Unit-4

Short Answer Questions

1. Define generating function.(or) What do you mean by generating function with an example?

Ans: **Generating function** is a formal power series whose coefficients represent elements of a sequence. It is used to study sequences in combinatorics, number theory, and recurrence relations.

The generating function for a sequence $\{a_n\}$ is: $G(x)=\sum_{n=0}^{\infty} a_n x^n$

Example:

For the sequence {1,1,1,1,… }, the generating function is:

$G(x)= \sum_{n=0}^{\infty} x^n=\frac{1}{1-x}$, for $|x|<1$.

2. Find the generating function for the finite sequence 1, 4, 16, 64, 256.

Ans: To find the generating function for a finite sequence $a_{0,1,\ldots,}a_k$ the formula is:

$G(x)=a_0+a_1x+a_2x^2+\cdots+a_k x^{k.}$ For the sequence $a_k=C(n,k)$ for $0\leqslant k\leqslant n$, the generating function will be: $G(x)=C(n,0)+C(n,1)x+C(n,2)x^2+\cdots+C(n,n)x^n$

By the Binomial Theorem, this generating function can be expressed as: $(1+x)^n$

The given sequence is 1, 4, 16, 64, 256, so the corresponding generating function can be stated as: $G(x)=1+4x+16x^2+64x^3+256x^4$.

This can be simplified as: $1+4x+(4x)^2+(4x)^3+(4x)^4= \frac{(4x)^5-1}{4x-1}$.

Therefore, the generating function for the given sequence is $\frac{(4x)^5-1}{4x-1}$.

3. Evaluate $\sum_{i=0}^{2} \quad \sum_{j=0}^{3} \quad (3i + 2j)$

Ans: (0 + 2 + 4 + 6) + (3 + 5 + 7 + 9) + (6 + 8 + 10 + 12) = 72

4. Find a sequence for the generating function $1/(1-4x)^n$.

Ans: Expanding using the **binomial theorem**:

$(1-4x)^{-n}=\sum_{k=0}^{\infty}\binom{n+k-1}{k}4x^k= \sum_{k=0}^{\infty}\binom{n+k-1}{k}4^k x^k$

Thus, the sequence corresponding to the generating function is:

$a_k=\binom{n+k-1}{k}4^k$

5. What is recurrence relation?

Ans: A recurrence relation (or difference equations) for the sequence {an} is an equation that expresses sequence a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

6. Write general form of linear homogenous recurrence relation

Ans: A linear homogeneous recurrence relation of order k has the form:

$a_n+c_1a_{n-1}+c_2a_{n-2}+\cdots+c_ka_{n-k}=0$ where $c_1,c_2,\ldots,c_k$ are constants.

Example**:** $a_n-5a_{n-1}+6a_{n-2}=0$.

7. Solve the recurrence relation $a_n = a_{n-1} + 2\, a_{n-2}$

Ans: Characteristic Equation: $r^2-r-2=0$

Factorizing:$(r-2)(r+1)=0$

Roots: $r=2,-1$.

General Solution:$a_n=A(2)^n+B(-1)^n$

8. What is inhomogeneous recurrence relation?

Ans: Suppose we wanted an explicit formula for a sequence a_n satisfying a_0=0, and a_{n+1}-$2a_n$=F_n for n>=0, where F_n is the Fibbonacci sequence as usual.

This is not a linear recurrence in the sense we have been talking about (because of the F_n on the right hand side instead of 0). A recurrence of this type, linear except for a function of *n* on the right hand side, is called an *inhomogeneous recurrence*.

Long Answer Questions

1. Solve Fibonacci's Recurrence relation with conditions F0 =1 and F1 =1

Ans: The sequence of Fibonacci numbers satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$. The roots of the characteristic equation $r^2 - r - 1 = 0$ are $r_1 = (1 +\sqrt{5})/2$ and $r_2 = (1 -\sqrt{5})/2$

Therefore by Theorem 1

$f_n = \alpha_1 ((1 +\sqrt{5})/2)^n + \alpha_2 ((1 -\sqrt{5})/2)^n$ for some constants α_1 and α_2.

Here the initial conditions are given as $f_0 = 1$ and f_1=1 , we have $f_0 = \alpha_1 + \alpha_2 = 1$, $f_1 = \alpha_1 ((1 +\sqrt{5})/2) + \alpha_2 ((1 -\sqrt{5})/2)$=1. Solving, we obtain α_1=$1/\sqrt{5}$, $\alpha_2 = -1/\sqrt{5}$.Hence, the Fibonacci numbers are given by

$f_n = 1/\sqrt{5}(1+\sqrt{5}/2)^n - 1/\sqrt{5}(1 -\sqrt{5}/2)^n$.

