Euler's Method

Suppose we have to solve the differential equation $\frac{dy}{dx} = f(x, y)$ subject to $y(x_0) = y_0$. $\rightarrow$ (1)

Let $x_0, x_1 = x_0 + h, x_2 = x_0 + 2h \ldots$ be equidistant values of x. Now, we want to find the approximate value y_1 corresponding to $x = x_1$.

In Euler's method, we use the property that, in a small interval, a curve is nearly a straight line. In the interval, x_0 to x_1 of x, we approximate the curve by the tangent at the point (x_0, y_0).

Then, the equation of the tangent at the point (x_0, y_0) is

$$y - y_0 = \left(\frac{dy}{dx}\right)_{(x_0, y_0)} (x - x_0)$$

$$\Rightarrow y - y_0 = f(x_0, y_0)(x - x_0) \qquad \text{(by (1))}$$

$$\Rightarrow y = y_0 + f(x_0, y_0)(x - x_0). \longrightarrow (2)$$

This is the value of the y coordinate of a point on the tangent. Since the curve is approximated by the tangent in the interval (x_0, x_1), the value of y on the curve corresponding to $x = x_1$ is given by the above value of y in (2) approximately.

Put $x = x_1$ in (2), we get

$$\Rightarrow y_1 = y_0 + f(x_0, y_0) \cdot (x_1 - x_0)$$

$$\Rightarrow y_1 = y_0 + h f(x_0, y_0)$$

Similarly, approximating the curve in the next interval (x_1, x_2) by a line through (x_1, y_1), whose slope is $f(x_1, y_1)$ we get,

$$y_2 = y_1 + h\, f(x_1, y_1)$$

Similarly, $y_3 = y_2 + h\, f(x_2, y_2)$.

In general, $\boxed{y_{m+1} = y_m + h\, f(x_m, y_m), \quad m = 0, 1, 2, \ldots}$ (3)

The above equation is called "Euler Algorithm."

(3) can also be written as $y(x+h) = y(x) + h\, f(x, y)$.

Note:

1) In Euler's method, the actual curve of solution is approximated by a sequence of short lines.

2) This process is very slow and to obtain reasonable accuracy with Euler's method, we have to take a smaller value of h.

Improved Euler's Method:

The formula for Improved Euler's Method is

$$\boxed{y_{m+1} = y_m + \frac{1}{2} h \left[f(x_m, y_m) + f(x_m + h,\ y_m + h\, f(x_m, y_m)) \right], \quad m = 0, 1, 2, \ldots}$$

The above equation can also be written as,

$$y(x+h) = y(x) + \frac{1}{2} h \left[f(x, y) + f(x+h,\ y + h f(x, y)) \right]$$

Modified Euler Method:

In the Improved Euler Method, we averaged slopes. Another method is to average points.

The formula for Modified Euler Method is, (3)

$$\boxed{y_{m+1} = y_m + h\left[f\left(x_m + h/2,\ y_m + h/2\, f(x_m, y_m)\right)\right], \quad m = 0,1,2,\ldots}$$

This equation can also be written as,

$$y(x+h) = y(x) + h\left[f\left(x + h/2,\ y + h/2\, f(x,y)\right)\right].$$

PROBLEM

1) Solve the equation $\frac{dy}{dx} = 1-y$ with the initial condition $x=0,\ y=0$ using Euler's algorithm and tabulate the solutions at $x = 0.1, 0.2, 0.3, 0.4$. Get the solutions by Euler's improved method and Euler's modified method. Also compare with the exact solution.

Solution:

The given differential equation is $\frac{dy}{dx} = 1-y = f(x,y)$. → (1)

By Euler's algorithm, is ~~are~~ ~~have~~

$$\boxed{y_{m+1} = y_m + h f(x_m, y_m)}, \quad m = 0,1,2,\ldots \to (2)$$

Given $x_0 = 0$ and $y_0 = 0$. Also $h = 0.1$.

