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Student Name	Student ID	Signature*
Abdulkadir Ahmed	93722	A.A.
Orson Marmon	41933	O.M.

(Note: remove the first 4 digits from your student ID)

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-Abstract

Having learnt and researched the ramifications of the topic surrounding graph theory containing its multiple facets, we display the knowledge we have accumulated throughout our research by detailing what graph theory entails, its numerous applications in real world problems and the mathematical structure used to model it in other fields and objects.

-INTRODUCTION

In this section, a portraiture of what a graph is will be detailed and illustrated. A *graph* $G = \{V, E\}$ consists of a set of points called *vertices* $V = \{v_1, v_2 ...\}$, which are connected by *edges* $E = \{e_1, e_2 ...\}$ [1]. The two vertices which are commonly denoted by dots are linked by edges and are referred to as endpoints [2]. The following is represented by a diagram which can often be referred to as a graph:

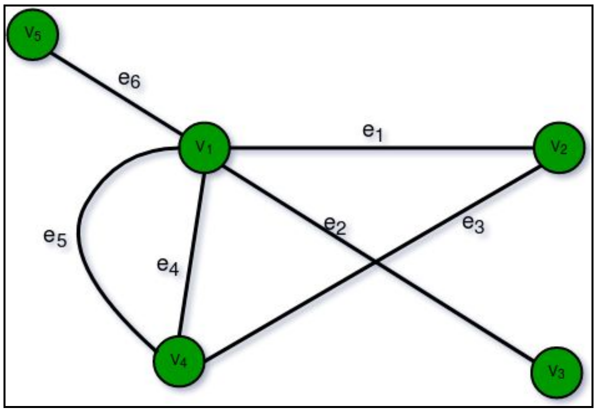

Figure 1.1: Graph with 5 vertices and 6 edges [11].

A graph is permitted to have loops (edges with the same endpoints) and parallel lines (multiple edges with the same endpoints) since it forms an important understudy for the research of graph theory [2]. However, a graph that does not include loops, or parallel edges is simply called a *simple graph [4].* A single graph can be expressed in multiple drawings, as long the prevalence of the vertex and edges match:

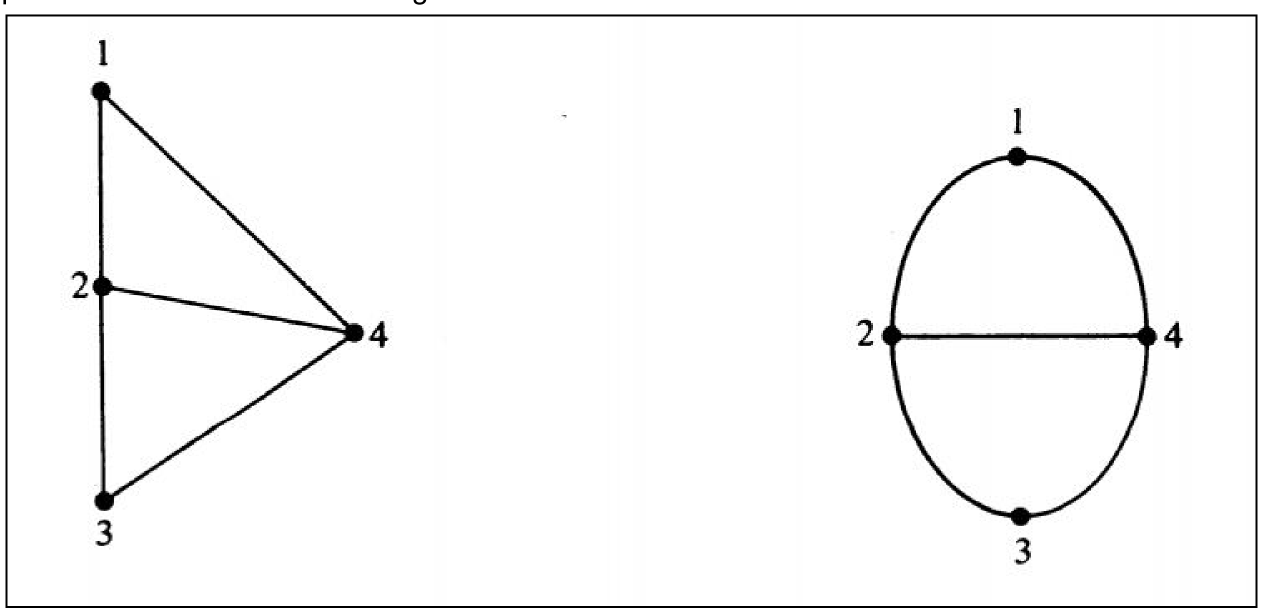

Figure 1.2: Similar graphs drawn differently - same amount of edges and vertices [2].

In certain diagrams of a graph like, it can also be depicted that two edges can not and will not ever share a common point if it is drawn in the same inner plane which causes it to be traversed in the same vertex. Two edges can only have a common point if it is illustrated in a planar way. The relationship which highlights the number of edges and number of vertices which are connected in a planar graph is called the Euler formula [2]. Furthermore, a vertex can commonly be described as a node or a junction and an edge can be interpreted as a branch or an element. The nature of this will be explained further on in the report.

-APPLICATIONS

Graphs and diagrams have various applications to solve distinct problems which require an expanded amount of operations.

One of the more popular problems being the *Konigsberg Bridge.* The problem circled around the question: What is the path required to cross 4 areas of the Prussian city using 7 bridges only once and returning to the starting place? This problem was solved by the mathematician Leonhard Euler after many years using Graph Theory [4]. Figure 2.1 (b) displays how Euler used vertices and edges to represent the areas and the bridges respectively.