2. Explain Towers of Hanoi Problem and find its recurrence relation

Ans: In the late nineteenth century, the French mathematician Édouard Lucas invented a puzzle consisting of three pegs on a board with disks of different sizes. Initially all of the disks are on the first peg in order of size, with the largest on the bottom.

Rules: You are allowed to move the disks one at a time from one peg to another as long as a larger disk is never placed on a smaller.

Goal: Using allowable moves, end up with all the disks on the second peg in order of size with largest on the bottom.

Let $\{H_n\}$ denote the number of moves needed to solve the Tower of Hanoi Puzzle with *n* disks. Begin with *n* disks on peg 1. We can transfer the top $n - 1$ disks, following the rules of the puzzle, to peg 3 using H_{n-1} moves. First, we use 1 move to transfer the largest disk to the second peg. Then we transfer the $n - 1$ disks from peg 3 to peg 2 using H_{n-1} additional moves. This cannot be done in fewer steps.

Hence, $H_n = 2H_{n-1} + 1$. The initial condition is H_1= 1 since a single disk can be transferred from peg 1 to peg 2 in one move.

We can use an iterative approach to solve this recurrence relation by repeatedly expressing H_n in terms of the previous terms of the sequence.

$H_n = 2H_{n-1} + 1$
$= 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1$
$= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1$
$\vdots$
$= 2^{n-1}H_1 + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1$
$= 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1$ because $H_1 = 1$
$= 2^n - 1$ using the formula for the sum of the terms of a geometric series

3. Find the co-efficient of x^{50} in $(x^7 + x^8 + x^9 + \ldots..)^6$.

Ans: It is clear that we can view this problem as asking for the coefficient of x^8 in $x^{42} (1 + x + x^2 + x^3 + \cdots)^6$, since each x^5 in the original is playing the role of *x* here. Since $(1 + x + x^2 + x^3 + \cdots)^3 = 1/(1- x)^6 = C(n + r-1, r-1)x^n$, the answer is clearly C(8 + 6-1, 6-1) = C(13, 8) = 1287.

4. Solve the recurrence relation a_n-5a_{n-1}+6a_{n-2}=0 where a_0=2 and a_1=5.

Ans: To solve the recurrence relation an = 5a_{n-1} - 6a_{n-2} with initial conditions a_0 = 2, a_1 = 5, , we can use the characteristic equation method.

The characteristic equation for this recurrence relation is:

r^2 - 5r +6 = 0

This equation can be factored as:

(r -2)(r-3) = 0

Therefore, the roots of the characteristic equation are r_1 = 2 and r_2=3.

Since two distinct root are there, the general solution to the recurrence relation is:

$a_n = (\alpha_1.2^n + \alpha_2.3^n)$

where A and B, are constants that can be determined from the initial conditions.

Substituting the initial conditions a_0 = 2, a_1 = 5, into the general solution, we get:

Using initial conditions:

1. $a_0=2\Rightarrow \alpha_1 (2^0)+ \alpha_2 (3^0)=2\Rightarrow \alpha_1+ \alpha_2=2$
2. $a_1=5\Rightarrow \alpha_1 (2^1)+ \alpha_2 (3^1)=5\Rightarrow 2 \alpha_1+3 \alpha_2=5$

Solving:

$\alpha_1+ \alpha_2=2$

$2 \alpha_1+3 \alpha_2=5$

Multiplying the first equation by 2 and subtracting:

$(2 \alpha_1+2 \alpha_2)-(2 \alpha_1+3 \alpha_2)=4-5$

$- \alpha_2=-1\Rightarrow \alpha_2=1$

$\alpha_1=1$

Thus, the solution is:

$a_n=1\cdot 2^n+1\cdot 3^n$

5. Solve the recurrence relation a_n-7a_{n-1}+10a_{n-2}=0 for $n\geq 2$ with initial conditions a_0=10, a_1=41

Ans: The characteristic equation of the recurrence relation is $r^2-7r+10=0$.

Its roots are r_1= 2 and r_2= 5. Hence the sequence {a_n} is a solution to the recurrence relation if and only if $a_n=\alpha_1 r_1^n+\alpha_2 r_2^n$ =) $a_n=\alpha_1 2^n+\alpha_2 5^n$ for some constant α_1 and α_2.

From the initial condition, it follows that $a_0=10=\alpha_1+\alpha_2$ and $a_1=41=2\alpha_1+5\alpha_2$.

Solving the equations, we get α_1=3, α_2=7.

