Solution to HW#3 (CE391F)

By: Abduallah Mohamed

To: Dr. Christian Claudel

Code: https://goo.gl/iss8go

Homework 3 (CE391F)

Due on November 8, 2017

I-Godunov scheme: We want to solve the classical LWR PDE on a grid, using the Godunov scheme. Consider a Trapezoidal fundamental diagram, defined by $\psi(k)=k\,v$ for $k<k_{c1}$, $\psi(k)=k_{c1}v$ for $k>k_{c1}$ and $k<k_{c2}$ and $\psi(k)=-w(k-k_{max})$ for $k>k_{c2}$, with parameters k_{c1}=0.02/m, k_{max}=0.2/m, v=30 m/s and w=5 m/s

1) Determine the value of k_{c2}, given that the fundamental diagram is continuous in k_{c2}.

$$\psi(k) = \begin{cases} kv, & if\ k \le k_{c1} \\ k_{c1}v, & if\ k > k_{c1}\ and\ k \le k_{c2} \\ -w(k - k_{max})\ , & if\ k > k_{c2} \end{cases}$$

A function f is continuous at $x = a$ if and only if [1]

$$f(a) = \lim_{x \to a} f(x) \quad \dots (1)$$

Applying it to the Trapezoidal fundamental diagram in domain where k >k_{c2}:

$$f(k_{c2}) = \lim_{k \to k_c 2} -w(k - k_{max})$$
$$= -w(k_{c2} - k_{max}) \dots (2)$$

If the fundamental diagram is continuous over the period $k_{c1} < k \le k_{c2}$,yields:

$$k_{c1}v = -w(k_{c2} - k_{max})$$
$$k_{c2} = \frac{k_{c1}v}{-w} + k_{max} \dots (3)$$

By substituting into (3) with the given parameters:

k_{c2} = 0.08/m will ensure that the fundamental diagram is continuous over in k_{c2}

Solving it using another method, from the geometry of Trapezoidal,
We can tell:

$$\psi(k_{c1}) = \psi(k_{c2}) \dots (4)$$

Leading to:

$$k_{c1}v = -w(k_{c2} - k_{max})$$
$$k_{c2} = \frac{k_{c1}v}{-w} + k_{max} \dots (5)$$

Which is the exact same solution we obtained in (3)

