MATHEMATICS 1

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Part I

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CHAPTER 1

Functions

The concept of function is one of the basic ideas in all mathematics. Almost every study that concerns the application of mathematics to practical problems or that involves the analysis of empirical data makes use of this mathematical concept.

A function expresses the idea of one quantity depending on or being determined by another.

1.1. Definition of a Function:

Let X and Y be two nonempty sets. Then a function from X to Y ($f: X \to Y$) is a rule that assigns to each element $x \in X$ a unique $y \in Y$. If a function assigns y to a particular x, we say that y is the value of the function at x.

A function is generally denoted by letter such as f, g, F or G. We read $f(x)$ as "f of x"; it is called the value of f at x. Note that $f(x)$ is not the product of f and x.
If a function f is expressed by a relation of the type $y = f(x)$, then x is called the independent variable or argument of f and y is called the dependent variable.

Generally, we shall encounter functions that are expressed by stating the value of the function by means of an algebraic formula in terms of the independent variable involved. For example, $f(x) = 5x^2 - 7x + 2$ and $g(p) = 2p^3 + 7/(p+1)$.

Example 1: Given $f(x) = 2x^2 - 5x + 1$, find the value of f when $x = a$, $x = 3$, $x = -2$ and $x = -\frac{1}{4}$; that is, find $f(a)$, $f(3)$, $f(-2)$ and $f\left(-\frac{1}{4}\right)$.

Solution :

$f(x) = 2x^2 - 5x + 1$
To find $f(a)$, we replace x by a, thus

$$f(a) = 2a^2 - 5a + 1$$

To evaluate $f(3)$, we substitute 3 for x on f(a).

$$f(3)=2(3)^2-5(3)+1=18-15+1=4$$

Similarly

$$f(-2)=2(-2)^2-5(-2)+1=19$$

and

$$f\left(-\frac{1}{4}\right)=2\left(-\frac{1}{4}\right)^2-5\left(-\frac{1}{4}\right)+1=\frac{19}{8}$$

1.2. The Domain and Range of a Function :

The set X for which f assigns a unique $y \in Y$ is called the domain of the function f. It is often denoted by D_f. The corresponding set of values $y \in Y$ is called the range of the function and is often denoted by R_f.

Example 2 : Evaluate $F(0)$, $F(1)$, and $F(4)$ for the function

F defined by $F(x)=\dfrac{x-4}{\sqrt{2-x}}$

Solution : First replace x by 0 :

$$F(0)=\frac{0-4}{\sqrt{2-0}}=\frac{-4}{\sqrt{2}}=-2\sqrt{2}$$

Next, replace x by 1 :

$$F(1)=\frac{1-4}{\sqrt{2-1}}=\frac{-3}{\sqrt{1}}=-3$$

Finally, replace x by 4 :

$$F(4)=\frac{4-4}{\sqrt{2-4}}=\frac{-0}{\sqrt{-2}} \text{ not defined}$$

$F(4)$ does not exist; in other words, 4 is not in the domain of F.

Three Reasons for Restricting the Domain :

1- You cannot have zero in a denominator.
2- You cannot take the square root (or fourth or six root, …etc.) of a negative number.
3- The domain is restricted by the nature of the applied problem under condition. For example, if $f(x)$ denotes the cost of buying x barrels of oil, then, necessarily, $x \geq 0$ because it is impossible to buy negative quantities.

Example 3 : Find the domain of g, where

$$g(x)=\frac{x+3}{x-2}$$

Solution :

Clearly, $g(x)$ is not a well-defined real number for $x=2$. For any other value of $x, g(x)$ is a well-defined real number. Thus the domain of g is the set of all real numbers except 2.

Example 4 : Find the domain of f if $f(x)=\sqrt{x-4}$

Solution : The domain of f is the set of all values of x for which the expression under the radical sign is nonnegative. That is,

$$x-4 \geq 0 \text{ or } x \geq 4$$

For $x<4, f(x)$ is not a real number, since the quantity beneath the square root, $x-4$, is negative.

Example 5 : Find the domain of h where

$$h(x)=\frac{x}{(x-2)\sqrt{x-1}}$$

Solution :

Here the radical is defined only for $x \geq 1$. But the denominator is zero if $x=1$ or $x=2$, so these two points must be excluded from the domain,
Therefore,

$$D_h=\{x : x>1, x \neq 2\}$$

1.3. Combination of Functions :

A variety of situations arise in which we have to combine two or more functions in one of several ways to get new functions. For example, let $f(t)$ and $g(t)$ denote the incomes of a person from two different sources at time t; then the combined income from the two sources is $f(t)+g(t)$. From the two functions f and g, we have in this way obtained a third function, the sum of f and g.

Defenition 1:

Given two functions f and g, the sum, difference, product, and quotient functions are defined as follows.

Sum: $(f+g)(x)=f(x)+g(x)$
Difference : $(f-g)(x)=f(x)-g(x)$
Product : $(f \cdot g)(x)=f(x) \cdot g(x)$
Quotient : $\left(\frac{f}{g}\right)(x)=f(x)+g(x)$, provided $g(x) \neq 0$

The domains of the sum, difference and product functions are all equal to the common part of the domains of f and g (the intersection of the domains of f and g), that is, the set of x at which both f and g are defined. In the case of the quotient function, the domain is the common part of the domains of f and g except for those values of x for which $g(x)=0$.

Example 6: Let $f(x)=1/(x-1)$ and $g(x)=\sqrt{x}$. Find $f+g$, $f-g$, $f\cdot g$, f/g and g/f. Determine the domain in each case.

Solution : We have :

$$(f+g)(x)=f(x)+g(x)=\frac{1}{x-1}+\sqrt{x} \quad (f-g)(x)=f(x)-g(x)=\frac{1}{x-1}-\sqrt{x}$$

$$(f\cdot g)(x)=f(x)\cdot g(x)=\frac{1}{x-1}\cdot\sqrt{x}=\frac{\sqrt{x}}{x-1}$$

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}=\frac{1/(x-1)}{\sqrt{x}}=\frac{1}{\sqrt{x}(x-1)}$$

$$\left(\frac{g}{f}\right)(x)=\frac{g(x)}{f(x)}=\frac{\sqrt{x}}{1/(x-1)}=\sqrt{x}(x-1)$$

Since its denominator becomes zero when $x=1$, $f(x)$ is not defined for $x=1$, so the domain of f is the set of all real numbers except 1.

Similarly, $g(x)$ id defined for the values of x for which the expression under the radical sign is nonnegative, that is, for $x\geq 0$. Thus

$$D_f=\{x: x\neq 1\} \text{ and } D_g=\{x: x\geq 0\}$$

The common part of D_f and D_g is

$$\{x: x\geq 0 \text{ and } x\neq 1\}$$

This set provides the domain of $f+g$, $f-g$ and $f\cdot g$.

Since $g(x)=\sqrt{x}$ is zero when $x=0$, this point must be excluded from the domain of f/g. Thus the domain of f/g is

$$\{x: x>0 \text{ and } x\neq 1\}$$

Since $f(x)$ is never zero, the domain of g/f is again the common part of D_f and D_g, namely $\{x: x\geq 0$ and $x\neq 1\}$. It appears from the algebraic formula for $(g/f)(x)$ that this function is well-defined when $x=1$. In spite of this, it is still necessary to exclude $x=1$ from the domain of this function, as g/f is defined only at points where both g and f are defined.

1.4 .Composition of Functions :

Another way in which two functions can be combined to yield a third function is called the composition of functions. Consider the following situation.

Defenition 2:

Let f and g be two functions. Let x belong to the domain of g and be such that $g(x)$ belongs to the domain of f. Then the composite function $f \circ g$ (read f circle g) is defined by

$$(f \circ g)(x) = f(g(x))$$

Example 6: Let $f(x) = 1/(x-2)$ and $g(x) = \sqrt{x}$. Evaluate

(a) $(f \circ g)(9)$; (b) $(f \circ g)(4)$; (c) $(f \circ g)(x)$;
(d) $(g \circ f)(6)$; (e) $(g \circ f)(1)$; (f) $(g \circ f)(x)$;

Solution :

(a) $g(9) = \sqrt{9} = 3$. Therefore
$(f \circ g)(9) = f(g(9)) = f(3) = 1/(3-2) = 1$
(b) $g(4) = \sqrt{4} = 2$. We have
$(f \circ g)(4) = f(g(4)) = f(2) = 1/(2-2)$
This is not defined. The value $x=4$ does not belong to the domain of $f \circ g$, so $(f \circ g)(4)$ cannot be found.
(c) $g(x) = \sqrt{x}$

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = \frac{1}{\sqrt{x}-2}$$

(d) $f(6) = 1/(6-2) = \frac{1}{4}$;

$$(g \circ f)(6) = g(f(6)) = g\left(\frac{1}{4}\right) = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

(e) $f(1) = 1/(1-2) = -1$;
$(g \circ f)(1) = g(f(1)) = g(-1) = \sqrt{-1}$
which is not a real number. We cannot evaluate $(g \circ f)(1)$ as 1 dose not belong to the domain of $g \circ f$.

(f) $f(x) = 1/(x-2)$;

$$(g \circ f)\,(x) = g\,(f\,(x)) = g\left(\frac{1}{x-2}\right) = \sqrt{\frac{1}{x-2}} = \frac{1}{\sqrt{x-2}}$$

The domain of $f \circ g$ is given by

$$D_{f \circ g} = \{x : x \in D_g \text{ and } g(x) \in D_f\}$$

It can be shown that, for the functions in Example 6,

$$D_{f \circ g} = \{x : x \geq 0 \text{ and } x \neq 4\}$$

and

$$D_{g \circ f} = \{x : x > 2\}$$

1.5. Implicit Relations and Inverse Functions :

When y is a given function of x, that is, $y = f(x)$, then we often say that y is an explicit function of the independent variable x. Examples of explicit functions are $y = 3x^2 - 7x + 5$, and $y = 5x + 1/(x-1)$.

Sometimes the fact that y is a function of x is expressed indirectly by means of some equations of the type $F(x, y) = 0$, in which both x and y appear as arguments of the functions F on the left side. An equation of this type is called an implicit relation between x and y.

Example 8 :

Consider $xy + 3y - 7 = 0$. In this equation, we have a function on the left involving both x and y, and the equation provides an implicit relation between x and y. In this case we can solve for y.

$$y(x+3) = 7$$

$$y = \frac{7}{x+3}$$

Thus we can express y as an explicit function. In this example, the given implicit relation is equivalent to a certain explicit function. This is not always the case, as the following examples show.

Example 9: Consider the implicit relation

$x^2 + y^2 = 4$. In this case, we can again solve for y.

$$y^2 = 4 - x^2$$

$y = +\sqrt{4-x^2}$ and $y = -\sqrt{4-x^2}$

These last two are explicit functions. Thus the implicit relation $x^2+y^2=4$ leads to the two explicit functions,

$y=+\sqrt{4-x^2}$ and $y=-\sqrt{4-x^2}$

Example 10 :

Consider the implicit relation $x^2+y^2+4=0$. If we try to solve for y, we obtain $y^2=-4-x^2$ Whatever the value of x, the right side of the equation is always negative, so we cannot take the square root. In this case, the implicit relation has no solution. (We say that its domain is empty)

Example 11:

$y^5+x^3-3xy=0$. This given relation does imply that y is a function of x, but we cannot solve for y in terms of x; that is, we cannot express y as an explicit function of x by means of any algebraic formula.

When the fact that y is a function of x is implied by some relation of the form $F(x,y)=0$, we speak of y as an implicit function of x. As in Example 11, this does not necessarily mean that we can actually find a formula expressing y as a function of x.

In general, let $y=f(x)$ be some given function. The equation $y-f(x)=0$ represents an implicit relation between x and y. If we regard x as the independent variable, we can solve this relation for y, obtaining our original function, $y=f(x)$. On the other hand, we may wish to regard y as the independent variable and to solve for x in terms of y. We will not always be able to do this, but if we can, the solution is written as $x=f^{-1}(y)$ and f^{-1} is called the inverse function of f.

Note:

1- $f^{-1}(y)$ is not be confused with the negative power

$$[f(y)]^{-1}=\frac{1}{f(y)}$$

2- If we take the composition of f and its inverse function, we find that

$$\left(f^{-1}\circ f\right)(x)=x \quad \text{and} \quad \left(f\circ f^{-1}\right)(y)=y$$

In other words, the composition of f and f^{-1} gives the identity function, that is, the functions which leaves the variable unchanged.

Example 12: Find the inverse of the function $f(x)=2x+1$.

Solution: Setting $y=f(x)=2x+1$, we must solve for x as a function of y.

$$2x=y-1$$

$$x = \frac{y-1}{2}$$

Therefore the inverse function is given by $f^{-1}(y) = (y-1)/2$

Example 13: Find the inverse of the function $f(x) = x^3$ and sketch is graph.

Solution: Setting $y = f(x) = x^3$, we solve for

x, obtaining $x = f^{-1}(y) = y^{1/3}$

1.6. Classification of Functions:

In this section we will discuss types of functions.

1- Polynomial Function :

A polynomial function is any function of the form

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + \ldots\ c_1 x + c_0$$

Where $c_n,\ c_{n-1},\ \ldots,\ c_1$ and c_0 are real numbers with $c_n \neq 0$ and where a is a nonnegative integer (called the degree of the polynomial . Zero-degree of polynomials are of the form

$$f(x) = c_0 \quad \text{with} \quad c_0 \neq 0$$

and are called constant functions. Thus f (x) = 5 provides an example of a constant function. By convention no degree is assigned to the constant polynomial 0. First - degree polynomials are of the form

$$f(x) = c_1 x + c_0 \quad \text{with} \quad c_1 \neq 0$$

and are called linear functions. The particular linear function defined by $f(x) = x$ is called the identity function.

2- Rational Function :

A rational function is a quotient of two polynomials in the form :

$$f(x) = \frac{q(x)}{h(x)}, \qquad h(x) \neq 0$$

Example 14:

Let $f(x) = \dfrac{x^2 + x - 2}{x^2 + 5x - 6}$, Find the domain of the rational function $f(x)$.

Solution:

Since $x^2+5x-6=(x-1)(x+6)$. The denominator is 0 for $x=1$ and $x=-6$. Thus the domain of f consists of all numbers except 1 and -6 i.e R- {1,-6}.

3- Exponential Function:

The exponential function with base a is defined by

$$f(x)=a^x$$

where a > 0, $a\neq 0$, and x is any real number.
(Notice that if a=1 $\rightarrow$ $f(x)=1^x=1$ is a constant function).
The domain of exponential function: is the set of all real numbers.
The range of exponential function: is the set of positive real numbers.

Properties of Exponential Function :

1- $a^0=1$
2- $a^u a^v=a^{u+v}$
3- $a^u/a^v=a^{u-v}$
4- $(a^u)^v=a^{uv}$
5- $a^u>0$
6- $a^{-u}=\dfrac{1}{a^u}$

Special Case : It is often useful to use as base the number denoted by e, which is given to five decimal places by $e=2.71828$. The corresponding exponential function is written e^x and is called the natural exponential function.

Note:

$$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

4- Logarithmic Function:

The logarithm of x with base a is defined by

$y=\log_a x$ if and only if $x=a^y$

where $a > 0$, $a \neq 1$ and $x > 0$

The domain of a logarithmic function: is the set of all positive number.

The range of a logarithmic function: is the set of all real numbers.

Properties of Exponential Function :

1- $\log_a 1 = 0 \Leftrightarrow a^0 = 1$
2- $\log_a a = 1 \Leftrightarrow a^1 = a$
3- $\log_a a^x = x \Leftrightarrow a^x = a^x$
4- $\log_a (UV) = \log_a U + \log_a V$
5- $\log_a (U/V) = \log_a U - \log_a V$
6- $\log_a (U)^C = C \log_a U$

5- Trigonometric Functions :

1. $y = \sin x$
Domain : $x \in]-\infty, \infty[\equiv R$
Range : $y \in [-1, 1]$

2. $y = \cos x$
Domain : $x \in R$
Range : $y \in [-1, 1]$

3. $y = \tan x = \dfrac{\sin x}{\cos x}$

Domain : $x \in R$ excepte : $x = 0, \ \pm\dfrac{\pi}{2}, \ \pm\dfrac{3\pi}{2}, \dots$

Range : $y \in R$

4. $y = \cot y = \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$
Domain : $x \in R$ excepte : $x = 0, \ \pm\pi, \ \pm 2\pi, \dots$
Range : $y \in R$

5. $y = \sec x = \dfrac{1}{\cos x}$

Domain : $x \in R$ excepte : $x = 0, \ \pm\dfrac{\pi}{2}, \ \pm\dfrac{3\pi}{2}, \dots$

Range **:** $y \in]-\infty,-1] \cup [1,\infty[$

6. $y = \cos ec x = \frac{1}{\sin x}$

Domain : $x \in R$ excepte : $x = 0, \pm\pi, \pm 2\pi,$

Range **:** $y \in]-\infty,-1] \cup [1,\infty[$

Some Important Formulas :

For any real numbers α, β and θ we have :

1- *Pythagorean Identities* :

$\sin^2\theta + \cos^2\theta = 1,$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \cos \boldsymbol{ec}^2\theta$

2- *Formulas for negatives*

$\sin(-\theta) = -\sin\theta$	,	$\cos \boldsymbol{ec}\,(-\theta) = -\cos \boldsymbol{ec}\,(\theta)$
$\cos\,(-\theta) = \cos\theta$	,	$\sec\,(-\theta) = \sec\,(\theta)$
$\tan(-\theta) = -\tan\theta$	,	$\cot\,(-\theta) = -\cot\,(\theta)$

3- *Addition and subtraction formulas for the sine and cosine*

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$

$\cos\,(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\beta\sin\alpha$

6- The Inverse Trigonometric Functions:

1. $y = \sin^{-1} x$ iff $x = \sin y$

The Domain : $[-1,1]$

The Range **:** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

2. $y = \cos^{-1} x$ iff $x = \cos y$

The Domain : $[-1,1]$

The Range **:** $[0,\pi]$

3. $y = \tan^{-1} x$ iff $x = \tan y$

The Domain : $]-\infty,\infty[$

The Range **:** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

4. $y=\cot^{-1}x$ iff $x=\cot y$

The Domain : $]-\infty,\infty[$

The Range : $[0,\pi]$

Note : $\sin^{-1}$ does not mean $\frac{1}{\sin x}$, which is equal $\cos ecx$. Hence, in in the inverse function $y=\sin^{-1}x$, y is an angle and x is the sin of an angle y .

7- Hyperbolic Functions :

1. $\sinh x=\dfrac{e^{x}-e^{-x}}{2}$
2. $\cosh x=\dfrac{e^{x}+e^{-x}}{2}$
3. $\tanh x=\dfrac{\sinh x}{\cosh x}=\dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$
4. $\coth x=\dfrac{\cosh x}{\sinh x}=\dfrac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$
5. $\sec hx=\dfrac{1}{\cosh x}=\dfrac{2}{e^{x}+e^{-x}}$
6. $\cos echx=\dfrac{1}{\sinh m}=\dfrac{2}{e^{x}-e^{-x}}$

Some Important Formulas:

1- $\cosh^2 x-\sinh^2 x=1$

2- $\sinh 2x=2\sinh x\cosh x$

3- $\cosh 2x=\cosh^2 x+\sinh^2 x$

$=2\cosh^2 x+1=2\sinh^2 x+1$

4- $\sinh(x\pm y)=\sinh x\cosh y\pm\cosh x\sinh y$

5- $\cosh(x\pm y)=\cosh x\cosh y\pm\sinh x\sinh y$

6- $\tanh 2x=\dfrac{2\tanh x}{1+\tanh^2 x}$

7- $\tanh(x\pm y)=\dfrac{\tanh x\pm\tanh y}{1\pm\tanh x\tanh y}$

8- $1-\tanh^2 x=\sec h^2 x$

9- $\coth^2 x-1=\cos ech^2 x$

8- The inverse of Hyperbolic Functions:

1. $\sinh^{-1} x = ln\left(x + \sqrt{x^2+1}\right)$; $-\infty < x < \infty$

2. $\cosh^{-1} x = ln\left(x \pm \sqrt{x^2-1}\right)$; $1 \le x < \infty$

3. $\tanh^{-1} x = \frac{1}{2} ln \frac{1+x}{1-x}$; $|x| < 1$

4. $\coth^{-1} x = \frac{1}{2} ln \frac{1-x}{1+x}$; $|x| > 1$

5. $\operatorname{sec}h^{-1} x = \pm ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$; $0 < x \le 1$

6. $\cos ech^{-1} x = \pm ln\left(\frac{1+\sqrt{x^2+1}}{x}\right)$; $x > 0$

EXERCISES (1)

1. **Given** $f(x)=3x+2,$ **find** $f(1)$**,** $f(-2)$**,** $f(x^2)$ **and** $f(x+h)$**.**
2. **Given** $f(x)=x^2,$ **find** $f(3)$**,** $f(-2)$**,** $f(a)$**,** $f(\sqrt{x})$ **and** $f(x+h)$**.**
3. **Given** $f(x)=\sqrt{x},$ **find** $f(4)$**,** $f(x^2)$ **and** $f(a^2+h^2)$**.**

4. **Find the domain of the following functions :**

   a) $f(x)=\sqrt{x-2}$ b) $f(x)=\dfrac{1}{x}$

5. **find the sum, different, product and quotient of the two functions** f **and** g **in each of the following exercises. Determine the domain of the resulting functions**

   a) $f(x)=x^2+1;$ $g(x)=\sqrt{x}$

6. **Given** $f(x)=x^2$ **and** $g(x)=\sqrt{x-1}$**, evaluate each of the following**

   a) $(f\circ g)\left(\dfrac{5}{4}\right)$ b) $(g\circ f)(-2)$

7. **If** $f(x)=1/(2x+1)$ **and** $g(x)=-\sqrt{x}$ **, evaluate each of the following**

   a) $(f\circ g)(-1)$ b) $(g\circ f)(0)$
   c) $(g\circ f)(-1)$

8. **Determine** $(f\circ g)(x)$ **and** $(g\circ f)(x)$ **in the following exercises**

   a) $f(x)=\dfrac{1}{x+1};$ $g(x)=\sqrt{x}+1$
   b) $f(x)=x^2+2;$ $g(x)=x-3$
   c) $f(x)=3x-1;$ $g(x)=\dfrac{x+1}{3}$

9. **Find the explicit function or functions corresponding to the following implicit relations**

   a) $5x-2y=20$ b) $3xy+2x-4y=1$
   c) $4x^2+9y^2=36$ d) $\sqrt{x}+\sqrt{y}=1$

10. **Find the inverse of each of the following functions**

    a) $y=x-1$ b) $y=\sqrt{\dfrac{1}{4}x}+2$
    c) $y=(3-2x)^2$ d) $y=\sqrt{x^2+1}$

Chapter 2

LIMITS AND CONTINUITY

In chapter 2 we study the notions of limits and continuity. These concepts are fundamental to the main subjects of calculus: The derivative and the integral.

2.1. Definition Of Limit :

Definition 1:

Let L be a real number and $f(x)$ be a function that is defined for all values of x close to a, except possibly at the point a itself. We say that the limit as x approaches to a of $f(x)$ is L, written

$$\lim_{x\to a} f(x) = L$$

The value of some limits are easy to guess. For example, if x gets close to 1, then $x+1$ gets close to 2.
This suggests that:

$$\lim_{x\to 1}(x+1) = 2$$

In the same manner, as x approaches 4, $\sqrt{x}$ approaches 2 and therefore $\sqrt{x}+3$ approaches 5, this means that $\dfrac{1}{\sqrt{x}+3}$ approaches $\dfrac{1}{5}$, so

$$\lim_{x\to 4}\frac{1}{\sqrt{x}+3} = \frac{1}{5}$$

In this preceding cases we were able to find the limits without much difficulty. However, finding certain limits requires some cleverness. The following example illustrates another case when direct substitution does not work.

Example 1 : If $f(x) = \dfrac{x^2-1}{x-1}$, find $\lim\limits_{x\to 1} f(x)$

Solution:

Since x^2-1 and $x-1$ both approach 0 as x approach 1, it might seem that the limit of quotient should be $\dfrac{0}{0}$. But $\dfrac{0}{0}$ is undefined. As a result, we must try another approach.
Notice that

$$\frac{x^2-1}{x-1}=\frac{(x-1)(x+1)}{(x-1)}=x+1, \text{ for } x\neq 1$$

Consequently it seems reasonable that

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}(x+1)=2$$

Example 2 : If $f(x)=\dfrac{2x^2-5x+2}{5x^2-7x-6}$, find $\lim_{x\to 2} f(x)$.

Solution : The number 2 is not in the domain of f since the meaningless expression $\frac{0}{0}$ is obtained if 2 is substituted for x. Factoring the numerator and the denominator gives us.

$$f(x)=\frac{(x-2)(2x-1)}{(x-2)(5x+3)}$$

We cannot cancel the factor $(x-2)$ at this stage; however, if we take the limit of $f(x)$ as $x\to 2$, this cancellation is allowed, because $x\neq 2$ and hence $x-2\neq 0$. Thus,

$$\lim_{x\to 2} f(x)=\lim_{x\to 2}\frac{2x^2-5x+2}{5x^2-7x-6}$$

$$=\lim_{x\to 2}\frac{(x-2)(2x-1)}{(x-2)(5x+3)}$$

$$=\lim_{x\to 2}\frac{2x-1}{5x+3}=\frac{3}{13}$$

Definition 2 : One-side limits

Let L be a real number.

(i) Suppose that $f(x)$ is defined near a for $x>a$ and that as x gets close to a (with $x>a$), $f(x)$ gets close to L. Then we say that L is the right-hand limit of $f(x)$ as x approaches a and we write

$$\lim_{x\to a^+} f(x)=L$$

(ii) Suppose that $f(x)$ is defined near a for $x<a$ and that x gets close to a (with $x<a$), $f(x)$ gets close to L. Then we say that L is the left-hand limit of $f(x)$ as x approaches a, and we write

$$\lim_{x\to a^-} f(x)=L$$

Example 3 :

Find $\lim_{x\to 0^+} \frac{|x|}{x}$ and $\lim_{x\to 0^-} \frac{|x|}{x}$

Solution :

Since,

$$|x| = \begin{cases} x & x \ge 0 \\ -x & x < 0 \end{cases}$$

Thus

$$\frac{|x|}{x} = \begin{cases} 1 & x \ge 0 \\ -1 & x < 0 \end{cases}$$

Consequently,

$\lim_{x\to 0^+} \frac{|x|}{x} = 1$ and $\lim_{x\to 0^-} \frac{|x|}{x} = -1$

Example 4 : If $f(x) = \begin{cases} x^2 & , x < 1 \\ x & , x \ge 1 \end{cases}$

Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$

Solution :

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x = 1$

and

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1$

In example 3, we saw that the $\lim_{x\to 0} \frac{|x|}{x}$ does not exist, since we get different values when $x \to 0$ from the left and from the right. But, in example 4 the $\lim_{x\to 1} f(x)$ exists because the right and left-hand limits are equal. In general, we have the following theorem

Theorem 1:

$\lim_{x\to a} f(x) = L$ exists if and only if the following hold:

(i) $\lim_{x\to a^+} f(x)$ exists.