Hence the solution is the sequence {a_n} with an = $(3).2^n + (7).5^n$

6. Find the solution to the recurrence relation a_n=-3a_{n-1} - 3a_{n-2} - a_{n-3} with initial conditions a_0 = 1, a_1 = -2 and a_2 = -1.

Ans: This is a third degree recurrence relation. The characteristic equation is $r^3 + 3r^2 + 3r + 1 = 0$. Because $r^3 + 3r^2 + 3r + 1 = (r + 1)^3$, there is a single root r = −1 of multiplicity three of the characteristic equation. By Theorem 4 the solutions of this recurrence relation are of the form

$a_n = \alpha_{1,0}(-1)^n + \alpha_{1,1}n(-1)^n + \alpha_{1,2}n^2(-1)^n$.

To find the constants $\alpha_{1,0}$, $\alpha_{1,1}$, and $\alpha_{1,2}$, use the initial conditions. This gives

$a_0 = 1 = \alpha_{1,0}$,

$a_1 = -2 = -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2}$,

$a_2 = -1 = \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2}$.

The simultaneous solution of these three equations is $\alpha_{1,0}$ = 1, $\alpha_{1,1}$ = 3, and $\alpha_{1,2}$ = −2.

Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence {a_n} with

$a_n = (1 + 3n - 2n^2)(-1)^n$.

7. Solve the non-homogeneous recurrence relation using characteristic roots method a_n-5a_{n-1} + 6a_{n-2} =2^n for n≥2, a_0=1 and a_1=3.

Ans**:** Solve the Homogeneous Equation: The associated homogeneous recurrence relation is:
r2−5r+6=0.
Factoring:
(r−2)(r−3)=0.
Thus, the characteristic roots are r_1=2 and r_2=3.
So, the general solution for the homogeneous part is:
$a_n(h)= \alpha_1 2^n+ \alpha_2 3^n$.
Next, Find a Particular Solution
Since the right-hand side is 2^n, we assume a solution of the form:
$a_n(p)=Cn2^n$.
(Substituting just $C2^n$ would not work since 2^n is already a solution to the homogeneous equation.)
Plugging into the recurrence:
$(Cn2^n)-5(C(n-1)2^{n-1})+6(C(n-2)2^{n-2})=2^n$.
Dividing everything by 2^n:
$Cn-5C(n-1)/2+6C(n-2)/4=1$.
Expanding for small values:
$Cn-5Cn/2+5C/2+6Cn/4-12C/4=1$.
Solving for C, we get C=2.
Thus, the particular solution is:
$a_n(p)=2n2^n$.
General Solution
$a_n= \alpha_1 2^n+ \alpha_2 3^n+2n2^n$.
Find Constants α_1 and α_2
Using the initial conditions:

1. $a_0=1 \Rightarrow \alpha_1 (2^0)+ \alpha_2 (3^0)+2(0)2^0=1 \Rightarrow \alpha_1+ \alpha_2=1$.
2. $a_1=3 \Rightarrow \alpha_1 (2^1)+ \alpha_2(3^1)+2(1)2^1=3 \Rightarrow 2\alpha_1+3\alpha_2+4=3$.

Solving:
$\alpha_1+ \alpha_2=1$,
$2\alpha_1+3\alpha_2+4=3 \Rightarrow 2\alpha_1+3\alpha_2=-1$.
Multiply the first equation by 2:
$2\alpha_1+2\alpha_2=2$.
Subtract:
$(2\alpha_1+3\alpha_2)-(2\alpha_1+2\alpha_2)=-1-2$.
$\alpha_2=-3$. $\alpha_1=4$.
Thus $a_n=4(2^n)-3(3^n)+2n2^n$.

8. Find all the solutions of the recurrence relation a_n= 5a_{n-1} - 6a_{n-2} + 7^n.

Ans: This is a linear nonhomogeneous recurrence relation.
The solutions of its associated homogeneous recurrence relation a_n = 5a_{n-1} - 6a_{n-2} are
$a_n^{(h)}= \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n$, where α_1 and α_2 are constants.
Because $F(n) = 7^n$, a reasonable trial solution is $a_n^{(p)} = C \cdot 7^n$, where C is a constant.
Substituting the terms of this sequence into the recurrence relation implies that
$C \cdot 7^n = 5C \cdot 7^{n-1} - 6C \cdot 7^{n-2} + 7^n$.
Factoring out 7^{n-2}, this equation becomes $49C = 35C - 6C + 49$, which implies that $20C$ = 49, or that $C = 49/20$.
Hence, $a_n^{(p)} = (49/20)7^n$ is a particular solution.
By Theorem 5, all solutions are of the form
$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + (49/20)7^n$.