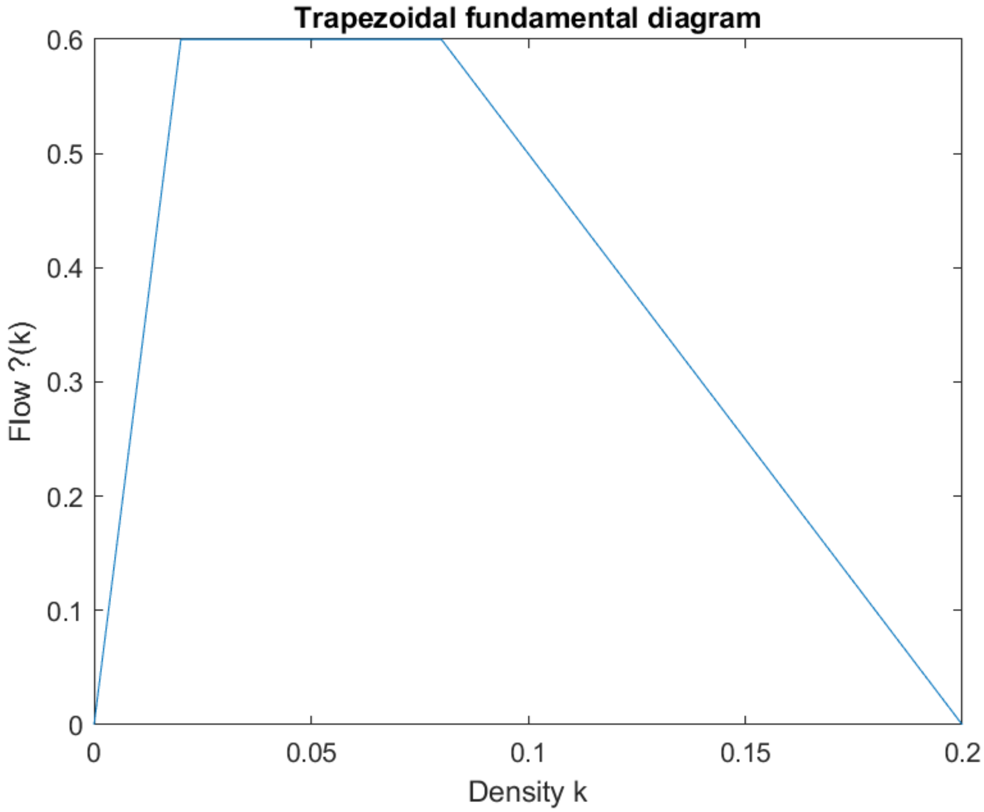


Figure 1:Plot of calculated fundamental diagram

2) Define the demand and supply functions at the boundaries for each cell

Demand function $D(k)$ corresponds to the flow that would be sent to the next cell assuming this cell has infinite storage.

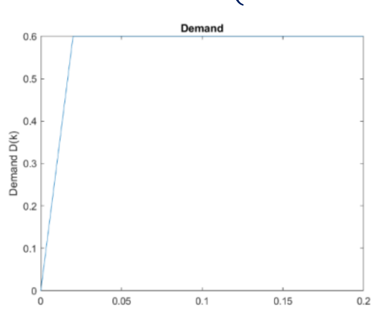
$$D(k) = \psi(\min(k, k_c)), where\ k_c \in \{k_{c1}, k_{c2}\} \ldots (6)$$ [2]

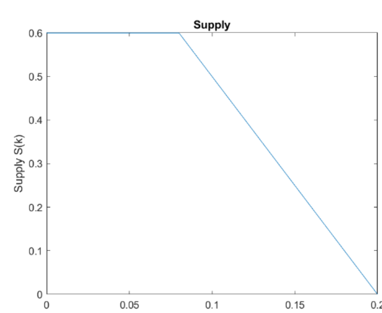
$$D(k) = \begin{cases} \psi(k), & if\ k \leq k_{c1} \\ \psi(k_{c1}) = vk_{c1}, & if\ k > k_{c1}\ and\ k \leq k_{c2} \\ \psi(k_{c2}) = vk_{c1}, & if\ k > k_{c2} \end{cases} \ldots (7)$$

Supply function $S(k)$ corresponds to the flow that would be received from the previous cell, assuming this cell has an infinite number of vehicles.

$$S(k) = \psi(\max(k, k_c)), where\ k_c \in \{k_{c1}, k_{c2}\} \ldots (8)$$

$$S(k) = \begin{cases} \psi(k_{c1}) = vk_{c1}, & if\ k \leq k_{c1} \\ \psi(k_{c2}) = vk_{c1}, & if\ k > k_{c1}\ and\ k \leq k_{c2} \\ \psi(k), & if\ k > k_{c2} \end{cases} \ldots (9)$$





3) Consider a network of 10 cells, each cell being 100 meters long. What is the minimal time step to consider according to the CFL condition?

The Courant–Friedrichs–Lewy (CFL) condition is a necessary condition for convergence while solving certain partial differential equations (usually hyperbolic PDEs) numerically by the method of finite differences.[3]

$$\left|v\frac{\Delta t}{\Delta x}\right| < 1 \dots (10)$$

$yileding: \Delta t < 3\frac{1}{3}$

4) Compute the solution to the following initial and boundary conditions problem (over 60s):