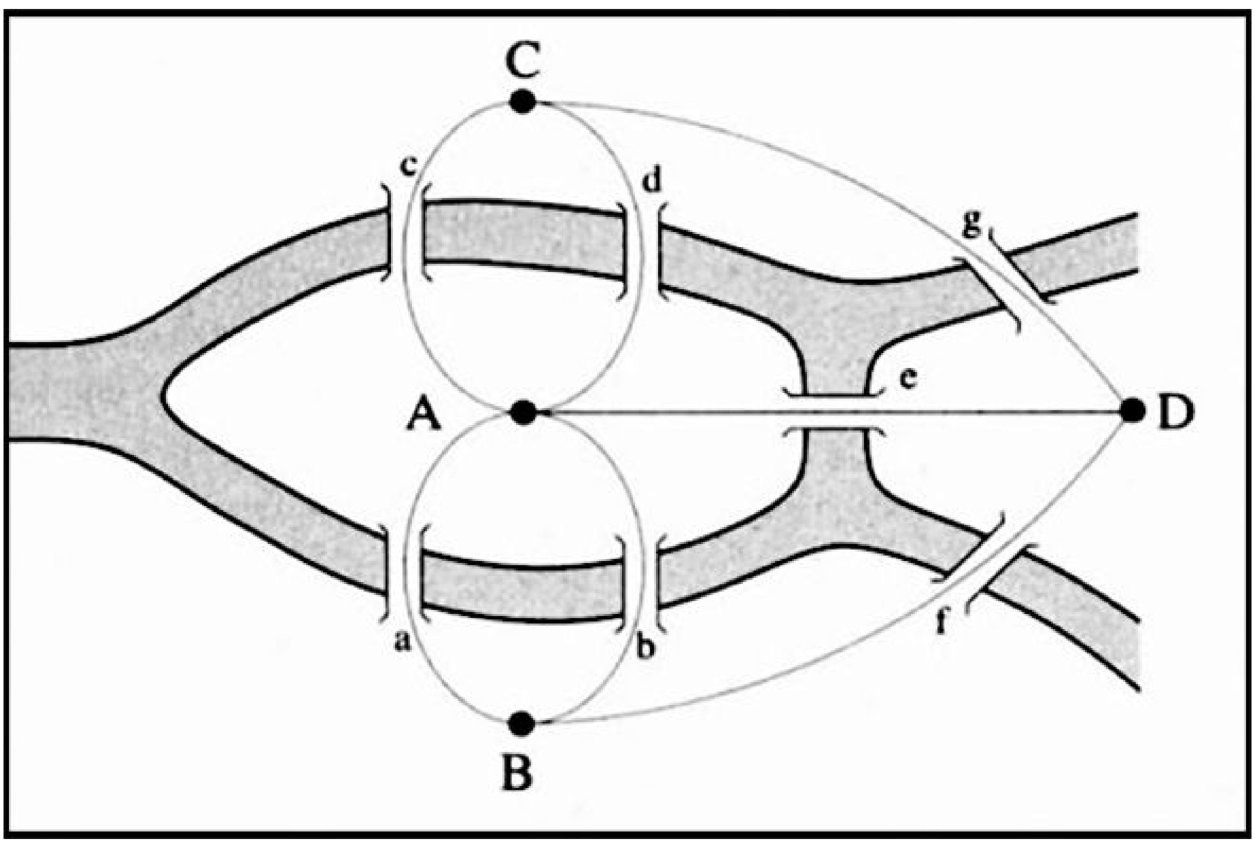

Figure 2.1 Konigsberg Bridge problem and graph [14]

Another interesting problem which uses Graph Theory is the *The Four Color Problem:* The theory to this problem states that each planar graph (a graph where the edges only intersect at the vertices) cannot be colored using no more than 4 colors where the regions are colored differently [13]. The theorem proved that the scope of Graph Theory became more prominent in the modelling of questions and developed into other fields such as engineering problems and linear systems and circuits.

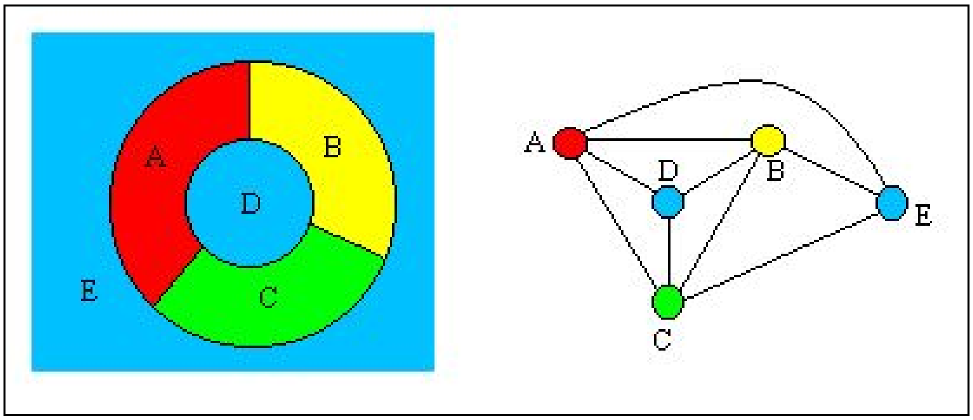

Figure 2.2: Four Color Theorem [13]

-HANDSHAKING LEMMA

As mentioned above, the degree of any vertex in a given set is the amount of edges that are joined at that particular vertex [1]. Having knowledge of this allows one to be able to count the number of directed edges going into the vertex and the number of directed edges going out of the vertex. Furthermore this notation of edges going into a vertex is called an in-degree whilst, edges directed out of a vertex is called an out-degree [16].

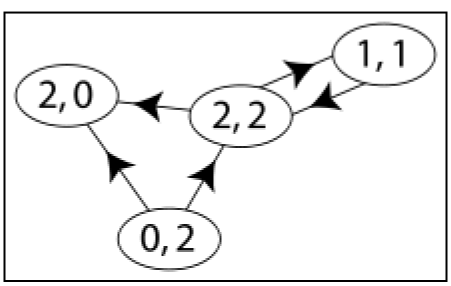

Figure 3.1 Directed graph [6]

As seen in Figure 3.1 the leftmost numbers represent the in-degree while the rightmost numbers represent the out-degree of said vertex. This displays that each edge is connected to two vertices. From this, Euler derived the Handshaking Lemma which states that the sum of the total in-degree and out-degree of all the vertices in the graph equals twice the number of directed edges [2].

