A Guide to the Design of a Trajectory Tracking Controller for a Thrust Vectored Vehicle

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1 Introduction

This manuscript presents the dynamical system model for a vectored thrust vehicle and the controller design for semi-global exponential stabilization of a given reference trajectory. We provide the proofs of the various stability results that accompany the controller design and, to take full advantage of this presentation, the reader should be familiar with the framework of hybrid dynamical systems, as presented in [**?**]. If, on the other hand, the reader wishes to skip such details, he or she may find a streamlined guide to the controller design in (C1) through (C4). To run the code that is provided in this repository, the user must have installed the hybrid equations solver and CVX that are available in [**?**] and [**?**], respectively. The reader will find that the functions and variables in this document are in an one-to-one relation with those in the source code.

2 Notation

$\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ denotes the set of natural numbers and zero, $\mathbb{R}_{>0} := \{x \in \mathbb{R} : x > 0\}$, $\mathbb{R}_{\geq 0} := \mathrm{cl}\,(\mathbb{R}_{>0})$ where $\mathrm{cl}\,(S)$ denotes the closure of a set S, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space equipped with the norm $|x| := \sqrt{\langle x, x\rangle}$ for each $x \in \mathbb{R}^n$, where $\langle u, v\rangle := u^\top v$ for each $u, v \in \mathbb{R}^n$. The canonical basis for $\mathbb{R}^n$ is denoted by $\{e_i\}_{1\leq i\leq n} \subset \mathbb{R}^n$ and $c + r\mathbb{B} := \{x \in \mathbb{R}^n : |x - c| \leq r\}$. If a real symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive (negative) definite, we write $A \in \mathbb{S}^n_{>0}$ ($A \in \mathbb{S}^n_{<0}$) or $A \succ 0$ ($A \prec 0$) if the dimensions can be inferred from context. If a real symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive (negative) semidefinite, we write $A \in \mathbb{S}^n_{\geq 0}$ ($A \in \mathbb{S}^n_{\leq 0}$) or $A \succeq 0$ ($A \preceq 0$) if the dimensions can be inferred from context. Given $A \in \mathbb{R}^{m\times n}$, $\sigma_{\max}(A)$ denotes the maximum singular value of A. The gradient of a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is given by $\nabla V(x) := \begin{bmatrix} \frac{\partial V}{\partial x_1}(x) & \ldots & \frac{\partial V}{\partial x_n}(x)\end{bmatrix}$ for each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. The derivative of a differentiable matrix function with matrix arguments $F : \mathbb{R}^{m\times n} \to \mathbb{R}^{k\times \ell}$ is given by $\mathcal{D}_X\,(F(X)) := \partial \mathrm{vec}\,(F(X))\,/\partial \mathrm{vec}\,(X)^\top$ for each $X \in \mathbb{R}^{m\times n}$. The domain of a set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is given by $\mathrm{dom}\, M := \{x \in \mathbb{R}^n : M(x) \neq \emptyset\}$. The range of M is the set $\mathrm{rge}\, M := \{y \in \mathbb{R}^n : \exists x \in \mathbb{R}^m \text{ such that } y \in M(x)\}$.

3 Controller Design

The dynamics of a thrust vectored vehicle such as a quadrotor can be described by

$$\begin{align} \dot{p} &= v \tag{1a}\\ \dot{v} &= Rru + g \tag{1b}\\ \dot{R} &= RS(\omega) \tag{1c} \end{align}$$

where $p \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$ denote the position and the velocity of the vehicle with respect to the inertial reference frame (in inertial coordinates), $R \in \mathrm{SO}(3) := \{R \in \mathbb{R}^{3\times 3} : R^\top R = I_3, \det(R) = 1\}$ is the rotation matrix that maps vectors in body-fixed coordinates to inertial coordinates, $g \in \mathbb{R}^3$ represents the gravity vector and $r \in \mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$ is the thrust vector in body-fixed coordinates. Furthermore, the inputs to (1) are $\omega \in \mathbb{R}^3$ and $u \in \mathbb{R}$ which represent the angular velocity in body-fixed coordinates and the magnitude of the thrust, respectively. The dynamical model (1) is a simplification of the one provided in [1] that better suits our experimental setup, since there the *Blade 200 QX* quadrotor that is used in the experiments has an embedded controller that tracks angular velocity and thrust commands. Furthermore, we assume that the reference trajectory satisfies the following assumption.

