) are no longer chi-squared distributions but a mixture of chi-squared distributions weighted by eigenvalues of the asymptotic covariance matrix of the vector of autocorrelations. Therefore for the asymptotic distributions of (22) and (23), the $\chi^2_{m-(p+q)}$ and $\chi^2_{mT-(p+q)T}$ approximations are no longer valid in the framework of weak $\mathrm{PARMA}(p;q)$ models.*

Theorem 3.6. *Under the assumptions of Proposition 2, when* **(A2)** *is replaced by the assumption that $(\epsilon_{nT+\nu})_{n\in\mathbb{Z}}$ is iid, we have*

$$\sqrt{N}\hat{\gamma}_m(\nu) \xrightarrow[N\to\infty]{d} \mathcal{N}\left(0, \nabla^{\mathrm{S}}_{\hat{\gamma}_m}(\nu,\nu)\right), \qquad \textit{where} \tag{40}$$

$\nabla^{\mathrm{S}}_{\hat{\gamma}_m}(\nu,\nu) = \sigma_\nu^4 I_m - \Psi_m(\nu) J(\beta_0,\omega^2)^{-1}\Psi_m'(\nu)$ *when* $p+q>0$ *and* $\nabla^{\mathrm{S}}_{\hat{\gamma}_m}(\nu,\nu) = \sigma_\nu^4 I_m$ *when* $p=q=0$.

Then:

$$\sqrt{N}\hat{\gamma}_{mT} \xrightarrow[N\to\infty]{d} \mathcal{N}\left(0, \nabla^{\mathrm{S}}_{\hat{\gamma}_m}\right) \tag{41}$$

where the asymptotic covariance matrix $\nabla^{\mathrm{S}}_{\hat{\gamma}_m}$ *is a block matrix, with the asymptotic variances given by* $\nabla^{\mathrm{S}}_{\hat{\gamma}_m}(\nu,\nu)$*, for* $\nu = 1,\dots,T$ *and the asymptotic covariances given by* $\nabla^{\mathrm{S}}_{\hat{\gamma}_m}(\nu,\nu') = -\Psi_m(\nu)J(\beta_0,\omega^2)^{-1}\Psi'_m(\nu)$*, for* $\nu \neq \nu'$*. We also have*

$$\sqrt{N}\hat{\rho}_m(\nu) \xrightarrow[N\to\infty]{d} \mathcal{N}\left(0, \nabla^{\mathrm{S}}_{\hat{\rho}_m}(\nu,\nu)\right) \quad \textit{where} \quad \nabla^{\mathrm{S}}_{\hat{\rho}_m}(\nu,\nu) = D_m^{-1}(\nu)\nabla^{\mathrm{S}}_{\hat{\gamma}_m}(\nu,\nu)D_m^{-1}(\nu) \tag{42}$$

$$\textit{and} \quad \sqrt{N}\hat{\rho}_{mT} \xrightarrow[N\to\infty]{d} \mathcal{N}\left(0, \nabla^{\mathrm{S}}_{\hat{\rho}_m}\right) \quad \textit{where} \quad \nabla^{\mathrm{S}}_{\hat{\rho}_m} = D_{mT}^{-1}\nabla^{\mathrm{S}}_{\hat{\gamma}_m}D_{mT}^{-1}. \tag{43}$$

The proof is the same as in Theorem 3.1 with some simplifications due to the independence assumption.

Remark 5. *Note that from* (40) *and* (41) *of Theorem 3.6 we retrieve the well-known results obtained by (Duchesne and Lafaye de Micheaux, 2013, Proposition 2).*

Theorem 3.7. *Under Assumptions of Theorem 3.1 and* (**H0**)*, when* **(A2)** *is replaced by the assumption that* $(\epsilon_{nT+\nu})_{n\in\mathbb{Z}}$ *is iid, the statistics* $Q_m^{\mathrm{LB}}(\nu)$ *and* $Q_m^{\mathrm{LBM}}(\nu)$ *defined in* (22) *converge in distribution, as* $N \to \infty$*, to*

$$Z_m^{\nu}(\xi_m^{\nu}) = \sum_{i=1}^{m} \xi_{i,m}^{\nu} Z_i^2$$

where $\xi_m^{\nu} = (\xi_{1,m}^{\nu},\dots,\xi_{m,m}^{\nu})'$ *is the vector of the eigenvalues of the matrix* $\nabla^{\mathrm{S}}_{\hat{\rho}_m}(\nu,\nu)$ *given in* (42) *and* $Z_1,\dots,Z_m$ *are independent* $\mathcal{N}(0,1)$ *variables.*

The asymptotic distribution of the global portmanteau test statistics (that takes into account all the seasons) Q_m^{LB} *and* Q_m^{LBM} *defined in* (23) *is also a weighted sum of chi-square random variables:*

$$Z_m(\xi_m) = \sum_{i=1}^{mT} \xi_{i,mT} Z_i^2$$

where $\xi_{mT} = (\xi_{1,mT},\dots,\xi_{mT,mT})'$ *denotes the vector of the eigenvalues of the matrix* $\nabla^{\mathrm{S}}_{\hat{\rho}_m}$ *given in* (43)*.*

Remark 6. *In view of Theorem 5 when* m *is large, the matrix*

$$\nabla^{\mathrm{S}}_{\hat{\rho}_m}(\nu,\nu) \simeq I_m - D_m^{-1}(\nu)\Psi_m(\nu)J(\beta_0,\omega^2)^{-1}\Psi'_m(\nu)D_m^{-1}(\nu)$$

(resp. $\nabla^{\mathrm{S}}_{\hat{\rho}_m} \simeq D^{-1}_{mT}\nabla_{\hat{\gamma}_m}D^{-1}_{mT}$) is close to a projection matrix. Its eigenvalues are therefore equal to 0 and 1. The number of eigenvalues equal to 1 is $\mathrm{Tr}(\nabla^{\mathrm{S}}_{\hat{\rho}_m}(\nu,\nu)) = \mathrm{Tr}(I_{m-s_0(\nu))}) = m - s_0(\nu)$ (resp. $\mathrm{Tr}(\nabla^{\mathrm{S}}_{\hat{\rho}_m}) = \mathrm{Tr}(I_{mT-s_0}) = mT - s_0$) and $s_0(\nu)$ (resp. s_0) eigenvalues equal to 0, where $s_0(\nu)$ is the number of estimated parameters for a specified season ν and $\mathrm{Tr}(\cdot)$ denotes the trace of a matrix. Therefore we can provide an answer to the open problem raised on portmanteau white noise tests by Lund et al. (2006). More precisely, under **(H0)** *and in the strong PARMA case, the asymptotic distributions of the statistics defined in* (22) *and* (23) *can be approximated by a $\chi^2_{m-s_0(\nu)}$ and $\chi^2_{mT-s_0}$. As a consequence to the existing results, the $\chi^2_{m-(p+q)}$ and $\chi^2_{mT-(p+q)T}$ approximations are not valid in the case of parsimonious PARMA models when s_0 is considerably smaller then $(p+q)T$. Note that this result and also the existing results are not applicable when $m \leq (p+q)$ in contrast to our results (see Theorem 3.7). In fact, the exact distributions of the BP or LBM portmanteau test statistics are better approximated by those of weighted sums of chi-square random variables. Our results show that in the general PARMA case, the weights in the asymptotic distributions of the BP or LBM procedures may be relatively far from zero, and thus, adjusting the degrees of freedom does not represent a solution in the present framework.*

Theorem 3.8. *Under the assumptions of Theorem 3.4 and when* **(A2)** *is replaced by the assumption that $(\epsilon_{nT+\nu})_{n\in\mathbb{Z}}$ is iid, for a specified season ν we have*

$$N\hat{\gamma}'_m(\nu)\hat{C}^{-1}_m(\nu,\nu)\hat{\gamma}_m(\nu) \xrightarrow[N\to\infty]{\mathrm{d}} \mathcal{U}_m \text{ and } Q^{\mathrm{SN}}_m(\nu) =: N\hat{\rho}'_m(\nu)\hat{D}_m(\nu)\hat{C}^{-1}_m(\nu,\nu)\hat{D}_m(\nu)\hat{\rho}_m(\nu) \xrightarrow[N\to\infty]{\mathrm{d}} \mathcal{U}_m.$$

We also have for the global seasons

$$N\hat{\gamma}'_{mT}\hat{C}^{-1}_{mT}\hat{\gamma}_{mT} \xrightarrow[N\to\infty]{\mathrm{d}} \mathcal{U}_{mT} \text{ and } Q^{\mathrm{SN}}_m =: N\hat{\rho}'_{mT}\hat{D}_{mT}\hat{C}^{-1}_{mT}\hat{D}_{mT}\hat{\rho}_{mT} \xrightarrow[N\to\infty]{\mathrm{d}} \mathcal{U}_{mT}.$$

The proof is the same as in Theorem 3.5.

4. Numerical illustration

By means of Monte Carlo experiments, we investigate the finite sample properties of the tests introduced in this paper. The numerical illustrations of this section are made with the free statistical R software (see https://www.r-project.org/).

We indicate the conventions that we adopt in the discussion and in the tables. One refers to

- $\mathrm{BP}_{\mathrm{S}}(\nu)$ for the standard Box-Pierce test using the statistic $Q_m^{\mathrm{BP}}(\nu)$;
- $\mathrm{LBM}_{\mathrm{S}}(\nu)$ for the standard Ljung-Box-McLeod test using the statistic $Q_m^{\mathrm{LBM}}(\nu)$;
- $\mathrm{BP}_{\mathrm{SN}}(\nu)$ for the modified test using the statistic $Q_m^{\mathrm{SN}}(\nu)$ with the values of the quantiles of $\mathcal{U}_m$ simulated in Table 1 of Lobato (2001);
- $\mathrm{LBM}_{\mathrm{SN}}(\nu)$ for the modified test using the statistic $\tilde{Q}_m^{\mathrm{SN}}(\nu)$, using the critical values $\mathcal{U}_m$;
- $\mathrm{BP}_{\mathrm{W}}(\nu)$ for the modified Box-Pierce test using the statistic $Q_m^{\mathrm{BP}}(\nu)$;
- $\mathrm{LBM}_{\mathrm{W}}(\nu)$ for the modified Ljung-Box-McLeod test using the statistic $Q_m^{\mathrm{LBM}}(\nu)$;
- BP_{S} for the standard global Box-Pierce test using the statistic Q_m^{BP};
- $\mathrm{LBM}_{\mathrm{S}}$ for the standard global Ljung-Box-McLeod test using the statistic Q_m^{LBM};
- $\mathrm{BP}_{\mathrm{SN}}$ for the modified global test using the statistic Q_m^{SN}, using the critical values $\mathcal{U}_{mT}$;
- $\mathrm{LBM}_{\mathrm{SN}}$ for the modified global test using the statistic $\tilde{Q}_m^{\mathrm{SN}}$, using the critical values $\mathcal{U}_{mT}$;
- BP_{W} for the modified global Box-Pierce test using the statistic Q_m^{BP};
- $\mathrm{LBM}_{\mathrm{W}}$ for the modified global Ljung-Box-McLeod test using the statistic Q_m^{LBM}.

We will see in the tables that the numerical results using the Ljung-Box-McLeod tests are very close to those of the Box-Pierce tests. Nevertheless they are still presented here for the sake of completeness.

We consider a $\mathrm{PARMA}_2(1,1)$ model of the form

$$X_{2n+\nu} - \phi(\nu)X_{2n+\nu-1} = \epsilon_{2n+\nu} - \theta(\nu)\epsilon_{2n+\nu-1}, \quad \nu = 1,2 \tag{44}$$

where the unknown parameter is $\alpha_0 = (\phi(1), \phi(2), \theta(1), \theta(2))'$. Two different periodic noises are considered. First we assume that in (44) the innovation process $(\epsilon_{2n+\nu})_{n\in\mathbb{Z}}$ is an iid centered Gaussian process with variance $\sigma_\nu^2 = \mathbb{E}(\epsilon_{2n+\nu})$ where $\sigma_1 = 0.9$ and $\sigma_2 = 1.5$ which corresponds to the strong PARMA case. For the weak PARMA case, we consider that in (44) the innovation process $(\epsilon_{2n+\nu})_{n\in\mathbb{Z}}$ follows a periodic ARCH(1) given by

$$\epsilon_{2n+\nu} = \sqrt{\omega(\nu) + a(\nu)\epsilon_{2n+\nu-1}^2}\,\eta_{2n+\nu}, \quad \nu = 1,2 \tag{45}$$

with $(\omega(1), \omega(2)) = (0.2, 0.4)$, $(a(1), a(2)) = (0.4, 0.45)$ and where $(\eta_{2n+\nu})_{n\in\mathbb{Z}}$ is a sequence of iid centered Gaussian random variables with variance 1.

For each of these two different periodic white noises 1,000 independent replications of length $(N+100)\times 2$ were generated. These sequences were plugged into the model (44) yielding 1,000 independent replications of the periodic process $(X_{2n+\nu})_n$ of length $(N+100)\times 2$. Initial values were set to zero and in order to achieve periodic stationarity the first 200 observations were dropped. For each replication of size $NT = N \times 2$ we use the QLS method to estimate the corresponding coefficient α_0. After estimating Model (44) we apply portmanteau test to the residuals for different values of $m \in \{1, \dots, 12\}$, where m is the number of autocorrelations used in the portmanteau test statistic. In the experiment we considered two values of N: 400 and 2,000. The nominal asymptotic level considered of the tests is $\alpha = 5\%$. Note that for this level, the empirical relative frequency of rejection size over the $1,000$ independent replications should vary within the confidences intervals $[3.6\%, 6.4\%]$ with probability 95% and $[3.3\%, 6.9\%]$ with probability 99%. When the relative rejection frequencies are outside the significant limits with probability 95% (resp. 99%), they are displayed in bold (resp. underlined) type in Tables 2 and 3.

4.1. Strong $\mathrm{PARMA}_2(1,1)$ *model case*

Table 2 displays relative frequencies (in %) of rejection (over the 1,000 replications) of the null hypothesis (**H0**) that the data generating process (DGP for short) follows a strong periodic white noise *i.e.* a strong $\mathrm{PARMA}_2(1,1)$ given by (44) with $\alpha_0 = (0,0,0,0)'$. For the strong periodic white noise, all the relative rejection frequencies (for global or a specified season ν) are inside the significant limits. Thus the error of first kind is well controlled by all the tests in this case. Similar simulation experiments, not reported here (see supplementary materials), reveal that the observed relative rejection frequency of the standard tests is very far from the nominal $\alpha = 5\%$ for small m when $\alpha_0 \neq (0,0,0,0)'$. The results are worse for $N = 2,000$ than for $N = 400$. This is in accordance with the results in the literature on the strong PARMA models. The theory that the asymptotic distributions of (22) and (23), namely, χ^2_{m-2} and χ^2_{2m-4} approximations are better for larger m is confirmed. In contrast, the proposed tests well control the error of first kind, even when m is small.

From these examples we draw the conclusion that the proposed versions are preferable to the standard ones, when the number m of autocorrelations used is small.

4.2. Weak $PARMA_2(1,1)$ model case

We now repeat the same experiment on Model (44) by assuming that the innovation process $(\epsilon_{2n+\nu})_n$ follows (45). Tables 3 displays relative frequencies (in %) of rejection of the null hypothesis (**H0**) that the DGP follows a weak periodic white noise *i.e.* the weak $\mathrm{PARMA}_2(1,1)$ given by (44)–(45) with $\alpha_0 = (0,0,0,0)'$. As expected the observed relative rejection frequencies of the standard tests are definitely outside the significant limits. Thus the standard tests reject very often the true weak periodic noise. By contrast, the error of first kind is globally well controlled by the proposed tests when N increases. Similar simulation experiments, not reported here (see supplementary materials), reveal the same situation when $\alpha_0 \neq (0,0,0,0)'$.

We draw the conclusion that for this particular weak PARMA model the proposed tests are clearly preferable to the standard ones. For this particular $\mathrm{PARMA}_2(1,1)$ model, we notice that the standard and our proposed tests have very similar powers (see supplementary materials).

5. Adequacy of weak PARMA models for real datasets

We now consider an application to the daily log returns (also simply called the returns) of two stock market indices (closing values): CAC 40 (Paris) and DAX (Frankfurt). The returns are defined by $r_t = 100 \log(p_t/p_{t-1})$ where p_t denotes the price index of the corresponding index at time t. The observations cover the period from January 4, 1999 to November 20, 2020. The data can be downloaded from the website Yahoo Finance: http://fr.finance.yahoo.com/. Because of the presence of holidays many weeks comprise less than five observations. We removed the entire weeks when there was less than five data. The effective number of observations used for each index is given in Table 1 and the periodicity is then $T = 5$.

In Financial Econometric the returns are often assumed to be a weak white noise (though they are not generally independent sequences). In view of the so-called volatility clustering it is well known that the strong white noise assumption is generally not adequate for these series (see for instance Francq and Zakoïan (2019); Lobato (2001); Boubacar Mainassara et al. (2012); Boubacar Maïnassara and Saussereau (2018)). The squares of the returns have often second-order moments close to those of an ARMA$(1,1)$ (which is compatible with a GARCH$(1,1)$ model for the returns). We will test if these hypotheses remind valid in the case of periodic models by fitting weak PARMA models on the returns and on their squares.

A day-of-the-week seasonality property of the stock markets returns series was largely investigated in the literature, by Aknouche et al. (2020); Francq et al. (2011); Regnard and Zakoian (2011); Bollerslev and Ghysels (1996); Franses and Paap (2000); Balaban et al. (2001) to name a few. Most of these studies focus on the description of day-of-the-week seasonality in returns and volatility. In particular the so-called Monday effect in the finance literature was observed in many stock markets.

First we apply portmanteau tests on each series of daily returns for checking the hypothesis that the returns constitute a periodic white noise. In this section we only present the results on the Ljung-Box-McLeod tests since they are very close to those of the Box-Pierce tests. Table 4 displays the $p-$values and the statistics (for the self-normalized versions) of the standard and modified LBM tests for the mean corrected returns of each index. The p-values less than 5% are in bold, those less than 1% are underlined. At the $\alpha = 5\%$ significance level, the hypothesis of strong periodic noise is (frequently) rejected by the standard global, even for a specified season $\nu \in \{1, \dots, 5\}$ LBM tests for all indices. This is not surprising because as above-mentioned the standard test required the iid assumption and, in particular in view of the so-called volatility clustering, it is well known that the strong white noise model is not adequate for these series (see for instance Francq and Zakoïan (2019); Boubacar Mainassara et al. (2012); Francq et al. (2011)). By contrast, the weak periodic white noise hypothesis is not rejected for these two indices by all the global proposed tests, even for a specified season ν. To summarize, the outputs of Table 4 are in accordance with the common belief that these series are not strong white noises but could be weak white noises.

Next, let us turn to the dynamics of the squared returns by fitting a weak PARMA$_5(1,1)$ model. Denoting by $(X_{5n+\nu})_{n\geq 0}$ the mean corrected series of the squared returns $(r^2_{5n+\nu})_{n\geq 0}$, we fit the model

$$X_{5n+\nu} - \phi(\nu)X_{5n+\nu-1} = \epsilon_{5n+\nu} - \theta(\nu)\epsilon_{5n+\nu-1}. \tag{46}$$

To check the stationary properties of $X_{5n+\nu}$ it is convenient to consider the solution to the characteristic equation of the autoregressive part of Equation (46), which with our notations in Section 2 can be shown to be equal to

$$\det(\Phi(z)) = \det(\Phi_0 - \Phi_1 z) = 1 - \phi(1)\phi(2)\phi(3)\phi(4)\phi(5)z = 0 \text{ for } |z| > 1 \tag{47}$$

where

$$\Phi_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\phi(2) & 1 & 0 & 0 & 0 \\ 0 & -\phi(3) & 1 & 0 & 0 \\ 0 & 0 & -\phi(4) & 1 & 0 \\ 0 & 0 & 0 & -\phi(5) & 1 \end{pmatrix} \text{ and } \Phi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & \phi(1) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The same result holds for the invertible Model (46) with $\phi(\nu)$ replaced by $\theta(\nu)$ in (47).

Table 1 presents the QLS estimated parameters of Model (46), their $p-$values (in parentheses) and their estimated standard errors (into brackets, under weak assumption on $\epsilon_{5n+\nu}$) of the squared returns of the CAC 40 and DAX indices. As expected all the estimated parameters are significant at any reasonable levels, except: $\hat{\mu}_4$ for the CAC 40 index; $\hat{\mu}_2$ (at 1% and 5% levels) and $\hat{\mu}_4$ for the DAX index. For all indices the mean $\hat{\mu}_1$ on Monday is positive and significant, it seems more possibly to talk of a global Monday effect. For the other days the means $\hat{\mu}_2$, $\hat{\mu}_3$, $\hat{\mu}_4$ and $\hat{\mu}_5$ are all negatives. Therefore Tuesday, Wednesday and Friday seem bad days for the CAC 40 and DAX (at 5% level) indices since they are significant.

For all indices the autoregressive coefficients $\hat{\phi}(\nu)$ are all positive and significant for all days. These coefficients are greater than one on Monday (for CAC 40 and DAX) and on Wednesday (for CAC 40). We also observe that the coefficients $\hat{\phi}(\nu)$ are the biggest on Monday for CAC 40 and DAX. Furthermore, with these two indices and the period considered, it is probably more appropriate to talk of a Monday effect. Additionally, Table 1 shows that there is evidence that the estimated noise standard deviations (estimated volatility) for all indices is considerably greater on Monday than the other days (Tuesday, Wednesday, Thursday and Friday), for which it is smaller and almost constant.

Note that from Table 1 and for all series, the product of the estimated coefficients $\hat{\phi}(\nu)$ (resp. $\hat{\theta}(\nu)$) are smaller than one. Thus in view of (47) $|z| > 1$ for all series (the smallest z verify

$|z| = 1.126 > 1$). So we think that the assumption **(A1)** is satisfied and thus our asymptotic normality theorem on the residual autocorrelations can be applied. After estimating the model (46), the next important step in the modeling consists in checking if the estimated model fits satisfactorily the data. Thus under the null hypothesis that the model has been correctly identified, the residuals are approximately a white noise.

We thus apply portmanteau tests to the residuals of Model (46) for each series. The results not reported here (see supplementary materials), reveal the conclusion that the strong $\mathrm{PARMA}_5(1,1)$ model is rejected by the standard global LBM test and even for a specified season ν at the nominal level $\alpha = 5\%$. By contrast a weak $\mathrm{PARMA}_5(1,1)$ model is not rejected. Note that for the first and second-order structures we found for the returns considered, namely a weak periodic white noise for the returns and a weak $\mathrm{PARMA}_5(1,1)$ model for the squares of the returns, seem compatible with a $\mathrm{PGARCH}(1,1)$ model.

To conclude our empirical investigations, a comparison of the two indices indicates that the DAX index is systematically more volatile on Tuesday and Thursday than the CAC 40 index.

TABLE 1

QLS estimates, their p−values (in parentheses) and their estimated standard errors (in brackets) of a weak $PARMA_5(1,1)$ model fitted to the mean corrected series of the squared returns of the CAC 40 and DAX indices.

Index	CAC 40				DAX			
NT	6830				6740			
Day	$\hat{\mu}_\nu$	$\hat{\phi}_\nu$	$\hat{\theta}_\nu$	$\hat{\sigma}_\nu$	$\hat{\mu}_\nu$	$\hat{\phi}_\nu$	$\hat{\theta}_\nu$	$\hat{\sigma}_\nu$
Monday	1.094 (0.000) [0.256]	2.183 (0.000) [0.225]	1.958 (0.000) [0.197]	8.844×10^{-8}	1.399 (0.000) [0.335]	2.062 (0.000) [0.315]	1.825 (0.000) [0.279]	11.964×10^{-8}
Tuesday	−0.363 (0.001) [0.113]	0.444 (0.004) [0.153]	0.411 (0.006) [0.148]	3.958×10^{-8}	−0.279 (0.098) [0.168]	0.954 (0.000) [0.182]	0.929 (0.000) [0.179]	5.458×10^{-8}
Wenesday	−0.325 (0.005) [0.116]	1.501 (0.007) [0.552]	1.501 (0.008) [0.567]	3.648×10^{-8}	−0.459 (0.000) [0.107]	0.702 (0.000) [0.095]	0.757 (0.000) [0.074]	3.367×10^{-8}
Thursday	−0.059 (0.730) [0.171]	0.840 (0.000) [0.219]	0.767 (0.000) [0.198]	5.979×10^{-8}	−0.214 (0.224) [0.176]	0.960 (0.000) [0.140]	0.846 (0.000) [0.140]	6.103×10^{-8}
Friday	−0.346 (0.007) [0.128]	0.727 (0.000) [0.165]	0.652 (0.000) [0.169]	4.491×10^{-8}	−0.447 (0.000) [0.115]	0.664 (0.000) [0.135]	0.595 (0.000) [0.132]	3.978×10^{-8}

TABLE 2

Empirical size of the standard and proposed tests: relative frequencies (in %) of rejection of a strong $PARMA_2(1,1)$ given by (44) with the parameter $\alpha_0 = (0,0,0,0)'$ and where the innovation $\epsilon_t \sim \mathcal{N}(0,1)$. When the relative rejection frequencies are outside the significant limits with probability 95% (resp. 99%), they are displayed in bold (resp. underlined) type.

Length N	Tests	Lag m											
		1	2	3	4	5	6	7	8	9	10	11	12
	$BP_s(1)$	5.0	4.5	5.2	4.7	5.2	5.2	5.5	5.7	5.7	5.8	5.9	6.0
	$BP_s(2)$	4.9	6.5	5.5	5.5	5.9	5.8	6.1	5.9	6.1	6.2	6.0	6.4
	$LBM_s(1)$	5.0	4.5	5.3	4.7	5.2	5.2	5.7	5.8	5.8	6.2	6.1	6.2
	$LBM_s(2)$	4.9	6.7	5.7	5.7	6.0	6.0	6.1	6.0	6.2	6.4	6.1	**6.7**
	$BP_W(1)$	4.3	3.6	3.6	3.5	3.6	4.0	4.2	4.0	3.8	4.4	4.4	4.1
	$BP_W(2)$	4.8	5.5	5.1	5.4	5.0	5.4	5.1	4.7	4.8	5.1	4.5	4.2
	$LBM_W(1)$	4.3	3.6	3.8	3.6	3.6	4.0	4.3	4.1	3.9	4.4	4.7	4.7
	$LBM_W(2)$	4.8	5.6	5.2	5.5	5.2	5.6	5.2	5.0	5.2	5.4	4.8	4.6
$N = 400$	$BP_{SN}(1)$	4.8	4.5	3.9	4.4	4.6	5.1	4.9	5.5	4.9	5.0	5.8	5.7
	$BP_{SN}(2)$	6.4	5.6	5.3	4.4	5.4	5.3	6.4	5.4	5.3	4.8	5.0	4.9
	$LBM_{SN}(1)$	4.8	4.5	4.0	4.4	4.7	5.1	5.0	5.7	4.9	5.0	5.9	6.2
	$LBM_{SN}(2)$	6.4	5.6	5.3	4.4	5.4	5.6	6.4	5.7	5.4	4.8	5.0	5.1
	BP_s	4.9	5.6	5.6	5.6	5.9	5.3	5.6	5.9	5.7	6.4	6.3	6.4
	LBM_s	4.9	5.6	5.6	5.7	6.2	5.6	5.7	6.4	6.1	$\underline{\mathbf{7.0}}$	**6.7**	**6.9**
	BP_W	4.3	4.4	4.2	4.4	4.2	3.6	3.7	3.7	3.9	3.7	$\underline{\mathbf{3.1}}$	3.4
	LBM_W	4.3	4.4	4.2	4.7	4.3	3.7	3.9	4.0	4.2	4.3	3.4	3.5
	BP_{SN}	5.9	4.6	4.9	5.4	5.4	5.4	5.8	5.5	5.4	4.9	5.4	4.8
	LBM_{SN}	6.1	4.7	5.1	5.4	5.4	5.8	5.8	5.8	5.5	5.1	5.5	5.2
		1	2	3	4	5	6	7	8	9	10	11	12
	$BP_s(1)$	5.8	5.8	5.0	5.3	5.7	5.3	6.0	5.6	5.4	5.6	5.9	**6.7**
	$BP_s(2)$	5.3	5.3	5.5	5.3	5.3	5.7	5.8	6.3	**6.5**	5.9	5.4	4.3
	$LBM_s(1)$	5.8	5.8	5.0	5.3	5.7	5.3	6.0	5.7	5.6	5.6	6.0	**6.7**
	$LBM_s(2)$	5.3	5.3	5.5	5.3	5.4	5.7	5.8	6.3	**6.6**	6.0	5.6	4.3
	$BP_W(1)$	5.5	5.9	4.9	5.5	5.5	4.9	5.6	5.6	5.0	5.4	5.7	5.8
	$BP_W(2)$	4.8	5.5	5.9	5.1	5.5	5.3	5.5	5.9	6.2	5.8	4.9	4.1
	$LBM_W(1)$	5.5	5.9	4.9	5.6	5.5	4.9	5.6	5.6	5.0	5.4	5.8	5.8
	$LBM_W(2)$	4.8	5.5	5.9	5.1	5.5	5.3	5.5	6.0	6.2	5.8	4.9	4.1
$N = 2,000$	$BP_{SN}(1)$	5.0	5.0	6.0	4.9	4.8	4.7	5.6	5.6	6.4	6.0	5.8	**6.7**
	$BP_{SN}(2)$	5.1	4.3	4.3	5.5	5.1	4.8	5.2	$\underline{\mathbf{7.3}}$	**6.9**	6.3	6.3	5.2
	$LBM_{SN}(1)$	5.0	5.0	6.0	5.0	4.8	4.7	5.6	5.7	6.4	6.0	5.8	**6.7**
	$LBM_{SN}(2)$	5.1	4.3	4.3	5.5	5.1	4.8	5.2	$\underline{\mathbf{7.3}}$	**6.9**	6.3	6.3	5.3
	BP_s	5.1	5.9	5.0	5.1	5.0	5.6	5.6	4.8	5.0	5.2	5.3	4.8
	LBM_s	5.1	5.9	5.1	5.1	5.0	5.6	5.7	4.8	5.0	5.3	5.4	4.8
	BP_W	5.6	6.0	4.8	5.0	5.2	5.3	4.9	4.6	4.3	4.6	4.9	4.3
	LBM_W	5.6	6.0	4.8	5.1	5.3	5.3	4.9	4.6	4.4	4.6	5.0	4.4
	BP_{SN}	5.2	4.5	5.2	4.8	4.6	4.9	4.7	5.5	6.0	**6.8**	**6.9**	5.6
	LBM_{SN}	5.2	4.5	5.2	4.8	4.6	5.0	4.8	5.6	6.0	**6.8**	**6.9**	5.7

TABLE 3

Empirical size of the standard and proposed tests: relative frequencies (in %) of rejection of a weak $PARMA_2(1,1)$ given by (44) with the parameter $\alpha_0 = (0,0,0,0)'$ and where the innovation $\epsilon_t \sim$ (45). When the relative rejection frequencies are outside the significant limits with probability 95% (resp. 99%), they are displayed in bold (resp. underlined) type.

