

Problem Set 3

Let's Simplex

Applied Mathematics 121 — Spring 2011

Due 5:00 PM, Friday, February 18, 2011

Announcements

- The assignment is due by 5:00 PM, Friday, February 18, 2011.
- Readings: Jensen and Bard, Sections 3.1-3.8.

Goals

This assignment has two goals. First, you will get practice solving linear programs using the Simplex method. Second, you will get a better understanding of linear programming through a better understanding of the Simplex method.

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But first, a distraction

In the last assignment, you got a bit of practice with modeling different objectives and constraints. But how do you know whether something is modelable via linear programming?

Exercise Task 1

Consider the following constraint:

$$\max(x_1, x_2) \geq 3 \quad (1)$$

In general, can we model this constraint with linear programming, without further knowledge of other constraints? (Hint: think about the geometry of the feasible region.)

End Task 1

1 Making it rain

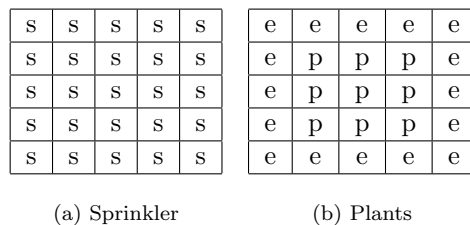


Figure 1: Plant and sprinkler system diagrams. s indicates a sprinkler, p indicates a plant, and e indicates a piece of land without any plants.

0.1	0.1	0.1
0.1	0.1	0.1
0.1	0.1	0.1

Figure 2: Spray delivered by an individual sprinkler

Taste Matters is growing impatient with all of your pen-and-paper theorizing—they want to see your work in action. They’ve thought about each of the formulations that you came up with in the previous assignment, and have decided to keep it simple. They want you to minimize the amount of solution used for spraying the plants, subject to the same arrangement of sprinklers and plants as before (Figure 1). They also remind you that each sprinkler delivers a uniform spray covering a 3 by 3 region centered over the plant (Figure 2), and that each plant has a different minimum amount of solution required:

27	15	40
30	20	20
17	40	32

Table 1: Minimum amounts of solution required

Exercise Task 2

For the following exercises, use the basic sprinkler model that you came up with in Task 7.1 of assignment 2. If you don't remember it off-hand, feel free to use the formulation used in the solutions to assignment 2.

1. Implement your model in AMPL using the data provided by Taste Matters. Run the model to find the sprinkling policy that minimizes the amount of solution used to spray the plants.
2. Taste Matters is happy with the result that you've provided, but their engineering department has made a mistake: the actual output from each sprinkler is spread over the 3 by 3 region in the following way:

0.07	0.1	0.05
0.1	0.3	0.1
0.04	0.1	0.06

Taste Matters doesn't think this mistake would nullify the policy you have found, but wants you to resolve the model at these rates to ensure that this is the case. Is it?

Hint: You should only have to change your data file.

End Task 2

2 Simplex by hand

Consider the following linear program:

$$\begin{array}{ll}\text{maximize} & x_1 + x_2 + 2x_3 \\ \text{subject to} & x_1 \leq 4 \\ & x_2 \leq 4 \\ & x_1 + x_2 \leq 6 \\ & -x_2 + 2x_3 \leq 2 \\ & x_1, x_2, x_3 \geq 0\end{array}$$

Exercise Task 3

1. Graph the 3-dimensional feasible region by hand. (Hint: you will find in the process of sketching that certain orientations of the region are easier to draw and visualize than others. Feel free to label and extend axes in directions that make graphing easier!)
2. Transform the linear program to standard equality form by introducing slack variables.
3. Solve the problem using the Simplex method, using the introduced slack variables as the initial feasible basis. In selecting an entering variable, choose the one with the smallest reduced cost. Record each tableau, feasible solution, and objective value you iterate through. As you solve the problem, trace the path the Simplex method takes on your diagram.

Note: To make your life easier, try doing pivots using mathematical software such as Maple.

4. Is your solution an unique optimal solution? Why or why not?

End Task 3

Exercise Task 4

1. Consider changing the objective function for the linear program in the previous task from $x_1 + x_2 + 2x_3$ to $2x_1 + x_2 + 2x_3$. Again solve the problem using the Simplex method, this time choosing the entering index based on the smallest subscript rule. Record each tableau, feasible solution, and objective value you iterate through. As you solve the problem, trace the path the Simplex method takes on your diagram (make sure to distinguish this trace from your trace from the previous task by either making a copy of your diagram or using some other device, say a different colored pen).
2. Is your solution an unique optimal solution? Why or why not?