Put $m=0$ in (2), we get

$$y_1 = y_0 + h f(x_0, y_0) = 0 + (0.1)(1-y_0) \quad \text{by (1)}$$

$$y_1 = 0 + (0.1)(1-0) = 0.1.$$

$$\boxed{y_1 = 0.1}$$

Formula : $\boxed{x_{m+1} = x_m + h}$ → (2A)

Put $m=0$ in (2A), we get

Now, $x_1 = x_0 + h = 0 + 0.1 = 0.1$

$$\therefore \boxed{(x_1, y_1) = (0.1, 0.1)}$$

Put $m=1$ in (2), we get

$$y_2 = y_1 + h f(x_1, y_1)$$

$$= (0.1) + (0.1)(1-y_1) = (0.1) + (0.1)(1-0.1)$$

$$= 0.1 + (0.1)(0.9) = 0.19$$

$$y_2 = 0.19.$$

Put $m=1$ in (2A), we get

Now, $x_2 = x_1 + h = (0.1) + 0.1 = 0.2$ (or)

$$x_2 = x_0 + 2h = 0 + 2(0.1) = 0.2.$$

$$\therefore \boxed{(x_2, y_2) = (0.2, 0.19)}$$

Put $m=2$ in (2), we get

$$y_3 = y_2 + h f(x_2, y_2).$$

$$= (0.19) + (0.1)(1-y_2) = (0.19) + (0.1)(1-0.19)$$

$$= (0.19) + (0.1)(0.81) = 0.271$$

$$y_3 = 0.271$$

Put $m=2$ in (2A), we get

Now, $x_3 = x_2 + h = 0.2 + 0.1 = 0.3$

$$\therefore \boxed{(x_3, y_3) = (0.3, 0.271).}$$

Put $m=3$ in (2), we get

$$y_4 = y_3 + h f(x_3, y_3).$$

$$= (0.271) + (0.1)(1-y_3) = 0.271 + (0.1)(1-0.271)$$

$$= 0.271 + (0.1)(0.729)$$

$$= 0.271 + 0.0729$$

$$y_4 = 0.344$$

Put $m=3$ in (2A), we get

Now, $x_4 = x_3 + h = 0.3 + 0.1 = 0.4$

$$\boxed{(x_4, y_4) = (0.4, 0.344)}$$

Euler's Improved Formula is,

$$\boxed{y_{m+1} = y_m + \frac{h}{2}\left[f(x_m, y_m) + f(x_m+h,\ y_m + h f(x_m, y_m))\right]} \rightarrow (3)$$

$\boxed{m = 0, 1, 2, \ldots}$

Since $f(x,y) = 1-y$, $f(x_m, y_m) = 1 - y_m$

$$f(x_m+h,\ y_m + h f(x_m, y_m)) = 1 - \left[y_m + h f(x_m, y_m)\right]$$
$$= 1 - y_m - h(1-y_m)$$
$$= 1 - y_m - h + h y_m$$
$$= (1-h) + y_m(h-1).$$
$$= (1-h) - y_m(1-h).$$
$$= (1-h)(1-y_m).$$

$\therefore f(x_m+h,\ y_m + h f(x_m, y_m)) = (1-h)(1-y_m).$

Then (3) becomes,

$$(3) \Rightarrow y_{m+1} = y_m + \frac{h}{2}\left[(1-y_m) + (1-h)(1-y_m)\right]$$
$$= y_m + \frac{h}{2}\left[(1-y_m)(1+1-h)\right]$$
$$y_{m+1} = y_m + \frac{h}{2}\left[(1-y_m)(2-h)\right]. \rightarrow (4)$$

Put $m=0$ in (4), we get

$$y_1 = y_0 + \frac{h}{2}\left[(1-y_0)(2-h)\right] \qquad [\because y_0 = 0;\ h = 0.1]$$
$$= 0 + \frac{0.1}{2}\left[(1-0)(2-0.1)\right]$$
$$= \frac{0.1}{2}(1.9)$$
$$= 0.095$$