Length N	Tests	Lag m											
		1	2	3	4	5	6	7	8	9	10	11	12
	$\mathrm{BP_s}(1)$	$\underline{\mathbf{13.2}}$	$\underline{\mathbf{12.5}}$	$\underline{\mathbf{10.8}}$	$\underline{\mathbf{9.5}}$	$\underline{\mathbf{9.4}}$	$\underline{\mathbf{9.0}}$	$\underline{\mathbf{9.0}}$	$\underline{\mathbf{8.2}}$	$\underline{\mathbf{7.3}}$	$\underline{\mathbf{7.5}}$	$\underline{\mathbf{7.4}}$	$\underline{\mathbf{7.4}}$
	$\mathrm{BP_s}(2)$	$\underline{\mathbf{10.5}}$	$\underline{\mathbf{8.9}}$	$\underline{\mathbf{8.0}}$	$\underline{\mathbf{7.2}}$	$\underline{\mathbf{7.9}}$	$\underline{\mathbf{8.1}}$	$\underline{\mathbf{7.9}}$	$\underline{\mathbf{7.5}}$	**6.9**	$\underline{\mathbf{7.6}}$	$\underline{\mathbf{7.7}}$	$\underline{\mathbf{7.9}}$
	$\mathrm{LBM_s}(1)$	$\underline{\mathbf{13.2}}$	$\underline{\mathbf{12.9}}$	$\underline{\mathbf{10.8}}$	$\underline{\mathbf{9.6}}$	$\underline{\mathbf{9.4}}$	$\underline{\mathbf{9.1}}$	$\underline{\mathbf{9.1}}$	$\underline{\mathbf{8.3}}$	$\underline{\mathbf{7.5}}$	$\underline{\mathbf{7.8}}$	$\underline{\mathbf{7.7}}$	$\underline{\mathbf{8.0}}$
	$\mathrm{LBM_s}(2)$	$\underline{\mathbf{10.5}}$	$\underline{\mathbf{8.9}}$	$\underline{\mathbf{8.3}}$	$\underline{\mathbf{7.2}}$	$\underline{\mathbf{8.1}}$	$\underline{\mathbf{8.1}}$	$\underline{\mathbf{8.0}}$	$\underline{\mathbf{7.7}}$	$\underline{\mathbf{7.3}}$	$\underline{\mathbf{7.8}}$	$\underline{\mathbf{8.3}}$	$\underline{\mathbf{8.4}}$
	$\mathrm{BP_w}(1)$	4.4	3.9	3.4	$\underline{\mathbf{3.1}}$	**3.3**	$\underline{\mathbf{3.0}}$	$\underline{\mathbf{2.5}}$	$\underline{\mathbf{2.7}}$	$\underline{\mathbf{2.8}}$	$\underline{\mathbf{2.6}}$	$\underline{\mathbf{2.6}}$	$\underline{\mathbf{3.0}}$
	$\mathrm{BP_w}(2)$	3.6	4.0	3.8	3.5	**3.3**	$\underline{\mathbf{3.2}}$	3.6	$\underline{\mathbf{3.2}}$	$\underline{\mathbf{3.0}}$	$\underline{\mathbf{2.4}}$	$\underline{\mathbf{2.7}}$	$\underline{\mathbf{2.9}}$
	$\mathrm{LBM_w}(1)$	4.4	4.0	3.5	**3.3**	3.6	$\underline{\mathbf{3.0}}$	$\underline{\mathbf{2.7}}$	$\underline{\mathbf{2.7}}$	$\underline{\mathbf{2.9}}$	$\underline{\mathbf{2.9}}$	$\underline{\mathbf{2.8}}$	$\underline{\mathbf{3.0}}$
	$\mathrm{LBM_w}(2)$	3.6	4.0	3.8	3.7	**3.3**	3.4	3.7	**3.3**	$\underline{\mathbf{3.0}}$	$\underline{\mathbf{2.7}}$	$\underline{\mathbf{2.8}}$	$\underline{\mathbf{3.0}}$
$N = 400$	$\mathrm{BP_{SN}}(1)$	4.8	4.3	4.4	5.2	4.6	4.4	5.2	4.5	3.9	3.7	3.6	**3.3**
	$\mathrm{BP_{SN}}(2)$	3.9	4.1	3.8	4.0	3.5	4.3	4.7	4.8	3.9	3.9	4.0	4.6
	$\mathrm{LBM_{SN}}(1)$	4.8	4.3	4.4	5.2	4.7	4.4	5.4	4.5	4.1	4.0	3.7	**3.3**
	$\mathrm{LBM_{SN}}(2)$	3.9	4.1	3.8	4.0	3.6	4.5	4.8	5.1	3.9	4.1	4.0	4.7
	$\mathrm{BP_s}$	$\underline{\mathbf{14.3}}$	$\underline{\mathbf{12.6}}$	$\underline{\mathbf{11.6}}$	$\underline{\mathbf{10.2}}$	$\underline{\mathbf{10.4}}$	$\underline{\mathbf{10.3}}$	$\underline{\mathbf{10.5}}$	$\underline{\mathbf{10.3}}$	$\underline{\mathbf{10.0}}$	$\underline{\mathbf{9.8}}$	$\underline{\mathbf{8.7}}$	$\underline{\mathbf{9.1}}$
	$\mathrm{LBM_s}$	$\underline{\mathbf{14.3}}$	$\underline{\mathbf{12.7}}$	$\underline{\mathbf{11.7}}$	$\underline{\mathbf{10.7}}$	$\underline{\mathbf{10.6}}$	$\underline{\mathbf{10.5}}$	$\underline{\mathbf{10.6}}$	$\underline{\mathbf{10.5}}$	$\underline{\mathbf{10.3}}$	$\underline{\mathbf{10.3}}$	$\underline{\mathbf{9.1}}$	$\underline{\mathbf{9.4}}$
	$\mathrm{BP_w}$	$\underline{\mathbf{3.2}}$	$\underline{\mathbf{2.7}}$	$\underline{\mathbf{2.3}}$	$\underline{\mathbf{2.4}}$	3.4	$\underline{\mathbf{2.7}}$	$\underline{\mathbf{1.7}}$	$\underline{\mathbf{2.0}}$	$\underline{\mathbf{2.5}}$	$\underline{\mathbf{1.5}}$	$\underline{\mathbf{2.2}}$	$\underline{\mathbf{1.5}}$
	$\mathrm{LBM_w}$	$\underline{\mathbf{3.2}}$	$\underline{\mathbf{2.8}}$	$\underline{\mathbf{2.3}}$	$\underline{\mathbf{2.4}}$	3.4	$\underline{\mathbf{2.8}}$	$\underline{\mathbf{2.0}}$	$\underline{\mathbf{2.2}}$	$\underline{\mathbf{2.5}}$	$\underline{\mathbf{2.1}}$	$\underline{\mathbf{2.5}}$	$\underline{\mathbf{1.7}}$
	$\mathrm{BP_{SN}}$	4.6	5.5	4.3	3.4	3.5	3.5	$\underline{\mathbf{3.1}}$	**3.3**	$\underline{\mathbf{2.8}}$	$\underline{\mathbf{2.0}}$	$\underline{\mathbf{1.4}}$	$\underline{\mathbf{1.2}}$
	$\mathrm{LBM_{SN}}$	4.7	5.5	4.4	3.5	3.6	3.7	3.6	3.4	$\underline{\mathbf{3.2}}$	$\underline{\mathbf{2.2}}$	$\underline{\mathbf{1.4}}$	$\underline{\mathbf{1.3}}$
		1	2	3	4	5	6	7	8	9	10	11	12
	$\mathrm{BP_s}(1)$	$\underline{\mathbf{16.6}}$	$\underline{\mathbf{14.8}}$	$\underline{\mathbf{12.9}}$	$\underline{\mathbf{11.7}}$	$\underline{\mathbf{12.1}}$	$\underline{\mathbf{11.3}}$	$\underline{\mathbf{11.8}}$	$\underline{\mathbf{11.9}}$	$\underline{\mathbf{10.9}}$	$\underline{\mathbf{10.6}}$	$\underline{\mathbf{10.2}}$	$\underline{\mathbf{10.3}}$
	$\mathrm{BP_s}(2)$	$\underline{\mathbf{12.8}}$	$\underline{\mathbf{10.5}}$	$\underline{\mathbf{9.9}}$	$\underline{\mathbf{9.9}}$	$\underline{\mathbf{8.8}}$	$\underline{\mathbf{8.5}}$	$\underline{\mathbf{8.1}}$	$\underline{\mathbf{7.6}}$	$\underline{\mathbf{7.1}}$	**6.9**	6.3	**6.6**
	$\mathrm{LBM_s}(1)$	$\underline{\mathbf{16.6}}$	$\underline{\mathbf{14.8}}$	$\underline{\mathbf{12.9}}$	$\underline{\mathbf{11.8}}$	$\underline{\mathbf{12.2}}$	$\underline{\mathbf{11.4}}$	$\underline{\mathbf{11.8}}$	$\underline{\mathbf{11.9}}$	$\underline{\mathbf{11.0}}$	$\underline{\mathbf{10.6}}$	$\underline{\mathbf{10.3}}$	$\underline{\mathbf{10.4}}$
	$\mathrm{LBM_s}(2)$	$\underline{\mathbf{12.8}}$	$\underline{\mathbf{10.5}}$	$\underline{\mathbf{9.9}}$	$\underline{\mathbf{9.9}}$	$\underline{\mathbf{8.9}}$	$\underline{\mathbf{8.6}}$	$\underline{\mathbf{8.1}}$	$\underline{\mathbf{7.6}}$	$\underline{\mathbf{7.1}}$	**6.9**	6.3	**6.7**
	$\mathrm{BP_w}(1)$	5.4	4.0	4.1	4.5	4.9	5.0	4.5	4.6	4.5	4.8	4.7	5.0
	$\mathrm{BP_w}(2)$	5.5	5.3	4.6	4.5	4.0	4.4	4.1	3.6	3.7	**3.3**	$\underline{\mathbf{3.2}}$	3.4
	$\mathrm{LBM_w}(1)$	5.4	4.0	4.1	4.5	4.9	5.0	4.5	4.6	4.5	4.8	4.8	5.1
	$\mathrm{LBM_w}(2)$	5.5	5.3	4.6	4.5	4.0	4.4	4.2	3.6	3.7	3.4	$\underline{\mathbf{3.2}}$	3.4
$N = 2,000$	$\mathrm{BP_{SN}}(1)$	5.7	4.3	4.7	4.6	5.1	5.0	5.4	5.6	5.4	5.5	5.2	5.4
	$\mathrm{BP_{SN}}(2)$	6.0	5.5	5.0	4.6	5.1	6.0	5.1	5.3	4.8	4.9	5.2	5.1
	$\mathrm{LBM_{SN}}(1)$	5.7	4.3	4.7	4.6	5.2	5.0	5.4	5.6	5.5	5.6	5.3	5.4
	$\mathrm{LBM_{SN}}(2)$	6.0	5.5	5.0	4.6	5.1	6.0	5.1	5.4	4.8	4.9	5.2	5.1
	$\mathrm{BP_s}$	$\underline{\mathbf{18.3}}$	$\underline{\mathbf{15.9}}$	$\underline{\mathbf{13.9}}$	$\underline{\mathbf{12.5}}$	$\underline{\mathbf{12.1}}$	$\underline{\mathbf{11.9}}$	$\underline{\mathbf{11.2}}$	$\underline{\mathbf{11.3}}$	$\underline{\mathbf{10.6}}$	$\underline{\mathbf{9.9}}$	$\underline{\mathbf{10.0}}$	$\underline{\mathbf{10.4}}$
	$\mathrm{LBM_s}$	$\underline{\mathbf{18.3}}$	$\underline{\mathbf{15.9}}$	$\underline{\mathbf{13.9}}$	$\underline{\mathbf{12.5}}$	$\underline{\mathbf{12.1}}$	$\underline{\mathbf{11.9}}$	$\underline{\mathbf{11.3}}$	$\underline{\mathbf{11.3}}$	$\underline{\mathbf{10.6}}$	$\underline{\mathbf{10.0}}$	$\underline{\mathbf{10.1}}$	$\underline{\mathbf{10.4}}$
	$\mathrm{BP_w}$	4.9	4.1	3.9	4.2	4.3	3.8	4.0	4.0	3.7	3.7	3.5	3.7
	$\mathrm{LBM_w}$	4.9	4.1	3.9	4.2	4.3	3.8	4.0	4.0	3.7	3.7	3.6	3.7
	$\mathrm{BP_{SN}}$	5.6	4.5	4.7	4.8	5.8	4.7	5.3	5.4	4.5	4.1	4.7	5.2
	$\mathrm{LBM_{SN}}$	5.6	4.5	4.7	4.8	5.8	4.7	5.4	5.5	4.5	4.1	4.7	5.2

Table 4

Modified and standard versions of portmanteau tests to check the null hypothesis that the returns follow a periodic white noise, based on m residuals autocorrelations. The p-values less than 5% are in bold, those less than 1% are underlined.

Index	Tests	Lag m											
		1	2	3	4	5	6	7	8	9	10	11	12
CAC 40	$\mathrm{LBM}_{\mathrm{S}}(1)$	0.083	0.113	0.210	0.335	0.468	0.210	0.277	0.178	0.225	0.286	0.351	0.431
	$\mathrm{LBM}_{\mathrm{S}}(2)$	**0.001**	**0.000**	**0.000**	**0.001**	**0.002**	**0.003**	**0.005**	**0.009**	**0.010**	**0.014**	**0.014**	**0.016**
	$\mathrm{LBM}_{\mathrm{S}}(3)$	0.617	0.806	0.508	0.675	0.051	0.068	0.079	0.104	0.116	0.147	0.189	0.083
	$\mathrm{LBM}_{\mathrm{S}}(4)$	0.504	0.570	0.276	0.424	0.516	0.508	0.253	0.248	0.310	0.393	0.461	0.436
	$\mathrm{LBM}_{\mathrm{S}}(5)$	0.423	0.674	0.324	0.079	0.071	0.075	**0.010**	**0.018**	**0.031**	**0.047**	0.066	**0.016**
	$\mathrm{LBM}_{\mathrm{W}}(1)$	0.263	0.435	0.570	0.710	0.797	0.641	0.695	0.619	0.664	0.711	0.755	0.817
	$\mathrm{LBM}_{\mathrm{W}}(2)$	**0.005**	**0.007**	**0.007**	**0.015**	**0.039**	0.054	0.085	0.111	0.129	0.170	0.159	0.176
	$\mathrm{LBM}_{\mathrm{W}}(3)$	0.782	0.905	0.766	0.872	0.374	0.414	0.436	0.501	0.523	0.591	0.654	0.522
	$\mathrm{LBM}_{\mathrm{W}}(4)$	0.631	0.760	0.607	0.745	0.833	0.840	0.671	0.675	0.740	0.814	0.848	0.823
	$\mathrm{LBM}_{\mathrm{W}}(5)$	0.604	0.842	0.678	0.433	0.428	0.419	0.236	0.295	0.345	0.389	0.446	0.293
	$\mathrm{LBM}_{\mathrm{SN}}(1)$	8.8	22.3	32.5	33.1	61.2	61.5	62.3	86.8	253.2	262.0	295.6	333.8
	$\mathrm{LBM}_{\mathrm{SN}}(2)$	75.0	152.0	156.6	370.1	406.7	410.1	445.7	905.0	906.0	907.0	1071.4	1208.8
	$\mathrm{LBM}_{\mathrm{SN}}(3)$	0.9	10.6	10.9	16.2	126.6	136.7	137.9	158.9	178.7	189.4	203.6	216.5
	$\mathrm{LBM}_{\mathrm{SN}}(4)$	1.3	11.6	49.7	57.6	60.7	150.3	375.1	382.4	383.3	383.3	430.5	462.0
	$\mathrm{LBM}_{\mathrm{SN}}(5)$	2.7	31.5	55.8	127.8	166.0	187.3	188.4	327.8	403.3	544.2	548.1	699.1
	$\mathrm{LBM}_{\mathrm{S}}$	**0.011**	**0.016**	**0.005**	**0.009**	**0.003**	**0.002**	**0.001**	**0.001**	**0.002**	**0.007**	**0.015**	**0.004**
	$\mathrm{LBM}_{\mathrm{W}}$	0.247	0.426	0.420	0.513	0.472	0.456	0.359	0.404	0.494	0.610	0.668	0.572
	$\mathrm{LBM}_{\mathrm{SN}}$	201.6	367.2	741.2	1334.8	1660.1	2363.3	2425.7	3318.8	4241.9	5046.5	5889.5	6504.8
		1	2	3	4	5	6	7	8	9	10	11	12
DAX	$\mathrm{LBM}_{\mathrm{S}}(1)$	0.166	0.217	0.247	0.360	0.498	0.524	0.602	0.499	0.506	0.600	0.367	0.368
	$\mathrm{LBM}_{\mathrm{S}}(2)$	**0.016**	**0.014**	**0.033**	0.062	0.067	**0.016**	**0.026**	**0.010**	**0.016**	**0.002**	**0.003**	**0.004**
	$\mathrm{LBM}_{\mathrm{S}}(3)$	0.298	0.408	0.393	0.328	0.147	0.217	0.298	0.194	0.266	0.127	0.075	0.085
	$\mathrm{LBM}_{\mathrm{S}}(4)$	0.205	**0.005**	**0.008**	**0.009**	**0.001**	**0.002**	**0.002**	**0.004**	**0.005**	**0.009**	**0.003**	**0.003**
	$\mathrm{LBM}_{\mathrm{S}}(5)$	0.256	0.148	0.236	0.295	0.052	0.087	0.070	0.089	0.128	0.090	0.103	0.018
	$\mathrm{LBM}_{\mathrm{W}}(1)$	0.350	0.496	0.546	0.674	0.773	0.806	0.849	0.800	0.799	0.859	0.751	0.773
	$\mathrm{LBM}_{\mathrm{W}}(2)$	**0.048**	0.123	0.175	0.273	0.329	0.205	0.271	0.198	0.230	0.116	0.137	0.199
	$\mathrm{LBM}_{\mathrm{W}}(3)$	0.555	0.685	0.738	0.749	0.643	0.733	0.798	0.725	0.787	0.690	0.628	0.650
	$\mathrm{LBM}_{\mathrm{W}}(4)$	0.437	0.140	0.190	0.237	0.177	0.210	0.227	0.253	0.279	0.322	0.247	0.246
	$\mathrm{LBM}_{\mathrm{W}}(5)$	0.438	0.437	0.560	0.651	0.356	0.420	0.410	0.443	0.503	0.446	0.475	0.283
	$\mathrm{LBM}_{\mathrm{SN}}(1)$	13.7	14.0	66.0	117.0	117.3	121.1	121.4	148.5	180.7	181.0	209.0	252.7
	$\mathrm{LBM}_{\mathrm{SN}}(2)$	18.9	40.8	43.7	62.6	62.6	63.0	113.3	116.1	117.2	179.0	182.7	187.7
	$\mathrm{LBM}_{\mathrm{SN}}(3)$	1.9	16.6	55.4	89.4	137.6	172.1	189.5	218.1	218.2	357.6	412.9	462.7
	$\mathrm{LBM}_{\mathrm{SN}}(4)$	1.9	50.2	86.8	140.6	140.9	170.3	171.5	222.7	300.2	316.6	319.5	390.1
	$\mathrm{LBM}_{\mathrm{SN}}(5)$	4.8	83.5	84.7	194.1	208.5	209.8	259.8	330.4	359.3	392.6	430.2	430.8
	$\mathrm{LBM}_{\mathrm{S}}$	**0.040**	**0.002**	**0.006**	**0.014**	**0.001**	**0.001**	**0.001**	**0.001**	**0.002**	**0.000**	**0.000**	**0.000**
	$\mathrm{LBM}_{\mathrm{W}}$	0.407	0.307	0.470	0.622	0.455	0.481	0.558	0.509	0.586	0.479	0.405	0.399
	$\mathrm{LBM}_{\mathrm{SN}}$	143.5	541.5	736.3	960.5	1644.0	2049.5	2363.4	3694.2	3982.4	5504.6	6356.8	6779.2

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Data availability statement

The data used in Section 5 are available in the supporting information of this article. It is openly available from the website Yahoo Finance: http://fr.finance.yahoo.com/.

Supporting information

Additional Supporting Information may be found online in the supporting information tab for this article.

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Portmanteau tests for periodic ARMA models with dependent errors

Supplementary Appendix

Appendix A: Proofs

A.1. Proof of Proposition 2

First we remark that the asymptotic normality of the joint distribution of $\sqrt{N}(\hat{\beta}_{\mathrm{LS}}'-\beta_0', \gamma_m'(\nu))'$ can be established along the same lines as the proof of Theorem 2 in Francq et al. (2011). The detailed proof is omitted. From (21) and (28) we have

$$\sqrt{N}\begin{pmatrix}\hat{\beta}_{\mathrm{LS}}-\beta_0\\ \gamma_m(\nu)\end{pmatrix} = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\begin{pmatrix}-J(\beta_0,\omega^2)^{-1}\sum_{\nu=1}^{T}\omega_\nu^{-2}\epsilon_{nT+\nu}\frac{\partial \epsilon_{nT+\nu}(\beta_0)}{\partial\beta}\\ (\epsilon_{nT+\nu-1},\dots,\epsilon_{nT+\nu-m})'\epsilon_{nT+\nu}\end{pmatrix}+\begin{pmatrix}\mathrm{o}_{\mathbb{P}}(1)\\ \mathbf{0}_m\end{pmatrix}$$
$$= \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}W_{nT+\nu}+\mathrm{o}_{\mathbb{P}}(1),$$

where $\mathbf{0}_m$ is the vector of $\mathbb{R}^{m\times 1}$ with zero components. It is clear that $W_{nT+\nu}$ is a measurable function of $\epsilon_{nT+\nu}, \epsilon_{nT+\nu-1}, \dots$ Thus by using the same arguments as in Francq et al. (2011) (see proof of Theorem 2), the central limit theorem (CLT) for strongly mixing processes $(W_{nT+\nu})_{n\in\mathbb{Z}}$ of Herrndorf (1984) implies that $(1/\sqrt{N})\sum_{n=0}^{N-1}W_{nT+\nu}$ has a limiting normal distribution with mean 0 and covariance matrix $\Xi(\nu)$.

In view of (28), by applying the CLT for mixing processes we directly obtain

$$\Omega_{\beta_0}^{\mathrm{LS}} = \lim_{N\to\infty}\mathrm{Var}\left(-\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}J(\beta_0,\omega^2)^{-1}\sum_{\nu=1}^{T}\omega_\nu^{-2}\epsilon_{nT+\nu}\frac{\partial\epsilon_{nT+\nu}(\beta_0)}{\partial\beta}\right)$$
$$:= J(\beta_0,\omega^2)^{-1}I(\beta_0,\omega^2)J(\beta_0,\omega^2)^{-1},$$

which gives the first block of the asymptotic covariance matrix of Proposition 2.

By the stationarity of $(\boldsymbol{\epsilon}_n)$ and Lebesgue's dominated convergence theorem, we obtain that for $l, l^{'} \geq 1$

$$\begin{aligned}
\mathrm{Cov}\left(\sqrt{N}\gamma(\nu,l),\sqrt{N}\gamma(\nu,l')\right) &= \mathrm{Cov}\left(\frac{1}{\sqrt{N}}\sum_{n=l+1}^{N}\epsilon_{nT+\nu}\epsilon_{nT+\nu-l},\frac{1}{\sqrt{N}}\sum_{n'=l'+1}^{N}\epsilon_{n'T+\nu}\epsilon_{n'T+\nu-l'}\right)\\
&= \frac{1}{N}\sum_{n=l+1}^{N}\sum_{n'=l'+1}^{N}\mathbb{E}\left(\epsilon_{nT+\nu}\epsilon_{nT+\nu-l}\epsilon_{n'T+\nu}\epsilon_{n'T+\nu-l'}\right)\\
&= \frac{1}{N}\sum_{h=-N+1}^{N-1}(N-|h|)\mathbb{E}\left(\epsilon_{nT+\nu}\epsilon_{nT+\nu-l}\epsilon_{(n-h)T+\nu}\epsilon_{(n-h)T+\nu-l'}\right)\\
&\xrightarrow[N\to\infty]{} \sum_{h=-\infty}^{\infty}\mathbb{E}\left(\epsilon_{nT+\nu}\epsilon_{nT+\nu-l}\epsilon_{(n-h)T+\nu}\epsilon_{(n-h)T+\nu-l'}\right) := \Gamma_{\nu}(l,l').
\end{aligned}$$

We thus have $\Sigma_{\gamma_m(\nu)} = [\Gamma_\nu(l,l^{'})]_{1\leq l,l'\leq m}$.

Let $Y_n = \sum_{\nu=1}^{T}\omega_\nu^{-2}\epsilon_{nT+\nu}\partial\epsilon_{nT+\nu}(\beta_0)/\partial\beta$. Finally by the stationarity of $(\boldsymbol{\epsilon}_n)$ and $(\epsilon_{nT+\nu}\partial\epsilon_{nT+\nu}(\beta_0)/\partial\beta)_{n\in\mathbb{Z}}$ we have

$$\begin{aligned}
&\mathrm{Cov}\left(-J(\beta_0,\omega^2)^{-1}\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}Y_n,\sqrt{N}\gamma(\nu,l)\right)\\
&\quad= \mathrm{Cov}\left(-J(\beta_0,\omega^2)^{-1}\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\sum_{\nu=1}^{T}\omega_\nu^{-2}\epsilon_{nT+\nu}\frac{\partial\epsilon_{nT+\nu}(\beta_0)}{\partial\beta},\frac{1}{\sqrt{N}}\sum_{n'=l}^{N-1}\epsilon_{n'T+\nu}\epsilon_{n'T+\nu-l}\right)\\
&\quad= -J(\beta_0,\omega^2)^{-1}\sum_{\nu'=1}^{T}\omega_\nu^{-2}\frac{1}{N}\sum_{n=0}^{N-1}\sum_{n'=l}^{N-1}\mathbb{E}\left(\epsilon_{nT+\nu}\frac{\partial\epsilon_{nT+\nu}(\beta_0)}{\partial\beta}\epsilon_{n'T+\nu}\epsilon_{n'T+\nu-l}\right)\\
&\quad= -J(\beta_0,\omega^2)^{-1}\sum_{\nu=1}^{T}\omega_\nu^{-2}\frac{1}{N}\sum_{h=-N+1}^{N-1}(N-|h|)\mathbb{E}\left(\epsilon_{nT+\nu}\frac{\partial\epsilon_{nT+\nu}(\beta_0)}{\partial\beta}\epsilon_{(n-h)T+\nu}\epsilon_{(n-h)T+\nu-l}\right)\\
&\quad\xrightarrow[N\to\infty]{} -J(\beta_0,\omega^2)^{-1}\sum_{\nu=1}^{T}\omega_\nu^{-2}\sum_{h=-\infty}^{\infty}\mathbb{E}\left(\epsilon_{nT+\nu}\frac{\partial\epsilon_{nT+\nu}(\beta_0)}{\partial\beta}\epsilon_{(n-h)T+\nu}\epsilon_{(n-h)T+\nu-l}\right) := \Sigma_{\hat{\beta},\gamma_m(\nu)}(\cdot,l),
\end{aligned}$$

by the dominated convergence theorem.

Note that the existence of the above matrices is a consequence of Assumption **(A4)** and of Davydov's inequality (see Davydov (1968)). The proof is then complete. □

A.2. Proof of Theorem 3.1

The joint asymptotic distribution of $\sqrt{N}\gamma_m(\nu)$ and $\sqrt{N}(\hat{\beta}_{\mathrm{LS}}-\beta_0)$ obtained in Proposition 2 shows that $\sqrt{N}\hat{\gamma}_m(\nu)$ has a limiting normal distribution with mean zero and covariance matrix

$$\begin{aligned}\lim_{N\to\infty}\mathrm{Var}\left(\sqrt{N}\hat{\gamma}_m(\nu)\right) &= \lim_{N\to\infty}\mathrm{Var}\left(\sqrt{N}\gamma_m(\nu)\right)+\Psi_m(\nu)\lim_{N\to\infty}\mathrm{Var}\left(\sqrt{N}(\hat{\beta}_{\mathrm{LS}}-\beta_0)\right)\Psi_m^{'}(\nu)\\ &\quad+\Psi_m(\nu)\lim_{N\to\infty}\mathrm{Cov}\left(\sqrt{N}(\hat{\beta}_{\mathrm{LS}}-\beta_0),\sqrt{N}\gamma_m(\nu)\right)\\ &\quad+\lim_{N\to\infty}\mathrm{Cov}\left(\sqrt{N}\gamma_m(\nu),\sqrt{N}(\hat{\beta}_{\mathrm{LS}}-\beta_0)\right)\Psi_m^{'}(\nu)\\ &=\Sigma_{\gamma_m(\nu)}+\Psi_m(\nu)\Omega_{\beta_0}^{\mathrm{LS}}\Psi_m^{'}(\nu)+\Psi_m(\nu)\Sigma_{\hat{\beta},\gamma_m(\nu)}+\Sigma_{\hat{\beta},\gamma_m(\nu)}^{'}\Psi_m^{'}(\nu).\end{aligned}$$

By the stationarity of $(\boldsymbol{\epsilon}_n)$ and Lebesgue's dominated convergence theorem, we obtain that for $l,l^{'}\geq 1$

$$\begin{aligned}\mathrm{Cov}\left(\sqrt{N}\gamma(\nu,l),\sqrt{N}\gamma(\nu',l')\right) &= \mathrm{Cov}\left(\frac{1}{\sqrt{N}}\sum_{n=l+1}^{N}\epsilon_{nT+\nu}\epsilon_{nT+\nu-l},\frac{1}{\sqrt{N}}\sum_{n'=l'+1}^{N}\epsilon_{n'T+\nu'}\epsilon_{n'T+\nu'-l'}\right)\\ &=\frac{1}{N}\sum_{n=l+1}^{N}\sum_{n'=l'+1}^{N}\mathbb{E}\left(\epsilon_{nT+\nu}\epsilon_{nT+\nu-l}\epsilon_{n'T+\nu'}\epsilon_{n'T+\nu'-l'}\right)\\ &=\frac{1}{N}\sum_{h=-N+1}^{N-1}(N-|h|)\mathbb{E}\left(\epsilon_{nT+\nu}\epsilon_{nT+\nu-l}\epsilon_{(n-h)T+\nu'}\epsilon_{(n-h)T+\nu'-l'}\right)\\ &\xrightarrow[N\to\infty]{}\sum_{h=-\infty}^{\infty}\mathbb{E}\left(\epsilon_{nT+\nu}\epsilon_{nT+\nu-l}\epsilon_{(n-h)T+\nu'}\epsilon_{(n-h)T+\nu'-l'}\right):=\Gamma_{\nu,\nu'}(l,l').\end{aligned}$$

We thus have $\Sigma_{\gamma_m}(\nu,\nu')=[\Gamma_{\nu,\nu'}(l,l^{'})]_{1\leq l,l'\leq m}$.