(ii) $\lim_{x \to a^-} f(x)$ exists.

(iii) $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$ exists.

Example 5 : Let $f(x) = \begin{cases} x+1 & , x > 0 \\ x-1 & , x \le 0 \end{cases}$

Compute $\lim_{x \to 0} f(x)$, if it exists, explain why there is no limit of it does not exists.

Solution :

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x+1) = 1$

and

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x-1) = -1$

Since these limits are different, $\lim_{x \to 0} f(x)$ does not exist.

2.2. Limit Theorems:

The purpose of this section is to introduce theorems that may be used to simplify problems involving limits.

The following theorem gives five basic values for finding limits of combinations of functions,

Theorem 2 :

If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then

(i) $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

(ii) $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

(iii) $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$, for any number c

(iv) $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

(v) $\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \lim_{x \to a} f(x) \Big/ \lim_{x \to a} g(x)$, Provide $\lim_{x \to a} g(x) \ne 0$.

Example 6 : Evaluate the following limits

(a) $\lim_{x\to -1} x^2$ (b) $\lim_{x\to -1}(4x+x^2)$ (c) $\lim_{x\to -1}\frac{x^2}{x+3}$

Solution :

(a) Since $\lim_{x\to -1} x=-1$, then by Theorem 2 (iv)

$\lim_{x\to -1} x^2=\lim_{x\to -1} x.\lim_{x\to -1} x=(-1)(-1)=1$.

(b) $\lim_{x\to -1}(4x+x^2)=4\lim_{x\to -1} x+\lim_{x\to -1} x^2$

$=4(-1)+(-1)^2=-4+1=-3$

(c) $\lim_{x\to -1}\frac{x^2}{x+3}=\lim_{x\to -1} x^2\Big/\lim_{x\to -1} x+3=\frac{1}{2}$

Example 7: Show that $\lim_{x\to 0}\frac{1}{x}$ does not exists

Solution

Supposed that $\lim_{x\to 0}\frac{1}{x}$ exists and let $L=\lim_{x\to 0}\frac{1}{x}$. Since $1=x\left(\frac{1}{x}\right)$, by Theorem 1. (iv)

$1=\lim_{x\to 0} 1=\lim_{x\to 0} x\left(\frac{1}{x}\right)=\lim_{x\to 0} x.\left(\lim_{x\to 0}\frac{1}{x}\right)=0.\ \ L=0$

Which is f else. Therefore $\lim_{x\to 0}\frac{1}{x}$ cannot exist.

Theorem 3:

If n is a positive integer, then

(i) $\lim_{x\to a} x^n=a^n$

(ii) $\lim_{x\to a}[f(x)]^n=[\lim_{x\to a} f(x)]^n$, provide $\lim_{x\to a} f(x)$ exists.

Example 8 : Find

(a) $\lim_{x\to 2}(3x+4)^2$ (b) $\lim_{x\to -2}(5x^3+3x^2-6)$

Solution :

(a) $\lim\limits_{x\to 2}(3x+4)^5 = [\lim\limits_{x\to 2} 3x+4]^5$

$= [3(2)+4]^5 = 10^5$

(b) $\lim\limits_{x\to -2}\left(5x^3+3x^2-6\right) = \lim\limits_{x\to -2} 5x^3 + \lim\limits_{x\to -2} 3x^2 + \lim\limits_{x\to -2} -6$

$= 5\lim\limits_{x\to -2} x^3 + 3\lim\limits_{x\to -2} x^2 - 6$

$= 5(-2)^3 + 3(-2) - 6 = -34$

Theorem 4 : If f is a polynomial and a is a real number then $\lim\limits_{x\to a} f(x) = f(a)$

Corollary 1

If q is a rational function and a in the domain of q, then

$$\lim_{x\to a} q(x) = q(a)$$

Remark : Corollary ١ also remains true if q is trigonometric, exponential, or logarithmic function.

Example 9 : Find $\lim\limits_{x\to -2}\dfrac{x^3+2x^2-x-2}{x^2-4}$

Solution :

Since $\lim\limits_{x\to -2} x^2-4=0$, thus

$$\lim_{x\to -2}\frac{x^3-2x^2-x-2}{x^2-4} = \lim_{x\to -2}\frac{(x+2)(x^2-1)}{(x+2)(x-2)}$$

$$= \lim_{x\to -2}\frac{x^2-1}{x-2} = \frac{-3}{4}$$

Theorem 5: If $a > 0$ and n is a positive integer, or if $a \leq 0$ and n is an odd positive integer, then

$$\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$$

Corollary 2:

If m, n are positive integer and $a > 0$, then

$$\lim_{x \to a} \left(\sqrt[n]{x}\right)^m = \left(\lim_{x \to a} \sqrt[n]{x}\right)^m = \left(\sqrt[n]{a}\right)^m$$

Example 10 : Find $\lim_{x \to 8} \dfrac{x^{2/3} + 3\sqrt{x}}{4 - (16/x)}$

Solution :

$$\lim_{x \to 8} \frac{x^{2/3} + 3\sqrt{3}}{4 - (16/x)} = \lim_{x \to 8}\left(x^{2/3} + 3\sqrt{x}\right) \Big/ \lim_{x \to 8}\left(4 - (16/x)\right)$$

$$= \left(\lim_{x \to 8} x^{2/3} + \lim_{x \to 8} 3\sqrt{x}\right) \Big/ \left(\lim_{x \to 8} 4 - \lim_{x \to 8} (16/x)\right)$$

$$= \frac{8^{2/3} + 3\sqrt{8}}{4 - 16/8} = \frac{4 + 6\sqrt{2}}{4 - 2} = 2 + 3\sqrt{2}$$

Example 11 : Evaluate $\lim_{x \to 9} \dfrac{x(\sqrt{x} - 3)}{x - 9}$

Solution :

$$\lim_{x \to 9} \frac{x(\sqrt{x} - 3)}{x - 9} = \lim_{x \to 9} \frac{x(\sqrt{x} - 3)}{(\sqrt{x} - 3)(\sqrt{x} + 3)}$$

$$= \lim_{x \to 9} \frac{x}{\sqrt{x} + 3} = \frac{3}{2}$$

Theorem 6:

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

Corollary 3 :

$(i)\ \lim_{x\to 0}\frac{\tan x}{x}=1$

$(ii)\ \lim_{x\to 0}\frac{\sin mx}{\sin nx}=\frac{m}{n}$

$(iii)\ \lim_{x\to 0}\frac{\tan mx}{\tan nx}=\frac{m}{n}$

Example 12 : Show that $\lim_{x\to 0}\frac{\cos x-1}{x}=0$

Solution : Notice that $\lim_{x\to 0}x=0$, so we cannot apply the Quotients Rule directly, However,

$$\lim_{x\to 0}\frac{\cos x-1}{x}=\lim_{x\to 0}\left(\frac{\cos x-1}{x}\right)\left(\frac{\cos x+1}{\cos x+1}\right)$$

$$=\lim_{x\to 0}\frac{\cos^2 x-1}{x(\cos x+1)}=\lim_{x\to 0}\frac{-\sin^2 x}{x(\cos x+1)}$$

$$=\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{-\sin x}{\cos x+1}$$

$$=1.\lim_{x\to 0}\frac{-\sin x}{\cos x+1}=1.\frac{0}{1+1}=0$$

Example 13 : Evaluate $\lim_{x\to 0}\frac{\sin x}{x^{2/3}}=0$

Solution :

Since $\frac{\sin x}{x^{2/3}}=\frac{\sin x}{x}\cdot x^{\frac{1}{3}}$

Then $\lim_{x\to 0}\frac{\sin x}{x^{2/3}}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}x^{\frac{1}{3}}$

$$=1\cdot 0=0$$

Theorem 8 :

If $\lim_{x\to a} f(x) = c$ and $\lim_{y\to c} g(y)$ exists Then $\lim_{x\to a} g(f(x)) = \lim_{y\to c} g(y)$

Example 13 : Evaluate $\lim_{x\to 2}\sqrt{x+\frac{1}{x}}$

Solution : Let $y = x + \frac{1}{x}$. Then

$$\lim_{x\to 2} y = \lim_{x\to 2}\left(x+\frac{1}{x}\right) = 5/2$$

Apply Theorem 8, we find that

$$\lim_{x\to 2}\sqrt{x+\frac{1}{x}} = \lim_{y\to 5/2}\sqrt{y} = \sqrt{5/2}$$

Example 15 : Evaluate $\lim_{x\to\frac{\Pi}{3}}\cos\left(x+\frac{\Pi}{6}\right)$

Solution :

Let $y = x + \frac{\Pi}{6}$. The

$$\lim_{x\to\frac{\Pi}{3}} y = \frac{\Pi}{3} + \frac{\Pi}{6} = \frac{\Pi}{2}$$

Apply Theorem 8,

$$\lim_{x\to\frac{\Pi}{3}}\cos\left(x+\frac{\Pi}{6}\right) = \lim_{x\to\frac{\Pi}{3}}\cos y = \cos\frac{\Pi}{2} = 0$$

Theorem 9:

$$\lim_{x\to a}\frac{x^n - a^n}{x-a} = na^{n-1}$$

Example 16 : Evaluate $\lim_{x\to 1}\frac{x^5 - 1}{\sqrt{x}-1}$

Solution :

Divided both numerator and denominator by $x-1$ and apply Theorem 9 to obtain

$$\lim_{x\to 1}\frac{x^5-1}{\sqrt{x}-1}=\lim_{x\to 1}\frac{x^5-1/x-1}{\sqrt{x}-1/x-1}=\frac{5}{1/2}=10$$

Corollary 4 :

$$\lim_{x\to a}\frac{x^m-a^m}{x^n-a^n}=\frac{m}{n}a^{m-n}$$

Example 17 : Evaluate

(a) $\lim_{x\to 3}\frac{x^3-27}{x^2-9}$

(b) $\lim_{x\to 2}\frac{\sqrt[3]{x}-\sqrt[3]{2}}{\sqrt{x}-\sqrt{2}}$

Solution :

(a) $\lim_{x\to 3}\frac{x^3-27}{x^2-9}=\lim_{x\to 3}\frac{x^3-(3)^3}{x^2-(3)^2}=\frac{3}{2}\quad 3^{3-2}=\frac{9}{2}$

(b) $\lim_{x\to 2}\frac{\sqrt[3]{x}-\sqrt[3]{2}}{\sqrt{x}-\sqrt{2}}=\lim_{x\to 2}\frac{(x)^{\frac{1}{3}}-(2)^{\frac{1}{3}}}{(x)^{\frac{1}{2}}-(2)^{\frac{1}{2}}}$

$$=\frac{1/3}{1/2}\quad 2^{\frac{1}{3}-\frac{1}{2}}=\frac{2}{3}\quad 2^{-\frac{1}{6}}$$

Theorem 10 :

Let a be a real number and supposed the following:

(a) $f(x)\le g(x)\le h(x)$

(b) $\lim_{x\to a}f(x)=\lim_{x\to a}h(x)=L$

Then, $\lim_{x\to a}g(x)=L$

Example 18 : Evaluate $\lim_{x\to\infty}\frac{3+\cos 2x}{x-1}$

Solution :

Since $-1 \le \cos 2x < 1$, then

$$\frac{2}{x-1} \le \frac{3+\cos 2x}{x-1} \le \frac{4}{x-1}$$

But $\lim\limits_{x\to\infty} \frac{2}{x-1} = 0$ and $\lim\limits_{x\to\infty} \frac{4}{x-1} = 0$. Thus

$$\lim_{x\to\infty} \frac{3+\cos 2x}{x-1} = 0$$

To find $\lim\limits_{x\to\infty} f(x)$ or $\lim\limits_{x\to-\infty} f(x)$ for a relation function $f(x)$, first divide the numerator and the denominator of $f(x)$ by x^n, where n is the highest power of x that appears in the denominator, and then use limit theorems. This technique is illustrated in the next examples,

Example 19 : Find $\lim\limits_{x\to-\infty} \frac{2x^2-5}{3x^2+x+2}$

Solution :

We divided the numerator and the denominator by x^2 and then use limit theorems.

Thus, $\lim\limits_{x\to-\infty} \frac{2x^2-5}{3x^2+x+2} = \lim\limits_{x\to-\infty} \frac{2-\frac{5}{x^2}}{3+\frac{1}{x}+\frac{2}{x^2}}$

$$= \frac{2-0}{3+0+0} = \frac{2}{3}$$

Example 20 : Find $\lim\limits_{x\to\infty} \frac{2x^3-5}{3x^2+x+2}$

Solution :

The highest power of x in the denominator is 2, so we have.

$$\lim_{x\to\infty} \frac{2x^3-5}{3x^2+x+2} = \frac{\lim\limits_{x\to\infty} 2x-\frac{5}{x^2}}{\lim\limits_{x\to\infty} 3+\frac{1}{x}+\frac{2}{x^2}} = \frac{\infty}{3} = \infty$$

2.3. Continuous Functions :

Definition 2:

A function f is continuous at a number a if the following conditions are satisfied :

(i) $f(a)$ is defined
(ii) $\lim\limits_{x \to a} f(x)$ exists
(iii) $\lim\limits_{x \to a} f(x) = f(a)$

A function f is discontinuous at a number a if f is not continuous.

Example 21 : Let $f(x) = \dfrac{x^2 - 5x + 4}{x^2 - 9}$. Determine the numbers at which f is continuous.

Solution :

Since the denominator of f is 0 for $x = 3$ and $x = -3$, f is defined for all numbers of R except 3 and -3. Consequently f is continuous at every number except 3 and -3.

Example 22 : **Example 22 :** Let $f(x) = \dfrac{x^2}{x^2 + 1}$. Determine the numbers at which f is continuous.

Solution :

The denominator $x^2 + 1 \neq 0$ for each real number. Thus f defined for all x, and therefore f is continuous at every real number.

The next theorems states that polynomial functions and rational functions (quotients of polynomial functions) are continuous at every number in their domains.

Theorem 11 :

(i) A polynomial function f is continuous at every real number a.

(ii) A rational function $q = \frac{f}{g}$ is continuous at every number except the numbers a such that $g(a) = 0$.

Similarly, each of the six trigonometric functions $\sin x$, $\cos x$, $\tan x$, $\sec x$, $\csc x$ and $\tan x$ is continuous at every point in its domain.

Theorem 12 :

Suppose f and g are continuous at a and c is any number. Then $f+g$, cf, fg are continuous at a. If $g(a) \neq 0$, then f/g is continuous at a.

Example 23 : Let $f(x) = x\sin x + 1$. Show that $f(x)$ is continuous at every number.

Solution :

Since $g(x) = x$, $h(x) = \sin x$, $k(x) = 1$ are continuous functions at every member. Then

$$f = gh + k$$

is continuous, therefore $f(x) = x\sin x + 1$ is continuous at every number.

EXERCISES (2)

1. **Use theorems on limits to find the limit, if it exists.**

1- $\lim\limits_{x\to 4}\dfrac{2x-1}{3x+1}$

2- $\lim\limits_{x\to 1}(-2x+5)^4$

3- $\lim\limits_{x\to 4}\dfrac{6x-1}{2x-9}$

4- $\lim\limits_{x\to 64}\left(\sqrt[3]{y}+\sqrt{y}\right)^2$

5- $\lim\limits_{x\to 1}\left(\dfrac{x^2}{x-1}-\dfrac{1}{x-1}\right)$

6- $\lim\limits_{x\to -1}\dfrac{x^2-1}{x+1}$

7- $\lim\limits_{x\to 0}\dfrac{x}{2x}$

8- $\lim\limits_{x\to 2}\dfrac{x^2-4}{x^3-8}$

9- $\lim\limits_{x\to -1}\dfrac{1+1/y}{y+1}$

10- $\lim\limits_{x\to 0}x\left(x-\dfrac{1}{x}\right)$

11- $\lim\limits_{x\to 0}\dfrac{x^2-2}{\cos x}$

12- $\lim\limits_{y\to 0}\dfrac{\Pi\sin y\cos y}{y}$

13- $\lim\limits_{y\to 0}\dfrac{\sin y-\tan y}{y}$

14- $\lim\limits_{y\to 0}y\cot y$

15- $\lim\limits_{x\to \Pi}\dfrac{\tan^2 x}{1+\sec x}$

16- $\lim\limits_{x\to 0}\dfrac{1-\cos 3x}{\sin 3x}$

17- $\lim\limits_{x\to \infty}\dfrac{-x^3+2x}{2x^2-3}$

18- $\lim\limits_{x\to \infty}\dfrac{2-x^2}{x+3}$

19- $\lim\limits_{x\to a+b}\dfrac{(x-a)^4-b^4}{x-a-b}$

20- $\lim\limits_{x\to a}\dfrac{x^{11}-a^{11}}{x^5-a^5}$

2. Show that f is continuous at a:

i- $f(x)=3x^2+7-\dfrac{1}{\sqrt{-x}}$; $a=-2$

3. Explain why f is discontinuous at a :

i- $f(x)=\dfrac{3}{x+2}$; $a=-2$

ii- $f(x)=\begin{cases}\dfrac{|x-3|}{x-3} & \text{if } x\neq 3\\ 1 & \text{if } x=3\end{cases}$

Chapter 3

THE DERIVATIVES

The derivatives is defined as a limit, and it is used initially to compute rates of changes and slopes of tangent lines to curves. The study of derivatives is called differential calculus. Derivatives can be used in sketching graphs in finding the extreme (largest and smallest) values of function, and in many other important applications.

3.1. Evaluation of the Derivatives :

We shall now give a formal definition of the derivative

Definition 1:

Let $y = f(x)$ be a given function. Then the derivative of y with respect to x, denoted by dy/dx, is defined to be

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \quad \text{or} \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Provided that this limit exists.

The derivative is also given the name differential coefficient, and the operation of calculating the derivative of a function is called differentiation.

If the derivative of a function f exists at a particular value of x, then we say that f is differentiable at that point.

The derivative of $y = f(x)$ with respect to x is also denoted by any one of the following symbols.

$$\frac{d}{dx}(y), \quad \frac{df}{dx}, \quad \frac{d}{dx}(f), \quad y', \quad f'(x), \quad D_x y, \quad D_x f$$

Every one of these notation means exactly the same thing as dy/dx.

Note : In order to calculate the derivative dy/dx, we can proceed as follows :

1- Calculate $y = f(x)$ and $y + \Delta y = f(x + \Delta x)$.
2- Subtract the first from the second to get Δy and simplify the result.
3- Divide Δy by Δx and then take the limit of the resulting expression as $\Delta x \to 0$.

Example 1 : Find $f'(x)$ if $f(x) = 2x^2 + 3x + 1$. Evaluate $f'(2)$ and $f'(-2)$.

Solution :

Let $y = f(x) = 2x^2 + 3x + 1$. Then

$$y + \Delta y = f(x + \Delta x) = 2(x + \Delta x)^2 + 3(x + \Delta x) + 1$$
$$= 2[x^2 + 2x \cdot \Delta x + (\Delta x)^2 + 3x + 3\Delta x + 1]$$
$$= 2x^2 + 4x \cdot \Delta x + 2(\Delta x)^2 + 3x + 3\Delta x + 1$$
$$= 2x^2 + 3x + 1 + \Delta x(4x + 3 + 2\Delta x).$$

Subtracting y from $y + \Delta y$, we have

$$\Delta y = \Delta x(4x + 3 + 2\Delta x)$$

and so $\Delta y / \Delta x = 4x + 3 + 2\Delta x$. Thus

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (4x + 3 + 2\Delta x) = 4x + 3$$

That is, $f'(x) = 4x + 3$.

When $x = 2$, $f'(2) = 4(2) + 3 = 11$;

When $x = -2$, $f'(-2) = 4(-2) + 3 = -5$.

Example 2 : Find $f'(x)$ if $f(x) = \sqrt{x}$.

Solution :

Let $y = f(x) = \sqrt{x}$. Then $y + \Delta y = f(x + \Delta x) = \sqrt{x + \Delta x}$,

So, $\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$

Thus

$$\frac{\Delta y}{\Delta x} = \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}$$

We wish to take limit as $\Delta x \to 0$; before doing so, we must rationalize the numerator. We do this by multiplying numerator and denominator by $\left(\sqrt{x + \Delta x} + \sqrt{x}\right)$.

$$\frac{\Delta y}{\Delta x} = \frac{\left(\sqrt{x + \Delta x} - \sqrt{x}\right)\left(\sqrt{x + \Delta x} + \sqrt{x}\right)}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)} = \frac{\left(\sqrt{x + \Delta x}\right)^2 - \left(\sqrt{x}\right)^2}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)}$$
$$= \frac{(x + \Delta x) - x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)} = \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}}$$

Therefore

$\displaystyle\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$ Hence $f'(x) = 1/2\sqrt{x}$.

3.2. Geometric Interpretation of the Derivative :

We have already seen that in case where the independent variable in a function $y = f(t)$ represents time, the derivative dy/dt gives the instantaneous rate of change of y. For example, if $s = f(t)$ represents that distance traveled by a moving object, then ds/dt provides the instantaneous velocity. A part from this kind of application of derivatives, however, they also have a very great geometrical significance.

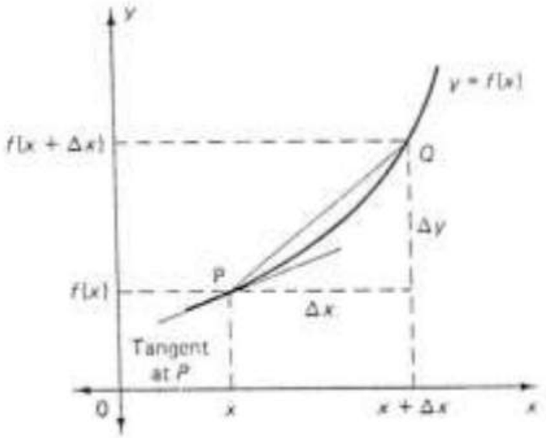
If P and Q are two points $(x, f(x))$ and $(x + \Delta x, f(x + \Delta x))$ on the graph of $y = f(x)$, then, the ratio

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Represents the slope of the line segment PQ. As Δx becomes smaller and smaller, the point Q moves closer and closer to P and the chord segment PQ becomes more and more nearly a tangent. As $\Delta x \to 0$, the slope of the chord PQ approaches the slope of the tangent line at P. Thus

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

Represents the slope of the tangent line to $y = f(x)$ at the point $P(x, f(x))$. As long as the curve $y = f(x)$ is "smooth" at P; that is, as long as we can draw a non vertical tangent at P, the limit will exist.



Thus, at any point a, $f'(a)$ determine the slope of the tangent line and the rate of change of a function.

- ***Slope of tangent line :*** The slope of the tangent line to the graph of the function $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$.
- ***Rate of change :*** If $y = f(x)$, the instantaneous rate of change of y with respect to x at a is $f'(a)$.

Example 3 :

Find the slope of the tangent line to the curve $y = x^3$ at the point (2,8).

Solution :

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - (x)^3}{\Delta x} = \lim_{x+\Delta x \to x} \frac{(x+\Delta x)^3 - (x)^3}{(x+\Delta x) - (x)} = 3x^{3-1} = 3x^2$$

$$\therefore \left(\frac{dy}{dx}\right)_{(2,8)} = 3(2)^2 = 12$$

Thus, the slope of the tangent line at (2,8) is 12.

3.3. General Rule of the Derivatives:

In the following theorem we give some general rules that simplify the task of finding derivative.

Theorem 1 :

Let f and g be differentiable functions and c is constant, then

(i) $\frac{d}{dx}(cf(x)) = c\frac{df(x)}{dx} = cf'(x)$

(ii) $\frac{d}{dx}(f(x) \pm g(x)) = \frac{df(x)}{dx} \pm \frac{dg(x)}{dx}$
$= f'(x) \pm g'(x)$

(iii) $\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$

(iv) $\frac{d}{dx}(f(x)/g(x)) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$, $\quad g(x) \neq 0$

(v) $\frac{d}{dx}x^n = nx^{n-1}$

Corollary 1 :

(i) $\frac{d}{dx}\frac{1}{x^n} = \frac{-n}{x^{n+1}}$

(ii) $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

Example 4 : Find the derivative of each of the following functions.

$(i)\ y = 3x^4 - \frac{1}{\sqrt[3]{x}}$ $\qquad (ii)\ y = \frac{3}{x^2} - \sqrt{x} + 7$

$(iii)\ y = (2x^2 + 3x)(3 - 2x)$ $\qquad (iv)\ y = \frac{\sqrt{x}}{\sqrt{x}+1}$

Solution :

$$(i)\ y = 3x^4 - \frac{1}{\sqrt[3]{x}} = 3x^4 - x^{\frac{-1}{3}}$$

$$\frac{dy}{dx} = \frac{d}{dx}(3x^4) - \frac{d}{dx}\left(x^{\frac{1}{3}}\right) = 3(4x^3) - \left(-\frac{1}{3}\right)x^{\frac{-4}{3}}$$

$$= 12x^3 + \frac{1}{3}x^{\frac{-4}{3}}$$

$$(ii)\ y = \frac{3}{x^2} - \sqrt{x} + 7$$

$$\frac{dy}{dx} = 3\frac{d}{dx}\frac{1}{x^2} - \frac{d}{dx}\sqrt{x} + \frac{d}{dx}7$$

$$= 3\left(\frac{-2}{x^3}\right) - \frac{1}{2\sqrt{x}} + 0 = \frac{-6}{x^3} - \frac{1}{2\sqrt{x}}$$

$$(iii)\ y = (2x^2 + 3x)(3 - 2x)$$

$$\frac{dy}{dx} = (2x^2 + 3x)\left(\frac{d}{dx}(3 - 2x)\right) + \left(\frac{d}{dx}(2x^2 + 3x)\right)(3 - 2x)$$

$$= (2x^2 + 3x)(-2) + (4x + 3)(3 - 2x) = -12x^2 + 9$$

$$(iv)\ y = \frac{\sqrt{x}}{\sqrt{x}+1}$$

$$\frac{dy}{dx} = \frac{d}{dx}\frac{\sqrt{x}}{\sqrt{x}+1}\frac{\sqrt{x}-1}{\sqrt{x}-1} = \frac{d}{dx}\frac{x-\sqrt{x}}{x-1}$$

$$= \frac{d}{dx}\frac{x-1+1-\sqrt{x}}{x-1} = \frac{d}{dx}\left(1 + \frac{1-\sqrt{x}}{x-1}\right) = \frac{d}{dx}(1) + \frac{d}{dx}\left(\frac{1-\sqrt{x}}{x-1}\right)$$

$$= 0 + \frac{(x-1)\frac{d}{dx}(1-\sqrt{x}) - (1-\sqrt{x})\frac{d}{dx}(x-1)}{(x-1)^2} = \frac{(x-1)\left(\frac{-1}{2\sqrt{x}}\right) - (1-\sqrt{x})}{(x-1)^2} = \frac{-x+1-(1-\sqrt{x})(2\sqrt{x})}{2\sqrt{x}(x-1)^2}$$

$$= \frac{-x+1-2\sqrt{x}+2x}{2\sqrt{x}(x-1)^2} = \frac{x-2\sqrt{x}+1}{2\sqrt{x}(x-1)^2} = \frac{(\sqrt{x}-1)^2}{2\sqrt{x}(x-1)^2}$$

3.4. The Derivatives of Composite Functions: (The chain rule)

Let f be differentiable functions and $y = f(u)$ and $u = g(x)$.

