Date: Day: 1

Question: 1

b) $L(x,y) = \frac{x^2+y^2}{x+y+1}$

Calculate partial derivative and find critical point.

$$Lx = \frac{2x(x+y+1) - (x^2+y^2)(1)}{(x+y+1)^2}$$

$$2x(x+y+1) - (x^2+y^2) = 2x^2 + 2xy + 2x - x^2 - y^2$$

$$= x^2 + 2xy + 2x - y^2$$

$$Lx = \frac{x^2 + 2xy - y^2 + 2x}{(x+y+1)^2}$$

set $Lx = 0$

$$x^2 + 2xy - y^2 + 2x = 0 \quad \text{— (1)}$$

$$Ly = \frac{2y(x+y+1) - (x^2+y^2)(1)}{(x+y+1)^2}$$

$$2y(x+y+1) - (x^2+y^2) =$$

$$2xy + y^2 + 2y - x^2$$

$$Ly = 2xy + y^2 + 2y - x^2/(x+y+1)^2$$

Date: Day: 2

Set $l_y = 0$

$$2xy + y^2 + 2y - x^2 = 0 \quad -(2)$$

Solve critical point.

$$\begin{cases} x^2 + 2xy + 2x - y^2 = 0 & -1 \\ 2xy + y^2 + 2y - x^2 = 0 & -2 \end{cases}$$

Eq (1) $\quad x^2 + 2xy + 2x - y^2 = 0$

$$y^2 = x^2 + 2xy + 2x \quad -(1')$$

Eq (2)

$$2xy + y^2 + 2y - x^2 = 0$$

$$x^2 = 2xy + y^2 + 2y \quad -(2')$$

Substitute (2') into (1')

from (2')

$$x^2 = 2xy + y^2 + 2y$$

plug into (1')

$$2xy + y^2 + 2y + 2xy + 2x = y^2$$

$$4xy + 2y + 2x = 0$$

$$\cancel{2(2xy} \; 2(2xy + y + x) = 0$$

$$2xy + y + x = 0$$

$$y(2x+1) + x = 0$$

$\cancel{y=0}$

$$y(2x+1+x)=0$$

$\cancel{3x+1=0}$

$$y(2x+1)+x=0$$

Since $x, y>0$, this can only hold if

$$x=0 \text{ or } y=0$$

But $x \geq 1, y \geq 1$, so no interior critical point.

c) Check boundaries.

Boundary 1: $x=1$

$$L(1,y)=\frac{1+y^2}{1+y+1}=\frac{1+y^2}{y+2}$$

Derivative w.r.t y

$$\frac{(2y)(y+2)-(1+y^2)(1)}{(y+2)^2}$$

$$=\frac{2y^2+4y-1-y^2}{(y+2)^2}=\frac{y^2+4y-1}{(y+2)^2}$$

for critical point.

$y^2 + 4y - 1 = 0$

$a = 1, \ b = 4, \ c = -1$

$y = \frac{-4 \pm \sqrt{16+4}}{2} = \frac{-4 \pm \sqrt{20}}{2}$

$y = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}$

only positive root.

$y = -2 + \sqrt{5} \approx -2 + 2.236$

$= -2 + 2.236 \approx 0.236 < 1$

Not in domain → Skip

Boundary 2: $y = 1$

$L(x, 1) = \frac{x^2+1}{x+1+1} = \frac{x^2+1}{x+2}$

Derivative w.r.t x

$\frac{(2x)(x+2) - (x^2+1)(1)}{(x+2)^2} = \frac{2x^2+4x-x^2-1}{(x+2)^2}$

$= \frac{x^2+4x-1}{(x+2)^2}$

for critical point.