Finally by the stationarity of $(\boldsymbol{\epsilon}_n)$ and $(\epsilon_{nT+\nu}\partial\epsilon_{nT+\nu}(\beta_0)/\partial\beta)_{n\in\mathbb{Z}}$ we have

$$\begin{aligned}&\mathrm{Cov}\left(\sqrt{N}(\hat{\beta}_{\mathrm{LS}}-\beta_0),\sqrt{N}\gamma(\nu',l)\right)\\ &\quad=\mathrm{Cov}\left(-J(\beta_0,\omega^2)^{-1}\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\sum_{\nu=1}^{T}\omega_\nu^{-2}\epsilon_{nT+\nu}\frac{\partial\epsilon_{nT+\nu}(\beta_0)}{\partial\beta},\frac{1}{\sqrt{N}}\sum_{n'=l}^{N-1}\epsilon_{n'T+\nu'}\epsilon_{n'T+\nu'-l}\right)\\ &\quad=-J(\beta_0,\omega^2)^{-1}\sum_{\nu'=1}^{T}\omega_\nu^{-2}\frac{1}{N}\sum_{n=0}^{N-1}\sum_{n'=l}^{N-1}\mathbb{E}\left(\epsilon_{nT+\nu}\frac{\partial\epsilon_{nT+\nu}(\beta_0)}{\partial\beta}\epsilon_{n'T+\nu'}\epsilon_{n'T+\nu'-l}\right)\\ &\quad=-J(\beta_0,\omega^2)^{-1}\sum_{\nu=1}^{T}\omega_\nu^{-2}\frac{1}{N}\sum_{h=-N+1}^{N-1}(N-|h|)\mathbb{E}\left(\epsilon_{nT+\nu}\frac{\partial\epsilon_{nT+\nu}(\beta_0)}{\partial\beta}\epsilon_{(n-h)T+\nu'}\epsilon_{(n-h)T+\nu'-l}\right)\\ &\quad\xrightarrow[N\to\infty]{}-J(\beta_0,\omega^2)^{-1}\sum_{\nu=1}^{T}\omega_\nu^{-2}\sum_{h=-\infty}^{\infty}\mathbb{E}\left(\epsilon_{nT+\nu}\frac{\partial\epsilon_{nT+\nu}(\beta_0)}{\partial\beta}\epsilon_{(n-h)T+\nu'}\epsilon_{(n-h)T+\nu'-l}\right):=\Sigma_{\hat{\beta},\gamma_m}(\nu,\nu')(\cdot,l),\end{aligned}$$

by the dominated convergence theorem. We then deduce that

$$\begin{aligned}
\lim_{N\to\infty} \mathrm{Cov}\left(\sqrt{N}\hat{\gamma}_m(\nu), \sqrt{N}\hat{\gamma}_m(\nu')\right) &= \lim_{N\to\infty} \mathrm{Cov}\left(\sqrt{N}\gamma_m(\nu), \sqrt{N}\gamma_m(\nu')\right) \\
&\quad + \Psi_m(\nu) \lim_{N\to\infty} \mathrm{Var}\left(\sqrt{N}(\hat{\beta}_{\mathrm{LS}} - \beta_0)\right) \Psi_m^{'}(\nu') \\
&\quad + \Psi_m(\nu) \lim_{N\to\infty} \mathrm{Cov}\left(\sqrt{N}(\hat{\beta}_{\mathrm{LS}} - \beta_0), \sqrt{N}\gamma_m(\nu')\right) \\
&\quad + \lim_{N\to\infty} \mathrm{Cov}\left(\sqrt{N}\gamma_m(\nu), \sqrt{N}(\hat{\beta}_{\mathrm{LS}} - \beta_0)\right) \Psi_m^{'}(\nu') \\
&= \Sigma_{\gamma_m}(\nu,\nu') + \Psi_m(\nu)\Omega_{\beta_0}^{\mathrm{LS}}\Psi_m^{'}(\nu') + \Psi_m(\nu)\Sigma_{\hat{\beta},\gamma_m}(\nu,\nu') \\
&\quad + \Sigma_{\hat{\beta},\gamma_m(\nu)}^{'}\Psi_m^{'}(\nu') \\
&=: \nabla_{\hat{\gamma}_m}(\nu,\nu').
\end{aligned}$$

We come back to the vector $\hat{\rho}_m(\nu)$ and observe that from (24), we have $\sqrt{N}(\hat{\gamma}(\nu,0) - \gamma(\nu,0)) = \mathrm{o}_{\mathbb{P}}(1)$. Applying the CLT for mixing processes (see Herrndorf (1984)) to the process $(\epsilon_t^2)_t$, we obtain

$$\sqrt{N}\left(\hat{\sigma}_\nu^2 - \sigma_\nu^2\right) = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\left(\epsilon_{nT+\nu}^2 - \mathbb{E}[\epsilon_{nT+\nu}^2]\right) + \mathrm{o}_{\mathbb{P}}(1) \xrightarrow[N\to\infty]{\text{in law}} \mathcal{N}\left(0, \sum_{h=-\infty}^{\infty}\mathrm{Cov}\left(\epsilon_\nu^2, \epsilon_{hT+\nu}^2\right)\right).$$

So we have $\sqrt{N}(\hat{\sigma}_\nu^2 - \sigma_\nu^2) = \mathrm{O}_{\mathbb{P}}(1)$ and $\sqrt{N}(\gamma(\nu,0) - \sigma_\nu^2) = \mathrm{O}_{\mathbb{P}}(1)$. Using the $\delta-$method we deduce that $\sqrt{N}(\sqrt{\hat{\sigma}_\nu^2} - \sqrt{\sigma_\nu^2}) = \mathrm{O}_{\mathbb{P}}(1)$ and $\sqrt{N}(\sqrt{\gamma(\nu,0)} - \sqrt{\sigma_\nu^2}) = \mathrm{O}_{\mathbb{P}}(1)$. Now using (30) and the ergodic theorem we obtain

$$N\left(\frac{\hat{\gamma}(\nu,h)}{\sqrt{\hat{\gamma}(\nu,0)\hat{\gamma}(\nu-h,0)}} - \frac{\hat{\gamma}(\nu,h)}{\sigma_\nu\sigma_{\nu-h}}\right) = \sqrt{N}\hat{\gamma}(\nu,h)\frac{\sqrt{N}\left(\sigma_\nu\sigma_{\nu-h} - \sqrt{\hat{\gamma}(\nu,0)\hat{\gamma}(\nu-h,0)}\right)}{\sigma_\nu\sigma_{\nu-h}\hat{\gamma}(\nu,0)} = \mathrm{O}_{\mathbb{P}}(1),$$

which means $\sqrt{N}\hat{\rho}(\nu,h) = \sqrt{N}\hat{\gamma}(\nu,h)/(\sigma_\nu\sigma_{\nu-h}) + \mathrm{O}_{\mathbb{P}}(N^{-1/2})$. For $h = 1,\dots,m$, it follows that

$$\sqrt{N}\hat{\rho}_m(\nu) = D_m^{-1}(\nu)\left(\sqrt{N}\hat{\gamma}(\nu,1),\dots,\sqrt{N}\hat{\gamma}(\nu,m)\right)^{'} + \mathrm{o}_{\mathbb{P}}(1) = \sqrt{N}D_m^{-1}(\nu)\hat{\gamma}_m(\nu) + \mathrm{o}_{\mathbb{P}}(1), \tag{48}$$

where $D_m(\nu) = \mathrm{Diag}(\sigma_\nu\sigma_{\nu-1},\dots,\sigma_\nu\sigma_{\nu-m})$. Thus from (48) the asymptotic distribution of the residual autocorrelations $\sqrt{N}\hat{\rho}_m(\nu)$ depends on the distribution of $\sqrt{N}\hat{\gamma}_m(\nu)$. Consequently we have

$$\lim_{N\to\infty}\mathrm{Var}\left(\sqrt{N}\hat{\rho}_m(\nu)\right) = \lim_{N\to\infty}\mathrm{Var}\left(D_m^{-1}(\nu)\sqrt{N}\hat{\gamma}_m(\nu)\right) = D_m^{-1}(\nu)\nabla_{\hat{\gamma}_m}(\nu,\nu)D_m^{-1}(\nu).$$

Observe that from (48) we deduce

$$\begin{aligned}\sqrt{N}\hat{\rho}_{mT} &= \left(D_m^{-1}(\nu)\sqrt{N}\hat{\gamma}_m(1), \dots, D_m^{-1}(\nu)\sqrt{N}\hat{\gamma}_m(T)\right)' + o_{\mathbb{P}}(1) \\ &= D_{mT}^{-1}\sqrt{N}\hat{\gamma}_{mT} + o_{\mathbb{P}}(1), \end{aligned} \tag{49}$$

where $D_{mT} = \mathrm{Diag}(\sigma_1\sigma_{1-1}, \dots, \sigma_1\sigma_{1-m}, \dots, \sigma_T\sigma_{T-1}, \dots, \sigma_T\sigma_{T-m})$.

Thus from (49) the asymptotic distribution of the residual autocorrelations $\sqrt{N}\hat{\rho}_{mT}$ depends on the distribution of $\sqrt{N}\hat{\gamma}_{mT}$. Consequently we have

$$\lim_{N\to\infty} \mathrm{Var}\left(\sqrt{N}\hat{\rho}_m\right) = \lim_{N\to\infty} \mathrm{Var}\left(D_{mT}^{-1}\sqrt{N}\hat{\gamma}_{mT}\right) = D_{mT}^{-1}\nabla_{\hat{\gamma}_m}D_{mT}^{-1} =: \nabla_{\hat{\rho}_m}.$$

The proof is completed. □

A.3. Proof of Proposition 3

The following proofs are quite technical and are adaptations of the arguments used in Boubacar Maïnassara and Saussereau (2018).

To prove the invertibility of the normalized matrix $C_m(\nu,\nu)$ we need to introduce the following notation. Let $S_n^i(\nu)$ be the i-th component of the vector $S_n(\nu) = \sum_{j=0}^n (\Lambda(\nu)W_{jT+\nu} - \gamma_m(\nu)) \in \mathbb{R}^m$. We remark that

$$S_{n-1}^i(\nu) = S_n^i(\nu) - \sum_{k=1}^{s_0} \delta_{i,k}(\nu)\omega_\nu^{-2}\epsilon_{nT+\nu}\frac{\partial}{\partial\beta_k}\epsilon_{nT+\nu}(\beta_0) - \epsilon_{nT+\nu}\epsilon_{nT+\nu-i} + \gamma(\nu,i), \tag{50}$$

where $\delta_{i,k}(\nu)$ is the $(i,k)-$th entry of the $m\times s_0$ matrix $G(\nu) := -\Psi_m(\nu)J(\beta_0,\omega^2)^{-1}$.

If the matrix $C_m(\nu,\nu)$ is not invertible, there exists some real constants $c_1,\dots,c_m$ not all equal to zero, such that we have

$$\sum_{i=1}^m\sum_{j=1}^m c_j[C_m(\nu,\nu)]_{ij}c_i = \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{i=1}^m\sum_{j=1}^m c_jS_n^j(\nu)S_n^i(\nu)c_i = \frac{1}{N^2}\sum_{n=0}^{N-1}\left(\sum_{i=1}^m c_iS_n^i(\nu)\right)^2 = 0,$$

which implies that $\sum_{i=1}^m c_iS_n^i(\nu) = 0$ for all $n\geq 0$.

Then by (50) it would imply that

$$\sum_{i=1}^m\sum_{k=1}^{s_0} c_i\delta_{i,k}(\nu)\omega_\nu^{-2}\epsilon_{nT+\nu}\frac{\partial}{\partial\beta_k}\epsilon_{nT+\nu}(\beta_0) + \sum_{i=1}^m c_i\epsilon_{nT+\nu}\epsilon_{nT+\nu-i} = \sum_{i=1}^m c_i\gamma(\nu,i). \tag{51}$$

By the ergodic Theorem, we also have $\sum_{i=1}^m c_i\gamma(\nu,i)\to 0$ almost-surely as N goes to infinity.

Consequently replacing this convergence in (51) implies that for all $n \geq 0$

$$\sum_{i=1}^{m}\sum_{k=1}^{s_0} c_i \sum_{\nu=1}^{T} \delta_{i,k}(\nu)\omega_\nu^{-2}\epsilon_{nT+\nu}\frac{\partial}{\partial\beta_k}\epsilon_{nT+\nu}(\beta_0) + \sum_{i=1}^{m} c_i\epsilon_{nT+\nu}\epsilon_{nT+\nu-i} = 0, \quad \text{a.s.}$$

Note that from (7) and under **(A1)** we have

$$\boldsymbol{\epsilon}_n(\beta_0) = \Phi^{-1}(L)\Theta(L)\boldsymbol{X}_n. \tag{52}$$

In view of (52) we deduce

$$\frac{\partial\boldsymbol{\epsilon}_n(\beta_0)}{\partial\beta_i} = \sum_{\ell=1}^{\infty} d'_\ell \boldsymbol{\epsilon}_{n-\ell}(\beta_0), \tag{53}$$

where the sequence of matrices $d_\ell = d_\ell(i)$ is such that $\|d_\ell\| \to 0$ at a geometric rate as $\ell \to \infty$ (see for instance (Francq et al., 2011, Lemma 7.11)). Using (53), the $\nu-$th component of $\partial\boldsymbol{\epsilon}_n(\beta_0)/\partial\beta_i$ is of the form:

$$\frac{\partial}{\partial\beta_i}\epsilon_{nT+\nu}(\beta_0) = \sum_{\ell=1}^{\infty}\sum_{j=1}^{T} d'_\ell(\nu,j)\epsilon_{nT+\nu-\ell}(\beta_0) \tag{54}$$

where $|d'_\ell(\nu,j)| = |d'_\ell(i,\nu,j)| \leq K\tau^\ell$ uniformly in $(i,\nu,j) \in \{1,\dots,s_0\}\times\{1,\dots,T\}^2$ and τ is a constant belonging to $[0,1)$ with $K > 0$. In view of (54) it yields that

$$\epsilon_{nT+\nu}\left\{\sum_{\ell\geq 1}\left(\sum_{i=1}^{m}\sum_{k=1}^{s_0} c_i\delta_{i,k}(\nu)\sum_{j=1}^{T} d'_\ell(\nu,j)\epsilon_{nT+\nu-\ell}\omega_\nu^{-2}\right)\epsilon_{nT+\nu-\ell} + \sum_{\ell=1}^{m} c_\ell\epsilon_{nT+\nu-\ell}\right\} = 0, \quad \text{a.s.}$$

Or equivalently,

$$\begin{aligned}\epsilon_{nT+\nu}&\left\{\sum_{\ell=1}^{m}\left(\sum_{i=1}^{m}\sum_{k=1}^{s_0} c_i\delta_{i,k}(\nu)\sum_{j=1}^{T} d'_\ell(\nu,j)\omega_\nu^{-2} + c_\ell\right)\epsilon_{nT+\nu-\ell}\right.\\ &\left. + \sum_{\ell\geq m+1}\left(\sum_{i=1}^{m}\sum_{k=1}^{s_0} c_i\delta_{i,k}(\nu)\sum_{j=1}^{T} d'_\ell(\nu,j)\omega_\nu^{-2}\right)\epsilon_{nT+\nu-\ell}\right\} = 0, \quad \text{a.s.}\end{aligned}$$

Thanks to Assumption **(A5)**, $(\boldsymbol{\epsilon}_n)_{n\in\mathbb{Z}}$ has a positive density in some neighborhood of zero and then $\epsilon_{nT+\nu} \neq 0$ almost-surely. Hence we obtain

$$\begin{aligned}&\sum_{\ell=1}^{m}\left(\sum_{i=1}^{m}\sum_{k=1}^{s_0} c_i\sum_{\nu=1}^{T}\delta_{i,k}(\nu)\sum_{j=1}^{T} d'_\ell(\nu,j)\omega_\nu^{-2} + c_\ell\right)\epsilon_{nT+\nu-\ell}\\ &\quad + \sum_{\ell\geq m+1}\left(\sum_{i=1}^{m}\sum_{k=1}^{s_0} c_i\sum_{\nu=1}^{T}\delta_{i,k}(\nu)\sum_{j=1}^{T} d'_\ell(\nu,j)\omega_\nu^{-2}\right)\epsilon_{nT+\nu-\ell} = 0, \quad \text{a.s.}\end{aligned}$$

Since the variance of the linear periodic innovation process in not equal to zero, we deduce that

$$\begin{cases} \sum_{i=1}^{m}\sum_{k=1}^{s_0} c_i\delta_{i,k}(\nu)\sum_{j=1}^{T} d_\ell'(\nu,j)\omega_\nu^{-2}+c_\ell=0 & \text{for all } \ell\in\{1,\dots,m\} \\ \\ \sum_{i=1}^{m}\sum_{k=1}^{s_0} c_i\delta_{i,k}(\nu)\sum_{j=1}^{T} d_\ell'(\nu,j)\omega_\nu^{-2}=0 & \text{for all } \ell\in\{m+1,\dots\}. \end{cases}$$

Then we would have $c_1=\cdots=c_m=0$ which is impossible. Thus we have a contradiction and the matrix $C_m(\nu,\nu)\in\mathbb{R}^{m\times m}$ is non singular.

The proof of the matrix $C_{mT}\in\mathbb{R}^{mT\times mT}$ follows the same lines that of $C_m(\nu,\nu)$. □

The proofs of Theorems 3.4 and 3.5 are similar to that given by Boubacar Maïnassara and Saussereau (2018) in the weak ARMA case.

A.4. Proof of Theorem 3.4

We recall that the Skorokhod space $\mathbb{D}^\ell[0,1]$ is the set of $\mathbb{R}^\ell-$valued functions on [0,1] which are right-continuous and has left limits everywhere. It is endowed with the Skorokhod topology and the weak convergence on $\mathbb{D}^\ell[0,1]$ is mentioned by $\xrightarrow{\mathbb{D}^\ell}$.

The proof is divided in two steps.

A.4.1. Functional central limit theorem for $(\Lambda(\nu)W_{nT+\nu})_{n\in\mathbb{Z}}$

In view of (26) and (29) we deduce that

$$\begin{aligned} \sqrt{N}\hat{\gamma}_m(\nu) &= \sqrt{N}\gamma_m(\nu)+\sqrt{N}\Psi_m(\nu)\left(\hat{\beta}_{\mathrm{LS}}-\beta_0\right)+\mathrm{o}_{\mathbb{P}}(1) \\ &= \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}W_{2,nT+\nu}+\Psi_m(\nu)\left(\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}W_{1,nT+\nu}+\mathrm{o}_{\mathbb{P}}(1)\right)+\mathrm{o}_{\mathbb{P}}(1) \\ &= \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\Lambda(\nu)W_{nT+\nu}+\mathrm{o}_{\mathbb{P}}(1). \end{aligned} \tag{55}$$

Now it is clear that the asymptotic behaviour of $\hat{\gamma}_m(\nu)$ is related to the limit distribution of $(W_{nT+\nu})_{n\in\mathbb{Z}}=(W_{1,nT+\nu}^{'},W_{2,nT+\nu}^{'})_{n\in\mathbb{Z}}^{'}$. Our first goal is to show that there exists a lower triangular matrix $\Pi(\nu)$ with nonnegative diagonal entries such that

$$\frac{1}{\sqrt{N}}\sum_{n=0}^{\lfloor Nr\rfloor-1}\Lambda(\nu)W_{nT+\nu}\xrightarrow[N\to\infty]{\mathbb{D}^m}\left(\Pi(\nu)\Pi^{'}(\nu)\right)^{1/2}B_m(r), \tag{56}$$

where $(B_m(r))_{r\geq 0}$ is a $m-$dimensional standard Brownian motion. Let $t = nT + \nu$ and using (54), W_t can be rewritten as

$$W_t = \left(- \left\{ \sum_{\ell=1}^{\infty} \sum_{j=1}^{T} d_\ell'(1,\nu,j)\epsilon_t\epsilon_{t-i}, \dots, \sum_{\ell=1}^{\infty} \sum_{j=1}^{T} d_\ell'(s_0,\nu,j)\epsilon_t\epsilon_{t-i} \right\}' J(\beta_0,\omega^2)^{-1'}, \epsilon_t\epsilon_{t-1}, \dots, \epsilon_t\epsilon_{t-m} \right)'.$$

The non-correlation between ϵ_t's implies that the process $(W_t)_t$ of $\mathbb{R}^{s_0+m}$ is centered. In order to apply the functional central limit theorem for strongly mixing process (see Herrndorf (1984)), we need to identify the asymptotic covariance matrix in the classical central limit theorem for the sequence $(W_t)_t$. It is proved in Proposition 2 that

$$\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} W_{nT+\nu} \xrightarrow[N\to\infty]{\text{in law}} \mathcal{N}\left(0, \Xi(\nu) := 2\pi f_W(0)\right),$$

where $f_W(0)$ is the spectral density of the stationary process $(W_{nT+\nu})_{n\in\mathbb{Z}}$ evaluated at frequency 0. The existence of the matrix $\Xi(\nu)$ is a consequence of Assumption **(A4)** and of Davydov's inequality (see Davydov (1968)). Since the matrix $\Xi(\nu)$ is positive definite it can be factorized as $\Xi(\nu) = \Upsilon(\nu)\Upsilon'(\nu)$, where the $(s_0 + m) \times (s_0 + m)$ lower triangular matrix $\Upsilon(\nu)$ has nonnegative diagonal entries. Therefore we have

$$\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \Lambda(\nu) W_{nT+\nu} \xrightarrow[n\to\infty]{\text{in law}} \mathcal{N}\left(0, \Lambda(\nu)\Xi(\nu)\Lambda'(\nu)\right),$$

and the new variance matrix can also been factorized as

$$\Lambda(\nu)\Xi(\nu)\Lambda'(\nu) = (\Lambda(\nu)\Upsilon(\nu))(\Lambda(\nu)\Upsilon(\nu))' := \Pi(\nu)\Pi'(\nu),$$

where $\Pi(\nu) \in \mathbb{R}^{m\times s_0}$. Thus

$$\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} (\Pi(\nu)\Pi'(\nu))^{-1/2} \Lambda(\nu) W_{nT+\nu} \xrightarrow[N\to\infty]{\text{in law}} \mathcal{N}(0, I_m),$$

where $(\Pi(\nu)\Pi'(\nu))^{-1/2}$ is the Moore-Penrose inverse (see Magnus and Neudecker (1999), p. 36) of $(\Pi(\nu)\Pi'(\nu))^{1/2}$.

Using the same arguments as in the proof of Theorem 2 in Francq et al. (2011) the asymptotic distribution of $N^{-1/2}\sum_{n=0}^{N-1} W_{nT+\nu}$ when N tends to infinity is obtained by introducing the random vector $W_t^k := W_{nT+\nu}^k$ defined for any strictly positive integer k by

$$W_t^k = \left(- \left\{ \sum_{\ell=1}^{k} \sum_{j=1}^{T} d_\ell'(1,\nu,j)\epsilon_t\epsilon_{t-i}, \dots, \sum_{\ell=1}^{k} \sum_{j=1}^{T} d_\ell'(s_0,\nu,j)\epsilon_t\epsilon_{t-i} \right\}' J(\beta_0,\omega^2)^{-1'}, \epsilon_t\epsilon_{t-1}, \dots, \epsilon_t\epsilon_{t-m} \right)'.$$

Since $W^k_{nT+\nu}$ depends on a finite number of values of the periodic noise process $(\epsilon_{nT+\nu})_{n\in\mathbb{Z}}$. It also satisfies a mixing property (see Theorem 14.1 in Davidson (1994), p. 210). Then applying the central limit theorem for strongly mixing process of Herrndorf (1984) shows that its asymptotic distribution is normal with zero mean and variance matrix $\Xi_k(\nu,\nu)$ that converges when k tends to infinity to $\Xi(\nu)$. More precisely we have

$$\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} W^k_{nT+\nu} \xrightarrow[N\to\infty]{\text{in law}} \mathcal{N}\left(0,\Xi_k(\nu)\right).$$

The above arguments also apply to matrix $\Xi_k(\nu)$ with some matrix $\Pi_k(\nu)$ which is defined analogously as $\Pi(\nu)$. Consequently we obtain

$$\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} \Lambda(\nu)W^k_{nT+\nu} \xrightarrow[n\to\infty]{\text{in law}} \mathcal{N}\left(0,\Lambda(\nu)\Xi_k(\nu)\Lambda^{'}(\nu)\right)$$

and we also have $N^{-1/2}\sum_{n=0}^{N-1}(\Pi_k(\nu)\Pi_k^{'}(\nu))^{-1/2}\Lambda(\nu)W^k_{nT+\nu} \xrightarrow[N\to\infty]{\text{in law}} \mathcal{N}(0,I_m)$.

Now we are able to apply the functional central limit theorem (see Herrndorf (1984)) and we obtain that

$$\frac{1}{\sqrt{N}}\sum_{n=0}^{\lfloor Nr\rfloor-1} (\Pi_k(\nu)\Pi_k^{'}(\nu))^{-1/2}\Lambda(\nu)W^k_{nT+\nu} \xrightarrow[N\to\infty]{\mathbb{D}^m} B_m(r).$$

Since for all $n\in\{0,\dots,\lfloor nr\rfloor-1\}$ we write

$$\begin{aligned}(\Pi(\nu)\Pi^{'}(\nu))^{-1/2}\Lambda(\nu)W^k_{nT+\nu} &= \left((\Pi(\nu)\Pi^{'}(\nu))^{-1/2}-(\Pi_k(\nu)\Pi_k^{'}(\nu))^{-1/2}\right)\Lambda(\nu)W^k_{nT+\nu}\\ &\quad+(\Pi_k(\nu)\Pi_k^{'}(\nu))^{-1/2}\Lambda(\nu)W^k_{nT+\nu},\end{aligned}$$

we obtain the following weak convergence on $\mathbb{D}^m[0,1]$:

$$\frac{1}{\sqrt{N}}\sum_{n=0}^{\lfloor Nr\rfloor-1} (\Pi(\nu)\Pi^{'}(\nu))^{-1/2}\Lambda(\nu)W^k_{nT+\nu} \xrightarrow[N\to\infty]{\mathbb{D}^m} B_m(r).$$

In order to conclude that (56) is true, it remains to observe that uniformly with respect to N

$$Y^k_N(r) := \frac{1}{\sqrt{N}}\sum_{n=0}^{\lfloor Nr\rfloor-1} (\Pi(\nu)\Pi^{'}(\nu))^{-1/2}\Lambda(\nu)Z^k_{nT+\nu} \xrightarrow[k\to\infty]{\mathbb{D}^m} 0, \tag{57}$$

where the random vector $Z^k_t := Z^k_{nT+\nu}$ is defined by

$$Z^k_t = \left(-\left\{\sum_{\ell=k+1}^{\infty}\sum_{j=1}^{T} d_\ell^{\prime}(1,\nu,j)\epsilon_t\epsilon_{t-i},\dots,\sum_{\ell=k+1}^{\infty}\sum_{j=1}^{T} d_\ell^{\prime}(s_0,\nu,j)\epsilon_t\epsilon_{t-i}\right\}^{\prime} J(\beta_0,\omega^2)^{-1^{\prime}},\epsilon_t\epsilon_{t-1},\dots,\epsilon_t\epsilon_{t-m}\right)^{\prime}.$$

Using the same arguments as those used in the proof of Theorem 2 in Francq et al. (2011) we obtain

$$\sup_N \mathrm{Var}\left(\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} Z^k_{nT+\nu}\right) \underset{k\to\infty}{\longrightarrow} 0$$

and since $\lfloor Nr\rfloor \leq N$ we have

$$\sup_{0\leq r\leq 1}\sup_N \left\{\left\| Y^k_N(r)\right\|\right\} \underset{k\to\infty}{\longrightarrow} 0.$$

Thus (57) is true and the proof of (56) is achieved.

A.4.2. Limit theorem

To conclude the prove of Theorem 3.4, we follow the arguments developed in Boubacar Maïnassara and Saussereau (2018). Note that the previous step ensures us that Assumption 1 in Lobato (2001) is satisfied for the sequence $(\Lambda(\nu)W_{nT+\nu})_{n\geq 0}$. Firstly from (56) we deduce that

$$\begin{aligned}\frac{1}{\sqrt{N}}S_{\lfloor Nr\rfloor}(\nu) &= \frac{1}{\sqrt{N}}\sum_{n=0}^{\lfloor Nr\rfloor -1}\Lambda(\nu)W_{nT+\nu} - \frac{\lfloor Nr\rfloor}{N}\left(\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\Lambda(\nu)W_{nT+\nu}\right)\\ &\overset{\mathbb{D}^m}{\underset{N\to\infty}{\longrightarrow}} (\Pi(\nu)\Pi^{'}(\nu))^{1/2}B_m(r) - r(\Pi(\nu)\Pi^{'}(\nu))^{1/2}B_m(1). \qquad (58)\end{aligned}$$

Observe now that the continuous mapping theorem implies

$$\begin{aligned}C_m(\nu,\nu) &= \frac{1}{N}\sum_{n=0}^{N-1}\left(\frac{1}{\sqrt{N}}S_n\right)\left(\frac{1}{\sqrt{N}}S_n\right)^{'}\\ &\overset{\mathbb{D}^m}{\underset{N\to\infty}{\longrightarrow}} (\Pi(\nu)\Pi^{'}(\nu))^{1/2}\left[\int_0^1\left\{B_m(r)-rB_m(1)\right\}\left\{B_m(r)-rB_m(1)\right\}^{'}\mathrm{dr}\right](\Pi(\nu)\Pi^{'}(\nu))^{1/2}\\ &= (\Pi(\nu)\Pi^{'}(\nu))^{1/2}V_m(\Pi(\nu)\Pi^{'}(\nu))^{1/2}.\end{aligned}$$

Using (55), (58) and again the continuous mapping theorem on the Skorokhod space, one finally obtains

$$\begin{aligned}N\hat{\gamma}^{'}_m(\nu)C_m^{-1}(\nu,\nu)\hat{\gamma}_m(\nu) &\overset{\mathbb{D}^m}{\underset{N\to\infty}{\longrightarrow}} \left\{(\Pi(\nu)\Pi^{'}(\nu))^{1/2}B_m(1)\right\}^{'}\left\{(\Pi(\nu)\Pi^{'}(\nu))^{1/2}V_m(\Pi(\nu)\Pi^{'}(\nu))^{1/2}\right\}^{-1}\\ &\qquad\times\left\{(\Pi(\nu)\Pi^{'}(\nu))^{1/2}B_m(1)\right\}\\ &= B^{'}_m(1)V_m^{-1}B_m(1) := \mathcal{U}_m.\end{aligned}$$

Consequently from (48) it follows that

$$N\hat{\rho}_m^{'}(\nu)D_m(\nu)C_m^{-1}(\nu,\nu)D_m(\nu)\hat{\rho}_m(\nu)\xrightarrow[N\to\infty]{\mathbb{D}^m}\mathcal{U}_m,$$

which completes the proof of the convergence of $\sqrt{N}\hat{\rho}_m(\nu)$.