Then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

In general, if $y = f(u),\ u = g(v),\ v = h(x),$ Then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$

Example 5: Let $y = (1+\sqrt[3]{x})^2$. Find $\frac{dy}{dx}$

Solution : Consider $u = 1+\sqrt[3]{x}$, then

$$y = u^2$$

$$\frac{du}{dx} = \frac{d}{dx}(1+\sqrt[3]{x}) = \frac{1}{3}x^{\frac{-2}{3}}$$

$$\frac{dy}{du} = 2u\ .$$

Since $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$\frac{dy}{dx} = (2u) \cdot \left(\frac{1}{3}x^{\frac{-2}{3}}\right)$$

$$= 2(1+\sqrt[3]{x})\left(\frac{1}{3}x^{\frac{-2}{3}}\right) = \frac{2}{3}x^{\frac{-2}{3}}(1+\sqrt[3]{x})$$

Corollary 2 :

If $y = [f(x)]^n$, then

$$\frac{dy}{dx} = n[f(x)]^{n-1}\frac{df(x)}{dx}$$

Example 6 : Find y' for the following functions

$(i)\ y = (5x^2+3x-1)^6$

$(ii)\ y=\sqrt{x^2+5x+1}$

Solution :

$(i)\ y=(5x^2+3x-1)^6$

$$y'=6(5x^2+3x-1)^5(10x+3)$$

$(ii)\ y=\sqrt{x^2+5x+1}$

$$y'=\frac{1}{2\sqrt{x^2+5x+1}}(2x+5)$$

Example 7: If $y=\dfrac{w-1}{w^2+1}$, $w=z^2-2$, $z=3x-4$ Find $\dfrac{dy}{dx}$.

Solution :

$$\frac{dy}{dx}=\frac{dy}{dw}\frac{dw}{dz}\frac{dz}{dx}=\frac{(w^2+1)(1)-(w-1)(2w)}{(w^2+1)^2}(2z)\cdot 3$$

$$=6(3x-4)\frac{1+2\left[(3x-4)^2-2\right]-\left[(3x-4)^2-2\right]^2}{\left[\left((3x-4)^2-2\right)^2+1\right]^2}$$

3.5. Implicit Differentiation:

An equation in which y appears alone on one side of the equality sign and no y appears on the other side is said to define y explicitly. Thus, each of the equations $y=\sin x$ or $y=x^2+2x+5$ define y explicitly. By contrast, the function which appears in the form $f(x,y)=0$ in which we can not express y explicitly as a function of the independent variable x. To find the derivative $\dfrac{dy}{dx}$ of this type of functions we differentiate $f(x,y)$ directly with respect to x to obtain the equation

$$f\left(x,y,\frac{dy}{dx}\right)=0$$

which is a linear equation and so we have

$$\frac{dy}{dx}=g(x,y)$$

The process of differentiating $f(x,y)$ in this case is called implicit differentiation.

Example 8 : If $y = f(x)$, where f is determined implicitly by the equation $y^4 + 3y - 4x^3 = 5x + 1$, find y'

Solution :

Differentiating both sides of the equation $y^4+3y-4x^3=5x+1$ whhith respect to x, obtaining:

$$\frac{d}{dx}\left(y^4+3y-4x^2\right)=\frac{d}{dx}(5x+1)$$

$$4y^3y'+3y'-8x=5$$

$$y'=\frac{8x+5}{4y^3+3},\ \text{if}\ y\neq-\sqrt[3]{\frac{3}{4}}$$

Example 9 : Find y' if $4x^2+9y^2-36=0$

Solution :

Differentiating both sides of the equation with respect to x yields.

$$\frac{d}{dx}\left(4x^2+9y^2-36\right)=\frac{d}{dx}(0)$$

$$8x+18yy'=0$$

Which implies $y'=\dfrac{-8x}{18y}=\dfrac{-4x}{9y}$

3.6. Higher Derivative:

Let $y=f(x)$ be given function of x with derivative $\dfrac{dy}{dx}=f'(x)$. We call this the **first derivative** of y with respect to x. If $f'(x)$ is differentiable function of x, its derivative is called the **second derivative** of y with respect to x. If the second derivative is a differentiable function of x, its derivative is called the **third derivative** of y, and so on

The first and all higher derivatives of y with respect to x are generally denoted by one of the following types of nation.

$$\frac{dy}{dx}\ ,\ \frac{d^2y}{dx^2}\ ,\ \frac{d^3y}{dx}\ ,\ \dots\ ,\ \frac{d^ny}{dx}$$

$$y'\ ,\ y''\ ,\ y'''\ ,\ \dots\ ,\ y^{(n)}$$

$$f'(x)\ ,\ f''(x)\ ,\ f'''(x)\ ,\ \dots\ ,\ f^{(n)}(x)$$

Example 10 : Find the third derivatives of $y=\dfrac{x}{x-3}$

Solution :

$$y' = \frac{(x-3)-x}{(x-3)^2} = \frac{-3}{(x-3)^2} = -3(x-3)^{-2}$$
$$y'' = 6(x-3)^{-3}$$
$$y''' = -18(x-3)^{-4} = \frac{-18}{(x-3)^4}$$

Example 11: If $y^2 - 2x^3 - 3x = 1$, prove that $yy'' + y'^2 - 6x = 0$

Solution :

$$y^2 - 2x^3 - 3x = 1$$
$$2yy' - 6x^2 - 3 = 0$$
$$2\left[yy'' + y'^2\right] - 12x = 0$$
$$yy'' + y'^2 - 6x = 0$$

3.7. Derivative of Trigonometric Functions:

a) If $y = \sin x$ then $\frac{dy}{dx} = \cos x$

Proof :

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \cdot 2\cos\left(\frac{x+\Delta x + x}{2}\right)\sin\left(\frac{x+\Delta x - x}{2}\right)$$
$$= \lim_{\Delta x \to 0} \cos\left(x + \frac{\Delta x}{2}\right) \cdot \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} = \cos x \cdot 1 = \cos x$$

b) If $y = \cos x$ then $\frac{dy}{dx} = -\sin x$

Proof :

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos(x+\Delta x) - \cos(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[-2\sin\frac{(x+\Delta x+x)}{2}\sin\frac{(x+\Delta x-x)}{\Delta x}\right]$$

$$= -\lim_{\Delta x \to 0} \sin\left(x + \frac{\Delta x}{2}\right) \cdot \frac{\sin(\frac{\Delta x}{2})}{\frac{\Delta x}{2}} = -\sin x \cdot 1 = -\sin x$$

d) If $y = \tan x$ then $\frac{dy}{dx} = \sec^2 x$

Proof :

$$\frac{dy}{dx} = \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

e) If $y = \cot x$ then $\frac{dy}{dx} = -\cos ec^2 x$

Proof :

$$y = \cot x = \frac{\cos x}{\sin x}$$

$$\frac{dy}{dx} = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}$$

$$= -\cos ec^2 x$$

f) If $y = \sec x$ then $\frac{dy}{dx} = \sec x \tan x$

Proof :

$$y = \sec x = \frac{1}{\cos x}$$

$$\frac{dy}{dx} = \frac{\cos x(0) - 1(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x}$$

$$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \cdot \tan x$$

g) If $y = \cos ec\, x$ then $\frac{dy}{dx} = -\cos ec\, x \ \cot x$

Proof :

$$y = \cos ec\, x = \frac{1}{\sin x}$$

$$\frac{dy}{dx} = \frac{\sin x(0) - 1 \cdot \cos x}{\sin^2 x} = -\frac{\cos x}{\sin^2 x}$$

$$= -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\cos ec\, x \cdot \cot x$$

Note: If $y = \sin u$, $u = u(x)$

Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \cos u \cdot \frac{du}{dx}$

y	y'
$y = \sin u$	$y' = \cos u \cdot \frac{du}{dx}$
$y = \cos u$	$y' = -\sin u \cdot \frac{du}{dx}$
$y = \tan u$	$y' = \sec^2 u \cdot \frac{du}{dx}$
$y = \cot u$	$y' = -\cos ec^2 u \cdot \frac{du}{dx}$
$y = \sec u$	$y' = \sec u \cdot \tan u \cdot \frac{du}{dx}$
$y = \cos ec\, u$	$y' = -\cos ec\, u \cdot \cot u \cdot \frac{du}{dx}$

Example 12 : Find the derivative for the following functions

(i) $y = \frac{\sin^2 x}{x+2}$ (ii) $y = \cos^2 x \sin x$

(iii) $y = \frac{x - \tan x}{x + \tan x}$ (iv) $y = (2\sec x - 3\tan x)$

(v) $y = \frac{\sqrt{x} + \cos ec\, x}{\sqrt{x} - \cos ec\, x}$

Solution :

(i) $y = \frac{\sin^2 x}{x+2}$

$$y'=\frac{(x+2)[2\sin x\cos x]-\sin^2 x\ (1)}{(x+2)^2}$$

(ii) $y=\cos^2 x\sin x \Rightarrow$

$$y'=\cos^2 x[\cos x]+(\sin x)[2\cos x(-\sin x)]$$

(iii) $y=\dfrac{x-\tan x}{x+\tan x}$

$$y'=\frac{(x+\tan x)(1-\sec^2 x)-[x-\tan x][1+\sec^2 x]}{[x+\tan x]^2}$$

(iv) $y=(2\sec x-3\tan x)$

$$y'=2\sec x\tan x-3\sec x^2$$

(v) $y=\dfrac{\sqrt{x}+\cos ecx}{\sqrt{x}-\cos ecx}$

$$y'=\frac{(\sqrt{x}-\operatorname{cosec} x)(\frac{1}{2\sqrt{x}}-\operatorname{cosec} x\cot x)}{[\sqrt{x}-\operatorname{cosec} x]^2}-$$

$$\frac{(\sqrt{x}+\operatorname{cosec} x)(\frac{1}{2\sqrt{x}}-\operatorname{cosec} x\cot x)}{[\sqrt{x}-\operatorname{cosec} x]^2}$$

Example 13 :

Find $\dfrac{dy}{dx}$ for the following functions :

(i) $y=\sin^2 3x+\cos^2\sqrt{x}$ (ii) $y=\tan\left(\dfrac{x}{1-x}\right)$

(iii) $y=\sqrt{\dfrac{1+\sin x}{1-\sin x}}$ (iv) $y=\sec^2\left(\dfrac{1}{\sqrt{x}}\right)+2\cos ec(3x)$

Solution :

(i) $y=\sin^2 3x+\cos^2\sqrt{x}$

$$y'=2\sin 3x\cdot\cos 3x(3)+2\cos\sqrt{x}\cdot(-\sin\sqrt{x})\cdot\frac{1}{2\sqrt{x}}$$

(ii) $y=\tan\left(\dfrac{x}{1-x}\right)$

$$y'=\sec^2\left(\frac{x}{1-x}\right)\cdot\frac{(1-x)\cdot 1-x(-1)}{(1-x)^2}$$

$$=\left[\sec^2\left(\frac{x}{1-x}\right)\right]\left[\frac{1}{(1-x)^2}\right]$$

(iii) $y=\sqrt{\dfrac{1+\sin x}{1-\sin x}}$

$$y'=\frac{1}{2\sqrt{\dfrac{1+\sin x}{1-\sin x}}}\cdot\frac{(1-\sin x)(\cos x)-(1+\sin x)(-\cos)}{(1-\sin x)^2}$$

(iv) $y=\sec^2\left(\dfrac{1}{\sqrt{x}}\right)+2\cos ec(3x)$

$$y'=2\sec\frac{1}{\sqrt{x}}\cdot\sec\frac{1}{\sqrt{x}}\cdot\tan\frac{1}{\sqrt{x}}\left(-\frac{1}{2\sqrt{x^3}}\right)$$

$$+2[-\cos ec3x\cot 3x.3]$$

$$=-x^{-3/2}\sec^2\left(\frac{1}{\sqrt{x}}\right)\tan\left(\frac{1}{\sqrt{x}}\right)-6\cos ec(3x)\cot(3x)$$

Example 14 : Find $\dfrac{dy}{dx}$:

(i) $\tan x+\tan y=1$ $\qquad$ (ii) $y=\sin(x-y)$

(iii) $x^2+y^2+3x+y+\sec y=0$ $\qquad$ (iv) $y=\cot(2x+y)$

Solution :

(i) $\tan x+\tan y=1\Rightarrow \sec^2 x+\sec^2 y\cdot y'=0\Rightarrow$

$$y'=-\frac{\sec^2 x}{\sec^2 y}$$

(ii) $y=\sin(x-y)\Rightarrow y'=\cos(x-y)[1-y']$

$\therefore y'+y'\cos(x-y)=\cos(x-y)$

$$y'=\frac{\cos(x-y)}{1+\cos(x-y)}$$

(iii) $x^2+y^2+3x+y+\sec y=0\Rightarrow$

$$2x+2yy'+3+y'+\sec y\tan y\cdot y'=0$$

$$(2y+1+\sec y\tan y)y'=-2x-3$$

$$\therefore y'=-\frac{(2x+3)}{2y+1+\sec y\tan y}$$

(iv) $y=\cot(2x+y)\Rightarrow$

$$y'=-\cos ec^2(2x+y)[2+y']$$

$$y'\left[1+\cos ec^2(2x+y)\right]=-2\cos ec^2(2x+y)$$

$$\therefore y'=\frac{-2\cos ec^2(2x+y)}{1+\cos ec^2(2x+y)}$$

3.8. Derivative of Inverse Trigonometric Functions:

(a) If $y=\sin^{-1}x$ then $\dfrac{dy}{dx}=\dfrac{1}{\sqrt{1-x^2}}$

Proof :

$$y=\sin^{-1}x \Rightarrow \sin y = x$$

$$\therefore \cos y\cdot y'=1 \Rightarrow y'=\frac{1}{\cos y}=\frac{1}{\sqrt{1-\sin^2 y}}$$

$$\therefore \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$$

(b) If $y=\cos^{-1}x$ then $\dfrac{dy}{dx}=\dfrac{-1}{\sqrt{1-x^2}}$

Proof : As (a)

(c) If $y=\tan^{-1}x$ then $\dfrac{dy}{dx}=\dfrac{1}{1+x^2}$

Proof :

$$y=\tan^{-1}x \Rightarrow \tan y = x$$

$$\sec^2 y\cdot y'=1 \Rightarrow y'=\frac{1}{\sec^2 y}$$

$$\therefore y'=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2}$$

(d) If $y=\cot^{-1}x$ then $\dfrac{dy}{dx}=\dfrac{-1}{1+x^2}$

Proof : As (c)

(e) If $y=\sec^{-1}x$ then $\dfrac{dy}{dx}=\dfrac{1}{x\sqrt{x^2-1}}$

Proof :

$$y=\sec^{-1}x \Rightarrow \sec y = x$$

$$\therefore \sec y\cdot\tan y\cdot y'=1 \Rightarrow y'=\frac{1}{\sec y\cdot\tan y}$$

$$\therefore y'=\frac{1}{\sec y\sqrt{\sec^2 y-1}}=\frac{1}{x\sqrt{x^2-1}}$$

(f) If $y=\cos^{-1}x$ then $\dfrac{dy}{dx}=\dfrac{-1}{x\sqrt{x^2-1}}$

Proof : As (e)

y	y'
$y=\sin^{-1}u$	$y'=\dfrac{1}{\sqrt{1-u^2}}\cdot u'$
$y=\cos^{-1}u$	$y'=\dfrac{-1}{\sqrt{1-u^2}}\cdot u'$
$y=\tan^{-1}u$	$y'=\dfrac{1}{1+u^2}\cdot u'$
$y=\cot^{-1}u$	$y'=\dfrac{-1}{1+u^2}\cdot u'$
$y=\sec^{-1}u$	$y'=\dfrac{1}{u\sqrt{u^2-1}}\cdot u'$
$y=\cos ec^{-1}u$	$y'=\dfrac{-1}{u\sqrt{u^2-1}}\cdot u'$

Example 15 :

Find $\dfrac{dy}{dx}$ for the following functions:

(i) $y=\sin^{-1}(3x+1)+\cos^{-1}\left(\sqrt{x}+\dfrac{1}{x}\right)$

(ii) $xy=\tan^{-1}(y/x)$

(iii) $y=\sec^{-1}\dfrac{\sqrt{1+x^2}}{x}$

(iv) $y=\cos ec^{-1}x^2+\cot-\sqrt{x}$

Solution :

$(i)\ \ y=\sin^{-1}(3x+1)+\cos^{-1}(\sqrt{x}+\frac{1}{x})$

$$y'=\frac{1}{\sqrt{1-(3x+1)^2}}(3)+\frac{-1}{\sqrt{1-\left(\sqrt{x}+\frac{1}{x}\right)^2}}\left(\frac{1}{2\sqrt{x}}-\frac{1}{x^2}\right)$$

$(ii)\ \ xy=\tan^{-1}(y/x)$

$$xy'+y\cdot 1=\frac{-1}{1+(y/x)^2}\cdot\frac{xy'-y\cdot 1}{x^2}$$

$$\therefore xy'+y=\frac{1}{x^2+y^2}(y-xy')$$

$$y'\left(x-\frac{x}{x^2+y^2}\right)=\frac{-y}{x^2+y^2}-y=\frac{-y}{x^2+y^2}\left(1+x^2+y^2\right)$$

$$\therefore y'=\frac{y\left(1+x^2+y^2\right)}{x\left(1-x^2-y^2\right)}$$

$(iii)\ \ y=\sec^{-1}\frac{\sqrt{1+x^2}}{x}\Rightarrow y=\sec^{-1}u,\quad (u=\frac{\sqrt{1+x^2}}{x})$

$$\therefore u'=\frac{x\dfrac{1}{2\sqrt{1+x^2}}(2x)-\sqrt{1+x^2}\cdot 1}{x^2}=\frac{-1}{x^2\sqrt{1+x^2}}$$

$$\therefore \frac{dy}{dx}=\frac{1}{u\sqrt{u^2-1}}\cdot u'$$

$$=\frac{1}{\dfrac{\sqrt{1+x^2}}{x}\sqrt{\dfrac{1+x^2}{x^2}-1}}\cdot\frac{-1}{x^2\sqrt{1+x^2}}$$

$$=\frac{-1}{\left(1+x^2\right)}$$

3.9. Derivative of the Logarithmic Functions:

If $y=\log_a x$ then $\dfrac{dy}{dx}=\dfrac{1}{x\ln a}$

Proof :

$\boldsymbol{y}=\log_{\boldsymbol{a}}\boldsymbol{x}$

$$\therefore y'=\lim_{\Delta x\to 0}\frac{\log_a(x+\Delta x)-\log_a x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \log_a \left(\frac{x + \Delta x}{x} \right)$$

$$= \lim_{\Delta x \to 0} \frac{1}{x} \cdot \frac{x}{\Delta x} \log_a \left(\frac{x + \Delta x}{x} \right)$$

$$= \lim_{\Delta x \to 0} \frac{1}{x} \log_a \left(1 + \frac{\Delta x}{x} \right)^{\frac{x}{\Delta x}}$$

$$= \frac{1}{x} \log_a e \quad \text{or} \quad y' = \frac{1}{x \ln a}$$

Special case :

If $a = e$, then $y = \ln x \Rightarrow y' = \frac{1}{x \ln e} = \frac{1}{x}$

3.10. Derivative of Exponential Functions:

If $y = a^x$ then $\frac{dy}{dx} = a^x \ln a$

Proof :

$$y = a^x \Rightarrow x = \log_a y$$

$$\therefore 1 = \frac{1}{y \ln a} \cdot y' \Rightarrow y' = y \ln a$$

$$\therefore y' = a^x \ln a$$

Special case :

If $y = e^x$, then $y' = e^x$

y	y'
$y = \log_a u$	$y' = \frac{1}{u} \log_a e \cdot u'$
$y = \ln u$	$y' = \frac{1}{u} \cdot u'$
$y = a^u$	$y' = a^u \ln a \cdot u'$
$y = e^u$	$y' = e^u \cdot u'$

Example 16 :

Find the first derivative for the following functions :

$(i)\ \ y=\ln\left(x^{2}+5\right)$ $\qquad (ii)\ \ y=\log_{a}\sin x,$

$(iii)\ y=e^{\tan^{-1}\sqrt{x}}$ $\qquad (iv)\ \ y=8^{\sin^{-1}x}$

Solution :

$$(i)\ \ y=\ln\left(x^{2}+5\right)\Rightarrow y'=\frac{1}{x^{2}+5}(2x)$$

$$\therefore y'=\frac{2x}{x^{2}+5}$$

$$(ii)\ \ y=\log_{a}\sin x\Rightarrow y'=\frac{1}{\sin x}\log_{a}e\cdot\cos x$$

$$(iii)\ \ y=e^{\tan^{-1}\sqrt{x}}\Rightarrow$$

$$y'=e^{\tan^{-1}\sqrt{x}}\cdot\frac{1}{1+\left(\sqrt{x}\right)^{2}}\cdot\frac{1}{2\sqrt{x}}$$

$$(iv)\ \ y=8^{\sin^{-1}x}\Rightarrow y'=8^{\sin^{-1}x}\cdot\ln 8\cdot\frac{1}{\sqrt{1-x^{2}}}$$

Example 17 : Find $\frac{dy}{dx}$ if

$e^{3y}+\ln(2x+5)=10x-7y$

Solution :

$$e^{3y}3y'+\frac{2}{2x+5}=10-7y'$$

$$y'\left(3e^{3y}+7\right)=10-\frac{2}{2x+5}=\frac{20x+50-2}{2x+5}$$

$$\therefore y'=\frac{4(5x+12)}{\left(3e^{3y}+7\right)(2x+5)}$$

Example 18 : Find the first derivative :

$(i)\ \ y=x^{a}+a^{x}$, $(ii)\ y=x^{x}$, $(iii)\ \ y=(\sin x)^{\cos x}$

Solution :

$(i)\ y = x^a + a^x \Rightarrow y' = ax^{a-1} + a^x \ln a$

$(ii)\ y = x^x \Rightarrow \ln y = x \ln x$

$$\therefore \frac{1}{y} \cdot y' = x \cdot \frac{1}{x} + \ln x \cdot 1$$

$$\therefore y' = y(1 + \ln x) = x^x (1 + \ln x)$$

$(iii)\ y = (\sin x)^{\cos x}$

$$\ln y = \cos x \ln \sin x$$

$$\frac{1}{y} \cdot y' = \cos x \cdot \frac{1}{\sin x} \cdot \cos x + (\ln \sin x)(-\sin x)$$

$$\therefore y' = (\sin x)^{\cos x} [\cos x \cot x - \sin x \ln \sin x]$$

3.11. Derivative of Hyperbolic Functions:

(a) If $y = \sinh x$ then $y' = \cosh x$

Proof :

$$y = \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\therefore y' = \frac{e^x - (-e^{-x})}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$

(b) If $y = \cosh x$ then $y' = \sinh x$

Proof :

$$y = \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\therefore y' = \frac{e^x + (-e^{-x})}{2} = \frac{e^x - e^{-x}}{2}$$

$$\therefore y' = \sinh x$$

(d) If $y = \tanh x$ then $y' = \mathrm{sec}h^2 x$

Proof :

$$y = \tanh x = \frac{\sinh x}{\cosh x} \Rightarrow y' = \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x}$$

$$\therefore y' = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \sec h^2 x$$

(e) If $y = \coth x$ then $y' = -\cos ech^2 x$

Proof :

$$y = \coth x = \frac{\cosh x}{\sinh x}$$

$$\therefore y' = \frac{\sinh x \sinh x - \cosh x \cosh x}{\sinh^2 x}$$

$$\therefore y' = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-1}{\sinh^2 x} = -\cos ech^2 x$$

(f) If $y = \sec hx$ then $y' = -\sec hx \tanh x$

Proof :

$$y = \sec hx = \frac{1}{\cosh x}$$

$$\therefore y' = \frac{\cosh x(0) - 1 \cdot \sinh x}{\cosh^2 x}$$

$$\therefore y' = -\frac{\sinh x}{\cosh x} \cdot \frac{1}{\cosh x} = -\sec hx \tanh x$$

(g) If $y = \cos echx$ then $y' = -\cos echx \coth x$

Proof :

$$y = \cos echx = \frac{1}{\sinh x}$$

$$\therefore y' = \frac{-\cosh x}{\sinh^2 x}$$

$$\therefore y' = -\frac{1}{\sinh x} \cdot \frac{\cosh}{\sinh x} = -\cos echx \coth x$$

y	y'
$y=\sinh u$	$y'=\cosh u\cdot u'$
$y=\cosh u$	$y'=\sinh u\cdot u'$
$y=\tanh u$	$y'=\operatorname{sec}h^2 u\cdot u'$
$y=\coth u$	$y'=-\cos ech^2 u\cdot u'$
$y=\sec hu$	$y'=-\sec hu\cdot\tanh u\cdot u'$
$y=\cos echu$	$y'=-\cos echu\cdot\coth u\cdot u'$

Example 18 : Find $\frac{dy}{dx}$ for the following functions

$(i)\sinh(x^2+4)$ $(ii)\sec h(e^{2\sin x^2})$ $(iii)\coth(3x+5)$ $(iv)\, e^{\cos echx^2}$ $(v)\tan^{-1}\tanh x$

Solution :

$(i)\, y=\sinh(x^2+4)\Rightarrow y'=\cosh(x^2+4)\cdot(2x)$

$(ii)\, y=\sec h(e^{2\sin x^2})\Rightarrow$

$y'=-\sec h(e^{2\sin x})\tanh(e^{2\sin x})\, e^{2\sin x}\cdot 2\cos x$

$(iii)\, y=\coth(3x+5)\Rightarrow y'=-\cos ech^2(3x+5)(3)$

$(iv)\, y=e^{\cos echx^2}\Rightarrow$

$y'=e^{\cos echx^2}\left(-\cos echx^2\coth x^2\right)\cdot 2x$

$(v)\, y=\tan^{-1}\tanh x\Rightarrow$

$$y'=\frac{1}{1+\tanh^2 x}\cdot\sec h^2 x=\frac{\sec h^2 x}{1+\tanh^2 x}$$

3.12. Derivative of Inverse Hyperbolic Functions:

(a) If $y=\sinh^{-1}x$ then $y'=\dfrac{1}{\sqrt{1+x^2}}$

Proof :

$$y=\sinh^{-1}x \Rightarrow \sinh y = x \Rightarrow \cosh y \cdot y' = 1$$

$$\therefore y' = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}}$$

$$\therefore y' = \frac{1}{\sqrt{1+x^2}}$$

(b) If $y=\cosh^{-1}x$ then $y'=\dfrac{1}{\sqrt{x^2-1}}$

Proof :

$$y=\cosh^{-1}x \Rightarrow \cosh y = x \qquad \therefore \sinh y \cdot y' = 1$$

$$\therefore y' = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2-1}}$$

(d) If $y=\tanh^{-1}x$ then $y'=\dfrac{1}{1-x^2}$

Proof :

$$y=\tanh^{-1}x \Rightarrow \tanh y = x \Rightarrow \operatorname{sec}h^2 y \cdot y' = 1$$

$$\therefore y' = \frac{1}{\operatorname{sec}h^2 y} = \frac{1}{1-\tanh^2 y} = \frac{1}{1-x^2}$$