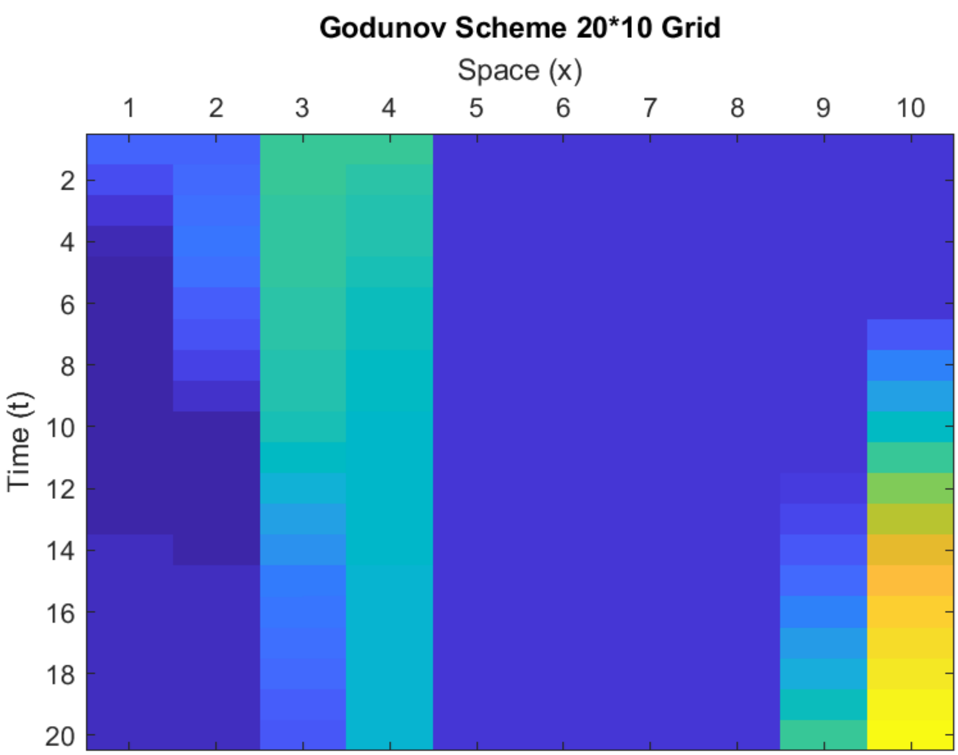
a. Initial condition: k(0,x)=0.04 if 0<x<300, k(0,x)=0.1/m for 300<x<500, and k(0,x)=0.02/m for x>500

I chose $\Delta t = 3$ which lead to having 20 cell in time axis vs 10 cells in space axis.

b. Upstream boundary demand: d(t)=0.3/s if 0s<t<40s, and d(t)=0.5 for 40s<t<60s

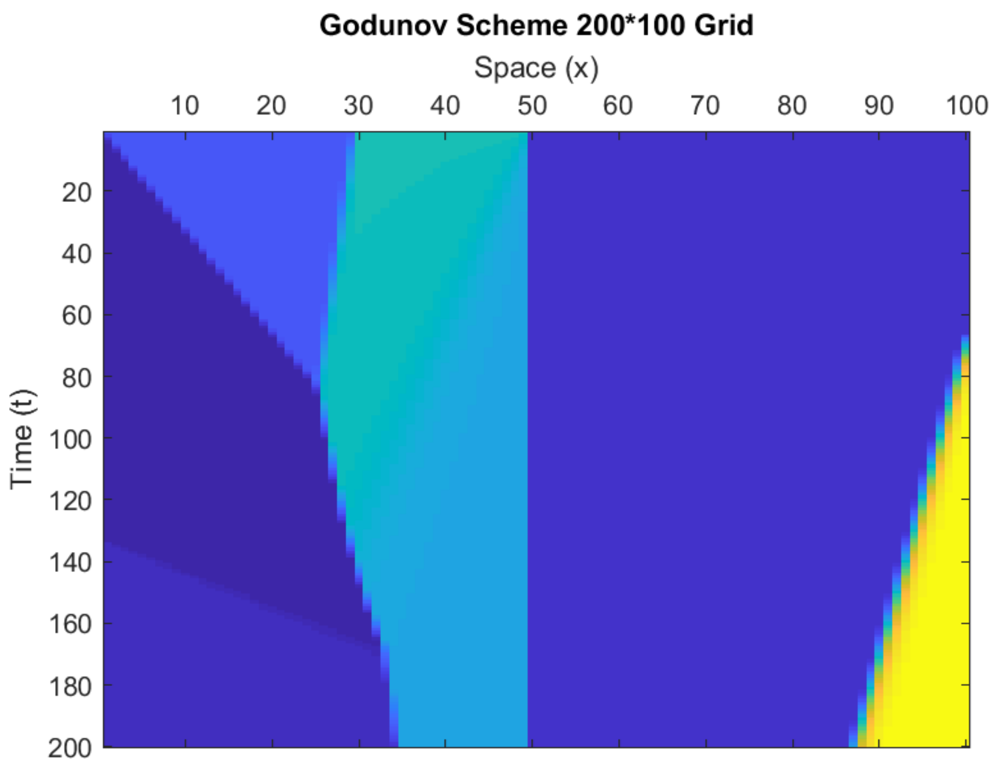
c. Downstream boundary supply: s(t)=0.6/s if 0s<t<20s, and s(t)=0.05/s for 20s<t<60s