The proof of the convergence of $\sqrt{N}\hat{\rho}_{mT}$ follows the same lines that of $\sqrt{N}\hat{\rho}_m(\nu)$ which completes the proof of Theorem 3.4. □

A.5. Proof of Theorem 3.5

We write $\hat{C}_m(\nu,\nu) = C_m(\nu,\nu) + \Upsilon_N(\nu,\nu)$ where $\Upsilon_N(\nu,\nu) = N^{-2}\sum_{n=0}^{N-1}\big(S_n(\nu)S_n'(\nu) - \hat{S}_n(\nu)\hat{S}_n'(\nu)\big)$. Now observe that there are three kinds of entries in the matrix Υ_N. The first one is a sum composed of

$$\upsilon_{nT+\nu}^{k,k'} = \epsilon_{nT+\nu}^2(\beta_0)\epsilon_{nT+\nu-k}(\beta_0)\epsilon_{nT+\nu-k'}(\beta_0) - e_{nT+\nu}^2(\hat{\beta}_{\text{LS}})e_{nT+\nu-k}(\hat{\beta}_{\text{LS}})e_{nT+\nu-k'}(\hat{\beta}_{\text{LS}})$$

for $(k,k')\in\{1,\dots,m\}^2$. By (Francq et al., 2011, Lemma 7.11) and the consistency of $\hat{\beta}_{\text{LS}}$, we have $\upsilon_{nT+\nu}^{k,k'} = \mathrm{o}_{\mathbb{P}}(1)$ almost surely. The two last kinds of entries of $\Upsilon_N(\nu,\nu)$ come from the following quantities for $i,j\in\{1,\dots,s_0\}$ and $k\in\{1,\dots,m\}$

$$\begin{aligned}\tilde{\upsilon}_{nT+\nu}^{k,i} &= \epsilon_{nT+\nu}^2(\beta_0)\epsilon_{nT+\nu-k}(\beta_0)\frac{\partial\epsilon_{nT+\nu}(\beta_0)}{\partial\beta_i} - e_{nT+\nu}^2(\hat{\beta}_{\text{LS}})e_{nT+\nu-k}(\hat{\beta}_{\text{LS}})\frac{\partial e_{nT+\nu}(\hat{\beta}_{\text{LS}})}{\partial\beta_i},\\ \bar{\upsilon}_{nT+\nu}^{i,j} &= \epsilon_{nT+\nu}^2(\beta_0)\frac{\partial\epsilon_{nT+\nu}(\beta_0)}{\partial\beta_i}\frac{\partial\epsilon_{nT+\nu}(\beta_0)}{\partial\beta_j} - e_{nT+\nu}^2(\hat{\beta}_{\text{LS}})\frac{\partial e_{nT+\nu}(\hat{\beta}_{\text{LS}})}{\partial\beta_i}\frac{\partial e_{nT+\nu}(\hat{\beta}_{\text{LS}})}{\partial\beta_j}\end{aligned}$$

and they also satisfy $\tilde{\upsilon}_{nT+\nu}^{k,i} + \bar{\upsilon}_{nT+\nu}^{i,j} = \mathrm{o}_{\mathbb{P}}(1)$ almost surely. Consequently, $\Upsilon_N(\nu,\nu) = \mathrm{o}_{\mathbb{P}}(1)$ almost surely as N goes to infinity. Thus one may find a matrix $\Upsilon_N^*(\nu,\nu)$, that tends to the null matrix almost surely, such that

$$\begin{aligned}N\,\hat{\gamma}_m'(\nu)\hat{C}_m^{-1}(\nu,\nu)\hat{\gamma}_m(\nu) &= N\,\hat{\gamma}_m'(\nu)(C_m(\nu,\nu)+\Upsilon_N(\nu,\nu))^{-1}\hat{\gamma}_m(\nu)\\ &= N\,\hat{\gamma}_m'(\nu)C_m^{-1}(\nu,\nu)\hat{\gamma}_m(\nu) + N\,\hat{\gamma}_m'(\nu)\Upsilon_N^*(\nu,\nu)\hat{\gamma}_m(\nu)\ .\end{aligned}$$

Thanks to the arguments developed in the proof of Theorem 3.4, $N\hat{\gamma}_m'(\nu)\hat{\gamma}_m$ converges in distribution. So $N\hat{\gamma}_m'(\nu)\Upsilon_N^*(\nu,\nu)\hat{\gamma}_m(\nu)$ tends to zero in distribution, hence in probability. Then $N\hat{\gamma}_m'(\nu)\hat{C}_m^{-1}(\nu,\nu)\hat{\gamma}_m(\nu)$ and $N\hat{\gamma}_m'(\nu)C_m^{-1}(\nu,\nu)\hat{\gamma}_m(\nu)$ have the same limit in distribution and the result is proved.

Turning to the global normalization matrix we write $\hat{C}_{mT} = C_{mT} + \Upsilon_N$ where $\Upsilon_N = N^{-2}\sum_{n=0}^{N-1}\left(S_n S_n' - \hat{S}_n\hat{S}_n'\right)$. The proof of the convergence of $\hat{C}_{mT}$ to C_{mT} follows the same lines that of of $\hat{C}_m(\nu,\nu)$ to $C_m(\nu,\nu)$ which completes the proof of Theorem 3.5. $\square$

Appendix B: Example of explicit calculation of $\Omega(\alpha_0, \sigma^2)$ and $\nabla_{\hat{\rho}_m}(\nu,\nu)$

The results of the previous Sections 2, 3 and 3.2 are particularized in the PARMA$_2(1,0)$ and PARMA$_2(0,1)$ cases. First we consider the case of a PARMA$_2(1,0)$ model of the form

$$\begin{cases} X_{2n+1} = \phi(1)X_{2n} + \epsilon_{2n+1} \\ X_{2n+2} = \phi(2)X_{2n+1} + \epsilon_{2n+2} \end{cases}, \tag{59}$$

where the unknown parameter is $\alpha_0 = (\phi(1), \phi(2))'$. For simplification, we assume that in (59) the innovation process $(\epsilon_t)_{t\in\mathbb{Z}} = (\epsilon_{2n+\nu})_{n\in\mathbb{Z}}$ is the GARCH$(1,1)$ process given by the model

$$\begin{cases} \epsilon_t = \sigma_t\eta_t \\ \sigma_t^2 = \omega + a_1\epsilon_{t-1}^2 + b_1\sigma_{t-1}^2, \end{cases} \tag{60}$$

with $\omega > 0$, $a_1 \geq 0$, $b_1 \geq 0$ and where $(\eta_t)_{t\in\mathbb{Z}}$ is a sequence of iid centered Gaussian random variables with variance 1. We also assume that in (60): $a_1^2\kappa_\eta + b_1^2 + 2a_1b_1 < 1$,[1] where $\kappa_\eta := \mathbb{E}\eta_1^4$ and we assume that $\kappa_\eta > 1$.

For the sake of simplicity we assume that the variables $(\eta_t)_{t\in\mathbb{Z}}$ involved in (60) have a symmetric distribution. More precisely, we have the following symmetry assumption

$$\mathbb{E}[\epsilon_{t_1}\epsilon_{t_2}\epsilon_{t_3}\epsilon_{t_4}] = 0 \qquad \text{when} \qquad t_1 \neq t_2,\ t_1 \neq t_3 \text{ and } t_1 \neq t_4, \tag{61}$$

made in Francq and Zakoïan (2009), Boubacar Mainassara et al. (2012). For this particular GARCH$(1,1)$ model with fourth-order moments and symmetric innovations satisfying (61), it can be shown that

$$\mathbb{E}\left[\epsilon_t\epsilon_{t-\ell}\epsilon_{t-h}\epsilon_{t-h-\ell'}\right] = \begin{cases} \mathbb{E}\left[\epsilon_t^2\epsilon_{t-\ell}^2\right] & \text{if } h = 0 \text{ and } \ell = \ell' \\ 0 & \text{otherwise.} \end{cases} \tag{62}$$

[1] This is a necessary and sufficient condition for the existence of a nonanticipative stationary solution process $(\epsilon_t)_{t\in\mathbb{Z}}$ with fourth-order moments (see (Francq and Zakoïan, 2019, Example 2.3)).

Now we need to compute the autocovariance structure of $(\epsilon_t^2)_{t\in\mathbb{Z}}$. We will use the fact that the GARCH process $(\epsilon_t)_{t\in\mathbb{Z}}$ is fourth-order stationary, then $(\epsilon_t^2)_{t\in\mathbb{Z}}$ is a solution of the following ARMA$(1,1)$ model

$$\epsilon_t^2 = \omega + (a_1+b_1)\epsilon_{t-1}^2 + v_t - b_1 v_{t-1}, \quad t\in\mathbb{Z} \tag{63}$$

where $v_t = \epsilon_t^2 - \sigma_t^2$ is the innovation of $(\epsilon_t^2)_{t\in\mathbb{Z}}$. From (63) the autocovariances of $(\epsilon_t^2)_{t\in\mathbb{Z}}$ take the form

$$\gamma_{\epsilon^2}(\ell) := \mathrm{Cov}(\epsilon_t^2, \epsilon_{t-\ell}^2) = \gamma_{\epsilon^2}(1)(a_1+b_1)^{\ell-1}, \qquad \ell \geq 1, \tag{64}$$

where

$$\gamma_{\epsilon^2}(1) = \frac{(\kappa_\eta - 1)(a_1 - a_1 b_1^2 - a_1^2 b_1)}{1 - b_1^2 - 2a_1 b_1 - a_1^2\kappa_\eta}\sigma^4, \quad \gamma_{\epsilon^2}(0) := \frac{(\kappa_\eta-1)(1-b_1^2-2a_1b_1)}{1-b_1^2-2a_1b_1-a_1^2\kappa_\eta}\sigma^4,$$

and $\sigma^2 := \mathbb{E}\epsilon_t^2 = \dfrac{\omega}{1-a_1-b_1}$.

From (62) and (64) we deduce that for any $\ell \geq 1$

$$\Gamma(\ell,\ell) = \mathbb{E}\left[\epsilon_t^2\epsilon_{t-\ell}^2\right] = \mathrm{Cov}(\epsilon_t^2,\epsilon_{t-\ell}^2) + \mathbb{E}\left[\epsilon_t^2\right]\mathbb{E}\left[\epsilon_{t-\ell}^2\right] = \left\{1 + \frac{1}{\sigma^4}\gamma_{\epsilon^2}(1)(a_1+b_1)^{\ell-1}\right\}\sigma^4. \tag{65}$$

B.1. Example of analytic computation of $\Omega(\alpha_0,\sigma^2)$

In view of Remark 2, the asymptotic covariance matrix of the QLS estimators obtained under independent errors is generally different from the one obtained under uncorrelated but dependent errors when $I(\alpha_0,\sigma^2) \neq J(\alpha_0,\sigma^2)$. Here, we give explicit expressions for the asymptotic covariance of the QLS estimator of a weak PAR$_2(1)$ model (59). For that sake, we need the following additional expressions. From (7) and using **(A1)** we deduce that:

$$\begin{cases} X_{2n+1} = \epsilon_{2n+1} + \frac{1}{\phi(2)}\sum_{i\geq 1}\phi^i(1)\phi^i(2)\epsilon_{2(n-i)+2} \\ X_{2n+2} = \phi(2)\epsilon_{2n+1} + \sum_{i\geq 0}\phi^i(1)\phi^i(2)\epsilon_{2(n-i)+2} \end{cases}, \tag{66}$$

It is classical that the noise derivatives in (59) can be represented as

$$\frac{\partial\epsilon_{2n+\nu}(\alpha_0)}{\partial\alpha} = \begin{pmatrix} \frac{\partial\epsilon_{2n+1}}{\partial\phi(1)} & \frac{\partial\epsilon_{2n+1}}{\partial\phi(2)} \\ \frac{\partial\epsilon_{2n+2}}{\partial\phi(1)} & \frac{\partial\epsilon_{2n+2}}{\partial\phi(2)} \end{pmatrix} = \begin{pmatrix} -X_{2n} & 0 \\ 0 & -X_{2n+1} \end{pmatrix}. \tag{67}$$

We compute the information matrices $J(\alpha_0, \sigma^2)$ and $I(\alpha_0, \sigma^2)$ by using (66) and (67). Then we have

$$J(\alpha_0, \sigma^2) = \frac{1}{1 - \phi^2(1)\phi^2(2)} \begin{pmatrix} \phi^2(2)[1 - \phi^2(1)\phi^2(2)] + 1 & 0 \\ 0 & 1 - \phi^2(1)\phi^2(2) + \phi^2(1) \end{pmatrix}. \quad (68)$$

A simple calculation implies that

$$J^{-1}(\alpha_0, \sigma^2) = \begin{pmatrix} \frac{1-\phi^2(1)\phi^2(2)}{\phi^2(2)[1-\phi^2(1)\phi^2(2)]+1} & 0 \\ 0 & \frac{1-\phi^2(1)\phi^2(2)}{1-\phi^2(1)\phi^2(2)+\phi^2(1)} \end{pmatrix}. \quad (69)$$

We now investigate a similar tractable expression for $I(\alpha_0, \sigma^2)$. Using (67) and (61) we have

$$I(\alpha_0, \sigma^2) = J(\alpha_0, \sigma^2) + \frac{(\kappa_\eta - 1)(a_1 - a_1 b_1^2 - a_1^2 b_1)}{1 - b_1^2 - 2a_1 b_1 - a_1^2 \kappa_\eta} \tilde{I}(\alpha_0, \sigma^2), \quad (70)$$

where

$$\tilde{I}(\alpha_0, \sigma^2) = \begin{pmatrix} \frac{\phi^2(2)[1-\phi^2(1)\phi^2(2)(a_1+b_1)^2](a_1+b_1)+1}{1-\phi^2(1)\phi^2(2)(a_1+b_1)^2} & 0 \\ 0 & \frac{1-\phi^2(1)\phi^2(2)(a_1+b_1)^2+\phi^2(1)(a_1+b_1)}{1-\phi^2(1)\phi^2(2)(a_1+b_1)^2} \end{pmatrix}.$$

Note that when $a_1 = b_1 = 0$ in (70) we retrieve the well know result: $I(\alpha_0, \sigma^2) = J(\alpha_0, \sigma^2)$ obtained in the strong PARMA case (see Remark 2). Direct computation of (68) and (70) lead to the following asymptotic covariance matrix of the QLS estimators of $\sqrt{N}(\hat{\phi}(1), \hat{\phi}(1))'$:

$$\Omega(\alpha_0, \sigma^2) = J^{-1}(\alpha_0, \sigma^2) + \frac{(\kappa_\eta - 1)(a_1 - a_1 b_1^2 - a_1^2 b_1)}{1 - b_1^2 - 2a_1 b_1 - a_1^2 \kappa_\eta} J^{-1}(\alpha_0, \sigma^2)\tilde{I}(\alpha_0, \sigma^2)J^{-1}(\alpha_0, \sigma^2). \quad (71)$$

It is obvious that (71) can be quite different from the asymptotic covariance matrix (69) corresponding to a strong PARMA model (see also Remark 2).

B.2. Example of analytic and numerical computations of $\nabla_{\hat{\rho}_m}(\nu, \nu)$

As mentioned before, the subject of this subsection is to give an explicit expression of the asymptotic variance of residual autocorrelations $\nabla_{\hat{\rho}_m}(\nu, \nu)$ defined in (32) in this particular case of model (59). Using (65) and under the symmetry assumption (61), the matrix $\Sigma_{\gamma_m(\nu)}$ takes the simple following diagonal form

$$\Sigma_{\gamma_m(\nu)} = \sigma^4 I_m + \sigma^4 \frac{(\kappa_\eta - 1)(a_1 - a_1 b_1^2 - a_1^2 b_1)}{1 - b_1^2 - 2a_1 b_1 - a_1^2 \kappa_\eta} \mathrm{diag}(1, (a_1 + b_1), \dots, (a_1 + b_1)^{m-1}). \quad (72)$$

In the sequel we consider $m = 4$. The matrix defined in (25) can be rewritten as

$$\Psi_4(2) = -\sigma^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & \phi(1) & 0 & \phi^2(1)\phi(2) \end{pmatrix}' \text{ and } \Psi_4(1) = -\sigma^2 \begin{pmatrix} 1 & \phi(2) & \phi(1)\phi(2) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}'. \tag{73}$$

Using (61), (66), (67) and (69), the matrix $\Sigma'_{\hat{\alpha},\gamma_4(\nu)}$ is given by

$$\Sigma'_{\hat{\alpha},\gamma_4(\nu)} = \frac{1-\phi^2(1)\phi^2(2)}{\sigma^2} \begin{pmatrix} \frac{\Sigma_{\gamma_4(\nu)}(1,1)}{\phi^2(2)[1-\phi^2(1)\phi^2(2)]+1} & \frac{\Sigma_{\gamma_4(\nu)}(1,1)}{1-\phi^2(1)\phi^2(2)+\phi^2(1)} \\ \frac{\phi(2)\Sigma_{\gamma_4(\nu)}(2,2)}{\phi^2(2)[1-\phi^2(1)\phi^2(2)]+1} & \frac{\phi(1)\Sigma_{\gamma_4(\nu)}(2,2)}{1-\phi^2(1)\phi^2(2)+\phi^2(1)} \\ \frac{\phi(1)\phi(2)\Sigma_{\gamma_4(\nu)}(3,3)}{\phi^2(2)[1-\phi^2(1)\phi^2(2)]+1} & 0 \\ 0 & \frac{\phi^2(1)\phi(2)\Sigma_{\gamma_4(\nu)}(4,4)}{1-\phi^2(1)\phi^2(2)+\phi^2(1)} \end{pmatrix}, \tag{74}$$

where for any $1 \le i,j \le 4$, $\Sigma_{\gamma_4(\nu)}(i,j)$ is given by (72).

From Remark 6 when m is large, in the strong PARMA case the asymptotic variance of residual autocorrelations takes a simpler form

$$\nabla^{\mathrm{S}}_{\hat{\rho}_4}(\nu,\nu) \simeq I_4 - D_4^{-1}(\nu)\Psi_4(\nu)J(\alpha_0,\omega^2)^{-1}\Psi_4'(\nu)D_4^{-1}(\nu).$$

More precisely, we have

$$\nabla^{\mathrm{S}}_{\hat{\rho}_4}(1,1) = I_4 - \frac{1-\phi^2(1)\phi^2(2)}{\phi^2(2)[1-\phi^2(1)\phi^2(2)]+1}R_1 \text{ and } \nabla^{\mathrm{S}}_{\hat{\rho}_4}(2,2) = I_4 - \frac{1-\phi^2(1)\phi^2(2)}{1-\phi^2(1)\phi^2(2)+\phi^2(1)}R_2,$$

where

$$R_1 = \begin{pmatrix} 1 & \phi(2) & \phi(1)\phi(2) & 0 \\ \phi(2) & \phi^2(2) & \phi(1)\phi^2(2) & 0 \\ \phi(1)\phi(2) & \phi(1)\phi^2(2) & \phi^2(1)\phi^2(2) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } R_2 = \begin{pmatrix} 1 & \phi(1) & 0 & \phi^2(1)\phi(2) \\ \phi(1) & \phi^2(1) & 0 & \phi^3(1)\phi(2) \\ 0 & 0 & 0 & 0 \\ \phi^2(1)\phi(2) & \phi^3(1)\phi(2) & 0 & \phi^4(1)\phi^2(2)) \end{pmatrix}.$$

From the above explicit expressions we deduce that the asymptotic variance of residual autocorrelations for this model is in the form

$$\begin{aligned}\nabla_{\hat{\rho}_4}(1,1) = \nabla^{\mathrm{S}}_{\hat{\rho}_4}(1,1) + \frac{(\kappa_\eta-1)(a_1-a_1b_1^2-a_1^2b_1)}{1-b_1^2-2a_1b_1-a_1^2\kappa_\eta} \Big[& (a_1+b_1)^{i-1}\mathbb{1}_{\{i=j\}} \\ & - \frac{1-\phi^2(1)\phi^2(2)}{\phi^2(2)[1-\phi^2(1)\phi^2(2)]+1}R_1(i,j)\left\{(a_1+b_1)^{i-1}+(a_1+b_1)^{j-1}\right\} \\ +R_1(i,j)\left\{\frac{1-\phi^2(1)\phi^2(2)}{\phi^2(2)[1-\phi^2(1)\phi^2(2)]+1}\right\}^2 & \frac{\phi^2(2)[1-\phi^2(1)\phi^2(2)(a_1+b_1)^2](a_1+b_1)+1}{1-\phi^2(1)\phi^2(2)(a_1+b_1)^2}\Big]_{1\le i,j\le 4},\end{aligned}$$

$$\nabla_{\hat{\rho}_4}(2,2) = \nabla^{\mathrm{S}}_{\hat{\rho}_4}(2,2) + \frac{(\kappa_\eta - 1)(a_1 - a_1 b_1^2 - a_1^2 b_1)}{1 - b_1^2 - 2a_1 b_1 - a_1^2\kappa_\eta}\Big[(a_1+b_1)^{i-1}\mathbb{1}_{\{i=j\}}$$
$$-\frac{1-\phi^2(1)\phi^2(2)}{1-\phi^2(1)\phi^2(2)+\phi^2(1)}R_2(i,j)\left\{(a_1+b_1)^{i-1}+(a_1+b_1)^{j-1}\right\}$$
$$+R_2(i,j)\left\{\frac{1-\phi^2(1)\phi^2(2)}{1-\phi^2(1)\phi^2(2)+\phi^2(1)}\right\}^2\frac{1-\phi^2(1)\phi^2(2)(a_1+b_1)^2+\phi^2(1)(a_1+b_1)}{1-\phi^2(1)\phi^2(2)(a_1+b_1)^2}\Big]_{1\le i,j\le 4},$$

For instance when $b_1 = 0$, $\kappa_\eta = 3$, $\omega = 1$, $\phi(1) = -0.55$ and $\phi(2) = 0.75$ we have

ν	α_0	$\nabla_{\hat{\rho}_4}(\nu,\nu)$	$\xi_4^\nu = (\xi_{1,4}^\nu, \xi_{2,4}^\nu, \xi_{3,4}^\nu, \xi_{4,4}^\nu)$	$Z_4^\nu(\xi_4^\nu)$
$\nu = 1$	$a_1 = 0$	$\begin{pmatrix} 0.43 & -0.42 & 0.23 & 0 \\ -0.42 & 0.68 & 0.18 & 0 \\ 0.23 & 0.18 & 0.90 & 0 \\ 0.00 & 0.00 & 0.00 & 1 \end{pmatrix}$	$(1.00, 1.00, 1.00, 0.02)$	$\chi_1^2 + \chi_1^2 + \chi_1^2 + 0.02\chi_1^2$
$\nu = 2$	$a_1 = 0$	$\begin{pmatrix} 0.27 & 0.40 & 0 & -0.17 \\ 0.40 & 0.78 & 0 & 0.09 \\ 0.00 & 0.00 & 1 & 0.00 \\ -0.17 & 0.09 & 0 & 0.96 \end{pmatrix}$	$(1.00, 1.00, 1.00, 0.01)$	$\chi_1^2 + \chi_1^2 + \chi_1^2 + 0.02\chi_1^2$
$\nu = 1$	$a_1 = 0.55$	$\begin{pmatrix} 4.06 & -4.35 & 1.71 & 0.00 \\ -4.35 & 5.98 & 0.34 & 0.00 \\ 1.71 & 0.34 & 4.69 & 0.00 \\ 0.00 & 0.00 & 0.00 & 2.98 \end{pmatrix}$	$(9.61, 5.08, 2.98, 0.04)$	$9.61\chi_1^2 + 5.08\chi_1^2 + 2.98\chi_1^2 + 0.04\chi_1^2$
$\nu = 2$	$a_1 = 0.55$	$\begin{pmatrix} 2.24 & 3.70 & 0.0 & -0.77 \\ 3.70 & 6.69 & 0.0 & -0.07 \\ 0.00 & 0.00 & 4.6 & 0.00 \\ -0.77 & -0.07 & 0.0 & 3.18 \end{pmatrix}$	$(8.82, 4.60, 3.27, 0.01)$	$8.82\chi_1^2 + 4.60\chi_1^2 + 3.27\chi_1^2 + 0.01\chi_1^2$

It is clear that for $a_1 = 0.55$, the McLeod (1994, 1995) approximation by a χ_3^2 distribution will be disastrous. The eigenvalues ξ_4^ν can be very different from those of strong PARMA models which are close to 1 or 0 when the lag m is large enough (see Remark 6).

The same result holds for $\mathrm{PARMA}_2(0,1)$ model with $\phi(\nu)$ replaced by $\theta(\nu)$ in α_0.

Appendix C: Additional Monte Carlo experiments and real datasets

C.1. Empirical size

Secondly a weak noise defined by a following PGARCH$(1,1)$ is also considered:

$$\begin{cases} \epsilon_{2n+\nu} = \sqrt{h_{2n+\nu}}\eta_{2n+\nu} \\ h_{2n+\nu} = \omega(\nu) + a(\nu)\epsilon^2_{2n+\nu-1} + b(\nu)h_{2n+\nu-1}, \quad \nu = 1,2 \end{cases} \tag{75}$$

with $(\omega(1), \omega(2)) = (0.2, 0.4)$, $(a(1), a(2)) = (0.1, 0.15)$ and $(b(1), b(2)) = (0.83, 0.82)$ and where $(\eta_{2n+\nu})_{n\in\mathbb{Z}}$ is a sequence of iid centered Gaussian random variables with variance 1.

When the relative rejection frequencies are outside the significant limits with probability 95% (resp. 99%), they are displayed in bold (resp. underlined) type in Tables 5, 6, 7,. . . , 11.

We now repeat the same experiment on Model (44) by assuming that the innovation process $(\epsilon_{2n+\nu})_n$ follows first (45) and secondly (75).

Table 5 displays relative frequencies (in %) of rejection of the null hypothesis (**H0**) that the DGP follows two weak periodic white noises *i.e.* the weak $\mathrm{PARMA}_2(1, 1)$ given by (44)–(75) with $\alpha_0 = (0, 0, 0, 0)'$. As expected the observed relative rejection frequencies of the standard tests are definitely outside the significant limits. Thus the standard tests reject very often the true weak periodic noise (see Table 5). By contrast, the error of first kind is globally well controlled by the proposed tests when N increases. We draw the conclusion that for this particular weak PARMA model the proposed tests are clearly preferable to the standard ones.

Table 6 (resp. Table 9) displays relative frequencies (in %) of rejection (over the 1,000 replications) of the null hypothesis (**H0**) that the DGP follows a strong $\mathrm{PARMA}_2(1, 1)$ given by (44) with the parameter $\alpha_0 = (0.8, 0.9, 0, 0)'$ (resp. $\alpha_0 = (0.8, 0.9, -0.5, -0.45)'$). Note that the empirical size is not available (n.a.) for the standard BP and LBM tests because these tests are not applicable to $m = 1$ (see Table 6) and $m \leq 2$ (see Table 9). As expected the observed relative rejection frequency of the standard tests is very far from the nominal $\alpha = 5\%$ for small m when $\alpha_0 \neq (0, 0, 0, 0)'$. The results are worse for $N = 2,000$ than for $N = 400$. This is in accordance with the results in the literature on the strong PARMA models. The theory that the asymptotic distributions of (22) and (23), namely, $\chi^2_{m-(p+q)}$ and $\chi^2_{2m-2(p+q)}$ approximations are better for larger m is confirmed. In contrast, the proposed tests well control the error of first kind, even when m is small. From these examples we draw the conclusion that the proposed versions are preferable to the standard ones, when the number m of autocorrelations used is small.

Tables 7 and 8 (resp. Tables 10 and 11) display relative frequencies (in %) of rejection (over the 1,000 replications) of the null hypothesis (**H0**) that the DGP follows two weak $\mathrm{PARMA}_2(1, 1)$ given by (44)–(45) (resp. (44)–(75)) with the parameter $\alpha_0 = (0.8, 0.9, 0, 0)'$

(resp. $\alpha_0 = (0.8, 0.9, -0.5, -0.45)'$). Note also that the empirical size is not available (n.a.) for the standard BP and LBM tests because these tests are not applicable to $m = 1$ (see Tables 7 and 8) and $m \leq 2$ (see Tables 10 and 11). As expected, Tables 7, 8, 10 and 11 show that the standard tests poorly perform to assess the adequacy of these weak PARMA models. The observed relative rejection frequencies of the standard tests are definitely outside the significant limits. Thus the standard tests reject very often the true weak $\text{PARMA}_2(1,1)$ models (see Tables 7 and 8, Tables 10 and 11). By contrast, the error of first kind is globally well controlled by the proposed tests when N increases. We draw the conclusion that for these particular weak PARMA models the proposed tests are clearly preferable to the standard ones.

TABLE 5

Empirical size of the standard and proposed tests: relative frequencies (in %) of rejection of a weak $PARMA_2(1,1)$ given by (44) with the parameter $\alpha_0 = (0,0,0,0)'$ and where the innovation $\epsilon_t \sim$ (75). When the relative rejection frequencies are outside the significant limits with probability 95% (resp. 99%), they are displayed in bold (resp. underlined) type.