(e) If $y=\coth^{-1}x$ then $y'=\dfrac{-1}{x^2-1}$

Proof :

$$y=\coth^{-1}x \Rightarrow \coth y = x \Rightarrow -\operatorname{cosech}^2 y \cdot y' = 1$$

$$\therefore y' = \frac{-1}{\cos ech^2 y} = -\frac{1}{\coth^2 y - 1} = -\frac{1}{x^2 - 1}$$

(f) If $y = \sec h^{-1} x$ then $y' = -\dfrac{1}{x\sqrt{1-x^2}}$

Proof :

$$y = \sec h^{-1} x \Rightarrow \sec hy = x \Rightarrow -\sec hy \cdot \tanh y \cdot y' = 1$$

$$\therefore y' = -\frac{1}{\sec hy \cdot \tanh y} = -\frac{1}{\sec hy\sqrt{1-\sec h^2 y}}$$

$$= -\frac{1}{x\sqrt{1-x^2}}$$

(g) If $y = \cos ech^{-1} x$ then $y' = \dfrac{-1}{x\sqrt{1+x^2}}$

Proof :

$$y = \cos ech^{-1} x \Rightarrow \cos echy = x$$

$$\Rightarrow -\cos echy \cdot \coth y \cdot y' = 1$$

$$\therefore y' = -\frac{1}{-\cos echy \cdot \coth y} = -\frac{1}{\cos echy\sqrt{1+\cos echh^2 y}}$$

$$\therefore y' = -\frac{1}{x\sqrt{1+x^2}}$$

y	y'
$y = \sinh^{-1} u$	$y' = \dfrac{1}{\sqrt{1+u^2}} \cdot u'$
$y = \cosh^{-1} u$	$y' = \dfrac{1}{\sqrt{u^2 - 1}} \cdot u'$
$y = \tanh^{-1} u$	$y' = \dfrac{1}{1-u^2} \cdot u'$
$y = \coth^{-1} u$	$y' = \dfrac{-1}{u^2 - 1} \cdot u'$
$y = \sec h^{-1} u$	$y' = -\dfrac{1}{u\sqrt{1-u^2}} \cdot u'$
$y = \cos ech^{-1} u$	$y' = \dfrac{-1}{u\sqrt{1+u^2}} \cdot u'$

Example 19 :

Find $\frac{dy}{dx}$ for the following functions :

$(i)\ y=\sinh^{-1}\left(4\ln x^2\right)$ $\quad (ii)\ y=\tanh^{-1}\frac{1}{1+x}$

$(iii)\ y=\tan\left(\tanh^{-1}x\right)$ $\quad (iv)\ y=3^{\sinh^{-1}x}+4^{\cosh^{-1}x}$

Solution :

$$(i)\ y=\sinh^{-1}\left(4\ln x^2\right)$$

$$y'=\frac{1}{\sqrt{1+\left(4\ln x^2\right)^2}}\cdot 4\cdot\frac{1}{x^2}\cdot 2x$$

$$(ii)\ y=\tanh^{-1}\frac{1}{1+x}$$

$$y'=\frac{1}{1-\left(\frac{1}{1+x^2}\right)^2}\cdot\left(\frac{-2x}{\left(1+x^2\right)^2}\right)$$

$$(iii)\ y=\tan\left(\tanh^{-1}x\right)$$

$$y'=\sec^2\left(\tanh^{-1}x\right)\cdot\frac{1}{1-x^2}$$

$$(iv)\ y=3^{\sinh^{-1}x}+4^{\cosh^{-1}x}$$

$$y'=3^{\sinh^{-1}x}\cdot\frac{\ln 3}{\sqrt{1+x^2}}+4^{\cosh^{-1}x}\cdot\frac{\ln 4}{\sqrt{x^2-1}}$$

Table of Derivatives

y	$\frac{dy}{dx}$
$y = c$ (c constant)	0
$y = x^n$	nx^{n-1}
$y = \frac{1}{x^n}$	$\frac{-n}{x^{n-1}}$
$y = \sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$y = e^x$	e^x
$y = a^x$	$a^x \ln a$
$y = \ln x$	$\frac{1}{x}$
$y = \log_a^x$	$\frac{1}{x \ln a}$
$y = \sin x$	$\cos x$
$y = \cos x$	$-\sin x$
$y = \tan x$	$\sec^2 x$
$y = \cot x$	$-\cos ec^2 x$
$y = \sec x$	$\sec x \tan x$
$y = \cos ecx$	$-\cos ecx \cot x$
$y = \sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$y = \cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$y = \tan^{-1} x$	$\frac{1}{1+x^2}$
$y = \cot^{-1} x$	$\frac{-1}{1+x^2}$

$y=\sec^{-1}x$	$\frac{1}{x\sqrt{1-x^2}}$
$y=\cos ec^{-1}x$	$\frac{-1}{x\sqrt{1-x^2}}$
$y=\sinh x$	$\cosh x$
$y=\cosh x$	$\sinh x$
$y=\tanh x$	$\sec h^2 x$
$y=\coth x$	$-\csc^2 x$
$y=\sec hx$	$-\sec hx\tanh x$
$y=\cos echx$	$-\cos echx\coth x$
$y=\sinh^{-1}x$	$\frac{1}{\sqrt{x^2+1}}$
$y=\cosh^{-1}x$	$\frac{1}{\sqrt{x^2-1}}, \quad x>1$
$y=\tanh^{-1}x$	$\frac{1}{1-x^2}, \quad \lvert x\rvert<1$
$y=\coth^{-1}x$	$\frac{1}{1-x^2}, \quad \lvert x\rvert<1$
$y=\sec h^{-1}x$	$\frac{-1}{x\sqrt{1-x^2}}, \quad \lvert x\rvert<1$
$y=\cos ech^{-1}x$	$\frac{-1}{x\sqrt{1+x^2}}$

3.13 Parametric Differentiation

If x, y are two continuous functions in variable t, it's mean $x = f(t); y = g(t)$ (parametric equations), if we delete t, we can get direct relation between x, y, let f, g have a continuous inverse functions, we obtain $\frac{dy}{dx}$ as the following:-

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta t \to 0} \frac{\Delta y / \Delta t}{\Delta x / \Delta t} = \frac{dy/dt}{dx/dt}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d(dy/dx)}{dt}}{dx/dt}$$

Example 20:-

(i) Find $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ if $x = a\cos t, y = a\sin t,$ *where* (a is constant)

$$\frac{dy}{dt} = a\cos t, \frac{dx}{dt} = -a\sin t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t}{-a\sin t} = -\cot t$$

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{dx/dt} = \frac{-\cot t}{-a\sin t} = \frac{\cos t}{a\sin^2 t}$$

(ii) Find $x = \sin t, y = \cos t$ at $t = \frac{\pi}{4}$

$$\frac{dy}{dt} = -\sin t, \frac{dx}{dt} = \cos t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\tan t$$

$$\frac{dy}{dx} = -\tan\frac{\pi}{4} = -1$$

(iii) Find $x = a(t - \sin t), y = a(1 - \cos t)$

$$\frac{dy}{dt} = a\sin t, \frac{dx}{dt} = a(1 - \cos t)$$

$$\frac{dy}{dx} = \frac{dy/dt}{dy/dt} = \frac{\sin t}{1 - \cos t} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}} = \cot\frac{t}{2}$$

EXERCISES (3)

1. Find the deirvative :

(a) $y=\frac{2}{x}$ (b) $y=\frac{1}{x-1}$ (c) $y=\sqrt{x-2}$

(d) $y=x^3$

2. Find the first deirvative :

(a) e^{5-x} (b) $e^{\sin x}$

3. Find $\frac{dy}{dx}$ for the following :

(a) $\log_a\left(x^2-5x+1\right)$ (b) $\log_a\left(x^2-2\right)^3$ (c) $\left(x^3+1\right)\log_{10} x$

4. Find $\frac{dy}{dx}$ for the following relations :

(a) $y=\ln\sqrt{\sin x}$ (b) $y=\ln\sqrt[5]{x^2-2}$

5. Find the deirvative :

(a) $e^{x+y}-\ln\left(y^3+2\right)=0$ (b) $2\ln(x+y)+y\ln\left(\frac{x+2}{y}\right)=0$

6. If $y=\sin^{-1}\frac{e^x}{\sqrt{1+e^{2x}}}$**, prove that** $\frac{dy}{dx}=\frac{e^x}{1+e^{2x}}$

7. Find the first deirvative for the following functions:

(a) $x^2-y^3=5x$ (b) $y=\ln\sqrt[5]{x^2-2x}=5t,\ y=t^2+5t+1$

(c) $y=z^4,\ z=\sqrt{x^2+2x-1}$

8. Find the first deirvative :

(a) $\sin x\cos x$ (b) $\sin\sqrt{x^2-3}$ (c) $x^2\sec x+\frac{1}{x}\cos ecx$

(d) $\sin(2x+5)$ (e) $\cos ec^{-1}(3x+1)+\ln\sec^{-1}\sqrt{x}$

9. Find the first derivative for the following function:

(a) $y=\ln(\tanh x)$ (b) $y=\sinh xe^{\cosh x}$

(c) $y = \operatorname{sec}h^2 3x$ (d) $y = \tan^{-1}(\tanh x)$

10. Find the first deirvative :

(a) $y = \sinh^{-1} 3x$ (b) $y = \ln \cosh^{-1} 2x$

(c) $y = \cosh^{-1}\left(\sin\sqrt{x}\right)$ (f) $y = \ln \sinh^{-1}(2x)$

(e) $y = \left(\sinh^{-1} 2x\right)^2$

11. Find the slope of tangent line to the following curves :

(a) $y = \sqrt{x}$ at $(1,-2)$

(b) $y = \dfrac{1}{\sqrt{x}}$ at $\left(4,\dfrac{1}{2}\right)$

12. Find $\dfrac{dy}{dx}$:

(a) $y = \sqrt[6]{x}$ (b) $y = \dfrac{1}{x^6}$

13. Find the deirvative :

(a) $y = (x^3-1)(x^2+2x-5)$ (b) $y = \left(\sqrt{x}-2\right)\left(\sqrt{x}+3\right)$ (c) $y = \dfrac{1}{(2x+4)^3}$

14. Find $\dfrac{dy}{dx}$ for the following functions :

(a) $y = \dfrac{1-2\sin x}{1+2\cos x}$ (b) $y = \dfrac{\tan x - 1}{\tan nx + 1}$ (c) $y = \dfrac{x - \tan x}{x + \tan x}$

15. Find $\dfrac{dy}{dx}$:

(a) $y = x\cos y - \sin(x+y)$ (b) $\sec^2 x + \cos ec^2 y = a^2$

16. Find the third deirvative for the function:

$$y = \frac{x}{x+2}$$

17. If $y = \left(x^3+2\right)(x-4)$ **Find** $\dfrac{d^2y}{dx^2}$

18. If $y = e^{x+y}$**, Find** y''

19. If $y = \dfrac{a+b\sin x}{b+a\sin x}$**, Prove that** $y' = \dfrac{\left(b^2-a^2\right)\cos x}{(b+a\sin x)^2}$

Chapter 4

APPLICATIONS OF DERIVATIVES

In the previous chapter we focused almost exclusively on the computa on of deriva ves. In this chapter will focus on applica ons of deriva ves. It is important to always remember that we didn't spend a whole chapter talking about compu ng deriva ves just to be talking about them. There are many very important applica ons to deriva ves.

Section 4-1 : Rates of Change

The purpose of this sec on is to remind us of one of the more important applica ons of deriva ves. That is the fact that $f'(x)$ represents the rate of change of $f(x)$. This is an applica on that we repeatedly saw in the previous chapter. Almost every sec on in the previous chapter contained at least one problem dealing with this applica on of deriva ves. While this applica on will arise occasionally in this chapter we are going to focus more on other applica ons in this chapter. So, to make sure that we don't forget about this application here is a brief set of examples concentra ng on the rate of change applica on of deriva ves.

Example 1: Determine all the points where the following function is not changing.

$f(x) = 5 - 6x - 10\cos\,(2x)$

Solution

First, we'll need to take the derivative of the function.

$f'(x) = -6 + 20\sin\,(2x)$

Now, the func on will not be changing if the rate of change is zero and so to answer this ques on we need to determine where the deriva ve is zero. So, let's set this equal to zero and solve.

$-6 + 20\sin\,(2x) = 0 \quad \Rightarrow \quad \sin(2x) = 0.3$

The solu on to this is then,

$$2x = 0.3047 + 2\pi n \qquad \text{OR} \qquad 2x = 2.8369 + 2\pi n \qquad n = 0, \pm1, \pm2, \ldots$$

$$x = 0.1524 + \pi n \qquad \text{OR} \qquad x = 1.4185 + \pi n \qquad n = 0, \pm1, \pm2, \ldots$$

Example 2 : Determine where the following func on is increasing and decreasing.

$A(t) = 27t^5 - 45t^4 - 130t^3 + 150$

Solution

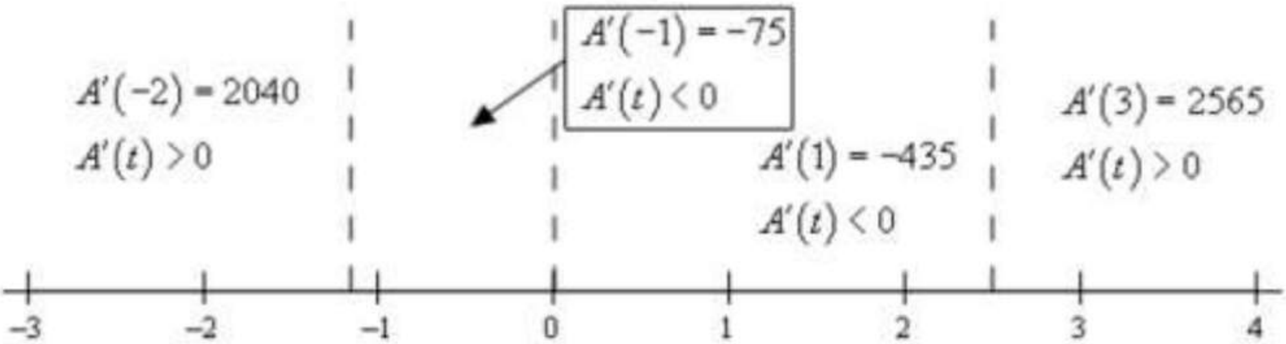
As with the rst problem we rst need to take the deriva ve of the func on.

$$A'(t) = 135t^4 - 180t^3 - 390t^2 = 15t^2(9t^2 - 12t - 26)$$

Next, we need to determine where the function isn't changing. This is at,

$$t = 0\ , \qquad t = \frac{2 \pm \sqrt{30}}{3} = -1.159\ ,\quad 2.492$$

So, the func on is not changing at three values of t. Finally, to determine where the func on is increasing or decreasing we need to determine where the deriva ve is posi ve or nega ve. Recall that if the deriva ve is posi ve then the func on must be increasing and if the deriva ve is nega ve then the func on must be decreasing. The following number line gives this informa on.



So, from this number line we can see that we have the following increasing and decreasing informa on.

Increasing : $-\infty < t < -1.159, \qquad 2.492 < t < \infty$

Decreasing : $-1.159 < t < 0, \qquad 0 < t < 2.492$

Section 4-2 : Critical Points

Cri cal points will show up throughout a majority of this chapter so we rst need to de ne them and work a few examples before ge ng into the sec ons that actually use them.

Definition :

We say that $x = c$ is a cri cal point of the func on $f(x)$ if

exists and if either of the following are true. $f(c)$
OR $f'(c)$ **doesn't exist** $f'(c) = 0$

Note that we require that $f(c)$ exists in order for $x = c$ to actually be a cri cal point. This is an important, and o en overlooked, point. What this is really saying is that all cri cal points must be in the domain of the func on. If a point is not in the domain of the func on then it is not a cri cal point.

The main point of this sec on is to work some examples nding cri cal points. So, let’s work some examples.

Example 1: Determine all the cri cal points for the func on.
$f(x) = 6x^5 + 33x^4 - 30x^3 + 100$

Solution
We rst need the deriva ve of the func on in order to nd the cri cal points and so let’s get that and notice that we’ll factor it as much as possible to make our life easier when we go to nd the cri cal points.

$$\begin{aligned} f'(x) &= 30x^4 + 132x^3 - 90x^2 \\ &= 6x^2\left(5x^2 + 22x - 15\right) \\ &= 6x^2\left(5x - 3\right)\left(x + 5\right) \end{aligned}$$

Now, our deriva ve is a polynomial and so will exist everywhere. Therefore, the only cri cal points will be those values of x **which make the deriva ve zero**. So, we must solve.
$6x^2(5x - 3)(x + 5) = 0$
Because this is the factored form of the derivative it’s pretty easy to iden fy the three cri cal points.
They are, $x = -5$, $x = 0$, $x = \frac{3}{5}$

Polynomials are usually fairly simple func ons to nd critical points for provided the degree doesn’t get so large that we have trouble nding the roots of the deriva ve.

Most of the more “interesting” functions for finding critical points aren’t polynomials however. So let’s take a look at some func ons that require a li le more e ort on our part.

Example 2: Determine all the cri cal points for the func on.

$$g(t) = \sqrt[3]{t^2}\ (2t-1)$$

Solution

To find the derivative it’s probably easiest to do a little simplification before we actually di eren ate. Let’s multiply the root through the parenthesis and simplify as much as possible. This will allow us to avoid using the product rule when taking the deriva ve.

$$g(t) = t^{\frac{2}{3}}(2t-1) = 2t^{\frac{5}{3}} - t^{\frac{2}{3}}$$

Now di eren ate.

$$g'(t) = \frac{10}{3}t^{\frac{2}{3}} - \frac{2}{3}t^{-\frac{1}{3}} = \frac{10t^{\frac{2}{3}}}{3} - \frac{2}{3t^{\frac{1}{3}}}$$

So, we’ve found one critical point (where the derivative doesn’t exist), but we now need to determine where the deriva ve is zero (provided it is of course…). To help with this it’s usually best to combine the two terms into a single ra onal expression. So, ge ng a common denominator and combining gives us,

$$g'(t) = \frac{10t-2}{3t^{\frac{1}{3}}}$$

No ce that we s ll have $t = 0$ as a cri cal point. Doing this kind of combining should never lose critical points, it’s only being done to help us find them. As we can see it’s now become much easier

So, in this case we can see that the numerator will be zero if $t = \frac{1}{5}$ and so there are **two cri cal points** for this func on.

and $t = \frac{1}{5}t = 0$

Example 3: Determine all the cri cal points for the func on.

$$y = 6x - 4\cos(3x)$$

Solution :

First get the derivative and don’t forget to use the chain rule on the second term.

$$y' = 6 + 12\, sin\,(3x)$$

Now, this will exist everywhere and so there won’t be any critical points for which the deriva ve doesn’t exist. The only cri cal points will come from points that make the deriva ve zero. We will need to solve,

$$6 + 12\, sin\,(3x) = 0$$

$$sin\,(3x) = \frac{1}{2}$$

Solving this equation gives the following.

$$3x = 3.6652 + 2\,\pi\, n, \qquad n = 0, \pm 1, \pm 2, ...$$

$$3x = 5.7696 + 2\,\pi\, n, \qquad n = 0, \pm 1, \pm 2, ...$$

Don’t forget the $2\pi\, n$ on these! There will be problems down the road in which we will miss solu ons without this! Also make sure that it gets put on at this stage! Now divide by 3 to get all the cri cal points for this func on.

$$x = 1.2217 + \frac{2\,\pi\, n}{3}, \qquad n = 0, \pm 1, \pm 2, ...$$

$$x = 1.9199 + \frac{2\,\pi\, n}{3}, \qquad n = 0, \pm 1, \pm 2, ...$$

Example 4: Determine all the cri cal points for the func on.

$$f(x) = x^2 \ln(3x) + 6$$

Solution :

Before ge ng the derivative let’s notice that since we can’t take the log of a nega ve number or zero we will only be able to look at > 0.

The deriva ve is then,

$$f'(x) = 2x\ln(3x) + x^2\left(\frac{3}{3x}\right) = x(2\ln(3x) + 1)$$

Now, this deriva ve will not exist if *x* is a nega ve number or if $= 0$, but then again neither will the func on and so these are not cri cal points.

Remember that the func on will only exist if $x > 0$and nicely enough the deriva ve will also only exist if $x > 0$ and so the only thing we need to worry about is where the deriva ve is zero.
First note that, despite appearances, the deriva ve will not be zero for x = 0 . As noted above the derivative doesn't exist at x = 0 because of the natural logarithm and so the derivative can't be zero there!
So, the deriva ve will only be zero if,

$$2\ln(3x) + 1 = 0 \qquad \Rightarrow \qquad \ln 3x = -\frac{1}{2}$$

Recall that we can solve this by exponen ating both sides.

$$x = \frac{1}{3}\, e^{\frac{-1}{2}} = \frac{1}{3\sqrt{e}}$$

Section 4-3 : Minimum and Maximum Values

Many of our applica ons in this chapter will revolve around minimum and maximum values of a function. While we can all visualize the minimum and maximum values of a func on we want to be a li le more speci c in our work here. In par cular, we want to di eren ate between two types of minimum or maximum values. The following de ni on gives the types of minimums and/or maximums values that we'll be looking at.

Definition:

1. We say that $f(x)$ has an **absolute (or global) maximum** at $x = c$ if $f(x) \le f(c)$ for every x in the domain we are working on.

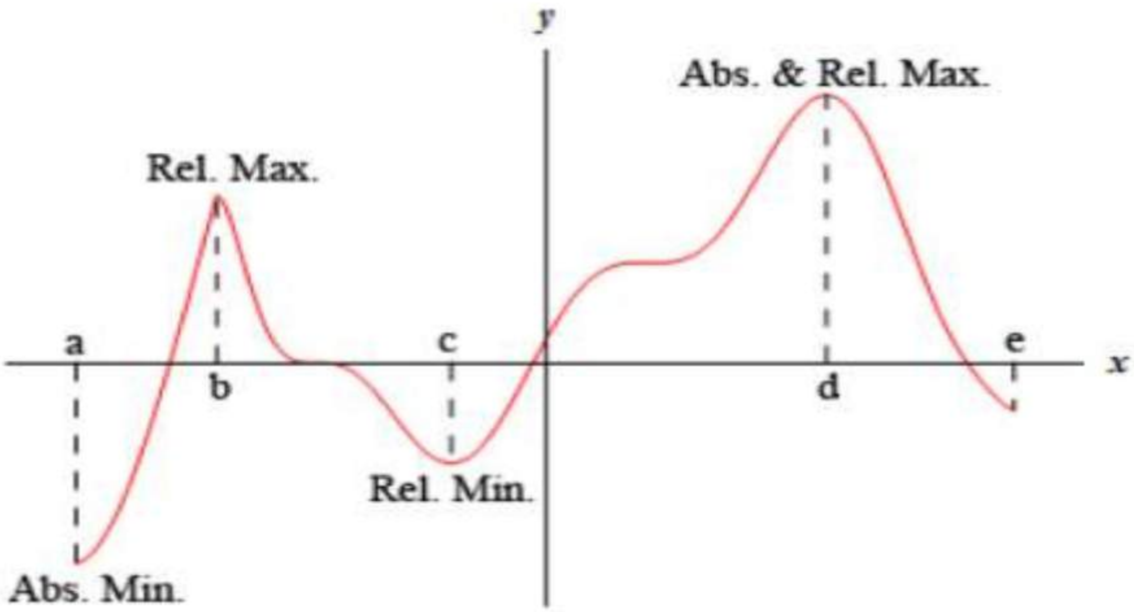
2. We say that $f(x)$ has a **relative (or local) maximum** at $x = c$ if $f(x) \le f(c)$ for every x in some open interval around $x = c$.

3. We say that $f(x)$ has an **absolute (or global) minimum** at x = c if $f(x) \ge f(c)$ for every x in the domain we are working on.

4. We say that $f(x)$ has a **relative (or local) minimum** at $x = c$ if $f(x) \ge f(c)$ for every x in some open interval around $x = c$.

We will collectively call the minimum and maximum points of a function the **extrema** of the function. So, relative extrema will refer to the relative minimums and maximums while absolute extrema refer to the absolute minimums and maximums.

A relative maximum or minimum is slightly different. All that's required for a point to be a relative maximum or minimum is for that point to be a maximum or minimum in some interval of $x's$ around $= c$. There may be larger or smaller values of the function at some other place, but relative to $= c$, or local to $x = c$, $f(c)$ is larger or smaller than all the other function values that are near it.
It's usually easier to get a feel for the definitions by taking a quick look at a graph. For



For the func on shown in this graph we have rela ve maximums at $x = b$ and $x = d$. Both of these points are rela ve maximums since they are interior to the domain shown and are the largest point on the graph in some interval around the point. We also have a rela ve minimum at $x = c$ since this point is interior to the domain and is the lowest point on the graph in an interval around it. The far-right end point, $x = e$, will not be a rela ve minimum since it is an end point.

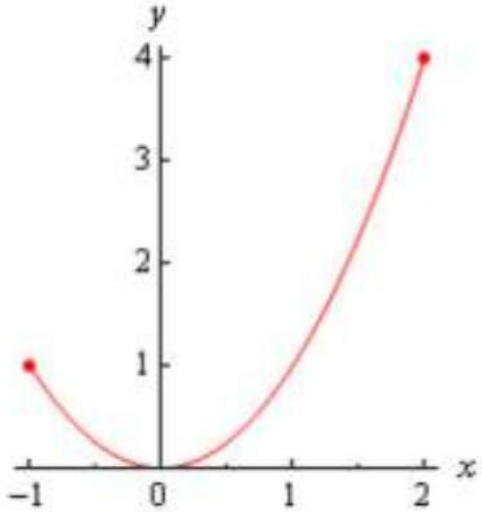
The func on will have an absolute maximum at $x = d$ and an absolute minimum at $x = a$. These two points are the largest and smallest that the func on will ever be. We can also no ce that the absolute extrema for a func on will occur at either the endpoints of the domain or at rela ve extrema. We will use this idea in later sections so it's more important than it might seem at the present me.

Let's take a quick look at some examples to make sure that we have the definitions of absolute extrema and rela ve extrema straight

Example 1: Iden fy the absolute extrema and rela ve extrema for the following func on. $f(x) = x^2$ on $[-1, 2]$

Solution

Since this function is easy enough to graph let's do that. However, we only want the graph on the interval $[-1,2]$. Here is the graph,



We can now identify the extrema from the graph. It looks like we've got a rela ve and absolute minimum of zero at $x = 0$ and an absolute maximum of four at $x = 2$. Note that $x = -1$ is not a rela ve maximum since it is at the end point of the interval.

This function doesn't have any relative maximums.