End Task 4

3 Simplex, the works

Considering the following linear program:

$$\begin{array}{ll}\text{maximize} & -3x_1 + 3x_2 - x_3 \\ \text{subject to} & 2x_1 + 2x_2 - 3x_3 \geq 3 \\ & 7x_1 + 2x_2 - 2x_3 \leq 14 \\ & x_1, x_2 \geq 0; x_3 \text{ free}\end{array}$$

Exercise Task 5

For this exercise, feel free to make use of Matlab or Maple to perform the pivot operations as you have done on the previous assignment. This will help to avoid manual error!

1. Transform the linear program to standard equality form. This will require a number of steps.
2. Perform Phase I of the Simplex method to arrive at a feasible tableau. In selecting an entering variable, choose the one with the smallest subscript among the x_i 's before any other introduced variables. Be sure to record all your steps.
3. Using the found tableau, solve the problem using Phase II of the Simplex method. In selecting an entering variable, choose the one with the smallest reduced cost. Record each tableau, feasible solution, and objective value you iterate through. You can error check by solving the problem in AMPL.

End Task 5

4 Learning from the tableau (by helping a friend...)

A friend of yours decides to play around with the following system of equations:

$$\begin{array}{cccccccccccc} z & & & & & & & x_5 & x_6 & x_7 & = & 0 \\ - & 2x_1 & - & 2x_2 & + & 4x_3 & + & 3x_4 & + & x_5 & & = & 4 \\ & 3x_1 & - & 2x_2 & + & x_3 & - & 4x_4 & & + & x_6 & & = & 3 \\ & 6x_1 & + & x_2 & + & 3x_3 & + & 5x_4 & & & + & x_7 & = & 5 \end{array}$$

He then performs a few pivots using Maple:

```
A := <<1,0,0,0>|<0,-2,3,6>|<0,-2,-2,1>|<0, 4, 1, 3>|  
      <0,3,-4,5>|<1,1,0,0>|<1,0,1,0>|<1,0,0,1>|<0,4,3,5>>;
```

```

A := pivot(A, 2, 6);
A := pivot(A, 3, 7);
A := pivot(A, 4, 8);
A := pivot(A, 4, 2);
A := mulrow(A, 4, 1/A[4,2]);
A := pivot(A, 2, 4);
A := mulrow(A, 2, 1/A[2,4]);

```

and arrives at the following tableau:

$$\left[\begin{array}{c|cccccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & b \\ \hline 1 & 0 & 8/3 & 0 & \frac{181}{30} & \frac{9}{10} & 0 & \frac{22}{15} & -\frac{16}{15} \\ 0 & 0 & -1/3 & 1 & \frac{14}{15} & 1/5 & 0 & 1/15 & \frac{17}{15} \\ 0 & 0 & -8/3 & 0 & -\frac{181}{30} & 1/10 & 1 & -\frac{7}{15} & \frac{16}{15} \\ 0 & 1 & 1/3 & 0 & \frac{11}{30} & -1/10 & 0 & 2/15 & \frac{4}{15} \end{array} \right]$$

But now your friend is clueless. Help.

Exercise Task 6

1. What are the basic variables? What is the corresponding basic solution?
2. Is the solution optimal for maximizing the value of z ?
3. What does the solution imply about the solution to the following system of equations?

$$\begin{array}{rrrrrr} - & 2x_1 & - & 2x_2 & + & 4x_3 & + & 3x_4 & = & 4 \\ & 3x_1 & - & 2x_2 & + & x_3 & - & 4x_4 & = & 3 \\ & 6x_1 & + & x_2 & + & 3x_3 & + & 5x_4 & = & 5 \end{array}$$

End Task 6

Your friend thinks you are awesome, and decides to play around with another linear program, knowing you will be there for backup as needed:

$$\begin{array}{ll} \text{maximize} & 2x_3 \\ \text{subject to} & x_2 + x_3 \leq 8 \\ & -x_2 + x_3 \leq 0 \\ & x_1 + x_3 \leq 8 \\ & -x_1 + x_3 \leq 0 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Your friends adds slack variables x_4, x_5, x_6 , and x_7 as the basis variables for an initial feasible tableau:

$$\left[\begin{array}{c|cccccccc|c} & z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & b \\ \hline z & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ x_4 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 8 \\ x_5 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ x_6 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 8 \\ x_7 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