Length N	Tests	Lag m											
		1	2	3	4	5	6	7	8	9	10	11	12
	$\mathrm{BP}_{\mathrm{s}}(1)$	$\underline{\mathbf{12.6}}$	$\underline{\mathbf{13.8}}$	$\underline{\mathbf{15.7}}$	$\underline{\mathbf{17.1}}$	$\underline{\mathbf{16.9}}$	$\underline{\mathbf{18.2}}$	$\underline{\mathbf{18.7}}$	$\underline{\mathbf{19.2}}$	$\underline{\mathbf{20.1}}$	$\underline{\mathbf{19.0}}$	$\underline{\mathbf{18.7}}$	$\underline{\mathbf{19.5}}$
	$\mathrm{BP}_{\mathrm{s}}(2)$	$\underline{\mathbf{9.6}}$	$\underline{\mathbf{12.1}}$	$\underline{\mathbf{12.7}}$	$\underline{\mathbf{15.1}}$	$\underline{\mathbf{14.9}}$	$\underline{\mathbf{15.1}}$	$\underline{\mathbf{15.3}}$	$\underline{\mathbf{15.8}}$	$\underline{\mathbf{16.5}}$	$\underline{\mathbf{17.1}}$	$\underline{\mathbf{16.6}}$	$\underline{\mathbf{16.3}}$
	$\mathrm{LBM}_{\mathrm{s}}(1)$	$\underline{\mathbf{12.6}}$	$\underline{\mathbf{14.0}}$	$\underline{\mathbf{15.7}}$	$\underline{\mathbf{17.3}}$	$\underline{\mathbf{17.2}}$	$\underline{\mathbf{18.5}}$	$\underline{\mathbf{18.9}}$	$\underline{\mathbf{19.4}}$	$\underline{\mathbf{20.4}}$	$\underline{\mathbf{19.6}}$	$\underline{\mathbf{19.2}}$	$\underline{\mathbf{20.1}}$
	$\mathrm{LBM}_{\mathrm{s}}(2)$	$\underline{\mathbf{9.6}}$	$\underline{\mathbf{12.2}}$	$\underline{\mathbf{12.7}}$	$\underline{\mathbf{15.1}}$	$\underline{\mathbf{15.4}}$	$\underline{\mathbf{15.4}}$	$\underline{\mathbf{15.5}}$	$\underline{\mathbf{16.4}}$	$\underline{\mathbf{16.8}}$	$\underline{\mathbf{17.7}}$	$\underline{\mathbf{17.2}}$	$\underline{\mathbf{16.8}}$
	$\mathrm{BP}_{\mathrm{w}}(1)$	5.1	4.4	3.7	**3.3**	3.5	$\underline{\mathbf{3.1}}$	**3.3**	3.4	$\underline{\mathbf{2.6}}$	$\underline{\mathbf{2.9}}$	$\underline{\mathbf{3.2}}$	$\underline{\mathbf{3.0}}$
	$\mathrm{BP}_{\mathrm{w}}(2)$	3.4	3.5	$\underline{\mathbf{2.7}}$	$\underline{\mathbf{2.7}}$	$\underline{\mathbf{2.7}}$	$\underline{\mathbf{3.1}}$	$\underline{\mathbf{3.2}}$	$\underline{\mathbf{3.1}}$	$\underline{\mathbf{2.5}}$	$\underline{\mathbf{2.6}}$	$\underline{\mathbf{2.4}}$	$\underline{\mathbf{2.4}}$
	$\mathrm{LBM}_{\mathrm{w}}(1)$	5.1	4.4	3.7	**3.3**	3.5	**3.3**	3.4	3.6	$\underline{\mathbf{3.0}}$	$\underline{\mathbf{2.9}}$	3.4	$\underline{\mathbf{3.2}}$
	$\mathrm{LBM}_{\mathrm{w}}(2)$	3.4	3.5	$\underline{\mathbf{2.8}}$	$\underline{\mathbf{2.8}}$	$\underline{\mathbf{2.7}}$	$\underline{\mathbf{3.2}}$	$\underline{\mathbf{3.2}}$	$\underline{\mathbf{3.2}}$	$\underline{\mathbf{2.6}}$	$\underline{\mathbf{2.7}}$	$\underline{\mathbf{2.7}}$	$\underline{\mathbf{2.5}}$
$N = 400$	$\mathrm{BP}_{\mathrm{SN}}(1)$	4.6	3.4	3.8	3.4	$\underline{\mathbf{3.1}}$	$\underline{\mathbf{2.4}}$	$\underline{\mathbf{2.5}}$	$\underline{\mathbf{2.2}}$	$\underline{\mathbf{2.0}}$	$\underline{\mathbf{1.1}}$	$\underline{\mathbf{1.5}}$	$\underline{\mathbf{1.4}}$
	$\mathrm{BP}_{\mathrm{SN}}(2)$	$\underline{\mathbf{2.8}}$	$\underline{\mathbf{2.9}}$	$\underline{\mathbf{2.7}}$	3.5	$\underline{\mathbf{2.1}}$	$\underline{\mathbf{2.2}}$	$\underline{\mathbf{2.6}}$	$\underline{\mathbf{2.1}}$	$\underline{\mathbf{1.5}}$	$\underline{\mathbf{1.4}}$	$\underline{\mathbf{1.4}}$	$\underline{\mathbf{1.4}}$
	$\mathrm{LBM}_{\mathrm{SN}}(1)$	4.6	3.4	3.9	3.4	$\underline{\mathbf{3.1}}$	$\underline{\mathbf{2.5}}$	$\underline{\mathbf{2.6}}$	$\underline{\mathbf{2.2}}$	$\underline{\mathbf{2.0}}$	$\underline{\mathbf{1.1}}$	$\underline{\mathbf{1.5}}$	$\underline{\mathbf{1.4}}$
	$\mathrm{LBM}_{\mathrm{SN}}(2)$	$\underline{\mathbf{2.8}}$	$\underline{\mathbf{2.9}}$	$\underline{\mathbf{2.8}}$	3.6	$\underline{\mathbf{2.1}}$	$\underline{\mathbf{2.2}}$	$\underline{\mathbf{2.6}}$	$\underline{\mathbf{2.1}}$	$\underline{\mathbf{1.6}}$	$\underline{\mathbf{1.5}}$	$\underline{\mathbf{1.4}}$	$\underline{\mathbf{1.6}}$
	BP_{s}	$\underline{\mathbf{13.6}}$	$\underline{\mathbf{15.9}}$	$\underline{\mathbf{20.0}}$	$\underline{\mathbf{20.9}}$	$\underline{\mathbf{23.6}}$	$\underline{\mathbf{23.9}}$	$\underline{\mathbf{24.7}}$	$\underline{\mathbf{25.9}}$	$\underline{\mathbf{26.8}}$	$\underline{\mathbf{26.4}}$	$\underline{\mathbf{25.9}}$	$\underline{\mathbf{25.6}}$
	$\mathrm{LBM}_{\mathrm{s}}$	$\underline{\mathbf{13.6}}$	$\underline{\mathbf{16.0}}$	$\underline{\mathbf{20.4}}$	$\underline{\mathbf{21.2}}$	$\underline{\mathbf{23.8}}$	$\underline{\mathbf{24.3}}$	$\underline{\mathbf{25.0}}$	$\underline{\mathbf{26.3}}$	$\underline{\mathbf{27.5}}$	$\underline{\mathbf{27.2}}$	$\underline{\mathbf{26.7}}$	$\underline{\mathbf{26.7}}$
	BP_{w}	3.4	$\underline{\mathbf{2.8}}$	$\underline{\mathbf{2.0}}$	$\underline{\mathbf{2.5}}$	$\underline{\mathbf{2.5}}$	$\underline{\mathbf{2.4}}$	$\underline{\mathbf{2.0}}$	$\underline{\mathbf{2.5}}$	$\underline{\mathbf{2.3}}$	$\underline{\mathbf{2.2}}$	$\underline{\mathbf{2.3}}$	$\underline{\mathbf{1.8}}$
	$\mathrm{LBM}_{\mathrm{w}}$	3.4	$\underline{\mathbf{2.8}}$	$\underline{\mathbf{2.2}}$	$\underline{\mathbf{2.5}}$	$\underline{\mathbf{2.5}}$	$\underline{\mathbf{2.7}}$	$\underline{\mathbf{2.1}}$	$\underline{\mathbf{2.5}}$	$\underline{\mathbf{2.3}}$	$\underline{\mathbf{2.3}}$	$\underline{\mathbf{2.3}}$	$\underline{\mathbf{2.0}}$
	$\mathrm{BP}_{\mathrm{SN}}$	**3.3**	$\underline{\mathbf{3.2}}$	$\underline{\mathbf{2.8}}$	$\underline{\mathbf{1.4}}$	$\underline{\mathbf{0.9}}$	$\underline{\mathbf{1.2}}$	$\underline{\mathbf{0.7}}$	$\underline{\mathbf{0.8}}$	$\underline{\mathbf{0.6}}$	$\underline{\mathbf{0.6}}$	$\underline{\mathbf{0.6}}$	$\underline{\mathbf{0.3}}$
	$\mathrm{LBM}_{\mathrm{SN}}$	**3.3**	$\underline{\mathbf{3.2}}$	$\underline{\mathbf{2.8}}$	$\underline{\mathbf{1.4}}$	$\underline{\mathbf{0.9}}$	$\underline{\mathbf{1.2}}$	$\underline{\mathbf{0.7}}$	$\underline{\mathbf{0.8}}$	$\underline{\mathbf{0.6}}$	$\underline{\mathbf{0.6}}$	$\underline{\mathbf{0.6}}$	$\underline{\mathbf{0.3}}$
		1	2	3	4	5	6	7	8	9	10	11	12
	$\mathrm{BP}_{\mathrm{s}}(1)$	$\underline{\mathbf{15.5}}$	$\underline{\mathbf{17.7}}$	$\underline{\mathbf{19.3}}$	$\underline{\mathbf{21.2}}$	$\underline{\mathbf{23.7}}$	$\underline{\mathbf{24.4}}$	$\underline{\mathbf{25.3}}$	$\underline{\mathbf{26.5}}$	$\underline{\mathbf{27.3}}$	$\underline{\mathbf{26.6}}$	$\underline{\mathbf{27.9}}$	$\underline{\mathbf{28.4}}$
	$\mathrm{BP}_{\mathrm{s}}(2)$	$\underline{\mathbf{13.1}}$	$\underline{\mathbf{15.6}}$	$\underline{\mathbf{18.5}}$	$\underline{\mathbf{19.9}}$	$\underline{\mathbf{20.4}}$	$\underline{\mathbf{21.9}}$	$\underline{\mathbf{22.9}}$	$\underline{\mathbf{22.0}}$	$\underline{\mathbf{23.3}}$	$\underline{\mathbf{22.2}}$	$\underline{\mathbf{23.9}}$	$\underline{\mathbf{23.8}}$
	$\mathrm{LBM}_{\mathrm{s}}(1)$	$\underline{\mathbf{15.5}}$	$\underline{\mathbf{17.7}}$	$\underline{\mathbf{19.3}}$	$\underline{\mathbf{21.3}}$	$\underline{\mathbf{23.7}}$	$\underline{\mathbf{24.5}}$	$\underline{\mathbf{25.3}}$	$\underline{\mathbf{26.5}}$	$\underline{\mathbf{27.3}}$	$\underline{\mathbf{26.7}}$	$\underline{\mathbf{28.4}}$	$\underline{\mathbf{28.6}}$
	$\mathrm{LBM}_{\mathrm{s}}(2)$	$\underline{\mathbf{13.1}}$	$\underline{\mathbf{15.7}}$	$\underline{\mathbf{18.5}}$	$\underline{\mathbf{20.0}}$	$\underline{\mathbf{20.4}}$	$\underline{\mathbf{21.9}}$	$\underline{\mathbf{22.9}}$	$\underline{\mathbf{22.0}}$	$\underline{\mathbf{23.4}}$	$\underline{\mathbf{22.6}}$	$\underline{\mathbf{24.1}}$	$\underline{\mathbf{23.9}}$
	$\mathrm{BP}_{\mathrm{w}}(1)$	5.1	5.2	4.5	5.0	4.4	4.0	4.6	4.5	4.7	5.2	4.9	4.0
	$\mathrm{BP}_{\mathrm{w}}(2)$	5.1	5.3	4.4	3.7	3.7	4.4	4.1	3.9	**3.3**	3.7	3.5	**3.3**
	$\mathrm{LBM}_{\mathrm{w}}(1)$	5.1	5.2	4.5	5.0	4.4	4.0	4.6	4.5	4.7	5.3	5.1	4.0
	$\mathrm{LBM}_{\mathrm{w}}(2)$	5.1	5.3	4.4	3.8	3.8	4.4	4.1	3.9	3.4	3.8	3.5	3.4
$N = 2,000$	$\mathrm{BP}_{\mathrm{SN}}(1)$	5.4	5.1	4.5	3.4	4.8	4.7	4.3	4.7	4.1	4.0	3.7	$\underline{\mathbf{3.2}}$
	$\mathrm{BP}_{\mathrm{SN}}(2)$	5.8	5.1	4.5	3.6	4.3	3.7	4.5	4.1	3.9	3.8	3.9	$\underline{\mathbf{3.2}}$
	$\mathrm{LBM}_{\mathrm{SN}}(1)$	5.4	5.1	4.5	3.5	4.8	4.7	4.3	4.7	4.1	4.0	3.7	$\underline{\mathbf{3.2}}$
	$\mathrm{LBM}_{\mathrm{SN}}(2)$	5.8	5.1	4.5	3.6	4.3	3.7	4.5	4.1	3.9	3.9	4.0	**3.3**
	BP_{s}	$\underline{\mathbf{19.1}}$	$\underline{\mathbf{21.9}}$	$\underline{\mathbf{26.1}}$	$\underline{\mathbf{29.0}}$	$\underline{\mathbf{30.1}}$	$\underline{\mathbf{30.7}}$	$\underline{\mathbf{31.3}}$	$\underline{\mathbf{34.1}}$	$\underline{\mathbf{34.7}}$	$\underline{\mathbf{34.6}}$	$\underline{\mathbf{36.4}}$	$\underline{\mathbf{36.5}}$
	$\mathrm{LBM}_{\mathrm{s}}$	$\underline{\mathbf{19.1}}$	$\underline{\mathbf{22.0}}$	$\underline{\mathbf{26.1}}$	$\underline{\mathbf{29.0}}$	$\underline{\mathbf{30.3}}$	$\underline{\mathbf{30.7}}$	$\underline{\mathbf{31.6}}$	$\underline{\mathbf{34.1}}$	$\underline{\mathbf{35.0}}$	$\underline{\mathbf{34.8}}$	$\underline{\mathbf{36.5}}$	$\underline{\mathbf{36.8}}$
	BP_{w}	5.3	5.0	4.2	3.8	3.9	3.9	4.1	4.1	4.0	3.4	3.4	3.9
	$\mathrm{LBM}_{\mathrm{w}}$	5.3	5.0	4.2	3.8	3.9	3.9	4.2	4.2	4.0	3.4	3.4	4.1
	$\mathrm{BP}_{\mathrm{SN}}$	6.2	5.0	4.4	4.1	3.7	4.0	3.9	3.8	$\underline{\mathbf{3.0}}$	$\underline{\mathbf{1.7}}$	$\underline{\mathbf{1.3}}$	$\underline{\mathbf{1.3}}$
	$\mathrm{LBM}_{\mathrm{SN}}$	6.2	5.0	4.4	4.1	3.7	4.0	3.9	3.8	$\underline{\mathbf{3.1}}$	$\underline{\mathbf{1.7}}$	$\underline{\mathbf{1.4}}$	$\underline{\mathbf{1.3}}$

Table 6

Empirical size of the standard and proposed tests: relative frequencies (in %) of rejection of a strong $PARMA_2(1,1)$ given by (44) with the parameter $\alpha_0 = (0.8, 0.9, 0, 0)'$ and where the innovation $\epsilon_t \sim \mathcal{N}(0,1)$. When the relative rejection frequencies are outside the significant limits with probability 95% (resp. 99%), they are displayed in bold (resp. underlined) type.

Length N	Tests	Lag m											
		1	2	3	4	5	6	7	8	9	10	11	12
	$\mathrm{BP_s}(1)$	n.a.	$\underline{\mathbf{11.8}}$	$\underline{\mathbf{9.1}}$	$\underline{\mathbf{8.5}}$	$\underline{\mathbf{7.6}}$	$\underline{\mathbf{7.2}}$	$\underline{\mathbf{7.5}}$	**6.5**	6.4	**6.8**	6.3	6.4
	$\mathrm{BP_s}(2)$	n.a.	$\underline{\mathbf{8.4}}$	$\underline{\mathbf{7.0}}$	**6.7**	**6.7**	$\underline{\mathbf{7.0}}$	$\underline{\mathbf{7.2}}$	$\underline{\mathbf{7.4}}$	**6.6**	$\underline{\mathbf{7.4}}$	**6.8**	**6.9**
	$\mathrm{LBM_s}(1)$	n.a.	$\underline{\mathbf{11.9}}$	$\underline{\mathbf{9.1}}$	$\underline{\mathbf{8.8}}$	$\underline{\mathbf{7.6}}$	$\underline{\mathbf{7.4}}$	$\underline{\mathbf{7.7}}$	**6.6**	**6.9**	$\underline{\mathbf{7.0}}$	$\underline{\mathbf{7.0}}$	**6.6**
	$\mathrm{LBM_s}(2)$	n.a.	$\underline{\mathbf{8.5}}$	$\underline{\mathbf{7.3}}$	**6.9**	**6.8**	$\underline{\mathbf{7.1}}$	$\underline{\mathbf{7.2}}$	$\underline{\mathbf{7.5}}$	**6.9**	$\underline{\mathbf{7.6}}$	$\underline{\mathbf{7.2}}$	$\underline{\mathbf{7.3}}$
	$\mathrm{BP_w}(1)$	5.5	3.9	3.8	$\underline{\mathbf{3.1}}$	$\underline{\mathbf{2.8}}$	$\underline{\mathbf{2.0}}$	$\underline{\mathbf{2.5}}$	$\underline{\mathbf{2.2}}$	$\underline{\mathbf{1.7}}$	$\underline{\mathbf{1.9}}$	$\underline{\mathbf{2.0}}$	$\underline{\mathbf{1.5}}$
	$\mathrm{BP_w}(2)$	4.2	4.2	4.4	$\underline{\mathbf{2.8}}$	$\underline{\mathbf{2.9}}$	$\underline{\mathbf{2.3}}$	$\underline{\mathbf{2.5}}$	$\underline{\mathbf{2.1}}$	$\underline{\mathbf{1.5}}$	$\underline{\mathbf{1.7}}$	$\underline{\mathbf{1.0}}$	$\underline{\mathbf{0.9}}$
	$\mathrm{LBM_w}(1)$	5.5	3.9	3.8	$\underline{\mathbf{3.1}}$	$\underline{\mathbf{2.8}}$	$\underline{\mathbf{2.1}}$	$\underline{\mathbf{2.6}}$	$\underline{\mathbf{2.3}}$	$\underline{\mathbf{1.8}}$	$\underline{\mathbf{2.0}}$	$\underline{\mathbf{2.1}}$	$\underline{\mathbf{1.8}}$
	$\mathrm{LBM_w}(2)$	4.2	4.5	4.4	$\underline{\mathbf{3.1}}$	$\underline{\mathbf{3.1}}$	$\underline{\mathbf{2.4}}$	$\underline{\mathbf{2.8}}$	$\underline{\mathbf{2.3}}$	$\underline{\mathbf{1.6}}$	$\underline{\mathbf{2.2}}$	$\underline{\mathbf{1.4}}$	$\underline{\mathbf{1.0}}$
$N = 400$	$\mathrm{BP_{SN}}(1)$	5.4	5.3	5.8	4.9	4.6	5.7	6.0	5.5	5.3	6.1	5.1	5.6
	$\mathrm{BP_{SN}}(2)$	4.8	6.0	5.9	5.8	5.6	5.6	5.2	6.1	5.1	5.4	5.9	6.3
	$\mathrm{LBM_{SN}}(1)$	5.4	5.4	5.8	4.9	4.6	5.9	6.3	5.6	5.5	6.2	5.3	5.7
	$\mathrm{LBM_{SN}}(2)$	4.8	6.0	6.3	5.9	5.6	5.7	5.3	6.1	5.3	5.6	6.2	6.5
	$\mathrm{BP_s}$	n.a.	$\underline{\mathbf{12.3}}$	$\underline{\mathbf{9.8}}$	$\underline{\mathbf{8.8}}$	$\underline{\mathbf{9.0}}$	$\underline{\mathbf{8.9}}$	$\underline{\mathbf{8.5}}$	$\underline{\mathbf{8.1}}$	$\underline{\mathbf{8.1}}$	$\underline{\mathbf{8.5}}$	$\underline{\mathbf{7.8}}$	$\underline{\mathbf{8.1}}$
	$\mathrm{LBM_s}$	n.a.	$\underline{\mathbf{12.4}}$	$\underline{\mathbf{9.9}}$	$\underline{\mathbf{9.0}}$	$\underline{\mathbf{9.1}}$	$\underline{\mathbf{8.9}}$	$\underline{\mathbf{8.8}}$	$\underline{\mathbf{8.6}}$	$\underline{\mathbf{8.3}}$	$\underline{\mathbf{8.9}}$	$\underline{\mathbf{8.6}}$	$\underline{\mathbf{8.5}}$
	$\mathrm{BP_w}$	4.4	4.0	**3.3**	3.4	$\underline{\mathbf{2.6}}$	$\underline{\mathbf{1.7}}$	$\underline{\mathbf{1.4}}$	$\underline{\mathbf{1.4}}$	$\underline{\mathbf{1.1}}$	$\underline{\mathbf{1.1}}$	$\underline{\mathbf{0.9}}$	$\underline{\mathbf{0.5}}$
	$\mathrm{LBM_w}$	4.4	4.1	**3.3**	3.5	$\underline{\mathbf{2.6}}$	$\underline{\mathbf{2.0}}$	$\underline{\mathbf{1.5}}$	$\underline{\mathbf{1.4}}$	$\underline{\mathbf{1.2}}$	$\underline{\mathbf{1.2}}$	$\underline{\mathbf{1.0}}$	$\underline{\mathbf{0.5}}$
	$\mathrm{BP_{SN}}$	5.0	5.8	6.4	5.6	5.9	5.9	5.3	5.5	6.0	5.9	4.9	5.0
	$\mathrm{LBM_{SN}}$	5.0	5.8	6.4	5.7	6.0	6.3	5.7	5.6	6.1	6.2	5.2	5.3
		1	2	3	4	5	6	7	8	9	10	11	12
	$\mathrm{BP_s}(1)$	n.a.	$\underline{\mathbf{13.1}}$	$\underline{\mathbf{9.6}}$	$\underline{\mathbf{9.7}}$	$\underline{\mathbf{8.3}}$	$\underline{\mathbf{7.1}}$	$\underline{\mathbf{7.5}}$	$\underline{\mathbf{7.2}}$	$\underline{\mathbf{7.2}}$	$\underline{\mathbf{7.7}}$	$\underline{\mathbf{8.2}}$	$\underline{\mathbf{8.3}}$
	$\mathrm{BP_s}(2)$	n.a.	$\underline{\mathbf{9.5}}$	$\underline{\mathbf{7.3}}$	**6.9**	**6.9**	$\underline{\mathbf{7.0}}$	6.3	6.4	6.4	5.3	4.6	4.5
	$\mathrm{LBM_s}(1)$	n.a.	$\underline{\mathbf{13.1}}$	$\underline{\mathbf{9.6}}$	$\underline{\mathbf{9.7}}$	$\underline{\mathbf{8.3}}$	$\underline{\mathbf{7.1}}$	$\underline{\mathbf{7.7}}$	$\underline{\mathbf{7.2}}$	$\underline{\mathbf{7.2}}$	$\underline{\mathbf{7.7}}$	$\underline{\mathbf{8.2}}$	$\underline{\mathbf{8.3}}$
	$\mathrm{LBM_s}(2)$	n.a.	$\underline{\mathbf{9.5}}$	$\underline{\mathbf{7.3}}$	**6.9**	$\underline{\mathbf{7.0}}$	$\underline{\mathbf{7.0}}$	6.4	6.4	**6.5**	5.4	4.8	4.6
	$\mathrm{BP_w}(1)$	**6.5**	5.3	4.8	5.0	4.9	4.3	3.9	4.1	4.3	4.6	5.4	5.7
	$\mathrm{BP_w}(2)$	5.1	5.6	5.8	4.8	5.4	5.2	5.1	4.7	4.3	$\underline{\mathbf{3.0}}$	$\underline{\mathbf{3.0}}$	$\underline{\mathbf{2.9}}$
	$\mathrm{LBM_w}(1)$	**6.5**	5.3	4.8	5.0	4.9	4.3	3.9	4.1	4.3	4.6	5.5	5.7
	$\mathrm{LBM_w}(2)$	5.1	5.6	5.8	4.8	5.4	5.2	5.1	4.8	4.4	$\underline{\mathbf{3.1}}$	$\underline{\mathbf{3.0}}$	$\underline{\mathbf{3.0}}$
$N = 2,000$	$\mathrm{BP_{SN}}(1)$	6.1	5.1	5.9	5.6	6.1	6.4	6.1	**6.7**	**6.5**	$\underline{\mathbf{7.0}}$	6.2	6.3
	$\mathrm{BP_{SN}}(2)$	5.2	5.5	4.2	5.3	5.7	6.3	6.0	6.0	6.1	6.2	6.4	6.1
	$\mathrm{LBM_{SN}}(1)$	6.1	5.1	5.9	5.6	6.1	6.4	6.1	**6.7**	**6.5**	$\underline{\mathbf{7.0}}$	6.2	6.4
	$\mathrm{LBM_{SN}}(2)$	5.2	5.5	4.2	5.3	5.7	6.3	6.1	6.0	6.2	6.4	6.4	6.1
	$\mathrm{BP_s}$	n.a.	$\underline{\mathbf{15.3}}$	$\underline{\mathbf{10.5}}$	$\underline{\mathbf{9.8}}$	$\underline{\mathbf{8.7}}$	$\underline{\mathbf{8.7}}$	$\underline{\mathbf{7.6}}$	$\underline{\mathbf{6.8}}$	$\underline{\mathbf{7.5}}$	$\underline{\mathbf{7.2}}$	$\underline{\mathbf{7.5}}$	$\underline{\mathbf{8.2}}$
	$\mathrm{LBM_s}$	n.a.	$\underline{\mathbf{15.3}}$	$\underline{\mathbf{10.5}}$	$\underline{\mathbf{9.8}}$	$\underline{\mathbf{8.9}}$	$\underline{\mathbf{8.7}}$	$\underline{\mathbf{7.7}}$	**6.9**	$\underline{\mathbf{7.5}}$	$\underline{\mathbf{7.3}}$	$\underline{\mathbf{7.5}}$	$\underline{\mathbf{8.3}}$
	$\mathrm{BP_w}$	5.9	5.5	4.9	4.8	4.4	4.0	4.2	4.1	3.5	4.0	**3.3**	4.0
	$\mathrm{LBM_w}$	5.9	5.5	4.9	4.8	4.4	4.0	4.2	4.1	3.6	4.2	3.4	4.0
	$\mathrm{BP_{SN}}$	**6.6**	5.6	4.5	4.7	5.0	6.0	6.0	6.2	5.3	5.9	6.3	**6.5**
	$\mathrm{LBM_{SN}}$	**6.6**	5.6	4.5	4.7	5.0	6.0	6.0	6.2	5.3	6.2	6.3	**6.5**

TABLE 7

Empirical size of the standard and proposed tests: relative frequencies (in %) of rejection of a weak $PARMA_2(1,1)$ given by (44) with the parameter $\alpha_0 = (0.8, 0.9, 0, 0)'$ and where the innovation $\epsilon_t \sim$ (45). When the relative rejection frequencies are outside the significant limits with probability 95% (resp. 99%), they are displayed in bold (resp. underlined) type.

Length N	Tests	Lag m											
		1	2	3	4	5	6	7	8	9	10	11	12
	$\mathrm{BP_S}(1)$	n.a.	**18.5**	**14.5**	**12.6**	**12.2**	**10.8**	**10.0**	**9.9**	**10.8**	**9.5**	**8.6**	**8.5**
	$\mathrm{BP_S}(2)$	n.a.	**15.9**	**13.5**	**9.6**	**9.8**	**9.2**	**9.4**	**9.3**	**8.6**	**8.2**	**8.9**	**8.6**
	$\mathrm{LBM_S}(1)$	n.a.	**18.6**	**14.5**	**13.0**	**12.3**	**11.0**	**10.2**	**10.0**	**11.1**	**9.7**	**8.9**	**9.1**
	$\mathrm{LBM_S}(2)$	n.a.	**16.1**	**13.5**	**9.7**	**10.0**	**9.3**	**9.4**	**9.6**	**8.8**	**9.2**	**9.0**	**8.8**
	$\mathrm{BP_W}(1)$	4.3	**3.1**	**2.5**	**2.9**	**2.7**	**2.1**	**1.8**	**1.9**	**1.4**	**1.2**	**1.1**	**1.0**
	$\mathrm{BP_W}(2)$	3.6	3.5	**2.8**	**2.6**	**1.9**	**1.9**	**2.0**	**1.3**	**1.1**	**1.1**	**0.8**	**0.7**
	$\mathrm{LBM_W}(1)$	4.3	**3.1**	**2.5**	**2.9**	**2.7**	**2.2**	**1.8**	**1.9**	**1.4**	**1.2**	**1.3**	**1.0**
	$\mathrm{LBM_W}(2)$	3.6	3.6	**2.8**	**2.7**	**2.1**	**1.9**	**2.2**	**1.3**	**1.2**	**1.2**	**1.0**	**1.1**
$N = 400$	$\mathrm{BP_{SN}}(1)$	4.4	3.7	3.8	4.4	3.9	4.2	5.0	4.5	3.9	4.3	3.7	3.5
	$\mathrm{BP_{SN}}(2)$	3.5	4.8	6.4	5.4	4.5	5.8	4.7	5.5	5.0	4.0	3.7	4.8
	$\mathrm{LBM_{SN}}(1)$	4.4	3.7	3.8	4.5	4.0	4.2	5.2	4.5	3.9	4.7	3.8	3.6
	$\mathrm{LBM_{SN}}(2)$	3.5	4.8	6.4	5.4	4.6	5.8	4.7	5.7	5.0	4.4	4.1	5.0
	$\mathrm{BP_S}$	n.a.	**24.4**	**19.1**	**16.0**	**14.6**	**13.6**	**13.3**	**13.6**	**12.4**	**12.5**	**11.5**	**11.1**
	$\mathrm{LBM_S}$	n.a.	**24.4**	**19.3**	**16.1**	**15.1**	**14.1**	**13.3**	**13.9**	**13.0**	**12.9**	**12.1**	**11.8**
	$\mathrm{BP_W}$	**2.7**	**2.0**	**2.0**	**2.2**	**1.6**	**1.1**	**0.8**	**0.9**	**0.7**	**0.5**	**0.4**	**0.4**
	$\mathrm{LBM_W}$	**2.7**	**2.0**	**2.0**	**2.2**	**1.7**	**1.1**	**0.8**	**0.9**	**0.8**	**0.5**	**0.4**	**0.4**
	$\mathrm{BP_{SN}}$	4.0	4.4	5.0	4.7	5.3	4.4	3.7	3.9	**3.2**	**2.6**	**1.4**	**1.8**
	$\mathrm{LBM_{SN}}$	4.0	4.5	5.0	4.8	5.5	4.4	4.0	4.0	3.4	**2.7**	**1.6**	**2.1**
		1	2	3	4	5	6	7	8	9	10	11	12
	$\mathrm{BP_S}(1)$	n.a.	**23.6**	**17.3**	**15.7**	**15.3**	**14.4**	**14.7**	**14.0**	**12.7**	**11.9**	**11.6**	**11.3**
	$\mathrm{BP_S}(2)$	n.a.	**18.4**	**14.2**	**12.2**	**11.3**	**10.2**	**8.8**	**8.9**	**8.5**	**7.5**	**7.5**	**7.3**
	$\mathrm{LBM_S}(1)$	n.a.	**23.6**	**17.4**	**15.7**	**15.3**	**14.4**	**14.7**	**14.1**	**13.0**	**12.1**	**11.6**	**11.3**
	$\mathrm{LBM_S}(2)$	n.a.	**18.4**	**14.2**	**12.2**	**11.3**	**10.2**	**8.8**	**9.0**	**8.5**	**7.5**	**7.5**	**7.3**
	$\mathrm{BP_W}(1)$	6.2	5.1	4.2	3.9	4.0	4.6	4.2	3.6	3.7	4.1	3.8	4.0
	$\mathrm{BP_W}(2)$	5.1	4.9	4.6	4.7	3.5	**3.2**	3.5	3.5	3.5	**2.9**	**2.6**	**3.0**
	$\mathrm{LBM_W}(1)$	6.2	5.1	4.2	3.9	4.0	4.6	4.3	3.6	3.8	4.1	3.8	4.1
	$\mathrm{LBM_W}(2)$	5.1	4.9	4.6	4.7	3.5	**3.3**	3.5	3.5	3.7	**3.0**	**2.6**	**3.2**
$N = 2,000$	$\mathrm{BP_{SN}}(1)$	4.9	4.8	5.3	5.4	5.8	5.5	5.9	5.7	5.4	5.4	5.5	5.3
	$\mathrm{BP_{SN}}(2)$	4.8	5.2	4.8	4.6	5.1	6.2	5.0	5.4	5.7	5.7	6.0	5.3
	$\mathrm{LBM_{SN}}(1)$	4.9	4.8	5.3	5.4	5.8	5.6	6.0	5.7	5.5	5.5	5.5	5.4
	$\mathrm{LBM_{SN}}(2)$	4.8	5.2	4.8	4.6	5.1	6.2	5.1	5.4	5.7	5.8	6.0	5.4
	$\mathrm{BP_S}$	n.a.	**30.8**	**23.4**	**20.1**	**17.3**	**16.3**	**15.5**	**14.1**	**12.5**	**12.3**	**11.7**	**12.1**
	$\mathrm{LBM_S}$	n.a.	**30.8**	**23.5**	**20.2**	**17.4**	**16.3**	**15.6**	**14.1**	**12.5**	**12.4**	**11.8**	**12.3**
	$\mathrm{BP_W}$	4.7	4.3	3.6	3.7	3.6	**2.9**	**3.1**	**2.6**	**2.8**	**2.9**	**3.0**	3.4
	$\mathrm{LBM_W}$	4.7	4.3	3.6	3.8	3.6	**2.9**	**3.1**	**2.6**	**2.8**	**2.9**	**3.0**	3.4
	$\mathrm{BP_{SN}}$	5.6	6.2	5.7	5.2	5.6	5.5	5.2	6.1	5.6	5.1	4.9	5.8
	$\mathrm{LBM_{SN}}$	5.6	6.2	5.7	5.2	5.6	5.5	5.2	6.2	5.7	5.1	4.9	5.9

Table 8

Empirical size of the standard and proposed tests: relative frequencies (in %) of rejection of a weak $PARMA_2(1,1)$ given by (44) with the parameter $\alpha_0 = (0.8, 0.9, 0, 0)'$ and where the innovation $\epsilon_t \sim$ (75). When the relative rejection frequencies are outside the significant limits with probability 95% (resp. 99%), they are displayed in bold (resp. underlined) type.