$x^2+4x-1=0$

$a=1, b=4, c=-1$

$\cancel{y=}\quad x = \frac{-4 \pm \sqrt{16+4}}{2} = \frac{-4 \pm \sqrt{20}}{2}$

$x = -2 \pm \sqrt{5}$

only positive root

$x = \cancel{+2+} \; -2+\sqrt{5} =$

$x = -2+2.236 \approx 0.236 < 1$

Not in domain → skip

Boundary 3: $x=20$

$L(20,y) = \frac{400+y^2}{20+y+1} = \frac{400+y^2}{21+y}$

Derivative w.r.t y

$2y(y+21) - (400+y^2)(1) / (21+y)^2$

$\frac{2y^2+42y-400-y^2}{(y+21)^2} = \frac{y^2+42y-400}{(y+21)^2}$

for critical point.

$y^2+42y-400=0$

$a=1, b=42, c=-400$

$y = \frac{-42 \pm \sqrt{1764+1600}}{2} = \frac{-42 \pm \sqrt{3364}}{2}$

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$y = \frac{-42 \pm 58}{2}$, only positive root

$y = \frac{-42+58}{2} = \frac{16}{2} = 8$

$P(20, 8)$

Boundary 4: $y = 20$

$P(x, 20) = \frac{x^2+400}{x+20+1} = \frac{x^2+400}{x+21}$

Derivative w.r.t x

$$\frac{2x(x+21) - (x^2+400)(1)}{(x+21)^2}$$

$$\frac{2x^2+42x-x^2-400}{(x+21)^2} = \frac{x^2+42x-400}{(x+21)^2}$$

for critical point

$$x^2 + 42x - 400 = 0$$

$a=1$, $b=42$, $c=-400$

$$x = \frac{-42 \pm \sqrt{42^2+1600}}{2} = \frac{-42 \pm 58}{2}$$

Positive root $x = \frac{-42+58}{2} = \frac{16}{2} = 8$

$x = 8$

$P(8, 20)$

c) Find optimal point.

We now compare.

$L(20,8)$
$L(8,20)$
corners: $(1,1), (1,20), (20,1), (20,20)$

Compute value

$$L(20,8) = \frac{400+64}{20+8+1} = \frac{464}{29} \approx 16$$

$$L(8,20) = \frac{64+400}{8+20+1} = \frac{464}{29} \approx 16$$

corner $(1,1)$

$$L(1,1) = \frac{1+1}{1+1+1} = \frac{2}{3} \approx 0.67$$

Corner $(1,20)$

$$L(1,20) = \frac{1+400}{1+20+1} = \frac{401}{22} \approx 18.2$$

Corner $(20,1)$

$$L(20,1) = \frac{400+1}{20+1+1} = \frac{401}{22} \approx 18.2$$

Corner $(20,20)$

$$L(20,20) = \frac{40+40}{20+20+1} = \frac{800}{41} \approx 19.5$$

optimal load distribution

$$x^* = 1, \; y^* = 1$$

Minimum latency: 0.67

Questions: 2

a) Production Function:

$$f(x,y) = 100x^{3/4}y^{1/4}$$

Cost constraint:

x = unit of labor (cost \$150/unit)
y = Unit of capital (cost \$250/unit)

$$150x + 250y = 50000$$

$$g(x,y) = 150x + 250y - 50000 = 0$$

$$\nabla f = \cancel{100}^{25} \times \frac{3}{\cancel{4}} x^{-1/4} y^{1/4} i + 100 \times \frac{1}{4} x^{3/4} y^{-3/4} j$$

$$\nabla f = 75x^{-1/4}y^{1/4} i + 25x^{3/4}y^{-3/4} j$$

$$\nabla g = 150i + 250j$$

$$\nabla f = \lambda \nabla g$$

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$$75x^{-1/4}y^{1/4}i + 25x^{3/4}y^{-3/4}j = 150\lambda i + 250\lambda j$$

$$75x^{-1/4}y^{1/4} = 150\lambda \quad | \quad 25x^{3/4}y^{-3/4} = 250\lambda$$

$$\lambda = \frac{75x^{-1/4}y^{1/4}}{150} \quad | \quad \lambda = \frac{25x^{3/4}y^{-3/4}}{250}$$

$$\lambda = \frac{1}{2}x^{-1/4}y^{1/4} \quad | \quad \lambda = \frac{1}{10}x^{3/4}y^{-3/4}$$