Example 2: Iden fy the absolute extrema and rela ve extrema for the following func on. $f(x) = x^2$ on $[-2, 2]$

Solution

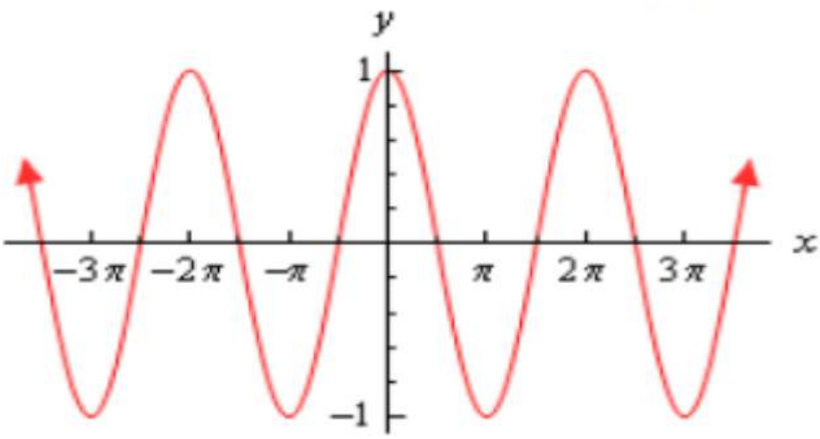
In this case we s ll have a rela ve and absolute minimum of zero at x = 0 . We also s ll have an absolute maximum of four. However, unlike the rst example this will occur at two points, x = −2 and x = 2 .

Again, the function doesn't have any relative maximums.

Example 3: Iden fy the absolute extrema and rela ve extrema for the following func on. $f(x) = \cos x$

Solution

We've not restricted the domain for this function. Here is the graph.



Cosine has extrema (relative and absolute) that occur at many points. Cosine has both relative and absolute maximums of 1 at

$x = \cdots . -4\pi , -2\pi , 0, 2\pi , 4\pi , ...$

Cosine also has both relative and absolute minimums of -1 at

$x = \cdots , -3\pi , -\pi , \pi , 3\pi , ...$

As this example has shown a graph can in fact have extrema occurring at a large number (infinite in this case) of points.

Section 4-4 : Finding Absolute Extrema

It's now time to see our first major application of derivatives in this chapter. Given a continuous function, (x) , on an interval $[a, b]$ we want to determine the absolute extrema of the function. To do this we will need many of the ideas that we looked at in the previous section.

Finding Absolute Extrema of $f(x)$ on $[a, b]$.

1- Verify that the function is continuous on the interval [*a*,*b*].
2- Find all critical points of *f* (*x*) that are in the interval [*a*,*b*]. This makes sense if you think about it. Since we are only interested in what the function is doing in this interval we don't care about critical points that fall outside the interval.
3- Evaluate the function at the critical points found in step 1 and the end points.
4- Identify the absolute extrema.

Example 1: Determine the absolute extrema for the following function and interval.

$$f(x) = 2x^3 + 3x^2 - 12x + 4 \quad \text{on} \quad [-4, 2]$$

Solution:

All we really need to do here is follow the procedure given above. So, rst no ce that this is a polynomial and so is con nuous everywhere and therefore is con nuous on the given interval.
Now, we need to get the deriva ve so that we can nd the cri cal points of the func on.

$$f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1)$$

It looks like we'll have two critical points, $x = -2$ and $x = 1$. Note that we actually want something more than just the cri cal points. We only want the cri cal points of the func on that lie in the interval in ques on. Both of these do fall in the interval as so we will use both of them. That may seem like a silly thing to men on at this point, but it is o en forgo en, usually when it becomes important, and so we will men on it at every opportunity to make sure it's not forgotten.
Now we evaluate the function at the cri cal points and the end points of the interval.

$$f(-2) = 24 \quad , \qquad f(1) = -3$$
$$f(-4) = -28 \quad , \qquad f(2) = 8$$

Absolute extrema are the largest and smallest the func on will ever be and these four points represent the only places in the interval where the absolute extrema can occur. So, from this list we see that the absolute maximum of $f(x)$ is 24 and it occurs at $x = -2$ (a cri cal point) and the absolute minimum of $f(x)$ is -28 which occurs at $x = -4$ (an endpoint).
In this example we saw that absolute extrema can and will occur at both endpoints and critical points. One of the biggest mistakes that students make with these problems is to forget to check the endpoints of the interval.
Example 2: Determine the absolute extrema for the following func on and interval.

$$f(x) = 2x^3 + 3x^2 - 12x + 4 \quad \text{on} \quad [0, 2]$$

Solution:

Note that this problem is almost iden cal to the rst problem. The only di erence is the interval that we're working on. This small change will

completely change our answer however. With this change we have excluded both of the answers from the rst example.
The rst step is to again nd the cri cal points. From the rst example we know these are $x = -2$ and $x = 1$. At this point it's important to recall that we only want the cri cal points that actually fall in the interval in ques on. This means that we only want x=1 since $x = -2$ falls outside the interval.

Now evaluate the function at the single cri cal point in the interval and the two endpoints.
$f(1) = -3$, $f(0) = 4$, $f(2) = 8$
From this list of values we see that the absolute maximum is 8 and will occur at $x = 2$ and the absolute minimum is -3 which occurs at $x = 1$.

Example 3: Determine the absolute extrema for the following func on and interval.

on $[-5, -1]\, f(x) = 3x(x+4)^{\frac{2}{3}}$
Solution:

Again, as with all the other examples here, this func on is con nuous on the given interval and so we know that this can be done.
First, we'll need the derivative and make sure you can do the simpli ca on that we did here to make the work for nding the cri cal points easier.

$$f'(x) = \frac{5x+12}{(x+4)^{\frac{1}{3}}}$$

So, it looks like we've got two critical points.
$x = -4$ Because the deriva ve doesn't exist here.
Because the deriva ve is zero here. $x = -\frac{12}{5}$
Both of these are in the interval so let's evaluate the function at these points and the end points of the interval.

$f(-4) = 0$, $f\left(-\frac{12}{5}\right) = -9.849$,
$f(-5) = -15$, $f(-1) = -6.241$

The function has an absolute maximum of **zero** at $x = -4$
and the function will have an absolute minimum of **-15** at $x = -5$.

So, if we had ignored or forgotten about the critical point where the derivative doesn't exist ($x = -4$) we would not have gotten the correct answer.

Exercise: Suppose that the population (in thousands) of a certain kind of insect after x months is given by the following formula.
$f(x) = 3x + \sin 4x + 100$

Determine the minimum and maximum population in the first 4 months.

Section 4-5 : **L'Hospital`s Rule and Indeterminate Forms**

Back in the chapter on Limits we saw methods for dealing with the following limits.
$\lim\limits_{x\to 4} \frac{x^2 - 16}{x - 4}$, $\lim\limits_{x\to\infty} \frac{4x^2 - 5x}{1 - 3x^2}$

In the first limit if we plugged in $x = 4$ we would get $0/0$ and in the second limit if we "plugged" in infinity we would get $\infty - \infty$ (recall that as x goes to infinity a polynomial will behave in the same fashion that its largest power behaves). Both of these are called **indeterminate forms**. In both of these cases there are competing interests or rules and it's not clear which will win out.

In the case of $0/0$ we typically think of a fraction that has a numerator of zero as being zero. However, we also tend to think of fractions in which the denominator is going to zero, in the limit, as infinity or might not exist at all. Likewise, we tend to think of a fraction in which the numerator and denominator are the same as one. So, which will win out? Or will neither win out and they all "cancel out" and the limit will reach some other value?

In the case of $\infty - \infty$ we have a similar set of problems. If the numerator of a fraction is going to infinity we tend to think of the whole fraction going to infinity. Also, if the denominator is going to infinity, in the limit, we tend to think of the fraction as going to zero. We also have the case of a fraction in which the numerator and denominator are the same (ignoring the minus sign) and so we might get -1. Again, it's not clear which of these will win out, if any of them will win out.

With the second limit there is the further problem that infinity isn't really a number and so we really shouldn't even treat it like a number. Much of the time it simply won't behave as we would expect it to if it was a number. To look a little more into this, check out the **Types of Infinity** section in the Extras chapter at the end of this document.

This is the problem with indeterminate forms. It's just not clear what is happening in the limit. There are other types of indeterminate forms as well. Some other types are,

$(0)(\pm\infty)\ ,\quad 1^{\infty}\ ,\qquad 0^{0}\ ,\qquad \infty^{0}\ ,\qquad \infty-\infty$

So, we can deal with some of these. However, what about the following two limits.

This first is a $0/0$ indeterminate form, but we can't factor this one. The second is an ∞/∞ indeterminate form, but we can't just factor an x^2 out of the numerator. So, nothing that we've got in our bag of tricks will work with these two limits.

This is where the subject of this section comes into play.

L'Hospital's Rule:

Suppose that we have one of the following cases,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \qquad OR \qquad \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

Where a can be any real number, infinity or negative infinity. In these cases we have,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

So, **L'Hospital's Rule** tells us that if we have an indeterminate form $0/0$ $or\ \infty/\infty$ all we need to do is differentiate the numerator and differentiate the denominator and then take the limit.

Let's work some examples.

Evaluate each of the following limits. : **Example 1**

(a) $\lim_{x \to 0} \frac{\sin x}{x}$

(b) $\lim_{x \to \infty} \frac{5x^4 - 4x^2 - 1}{10 - x - 9x^3}$

(c) $\lim_{x \to \infty} \frac{e^x}{x^2}$

SOLUTION

(a) $\lim_{x \to 0} \frac{\sin x}{x}$

So, we have already established that this is a $0/0$ indeterminate form so let's just apply L'Hospital's Rule.

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \frac{1}{1} = 1$$

(b) $\lim_{x \to \infty} \frac{5x^4 - 4x^2 - 1}{10 - x - 9x^3}$

In this case we also have a 0/0 indeterminate form and if we were really good at factoring we could factor the numerator and denominator, simplify and take the limit. However, that's going to be more work than just using L'Hospital's Rule.

$$\lim_{x\to\infty} \frac{5x^4 - 4x^2 - 1}{10 - x - 9x^3} = \lim_{x\to\infty} \frac{20x^3 - 8x}{-1 - 27x^2} = -\frac{3}{7}$$

(c) $\lim_{x\to\infty} \frac{e^x}{x^2}$

This was the other limit that we started off looking at and we know that it's the indeterminate form ∞/∞ so let's apply L'Hospital's Rule.

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x}$$

Now we have a small problem. This new limit is also a ∞/ ∞ indeterminate form. However, it's not really a problem. We know how to deal with these kinds of limits. Just apply L'Hospital's Rule.

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \frac{\infty}{2} = \infty$$

Sometimes we will need to apply L'Hospital's Rule more than once.

Evaluate the following limit. : **Example 2**

$$\lim_{x\to 0^+} x \ln x$$

SOLUTION

Note that we really do need to do the right-hand limit here. We know that the natural logarithm is only defined for positive x and so this is the only limit that makes any sense.

Now, in the limit, we get the indeterminate form $(0)(-\infty)$. L'Hospital's Rule won't work on products, it only works on quotients. However, we can turn this into a fraction if we rewrite things a little.

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x}$$

The function is the same, just rewritten, and the limit is now in the form $-\infty/\infty$ and we can now use L'Hospital's Rule.

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2}$$

Now, this is a mess, but it cleans up nicely.

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0$$

Evaluate the following limit. : **Example 3**

$$\lim_{x\to -\infty} x\ e^x$$

SOLUTION

So, it's in the form $(\infty)(0)$. This means that we'll need to write it as a quotient. Moving the x to the denominator worked in the previous example so let's try that with this problem as well.

$$\lim_{x\to -\infty} x\ e^x = \lim_{x\to -\infty} \frac{x}{1/e^x} = \lim_{x\to -\infty} \frac{x}{e^{-x}} = \lim_{x\to -\infty} \frac{1}{-e^{-x}} = 0$$

So, when faced with a product $(0)(\pm\infty)$ we can turn it into a quotient that will allow us to use L'Hospital's Rule. However, as we saw in the last example we need to be careful with how we do that on occasion. Sometimes we can use either quotient and in other cases only one will work.

Let's now take a look at the indeterminate forms,

$$1^\infty , \quad 0^0 , \quad \infty^0$$

These can all be dealt with in the following way so we'll just work one example.

Evaluate the following limit. : **Example 3**

$$\lim_{x\to\infty} x^{\frac{1}{x}}$$

SOLUTION

In the limit this is the indeterminate form ∞^0 . We're actually going to spend most of this problem on a different limit. Let's first define the following.

$$y = x^{\frac{1}{x}}$$

Now, if we take the natural log of both sides we get,

$$\ln\ y = \ln\ x^{\frac{1}{x}} = \frac{1}{x}\ln x = \frac{\ln x}{x}$$

Let's now take a look at the following limit.

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

Well first notice that,

$$e^{lny} = y$$

and so our limit could be written as,

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} y = \lim_{x\to\infty} e^{lny}$$

We can now use the limit above to finish this problem.

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} y = \lim_{x\to\infty} e^{lny} = e^{\lim_{x\to\infty} \ln y} = e^0 = 1$$

EXERCISES

For each of the following problems determine the absolute extrema of the given function on the specified interval.

1. $f(x) = 8x^3 + 81x^2 - 42x - 8$ on $[-8, 2]$

2. $f(x) = 8x^3 + 81x^2 - 42x - 8$ on $[-4, 2]$

3. $R(t) = 1 + 80t^3 + 5t^4 - 2t^5$ on $[-4.5, 4]$

4. $R(t) = 1 + 80t^3 + 5t^4 - 2t^5$ on $[0, 7]$

5. $h(z) = 4z^3 - 3z^2 + 9z + 12$ on $[-2, 1]$

6. $g(x) = 3x^4 - 26x^3 + 60x^2 - 11$ on $[1, 5$

7. $Q(x) = (2 - 8x)^4 (x^2 - 9)^3$ on $[-3, 3]$

8. $h(w) = 2w^3 (w + 2)^5$ on $\left[-\frac{5}{2}, \frac{1}{2}\right]$

9. $f(z) = \dfrac{z + 4}{2z^2 + z + 8}$ on $[-10, 0]$

10. $A(t) = t^2 (10 - t)^{\frac{2}{3}}$ on $[2, 10.5]$

11. $f(y) = \sin\left(\frac{y}{3}\right) + \frac{2y}{9}$ on $[-10, 15]$

12. $g(w) = \mathbf{e}^{w^3 - 2w^2 - 7w}$ on $\left[-\frac{1}{2}, \frac{5}{2}\right]$

13. $R(x) = \ln\left(x^2 + 4x + 14\right)$ on $[-4, 2]$

Use L'Hospital's Rule to evaluate each of the following limits.

1. $\lim\limits_{x \to 2} \dfrac{x^3 - 7x^2 + 10x}{x^2 + x - 6}$

2. $\lim\limits_{w \to -4} \dfrac{\sin(\pi w)}{w^2 - 16}$

3. $\lim\limits_{t \to \infty} \dfrac{\ln(3t)}{t^2}$

4. $\lim\limits_{z \to 0} \dfrac{\sin(2z) + 7z^2 - 2z}{z^2(z+1)^2}$

5. $\lim\limits_{x \to -\infty} \dfrac{x^2}{\mathbf{e}^{1-x}}$

6. $\lim\limits_{z \to \infty} \dfrac{z^2 + \mathbf{e}^{4z}}{2z - \mathbf{e}^z}$

7. $\lim\limits_{t \to \infty} \left[t \ln\left(1 + \dfrac{3}{t}\right) \right]$

8. $\lim\limits_{w \to 0^+} \left[w^2 \ln(4w^2) \right]$

9. $\lim\limits_{x \to 1^+} \left[(x-1) \tan\left(\frac{\pi}{2} x\right) \right]$

10. $\lim\limits_{y \to 0^+} \left[\cos(2y) \right]^{1/y^2}$

11. $\lim\limits_{x \to \infty} \left[\mathbf{e}^x + x \right]^{1/x}$

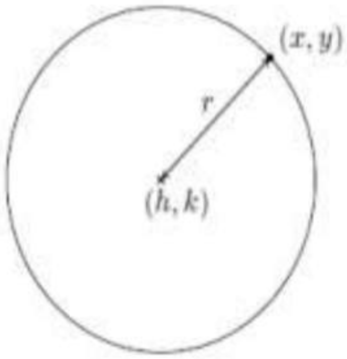
Chapter 5

GEOMETRY

7.2 Circles

Recall from Geometry that a circle can be determined by fixing a point (called the center) and a positive number (called the radius) as follows.

> **Definition 7.1.** A **circle** with center (h, k) and radius $r > 0$ is the set of all points (x, y) in the plane whose distance to (h, k) is r.



From the picture, we see that a point (x, y) is on the circle if and only if its distance to (h, k) is r. We express this relationship algebraically using the Distance Formula, Equation 1.1, as

$$r = \sqrt{(x - h)^2 + (y - k)^2}$$

By squaring both sides of this equation, we get an equivalent equation (since $r > 0$) which gives us the standard equation of a circle.

> **Equation 7.1. The Standard Equation of a Circle:** The equation of a circle with center (h, k) and radius $r > 0$ is $(x - h)^2 + (y - k)^2 = r^2$.

Example 7.2.1. Write the standard equation of the circle with center $(-2, 3)$ and radius 5.

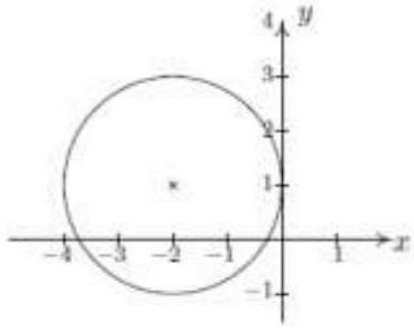
Solution. Here, $(h, k) = (-2, 3)$ and $r = 5$, so we get

$$\begin{aligned} (x - (-2))^2 + (y - 3)^2 &= (5)^2 \\ (x + 2)^2 + (y - 3)^2 &= 25 \end{aligned}$$

□

Example 7.2.2. Graph $(x + 2)^2 + (y - 1)^2 = 4$. Find the center and radius.

Solution. From the standard form of a circle, Equation 7.1, we have that $x + 2$ is $x - h$, so $h = -2$ and $y - 1$ is $y - k$ so $k = 1$. This tells us that our center is $(-2, 1)$. Furthermore, $r^2 = 4$, so $r = 2$. Thus we have a circle centered at $(-2, 1)$ with a radius of 2. Graphing gives us



□

If we were to expand the equation in the previous example and gather up like terms, instead of the easily recognizable $(x+2)^2+(y-1)^2=4$, we'd be contending with $x^2+4x+y^2-2y+1=0$. If we're given such an equation, we can complete the square in each of the variables to see if it fits the form given in Equation 7.1 by following the steps given below.

To Write the Equation of a Circle in Standard Form

1. Group the same variables together on one side of the equation and position the constant on the other side.
2. Complete the square on both variables as needed.
3. Divide both sides by the coefficient of the squares. (For circles, they will be the same.)

Example 7.2.3. Complete the square to find the center and radius of $3x^2-6x+3y^2+4y-4=0$.

Solution.

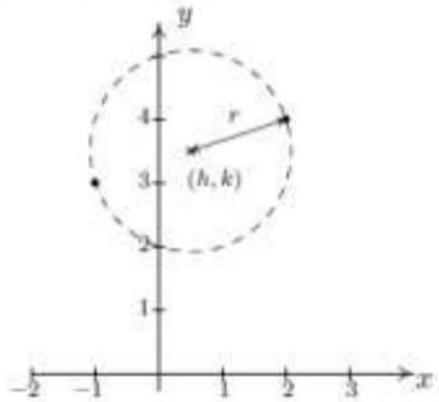
$$\begin{array}{rcll}
3x^2-6x+3y^2+4y-4 & = & 0 & \\
3x^2-6x+3y^2+4y & = & 4 & \text{add 4 to both sides} \\
3\left(x^2-2x\right)+3\left(y^2+\frac{4}{3}y\right) & = & 4 & \text{factor out leading coefficients} \\
3\left(x^2-2x+\underline{1}\right)+3\left(y^2+\frac{4}{3}y+\underline{\underline{\frac{4}{9}}}\right) & = & 4+3\underline{(1)}+3\underline{\underline{\left(\frac{4}{9}\right)}} & \text{complete the square in } x,\ y \\
3(x-1)^2+3\left(y+\frac{2}{3}\right)^2 & = & \frac{25}{3} & \text{factor} \\
(x-1)^2+\left(y+\frac{2}{3}\right)^2 & = & \frac{25}{9} & \text{divide both sides by 3}
\end{array}$$

From Equation 7.1, we identify $x-1$ as $x-h$, so $h=1$, and $y+\frac{2}{3}$ as $y-k$, so $k=-\frac{2}{3}$. Hence, the center is $(h,k)=\left(1,-\frac{2}{3}\right)$. Furthermore, we see that $r^2=\frac{25}{9}$ so the radius is $r=\frac{5}{3}$. □

It is possible to obtain equations like $(x-3)^2+(y+1)^2=0$ or $(x-3)^2+(y+1)^2=-1$, neither of which describes a circle. (Do you see why not?) The reader is encouraged to think about what, if any, points lie on the graphs of these two equations. The next example uses the Midpoint Formula, Equation 1.2, in conjunction with the ideas presented so far in this section.

Example 7.2.4. Write the standard equation of the circle which has $(-1,3)$ and $(2,4)$ as the endpoints of a diameter.

Solution. We recall that a diameter of a circle is a line segment containing the center and two points on the circle. Plotting the given data yields



Since the given points are endpoints of a diameter, we know their midpoint (h,k) is the center of the circle. Equation 1.2 gives us

$$\begin{aligned}(h,k) &= \left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)\\ &= \left(\frac{-1+2}{2},\frac{3+4}{2}\right)\\ &= \left(\frac{1}{2},\frac{7}{2}\right)\end{aligned}$$

The diameter of the circle is the distance between the given points, so we know that half of the distance is the radius. Thus,

$$\begin{aligned}r &= \frac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\ &= \frac{1}{2}\sqrt{(2-(-1))^2+(4-3)^2}\\ &= \frac{1}{2}\sqrt{3^2+1^2}\\ &= \frac{\sqrt{10}}{2}\end{aligned}$$

Finally, since $\left(\frac{\sqrt{10}}{2}\right)^2=\frac{10}{4}$, our answer becomes $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{7}{2}\right)^2=\frac{10}{4}$ □

We close this section with the most important[1] circle in all of mathematics: the **Unit Circle**.

> **Definition 7.2.** The **Unit Circle** is the circle centered at $(0, 0)$ with a radius of 1. The standard equation of the Unit Circle is $x^2 + y^2 = 1$.

Example 7.2.5. Find the points on the unit circle with y-coordinate $\frac{\sqrt{3}}{2}$.

Solution. We replace y with $\frac{\sqrt{3}}{2}$ in the equation $x^2 + y^2 = 1$ to get

$$\begin{aligned} x^2 + y^2 &= 1 \\ x^2 + \left(\frac{\sqrt{3}}{2}\right)^2 &= 1 \\ \frac{3}{4} + x^2 &= 1 \\ x^2 &= \frac{1}{4} \\ x &= \pm\sqrt{\frac{1}{4}} \\ x &= \pm\frac{1}{2} \end{aligned}$$

Our final answers are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. □

7.2.1 Exercises

In Exercises 1 - 6, find the standard equation of the circle and then graph it.

1. Center $(-1,-5)$, radius 10
2. Center $(4,-2)$, radius 3
3. Center $\left(-3,\frac{7}{13}\right)$, radius $\frac{1}{2}$
4. Center $(5,-9)$, radius $\ln(8)$
5. Center $\left(-e,\sqrt{2}\right)$, radius π
6. Center (π,e^2), radius $\sqrt[3]{91}$

In Exercises 7 - 12, complete the square in order to put the equation into standard form. Identify the center and the radius or explain why the equation does not represent a circle.

7. $x^2-4x+y^2+10y=-25$
8. $-2x^2-36x-2y^2-112=0$
9. $x^2+y^2+8x-10y-1=0$
10. $x^2+y^2+5x-y-1=0$
11. $4x^2+4y^2-24y+36=0$
12. $x^2+x+y^2-\frac{6}{5}y=1$

In Exercises 13 - 16, find the standard equation of the circle which satisfies the given criteria.

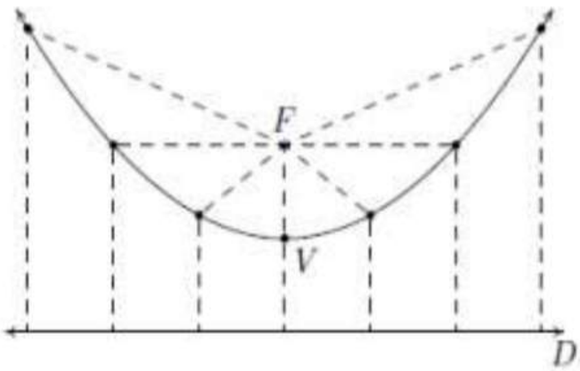
13. center $(3,5)$, passes through $(-1,-2)$
14. center $(3,6)$, passes through $(-1,4)$
15. endpoints of a diameter: $(3,6)$ and $(-1,4)$
16. endpoints of a diameter: $\left(\frac{1}{2},4\right)$, $\left(\frac{3}{2},-1\right)$

7.3 Parabolas

We have already learned that the graph of a quadratic function $f(x) = ax^2 + bx + c$ $(a \neq 0)$ is called a **parabola**. To our surprise and delight, we may also define parabolas in terms of distance.

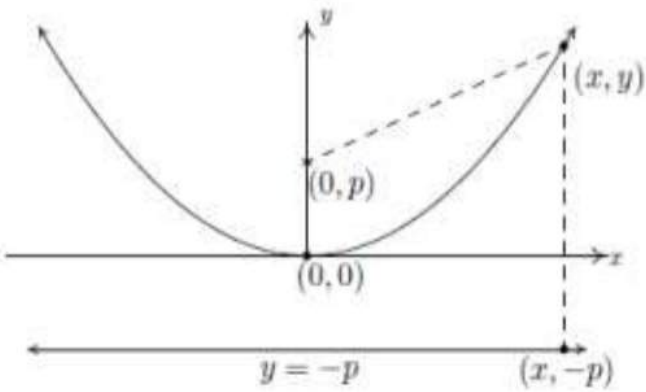
> **Definition 7.3.** Let F be a point in the plane and D be a line not containing F. A **parabola** is the set of all points equidistant from F and D. The point F is called the **focus** of the parabola and the line D is called the **directrix** of the parabola.

Schematically, we have the following.



Each dashed line from the point F to a point on the curve has the same length as the dashed line from the point on the curve to the line D. The point suggestively labeled V is, as you should expect, the **vertex**. The vertex is the point on the parabola closest to the focus.