Length N	Tests	Lag m											
		1	2	3	4	5	6	7	8	9	10	11	12
$N = 400$	BP$_{\mathrm{s}}$(1)	n.a.	**22.2**	**21.5**	**21.2**	**20.4**	**21.2**	**21.3**	**21.0**	**20.7**	**20.5**	**21.4**	**20.3**
	BP$_{\mathrm{s}}$(2)	n.a.	**20.9**	**18.3**	**18.3**	**19.4**	**19.5**	**18.4**	**19.5**	**19.0**	**19.0**	**19.6**	**18.5**
	LBM$_{\mathrm{s}}$(1)	n.a.	**22.2**	**21.6**	**21.6**	**20.5**	**21.6**	**21.3**	**21.8**	**21.4**	**20.9**	**21.7**	**21.3**
	LBM$_{\mathrm{s}}$(2)	n.a.	**20.9**	**18.3**	**18.4**	**19.7**	**19.8**	**18.5**	**19.7**	**19.3**	**19.6**	**20.0**	**19.2**
	BP$_{\mathrm{w}}$(1)	5.0	**3.1**	3.8	**2.8**	**1.9**	**1.5**	**1.4**	**1.3**	**0.9**	**0.8**	**0.6**	**0.3**
	BP$_{\mathrm{w}}$(2)	4.1	**3.1**	**2.5**	**2.1**	**1.4**	**0.8**	**0.8**	**0.9**	**0.5**	**0.4**	**0.6**	**0.2**
	LBM$_{\mathrm{w}}$(1)	5.0	**3.2**	3.8	**3.0**	**2.1**	**1.7**	**1.5**	**1.4**	**0.9**	**1.0**	**0.8**	**0.6**
	LBM$_{\mathrm{w}}$(2)	4.1	**3.1**	**2.5**	**2.1**	**1.5**	**0.8**	**0.8**	**1.0**	**0.7**	**0.6**	**0.6**	**0.2**
	BP$_{\mathrm{SN}}$(1)	4.2	3.7	3.6	**3.3**	3.4	**2.3**	**2.7**	**2.7**	**1.9**	**1.5**	**1.7**	**1.7**
	BP$_{\mathrm{SN}}$(2)	3.8	**3.3**	**3.3**	3.7	**2.9**	**2.4**	**3.2**	**2.9**	**1.8**	**1.7**	**1.6**	**1.7**
	LBM$_{\mathrm{SN}}$(1)	4.2	3.7	3.6	3.4	3.4	**2.4**	**2.7**	**2.7**	**1.9**	**1.6**	**1.8**	**1.8**
	LBM$_{\mathrm{SN}}$(2)	3.8	3.4	**3.3**	3.7	**2.9**	**2.5**	**3.3**	**3.3**	**1.8**	**1.7**	**1.7**	**1.8**
	BP$_{\mathrm{s}}$	n.a.	**31.4**	**28.8**	**28.2**	**28.4**	**29.3**	**28.3**	**28.5**	**29.3**	**28.7**	**29.4**	**29.0**
	LBM$_{\mathrm{s}}$	n.a.	**31.5**	**28.9**	**28.5**	**28.7**	**29.8**	**28.4**	**29.4**	**30.3**	**29.4**	**30.4**	**30.1**
	BP$_{\mathrm{w}}$	3.9	**2.2**	**2.1**	**1.7**	**0.6**	**0.7**	**0.1**	**0.4**	**0.3**	**0.1**	**0.1**	**0.0**
	LBM$_{\mathrm{w}}$	3.9	**2.2**	**2.2**	**1.9**	**0.7**	**0.7**	**0.1**	**0.5**	**0.3**	**0.1**	**0.1**	**0.0**
	BP$_{\mathrm{SN}}$	**3.1**	3.9	**2.9**	**2.4**	**2.1**	**1.5**	**1.0**	**1.2**	**0.6**	**0.5**	**0.5**	**0.5**
	LBM$_{\mathrm{SN}}$	**3.1**	3.9	**2.9**	**2.4**	**2.2**	**1.6**	**1.0**	**1.2**	**0.6**	**0.6**	**0.6**	**0.5**
		1	2	3	4	5	6	7	8	9	10	11	12
$N = 2,000$	BP$_{\mathrm{s}}$(1)	n.a.	**24.4**	**23.5**	**25.5**	**26.8**	**26.8**	**27.5**	**28.2**	**29.2**	**28.6**	**29.6**	**30.1**
	BP$_{\mathrm{s}}$(2)	n.a.	**24.1**	**24.3**	**24.7**	**22.8**	**23.5**	**24.3**	**25.2**	**25.3**	**25.3**	**25.6**	**25.5**
	LBM$_{\mathrm{s}}$(1)	n.a.	**24.6**	**23.5**	**25.5**	**26.9**	**27.0**	**27.5**	**28.2**	**29.6**	**28.6**	**29.7**	**30.2**
	LBM$_{\mathrm{s}}$(2)	n.a.	**24.1**	**24.3**	**24.8**	**22.8**	**23.6**	**24.4**	**25.3**	**25.3**	**25.4**	**25.7**	**25.5**
	BP$_{\mathrm{w}}$(1)	4.6	5.3	4.7	3.7	3.7	**2.9**	**3.1**	**3.2**	**2.7**	3.6	**3.3**	**2.8**
	BP$_{\mathrm{w}}$(2)	5.4	4.9	4.5	4.0	**3.3**	3.5	**3.0**	**3.2**	**2.8**	**2.3**	**1.8**	**1.7**
	LBM$_{\mathrm{w}}$(1)	4.6	5.3	4.7	3.7	3.7	**2.9**	**3.1**	**3.2**	**2.7**	3.7	**3.3**	**2.8**
	LBM$_{\mathrm{w}}$(2)	5.4	4.9	4.5	4.0	**3.3**	3.6	**3.0**	**3.2**	**2.8**	**2.4**	**1.8**	**1.7**
	BP$_{\mathrm{SN}}$(1)	5.0	4.8	4.2	4.6	5.0	4.6	4.5	5.1	4.5	3.8	3.9	3.7
	BP$_{\mathrm{SN}}$(2)	4.5	4.2	4.5	4.3	4.4	3.9	3.8	4.1	3.6	4.0	3.8	**3.0**
	LBM$_{\mathrm{SN}}$(1)	5.0	4.8	4.2	4.6	5.0	4.6	4.5	5.1	4.5	3.8	3.9	3.7
	LBM$_{\mathrm{SN}}$(2)	4.5	4.3	4.5	4.3	4.4	3.9	3.9	4.1	3.7	4.1	3.9	**3.1**
	BP$_{\mathrm{s}}$	n.a.	**38.2**	**36.2**	**37.1**	**37.3**	**36.7**	**36.5**	**38.7**	**39.9**	**39.2**	**39.5**	**38.5**
	LBM$_{\mathrm{s}}$	n.a.	**38.2**	**36.4**	**37.2**	**37.5**	**37.0**	**36.6**	**38.8**	**40.0**	**39.3**	**39.7**	**38.6**
	BP$_{\mathrm{w}}$	5.3	4.7	**3.1**	**2.7**	**2.8**	**2.8**	**2.2**	**2.3**	**2.2**	**1.9**	**1.8**	**2.0**
	LBM$_{\mathrm{w}}$	5.3	4.7	**3.1**	**2.7**	**2.8**	**2.8**	**2.2**	**2.3**	**2.2**	**1.9**	**1.9**	**2.0**
	BP$_{\mathrm{SN}}$	5.6	5.3	4.6	5.0	4.2	3.8	3.9	3.4	**2.9**	**2.2**	**1.7**	**1.9**
	LBM$_{\mathrm{SN}}$	5.6	5.3	4.6	5.0	4.4	3.8	3.9	3.4	**2.9**	**2.2**	**1.8**	**1.9**

Table 9

Empirical size of the standard and proposed tests: relative frequencies (in %) of rejection of a strong $PARMA_2(1,1)$ given by (44) with the parameter $\alpha_0 = (0.8, 0.9, -0.5, -0.45)'$ and where the innovation $\epsilon_t \sim \mathcal{N}(0,1)$. When the relative rejection frequencies are outside the significant limits with probability 95% (resp. 99%), they are displayed in bold (resp. underlined) type.

Length N	Tests	Lag m											
		1	2	3	4	5	6	7	8	9	10	11	12
	$BP_s(1)$	n.a.	n.a.	**17.2**	**13.3**	**10.9**	**10.2**	**10.0**	**9.1**	**9.8**	**9.3**	**9.2**	**8.6**
	$BP_s(2)$	n.a.	n.a.	**19.9**	**13.5**	**12.1**	**11.3**	**11.2**	**10.1**	**9.8**	**10.2**	**10.2**	**9.4**
	$LBM_s(1)$	n.a.	n.a.	**17.3**	**13.4**	**11.1**	**10.2**	**10.1**	**9.4**	**10.3**	**9.9**	**9.9**	**9.3**
	$LBM_s(2)$	n.a.	n.a.	**20.1**	**13.7**	**12.3**	**11.6**	**11.5**	**10.3**	**10.2**	**10.3**	**11.0**	**9.6**
	$BP_w(1)$	4.1	3.9	3.5	**2.8**	**2.6**	**2.7**	**2.7**	**2.5**	**2.1**	**2.8**	**1.9**	**1.9**
	$BP_w(2)$	5.0	5.4	5.2	3.9	3.5	**3.1**	**3.2**	**2.8**	**2.0**	**2.0**	**1.9**	**1.6**
	$LBM_w(1)$	4.1	3.9	3.7	**2.8**	**2.8**	**2.7**	**2.7**	**2.8**	**2.5**	**3.1**	**2.0**	**2.0**
	$LBM_w(2)$	5.0	5.4	5.2	3.9	3.6	**3.1**	3.4	**2.9**	**2.3**	**2.0**	**1.9**	**1.6**
$N = 400$	$BP_{SN}(1)$	5.3	5.3	4.6	3.9	4.3	5.2	5.1	5.0	5.4	5.3	6.0	6.2
	$BP_{SN}(2)$	6.0	5.6	5.3	5.1	4.6	6.6	5.8	5.6	5.0	5.6	5.1	5.5
	$LBM_{SN}(1)$	5.3	5.3	4.6	3.9	4.3	5.2	5.2	5.2	5.6	5.3	6.3	**6.5**
	$LBM_{SN}(2)$	6.0	5.7	5.4	5.1	4.7	**6.9**	6.2	5.8	5.1	5.9	5.3	5.9
	BP_s	n.a.	n.a.	**26.8**	**20.3**	**17.1**	**16.3**	**15.8**	**13.0**	**12.1**	**13.3**	**12.0**	**11.5**
	LBM_s	n.a.	n.a.	**27.2**	**20.6**	**17.5**	**16.8**	**16.5**	**13.5**	**13.0**	**14.2**	**12.6**	**12.8**
	BP_w	4.8	3.7	3.8	**2.8**	**2.3**	**1.9**	**1.8**	**1.1**	**1.6**	**1.5**	**1.1**	**0.8**
	LBM_w	4.8	3.7	3.9	**3.0**	**2.3**	**1.9**	**1.8**	**1.2**	**1.8**	**1.5**	**1.2**	**0.9**
	BP_{SN}	3.9	3.6	4.9	4.9	5.4	5.9	6.1	**6.7**	5.8	4.8	5.1	4.5
	LBM_{SN}	3.9	3.8	4.9	4.9	5.4	6.0	6.2	**6.7**	6.0	5.1	5.2	5.0
		1	2	3	4	5	6	7	8	9	10	11	12
	$BP_s(1)$	n.a.	n.a.	**17.9**	**12.1**	**10.2**	**8.2**	**8.0**	**8.5**	**7.6**	**8.1**	**7.8**	**8.1**
	$BP_s(2)$	n.a.	n.a.	**18.3**	**12.9**	**11.9**	**11.2**	**9.5**	**10.1**	**9.2**	**9.2**	**8.7**	**8.2**
	$LBM_s(1)$	n.a.	n.a.	**18.0**	**12.1**	**10.2**	**8.3**	**8.0**	**8.6**	**7.6**	**8.2**	**7.8**	**8.1**
	$LBM_s(2)$	n.a.	n.a.	**18.3**	**13.0**	**11.9**	**11.2**	**9.6**	**10.1**	**9.3**	**9.2**	**8.9**	**8.2**
	$BP_w(1)$	4.3	4.9	4.5	4.7	4.3	3.9	3.8	4.4	3.6	3.7	4.1	5.1
	$BP_w(2)$	5.4	5.9	5.3	4.4	4.7	4.6	5.3	4.9	5.0	4.7	3.9	**3.0**
	$LBM_w(1)$	4.3	4.9	4.5	4.7	4.3	3.9	3.8	4.4	3.6	3.7	4.1	5.1
	$LBM_w(2)$	5.4	5.9	5.3	4.4	4.7	4.6	5.3	4.9	5.0	4.9	4.0	**3.0**
$N = 2,000$	$BP_{SN}(1)$	4.7	4.8	5.4	4.2	5.2	5.6	5.7	5.5	5.2	5.2	5.6	5.6
	$BP_{SN}(2)$	5.9	5.0	4.1	5.7	5.5	4.9	5.5	6.1	6.5	6.4	5.9	5.3
	$LBM_{SN}(1)$	4.7	4.8	5.4	4.2	5.2	5.6	5.8	5.5	5.2	5.2	5.9	5.7
	$LBM_{SN}(2)$	5.9	5.0	4.1	5.7	5.5	4.9	5.5	6.1	**6.5**	6.4	6.1	5.3
	BP_s	n.a.	n.a.	**26.5**	**18.3**	**15.2**	**13.1**	**13.4**	**11.7**	**9.4**	**10.2**	**9.8**	**8.7**
	LBM_s	n.a.	n.a.	**26.5**	**18.3**	**15.2**	**13.1**	**13.6**	**11.7**	**9.4**	**10.3**	**10.0**	**8.7**
	BP_w	4.6	4.9	4.2	4.3	4.2	4.0	3.6	**3.2**	3.5	3.6	3.4	**3.0**
	LBM_w	4.6	4.9	4.2	4.3	4.2	4.0	3.6	**3.3**	3.7	3.7	3.4	**3.0**
	BP_{SN}	4.7	5.4	5.4	4.7	4.6	4.7	5.4	5.3	4.9	**6.7**	**6.8**	**6.5**
	LBM_{SN}	4.7	5.4	5.4	4.7	4.6	4.8	5.4	5.4	5.0	**6.7**	**6.9**	**6.5**

TABLE 10

Empirical size of the standard and proposed tests: relative frequencies (in %) of rejection of a weak $PARMA_2(1,1)$ given by (44) with the parameter $\alpha_0 = (0.8, 0.9, -0.5, -0.45)'$ and where the innovation $\epsilon_t \sim$ (45). When the relative rejection frequencies are outside the significant limits with probability 95% (resp. 99%), they are displayed in bold (resp. underlined) type.

Length N	Tests	Lag m											
		1	2	3	4	5	6	7	8	9	10	11	12
	$\mathrm{BP_s}(1)$	n.a.	n.a.	**24.8**	**16.9**	**14.2**	**11.3**	**10.1**	**10.4**	**10.8**	**10.4**	**9.8**	**9.5**
	$\mathrm{BP_s}(2)$	n.a.	n.a.	**22.5**	**17.5**	**13.6**	**12.4**	**12.1**	**12.1**	**11.1**	**10.8**	**10.4**	**10.3**
	$\mathrm{LBM_s}(1)$	n.a.	n.a.	**24.8**	**17.2**	**14.2**	**11.5**	**10.2**	**10.6**	**11.1**	**10.7**	**10.2**	**9.8**
	$\mathrm{LBM_s}(2)$	n.a.	n.a.	**22.6**	**17.7**	**13.8**	**12.8**	**12.3**	**12.3**	**11.2**	**11.1**	**10.9**	**10.7**
	$\mathrm{BP_w}(1)$	4.6	3.4	**3.0**	**2.8**	**3.1**	**2.8**	**2.0**	**2.3**	**2.1**	**1.7**	**1.3**	**0.9**
	$\mathrm{BP_w}(2)$	3.4	**3.2**	3.5	**3.2**	**2.6**	**2.6**	**2.0**	**1.2**	**1.2**	**0.8**	**1.0**	**0.8**
	$\mathrm{LBM_w}(1)$	4.6	3.4	**3.1**	**2.8**	**3.2**	**2.9**	**2.1**	**2.3**	**2.2**	**2.0**	**1.4**	**1.0**
	$\mathrm{LBM_w}(2)$	3.4	**3.2**	3.6	**3.3**	**2.6**	**2.8**	**2.2**	**1.5**	**1.3**	**0.9**	**1.1**	**0.9**
$N = 400$	$\mathrm{BP_{SN}}(1)$	4.5	4.0	4.4	4.1	4.6	4.4	4.6	4.1	4.7	4.4	3.9	3.8
	$\mathrm{BP_{SN}}(2)$	4.0	4.5	4.4	5.0	4.8	4.5	5.2	5.5	5.4	4.8	4.1	4.4
	$\mathrm{LBM_{SN}}(1)$	4.5	4.0	4.5	4.1	4.7	4.6	4.9	4.3	4.7	4.4	4.1	3.9
	$\mathrm{LBM_{SN}}(2)$	4.0	4.6	4.4	5.1	4.9	4.6	5.2	5.7	5.7	5.0	4.2	4.5
	$\mathrm{BP_s}$	n.a.	n.a.	**35.1**	**24.7**	**20.2**	**17.7**	**16.6**	**14.9**	**15.2**	**15.0**	**14.2**	**13.5**
	$\mathrm{LBM_s}$	n.a.	n.a.	**35.1**	**24.8**	**20.3**	**18.1**	**16.9**	**15.4**	**15.4**	**15.7**	**15.0**	**14.8**
	$\mathrm{BP_w}$	3.4	**2.7**	**2.8**	**2.5**	**1.4**	**1.3**	**1.2**	**1.1**	**0.8**	**0.5**	**0.7**	**0.4**
	$\mathrm{LBM_w}$	3.4	**2.7**	**2.9**	**2.5**	**1.5**	**1.4**	**1.2**	**1.1**	**0.8**	**0.7**	**0.7**	**0.5**
	$\mathrm{BP_{SN}}$	3.4	4.7	4.0	3.8	3.6	3.9	**3.3**	3.4	**2.8**	**2.5**	**1.9**	**1.5**
	$\mathrm{LBM_{SN}}$	3.4	4.7	4.0	3.9	3.7	3.9	3.4	3.4	**3.0**	**2.7**	**2.1**	**1.8**
		1	2	3	4	5	6	7	8	9	10	11	12
	$\mathrm{BP_s}(1)$	n.a.	n.a.	**28.2**	**21.3**	**18.8**	**16.5**	**16.6**	**14.1**	**13.4**	**14.4**	**12.8**	**12.6**
	$\mathrm{BP_s}(2)$	n.a.	n.a.	**25.8**	**19.3**	**15.1**	**13.3**	**12.6**	**10.7**	**10.2**	**10.0**	**8.7**	**8.2**
	$\mathrm{LBM_s}(1)$	n.a.	n.a.	**28.2**	**21.3**	**18.8**	**16.5**	**16.6**	**14.1**	**13.7**	**14.4**	**13.1**	**12.6**
	$\mathrm{LBM_s}(2)$	n.a.	n.a.	**25.8**	**19.3**	**15.1**	**13.5**	**12.6**	**10.7**	**10.2**	**10.0**	**8.7**	**8.2**
	$\mathrm{BP_w}(1)$	5.0	4.3	4.3	4.6	4.7	4.6	4.4	4.3	4.3	4.1	4.3	4.8
	$\mathrm{BP_w}(2)$	5.0	5.0	4.3	4.4	3.7	3.8	3.5	**3.2**	3.5	**2.8**	**3.2**	3.5
	$\mathrm{LBM_w}(1)$	5.0	4.3	4.3	4.6	4.7	4.6	4.4	4.4	4.3	4.3	4.3	4.8
	$\mathrm{LBM_w}(2)$	5.0	5.1	4.3	4.4	3.8	3.9	3.5	**3.2**	3.5	**2.8**	**3.2**	3.5
$N = 2,000$	$\mathrm{BP_{SN}}(1)$	5.5	4.9	5.0	4.3	5.5	5.2	5.5	5.3	5.3	4.9	4.1	5.0
	$\mathrm{BP_{SN}}(2)$	5.3	5.1	5.7	5.3	5.3	5.3	5.2	5.7	5.4	5.7	5.8	5.6
	$\mathrm{LBM_{SN}}(1)$	5.5	5.0	5.0	4.4	5.5	5.2	5.6	5.3	5.3	5.0	4.1	5.0
	$\mathrm{LBM_{SN}}(2)$	5.3	5.1	5.7	5.3	5.3	5.3	5.2	5.7	5.5	5.7	5.8	5.6
	$\mathrm{BP_s}$	n.a.	n.a.	**40.4**	**29.1**	**24.8**	**22.3**	**19.7**	**18.4**	**16.2**	**15.1**	**15.1**	**14.9**
	$\mathrm{LBM_s}$	n.a.	n.a.	**40.5**	**29.1**	**24.9**	**22.5**	**19.8**	**18.5**	**16.2**	**15.1**	**15.1**	**15.3**
	$\mathrm{BP_w}$	4.8	4.0	3.9	3.9	3.6	**2.9**	**3.1**	**2.3**	**2.4**	**2.9**	**3.0**	3.8
	$\mathrm{LBM_w}$	4.8	4.0	3.9	3.9	3.7	**2.9**	**3.1**	**2.4**	**2.4**	**2.9**	**3.0**	3.8
	$\mathrm{BP_{SN}}$	5.5	4.8	5.0	5.0	5.3	5.1	5.6	5.1	4.6	4.4	4.6	5.0
	$\mathrm{LBM_{SN}}$	5.5	4.8	5.0	5.0	5.3	5.1	5.7	5.1	4.6	4.5	4.6	5.0

Table 11

Empirical size of the standard and proposed tests: relative frequencies (in %) of rejection of a weak $PARMA_2(1,1)$ given by (44) with the parameter $\alpha_0 = (0.8, 0.9, -0.5, -0.45)'$ and where the innovation $\epsilon_t \sim$ (75). When the relative rejection frequencies are outside the significant limits with probability 95% (resp. 99%), they are displayed in bold (resp. underlined) type.

Length N	Tests	Lag m											
		1	2	3	4	5	6	7	8	9	10	11	12
	$\mathrm{BP_s}(1)$	n.a.	n.a.	**35.7**	**27.8**	**25.9**	**25.4**	**25.1**	**23.5**	**23.6**	**22.6**	**22.7**	**23.1**
	$\mathrm{BP_s}(2)$	n.a.	n.a.	**31.3**	**26.8**	**23.7**	**23.4**	**23.0**	**21.6**	**21.1**	**21.0**	**21.3**	**20.2**
	$\mathrm{LBM_s}(1)$	n.a.	n.a.	**35.7**	**28.2**	**26.0**	**25.7**	**25.4**	**24.1**	**23.9**	**23.0**	**23.0**	**23.4**
	$\mathrm{LBM_s}(2)$	n.a.	n.a.	**31.4**	**26.8**	**24.3**	**23.9**	**23.3**	**21.9**	**21.8**	**21.5**	**21.8**	**21.1**
	$\mathrm{BP_w}(1)$	4.0	3.8	3.5	**3.1**	**2.3**	**2.4**	**1.5**	**1.5**	**1.0**	**0.9**	**0.7**	**0.4**
	$\mathrm{BP_w}(2)$	**3.3**	**2.5**	**2.1**	**1.5**	**1.0**	**1.0**	**1.1**	**0.8**	**0.6**	**0.6**	**0.5**	**0.3**
	$\mathrm{LBM_w}(1)$	4.0	3.9	3.6	**3.1**	**2.4**	**2.5**	**1.6**	**1.7**	**1.0**	**0.9**	**0.8**	**0.6**
	$\mathrm{LBM_w}(2)$	**3.3**	**2.5**	**2.2**	**1.7**	**1.2**	**1.0**	**1.2**	**0.9**	**0.6**	**0.6**	**0.6**	**0.3**
$N = 400$	$\mathrm{BP_{SN}}(1)$	4.0	3.4	**3.3**	3.4	3.4	**2.4**	**2.9**	**2.0**	**1.8**	**1.4**	**1.7**	**1.5**
	$\mathrm{BP_{SN}}(2)$	**3.0**	**2.9**	**3.3**	**3.2**	**2.4**	**2.6**	**3.0**	**3.0**	**2.1**	**1.9**	**1.7**	**1.8**
	$\mathrm{LBM_{SN}}(1)$	4.0	3.5	**3.3**	3.5	3.5	**2.4**	**2.9**	**2.1**	**1.8**	**1.4**	**1.7**	**1.5**
	$\mathrm{LBM_{SN}}(2)$	**3.0**	**2.9**	3.4	**3.2**	**2.4**	**2.6**	**3.0**	**3.0**	**2.1**	**2.0**	**1.7**	**1.9**
	$\mathrm{BP_s}$	n.a.	n.a.	**48.3**	**40.4**	**36.7**	**36.4**	**34.5**	**33.8**	**34.4**	**33.6**	**31.4**	**31.6**
	$\mathrm{LBM_s}$	n.a.	n.a.	**48.5**	**40.8**	**37.0**	**37.1**	**34.8**	**34.2**	**35.3**	**34.4**	**32.4**	**33.1**
	$\mathrm{BP_w}$	**2.7**	**2.6**	**2.2**	**2.0**	**1.2**	**0.8**	**0.4**	**0.4**	**0.3**	**0.2**	**0.0**	**0.1**
	$\mathrm{LBM_w}$	**2.7**	**2.6**	**2.3**	**2.0**	**1.3**	**0.8**	**0.4**	**0.5**	**0.3**	**0.2**	**0.1**	**0.1**
	$\mathrm{BP_{SN}}$	**2.7**	**2.8**	**2.4**	**1.9**	**1.4**	**1.3**	**0.9**	**0.8**	**0.4**	**0.6**	**0.5**	**0.5**
	$\mathrm{LBM_{SN}}$	**2.8**	**2.9**	**2.5**	**1.9**	**1.4**	**1.4**	**0.9**	**0.8**	**0.5**	**0.6**	**0.6**	**0.5**
		1	2	3	4	5	6	7	8	9	10	11	12
	$\mathrm{BP_s}(1)$	n.a.	n.a.	**36.4**	**33.5**	**33.6**	**32.8**	**32.9**	**32.5**	**33.8**	**33.8**	**34.6**	**34.7**
	$\mathrm{BP_s}(2)$	n.a.	n.a.	**36.0**	**31.9**	**28.1**	**29.0**	**28.6**	**29.2**	**29.3**	**28.5**	**28.8**	**27.7**
	$\mathrm{LBM_s}(1)$	n.a.	n.a.	**36.4**	**33.6**	**33.6**	**32.9**	**32.9**	**32.5**	**33.9**	**33.8**	**34.6**	**34.7**
	$\mathrm{LBM_s}(2)$	n.a.	n.a.	**36.0**	**31.9**	**28.1**	**29.1**	**28.7**	**29.2**	**29.5**	**28.5**	**28.8**	**27.9**
	$\mathrm{BP_w}(1)$	5.4	5.4	4.7	3.8	**3.3**	**2.9**	**3.3**	**3.0**	**3.0**	3.5	**2.8**	**2.9**
	$\mathrm{BP_w}(2)$	6.0	5.1	4.6	3.4	**3.2**	3.8	3.5	3.5	**3.3**	**3.0**	**2.7**	**2.3**
	$\mathrm{LBM_w}(1)$	5.4	5.4	4.7	3.9	**3.3**	**2.9**	**3.3**	**3.1**	**3.0**	3.5	**2.9**	**2.9**
	$\mathrm{LBM_w}(2)$	6.0	5.1	4.6	3.4	**3.2**	3.8	3.5	3.5	**3.3**	**3.0**	**2.8**	**2.3**
$N = 2{,}000$	$\mathrm{BP_{SN}}(1)$	4.2	4.3	4.4	4.3	5.3	5.2	4.5	4.2	4.2	3.7	3.8	**3.1**
	$\mathrm{BP_{SN}}(2)$	4.4	4.7	5.2	4.6	3.9	4.1	3.6	**3.3**	3.4	**3.3**	**3.3**	**3.2**
	$\mathrm{LBM_{SN}}(1)$	4.2	4.3	4.4	4.3	5.3	5.2	4.5	4.2	4.3	3.7	3.9	**3.2**
	$\mathrm{LBM_{SN}}(2)$	4.4	4.7	5.2	4.6	3.9	4.1	3.6	**3.3**	3.4	**3.3**	**3.3**	**3.2**
	$\mathrm{BP_s}$	n.a.	n.a.	**53.9**	**49.5**	**47.3**	**45.0**	**44.6**	**47.1**	**46.8**	**47.0**	**46.9**	**44.5**
	$\mathrm{LBM_s}$	n.a.	n.a.	**53.9**	**49.6**	**47.4**	**45.1**	**44.6**	**47.2**	**47.2**	**47.2**	**47.2**	**44.6**
	$\mathrm{BP_w}$	5.5	4.6	**3.0**	**2.9**	**2.6**	**2.6**	**2.8**	**2.4**	**2.2**	**1.6**	**1.9**	**2.0**
	$\mathrm{LBM_w}$	5.5	4.7	**3.0**	**2.9**	**2.6**	**2.6**	**2.8**	**2.4**	**2.2**	**1.6**	**1.9**	**2.0**
	$\mathrm{BP_{SN}}$	4.5	3.7	4.6	3.7	3.8	**3.1**	**3.0**	**3.0**	**3.3**	**1.7**	**1.8**	**1.4**
	$\mathrm{LBM_{SN}}$	4.5	3.7	4.6	3.7	3.8	**3.1**	**3.0**	**3.0**	**3.3**	**1.7**	**1.8**	**1.4**

C.2. Empirical power

We now repeat the same experiments for $N = 1{,}000$ to examine the empirical power of the standard and proposed tests: first for the null hypothesis of a periodic white noise against a $\mathrm{PARMA}_2(1,1)$ with $\alpha_0 = (0.8, 0.9, 0.0, 0.0)')$ alternative given by (44). Second for the null hypothesis of a $\mathrm{PAR}_2(1)$ against a $\mathrm{PARMA}_2(1,1)$ alternative given by (44) with $\alpha_0 = (0.8, 0.9, 0.5, 0.45)'$.