Assumption 1 *The reference trajectory $t \mapsto p_d(t)$ is defined for each $t \geq 0$ and there exist $M_2 \in (0, |g|)$ and $M_3 > 0$ such that*

$$|\ddot{p}_d(t)| \leq M_2 \text{ and } p_d^{(3)}(t) \leq M_3$$

for all $t \geq 0$.

Given a path that satisfies Assumption 1, we define the tracking errors as

$$\begin{aligned} \widetilde{p} &:= p - p_d\\ \widetilde{v} &:= v - \dot{p}_d \end{aligned}$$

whose dynamics can be derived from (1) and are given by:

$$\begin{aligned} \dot{\widetilde{p}} &= \widetilde{v},\\ \dot{\widetilde{v}} &= Rru + g - \ddot{p}_d. \end{aligned} \tag{2}$$

Given $w : \mathbb{R}^6 \to \mathbb{R}^3$, we define

$$\begin{align} \rho(z) &:= \frac{w(\widetilde{p},\widetilde{v}) - g + \ddot{p}_d}{|w(\widetilde{p},\widetilde{v}) - g + \ddot{p}_d|} && \forall z \in Z, \tag{3a}\\ \kappa_u(z,R) &:= r^\top R^\top (w(\widetilde{p},\widetilde{v}) - g + \ddot{p}_d) && \forall (z,R) \in Z \times \mathrm{SO}(3), \tag{3b} \end{align}$$

where $z := (\ddot{p}_d, \widetilde{p}, \widetilde{v}) \in Z := M_2\mathbb{B} \times \mathbb{R}^6$. Note that, if $Rr = \rho(z)$ and $u = \kappa_u(z,R)$, we obtain

$$\begin{aligned} \dot{\widetilde{p}} &= \widetilde{v},\\ \dot{\widetilde{v}} &= w(\widetilde{p},\widetilde{v}) \end{aligned} \tag{4}$$

from (2), provided that

$$w(\widetilde{p},\widetilde{v}) - g + \ddot{p}_d \neq 0. \tag{5}$$

On the other hand, if $Rr \neq \rho(z)$, then $\kappa_u(z, R)$ is the solution to the least-squares problem:

$$\min\{|Rru + g - \ddot{p}_d - w(\widetilde{p}, \widetilde{v})|^2 : u \in \mathbb{R}\}.$$

The mismatch between Rr and $\rho(z)$ may lead to an increase in the the position and velocity tracking errors until the thrust vector Rr is aligned with $\rho(z)$. The controller design is a two-step process, where we start by designing a feedback law $(\widetilde{p}, \widetilde{v}) \mapsto w(\widetilde{p}, \widetilde{v})$ that exponentially stabilizes the origin of (4) and then we design a partial attitude tracking controller that exponentially stabilizes $\rho(z)$ for the dynamics of the thrust vector Rr.

3.1 Controller for the Position Subsystem

We show next that it is possible to satisfy (5) and exponentially stabilize the origin of (4) from an arbitrary compact set U using a saturated linear feedback law

$$w(\widetilde{p}, \widetilde{v}) := \mathrm{Sat}_b\left(K\begin{bmatrix}\widetilde{p}^\top & \widetilde{v}^\top\end{bmatrix}^\top\right) \tag{6}$$

where $b \in (0, |g| - M_2)$ and $v \mapsto \mathrm{Sat}_b(v) := \begin{bmatrix}\mathrm{sat}_b(v_1) & \dots & \mathrm{sat}_b(v_n)\end{bmatrix}^\top$ for each $v \in \mathbb{R}^n$ with $\mathrm{sat}_b : \mathbb{R} \to \mathbb{R}$ continuously differentiable and nondecreasing, satisfying $\mathrm{sat}_b(x) = x$ for all $x \in [-b, b]$ and $\mathrm{sat}_b(x) \in (M_2 - |g|, |g| - M_2)$ for each $x \in \mathbb{R}$. Defining $\Omega_P(\ell) := \{x \in \mathbb{R}^n : x^\top P x \leq \ell\}$ for $P \in \mathbb{R}^{n\times n}$ and $\ell > 0$, if we select $(H, \ell_H) \in \mathbb{S}^6_{>0}$ such that $\Omega_H(\ell_H)$ is a bounding ellipsoid for U, then we guarantee that the bounds $\pm b$ of the saturation function (6) are not reached if there exists $(P, \ell_P) \in \mathbb{S}^6_{>0} \times \mathbb{R}_{>0}$ and $K \in \mathbb{R}^{3\times 6}$, such that $\Omega_P(\ell_P)$ is forward invariant for every solution to (4) from U and

$$U \subset \Omega_H(\ell_H) \subset \Omega_P(\ell_P) \subset \Omega_{K^\top K}(b^2). \tag{7}$$