In this example, one can add up the total number of degrees and this should inherently equal twice the number of edges represented in the graph.

$$\sum_{i=1}^{n} d(v_i) = 2e$$
$$(Total\ in-degree) + (Total\ out-degree) = 2(edges)$$
$$(2+0+2+1) + (0+2+2+1) = 2(5)$$
$$10 = 10$$

Figure 3.2 Proof of Euler's Handshaking Lemma using directed graph from Figure 3.1 and summation equation [2]

Thus from the Handshaking Lemma one can see that the in-degree equals the out-degree. The Handshaking Lemma is applicable in many areas of math and science - in particular electrical circuit analysis. For instance, the Kirchhoff Current Law - a law developed by Kirchhoff states the electrical current around a closed loop must equate to zero amperes. Otherwise stated the electrical current entering a node equates the electrical current leaving a node [5]. If one were to assume the in-degree to be entering currents and out-degree being exiting currents at a node [16]. This is a decorative way of saying the in-degree equates the out-degree [1]. This is seen explicitly in Figure 3.3.

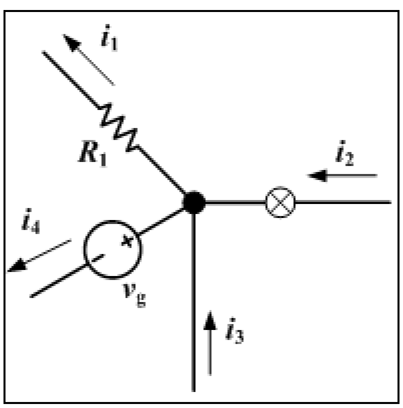

Figure 3.3 Currents entering and exiting a node [7]

Applying Kirchhoff Current Law (KCL) $i_1 + i_4 = i_3 + i_2$. If the node in Figure 3.3 is represented as a vertex it is evident that the in-degree (currents $i_3 + i_2$) equals the out-degree (currents $i_1 + i_4$). Thus through the Handshaking Lemma Theorem it had been used to deduce substantial laws that are used heavily in electric circuit analysis [2].

-PLANARITY

A graph that is drawn or can be reconfigured in such a way that allows for no edges in the graph to intersect is said to be a planar graph [15]. Hence a graph is said to be non-planar if it is not conceivable to reconstruct the given graph in a way that allows for no intersecting of edges [16].