$$\frac{1}{2}x^{-1/4}y^{1/4} = \frac{1}{10}x^{3/4}y^{-3/4}$$

Multiply both side 10

$$5x^{-1/4}y^{1/4} = x^{3/4}y^{-3/4}$$

$$5 = \frac{x^{3/4}y^{-3/4}}{x^{-1/4}y^{1/4}} = x^{3/4+\frac{1}{4}}y^{-3/4-\frac{1}{4}}$$

$$5 = x^{1}y^{-1} \quad . \quad 5 = \frac{x}{y}$$

$$\boxed{x = 5y}$$

Apply the cost constraint.

$$150x + 250y = 50000$$

substitute $x = 5y$

$$150(5y) + 250y = 50000$$
$$750y + 250y = 50000$$
$$1000y = 50000$$
$$\boxed{y = 50}$$

$$x = 5y = 5 \times 50 = \cancel{1000}$$

$$\boxed{x = 250}$$

Find Maximum Producation

$$f(x,y) = 100x^{3/4}y^{1/4}$$
$$f(250,50) =$$

Compute numeric Value

$$250^{1/4} = \sqrt{\sqrt{250}} \approx \sqrt{15.81} \approx 3.98$$

$$250^{3/4} = (250^{1/4})^3 \approx (3.98)^3 \approx 62.9$$

$$50^{1/4} = \sqrt{\sqrt{50}} \approx \sqrt{7.07} \approx 2.66$$

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Final Maximum Producation

$f(250, 50) = 100 \times 62.9 \times 2.66 \approx 100 \times 167.3$

$\boxed{f(250, 50) = 16,730 \text{ units}}$

Maximum Producation: $f(250, 50) = 16,730$

Question: 3

$P(x,y) = 8x + 10y - 0.001(x^2 + xy + y^2) - 10000$

2) Calculate first and second order partial derivatives.

first order:

$P_x = 8 - 0.001(2x + y)$

$= 8 - 0.001(2x + y)$

$\boxed{P_x = 8 - 0.002x - 0.001y}$

$P_y = 10 - 0.001(x + 2y)$

$\boxed{P_y = 10 - 0.001x - 0.002y}$

Second order

$\boxed{P_{xx} = -0.002}$

$\boxed{P_{yy} = -0.002}$

$\boxed{P_{xy} = -0.001}$

3) Find the production level for Maximum ~~revel~~ profit.

$P_x = 8 - 0.002x - 0.001y$

$P_y = 10 - 0.001x - 0.002y$

For critical point

$P_x = 0 \Rightarrow 8 - 0.002x - 0.001y = 0$ — (1)

$P_y = 0 \Rightarrow 10 - 0.001x - 0.002y = 0$ — (2)

$8 = 0.002x + 0.001y$ — (1)

$10 = 0.001x + 0.002y$ — (2)

Multiply eq (1) by 2

$16 = 0.004x + 0.002y$

subtract eq (2)

$(16-10) = (0.004x + 0.002y) - (0.001x + 0.002y)$

$6 = 0.003x$

$x = 6/0.003 = 2000$

$\boxed{x = 2000}$

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find y

plug $x = 2000$ into eq (7)

$$8 = 0.002(2000) + 0.001y$$
$$8 = 4 + 0.001y$$
$$4 = 0.001y \Rightarrow y = \frac{4}{0.001}$$

$$\boxed{y = 4000}$$

Calculate Maximum Profit.

$P(2000, 4000) =$

$x^2 = 4,000,000$

$xy = 8,000,000$

$y^2 = 16,000,000$

$$P(2000,4000) = 8(2000) + 10(4000) - 0.001(4,000,000 + 8,000,000 + 16,000,000) - 10000$$
$$= 16000 + 40000 - 0.001(28,000,000) - 10000$$
$$= 56000 - 28,000 - 10000$$
$$= 56000 - 38000$$
$$= 18,000 \text{ dollars}$$

$$\boxed{P(2000,4000) = 18,000 \text{ dollars}}$$

↳ Maximum Profit.