For each compact set $U \subset \mathbb{R}^6$, it is possible to select controller parameters that satisfy (7) as well as

$$(A + BK)^\top P + P(A + BK) \preceq -\widehat{Q} - K^\top \widehat{R} K \tag{8}$$

where

$$A := \begin{bmatrix} 0 & I_3 \\ 0 & 0 \end{bmatrix}, \quad B := \begin{bmatrix} 0 \\ I_3 \end{bmatrix}.$$

thus guaranteeing also the exponential stability of the origin of (4) from U, as stated in the proposition below.

Proposition 1 *For each compact set $U \subset \mathbb{R}^6$, $b > 0$, $\ell_P > 0$, $\widehat{R} \in \mathbb{S}^3_{>0}$ and $H \in \mathbb{S}^6_{>0}$, there exists $K \in \mathbb{R}^{3\times 6}$ and $\widehat{Q}, P \in \mathbb{S}^6_{>0}$ such that the conditions* (7) *and* (8) *are satisfied.*

Proof: From [2, Lemma 4.20], it follows that there is a function $\epsilon \mapsto \widehat{Q}(\epsilon)$, with the properties

$$\widehat{Q}(\epsilon) \succ 0 \quad \text{and} \quad \frac{d\widehat{Q}(\epsilon)}{d\epsilon} \succ 0$$

for each $\epsilon \in (0, 1]$, that generates a unique solution $P(\epsilon) \succ 0$ to

$$A^\top P(\epsilon) + P(\epsilon)A - P(\epsilon)B\widehat{R}^{-1}B^\top P(\epsilon) + \widehat{Q}(\epsilon) = 0 \tag{9}$$

satisfying $P(\epsilon) \to 0$ as $\epsilon \to 0$.

Choosing $K = -\widehat{R}^{-1}B^\top P(\epsilon)$, it follows that

$$\begin{aligned}(A+BK)^\top P(\epsilon) + P(\epsilon)(A+BK)\\ = A^\top P(\epsilon) + P(\epsilon)A - 2P(\epsilon)B\widehat{R}^{-1}B^\top P(\epsilon).\end{aligned} \tag{10}$$

Adding and subtracting $\widehat{Q}(\epsilon)$ to the right hand side of (10) and using (9) yields

$$\begin{aligned}A^\top P(\epsilon) + P(\epsilon)A + K^\top B^\top P(\epsilon) + P(\epsilon)BK =\\ - P(\epsilon)B\widehat{R}^{-1}B^\top P(\epsilon) - \widehat{Q}(\epsilon).\end{aligned}$$

Therefore, the condition (8) is satisfied with $P = P(\epsilon)$ and $\widehat{Q} = \widehat{Q}(\epsilon)$. The condition $\Omega_H(\ell_H) \subset \Omega_P(\ell_P)$ is satisfied if and only if

$$\frac{P(\epsilon)}{\ell_P} \preceq \frac{H}{\ell_H}, \tag{11}$$

and the condition $\Omega_P(\ell_P) \subset \Omega_{K^\top K}(b^2)$ is satisfied if and only if $\frac{K^\top K}{b^2} \preceq \frac{P(\epsilon)}{\ell_P} \iff P(\epsilon)^{\frac{1}{2}}B\widehat{R}^{-2}B^\top P(\epsilon)^{\frac{1}{2}} \preceq \frac{b^2}{\ell_P}I_6$, which is equivalent to

$$\left|\widehat{R}^{-1}B^\top P(\epsilon)^{\frac{1}{2}}v\right|^2 \leq \frac{b^2}{\ell_P}|v|^2 \quad \forall v \in \mathbb{R}^6. \tag{12}$$

From [3, Proposition 9.4.9], it follows that (12) is satisfied if

$$\sigma_{\max}(\widehat{R}^{-1}B^\top P(\epsilon)^{\frac{1}{2}})^2 \leq \frac{b^2}{\ell_P} \tag{13}$$

holds. Since $P(\epsilon)$ can be made arbitrarily close to zero by choosing a small enough ϵ, it follows that, for each $\ell_P > 0$, $\ell_H > 0$, $\widehat{R} \succ 0$ and $b > 0$, it is possible to find $P(\epsilon)$ satisfying both (13) and (11). □

The previous proposition is very important to the following theorem, which constitutes the main result of this section.