We want to use only the distance definition of parabola to derive the equation of a parabola and, if all is right with the universe, we should get an expression much like those studied in Section 2.3. Let p denote the directed[1] distance from the vertex to the focus, which by definition is the same as the distance from the vertex to the directrix. For simplicity, assume that the vertex is $(0, 0)$ and that the parabola opens upwards. Hence, the focus is $(0, p)$ and the directrix is the line $y = -p$. Our picture becomes



From the definition of parabola, we know the distance from $(0, p)$ to (x, y) is the same as the distance from $(x, -p)$ to (x, y). Using the Distance Formula, Equation 1.1, we get

$$
\begin{aligned}
\sqrt{(x-0)^2+(y-p)^2} &= \sqrt{(x-x)^2+(y-(-p))^2} & \\
\sqrt{x^2+(y-p)^2} &= \sqrt{(y+p)^2} & \\
x^2+(y-p)^2 &= (y+p)^2 & \text{square both sides} \\
x^2+y^2-2py+p^2 &= y^2+2py+p^2 & \text{expand quantities} \\
x^2 &= 4py & \text{gather like terms}
\end{aligned}
$$

Solving for y yields $y = \frac{x^2}{4p}$, which is a quadratic function of the form found in Equation 2.4 with $a = \frac{1}{4p}$ and vertex $(0,0)$.

We know from previous experience that if the coefficient of x^2 is negative, the parabola opens downwards. In the equation $y = \frac{x^2}{4p}$ this happens when $p < 0$. In our formulation, we say that p is a 'directed distance' from the vertex to the focus: if $p > 0$, the focus is above the vertex; if $p < 0$, the focus is below the vertex. The **focal length** of a parabola is $|p|$.

If we choose to place the vertex at an arbitrary point (h,k), we arrive at the following formula using either transformations from Section 1.7 or re-deriving the formula from Definition 7.3.

Equation 7.2. The Standard Equation of a Vertical[a] Parabola: The equation of a (vertical) parabola with vertex (h,k) and focal length $|p|$ is

$$(x-h)^2 = 4p(y-k)$$

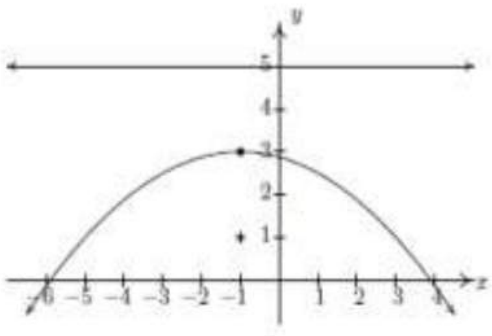
If $p > 0$, the parabola opens upwards; if $p < 0$, it opens downwards.

[a]That is, a parabola which opens either upwards or downwards.

Notice that in the standard equation of the parabola above, only one of the variables, x, is squared. This is a quick way to distinguish an equation of a parabola from that of a circle because in the equation of a circle, both variables are squared.

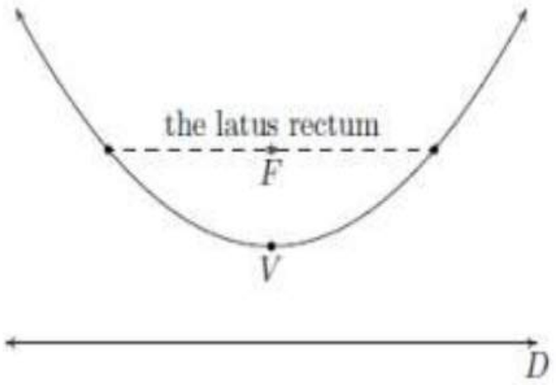
Example 7.3.1. Graph $(x+1)^2 = -8(y-3)$. Find the vertex, focus, and directrix.

Solution. We recognize this as the form given in Equation 7.2. Here, $x-h$ is $x+1$ so $h=-1$, and $y-k$ is $y-3$ so $k=3$. Hence, the vertex is $(-1,3)$. We also see that $4p=-8$ so $p=-2$. Since $p<0$, the focus will be below the vertex and the parabola will open downwards.



The distance from the vertex to the focus is $|p| = 2$, which means the focus is 2 units below the vertex. From $(-1, 3)$, we move down 2 units and find the focus at $(-1, 1)$. The directrix, then, is 2 units above the vertex, so it is the line $y = 5$. □

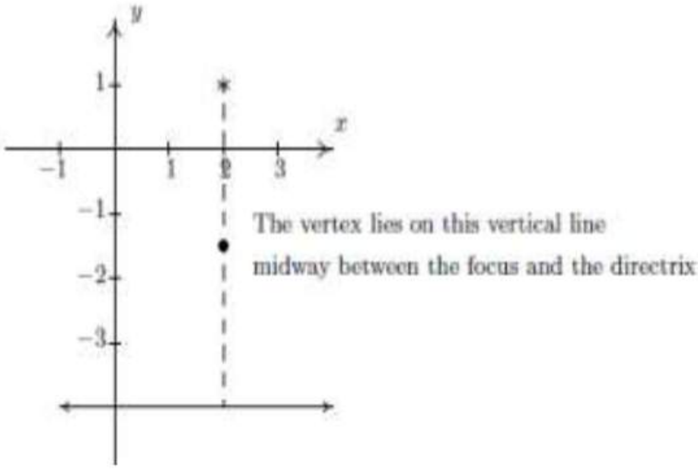
Of all of the information requested in the previous example, only the vertex is part of the graph of the parabola. So in order to get a sense of the actual shape of the graph, we need some more information. While we could plot a few points randomly, a more useful measure of how wide a parabola opens is the length of the parabola's latus rectum.[2] The **latus rectum** of a parabola is the line segment parallel to the directrix which contains the focus. The endpoints of the latus rectum are, then, two points on 'opposite' sides of the parabola. Graphically, we have the following.



It turns out[3] that the length of the latus rectum, called the **focal diameter** of the parabola is $|4p|$, which, in light of Equation 7.2, is easy to find. In our last example, for instance, when graphing $(x + 1)^2 = -8(y - 3)$, we can use the fact that the focal diameter is $|-8| = 8$, which means the parabola is 8 units wide at the focus, to help generate a more accurate graph by plotting points 4 units to the left and right of the focus.

Example 7.3.2. Find the standard form of the parabola with focus $(2, 1)$ and directrix $y = -4$.

Solution. Sketching the data yields,

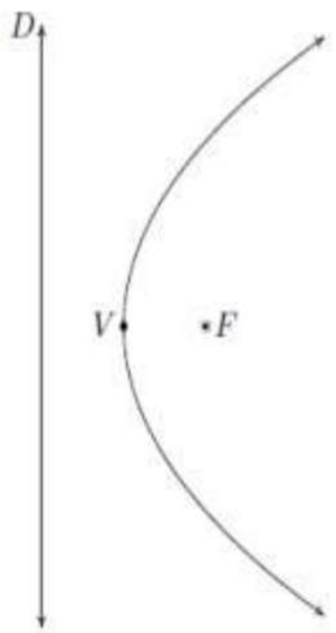


From the diagram, we see the parabola opens upwards. (Take a moment to think about it if you don't see that immediately.) Hence, the vertex lies below the focus and has an x-coordinate of 2. To find the y-coordinate, we note that the distance from the focus to the directrix is $1-(-4)=5$, which means the vertex lies $\frac{5}{2}$ units (halfway) below the focus. Starting at $(2,1)$ and moving down $5/2$ units leaves us at $(2,-3/2)$, which is our vertex. Since the parabola opens upwards, we know p is positive. Thus $p=5/2$. Plugging all of this data into Equation 7.2 give us

$$\begin{aligned}(x-2)^2 &= 4\left(\frac{5}{2}\right)\left(y-\left(-\frac{3}{2}\right)\right)\\ (x-2)^2 &= 10\left(y+\frac{3}{2}\right)\end{aligned}$$

□

If we interchange the roles of x and y, we can produce 'horizontal' parabolas: parabolas which open to the left or to the right. The directrices[4] of such animals would be vertical lines and the focus would either lie to the left or to the right of the vertex, as seen below.



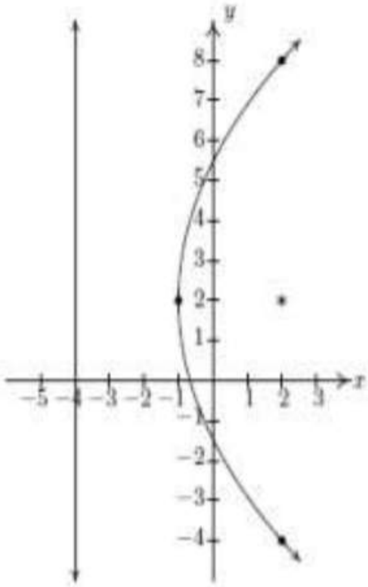
Equation 7.3. The Standard Equation of a Horizontal Parabola: The equation of a (horizontal) parabola with vertex (h,k) and focal length $|p|$ is

$$(y-k)^2 = 4p(x-h)$$

If $p>0$, the parabola opens to the right; if $p<0$, it opens to the left.

Example 7.3.3. Graph $(y-2)^2 = 12(x+1)$. Find the vertex, focus, and directrix.

Solution. We recognize this as the form given in Equation 7.3. Here, $x-h$ is $x+1$ so $h=-1$, and $y-k$ is $y-2$ so $k=2$. Hence, the vertex is $(-1,2)$. We also see that $4p=12$ so $p=3$. Since $p>0$, the focus will be the right of the vertex and the parabola will open to the right. The distance from the vertex to the focus is $|p|=3$, which means the focus is 3 units to the right. If we start at $(-1,2)$ and move right 3 units, we arrive at the focus $(2,2)$. The directrix, then, is 3 units to the left of the vertex and if we move left 3 units from $(-1,2)$, we'd be on the vertical line $x=-4$. Since the focal diameter is $|4p|=12$, the parabola is 12 units wide at the focus, and thus there are points 6 units above and below the focus on the parabola.



□

As with circles, not all parabolas will come to us in the forms in Equations 7.2 or 7.3. If we encounter an equation with two variables in which exactly one variable is squared, we can attempt to put the equation into a standard form using the following steps.

To Write the Equation of a Parabola in Standard Form

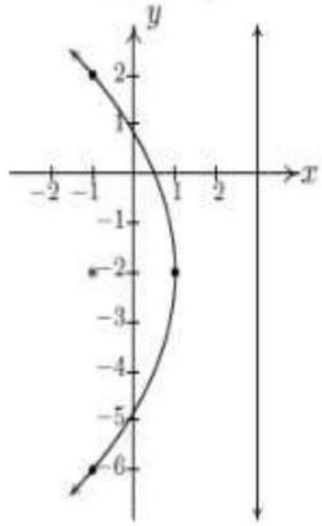
1. Group the variable which is squared on one side of the equation and position the non-squared variable and the constant on the other side.
2. Complete the square if necessary and divide by the coefficient of the perfect square.
3. Factor out the coefficient of the non-squared variable from it and the constant.

Example 7.3.4. Consider the equation $y^2+4y+8x=4$. Put this equation into standard form and graph the parabola. Find the vertex, focus, and directrix.

Solution. We need a perfect square (in this case, using y) on the left-hand side of the equation and factor out the coefficient of the non-squared variable (in this case, the x) on the other.

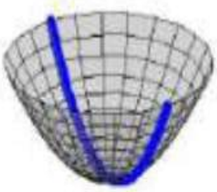
$$\begin{aligned} y^2+4y+8x &= 4 \\ y^2+4y &= -8x+4 \\ y^2+4y+4 &= -8x+4+4 \quad \text{complete the square in } y \text{ only} \\ (y+2)^2 &= -8x+8 \quad \text{factor} \\ (y+2)^2 &= -8(x-1) \end{aligned}$$

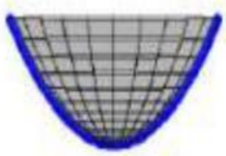
Now that the equation is in the form given in Equation 7.3, we see that $x-h$ is $x-1$ so $h=1$, and $y-k$ is $y+2$ so $k=-2$. Hence, the vertex is $(1,-2)$. We also see that $4p=-8$ so that $p=-2$. Since $p<0$, the focus will be the left of the vertex and the parabola will open to the left. The distance from the vertex to the focus is $|p|=2$, which means the focus is 2 units to the left of 1, so if we start at $(1,-2)$ and move left 2 units, we arrive at the focus $(-1,-2)$. The directrix, then, is 2 units to the right of the vertex, so if we move right 2 units from $(1,-2)$, we'd be on the vertical line $x=3$. Since the focal diameter is $|4p|$ is 8, the parabola is 8 units wide at the focus, so there are points 4 units above and below the focus on the parabola.



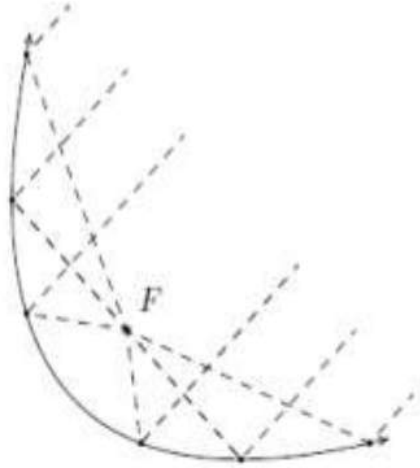
□

In studying quadratic functions, we have seen parabolas used to model physical phenomena such as the trajectories of projectiles. Other applications of the parabola concern its 'reflective property' which necessitates knowing about the focus of a parabola. For example, many satellite dishes are formed in the shape of a **paraboloid of revolution** as depicted below.



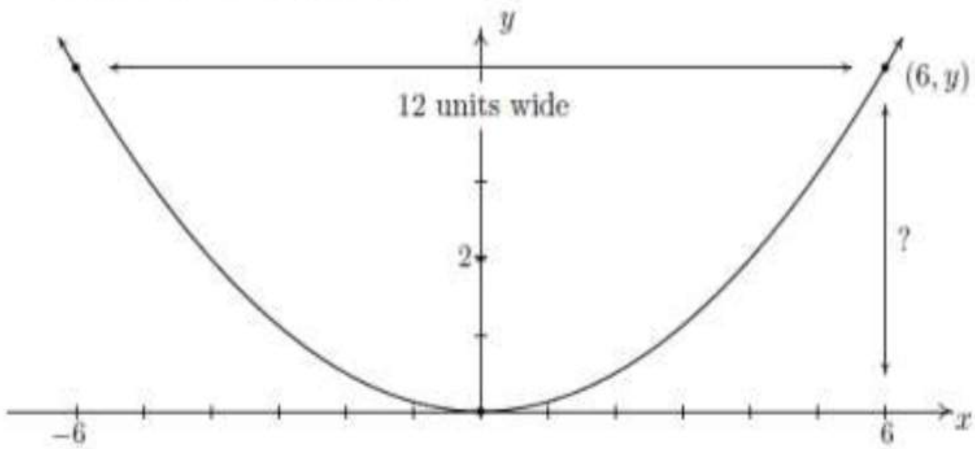


Every cross section through the vertex of the paraboloid is a parabola with the same focus. To see why this is important, imagine the dashed lines below as electromagnetic waves heading towards a parabolic dish. It turns out that the waves reflect off the parabola and concentrate at the focus which then becomes the optimal place for the receiver. If, on the other hand, we imagine the dashed lines as emanating from the focus, we see that the waves are reflected off the parabola in a coherent fashion as in the case in a flashlight. Here, the bulb is placed at the focus and the light rays are reflected off a parabolic mirror to give directional light.



Example 7.3.5. A satellite dish is to be constructed in the shape of a paraboloid of revolution. If the receiver placed at the focus is located 2 ft above the vertex of the dish, and the dish is to be 12 feet wide, how deep will the dish be?

Solution. One way to approach this problem is to determine the equation of the parabola suggested to us by this data. For simplicity, we'll assume the vertex is $(0, 0)$ and the parabola opens upwards. Our standard form for such a parabola is $x^2 = 4py$. Since the focus is 2 units above the vertex, we know $p = 2$, so we have $x^2 = 8y$. Visually,



Since the parabola is 12 feet wide, we know the edge is 6 feet from the vertex. To find the depth, we are looking for the y value when $x = 6$. Substituting $x = 6$ into the equation of the parabola yields $6^2 = 8y$ or $y = \frac{36}{8} = \frac{9}{2} = 4.5$. Hence, the dish will be 4.5 feet deep. □

7.3.1 EXERCISES

In Exercises 1 - 8, sketch the graph of the given parabola. Find the vertex, focus and directrix. Include the endpoints of the latus rectum in your sketch.

1. $(x-3)^2 = -16y$
2. $\left(x+\frac{7}{3}\right)^2 = 2\left(y+\frac{5}{2}\right)$
3. $(y-2)^2 = -12(x+3)$
4. $(y+4)^2 = 4x$
5. $(x-1)^2 = 4(y+3)$
6. $(x+2)^2 = -20(y-5)$
7. $(y-4)^2 = 18(x-2)$
8. $\left(y+\frac{3}{2}\right)^2 = -7\left(x+\frac{9}{2}\right)$

In Exercises 9 - 14, put the equation into standard form and identify the vertex, focus and directrix.

9. $y^2 - 10y - 27x + 133 = 0$
10. $25x^2 + 20x + 5y - 1 = 0$
11. $x^2 + 2x - 8y + 49 = 0$
12. $2y^2 + 4y + x - 8 = 0$
13. $x^2 - 10x + 12y + 1 = 0$
14. $3y^2 - 27y + 4x + \frac{211}{4} = 0$

In Exercises 15 - 18, find an equation for the parabola which fits the given criteria.

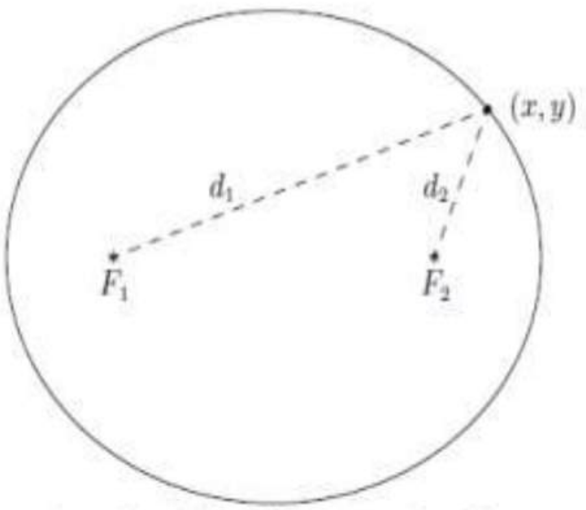
15. Vertex $(7,0)$, focus $(0,0)$
16. Focus $(10,1)$, directrix $x = 5$
17. Vertex $(-8,-9)$; $(0,0)$ and $(-16,0)$ are points on the curve
18. The endpoints of latus rectum are $(-2,-7)$ and $(4,-7)$

7.4 Ellipses

In the definition of a circle, Definition 7.1, we fixed a point called the **center** and considered all of the points which were a fixed distance r from that one point. For our next conic section, the ellipse, we fix two distinct points and a distance d to use in our definition.

Definition 7.4. Given two distinct points F_1 and F_2 in the plane and a fixed distance d, an **ellipse** is the set of all points (x, y) in the plane such that the sum of each of the distances from F_1 and F_2 to (x, y) is d. The points F_1 and F_2 are called the **foci**[a] of the ellipse.

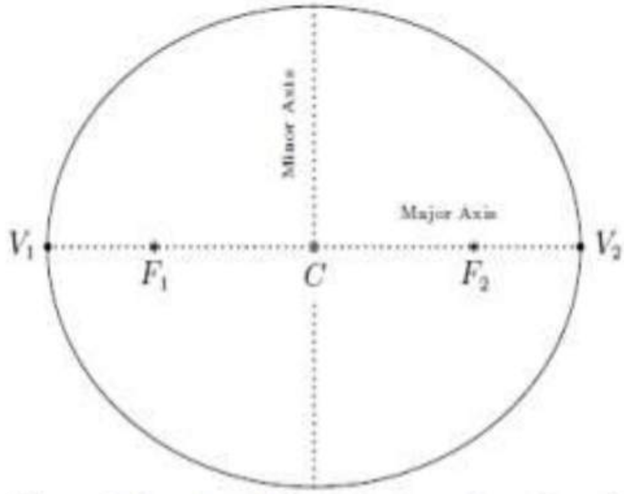
[a] the plural of 'focus'



$d_1 + d_2 = d$ for all (x, y) on the ellipse

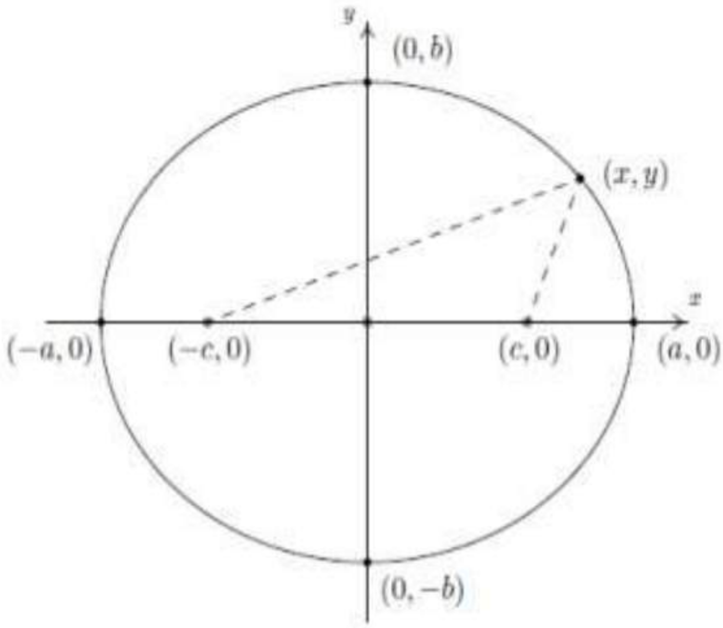
We may imagine taking a length of string and anchoring it to two points on a piece of paper. The curve traced out by taking a pencil and moving it so the string is always taut is an ellipse.

The **center** of the ellipse is the midpoint of the line segment connecting the two foci. The **major axis** of the ellipse is the line segment connecting two opposite ends of the ellipse which also contains the center and foci. The **minor axis** of the ellipse is the line segment connecting two opposite ends of the ellipse which contains the center but is perpendicular to the major axis. The **vertices** of an ellipse are the points of the ellipse which lie on the major axis. Notice that the center is also the midpoint of the major axis, hence it is the midpoint of the vertices. In pictures we have,



An ellipse with center C; foci F_1, F_2; and vertices V_1, V_2

Note that the major axis is the longer of the two axes through the center, and likewise, the minor axis is the shorter of the two. In order to derive the standard equation of an ellipse, we assume that the ellipse has its center at $(0,0)$, its major axis along the x-axis, and has foci $(c,0)$ and $(-c,0)$ and vertices $(-a,0)$ and $(a,0)$. We will label the y-intercepts of the ellipse as $(0,b)$ and $(0,-b)$ (We assume a, b, and c are all positive numbers.) Schematically,



Note that since $(a,0)$ is on the ellipse, it must satisfy the conditions of Definition 7.4. That is, the distance from $(-c,0)$ to $(a,0)$ plus the distance from $(c,0)$ to $(a,0)$ must equal the fixed distance d. Since all of these points lie on the x-axis, we get

$$\begin{aligned} \text{distance from } (-c,0) \text{ to } (a,0) + \text{distance from } (c,0) \text{ to } (a,0) &= d \\ (a+c)+(a-c) &= d \\ 2a &= d \end{aligned}$$

In other words, the fixed distance d mentioned in the definition of the ellipse is none other than the length of the major axis. We now use that fact $(0, b)$ is on the ellipse, along with the fact that $d = 2a$ to get

$$\begin{aligned}
\text{distance from } (-c,0) \text{ to } (0,b) + \text{distance from } (c,0) \text{ to } (0,b) &= 2a \\
\sqrt{(0-(-c))^2+(b-0)^2} + \sqrt{(0-c)^2+(b-0)^2} &= 2a \\
\sqrt{b^2+c^2} + \sqrt{b^2+c^2} &= 2a \\
2\sqrt{b^2+c^2} &= 2a \\
\sqrt{b^2+c^2} &= a
\end{aligned}$$

From this, we get $a^2 = b^2 + c^2$, or $b^2 = a^2 - c^2$, which will prove useful later. Now consider a point (x, y) on the ellipse. Applying Definition 7.4, we get

$$\begin{aligned}
\text{distance from } (-c,0) \text{ to } (x,y) + \text{distance from } (c,0) \text{ to } (x,y) &= 2a \\
\sqrt{(x-(-c))^2+(y-0)^2} + \sqrt{(x-c)^2+(y-0)^2} &= 2a \\
\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} &= 2a
\end{aligned}$$

In order to make sense of this situation, we need to make good use of Intermediate Algebra.

$$\begin{aligned}
\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} &= 2a \\
\sqrt{(x+c)^2+y^2} &= 2a - \sqrt{(x-c)^2+y^2} \\
\left(\sqrt{(x+c)^2+y^2}\right)^2 &= \left(2a - \sqrt{(x-c)^2+y^2}\right)^2 \\
(x+c)^2+y^2 &= 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2+y^2 \\
4a\sqrt{(x-c)^2+y^2} &= 4a^2 + (x-c)^2 - (x+c)^2 \\
4a\sqrt{(x-c)^2+y^2} &= 4a^2 - 4cx \\
a\sqrt{(x-c)^2+y^2} &= a^2 - cx \\
\left(a\sqrt{(x-c)^2+y^2}\right)^2 &= \left(a^2 - cx\right)^2 \\
a^2\left((x-c)^2+y^2\right) &= a^4 - 2a^2cx + c^2x^2 \\
a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 &= a^4 - 2a^2cx + c^2x^2 \\
a^2x^2 - c^2x^2 + a^2y^2 &= a^4 - a^2c^2 \\
\left(a^2-c^2\right)x^2 + a^2y^2 &= a^2\left(a^2-c^2\right)
\end{aligned}$$

We are nearly finished. Recall that $b^2 = a^2 - c^2$ so that

$$\begin{aligned}
\left(a^2-c^2\right)x^2 + a^2y^2 &= a^2\left(a^2-c^2\right) \\
b^2x^2 + a^2y^2 &= a^2b^2 \\
\frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1
\end{aligned}$$

This equation is for an ellipse centered at the origin. To get the formula for the ellipse centered at (h, k), we could use the transformations from Section 1.7 or re-derive the equation using Definition 7.4 and the distance formula to obtain the formula below.