9. Solve the recurrence relation: $a_{r+2} - 3a_{r+1} + 2a_r = 0$ by the method of generating functions with the initial conditions$a_0 = 2$, $a_1 = 3$.

Ans: Let us assume that

$G(x)=\sum_{r=0}^{\infty} a_r\, x^r$ …..eq(1)

Multiply equation (i) by x^r and summing from r = 0 to ∞, we have

$$\sum_{r=0}^{\infty}(a_{r+2} - 3a_{r+1} + 2a_r)x^r = 0$$

We can rewrite the summations as

$$\sum_{r=0}^{\infty}(a_{r+2})x^r - 3\sum_{r=0}^{\infty} a_{r+1}\, x^r + 2\sum_{r=0}^{\infty} a_r\, x^r = 0$$

Shifting indices:

- In the first sum, let k=r+2, so $a_{r+2}x^r$ becomes $\sum_{k=0}^{\infty} a_k x^{k-2}$.
- In the second sum, let k=r+1, so a_{r+1} becomes$\sum_{k=1}^{\infty} a_k x^{k-1}$.
- The third sum remains unchanged.

We can rewrite it as

$$\sum_{k=2}^{\infty}(a_k)x^{k-2} - 3\sum_{k=1}^{\infty} a_k\, x^{k-1} + 2\sum_{k=0}^{\infty} a_k\, x^k = 0$$

Express it in terms of G(x)

$(a_2+a_3\, x+a_4x^2+\cdots)-3(a_1+a_2\, x+a_3\, x^2+\cdots)+2(a_0+a_1\, x+a_2\, x^2+\cdots)=0$ [Since $G(t)=a_0+a_1\, x+a_2\, x^2+\cdots$]

Therefore

$\frac{G(x)-a0-a1x}{x^2}$ $-3(\frac{G(x)-a0}{X})+2G(X)=0$…….eq(2)

Now, put $a_0=2$ and $a_1=3$ in equation (ii) and solving, we get

$\frac{G(x)-2-3x}{x^2}$ $-3\frac{G(x)-2}{x}$ $+2G(x)=0$.

Multiplying by x^2 to clear denominators

$G(x)-2-3x-3x(G(x)-2)+2x^2G(x)=0$.

Expanding:

$G(x)-2-3x-3xG(x)+6x+2x^2G(x)=0$.

Rearranging:

$G(x)-3xG(x)+2x^2G(x)=2+3x-6x$.

$G(x)(1-3x+2x^2)=2-3x$.

$G(x)\ (1 - 3x + 2x^2) = 2 - 3x$.

$G(x)(1-3x+2x2)=2-3x$.

$G(x) = \frac{2-3x}{1-3x+2x^2}$.

Factor the denominator:

$1-3x+2x^2=(1-x)(1-2x)$.

Thus,

$G(x) =\frac{2-3x}{(1-x)(1-2x)}$

Using partial fractions:

$\frac{2-3x}{(1-x)(1-2x)} = \frac{A}{1-x} + \frac{B}{1-2x}$.

Multiplying both sides by (1−x)(1−2x),
2−3x=A(1−2x)+B(1−x).
Expanding:
2−3x=A−2Ax+B−Bx.
2−3x=(A+B)+(−2A−B)x.
Equating coefficients:

1. A+B=2.
2. −2A−B=−3.

Solving:

- From B=2−A, substitute into −2A−B=−3
- −2A−(2−A)=−3.
- -2A - 2 + A = -3.
- −A−2=−3.
- −A=−1⇒A=1.
- From B=2−A:
  B=2−1=1.

Thus,
G(x) $=\frac{1}{1-x}+\frac{1}{1-2x}$
Expand as a Power Series
Using the known series:
$\frac{1}{1-x} = \sum_{r=0}^{\infty} x^r$, $\frac{1}{1-2x} = \sum_{r=0}^{\infty} 2^r x^r$
Thus,
G(x)= $\sum_{r=0}^{\infty} x^r + \sum_{r=0}^{\infty} 2^r x^r$
G(x)= $\sum_{r=0}^{\infty}(1 + 2^r)x^r$
Extract the Sequence:
Comparing terms, we get:
a_r=1+2^r.
This satisfies the given initial conditions:

- $a_0=1+2^0=1+1=2$
- $a_1=1+2^1=1+2=3$

Thus, the solution to the recurrence relation is: $a_r=1+2^r$.