Theorem 1 *For each $b \in (0, |g| - M_2)$, and each compact set $U \subset \mathbb{R}^6$, there exists $K \in \mathbb{R}^{3\times 6}$ such that the origin of the closed-loop system resulting from the interconnection between (4) and (6) is exponentially stable from U. Moreover, each solution to (4) from U, denoted by $t \mapsto (\widetilde{p}, \widetilde{v})(t)$, satisfies $\left|K\begin{bmatrix}\widetilde{p}(t)^\top & \widetilde{v}(t)^\top\end{bmatrix}^\top\right| \leq b$ for each $t \geq 0$.*

Proof: Choosing a positive definite matrix $H \in \mathbb{S}^6_{>0}$ such that $\Omega_H(1)$ is a bounding ellipsoid for U, it follows from Proposition 1 that there exists $K \in \mathbb{R}^{3\times 6}$ and $\widehat{Q}, P \in \mathbb{S}^6_{>0}$ such that the conditions (7) and (8) hold for any positive definite matrix $\widehat{R} \in \mathbb{R}^{3\times 3}$ and any $\ell_P > 0$. Let $V_p(\widetilde{p}, \widetilde{v}) := \begin{bmatrix}\widetilde{p}^\top & \widetilde{v}^\top\end{bmatrix} P \begin{bmatrix}\widetilde{p}^\top & \widetilde{v}^\top\end{bmatrix}^\top$, for each $(\widetilde{p}, \widetilde{v}) \in \mathbb{R}^6$, which is a positive definite function relative to $\{(\widetilde{p}, \widetilde{v}) \in \mathbb{R}^6 : \widetilde{p} = \widetilde{v} = 0\}$ and satisfies: for each $(\widetilde{p}, \widetilde{v}) \in \Omega_P(\ell_P)$

$$\langle \nabla V_p(\widetilde{p}, \widetilde{v}), f_p(\widetilde{p}, \widetilde{v})\rangle \leq -\begin{bmatrix}\widetilde{p}\\ \widetilde{v}\end{bmatrix}^\top (\widehat{Q} + K^\top \widehat{R}K)\begin{bmatrix}\widetilde{p}\\ \widetilde{v}\end{bmatrix}, \tag{14}$$

where $f_p(\widetilde{p},\widetilde{v}) := \begin{bmatrix}\widetilde{v}^\top & \begin{bmatrix}\widetilde{p}^\top & \widetilde{v}^\top\end{bmatrix} K^\top\end{bmatrix}$. It follows from the conditions (7), that every solution $t \mapsto (\widetilde{p},\widetilde{v})(t)$ from U satisfies $\left| K \begin{bmatrix}\widetilde{p}(t) \\ \widetilde{v}(t)\end{bmatrix} \right| \le b < |g| - M_2$ for all $t \ge 0$. $\square$

In the next lemma, we show that the controller gain K in (6) can be computed from an optimization problem that provides a sub-optimal solution to the H_2-minimization problem (c.f. [4, Proposition 3.11]).

Lemma 1 *Given $\ell_H > 0$, $\ell_P > 0$, $\widehat{Q} \in \mathbb{S}^6_{>0}$ and $\widehat{R} \in \mathbb{S}^3_{>0}$, there exists a solution $P \in \mathbb{S}^6_{>0}$, $K \in \mathbb{R}^{3\times 6}$ to (7) and (8) if and only if there exists a solution $Y \in \mathbb{S}^6_{>0}$ and $L \in \mathbb{R}^{3\times 6}$ to*

$$\begin{array}{ll} \textit{minimize} & \operatorname{trace}(Y^{-1}) \\ \textit{subject to} & (Y, L) \in \chi_{LMI} \end{array}$$

where χ_{LMI} is the set of linear matrix inequalities:

$$\begin{bmatrix} -(AY+BL)^\top - (AY+BL) & Y & L^\top \\ Y & \widehat{Q}^{-1} & 0 \\ L & 0 & \widehat{R}^{-1} \end{bmatrix} \succeq 0 \tag{15a}$$

$$Y \succeq \frac{\ell_H}{\ell_P} H^{-1} \tag{15b}$$

$$\begin{bmatrix} Y & L \\ L^\top & \frac{b^2}{\ell_P} I_6 \end{bmatrix} \succeq 0 \tag{15c}$$

in which case $P = Y^{-1}$ and $K = LY^{-1}$.