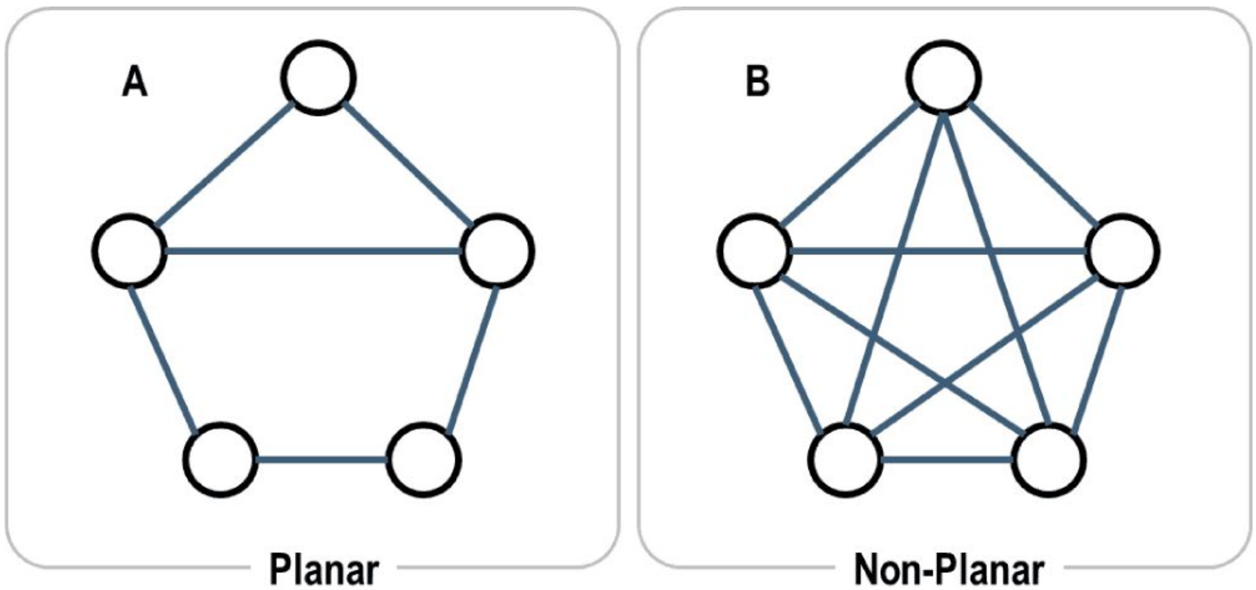

Figure 4.1 Image of planar and non-planar graphs [8]

From Figure 4.1 graph A is planar because there is no apparent edge-crossing. However, graph B is non-planar because there are multiple edge-crossings. Furthermore it must be noted that these edge-crossings in graph B cannot be uncrossed thus making graph B non-planar.

The planarity of a graph can be checked by using Kuratowski's Theorem. This Theorem states that if the two Kuratowski's graphs - $K_{3,3}$ or K_5 [16] - exist as a subgraph within an existing graph. The graph is said to be non-planar [2].

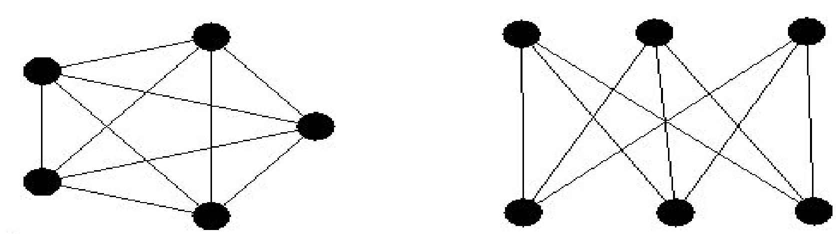

Figure 4.2 Kuratowski's two graphs $K_{3,3}$ & K_5 *[9]*

Planarity in electric circuit analysis plays an important role in determining the optimal way to layout a circuit [2]. This is due to the fact that the circuit (assuming if planar) can be reconfigured in a way that allows no edge-crossings or some edge-crossings to optimize space when designing an electric circuit. Furthermore, when analysing circuits it can present fewer difficulties if one is able to analyse the circuit in its planar form. Therefore, it is common practice amongst technicians and engineers to draw circuits without edge-crossings to make it easier to read while analyzing circuits [2].