Equation 7.4. The Standard Equation of an Ellipse: For positive unequal numbers a and b, the equation of an ellipse with center (h, k) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Some remarks about Equation 7.4 are in order. First note that the values a and b determine how far in the x and y directions, respectively, one counts from the center to arrive at points on the ellipse. Also take note that if $a > b$, then we have an ellipse whose major axis is horizontal, and hence, the foci lie to the left and right of the center. In this case, as we've seen in the derivation, the distance from the center to the focus, c, can be found by $c = \sqrt{a^2 - b^2}$. If $b > a$, the roles of the major and minor axes are reversed, and the foci lie above and below the center. In this case, $c = \sqrt{b^2 - a^2}$. In either case, c is the distance from the center to each focus, and $c = \sqrt{\text{bigger denominator} - \text{smaller denominator}}$. Finally, it is worth mentioning that if we take the standard equation of a circle, Equation 7.1, and divide both sides by r^2, we get

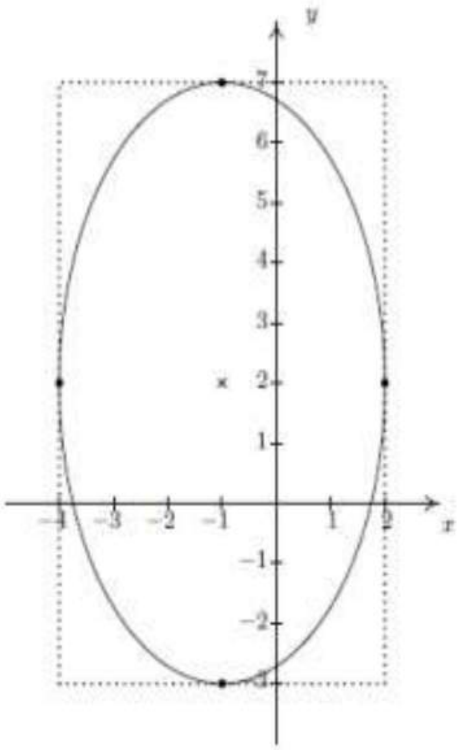
Equation 7.5. The Alternate Standard Equation of a Circle: The equation of a circle with center (h, k) and radius $r > 0$ is

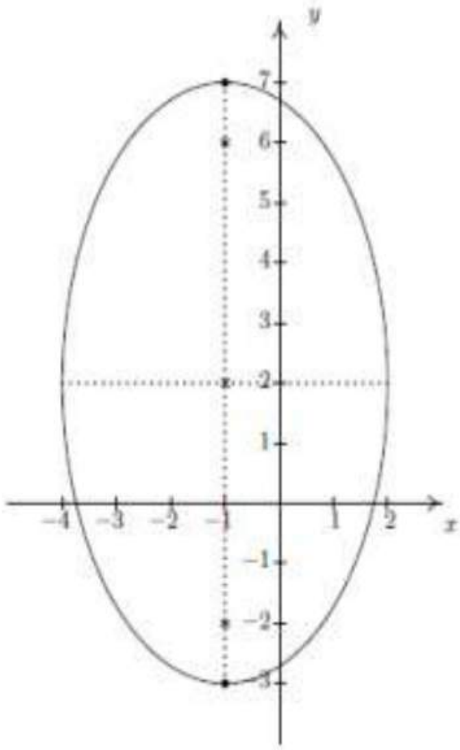
$$\frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{r^2} = 1$$

Notice the similarity between Equation 7.4 and Equation 7.5. Both equations involve a sum of squares equal to 1; the difference is that with a circle, the denominators are the same, and with an ellipse, they are different. If we take a transformational approach, we can consider both Equations 7.4 and 7.5 as shifts and stretches of the Unit Circle $x^2 + y^2 = 1$ in Definition 7.2. Replacing x with $(x - h)$ and y with $(y - k)$ causes the usual horizontal and vertical shifts. Replacing x with $\frac{x}{a}$ and y with $\frac{y}{b}$ causes the usual vertical and horizontal stretches. In other words, it is perfectly fine to think of an ellipse as the deformation of a circle in which the circle is stretched farther in one direction than the other.[1]

Example 7.4.1. Graph $\frac{(x+1)^2}{9} + \frac{(y-2)^2}{25} = 1$. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, and the foci.

Solution. We see that this equation is in the standard form of Equation 7.4. Here $x - h$ is $x + 1$ so $h = -1$, and $y - k$ is $y - 2$ so $k = 2$. Hence, our ellipse is centered at $(-1, 2)$. We see that $a^2 = 9$ so $a = 3$, and $b^2 = 25$ so $b = 5$. This means that we move 3 units left and right from the center and 5 units up and down from the center to arrive at points on the ellipse. As an aid to sketching, we draw a rectangle matching this description, called a **guide rectangle**, and sketch the ellipse inside this rectangle as seen below on the left.

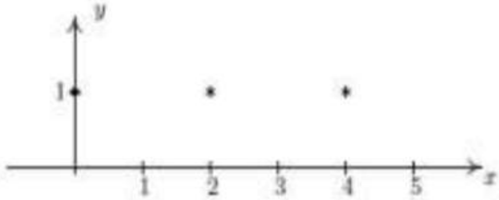




Since we moved farther in the y direction than in the x direction, the major axis will lie along the vertical line $x = -1$, which means the minor axis lies along the horizontal line, $y = 2$. The vertices are the points on the ellipse which lie along the major axis so in this case, they are the points $(-1, 7)$ and $(-1, -3)$, and the endpoints of the minor axis are $(-4, 2)$ and $(2, 2)$. (Notice these points are the four points we used to draw the guide rectangle.) To find the foci, we find $c = \sqrt{25 - 9} = \sqrt{16} = 4$, which means the foci lie 4 units from the center. Since the major axis is vertical, the foci lie 4 units above and below the center, at $(-1, -2)$ and $(-1, 6)$. Plotting all this information gives the graph seen above on the right. □

Example 7.4.2. Find the equation of the ellipse with foci $(2, 1)$ and $(4, 1)$ and vertex $(0, 1)$.

Solution. Plotting the data given to us, we have



From this sketch, we know that the major axis is horizontal, meaning $a > b$. Since the center is the midpoint of the foci, we know it is $(3, 1)$. Since one vertex is $(0, 1)$ we have that $a = 3$, so $a^2 = 9$. All that remains is to find b^2. Since the foci are 1 unit away from the center, we know $c = 1$. Since $a > b$, we have $c = \sqrt{a^2 - b^2}$, or $1 = \sqrt{9 - b^2}$, so $b^2 = 8$. Substituting all of our findings into the equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we get our final answer to be $\frac{(x-3)^2}{9} + \frac{(y-1)^2}{8} = 1$. □

As with circles and parabolas, an equation may be given which is an ellipse, but isn't in the standard form of Equation 7.4. In those cases, as with circles and parabolas before, we will need to massage the given equation into the standard form.

To Write the Equation of an Ellipse in Standard Form

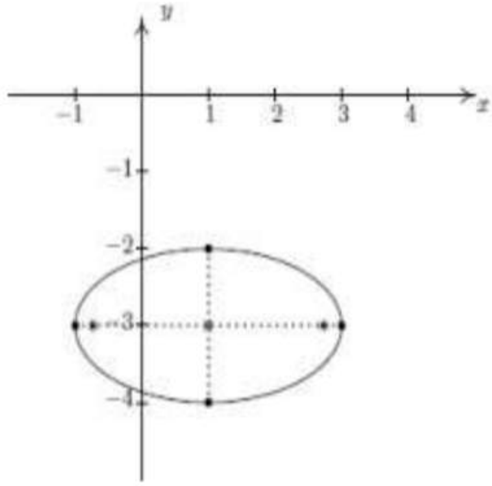
1. Group the same variables together on one side of the equation and position the constant on the other side.
2. Complete the square in both variables as needed.
3. Divide both sides by the constant term so that the constant on the other side of the equation becomes 1.

Example 7.4.3. Graph $x^2+4y^2-2x+24y+33=0$. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, and the foci.

Solution. Since we have a sum of squares and the squared terms have unequal coefficients, it's a good bet we have an ellipse on our hands.[2] We need to complete both squares, and then divide, if necessary, to get the right-hand side equal to 1.

$$\begin{aligned}
x^2+4y^2-2x+24y+33 &= 0 \\
x^2-2x+4y^2+24y &= -33 \\
x^2-2x+4\left(y^2+6y\right) &= -33 \\
\left(x^2-2x+1\right)+4\left(y^2+6y+9\right) &= -33+1+4(9) \\
(x-1)^2+4(y+3)^2 &= 4 \\
\frac{(x-1)^2+4(y+3)^2}{4} &= \frac{4}{4} \\
\frac{(x-1)^2}{4}+(y+3)^2 &= 1 \\
\frac{(x-1)^2}{4}+\frac{(y+3)^2}{1} &= 1
\end{aligned}$$

Now that this equation is in the standard form of Equation 7.4, we see that $x-h$ is $x-1$ so $h=1$, and $y-k$ is $y+3$ so $k=-3$. Hence, our ellipse is centered at $(1,-3)$. We see that $a^2=4$ so $a=2$, and $b^2=1$ so $b=1$. This means we move 2 units left and right from the center and 1 unit up and down from the center to arrive at points on the ellipse. Since we moved farther in the x direction than in the y direction, the major axis will lie along the horizontal line $y=-3$, which means the minor axis lies along the vertical line $x=1$. The vertices are the points on the ellipse which lie along the major axis so in this case, they are the points $(-1,-3)$ and $(3,-3)$, and the endpoints of the minor axis are $(1,-2)$ and $(1,-4)$. To find the foci, we find $c=\sqrt{4-1}=\sqrt{3}$, which means the foci lie $\sqrt{3}$ units from the center. Since the major axis is horizontal, the foci lie $\sqrt{3}$ units to the left and right of the center, at $(1-\sqrt{3},-3)$ and $(1+\sqrt{3},-3)$. Plotting all of this information gives



□

As you come across ellipses in the homework exercises and in the wild, you'll notice they come in all shapes in sizes. Compare the two ellipses below.

Certainly, one ellipse is more round than the other. This notion of 'roundness' is quantified below.

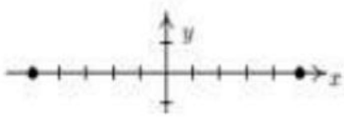
Definition 7.5. The **eccentricity** of an ellipse, denoted e, is the following ratio:

$$e = \frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}}$$

In an ellipse, the foci are closer to the center than the vertices, so $0 < e < 1$. The ellipse above on the left has eccentricity $e \approx 0.98$; for the ellipse above on the right, $e \approx 0.66$. In general, the closer the eccentricity is to 0, the more 'circular' the ellipse; the closer the eccentricity is to 1, the more 'eccentric' the ellipse.

Example 7.4.4. Find the equation of the ellipse whose vertices are $(\pm 5, 0)$ with eccentricity $e = \frac{1}{4}$.

Solution. As before, we plot the data given to us



From this sketch, we know that the major axis is horizontal, meaning $a > b$. With the vertices located at $(\pm 5, 0)$, we get $a = 5$ so $a^2 = 25$. We also know that the center is $(0, 0)$ because the center is the midpoint of the vertices. All that remains is to find b^2. To that end, we use the fact that the eccentricity $e = \frac{1}{4}$ which means

$$e = \frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}} = \frac{c}{a} = \frac{c}{5} = \frac{1}{4}$$

from which we get $c = \frac{5}{4}$. To get b^2, we use the fact that $c = \sqrt{a^2 - b^2}$, so $\frac{5}{4} = \sqrt{25 - b^2}$ from which we get $b^2 = \frac{375}{16}$. Substituting all of our findings into the equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, yields our final answer $\frac{x^2}{25} + \frac{16y^2}{375} = 1$. $\square$

7.4.1 Exercises

In Exercises 1 - 8, graph the ellipse. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, the foci and the eccentricity.

1. $\dfrac{x^2}{169}+\dfrac{y^2}{25}=1$
2. $\dfrac{x^2}{9}+\dfrac{y^2}{25}=1$
3. $\dfrac{(x-2)^2}{4}+\dfrac{(y+3)^2}{9}=1$
4. $\dfrac{(x+5)^2}{16}+\dfrac{(y-4)^2}{1}=1$
5. $\dfrac{(x-1)^2}{10}+\dfrac{(y-3)^2}{11}=1$
6. $\dfrac{(x-1)^2}{9}+\dfrac{(y+3)^2}{4}=1$
7. $\dfrac{(x+2)^2}{16}+\dfrac{(y-5)^2}{20}=1$
8. $\dfrac{(x-4)^2}{8}+\dfrac{(y-2)^2}{18}=1$

In Exercises 9 - 14, put the equation in standard form. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, the foci and the eccentricity.

9. $9x^2+25y^2-54x-50y-119=0$
10. $12x^2+3y^2-30y+39=0$
11. $5x^2+18y^2-30x+72y+27=0$
12. $x^2-2x+2y^2-12y+3=0$
13. $9x^2+4y^2-4y-8=0$
14. $6x^2+5y^2-24x+20y+14=0$

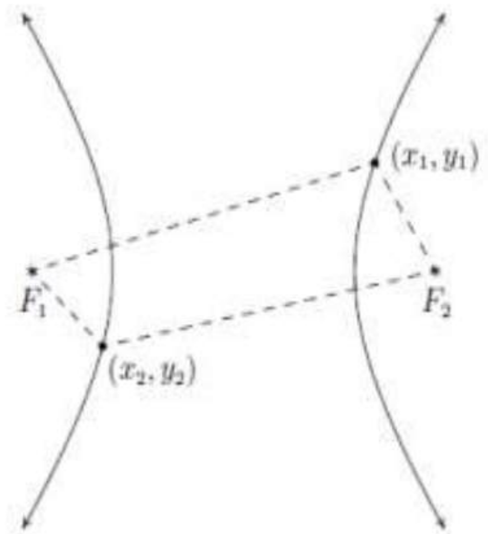
In Exercises 15 - 20, find the standard form of the equation of the ellipse which has the given properties.

15. Center $(3,7)$, Vertex $(3,2)$, Focus $(3,3)$
16. Foci $(0,\pm 5)$, Vertices $(0,\pm 8)$.
17. Foci $(\pm 3,0)$, length of the Minor Axis 10
18. Vertices $(3,2)$, $(13,2)$; Endpoints of the Minor Axis $(8,4)$, $(8,0)$
19. Center $(5,2)$, Vertex $(0,2)$, eccentricity $\frac{1}{2}$

7.5 HYPERBOLAS

In the definition of an ellipse, Definition 7.4, we fixed two points called foci and looked at points whose distances to the foci always **added** to a constant distance d. Those prone to syntactical tinkering may wonder what, if any, curve we'd generate if we replaced **added** with **subtracted**. The answer is a hyperbola.

Definition 7.6. Given two distinct points F_1 and F_2 in the plane and a fixed distance d, a **hyperbola** is the set of all points (x, y) in the plane such that the absolute value of the difference of each of the distances from F_1 and F_2 to (x, y) is d. The points F_1 and F_2 are called the **foci** of the hyperbola.



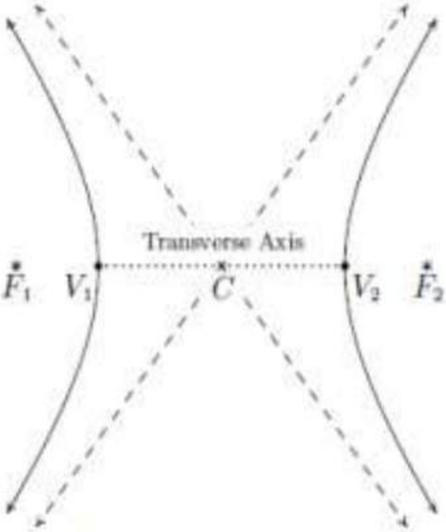
In the figure above:

$$\text{the distance from } F_1 \text{ to } (x_1, y_1) - \text{the distance from } F_2 \text{ to } (x_1, y_1) \;=\; d$$

and

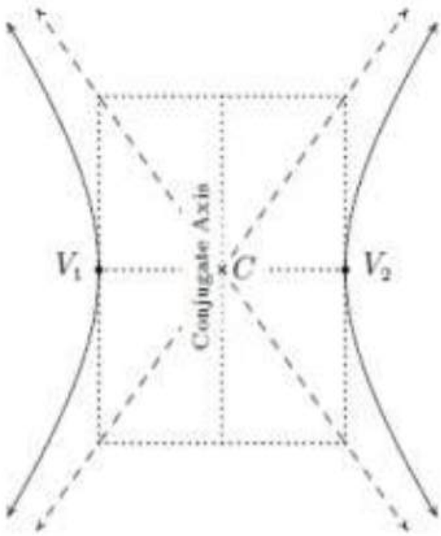
$$\text{the distance from } F_2 \text{ to } (x_2, y_2) - \text{the distance from } F_1 \text{ to } (x_2, y_2) \;=\; d$$

Note that the hyperbola has two parts, called **branches**. The **center** of the hyperbola is the midpoint of the line segment connecting the two foci. The **transverse axis** of the hyperbola is the line segment connecting two opposite ends of the hyperbola which also contains the center and foci. The **vertices** of a hyperbola are the points of the hyperbola which lie on the transverse axis. In addition, we will show momentarily that there are lines called **asymptotes** which the branches of the hyperbola approach for large x and y values. They serve as guides to the graph. In pictures,



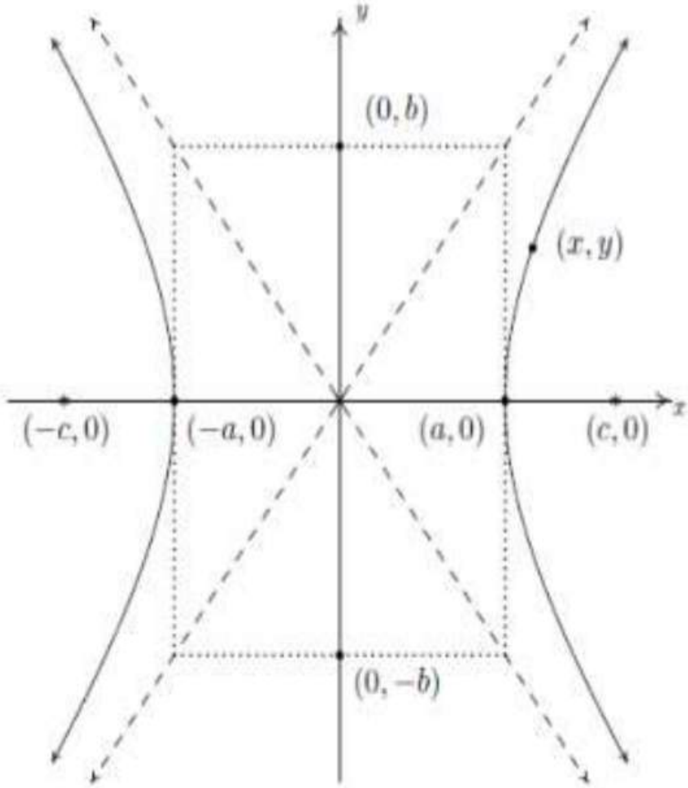
A hyperbola with center C; foci F_1, F_2; and vertices V_1, V_2 and asymptotes (dashed)

Before we derive the standard equation of the hyperbola, we need to discuss one further parameter, the **conjugate axis** of the hyperbola. The conjugate axis of a hyperbola is the line segment through the center which is perpendicular to the transverse axis and has the same length as the line segment through a vertex which connects the asymptotes. In pictures we have



Note that in the diagram, we can construct a rectangle using line segments with lengths equal to the lengths of the transverse and conjugate axes whose center is the center of the hyperbola and whose diagonals are contained in the asymptotes. This **guide rectangle**, much akin to the one we saw Section 7.4 to help us graph ellipses, will aid us in graphing hyperbolas.

Suppose we wish to derive the equation of a hyperbola. For simplicity, we shall assume that the center is $(0, 0)$, the vertices are $(a, 0)$ and $(-a, 0)$ and the foci are $(c, 0)$ and $(-c, 0)$. We label the endpoints of the conjugate axis $(0, b)$ and $(0, -b)$. (Although b does not enter into our derivation, we will have to justify this choice as you shall see later.) As before, we assume a, b, and c are all positive numbers. Schematically we have



Since $(a, 0)$ is on the hyperbola, it must satisfy the conditions of Definition 7.6. That is, the distance from $(-c, 0)$ to $(a, 0)$ minus the distance from $(c, 0)$ to $(a, 0)$ must equal the fixed distance d. Since all these points lie on the x-axis, we get

$$\begin{aligned} \text{distance from } (-c,0) \text{ to } (a,0) - \text{distance from } (c,0) \text{ to } (a,0) &= d \\ (a+c)-(c-a) &= d \\ 2a &= d \end{aligned}$$

In other words, the fixed distance d from the definition of the hyperbola is actually the length of the transverse axis! (Where have we seen that type of coincidence before?) Now consider a point (x, y) on the hyperbola. Applying Definition 7.6, we get

$$\begin{aligned} \text{distance from } (-c,0) \text{ to } (x,y) - \text{distance from } (c,0) \text{ to } (x,y) &= 2a \\ \sqrt{(x-(-c))^2+(y-0)^2} - \sqrt{(x-c)^2+(y-0)^2} &= 2a \\ \sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} &= 2a \end{aligned}$$

Using the same arsenal of Intermediate Algebra weaponry we used in deriving the standard formula of an ellipse, Equation 7.4, we arrive at the following.[1]

$$\left(a^2-c^2\right)x^2+a^2y^2 = a^2\left(a^2-c^2\right)$$

What remains is to determine the relationship between a, b and c. To that end, we note that since a and c are both positive numbers with $a < c$, we get $a^2 < c^2$ so that $a^2 - c^2$ is a negative number. Hence, $c^2 - a^2$ is a positive number. For reasons which will become clear soon, we re-write the equation by solving for y^2/x^2 to get

$$\begin{aligned}
\left(a^2-c^2\right)x^2+a^2y^2 &= a^2\left(a^2-c^2\right)\\
-\left(c^2-a^2\right)x^2+a^2y^2 &= -a^2\left(c^2-a^2\right)\\
a^2y^2 &= \left(c^2-a^2\right)x^2-a^2\left(c^2-a^2\right)\\
\frac{y^2}{x^2} &= \frac{\left(c^2-a^2\right)}{a^2}-\frac{\left(c^2-a^2\right)}{x^2}
\end{aligned}$$

As x and y attain very large values, the quantity $\frac{(c^2-a^2)}{x^2} \to 0$ so that $\frac{y^2}{x^2} \to \frac{(c^2-a^2)}{a^2}$. By setting $b^2 = c^2 - a^2$ we get $\frac{y^2}{x^2} \to \frac{b^2}{a^2}$. This shows that $y \to \pm\frac{b}{a}x$ as $|x|$ grows large. Thus $y = \pm\frac{b}{a}x$ are the asymptotes to the graph as predicted and our choice of labels for the endpoints of the conjugate axis is justified. In our equation of the hyperbola we can substitute $a^2 - c^2 = -b^2$ which yields

$$\begin{aligned}
\left(a^2-c^2\right)x^2+a^2y^2 &= a^2\left(a^2-c^2\right)\\
-b^2x^2+a^2y^2 &= -a^2b^2\\
\frac{x^2}{a^2}-\frac{y^2}{b^2} &= 1
\end{aligned}$$

The equation above is for a hyperbola whose center is the origin and which opens to the left and right. If the hyperbola were centered at a point (h,k), we would get the following.

Equation 7.6. The Standard Equation of a Horizontal[a] Hyperbola For positive numbers a and b, the equation of a horizontal hyperbola with center (h,k) is

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

[a]That is, a hyperbola whose branches open to the left and right

If the roles of x and y were interchanged, then the hyperbola's branches would open upwards and downwards and we would get a 'vertical' hyperbola.

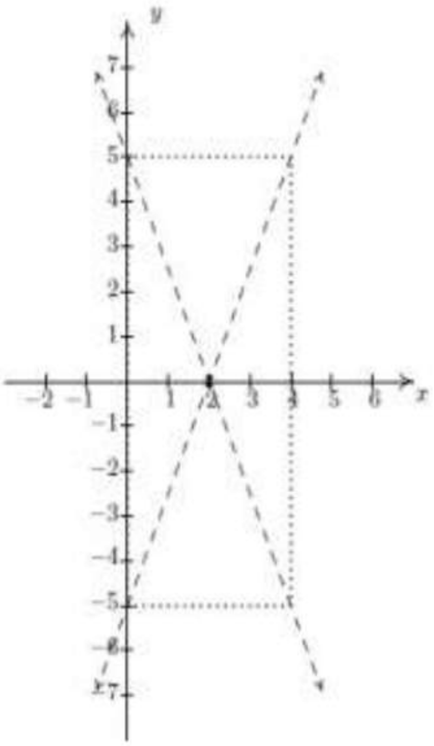
Equation 7.7. The Standard Equation of a Vertical Hyperbola For positive numbers a and b, the equation of a vertical hyperbola with center (h,k) is:

$$\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$$

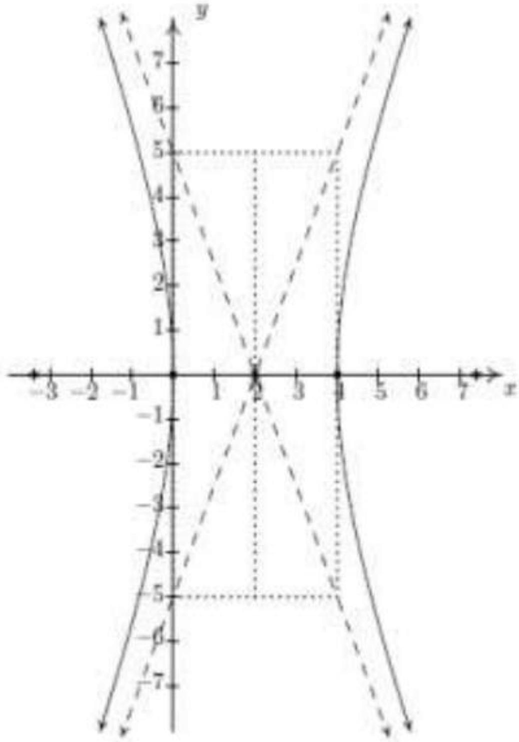
The values of a and b determine how far in the x and y directions, respectively, one counts from the center to determine the rectangle through which the asymptotes pass. In both cases, the distance from the center to the foci, c, as seen in the derivation, can be found by the formula $c = \sqrt{a^2 + b^2}$. Lastly, note that we can quickly distinguish the equation of a hyperbola from that of a circle or ellipse because the hyperbola formula involves a **difference** of squares where the circle and ellipse formulas both involve the **sum** of squares.

Example 7.5.1. Graph the equation $\frac{(x-2)^2}{4} - \frac{y^2}{25} = 1$. Find the center, the lines which contain the transverse and conjugate axes, the vertices, the foci and the equations of the asymptotes.

Solution. We first see that this equation is given to us in the standard form of Equation 7.6. Here $x - h$ is $x - 2$ so $h = 2$, and $y - k$ is y so $k = 0$. Hence, our hyperbola is centered at $(2, 0)$. We see that $a^2 = 4$ so $a = 2$, and $b^2 = 25$ so $b = 5$. This means we move 2 units to the left and right of the center and 5 units up and down from the center to arrive at points on the guide rectangle. The asymptotes pass through the center of the hyperbola as well as the corners of the rectangle. This yields the following set up.