Plot the solutions using Matlab

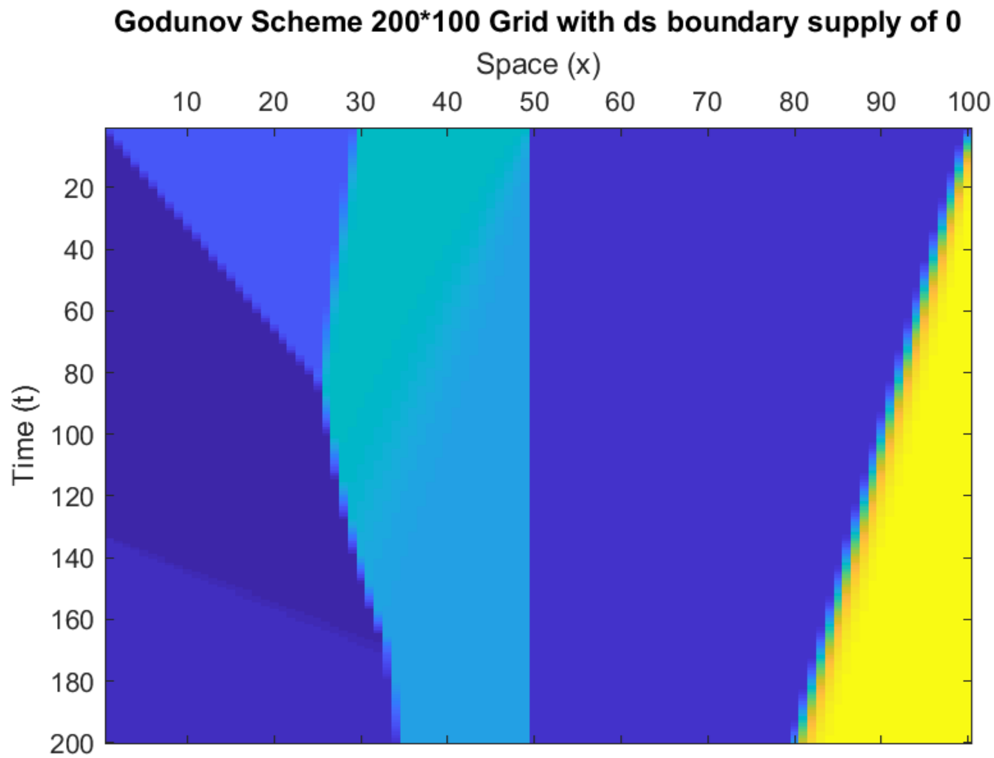


5) Repeat the above calculation, for a network containing 100 cells (modify the time step accordingly), and plot the corresponding solutions

   Keeping the space at 1000 m and applying the CFL condition $\Delta t < \frac{1}{3}$, I chose$\Delta t = 0.3$, thus we have a grid with 100 cell in space and 200 cell in time.



6) Now, plot the solution corresponding to a downstream boundary supply s(t)=0/s if 0<t<60s, and interpret the solution.



The 0 downstream boundary supply condition, allowed more flow from the end of the grid, it's like a traffic light with green signal, unlike the previous grid it was like a traffic light with red signal for a period, then started to be green, it's similar to a shockwave where higher density clashes with lower density .

References

[1] Continuous Functions, Lawrence S. Husch and University of Tennessee, Knoxville, Mathematics Department, http://archives.math.utk.edu/visual.calculus/1/continuous.5/

[2] Wen-Long Jin et. al ,"Supply-demand diagrams and a new framework for analyzing the inhomogeneous Lighthill-Whitham-Richards model",2010

[3] Courant–Friedrichs–Lewy condition, https://en.wikipedia.org/wiki/Courant%E2%80%93Friedrichs%E2%80%93Lewy_condition