The basis solution is $\{0, 0, 0, 8, 0, 8, 0\}$, $z = 0$. Your friend performs a pivot by letting x_3 enter, leaving on x_5 :

$$\begin{bmatrix} & z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & b \\ z & 1 & 0 & -2 & 0 & 0 & 2 & 0 & 0 & 0 \\ x_4 & 0 & 0 & 2 & 0 & 1 & -1 & 0 & 0 & 8 \\ x_3 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ x_6 & 0 & 1 & 1 & 0 & 0 & -1 & 1 & 0 & 8 \\ x_7 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}$$

The basis solution is $\{0, 0, 0, 8, 0, 8, 0\}$, $z = 0$. “But this is exactly the same solution as I started with!” your friend says. “Help!”

You help by continuing the Simplex method, following the smallest subscript rule, yielding the following tableaux:

$$\begin{bmatrix} & z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & b \\ z & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ x_4 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & -2 & 8 \\ x_3 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ x_6 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & -1 & 8 \\ x_2 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} & z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & b \\ z & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 8 \\ x_1 & 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & -1 & 4 \\ x_3 & 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 4 \\ x_6 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 \\ x_2 & 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 4 \end{bmatrix}$$

Exercise Task 7

1. Indicate the basis solution and objective values of the final two tableaux above. In addition, explain how the entering and leaving variables were chosen.
2. Sketch the feasible region in 3D. Trace the path that the Simplex algorithm took. Explain in a sentence why the same basis solution would show up more than once in this problem (Hint: think about the Simplex algorithm geometrically).

End Task 7

Having been so helpful to your friend, you decide to go the extra mile and come up with an example tableau to really help your friend understand the Simplex method. You have declared all variables as non-negative, and have left some elements of the tableau as lettered parameters:

	x_1	x_2	x_3	x_4	x_5	x_6	
z	3	0	a	0	0	b	8
x_2	c	1	d	0	0	-3	4
x_4	-3	0	-2	1	0	e	1
x_5	4	0	2	0	1	-1	f

You wish to explain to your friend the conditions under which the different types of scenarios occur in solving maximization linear programs. To avoid tarnishing your pristine reputation (to your friend, anyways), you review the conditions first yourself.

Exercise Task 8

For each of the following tasks, specify the conditions on the parameters that make the statement true. Aim to be as general as possible (that is, providing an example scenario is insufficient).

1. The current basic solution is optimal.
2. The current basic solution is infeasible.
3. The current basic solution is a degenerate basic feasible solution.
4. The current basic solution is feasible, but the objective value is unbounded.
5. The current basic solution is feasible, but can be improved by x_3 entering the basis, and x_5 leaving the basis.

End Task 8

5 Termination

Without the issue of degeneracy, we can show that the Simplex algorithm terminates by using the guaranteed increase in the objective value at each iteration to show that no basis (from a finite set of bases) will be repeated. With degeneracy, we are not guaranteed that the objective value will increase at each iteration. Nevertheless, we can prove the following:

Theorem 5.1. *Given an initial feasible tableau, the Simplex method using the smallest subscript rule will terminate in a finite number of iterations.*

We will prove this theorem in the following exercise.

Exercise Task 9

1. Place a bound on the maximum number of possible basis solutions one may encounter during the Simplex algorithm.
2. Given that two tableaux corresponding to the same basis can differ only in the order in which the equations are written, place a bound on the maximum number of possible tableaux one may encounter during the Simplex algorithm.
3. Deduce the theorem using the bound and Bland's Theorem (which states that the Simplex method using the smallest subscript rule for entering and leaving variables cannot cycle).

End Task 9

6 Turning in your assignment

Final Task 10

You must turn in a paper copy of your assignment by 5:00 PM, Friday, February 18, 2011. For AMPL exercises, this includes any model and data files you have created. When writing down the solution from AMPL (on paper), always include both the objective value and the values assigned to variables (when the program is feasible and bounded, of course!)

Gather the AMPL model and data files you have created for this assignment, as well as a script file containing the AMPL commands you used to solve the problems. Put all these files in a zip archive, and email the archive to apm121@fas.harvard.edu *AND* your teaching fellow before 5:00 PM, Friday, February 18, 2011. For the subject line, type: `submit <asst #> username`. If your user name is foobar, you would type `submit 3 foobar` for this particular assignment. Please note that submitting this zip archive is necessary **in addition** to your written submission.

End Task 10

Congratulations on completing your third AM121 assignment!