Tortoise and Hare Problem

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1 Introduction

The Tortoise and Hare approach is usually having two walkers, one slow and one fast, to find a cycle in a linked list. Prove that using tortoise and hare approach you can prove a cycle exists. Also show how to find the start of the cycle.

1.1 Proving cycle exists

Let there be a tortoise walking at the speed of 1 unit, and hare walking at the speed of 2 units. We need to prove that after some steps if the tortoise and hare meet a cycle exists.

Assuming that a cycle exists, let k be the unit of distance from the start of the cycle, and the number of steps in linked list traversed by tortoise be i.

The tortoise travels a distance of $i = m + q * n + k$ in and hare travels $m + p * n + k$, where q, p is the number of cycles taken by the tortoise and hare respectively.

Since hare is at $2x$ the speed of tortoise, the hare would've traveled a total of $2i$, therefore we have

$$i = m + p * n + k \tag{1}$$

$$2i = m + q * n + k \tag{2}$$

Equating both we can get at

$$m + k = (q - 2p) * n \tag{3}$$

Since the values of m, n are set. We need to find values of k, q, p that satisfies the equation above to prove that hypothesis is correct.

If we set $q = m$, and $p = 0$, and $k = mn - m$ we can have
$m + (mn - m) = (m - 2 * 0) * n$
$mn = mn$

Since a set of points exist to satisfy the equation above, we can say that if hare and tortoise meet a cycle exists.

1.2 Finding the start of the cycle

From the Eq 3 we have $q - 2p$ i.e the number of loops done in the cycle, for simplicity let's say
$\pi = q - 2p$ i.e

$$m + k = \pi * n \tag{4}$$

We know the start of the cycles is located m steps away from start of the linked list. So if we were to move the tortoise one step at a time from start for m steps we'll reach at the start of list. However we don't know what m is.

Also, from Eq 4 we know that walking m steps from the k^{th} point completes π cycles in the circle. A completion of the cycle would indicate that we are at the starting point of the cycle. Therefore, walking m steps from the k^{th} we can find the start of the cycle. However, again we don't know what m is.

Thus if we place the tortoise back at the starting position, and walk single-steps, and place hare at the meeting point at k^{th} spot in the cycle, and walk single-steps, they both **will meet at the start of the cycle**, and the number of steps taken would've been m.