10. Find number of solutions of e1+e2+e3=17 where e1,e2,e3 are non-negative integers with 2<=e1<=5, 3<=e2<=6, 4<=e3<=7

Ans: The number of solutions is the coefficient of x^{17} in the expansion of $(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7)$.
This follows because a term equal to x^{17} is obtained in the product by picking a term in the first sum x^{e1}, a term in the second sum x^{e2}, and a term in the third sum x^{e3}, where $e_1 + e_2 + e_3 = 17$.
There are three solutions since the coefficient of x^{17} in the product is 3
(or)
Define new variables to shift the lower bounds:
$x_1=e_1-2, x_2=e_2-3, x_3=e_3-4$
Since e_1,e_2,e_3 are bounded, these new variables satisfy:
$0\le x_1\le 3, 0\le x_2\le 3, 0\le x_3\le 3$.
Rewriting the original equation in terms of x_1,x_2,x_3:
$(x_1+2)+(x_2+3)+(x_3+4)=17$.
Simplify:

$x_1+x_2+x_3+9=17$.
$x_1+x_2+x_3=8$.
Counting Solutions with Bounded Values:
We need to count the number of solutions to:
$x_1+x_2+x_3=8, 0\le x1, x2, x3 \le 3$.
Using the stars and bars method without constraints, the total number of non-negative integer solutions is:
Total solutions$=\binom{8+3-1}{3-1}={}^{10}C_2=45$.
Now, we apply the inclusion-exclusion principle to subtract cases where any x_i exceeds 3.
Subtract Over-counted Cases:
Each x_i can be at most 3, so we remove cases where any one variable exceeds 3.
If $x_1\ge4$, set $y_1=x_1-4$, where $y_1\ge0$. Then the equation becomes:
$y_1+x_2+x_3=4$.
Using stars and bars:
$\binom{4+2}{2}=({}^6C_2)=15$.
Since this over-counting applies symmetrically to x_1,x_2,x_3, the total number to subtract is:
$3\times15=45$.
Add Back Cases Where Two Variables Exceed 3:
If both $x_1,x_2\ge4$, define $y_1=x_1-4$ and $y_2=x_2-4$, then:
$y_1+y_2+x_3=0$.
There is only **one** solution: (y1,y2,x3)=(0,0,0). The same applies for any pair of x_i, so we add back: $3\times1=3$.
Cases Where All Three Variables Exceed 3:
If all three $x_1,x_2,x_3\ge4$, then there are no solutions, so no need for further correction.
Thus
Applying inclusion-exclusion:
$45-45+3=3$.
Thus, the number of solutions is:3

11. In how many different ways can 8 identical cookies be distributed among 3 children if each child receives at least 2 cookies and no more than 4 cookies?

Ans: Because each child receives at least two but no more than four cookies, for each child there is a factor equal to $(x^2 + x^3 + x^4)$ in the generating function for the sequence $\{c_n\}$, where c_n is the number of ways to distribute n cookies. Because there are three children, this generating function is $(x^2 + x^3 + x^4)^3$.
We need the coefficient of x^8 in this product. The reason is that the x^8 terms in the expansion correspond to the ways that three terms can be selected, with one from each factor, that have exponents adding up to 8.
Furthermore, the exponents of the term from the first, second, and third factors are the numbers of cookies the first, second, and third children receive, respectively.
Computation shows that this coefficient equals 6.
{(2,2,4),(2,3,3),(2,4,2),(3,2,3),(3,3,2),(4,2,2)}.
Hence, there are six ways to distribute the cookies so that each child receives at least two, but no more than four, cookies.

12. Using Generating functions solve the recurrence relation $a_k = 3a_{k-1}$ and initial condition $a_0 = 2$.

Ans**:** Let $G(x)$ be the associated generating function for the sequence $\{a_k\}$,that is,
$G(x) = \sum_{k=0}^{\infty} akx^k$.

Note that $xG(x) = \sum_{k=0}^{\infty} akx^{k+1} = \sum_{k=1}^{\infty} ak-1x^k$,
Hence the recurrence relation becomes

$G(x) - 3xG(x) = \sum_{k=0}^{\infty} akx^k$ $-3\sum_{k=1}^{\infty} ak-1x^k$

$= a_0 + \sum_{k=1}^{\infty}(ak-3ak-1)x^k$

$= 2.$

because $a_0 = 2$ and $a_k = 3a_{k-1}$. Thus, $G(x) - 3xG(x) = (1 - 3x)G(x)=2$
Solving for $G(x)$ shows that $G(x) = 2/(1 - 3x)$.

Using the identity $1/(1 - ax)= \sum_{k=0}^{\infty} akx^k$ from Table 1, we have

$G(x) = 2 \sum_{k=0}^{\infty} 3kx^k = \sum_{k=0}^{\infty} 2 \cdot 3kx^k$
Consequently, $a_k = 2 \cdot 3^k$.