Tables 12 and 13 compare the empirical powers of Model (44) for the three different periodic noises above introduced over the $N \times 2$ independent replications at the asymptotic level $\alpha = 5\%$. Model I with innovation $\epsilon_t \sim \mathcal{N}(0,1)$; Model II with innovation $\epsilon_t \sim$ (45) and Model III with innovation $\epsilon_t \sim$ (75).

For these particular $\mathrm{PARMA}_2(1,1)$ models, we notice that the standard and our proposed tests have very similar powers except for $\mathrm{BP}_{\mathrm{SN}}(\nu)$ (resp. $\mathrm{BP}_{\mathrm{SN}}$) and $\mathrm{LBM}_{\mathrm{SN}}(\nu)$ (resp. $\mathrm{LBM}_{\mathrm{SN}}$) in the case of Model III.

Table 12

Empirical power (in %) of the modified and standard versions of the LBM and BP tests for the null hypothesis of a periodic noise against a $PARMA_2(1,1)$ given by (44) with the parameter $\alpha_0 = (0.8, 0.9, 0.0, 0.0)'$. Model I with innovation $\epsilon_t \sim \mathcal{N}(0,1)$; Model II with innovation $\epsilon_t \sim$ (45) and Model III with innovation $\epsilon_t \sim$ (75). The nominal asymptotic level of the tests is $\alpha = 5\%$.

Model	Tests	Lag m											
		1	2	3	4	5	6	7	8	9	10	11	12
	$\mathrm{BP}_{\mathrm{S}}(1)$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{BP}_{\mathrm{S}}(2)$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{LBM}_{\mathrm{S}}(1)$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{LBM}_{\mathrm{S}}(2)$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{BP}_{\mathrm{W}}(1)$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{BP}_{\mathrm{W}}(2)$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{LBM}_{\mathrm{W}}(1)$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{LBM}_{\mathrm{W}}(2)$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
I	$\mathrm{BP}_{\mathrm{SN}}(1)$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{BP}_{\mathrm{SN}}(2)$	99.9	100.0	100.0	99.9	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{LBM}_{\mathrm{SN}}(1)$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{LBM}_{\mathrm{SN}}(2)$	99.9	100.0	100.0	99.9	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	BP_{S}	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{LBM}_{\mathrm{S}}$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	BP_{W}	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{LBM}_{\mathrm{W}}$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{BP}_{\mathrm{SN}}$	99.8	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{LBM}_{\mathrm{SN}}$	99.8	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
		1	2	3	4	5	6	7	8	9	10	11	12
	$\mathrm{BP}_{\mathrm{S}}(1)$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{BP}_{\mathrm{S}}(2)$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{LBM}_{\mathrm{S}}(1)$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{LBM}_{\mathrm{S}}(2)$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{BP}_{\mathrm{W}}(1)$	100.0	100.0	100.0	99.9	99.9	99.8	99.9	99.8	99.8	99.9	99.9	99.9
	$\mathrm{BP}_{\mathrm{W}}(2)$	100.0	99.9	99.9	99.9	99.9	99.9	99.9	99.9	99.9	99.9	99.9	99.9
	$\mathrm{LBM}_{\mathrm{W}}(1)$	100.0	100.0	100.0	99.9	99.9	99.8	99.9	99.8	99.8	99.9	99.9	99.9
	$\mathrm{LBM}_{\mathrm{W}}(2)$	100.0	99.9	99.9	99.9	99.9	99.9	99.9	99.9	99.9	99.9	99.9	99.9
II	$\mathrm{BP}_{\mathrm{SN}}(1)$	99.1	98.8	98.8	99.0	98.9	98.8	98.7	98.5	98.6	98.1	98.3	98.2
	$\mathrm{BP}_{\mathrm{SN}}(2)$	99.1	98.8	98.9	98.8	98.9	98.8	98.2	98.3	98.4	98.1	97.9	97.7
	$\mathrm{LBM}_{\mathrm{SN}}(1)$	99.1	98.8	98.7	99.0	98.7	98.7	98.6	98.5	98.5	98.1	98.3	98.2
	$\mathrm{LBM}_{\mathrm{SN}}(2)$	99.1	98.8	98.9	98.7	98.9	98.8	98.2	98.2	98.4	98.1	97.9	97.7
	BP_{S}	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{LBM}_{\mathrm{S}}$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	BP_{W}	99.9	99.9	99.9	99.9	99.8	99.7	99.8	99.8	99.8	99.8	99.8	99.8
	$\mathrm{LBM}_{\mathrm{W}}$	99.9	99.9	99.9	99.9	99.8	99.7	99.8	99.8	99.8	99.8	99.8	99.8
	$\mathrm{BP}_{\mathrm{SN}}$	97.6	98.4	98.8	98.9	98.6	98.5	98.5	98.3	98.4	98.1	97.5	97.1
	$\mathrm{LBM}_{\mathrm{SN}}$	97.7	98.4	98.8	98.9	98.5	98.5	98.4	98.3	98.2	98.1	97.3	97.0
		1	2	3	4	5	6	7	8	9	10	11	12
	$\mathrm{BP}_{\mathrm{S}}(1)$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{BP}_{\mathrm{S}}(2)$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{LBM}_{\mathrm{S}}(1)$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{LBM}_{\mathrm{S}}(2)$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{BP}_{\mathrm{W}}(1)$	98.9	97.9	97.3	96.8	96.6	96.0	95.8	95.4	95.1	94.7	94.3	93.7
	$\mathrm{BP}_{\mathrm{W}}(2)$	98.6	97.6	97.1	97.2	96.6	95.9	95.8	95.3	94.8	94.4	93.8	93.7
	$\mathrm{LBM}_{\mathrm{W}}(1)$	98.9	97.9	97.3	96.8	96.6	96.1	95.8	95.4	95.1	94.7	94.3	93.7
	$\mathrm{LBM}_{\mathrm{W}}(2)$	98.6	97.7	97.1	97.2	96.6	95.9	95.9	95.3	94.8	94.5	93.8	93.7
III	$\mathrm{BP}_{\mathrm{SN}}(1)$	84.6	80.3	78.1	77.8	76.3	74.4	72.0	69.8	66.5	64.5	63.4	62.0
	$\mathrm{BP}_{\mathrm{SN}}(2)$	84.7	81.2	77.8	76.4	73.9	72.5	71.3	69.3	67.9	66.9	65.2	63.4
	$\mathrm{LBM}_{\mathrm{SN}}(1)$	84.6	80.2	78.0	77.6	76.5	73.8	71.4	69.5	66.3	64.2	62.9	61.7
	$\mathrm{LBM}_{\mathrm{SN}}(2)$	84.7	81.3	77.9	76.1	73.4	72.1	71.2	69.1	67.3	66.7	65.1	62.8
	BP_{S}	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$\mathrm{LBM}_{\mathrm{S}}$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	BP_{W}	98.2	96.8	96.4	96.5	95.9	95.2	94.8	94.9	94.2	93.5	92.7	92.4
	$\mathrm{LBM}_{\mathrm{W}}$	98.2	96.8	96.4	96.5	95.9	95.3	94.8	94.9	94.3	93.5	92.8	92.4
	$\mathrm{BP}_{\mathrm{SN}}$	76.4	76.0	73.5	71.2	68.7	64.4	60.3	56.7	51.1	46.8	41.2	37.4
	$\mathrm{LBM}_{\mathrm{SN}}$	76.6	76.0	73.8	71.1	68.8	63.7	59.9	56.2	50.1	46.0	40.6	37.1

TABLE 13

Empirical power (in %) of the modified and standard versions of the LBM and BP tests for the null hypothesis of a $PARMA_2(1,0)$ against a $PARMA_2(1,1)$ given by (44) with the parameter $\alpha_0 = (0.8, 0.9, 0.5, 0.45)'$. Model I with innovation $\epsilon_t \sim \mathcal{N}(0,1)$; Model II with innovation $\epsilon_t \sim$ (45) and Model III with innovation $\epsilon_t \sim$ (75). The nominal asymptotic level of the tests is $\alpha = 5\%$.

Model	Tests	Lag m											
		1	2	3	4	5	6	7	8	9	10	11	12
	$BP_S(1)$	n.a.	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$BP_S(2)$	n.a.	90.1	87.4	84.8	82.4	80.1	77.4	74.9	73.3	69.6	67.7	66.1
	$LBM_S(1)$	n.a.	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$LBM_S(2)$	n.a.	90.1	87.4	84.8	82.5	80.2	77.4	75.0	73.5	70.0	67.9	66.3
	$BP_W(1)$	100.0	100.0	100.0	100	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$BP_W(2)$	72.8	79.5	79.5	77.0	73.7	71.2	67.1	64.3	60.2	55.6	52.7	50.8
	$LBM_W(1)$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$LBM_W(2)$	72.8	79.5	79.5	77.1	73.7	71.4	67.2	64.4	60.4	55.8	53.1	51.2
I	$BP_{SN}(1)$	97.9	94.4	93.5	90.9	90.6	90.0	89.5	88.7	87.8	88.4	87.9	88.0
	$BP_{SN}(2)$	54.1	54.2	51.1	50.0	45.3	43.8	42.4	41.5	39.0	37.1	37.3	35.3
	$LBM_{SN}(1)$	97.9	94.4	93.5	90.9	90.7	90.1	89.6	88.8	87.9	88.2	87.9	88.1
	$LBM_{SN}(2)$	54.1	54.2	51.1	50.0	45.3	43.8	42.4	41.5	39.4	37.2	37.4	35.6
	BP_S	n.a.	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	LBM_S	n.a.	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	BP_W	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	LBM_W	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	BP_{SN}	96.0	95.6	94.9	94.8	94.1	94.6	94.4	93.7	92.4	91.7	90.6	90.1
	LBM_{SN}	96.0	95.6	94.9	94.8	94.1	94.6	94.6	93.7	92.5	91.7	90.6	90.2
		1	2	3	4	5	6	7	8	9	10	11	12
	$BP_S(1)$	n.a.	100.0	100.0	100.0	99.9	100.0	100.0	100.0	100.0	99.9	99.9	99.9
	$BP_S(2)$	n.a.	94.4	93.9	92.7	91.8	89.7	89.7	86.8	84.9	84.0	82.8	81.9
	$LBM_S(1)$	n.a.	100.0	100.0	100.0	99.9	100.0	100.0	100.0	100.0	99.9	99.9	99.9
	$LBM_S(2)$	n.a.	94.4	93.9	92.7	91.8	89.7	89.7	86.8	85.0	84.0	83.0	82.0
	$BP_W(1)$	97.7	99.2	99.1	99.0	99.0	99.0	98.9	98.8	98.7	98.5	98.5	98.2
	$BP_W(2)$	68.9	80.1	81.2	79.8	78.1	76.2	73.1	70.6	68.9	66.2	64.3	61.5
	$LBM_W(1)$	97.7	99.2	99.1	99.0	99.0	99.0	98.9	98.8	98.7	98.5	98.5	98.2
	$LBM_W(2)$	68.9	80.1	81.2	79.9	78.1	76.5	73.2	70.7	69.3	66.7	64.6	61.8
II	$BP_{SN}(1)$	83.8	85.5	82.5	80.6	78.8	76.9	75.6	74.0	71.3	69.3	68.6	65.2
	$BP_{SN}(2)$	48.6	48.5	49.3	46.3	44.8	41.9	39.5	35.9	34.3	32.5	30.5	28.9
	$LBM_{SN}(1)$	83.8	85.5	82.5	80.6	78.8	76.9	75.5	74.1	71.5	69.5	68.8	65.3
	$LBM_{SN}(2)$	48.6	48.5	49.3	46.4	45.1	41.9	39.6	36.1	34.5	32.7	30.7	29.1
	BP_S	n.a.	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	LBM_S	n.a.	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	BP_W	97.9	99.2	99.2	99.2	99.2	99.1	99.2	99.0	98.8	99.0	98.9	98.9
	LBM_W	97.9	99.2	99.2	99.2	99.2	99.1	99.2	99.0	99.0	99.0	98.9	98.9
	BP_{SN}	82.1	87.2	87.3	85.4	84.7	82.9	81.8	80.5	77.9	75.0	72.8	70.6
	LBM_{SN}	82.2	87.2	87.5	85.4	84.7	83.0	82.0	80.5	78.0	75.0	73.1	70.7
		1	2	3	4	5	6	7	8	9	10	11	12
	$BP_S(1)$	n.a.	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$BP_S(2)$	n.a.	96.3	96.7	96.3	96.1	95.2	94.5	93.1	92.7	92.1	91.7	90.4
	$LBM_S(1)$	n.a.	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	$LBM_S(2)$	n.a.	96.3	96.7	96.3	96.1	95.2	94.6	93.3	92.7	92.1	91.7	90.5
	$BP_W(1)$	96.2	97.8	97.7	96.8	96.2	95.5	94.9	94.0	92.6	91.3	90.6	89.9
	$BP_W(2)$	72.8	78.6	79.3	75.7	71.7	66.9	64.3	60.8	57.2	54.8	51.9	48.6
	$LBM_W(1)$	96.2	97.8	97.8	96.8	96.2	95.5	95.0	94.0	92.9	91.5	90.6	90.0
	$LBM_W(2)$	72.8	78.8	79.3	75.7	71.9	67.0	64.4	61.1	57.2	54.9	52.0	48.9
III	$BP_{SN}(1)$	82.7	77.4	75.5	71.5	69.1	65.1	64.2	60.7	58.7	56.4	52.7	49.8
	$BP_{SN}(2)$	53.9	48.4	49.1	42.6	40.4	35.8	32.9	29.9	27.1	26.4	25.1	22.0
	$LBM_{SN}(1)$	82.7	77.4	75.5	71.5	69.0	65.1	64.2	60.8	58.7	56.6	53.1	49.9
	$LBM_{SN}(2)$	53.9	48.5	49.2	42.6	40.5	35.8	33.0	29.9	27.2	26.4	25.2	22.1
	BP_S	n.a.	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	LBM_S	n.a.	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	BP_W	97.0	98.0	97.5	97.3	95.9	94.9	93.4	92.5	91.8	89.9	89.6	88.3
	LBM_W	97.0	98.0	97.5	97.3	95.9	94.9	93.4	92.6	91.8	90.1	89.6	88.4
	BP_{SN}	80.0	81.4	79.1	76.4	73.7	67.2	63.0	59.6	53.8	50.7	46.0	41.2
	LBM_{SN}	80.0	81.5	79.2	76.5	73.7	67.5	63.0	59.8	54.0	51.0	46.3	41.4

C.3. Real datasets

C.3.1. The SP500 and Nikkei indices as an illustrative example

As we did for the CAC 40 and DAX, we also consider an application to the daily log returns (also simply called the returns) of the SP500 and Nikkei indices (closing values): Nikkei (Osaka) and SP500 (New York). The observations cover the period from January 4, 1999 to November 20, 2020. The data can be downloaded from the website Yahoo Finance: http://fr.finance.yahoo.com/. Because of the presence of holidays many weeks comprise less than five observations. We removed the entire weeks when there was less than five data. The effective number of observations used for each index is given in Table 15 and the periodicity is then $T = 5$.

First we apply portmanteau tests on each series of daily returns for checking the hypothesis that the returns constitute a periodic white noise. In this section we only present the results on the Ljung-Box-McLeod tests since they are very close to those of the Box-Pierce tests. Table 14 displays the $p-$values and the statistics (for the self-normalized versions) of the standard and modified LBM tests for the mean corrected returns of each index. The p-values less than 5% are in bold, those less than 1% are underlined. At the $\alpha = 5\%$ significance level, the hypothesis of strong periodic noise is (frequently) rejected by the standard global, even for a specified season $\nu \in \{1, \dots, 5\}$ LBM tests for SP500 index. For the Nikkei index the strong periodic white noise assumption is not rejected. But since the class of strong periodic noises is a subset of the class of weak periodic noises, these results show that the standard inference based on the assumption of a strong periodic noise can be misleading (see for instance Francq et al. (2011)). By contrast, the weak periodic white noise hypothesis is not rejected for the two indices by all the global proposed tests, even for a specified season ν. To summarize, the outputs of Table 14 are in accordance with the common belief that these series are not strong white noises but could be weak white noises.

Next, let us turn to the dynamics of the squared returns by fitting a weak $\mathrm{PARMA}_5(1,1)$ model (46). To check the stationary properties of $X_{5n+\nu}$ it is convenient to consider the solution to the characteristic equation of the autoregressive part of Equation (46), which with our notations in Section 2 can be shown to be equal to (47) where

$$\Phi_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\phi(2) & 1 & 0 & 0 & 0 \\ 0 & -\phi(3) & 1 & 0 & 0 \\ 0 & 0 & -\phi(4) & 1 & 0 \\ 0 & 0 & 0 & -\phi(5) & 1 \end{pmatrix} \text{ and } \Phi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & \phi(1) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The same result holds for the invertible Model (46) with $\phi(\nu)$ replaced by $\theta(\nu)$ in (47).

Table 15 presents the QLS estimated parameters of Model (46), their $p-$values (in parentheses) and their estimated standard errors (into brackets, under weak assumption on $\epsilon_{5n+\nu}$) of the squared returns of the SP500 and Nikkei indices. As expected all the estimated parameters are significant at any reasonable levels, except: $\hat{\mu}_2$, $\hat{\mu}_3$ and $\hat{\mu}_4$ for the SP500 index. For these two indices the mean $\hat{\mu}_1$ on Monday is positive and significant, it seems more possibly to talk of a global Monday effect. For the other days the means $\hat{\mu}_2$, $\hat{\mu}_3$, $\hat{\mu}_4$ and $\hat{\mu}_5$ are all negatives. Therefore Tuesday, Wednesday, Thursday and Friday seem the bad days for the Nikkei index since they are significant. Friday seems a particularly bad day for the SP500 index.

For these indices the autoregressive coefficients $\hat{\phi}(\nu)$ are all positive and significant for all days. These coefficients are greater than one on Monday (for SP500), on Wednesday (for SP500 and Nikkei) and on Friday for Nikkei index. We also observe that the coefficients $\hat{\phi}(\nu)$ are the biggest on Monday (for SP500) and on Wednesday (for Nikkei). Furthermore, with the SP500 index and the period considered, it is probably more appropriate to talk of a Monday effect. By contrast for the Nikkei index, it is probably more appropriate to talk of a Wednesday effect rather than a Monday effect. Additionally, Table 15 shows that there is evidence that the estimated noise standard deviations (estimated volatility) for these indices is considerably greater on Monday than the other days (Tuesday, Wednesday, Thursday and Friday), for which it is smaller and almost constant.

Note that from Table 15 and for these two series, the product of the estimated coefficients $\hat{\phi}(\nu)$ (resp. $\hat{\theta}(\nu)$) are smaller than one. Thus in view of (47) $|z| > 1$ for all series (the smallest z verify $|z| = 1.236 > 1$). So we think that the assumption **(A1)** is satisfied and thus our asymptotic normality theorem on the residual autocorrelations can be applied.

We thus apply portmanteau tests to the residuals of Model (46) for each series. The results reveal the conclusion that the strong $\mathrm{PARMA}_5(1,1)$ model is rejected by the standard global LBM test and even for a specified season ν at the nominal level $\alpha = 5\%$. By contrast a weak $\mathrm{PARMA}_5(1,1)$ model is not rejected. Note that for the first and second-order structures we found for the returns considered, namely a weak periodic white noise for the returns and a weak $\mathrm{PARMA}_5(1,1)$ model for the squares of the returns, seem compatible with a $\mathrm{PGARCH}(1,1)$ model.

Table 14

Modified and standard versions of portmanteau tests to check the null hypothesis that the returns follow a periodic white noise, based on m residuals autocorrelations. The p-values less than 5% are in bold, those less than 1% are underlined.

Index	Tests	Lag m											
		1	2	3	4	5	6	7	8	9	10	11	12
	$\mathrm{LBM_S}(1)$	0.496	0.349	0.097	0.124	0.195	0.272	0.075	0.084	**0.015**	**0.005**	**0.008**	**0.002**
	$\mathrm{LBM_S}(2)$	**0.000**	**0.000**	**0.000**	**0.001**	**0.002**	**0.003**	**0.007**	**0.012**	**0.020**	**0.027**	**0.028**	**0.028**
	$\mathrm{LBM_S}(3)$	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
	$\mathrm{LBM_S}(4)$	0.960	0.822	**0.000**	**0.000**	**0.001**	**0.001**	**0.001**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
	$\mathrm{LBM_S}(5)$	0.224	0.470	0.100	0.116	0.192	**0.012**	**0.007**	**0.002**	**0.004**	**0.003**	**0.001**	**0.001**
	$\mathrm{LBM_W}(1)$	0.686	0.765	0.585	0.657	0.732	0.783	0.608	0.638	0.443	0.343	0.381	0.294
	$\mathrm{LBM_W}(2)$	0.027	0.048	0.157	0.215	0.264	0.340	0.409	0.476	0.544	0.583	0.589	0.607
	$\mathrm{LBM_W}(3)$	0.152	0.193	0.425	0.363	0.299	0.325	0.331	0.343	0.329	0.316	0.318	0.297
SP500	$\mathrm{LBM_W}(4)$	0.983	0.948	0.133	0.174	0.247	0.231	0.289	0.252	0.213	0.219	0.148	0.172
	$\mathrm{LBM_W}(5)$	0.365	0.666	0.315	0.407	0.504	0.152	0.130	0.104	0.127	0.130	0.080	0.103
	$\mathrm{LBM_{SN}}(1)$	0.4	34.8	115.4	140.6	155.1	172.6	262.9	267.9	294.9	326.4	334.8	582.8
	$\mathrm{LBM_{SN}}(2)$	25.9	37.9	142.6	166.0	185.5	192.0	284.5	295.9	337.4	343.7	375.5	384.6
	$\mathrm{LBM_{SN}}(3)$	13.5	19.3	25.9	45.3	45.3	203.9	237.5	243.1	348.0	350.4	354.3	366.3
	$\mathrm{LBM_{SN}}(4)$	0.0	1.7	93.4	93.6	93.8	94.9	102.4	140.6	213.8	216.8	233.0	233.1
	$\mathrm{LBM_{SN}}(5)$	12.8	14.3	73.2	93.6	153.2	234.0	244.7	300.1	419.2	426.1	441.9	462.7
	$\mathrm{LBM_S}$	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
	$\mathrm{LBM_W}$	0.149	0.354	0.279	0.361	0.359	0.342	0.324	0.322	0.292	0.284	0.258	0.263
	$\mathrm{LBM_{SN}}$	205.3	384.9	658.0	1445.5	1563.3	1787.3	1937.5	2006.4	2458.9	2797.1	3173.7	3848.8
		1	2	3	4	5	6	7	8	9	10	11	12
	$\mathrm{LBM_S}(1)$	**0.015**	**0.048**	**0.037**	0.075	0.119	0.138	0.204	0.262	0.290	0.371	0.441	0.394
	$\mathrm{LBM_S}(2)$	0.089	0.222	0.196	0.319	0.219	0.139	0.158	0.205	0.127	0.178	0.208	0.213
	$\mathrm{LBM_S}(3)$	0.571	0.832	0.697	0.837	0.851	0.920	0.952	0.974	0.986	0.993	0.992	0.996
	$\mathrm{LBM_S}(4)$	0.069	0.157	0.265	0.336	0.307	0.404	0.503	0.551	0.622	0.477	0.556	0.567
	$\mathrm{LBM_S}(5)$	0.199	0.412	0.273	0.128	0.121	**0.023**	**0.040**	**0.022**	**0.032**	**0.049**	0.067	0.083
	$\mathrm{LBM_W}(1)$	**0.045**	0.117	0.079	0.137	0.209	0.256	0.355	0.403	0.431	0.525	0.595	0.567
	$\mathrm{LBM_W}(2)$	0.118	0.343	0.438	0.537	0.461	0.396	0.413	0.471	0.376	0.432	0.458	0.462
	$\mathrm{LBM_W}(3)$	0.652	0.898	0.805	0.897	0.910	0.956	0.972	0.987	0.993	0.997	0.996	0.998
Nikkei	$\mathrm{LBM_W}(4)$	0.279	0.445	0.574	0.622	0.577	0.639	0.701	0.740	0.778	0.678	0.758	0.755
	$\mathrm{LBM_W}(5)$	0.620	0.769	0.696	0.568	0.573	0.398	0.449	0.373	0.405	0.436	0.460	0.488
	$\mathrm{LBM_{SN}}(1)$	54.4	54.5	126.8	127.8	221.0	226.9	579.8	605.1	609.4	919.6	923.9	979.8
	$\mathrm{LBM_{SN}}(2)$	14.0	14.0	85.0	123.7	171.5	176.9	177.6	215.6	240.1	240.2	319.8	320.4
	$\mathrm{LBM_{SN}}(3)$	2.1	2.4	5.2	10.9	21.9	23.1	25.5	53.4	53.4	71.2	98.1	98.3
	$\mathrm{LBM_{SN}}(4)$	5.3	6.1	54.0	94.8	100.5	115.2	126.8	127.5	127.7	176.7	324.8	345.9
	$\mathrm{LBM_{SN}}(5)$	2.7	3.6	18.5	40.3	111.3	133.8	229.6	291.9	292.8	308.1	340.9	499.5
	$\mathrm{LBM_S}$	**0.015**	0.135	0.096	0.155	0.145	0.069	0.153	0.182	0.211	0.291	0.409	0.444
	$\mathrm{LBM_W}$	0.323	0.609	0.597	0.641	0.630	0.545	0.640	0.671	0.689	0.742	0.809	0.821
	$\mathrm{LBM_{SN}}$	343.6	426.6	1013.1	1309.4	1602.2	2142.9	2931.9	3909.1	4737.3	5585.1	7404.2	8118.3

TABLE 15

QLS estimates, their p−values (in parentheses) and their estimated standard errors (in brackets) of a weak $PARMA_5(1,1)$ model fitted to the mean corrected series of the squared returns of the SP500 and Nikkei indices.

Index	SP500				Nikkei			
NT	6550				5920			
Day	$\hat{\mu}_\nu$	$\hat{\phi}_\nu$	$\hat{\theta}_\nu$	$\hat{\sigma}_\nu$	$\hat{\mu}_\nu$	$\hat{\phi}_\nu$	$\hat{\theta}_\nu$	$\hat{\sigma}_\nu$
Monday	0.718 (0.001) [0.213]	1.900 (0.000) [0.435]	1.613 (0.008) [0.611]	7.163×10^{-8}	2.716 (0.000) [0.418]	0.738 (0.000) [0.109]	0.529 (0.000) [0.128]	14.391×10^{-8}
Tuesday	−0.149 (0.250) [0.129]	0.871 (0.000) [0.060]	0.747 (0.000) [0.071]	4.093×10^{-8}	−0.994 (0.000) [0.138]	0.491 (0.000) [0.093]	0.472 (0.000) [0.093]	4.690×10^{-8}
Wenesday	−0.116 (0.502) [0.173]	1.518 (0.000) [0.283]	1.362 (0.000) [0.185]	4.958×10^{-8}	−0.571 (0.000) [0.154]	1.669 (0.000) [0.159]	1.551 (0.000) [0.172]	4.817×10^{-8}
Thursday	−0.061 (0.620) [0.123]	0.648 (0.000) [0.118]	0.490 (0.000) [0.109]	3.687×10^{-8}	−0.646 (0.000) [0.150]	0.900 (0.000) [0.104]	0.654 (0.000) [0.093]	4.670×10^{-8}
Friday	−0.391 (0.000) [0.074]	0.497 (0.000) [0.037]	0.402 (0.000) [0.042]	2.418×10^{-8}	−0.500 (0.006) [0.182]	1.357 (0.000) [0.241]	1.397 (0.000) [0.363]	5.360×10^{-8}

C.3.2. Portmanteau tests to check the null hypothesis that the squared returns follow a weak $PARMA_5(1,1)$ models for each of the four indices considered

We thus apply portmanteau tests to the residuals of Model (46) for each series. Tables 16 and 17 display the p−values and the statistics (for the self-normalized versions) of the standard and modified LBM tests for the mean corrected returns of each index. The p-values less than 5% are in bold, those less than 1% are underlined. From these tables we draw the conclusion that the strong $\mathrm{PARMA}_5(1,1)$ model is rejected by the standard global LBM test and even for a specified season ν at the nominal level $\alpha = 5\%$. By contrast a weak $\mathrm{PARMA}_5(1,1)$ model is not rejected. Note that for the first and second-order structures we found for the returns considered, namely a weak periodic white noise for the returns and a weak $\mathrm{PARMA}_5(1,1)$ model for the squares of the returns, are compatible with a PGARCH(1,1) model.

Figures 1, 2, 3 and 4 display the residual autocorrelations and their 5% significance limits under the strong and weak periodic noises assumptions.

Figures 5, 6, 7 and 8 display the residual autocorrelations and their 5% significance limits under the strong $\mathrm{PARMA}_5(1,1)$ and weak $\mathrm{PARMA}_5(1,1)$ assumptions. In view of Figures 1, 2, 3 and 4 (resp. Figures 5, 6, 7 and 8), the diagnostic checking of residuals does not indicate any inadequacy for the proposed tests. All of the sample autocorrelations should lie between the bands (at 95%) shown as dashed lines (green color) and solid lines (red color) for the modified tests, while the horizontal dotted (blue color) for standard test indicate that strong periodic noise (resp. strong $\mathrm{PARMA}_5(1,1)$) is not adequate. Figures 1, 2, 3 and 4 (resp. Figures 5, 6, 7 and 8) confirm the conclusions drawn from Tables 4 and 14 (resp. Tables 16 and 17).

To conclude our empirical investigations, a comparison of the four indices (CAC 40, DAX, Nikkei and SP500) indicates that Nikkei is systematically more volatile over two days of the week (Monday and Friday) than the other three. By contrast the DAX index is systematically more volatile on Tuesday and Thursday than the other three. Finally SP500 index is the most volatile on Wednesday.

TABLE 16

Modified and standard versions of portmanteau tests to check the null hypothesis that the squared returns follow a weak $PARMA_5(1,1)$ models, based on m residuals autocorrelations. The p-values less than 5% are in bold, those less than 1% are underlined.