$$\boxed{y_1 = 0.095}$$

Put $m=1$ in (4), we get

$$y_2 = y_1 + \frac{h}{2}\left[(1-y_1)(2-h)\right]$$

$$= 0.095 + \frac{0.1}{2}\left[(1-0.095)(2-0.1)\right]$$

$$= 0.095 + \frac{0.1}{2}\left[0.905 \times 1.9\right]$$

$$= 0.095 + \frac{0.1}{2}\left[1.7195\right]$$

$$= 0.095 + 0.085975$$

$$y_2 = 0.180975$$

$$\boxed{y_2 = 0.1810}$$

Put $m=2$ in (4), we get

$$y_3 = y_2 + \frac{h}{2}\left[(1-y_2)(2-h)\right]$$

$$= 0.1810 + \frac{0.1}{2}\left[(1-0.1810)(2-0.1)\right]$$

$$= 0.1810 + \frac{0.1}{2}\left[0.819 \times 1.9\right]$$

$$= 0.1810 + \frac{0.1}{2}\left[1.556\right]$$

$$= 0.1810 + 0.077805$$

$$y_3 = 0.258805.$$

$$\boxed{y_3 = 0.2588}$$

Put $m=3$ in (4), we get

$$y_4 = y_3 + \frac{h}{2}\left[(1-y_3)(2-h)\right]$$

$$= 0.2588 + \frac{0.1}{2}\left[(1-0.2588)(2-0.1)\right]$$

$$= 0.2588 + \frac{0.1}{2}\left[0.7412 \times 1.9\right]$$

$= 0.2588 + \frac{0.1}{2}[1.40828]$

$= 0.2588 + 0.070414$

$y_4 = 0.329214.$

$$\boxed{y_4 = 0.3292}$$

Euler's Modified formula is,

$$\boxed{y_{m+1} = y_m + h\left[f\left(x_m + h/2,\ y_m + h/2\, f(x_m, y_m)\right)\right]}, m = 0, 1, \ldots \rightarrow (5)$$

$f(x_m + h/2,\ y_m + h/2\, f(x_m, y_m)) = 1 - \left[y_m + h/2\, f(x_m, y_m)\right]$

$= 1 - y_m - h/2\, f(x_m, y_m).$

$= 1 - y_m - h/2\,(1 - y_m);$

$= 1 - y_m - h/2 + h/2\, y_m.$

$= (1 - h/2) + [y_m(h/2 - 1)]$

$= (1 - h/2) - y_m(1 - h/2)$

$= (1 - h/2)(1 - y_m).$

$= 1/2\,(2 - h)(1 - y_m).$

Then (5) becomes,

$(5) \Rightarrow y_{m+1} = y_m + h\left[1/2\,(2 - h)(1 - y_m)\right]$

$= y_m + h/2\left[(2 - h)(1 - y_m)\right].$

$\therefore y_{m+1} = y_m + h/2\,(2 - h)(1 - y_m) \longrightarrow (6)$

Here (6) is similar to (4), so the values of y_1, y_2, y_3 and y_4 are same in Euler's modified method also.

Therefore we get, (8)

$y_1 = 0.095$

$y_2 = \cancel{0.2588}\ 0.1810$

$y_3 = 0.2588$

$y_4 = 0.3292$

Now, we find the exact solution of the given differential equation, $\frac{dy}{dx} + y = 1$.

This above differential equation is of the form $\boxed{\frac{dy}{dx} + Py = Q}$.

The general solution of the above form is

$$\boxed{y e^{\int P dx} = \int Q e^{\int P dx} dx + c.} \rightarrow (7)$$

Now, we find the value of the integrating factor $e^{\int P dx}$.

Compare the given DE to the above form, we get $P = 1$ and $Q = 1$.

$$I.F = e^{\int P dx} = e^{\int dx} = e^{x}$$

$$(7) \Rightarrow y e^{x} = \int (1)\, e^{x} dx + c$$

$$\Rightarrow y e^{x} = e^{x} + c. \rightarrow (8)$$

Put $x=0$ and $y=0$ in (8), we get

9

(8) $\Rightarrow 0 = e^0 + c.$

$\Rightarrow 0 = 1 + c.$

$\Rightarrow \boxed{c = -1}$

(8) $\Rightarrow y e^x = e^x - 1.$

$\Rightarrow y = \dfrac{e^x - 1}{e^x}.$

$\Rightarrow y = 1 - 1/e^x = 1 - e^{-x}.$

$\Rightarrow \boxed{y = 1 - e^{-x}.} \longrightarrow (9')$

Put the values of x in (9'), we get.

$y(0.1) = 1 - e^{-0.1} = 1 - 0.9048 = 0.0952$

$y(0.2) = 1 - e^{-0.2} = 1 - 0.8187 = 0.1813$

$y(0.3) = 1 - e^{-0.3} = 1 - 0.7408 = 0.2592$

$y(0.4) = 1 - e^{-0.4} = 1 - 0.6703 = 0.3297.$

Now, we tabulate the values,

x	Solution by Euler	Solution by Improved Euler	Solution by Modified Euler	Exact solution
0	0	0	0	0
0.1	0.1	0.095	0.095	0.0952
0.2	0.19	0.1810	0.1810	0.1813
0.3	0.271	0.2588	0.2588.	0.2592
0.4	0.344	0.3292	0.3292	0.3297.