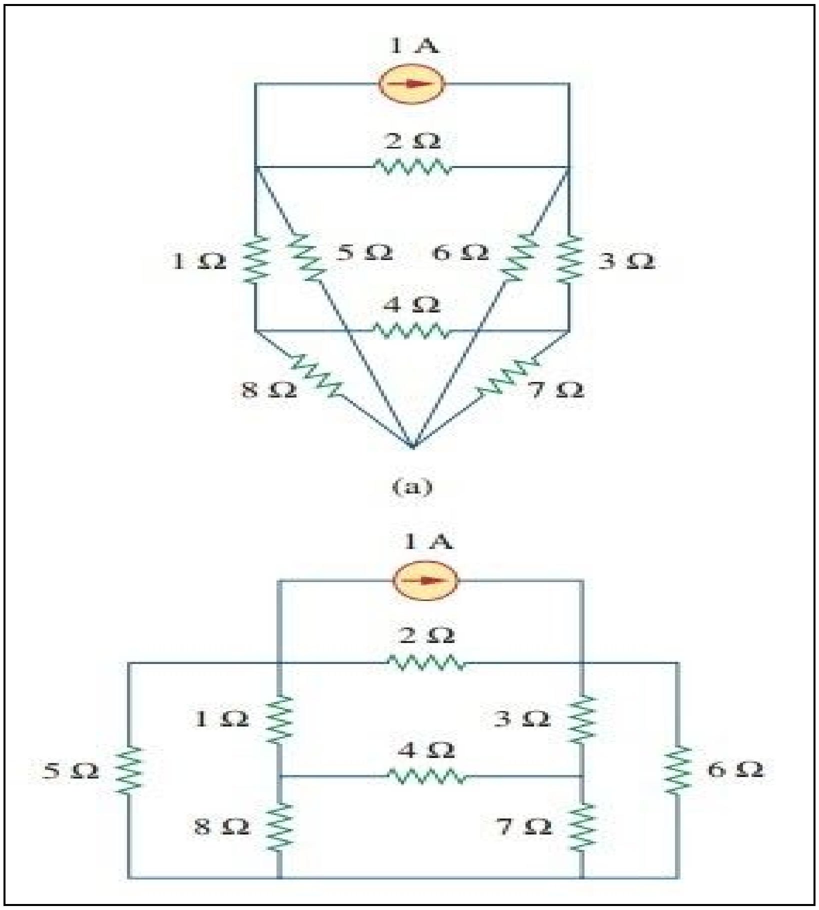

Figure 4.3 Planar electric circuit [10]

Table 4.1 Calculating the voltages at the nodes from circuit in Figure 4.3 using Kirchoff Current Law

$KCL\ V1$ $\frac{V1-V2}{2}+\frac{V1-V3}{1}+\frac{V1}{5}+1=0$ $KCL\ V2$ $\frac{V2-V1}{2}+\frac{V2-V4}{3}+\frac{V2}{6}-1=0$ $KCL\ V3$ $\frac{V3-V4}{4}+\frac{V3-V1}{1}+\frac{V3}{8}=0$ $KCL\ V4$ $\frac{V4-V2}{3}+\frac{V4-V3}{4}+\frac{V4}{7}=0$	*- Simplified system equations solved using MATLAB -* $17V1-5V2-10V3+0V4\ =\ -10$ $-3V1+6V2+0V3-2V4\ =\ 6$ $-8V1+0V2+11V3-2V4\ =0$ $0V1-14V2-10.5V3+30.5V4\ =0$ *-Solutions to system of equations -* $[V1\ \approx\ -581\ mV]$ $[V2\ \approx\ 786mV]$ $[V3\ \approx\ -380mV]$ $[V4\ \approx\ 230mV]$ $Note:\ mV\ -millivolt$

In Table 4.1 a sample calculation is provided of Kirchhoff Current Law (KCL). The voltages at each node - a place where two or more electrical branches are connected [3] - have been calculated. However, the voltage at the ground node - reference node - is chosen to be zero volts [3]. The node $V1$ is the node that connects the two ohm resistor, one ohm resistor, five ohm resistor, and the negative terminal of the one amp current source. The node $V2$ is the node that connects the two ohm resistor, three ohm resistor, six ohm resistor, and the positive side of the one amp current source. The node $V3$ is the node that connects the eight ohm resistor, the four ohm resistor, and one ohm resistor. The node $V4$ is the node that connects the seven ohm resistor, the three ohm resistor, and the four ohm resistor. Lastly, the voltages of the nodes are as follows: -581mV, 786mV, -380mv, and 230mV (Table 4.1).

-CONCLUSION

Throughout this report, one can see that graph theory has been used to solve many problems mentioned beforehand such as the Konigsberg Bridge and The Four Colour Problem. More importantly, the derivation of the Handshaking Lemma and Planarity allowed for the development of a multitude of techniques that are used for electric circuit analysis. Many of the laws used to solve electric circuits such as Kirchoff Current Law have their origin from the application of graph theory. Thus, through these examples it can be deduced the significance that graph theory has on the advancement of mathematical modelling whether it is theoretical or practical.

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