Proof: Since $\widehat{Q} \succ 0$, we can rewrite (8) as follows:

$$\begin{bmatrix} X - Y\widehat{Q}Y - L^\top \widehat{R} L & 0 \\ 0 & \widehat{Q}^{-1} \end{bmatrix} \succeq 0 \tag{16}$$

where $X := -(AY+BL)^\top - (AY+BL)$. Equivalently, we obtain

$$\begin{bmatrix} X & Y \\ Y & \widehat{Q}^{-1} \end{bmatrix} - \begin{bmatrix} L^\top \\ 0 \end{bmatrix} \widehat{R} \begin{bmatrix} L & 0 \end{bmatrix} \succeq 0 \tag{17}$$

from (16) by means of a congruence transformation. The application of [3, Proposition 8.2.4.iii)] to (17) yields (15a). The condition (15b) follows from the fact that $\Omega_H(\ell_H) \subset \Omega_P(\ell_P)$ if and only if $P \preceq \frac{\ell_P}{\ell_H} H$, using $Y = P^{-1}$. The condition (15c) follows from the fact that $\Omega_P(\ell_P) \subset \Omega_{K^\top K}(b^2)$ if and only if $\frac{K^\top K}{b^2} \preceq \frac{P}{\ell_P}$ and [3, Proposition 8.2.4] as follows:

$$\begin{aligned} \frac{K^\top K}{b^2} \preceq \frac{P}{\ell_P} &\iff \frac{YK^\top K Y}{b^2} \preceq \frac{Y}{\ell_p} \\ &\iff \frac{Y}{\ell_P} - \frac{L^\top L}{b^2} \succeq 0 \end{aligned}$$

where we have, once again, used $Y = P^{-1}$ and $L = KY$. $\square$

3.2 Partial attitude tracking

In this section, we develop a controller for (1) that tracks a reference trajectory satisfying Assumption 1. The desired acceleration, imposed by the reference trajectory upon the vehicle, is achieved by aligning the thrust vector Rr with the direction of the desired acceleration. We refer to this as partial attitude tracking because we do not control rotations around the thrust vector.

Given $k > 0$ and $\gamma \in (-1, 1)$, we define

$$V(x, y) := \frac{1 - r^\top x}{1 - r^\top x + k(1 - x^\top y)}$$

for each $(x, y) \in \mathbb{S}^2 \times \mathcal{Q}$, where $\mathcal{Q} := \{y \in \mathbb{S}^2 : y^\top r \leq \gamma\}$. Defining

$$\underline{\alpha} = \frac{1}{2(1 + k + \sqrt{1 + 2k\gamma + k^2})}, \tag{18a}$$

$$\overline{\alpha} = \frac{1}{2(1 + k - \sqrt{1 + 2k\gamma + k^2})}, \tag{18b}$$

$$\lambda_1 = \frac{2k(1 - V^*)(1 - \gamma)}{\left(1 + k + \sqrt{1 + 2k\gamma + k^2}\right)^2}. \tag{18c}$$

and

$$C := \{(x, y) \in \mathbb{S}^2 \times \mathcal{Q} : \mu(x, y) \leq \delta\}, \tag{19a}$$

$$D := \{(x, y) \in \mathbb{S}^2 \times \mathcal{Q} : \mu(x, y) \geq \delta\}, \tag{19b}$$

where $\mu(x, y) := V(x, y) - \min\{V(x, , y) : y \in \mathcal{Q}\}$ for each $(x, y) \in \mathbb{S}^2 \times \mathcal{Q}$ and $V^* := \max\{V(x, y) : (x, y) \in C\}$, it is possible to verify that

$$\begin{aligned} \underline{\alpha}\, |x - r|^2 \leq V(x, y) \leq \overline{\alpha}\, |x - r|^2 \quad & \forall (x, y) \in C \cup D = \mathbb{S}^2 \times \mathcal{Q} \\ |\Pi(x) \nabla V^y(x)|^2 \geq \lambda_1 V(x, y) \qquad & \forall (x, y) \in C. \end{aligned}$$