Index	Tests	Lag m											
		1	2	3	4	5	6	7	8	9	10	11	12
	$\mathrm{LBM_S}(1)$	n.a.	n.a.	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
	$\mathrm{LBM_S}(2)$	n.a.	n.a.	0.063	0.177	0.317	0.474	0.209	0.306	0.195	0.271	0.238	0.247
	$\mathrm{LBM_S}(3)$	n.a.	n.a.	**0.002**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
	$\mathrm{LBM_S}(4)$	n.a.	n.a.	**0.034**	**0.019**	**0.044**	0.078	0.127	0.197	0.280	**0.038**	0.061	0.080
	$\mathrm{LBM_S}(5)$	n.a.	n.a.	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
	$\mathrm{LBM_W}(1)$	0.652	0.401	0.275	0.336	0.307	0.307	0.328	0.382	0.386	0.382	0.362	0.372
	$\mathrm{LBM_W}(2)$	0.509	0.682	0.725	0.771	0.849	0.892	0.767	0.806	0.804	0.838	0.818	0.806
	$\mathrm{LBM_W}(3)$	**0.047**	0.414	0.460	0.391	0.430	0.392	0.374	0.366	0.369	0.356	0.358	0.347
CAC 40	$\mathrm{LBM_W}(4)$	0.291	0.311	0.550	0.462	0.524	0.517	0.544	0.581	0.590	0.478	0.491	0.488
	$\mathrm{LBM_W}(5)$	0.409	0.555	0.260	0.254	0.267	0.240	0.253	0.203	0.215	0.249	0.268	0.274
	$\mathrm{LBM_{SN}}(1)$	1.0	15.1	23.8	23.9	36.6	42.1	49.2	159.3	196.7	277.4	295.0	313.7
	$\mathrm{LBM_{SN}}(2)$	2.6	2.6	6.2	29.7	29.7	35.2	96.3	106.1	137.0	142.3	228.6	250.8
	$\mathrm{LBM_{SN}}(3)$	118.7	128.9	150.7	152.5	153.8	155.9	177.2	182.5	241.4	243.0	244.9	264.7
	$\mathrm{LBM_{SN}}(4)$	3.0	20.7	23.9	58.2	85.7	117.0	122.5	126.4	126.8	129.7	134.4	403.5
	$\mathrm{LBM_{SN}}(5)$	4.9	8.1	15.9	19.4	75.1	113.2	113.6	254.1	269.6	643.0	659.9	742.0
	$\mathrm{LBM_S}$	n.a.	n.a.	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
	$\mathrm{LBM_W}$	0.523	0.558	0.416	0.406	0.414	0.388	0.345	0.329	0.317	0.314	0.332	0.358
	$\mathrm{LBM_{SN}}$	239.6	483.0	641.0	771.3	830.9	1068.7	1300.6	1632.9	2460.4	2814.5	3299.4	6016.9
		1	2	3	4	5	6	7	8	9	10	11	12
	$\mathrm{LBM_S}(1)$	n.a.	n.a.	0.131	0.319	0.346	0.483	0.627	0.735	0.796	0.776	0.701	0.744
	$\mathrm{LBM_S}(2)$	n.a.	n.a.	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
	$\mathrm{LBM_S}(3)$	n.a.	n.a.	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
	$\mathrm{LBM_S}(4)$	n.a.	n.a.	**0.027**	0.085	0.112	0.198	0.228	0.175	0.254	0.344	0.415	0.470
	$\mathrm{LBM_S}(5)$	n.a.	n.a.	**0.002**	**0.008**	**0.023**	**0.010**	**0.016**	**0.005**	**0.008**	**0.014**	**0.012**	**0.000**
	$\mathrm{LBM_W}(1)$	0.530	0.772	0.380	0.473	0.361	0.426	0.503	0.567	0.573	0.648	0.610	0.716
	$\mathrm{LBM_W}(2)$	0.234	0.291	0.070	0.138	0.130	0.157	0.196	0.190	0.187	0.206	0.242	0.255
	$\mathrm{LBM_W}(3)$	0.650	0.375	0.391	0.427	0.515	0.548	0.419	0.453	0.401	0.345	0.341	0.355
DAX	$\mathrm{LBM_W}(4)$	0.503	0.583	0.479	0.539	0.559	0.584	0.586	0.483	0.526	0.569	0.574	0.576
	$\mathrm{LBM_W}(5)$	0.177	0.345	0.315	0.391	0.445	0.358	0.438	0.361	0.366	0.396	0.386	0.272
	$\mathrm{LBM_{SN}}(1)$	1.0	13.5	13.5	14.6	46.1	99.0	123.6	170.0	235.9	258.6	381.2	392.2
	$\mathrm{LBM_{SN}}(2)$	19.7	21.5	80.0	80.4	80.4	112.1	134.8	157.3	159.6	182.2	199.1	204.6
	$\mathrm{LBM_{SN}}(3)$	2.0	25.1	45.4	48.8	98.5	104.5	108.4	109.1	117.2	170.6	178.5	182.8
	$\mathrm{LBM_{SN}}(4)$	1.2	2.5	12.4	19.1	78.0	79.1	88.5	166.2	168.0	169.9	173.3	173.3
	$\mathrm{LBM_{SN}}(5)$	7.8	17.3	31.6	40.0	42.7	171.4	181.6	194.3	194.8	194.8	285.6	514.8
	$\mathrm{LBM_S}$	n.a.	n.a.	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
	$\mathrm{LBM_W}$	0.428	0.442	0.312	0.369	0.425	0.426	0.387	0.394	0.36	0.332	0.364	0.438
	$\mathrm{LBM_{SN}}$	77.0	105.9	343.8	427.1	548.1	707.6	887.5	1314.5	1542.8	1850.3	2371.7	3368.8

Table 17

Modified and standard versions of portmanteau tests to check the null hypothesis that the squared returns follow a weak $PARMA_5(1,1)$ models, based on m residuals autocorrelations. The p-values less than 5% are in bold, those less than 1% are underlined.

Index	Tests	Lag m											
		1	2	3	4	5	6	7	8	9	10	11	12
SP500	$\mathrm{LBM_S}(1)$	n.a.	n.a.	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
	$\mathrm{LBM_S}(2)$	n.a.	n.a.	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
	$\mathrm{LBM_S}(3)$	n.a.	n.a.	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
	$\mathrm{LBM_S}(4)$	n.a.	n.a.	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
	$\mathrm{LBM_S}(5)$	n.a.	n.a.	**0.006**	**0.018**	**0.010**	**0.001**	**0.003**	**0.004**	**0.000**	**0.001**	**0.001**	**0.000**
	$\mathrm{LBM_W}(1)$	0.632	0.378	0.386	0.412	0.455	0.386	0.396	0.377	0.343	0.342	0.365	0.372
	$\mathrm{LBM_W}(2)$	0.097	0.299	0.312	0.298	0.394	0.401	0.384	0.381	0.388	0.402	0.408	0.407
	$\mathrm{LBM_W}(3)$	0.126	0.231	0.270	0.276	0.231	0.257	0.248	0.249	0.257	0.253	0.233	0.240
	$\mathrm{LBM_W}(4)$	0.534	0.489	0.366	0.423	0.407	0.425	0.434	0.425	0.418	0.409	0.412	0.412
	$\mathrm{LBM_W}(5)$	0.733	0.878	0.659	0.713	0.646	0.522	0.545	0.517	0.383	0.456	0.467	0.453
	$\mathrm{LBM_{SN}}(1)$	0.5	13.4	19.9	20.4	39.2	52.7	116.0	140.1	141.1	142.3	169.4	187.9
	$\mathrm{LBM_{SN}}(2)$	10.7	14.2	16.2	16.6	27.0	28.6	37.7	46.5	87.8	88.7	91.5	164.3
	$\mathrm{LBM_{SN}}(3)$	9.9	10.0	15.4	25.8	34.4	43.9	44.7	146.5	147.4	150.7	153.1	192.4
	$\mathrm{LBM_{SN}}(4)$	1.6	13.7	14.0	44.0	51.3	52.8	72.9	126.5	126.7	134.9	211.9	217.7
	$\mathrm{LBM_{SN}}(5)$	0.5	3.9	18.1	125.7	136.4	144.2	146.3	159.5	243.3	256.4	257.9	322.6
	$\mathrm{LBM_S}$	n.a.	n.a.	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
	$\mathrm{LBM_W}$	0.526	0.345	0.344	0.369	0.400	0.389	0.394	0.388	0.335	0.332	0.352	0.361
	$\mathrm{LBM_{SN}}$	51.1	234.4	303.2	629.0	882.3	1263.3	2209.1	3376.7	4499.5	4661.4	7230.0	7631.6
		1	2	3	4	5	6	7	8	9	10	11	12
Nikkei	$\mathrm{LBM_S}(1)$	n.a.	n.a.	0.735	0.925	0.829	0.795	0.853	0.919	0.958	0.906	0.932	0.947
	$\mathrm{LBM_S}(2)$	n.a.	n.a.	**0.001**	**0.002**	**0.005**	**0.005**	**0.009**	**0.017**	**0.029**	**0.047**	0.055	0.083
	$\mathrm{LBM_S}(3)$	n.a.	n.a.	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
	$\mathrm{LBM_S}(4)$	n.a.	n.a.	**0.000**	**0.000**	**0.000**	**0.001**	**0.002**	**0.003**	**0.006**	**0.004**	**0.008**	**0.010**
	$\mathrm{LBM_S}(5)$	n.a.	n.a.	**0.021**	0.062	0.131	0.211	0.179	0.205	0.292	0.283	0.362	0.052
	$\mathrm{LBM_W}(1)$	0.781	0.967	0.976	0.987	0.892	0.863	0.885	0.910	0.931	0.922	0.933	0.949
	$\mathrm{LBM_W}(2)$	0.228	0.256	0.281	0.281	0.292	0.272	0.279	0.289	0.294	0.291	0.303	0.312
	$\mathrm{LBM_W}(3)$	0.586	0.604	0.220	0.192	0.195	0.195	0.186	0.210	0.210	0.207	0.212	0.214
	$\mathrm{LBM_W}(4)$	0.096	0.113	0.476	0.488	0.497	0.501	0.517	0.515	0.519	0.518	0.533	0.519
	$\mathrm{LBM_W}(5)$	0.270	0.634	0.640	0.647	0.672	0.692	0.634	0.648	0.672	0.642	0.666	0.486
	$\mathrm{LBM_{SN}}(1)$	0.0	0.3	1.4	2.1	10.5	19.6	32.4	32.5	37.3	160.2	213.0	216.1
	$\mathrm{LBM_{SN}}(2)$	5.4	25.7	34.4	56.7	88.6	88.8	93.3	234.4	299.0	308.9	345.4	459.1
	$\mathrm{LBM_{SN}}(3)$	1.2	6.3	20.1	214.9	215.6	220.8	283.7	289.1	377.6	533.5	654.6	782.1
	$\mathrm{LBM_{SN}}(4)$	33.2	53.3	70.1	87.6	99.5	125.9	127.2	127.3	139.5	174.4	182.0	363.1
	$\mathrm{LBM_{SN}}(5)$	10.9	23.6	23.8	25.6	47.4	50.5	163.3	163.8	267.4	354.6	395.2	516.7
	$\mathrm{LBM_S}$	n.a.	n.a.	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
	$\mathrm{LBM_W}$	0.150	0.356	0.406	0.388	0.400	0.404	0.397	0.417	0.425	0.452	0.455	0.440
	$\mathrm{LBM_{SN}}$	102.2	140.3	1062.8	1962.3	2536.5	3060.9	3719.6	4521.2	5420.0	6002.4	6715.0	8404.3

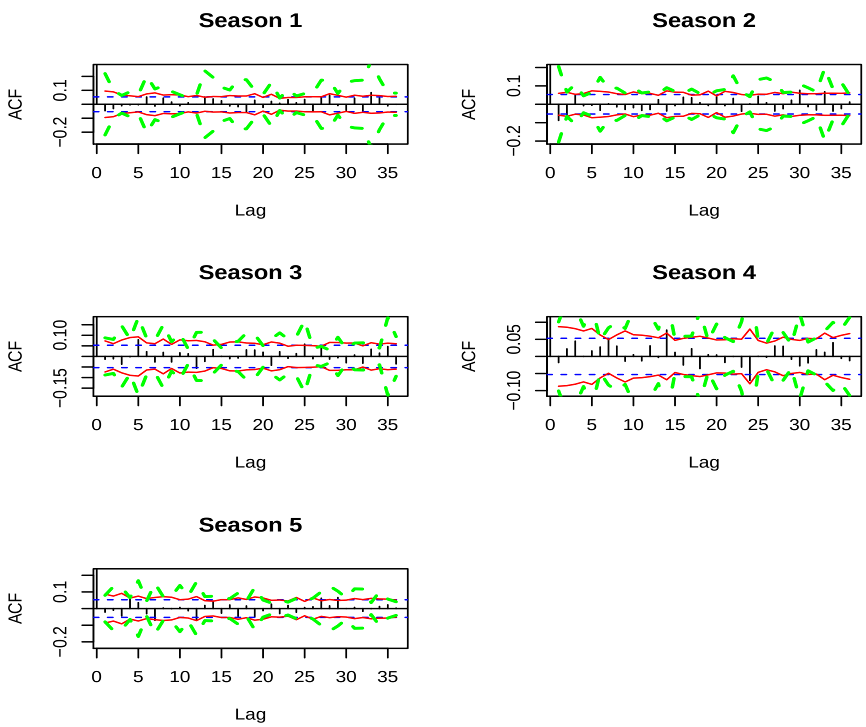

Figure 1. *Autocorrelation of the periodic noise for the CAC 40 returns. The horizontal dotted lines (blue color) correspond to the 5% significant limits obtained under the strong periodic noise assumption. The solid lines (red color) and dashed lines (green color) correspond also to the 5% significant limits under the weak periodic noise assumption. The full lines correspond to the asymptotic significance limits for the residual autocorrelations obtained in Theorem 3.1. The dashed lines (green color) correspond to the self-normalized asymptotic significance limits for the residual autocorrelations obtained in Theorem 3.5.*

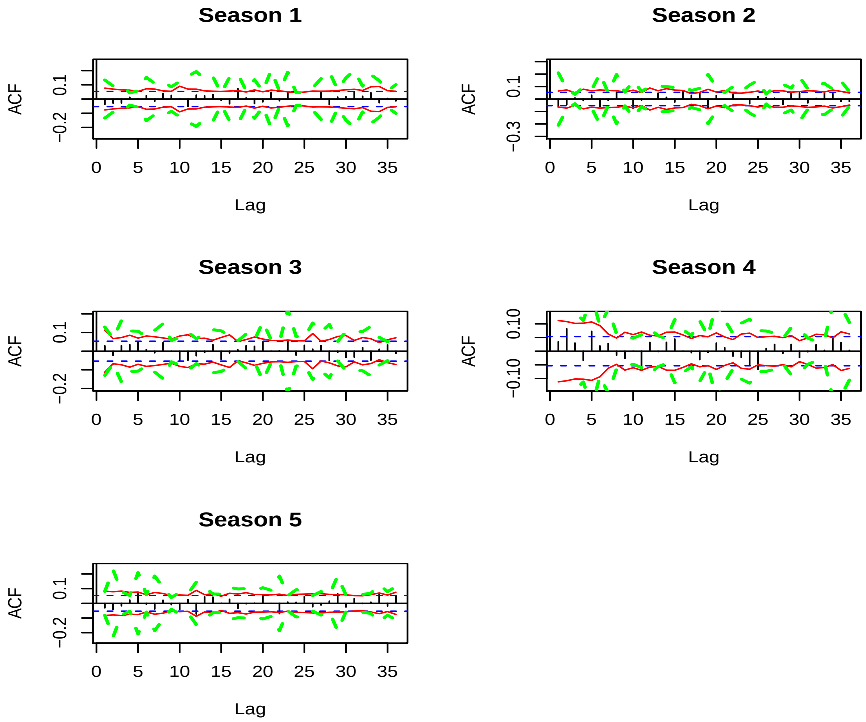

FIGURE 2. *Autocorrelation of the periodic noise for the DAX returns. The horizontal dotted lines (blue color) correspond to the 5% significant limits obtained under the strong periodic noise assumption. The solid lines (red color) and dashed lines (green color) correspond also to the 5% significant limits under the weak periodic noise assumption. The full lines correspond to the asymptotic significance limits for the residual autocorrelations obtained in Theorem 3.1. The dashed lines (green color) correspond to the self-normalized asymptotic significance limits for the residual autocorrelations obtained in Theorem 3.5.*

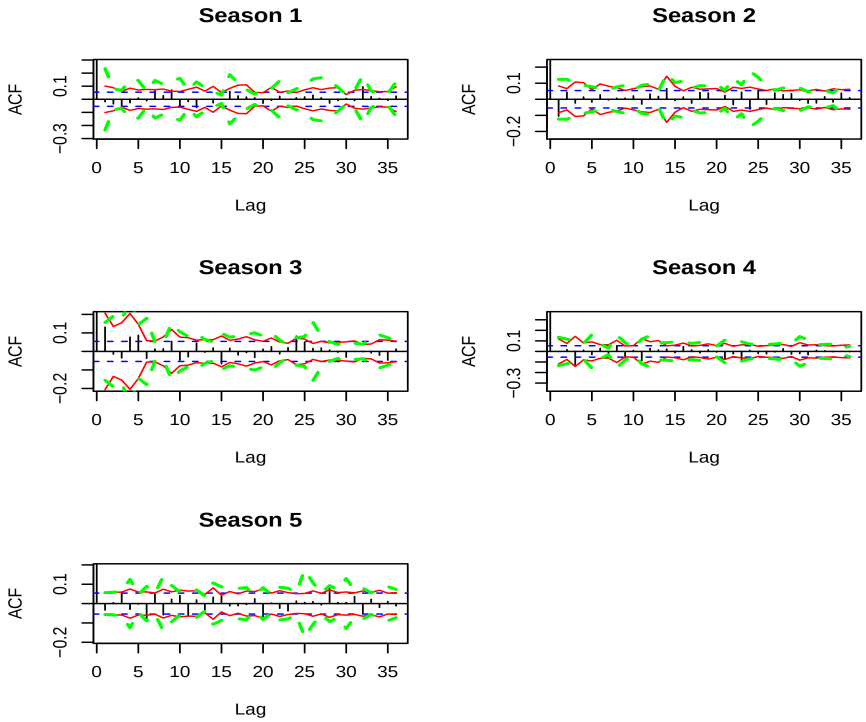

FIGURE 3. *Autocorrelation of the periodic noise for the SP500 returns. The horizontal dotted lines (blue color) correspond to the 5% significant limits obtained under the strong periodic noise assumption. The solid lines (red color) and dashed lines (green color) correspond also to the 5% significant limits under the weak periodic noise assumption. The full lines correspond to the asymptotic significance limits for the residual autocorrelations obtained in Theorem 3.1. The dashed lines (green color) correspond to the self-normalized asymptotic significance limits for the residual autocorrelations obtained in Theorem 3.5.*

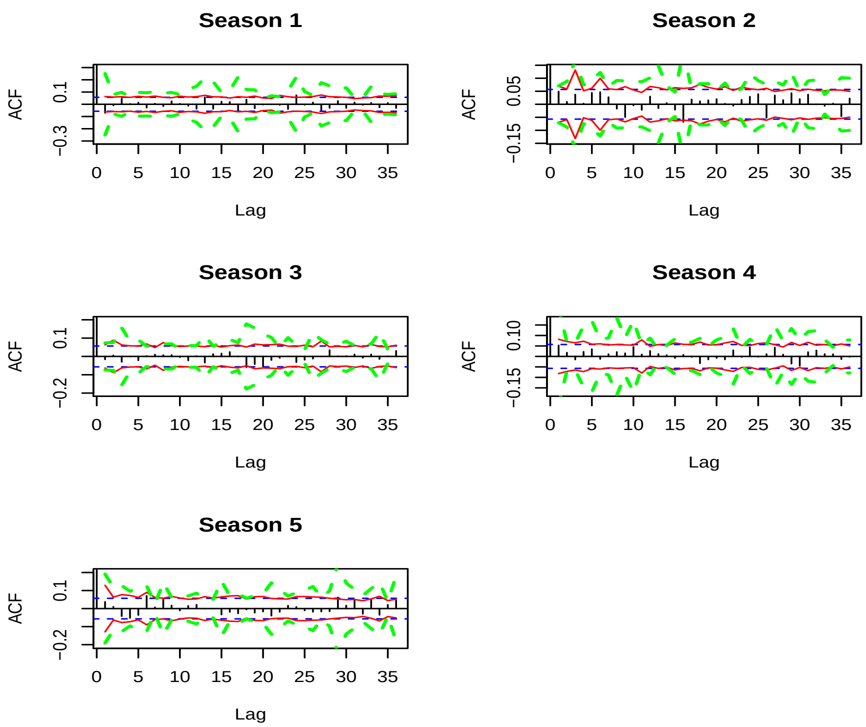

FIGURE 4. *Autocorrelation of the periodic noise for the Nikkei returns. The horizontal dotted lines (blue color) correspond to the 5% significant limits obtained under the strong periodic noise assumption. The solid lines (red color) and dashed lines (green color) correspond also to the 5% significant limits under the weak periodic noise assumption. The full lines correspond to the asymptotic significance limits for the residual autocorrelations obtained in Theorem 3.1. The dashed lines (green color) correspond to the self-normalized asymptotic significance limits for the residual autocorrelations obtained in Theorem 3.5.*

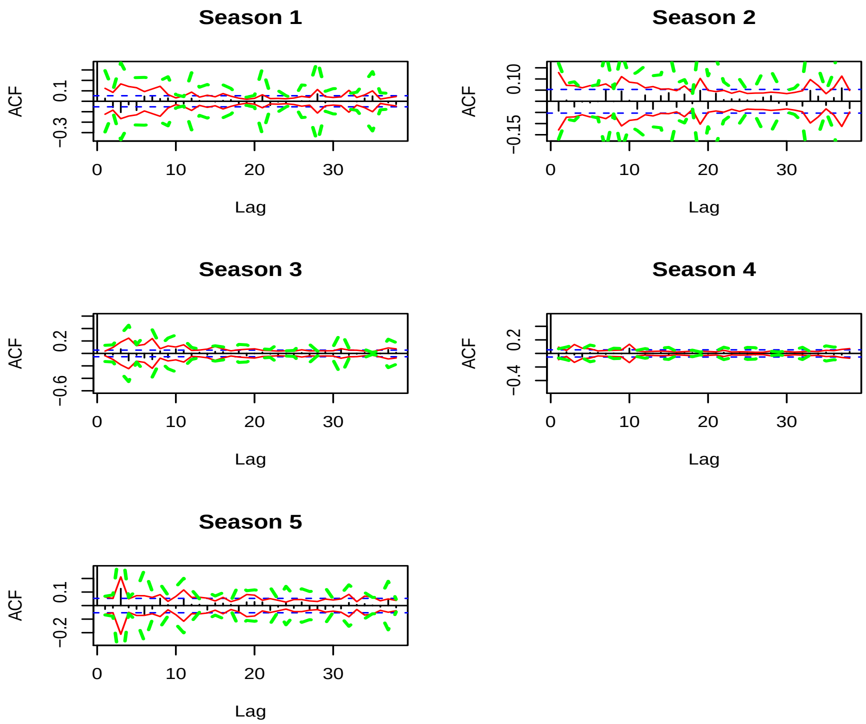

FIGURE 5. *Autocorrelation of the $PARMA_5(1,1)$ residuals for the squares of the CAC 40 returns. The horizontal dotted lines (blue color) correspond to the 5% significant limits obtained under the strong $PARMA_5(1,1)$ assumption. The solid lines (red color) and dashed lines (green color) correspond also to the 5% significant limits under the weak $PARMA_5(1,1)$ assumption. The full lines correspond to the asymptotic significance limits for the residual autocorrelations obtained in Theorem 3.1. The dashed lines (green color) correspond to the self-normalized asymptotic significance limits for the residual autocorrelations obtained in Theorem 3.5.*

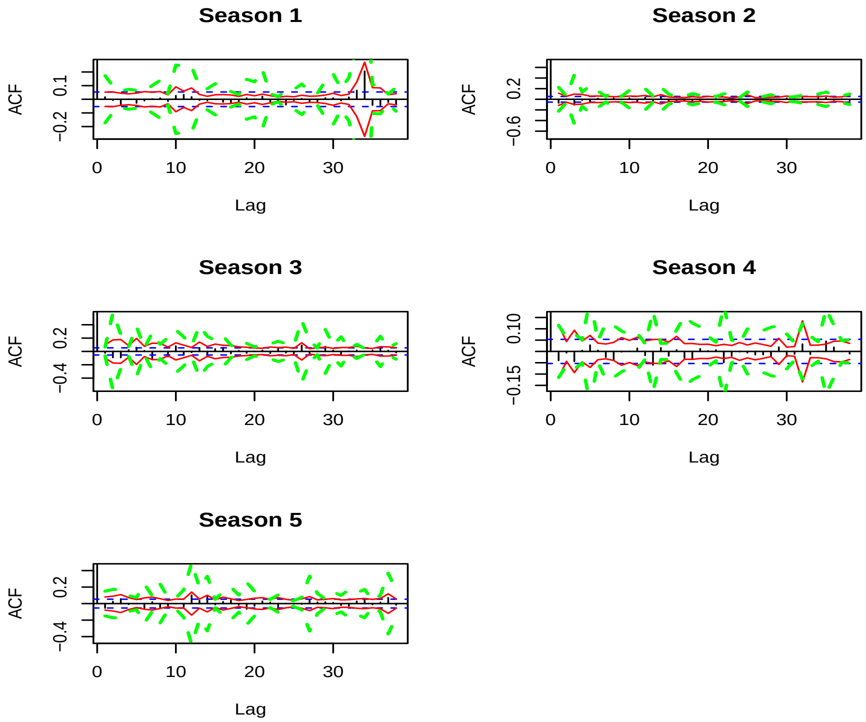

Figure 6. *Autocorrelation of the $PARMA_5(1,1)$ residuals for the squares of the DAX returns. The horizontal dotted lines (blue color) correspond to the 5% significant limits obtained under the strong $PARMA_5(1,1)$ assumption. The solid lines (red color) and dashed lines (green color) correspond also to the 5% significant limits under the weak $PARMA_5(1,1)$ assumption. The full lines correspond to the asymptotic significance limits for the residual autocorrelations obtained in Theorem 3.1. The dashed lines (green color) correspond to the self-normalized asymptotic significance limits for the residual autocorrelations obtained in Theorem 3.5.*

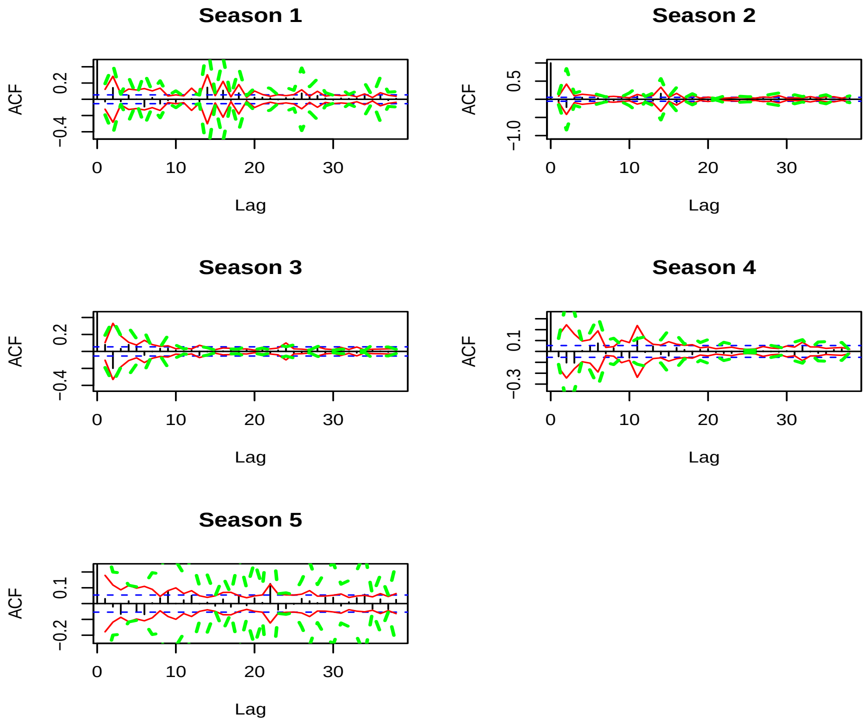

FIGURE 7. *Autocorrelation of the* $PARMA_5(1,1)$ *residuals for the squares of the SP500 returns. The horizontal dotted lines (blue color) correspond to the 5% significant limits obtained under the strong* $PARMA_5(1,1)$ *assumption. The solid lines (red color) and dashed lines (green color) correspond also to the 5% significant limits under the weak* $PARMA_5(1,1)$ *assumption. The full lines correspond to the asymptotic significance limits for the residual autocorrelations obtained in Theorem 3.1. The dashed lines (green color) correspond to the self-normalized asymptotic significance limits for the residual autocorrelations obtained in Theorem 3.5.*

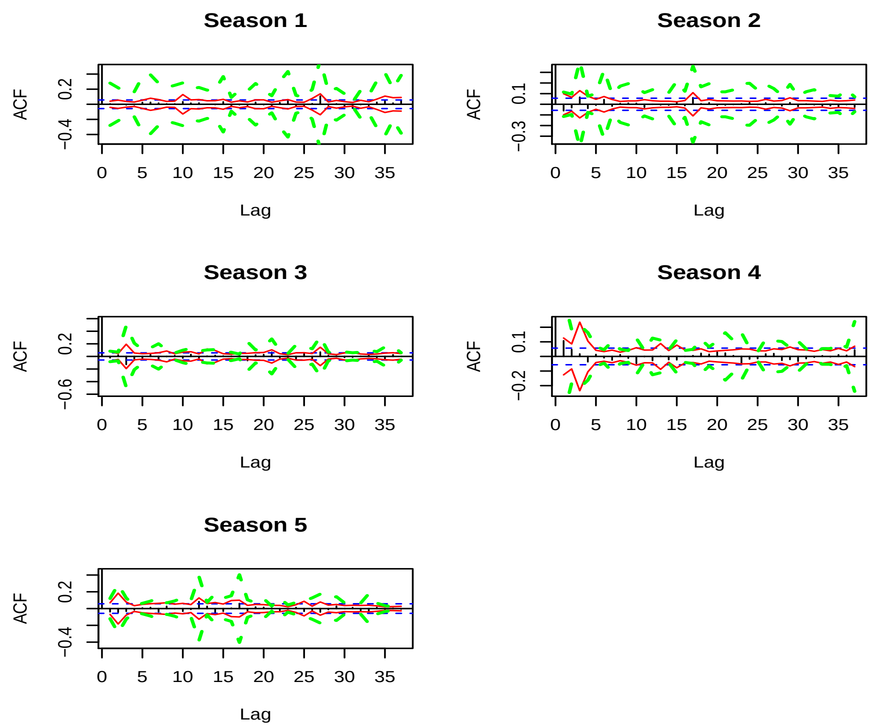

FIGURE 8. *Autocorrelation of the $PARMA_5(1,1)$ residuals for the squares of the Nikkei returns. The horizontal dotted lines (blue color) correspond to the 5% significant limits obtained under the strong $PARMA_5(1,1)$ assumption. The solid lines (red color) and dashed lines (green color) correspond also to the 5% significant limits under the weak $PARMA_5(1,1)$ assumption. The full lines correspond to the asymptotic significance limits for the residual autocorrelations obtained in Theorem 3.1. The dashed lines (green color) correspond to the self-normalized asymptotic significance limits for the residual autocorrelations obtained in Theorem 3.5.*