Using the previous construction, we define the the closed-loop system $\mathcal{H}_1 := (C_1, F_1, D_1, G_1)$ with state $\zeta := (z, R, y) \in \mathcal{Z} := Z \times \mathrm{SO}(3) \times \mathbb{S}^2$, given by

$$\begin{aligned} F_1(\zeta) &:= \left\{ \begin{pmatrix} F_p(p_d^{(3)}, \zeta) \\ RS(\kappa_1(p_d^{(3)}, \zeta)) \\ 0 \end{pmatrix} : p_d^{(3)} \in M_3 \mathbb{B} \right\} \\ & \qquad \qquad \forall \zeta \in C_1 := \{\zeta \in \mathcal{Z} : (R^\top \rho(z), y) \in C\} \\ G_1(\zeta) &:= \begin{pmatrix} z \\ R \\ \varrho(R^\top \rho(z)) \end{pmatrix} \quad \forall \zeta \in D_1 := \{\zeta \in \mathcal{Z} : (R^\top \rho(z), y) \in D\} \end{aligned} \tag{20}$$

where $\varrho(x) := \arg\min\{V(x, y) : y \in \mathcal{Q}\}$ for each $x \in \mathbb{S}^2$,

$$F_p(p_d^{(3)}, \zeta) := \begin{bmatrix} p_d^{(3)} \\ \widetilde{v} \\ Rr\kappa_u(z, R) + g - \ddot{p}_d \end{bmatrix}$$

for each $(p_d^{(3)}, \zeta) \in M_3\mathbb{B} \times \mathcal{Z}$, the inputs u and ω were assigned to $\kappa_u(z, \mathcal{R}(\zeta))$ and

$$\begin{aligned}\kappa_1(p_d^{(3)}, \zeta) := S(R^\top \rho(z))\big(&R^\top \mathcal{D}_z\left(\rho(z)\right) F_p(p_d^{(3)}, \zeta)\\ &+ (k_1 + k_p \nu^\star(z))\nabla V^y(R^\top \rho(z))\big),\end{aligned} \tag{21}$$

for each $(p_d^{(3)}, \zeta) \in M_3\mathbb{B} \times \mathcal{Z}$, respectively, with $k_1 > 0$, $k_p > 0$ and

$$\nu^\star(z) := \frac{2}{\sqrt{\underline{\alpha}}}\sigma_{\max}\left(\begin{bmatrix}0 & I_3\end{bmatrix} P^{\frac{1}{2}}\right)\left|w(\widetilde{p}, \widetilde{v}) - g + \ddot{p}_d\right|, \tag{22}$$

for each $z \in Z$. The function $\nu^\star$ is used to increase the gain of the attitude controller as the position error increases.

Given a reference trajectory satisfying Assumption 1 and a compact set of initial conditions $U \subset \mathbb{R}^6$ for the position and velocity errors the controller design is as follows:

(C1) Select $H \in \mathbb{S}^6_{>0}$ such that $\Omega_H(1)$ is a bounding ellipsoid for U;

(C2) Select $k_p, \bar{k}_1 > 0$ so that

$$k_p \bar{k}_1 \lambda_1 > 1 \tag{23}$$

(C3) Given $b \in (0, |g| - M_2)$ and $\widehat{R} \in \mathbb{S}^6_{>0}$, select $\widehat{Q} \in \mathbb{S}^6_{>0}$ small enough and $\ell_H = \ell_P = (1 + \bar{\nu})^2$, where

$$\bar{\nu} := \bar{k}_1 \sqrt{\max_{(x,y)\in\mathbb{S}^2\times\mathcal{Q}} V(x, y)};$$

(C4) Compute the controller gain K by means of the optimization problem in Lemma 1.

This controller design enables exponential stability, as proved next.

Theorem 2 *Let Assumption 1 hold. For each compact set $U \subset \mathbb{R}^6$, if (C1)–(C4) are satisfied, then $\mathcal{A}_1 := \{(\zeta, y) \in \mathcal{Z} \times \mathcal{Q} : \widetilde{p} = \widetilde{v} = 0, \rho(z) = Rr\}$ is exponentially stable in the t-direction from $U \times \mathrm{SO}(3) \times \mathcal{Q}$ for the hybrid system* (20)*.*

Proof:

Let

$$W_1(\zeta) := \sqrt{V_p(\widetilde{p}, \widetilde{v})} + \bar{k}_1\sqrt{V(x, y)} \quad \forall \zeta \in \mathcal{Z} \tag{24}$$

with $P := Y^{-1}$ and $x := R^\top \rho(z)$. We have that

$$\begin{aligned}\min\{\sqrt{\lambda_{\min}(P)}, \bar{k}_1\sqrt{\underline{\alpha}}\}\left|(\widetilde{p}, \widetilde{v}, x - r)\right| &\le W_1(\zeta)\\ &\le \sqrt{\lambda_{\max}(P) + \bar{k}_1^2\overline{\alpha}}\left|(\widetilde{p}, \widetilde{v}, x - r)\right|.\end{aligned}$$

for each $\zeta \in C_1 \cup D_1$. It follows from the Assumptions that the time derivative of (24) is given by

$$\begin{aligned}\langle W_1(\zeta), f_1\rangle &= \frac{1}{2\sqrt{V_p(\widetilde{p}, \widetilde{v})}}\nabla V_p(\widetilde{p}, \widetilde{v})^\top \begin{bmatrix}\widetilde{v}\\ Rr\kappa_u(z, R) + g - \ddot{p}_d\end{bmatrix}\\ &+ \frac{\bar{k}_1}{2\sqrt{V(x, y)}}\nabla V^y(x)^\top\left(S(x)\kappa_1(\zeta, y) - R^\top \mathcal{D}_z\left(\rho(z)\right) F_p(p_d^{(3)}, \zeta)\right)\end{aligned} \tag{25}$$

for each $f_1 \in F_1(\zeta)$, $\zeta \in \widehat{\Omega} := \{\zeta \in \mathcal{Z} : W_1(\zeta) \leq 1 + \bar{\nu}\}$, because $W_1(\zeta) \leq 1 + \bar{\nu}$ implies that $V_p(\widetilde{p}, \widetilde{v}) \leq (1 + \bar{\nu})^2$ and, by the construction (C1)-(C3), the saturation bound b is not exceeded in this set since it satisfies (7). Using $xx^\top = S(x)^2 + I_3$ for each $x \in \mathbb{S}^2$, and replacing (3b) and (21) into (25) yields

$$
\begin{aligned}
W_1^o(\zeta; f_1) &= \frac{1}{2\sqrt{V_p(\widetilde{p}, \widetilde{v})}} \nabla V_p(\widetilde{p}, \widetilde{v})^\top \begin{bmatrix} \widetilde{v} \\ w(\widetilde{p}, \widetilde{v}) \end{bmatrix} \\
&+ \frac{1}{2\sqrt{V_p(\widetilde{p}, \widetilde{v})}} \nabla V_p(\widetilde{p}, \widetilde{v})^\top \begin{bmatrix} 0 \\ I_3 \end{bmatrix} S(Rr)^2 (w(\widetilde{p}, \widetilde{v}) - g + \ddot{p}_d) \\
&- \frac{\bar{k}_1(k_1 + k_p \nu^\star(z))}{2\sqrt{V(x, y)}} \left| S(x)^2 \nabla V^y(x) \right|^2
\end{aligned}
\tag{26}
$$

for each $f_1 \in F_1(\zeta)$, $\zeta \in \widehat{\Omega}$. Using $\nabla V_p(\widetilde{p}, \widetilde{v}) = 2P \begin{bmatrix} \widetilde{p}^\top & \widetilde{v}^\top \end{bmatrix}^\top$ and (14) it follows from (26) that

$$
\begin{aligned}
W_1^o(\zeta; f_1) &\leq -\frac{1}{2\sqrt{V_p(\widetilde{p}, \widetilde{v})}} \begin{bmatrix} \widetilde{p} \\ \widetilde{v} \end{bmatrix}^\top (\widehat{Q} + K^\top \widehat{R} K) \begin{bmatrix} \widetilde{p} \\ \widetilde{v} \end{bmatrix} \\
&+ \frac{1}{\sqrt{V_p(\widetilde{p}, \widetilde{v})}} \left| \begin{bmatrix} 0 & I_3 \end{bmatrix} P \begin{bmatrix} \widetilde{p} \\ \widetilde{v} \end{bmatrix} \right| \left| S(Rr)^2 (w(\widetilde{p}, \widetilde{v}) - g + \ddot{p}_d) \right| \\
&- \frac{\bar{k}_1(k_1 + k_p \nu^\star(z))}{2\sqrt{V(x, y)}} \left| S(x)^2 \nabla V^y(x) \right|^2
\end{aligned}
\tag{27}
$$

for each $f_1 \in F_1(\zeta)$ $\zeta \in \widehat{\Omega}$. Since $\left| S(Rr)^2 (w(\widetilde{p}, \widetilde{v}) - g + \ddot{p}_d) \right| \leq |x - r| \, |w(\widetilde{p}, \widetilde{v}) - g + \ddot{p}_d|$ for each $\zeta \in C_1 \cup D_1$ and

$$
\frac{\left| \begin{bmatrix} 0 & I_3 \end{bmatrix} P z \right|}{\sqrt{z^\top P z}} = \frac{\left| \begin{bmatrix} 0 & I_3 \end{bmatrix} P^{1/2} \widetilde{z} \right|}{|\widetilde{z}|} \leq \sigma_{\max} \left(\begin{bmatrix} 0 & I_3 \end{bmatrix} P^{1/2} \right)
$$

for each $z \in \mathbb{R}^6 \backslash \{0\}$ where $\widetilde{z} := P^{1/2} z$, it follows (27) that

$$
\begin{aligned}
W_1^o(\zeta; f_1) &\leq -\frac{\lambda_{\min}(\widehat{Q} + K^\top \widehat{R} K)}{2\sqrt{\lambda_{\max}(P)}} \sqrt{V_p(\widetilde{p}, \widetilde{v})} \\
&+ \sigma_{\max} \left(\begin{bmatrix} 0 & I_3 \end{bmatrix} P^{1/2} \right) |x - r| \, |w(\widetilde{p}, \widetilde{v}) - g + \ddot{p}_d| \\
&- \frac{\bar{k}_1(k_1 + k_p \nu^\star(z))}{2\sqrt{V(x, y)}} \left| S(x)^2 \nabla V^y(x) \right|^2
\end{aligned}
\tag{28}
$$

for each $f_1 \in F_1(\zeta)$, $\zeta \in \widehat{\Omega}$. Replacing (22) into (28), we have that

$$
\begin{aligned}
W_1^o(\zeta; f_1) \leq &-\frac{\lambda_{\min}(\widehat{Q} + K^\top \widehat{R} K)}{2\sqrt{\lambda_{\max}(P)}} \sqrt{V_p(\widetilde{p}, \widetilde{v})} \\
&- \frac{\bar{k}_1(k_1 + k_p \nu^\star(z)) \lambda_1}{2} \sqrt{V(x, y)} \\
&+ \frac{\nu^\star(z)}{2} \sqrt{V(x, y)}
\end{aligned}
$$

for each $f_1 \in F_1(\zeta)$, $\zeta \in \widehat{\Omega}$. It follows from (C2) that

$$
\begin{aligned}
W_1^o(\zeta; f_1) \leq & -\frac{\lambda_{\min}(\widehat{Q} + K^\top \widehat{R} K)}{2\sqrt{\lambda_{\max}(P)}} \sqrt{V_p(\widetilde{p}, \widetilde{v})} \\
& - \frac{\bar{k}_1 k_1 \lambda_1 \sqrt{V(x,y)}}{2}
\end{aligned} \tag{29}
$$

for each $f_1 \in F_1(\zeta)$, $\zeta \in \widehat{\Omega}$. For each solution from U, we have $V_p(\widetilde{p}(0,0), \widetilde{v}(0,0)) \leq 1$ and, consequently, $W_1(\zeta(0,0)) \leq 1 + \bar{\nu}$. From (23) and (29), we have $W_1^o(\zeta; f_1) \leq -\lambda W_1(\zeta)$ for each $f_1 \in F_1(\zeta)$, $\zeta \in \widehat{\Omega}$ with $\lambda := \min\left\{\frac{\lambda_{\min}(\widehat{Q}+K^\top \widehat{R} K)}{2\sqrt{\lambda_{\max}(P)}}, \frac{\bar{k}_1 k_1 \lambda_1}{2}\right\}$.

The desired result follows from [5, Theorem 1] by noting that V is strictly decreasing during jumps of the closed-loop system.□

It should be pointed out that, underlying the controller design described in (C1) through (C4), there is a trade-off to be resolved: if $k_p << 1$, then the position controller is going to have a low gain K which results in large deviations from the reference; on the other hand, if controller gain K is large, the function $\nu^\star$ might increase the gain on the attitude controller beyond what is acceptable in practical terms.

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