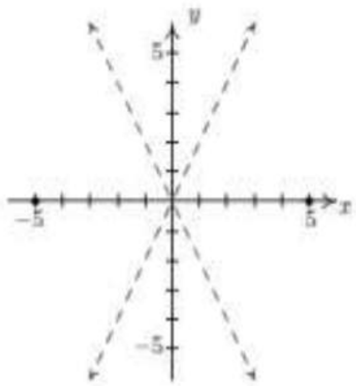
Since the y^2 term is being subtracted from the x^2 term, we know that the branches of the hyperbola open to the left and right. This means that the transverse axis lies along the x-axis. Hence, the conjugate axis lies along the vertical line $x = 2$. Since the vertices of the hyperbola are where the hyperbola intersects the transverse axis, we get that the vertices are 2 units to the left and right of $(2, 0)$ at $(0, 0)$ and $(4, 0)$. To find the foci, we need $c = \sqrt{a^2 + b^2} = \sqrt{4 + 25} = \sqrt{29}$. Since the foci lie on the transverse axis, we move $\sqrt{29}$ units to the left and right of $(2, 0)$ to arrive at $(2 - \sqrt{29}, 0)$ (approximately $(-3.39, 0)$) and $(2 + \sqrt{29}, 0)$ (approximately $(7.39, 0)$). To determine the equations of the asymptotes, recall that the asymptotes go through the center of the hyperbola, $(2, 0)$, as well as the corners of guide rectangle, so they have slopes of $\pm\frac{b}{a} = \pm\frac{5}{2}$. Using the point-slope equation of a line, Equation 2.2, yields $y - 0 = \pm\frac{5}{2}(x-2)$, so we get $y = \frac{5}{2}x - 5$ and $y = -\frac{5}{2}x + 5$. Putting it all together, we get



□

Example 7.5.2. Find the equation of the hyperbola with asymptotes $y = \pm 2x$ and vertices $(\pm 5, 0)$.

Solution. Plotting the data given to us, we have



This graph not only tells us that the branches of the hyperbola open to the left and to the right, it also tells us that the center is $(0,0)$. Hence, our standard form is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Since the vertices are $(\pm 5, 0)$, we have $a = 5$ so $a^2 = 25$. In order to determine b^2, we recall that the slopes of the asymptotes are $\pm\frac{b}{a}$. Since $a = 5$ and the slope of the line $y = 2x$ is 2, we have that $\frac{b}{5} = 2$, so $b = 10$. Hence, $b^2 = 100$ and our final answer is $\frac{x^2}{25} - \frac{y^2}{100} = 1$. □

As with the other conic sections, an equation whose graph is a hyperbola may not be given in either of the standard forms. To rectify that, we have the following.

To Write the Equation of a Hyperbola in Standard Form

1. Group the same variables together on one side of the equation and position the constant on the other side
2. Complete the square in both variables as needed
3. Divide both sides by the constant term so that the constant on the other side of the equation becomes 1

Example 7.5.3. Consider the equation $9y^2 - x^2 - 6x = 10$. Put this equation in to standard form and graph. Find the center, the lines which contain the transverse and conjugate axes, the vertices, the foci, and the equations of the asymptotes.

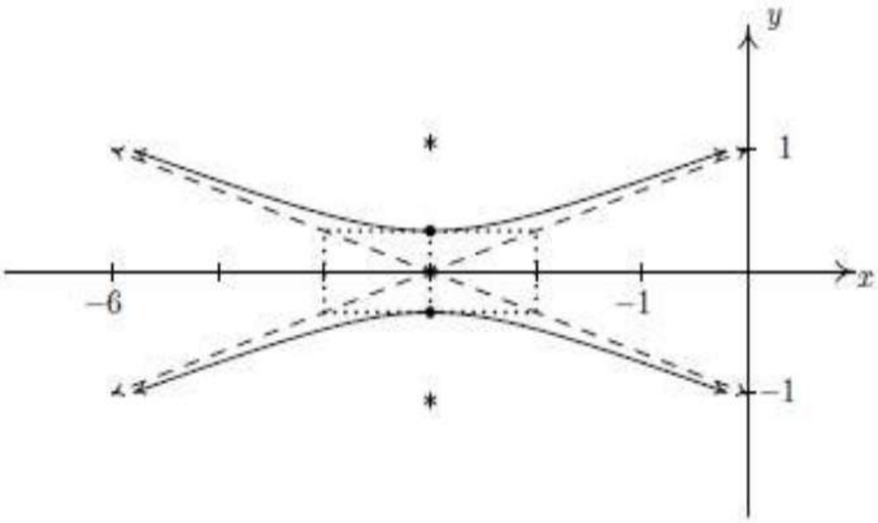
Solution. We need only complete the square on x:

$$\begin{aligned} 9y^2 - x^2 - 6x &= 10 \\ 9y^2 - 1\left(x^2 + 6x\right) &= 10 \\ 9y^2 - \left(x^2 + 6x + 9\right) &= 10 - 1(9) \\ 9y^2 - (x+3)^2 &= 1 \\ \frac{y^2}{\frac{1}{9}} - \frac{(x+3)^2}{1} &= 1 \end{aligned}$$

Now that this equation is in the standard form of Equation 7.7, we see that $x-h$ is $x+3$ so $h=-3$, and $y-k$ is y so $k=0$. Hence, our hyperbola is centered at $(-3,0)$. We find that $a^2 = 1$ so $a=1$, and $b^2 = \frac{1}{9}$ so $b = \frac{1}{3}$. This means that we move 1 unit to the left and right of the center and $\frac{1}{3}$ units up and down from the center to arrive at points on the guide rectangle. Since the x^2 term is being subtracted from the y^2 term, we know the branches of the hyperbola open upwards and downwards. This means the transverse axis lies along the vertical line $x=-3$ and the conjugate axis lies along the x-axis. Since the vertices of the hyperbola are where the hyperbola intersects the transverse axis, we get that the vertices are $\frac{1}{3}$ of a unit above and below $(-3,0)$ at $\left(-3, \frac{1}{3}\right)$ and $\left(-3, -\frac{1}{3}\right)$. To find the foci, we use

$$c = \sqrt{a^2+b^2} = \sqrt{\frac{1}{9}+1} = \frac{\sqrt{10}}{3}$$

Since the foci lie on the transverse axis, we move $\frac{\sqrt{10}}{3}$ units above and below $(-3,0)$ to arrive at $\left(-3, \frac{\sqrt{10}}{3}\right)$ and $\left(-3, -\frac{\sqrt{10}}{3}\right)$. To determine the asymptotes, recall that the asymptotes go through the center of the hyperbola, $(-3,0)$, as well as the corners of guide rectangle, so they have slopes of $\pm\frac{b}{a} = \pm\frac{1}{3}$. Using the point-slope equation of a line, Equation 2.2, we get $y = \frac{1}{3}x + 1$ and $y = -\frac{1}{3}x - 1$. Putting it all together, we get



Each of the conic sections we have studied in this chapter result from graphing equations of the form $Ax^2+Cy^2+Dx+Ey+F=0$ for different choices of A, C, D, E, and[5] F. While we've seen examples[6] demonstrate *how* to convert an equation from this general form to one of the standard forms, we close this chapter with some advice about *which* standard form to choose.[7]

Strategies for Identifying Conic Sections

Suppose the graph of equation $Ax^2+Cy^2+Dx+Ey+F=0$ is a non-degenerate conic section.[a]

- If just *one* variable is squared, the graph is a parabola. Put the equation in the form of Equation 7.2 (if x is squared) or Equation 7.3 (if y is squared).

If *both* variables are squared, look at the coefficients of x^2 and y^2, A and B.

- If $A=B$, the graph is a circle. Put the equation in the form of Equation 7.1.
- If $A \neq B$ but A and B have the *same sign*, the graph is an ellipse. Put the equation in the form of Equation 7.4.
- If A and B have the *different signs*, the graph is a hyperbola. Put the equation in the form of either Equation 7.6 or Equation 7.7.

[a]That is, a parabola, circle, ellipse, or hyperbola – see Section 7.1.

7.5.1 Exercises

In Exercises 1 - 8, graph the hyperbola. Find the center, the lines which contain the transverse and conjugate axes, the vertices, the foci and the equations of the asymptotes.

1. $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$
2. $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$
3. $\dfrac{(x-2)^2}{4} - \dfrac{(y+3)^2}{9} = 1$
4. $\dfrac{(y-3)^2}{11} - \dfrac{(x-1)^2}{10} = 1$
5. $\dfrac{(x+4)^2}{16} - \dfrac{(y-4)^2}{1} = 1$
6. $\dfrac{(x+1)^2}{9} - \dfrac{(y-3)^2}{4} = 1$
7. $\dfrac{(y+2)^2}{16} - \dfrac{(x-5)^2}{20} = 1$
8. $\dfrac{(x-4)^2}{8} - \dfrac{(y-2)^2}{18} = 1$

In Exercises 9 - 12, put the equation in standard form. Find the center, the lines which contain the transverse and conjugate axes, the vertices, the foci and the equations of the asymptotes.

9. $12x^2 - 3y^2 + 30y - 111 = 0$
10. $18y^2 - 5x^2 + 72y + 30x - 63 = 0$
11. $9x^2 - 25y^2 - 54x - 50y - 169 = 0$
12. $-6x^2 + 5y^2 - 24x + 40y + 26 = 0$

In Exercises 13 - 18, find the standard form of the equation of the hyperbola which has the given properties.

13. Center $(3, 7)$, Vertex $(3, 3)$, Focus $(3, 2)$
14. Vertex $(0, 1)$, Vertex $(8, 1)$, Focus $(-3, 1)$
15. Foci $(0, \pm 8)$, Vertices $(0, \pm 5)$.
16. Foci $(\pm 5, 0)$, length of the Conjugate Axis 6
17. Vertices $(3, 2)$, $(13, 2)$; Endpoints of the Conjugate Axis $(8, 4)$, $(8, 0)$
18. Vertex $(-10, 5)$, Asymptotes $y = \pm\frac{1}{2}(x-6) + 5$

In Exercises 19 - 28, find the standard form of the equation using the guidelines on page 540 and then graph the conic section.

19. $x^2 - 2x - 4y - 11 = 0$
20. $x^2 + y^2 - 8x + 4y + 11 = 0$
21. $9x^2 + 4y^2 - 36x + 24y + 36 = 0$
22. $9x^2 - 4y^2 - 36x - 24y - 36 = 0$

23. $y^2 + 8y - 4x + 16 = 0$

24. $4x^2 + y^2 - 8x + 4 = 0$

25. $4x^2 + 9y^2 - 8x + 54y + 49 = 0$

26. $x^2 + y^2 - 6x + 4y + 14 = 0$

27. $2x^2 + 4y^2 + 12x - 8y + 25 = 0$

28. $4x^2 - 5y^2 - 40x - 20y + 160 = 0$

Chapter 6

Introduction to Integration techniques

Antiderivatives

Antiderivatives are the inverse operations of derivatives or the backward operation which goes from the derivative of a function to the original function itself in addition with a constant. Mathematically, the antiderivative of a function on an interval *I* is stated as

$F'(x) = f(x)\ \forall\, x \in interval\ I$

> DEFINITION A function F is an antiderivative of f on an interval I if $F`(x) = f(x)$ for all x in I

EXAMPLE 1

Find an antiderivative for each of the following functions.

$a)\ f(x) = 2x \qquad b)\ g(x) = \cos x \qquad c)\ h(x) = \frac{1}{x} + 2e^{2x}$

Solution

We need to think backward here: What function do we know has a derivative equal to the given function?

Each answer can be checked by differentiating.

The derivative of $F(x) = x^2$ is $f(x) = 2x$

The derivative of $G(x) = sinx$ is $g(x) = cosx$

and the derivative of $H(x) = \ln|x| + e^{2x}$ is $h(x) = \frac{1}{x} + 2e^{2x}$

then

$a)\ F(x) = x^2 \qquad b)\ G(x) = \sin x \qquad c)\ H(x) = \ln|x| + e^{2x}$

<u>Note</u>

The function $F(x) = x^2$ is not the only function whose derivative is 2x. The function $F(x) = x^2 + 1$ has the same derivative. So does $F(x) = x^2 + C$ for any constant C.

There for the solution of the last example will be

$a)\ F(x) = x^2 + C \qquad b)\ G(x) = \sin x + C \qquad c)\ H(x) = \ln|x| + e^{2x} + C$

EXAMPLE 2

Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$

Solution

Since the derivative of x^3 is $3x^2$, the general antiderivative

$F(x) = x^3 + C$

The condition $F(1) = -1$ determines a specific value for C.

Substituting $x = 1$ into $F(x) = x^3 + C$ gives $C = -2$ So

$F(x) = x^3 - 2$

EXAMPLE 3

$f(x) = 2x - \frac{1}{x^2}$ Find an antiderivative of the function

Solution

and $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$ Since $\frac{d}{dx}(x^2) = 2x$

we see that one antiderivative of $f(x)$ is given by

$$F(x) = x^2 + \frac{1}{x} + C$$

However, any function of the form $x^2 + \frac{1}{x} + c$, c a constant, will do, since

$$\frac{d}{dx}(x^2+\frac{1}{x}+c)=2x-\frac{1}{x^2}+0=2x-\frac{1}{x^2}$$

In general, let the function $f(x)$ be continuous on the closed interval $[a,b]$. Then, the antiderivative for $f\in[a,b]$ is $F(x)$ if and only if $F(x)$ is a continuous function on the closed interval $[a,b]$ and $F'(x)=f(x)$ for all $x\in(a,b)$. This is commonly named as "indefinite integral", which is given below:

$$\int f(x)dx=F(x)+c$$

Where, $f(x)$ is the function on an interval I, $F(x)$ is an antiderivative of $f(x)$.

The symbol $\int$ is an integral sign. We call $\int f(x)dx$, the indefinite integral of $f(x)$. The expression $f(x)$ is the integrand and c is the constant of integration. The process of finding $F(x)+c$, when given $\int f(x)dx$, is referred to as indefinite integration, evaluating the integral, or integrating $f(x)$. We regard dx merely as symbol that specifies the independent variable x, which we refer to as the variable of integration. If we use a different variable of integration, such as t, we write

$$\int f(t)dt=F(t)+c \ where \ \ F`(t)=f(t).$$

Table of Integrals :

Every formula of differentiation becomes a corresponding formula of integration. Thus, we give a brief table of indefinite integrals.

Derivative $\frac{d}{dx}f(x)$	**Indefinite Integral** $\int\left(\frac{d}{dx}f(x)dx\right)=f(x)+c$
$\frac{d}{dx}(x)=1$	$\int 1dx=x+C$
$\frac{d}{dx}\left(\frac{x^{r+1}}{x+1}\right)=x^r, r\neq -1$	$\int x^r\,dx=\frac{x^{r+1}}{r+1}+C$
$\frac{d}{dx}(\sin kx)=k\cos kx$	$\int \cos kx\;dx=\frac{1}{k}\sin kx+C$
$\frac{d}{dx}(-\cos kx)=k\sin kx$	$\int \sin kx\;dx=-\frac{1}{k}\cos kx+C$
$\frac{d}{dx}(\tan kx)=k\,sec^2\,kx$	$\int sec^2\,kx\;dx=\frac{1}{k}\tan kx+C$
$\frac{d}{dx}(-\cot kx)=k\,csc^2\,kx$	$\int csc^2\,kx\;dx=-\frac{1}{k}\cot kx+C$
$\frac{d}{dx}(\sec kx)=k\,sec\,kx\tan kx$	$\int sec\,kx\tan kx\;dx=\frac{1}{k}\sec kx+C$
$\frac{d}{dx}(\csc kx)=-k\,csc\,kx\cot kx$	$\int csc\,kx\cot kx\;dx=-\frac{1}{k}\csc kx+C$
$\frac{d}{dx}(\ln kx)=\frac{1}{x}$	$\int \frac{1}{x}\,dx=\ln x+C$
$\frac{d}{dx}e^{kx}=ke^{kx}$	$\int e^{kx}\,dx=\frac{1}{k}e^{kx}+C$
$\frac{d}{dx}a^{kx}=ka^{kx}\ln a$	$\int a^{kx}\,dx=\frac{1}{k\ln a}a^{kx}+C$
$\frac{d}{dx}sin^{-1}x=\frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,dx=sin^{-1}x+C$

$\frac{d}{dx}tan^{-1}x = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = tan^{-1}x + C$
$\frac{d}{dx}sec^{-1}x = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\int \frac{1}{\lvert x\rvert\sqrt{x^2-1}}\,dx = sec^{-1}x + C$
$\frac{d}{dx}sinh^{-1}x = \frac{1}{\sqrt{1+x^2}}$	$\int \frac{1}{\sqrt{1+x^2}}\,dx = sinh^{-1}x + C$
$\frac{d}{dx}cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}$	$\int \frac{1}{\sqrt{x^2-1}}\,dx = cosh^{-1}x + C$
$\frac{d}{dx}tanh^{-1}x = \frac{1}{1-x^2}$	$\int \frac{1}{1-x^2}\,dx = tanh^{-1}x + C$
$\frac{d}{dx}coth^{-1}x = \frac{1}{1-x^2}$	$\int \frac{1}{1-x^2}\,dx = coth^{-1}x + C$
$\frac{d}{dx}sech^{-1}x = -\frac{1}{x\sqrt{1-x^2}}$	$\int \frac{1}{x\sqrt{1-x^2}}\,dx = -sech^{-1}x + C$
$\frac{d}{dx}csch^{-1}x = \frac{1}{\lvert x\rvert\sqrt{1+x^2}}$	$\int \frac{1}{\lvert x\rvert\sqrt{1+x^2}}\,dx = -csch^{-1}x + C$
$\frac{d}{dx}\sinh kx = k\cosh kx$	$\int \cosh kx\,dx = \frac{1}{k}\sinh kx + C$
$\frac{d}{dx}\cosh kx = k\sinh kx$	$\int \sinh kx\,dx = \frac{1}{k}\cosh kx + C$
$\frac{d}{dx}\tanh kx = k sech^2 kx$	$\int sech^2\,kx\,dx = \frac{1}{k}\tanh kx + C$
$\frac{d}{dx}\coth kx = -k\,csch^2 kx$	$\int csch^2\,kx\,dx = -\frac{1}{k}\coth kx + C$
$\frac{d}{dx}\operatorname{sech} kx = -k\operatorname{sech} kx\tanh kx$	$\int \operatorname{sech} kx\tanh kx\;dx = -\frac{1}{k}\operatorname{sech} kx + C$
$\frac{d}{dx}\operatorname{csch} kx = -k\operatorname{csch} kx\coth kx$	$\int \operatorname{csch} kx\coth kx\;dx = -\frac{1}{k}\operatorname{csch} kx + C$

EXAMPLE 4
Find the general antiderivative of each of the following functions.

(a) $f(x)=x^5$ (b) $g(x)=\dfrac{1}{\sqrt{x}}$

(c) $h(x)=\sin 2x$ (d) $i(x)=\cos(x/2)$

(e) $j(x)=e^{-3x}$ (f) $k(x)=2^x$

Solution

(a) $F(x)=\dfrac{x^6}{6}+C$ (b) $G(x)=-2\sqrt{x}+C$

(c) $H(x)=-\dfrac{1}{2}\cos 2x+C$ (d) $I(x)=2\sin(x/2)+C$

(e) $J(x)=\dfrac{1}{-3}e^{-3x}+C$ (f) $K(x)=\dfrac{1}{\ln 2}2^x+C$

EXAMPLE 5
Discuss whether the following integrals are right or wrong?

$$\int(\cos(\ln x))\,dx=\frac{x}{2}(\cos(\ln x)+\sin(\ln x))+C$$

Solution

$$\because \int f(x)\,dx=F(x)+c \quad \Leftrightarrow \quad \therefore \frac{d}{dx}\big(F(x)+c\big)=f(x)$$

$$\frac{d}{dx}\left[\frac{x}{2}(\cos(\ln x)+\sin(\ln x))+C\right]$$

$$=\frac{1}{2}(\cos(\ln x)+\sin(\ln x))+\frac{x}{2}\left(-\frac{1}{x}\sin(\ln x)+\frac{1}{x}\cos(\ln x)\right)$$

$$=\frac{1}{2}\big(\cos(\ln x)+\sin(\ln x)-\sin(\ln x)+\cos(\ln x)\big)$$

$$=\frac{1}{2}\Big(\cos(\ln x)+\cos\big(\ln x\big)\Big)=\cos\big(\ln x\big)$$

$$\therefore \int\big(\cos(\ln x)\big)\,dx=\frac{x}{2}\big(\cos(\ln x)+\sin(\ln x)\big)+C$$

Some Rules for Calculating Integrals

Rules for operating with integrals are derived from the rules for operating with derivatives. So, because

for any constant c, $\frac{d}{dx}\left(C\, f(x)\right) = C\,\frac{d}{dx}\left(f(x)\right)$,

we hasve Rule 1

for any constant c, $\int \left(C\, f(x)\right) dx = C\int \left(f(x)\right) dx$,

For example

$\int 10\,\cos x\, dx = 10\int \cos x\, dx = 10\sin x + C.$

It sometimes helps people to understand and remember rules like this if they say them in words. The rule given above says: The integral of a constant multiple of a function is a constant multiple of the integral of the function. Another way of putting it is You can move a constant past the integral sign without changing the value of the expression.

Similarly, from

$$\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx}\left(f(x)\right) + \frac{d}{dx}\left(g(x)\right),$$

we can derive the rule Rule 2

$$\int \left(f(x) + g(x)\right) dx = \int \left(f(x)\right) dx + \int \left(g(x)\right) dx,$$

For example,

$$\int \left(e^x + 2x\right) dx = \int e^x\, dx + \int 2x\, dx = e^x + x^2 + C.$$

In words, the integral of the sum of two functions is the sum of their integrals. We can easily extend this rule to include differences as well as sums, and to the case where there are more than two terms in the sum.

EXAMPLE 6
Evaluate each of the following Integrals.

(c) $\int \frac{dx}{x+3}$ (b) $\int \frac{\tan x}{\sec x} dx$ (a) $\int (3\sqrt{x} - 4x) dx$

(e) $\int \frac{du}{\cos u \cot u}$ (d) $\int \frac{(x^2-1)^2}{x^2} dx$

Solution

(a) $\int (3\sqrt{x} - 4x) dx = \int (3x^{\frac{1}{2}} - 4x) dx$

$$= (3)(\frac{2}{3}) x^{\frac{3}{2}} - \frac{4x^2}{2} + c$$

$$= 2x^{\frac{3}{2}} - 2x^2 + c$$

(b) $\int \frac{\tan x}{\sec x} dx = \int \frac{\sin x}{\cos x} \frac{\cos}{1} dx$

$$= \int \sin x dx = -\cos x + c$$

(c) $\frac{dx}{x+3} = \ln|x+3| + c$

(d) $\int \frac{(x^2-1)^2}{x^2} dx = \int \frac{x^4 - 2x^2 + 1}{x^2} dx$

$$= \int \left(x^2 - 2 + x^{-2} \right) dx = \frac{x^3}{3} - 2x - \frac{1}{x} + c$$

(e) $\displaystyle\int \frac{du}{\cos u \cot u} = \int \sec u \tan u\, du = \sec u + c$

EXAMPLE 7
Evaluate each of the following Integrals.

$b)\ \displaystyle\int \left(\frac{3}{\sqrt{1-x^2}} + \frac{4}{1+x^2} \right) dx$ $\qquad$ $a)\ \displaystyle\int \left(x^3 + 3\sqrt{x} - \frac{1}{2x} \right) dx$

$c)\ \displaystyle\int e^{2\ln(x+3)}\ dx$

Solution

$$a)\ \int \left(x^3 + 3\sqrt{x} - \frac{1}{2x} \right) dx = \int \left(x^3 + 3x^{\frac{1}{2}} - \frac{1}{2}\frac{1}{x} \right) dx$$
$$= \frac{x^4}{4} + 3\frac{x^{3/2}}{3/2} - \frac{1}{2}\ln x + C$$

$$b)\ \int \left(\frac{3}{\sqrt{1-x^2}} + \frac{4}{1+x^2} \right) dx = \int \left(3\frac{1}{\sqrt{1-x^2}} + 4\frac{1}{1+x^2} \right) dx$$
$$= 3\sin^{-1} x + 4\tan^{-1} x + C$$

$$c)\ \int e^{2\ln(x+3)}\ dx = \int e^{\ln(x+3)^2}\ dx = \int (x+3)^2\ dx$$
$$= \frac{x^3}{3} + 6\frac{x^2}{2} + 9x + C$$

EXAMPLE 8
Evaluate each of the following Integrals.

$b)\ \displaystyle\int \frac{(1+\sec x)\cot x}{\csc x}\, dx$ $\qquad$ $a)\ \displaystyle\int \sinh(\ln x)\ dx$

$$c)\int \frac{\tan x}{\sec x+\tan x}\,dx$$

Solution

$$a)\int \sinh(\ln x)\,dx=\int \frac{e^{\ln x}-e^{-\ln x}}{2}dx$$

$$=\frac{1}{2}\int\left(x-e^{\ln x^{-1}}\right)dx=\frac{1}{2}\int\left(x-\frac{1}{x}\right)dx$$

$$=\frac{1}{2}\left(\frac{x^2}{2}-\ln x\right)+C$$

$$b)\int \frac{(1+\sec x\ )\cot x}{\csc x}\,dx=\int \frac{(1+\dfrac{1}{\cos x}\ )\ \dfrac{\cos x}{\sin x}}{\dfrac{1}{\sin x}}\,dx$$

$$=\int (1+\frac{1}{\cos x}\ )\cos x\ dx=\int (\cos x+1\ )\ dx$$

$$=\sin x+x+C$$

$$c)\int \frac{\tan x}{\sec x+\tan x}\,dx=\int \frac{\tan x}{\sec x+\tan x}\,\frac{\sec x-\tan x}{\sec x-\tan x}\,dx$$

$$=\int \frac{\tan x(\ \sec x-\tan x\ )}{\sec^2 x-\tan^2 x}\,dx=\int\left(\ \sec x\tan x-\tan^2 x\right)dx$$

$$=\int\left(\sec x\tan x-\left(\sec^2 x-1\right)\right)dx=\sec x-\tan x+x+C$$

EXAMPLE 9

Find the integral $\displaystyle\int \frac{2+\sin x}{1+\cos x}\,dx$

Solution

$$\int \frac{2+\sin x}{1+\cos x}\,dx=\int \frac{\sin x}{1+\cos x}dx+\int \frac{2}{1+\cos x}dx$$

$$=-\ln\left(1+\cos x\right)+C+\int \frac{2}{1+\cos x}\,\frac{1-\cos x}{1-\cos x}dx$$

$$= -\ln(1+\cos x) + C + 2\int \frac{1-\cos x}{1-\cos^2 x} dx$$

$$= -\ln(1+\cos x) + C + 2\int \frac{1-\cos x}{\sin^2 x} dx$$

$$= -\ln(1+\cos x) + C + 2\int \frac{1}{\sin^2 x} dx - 2\int \frac{\cos x}{\sin x}\frac{1}{\sin x} dx$$

$$= -\ln(1+\cos x) + C + 2\int \csc^2 x\, dx - 2\int \cot x \csc x dx$$

$$= -\ln(1+\cos x) - 2\cot x + 2\csc x + C$$

Integration of Square Trigonometric Functions:

We must remember trigonometric identity first

I. Pythagorean Identities

A. $\sin^2\theta + \cos^2\theta = 1$

B. $\tan^2\theta + 1 = \sec^2\theta$

C. $\cot^2\theta + 1 = \csc^2\theta$

II. Sum and Difference of Angles Identities

A. $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

B. $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

C. $\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

III. Double Angle Identities

A. $\sin(2\theta) = 2\sin\theta\cos\theta$

B. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$=\cos(2\theta)=2\cos^2\theta-1$

$=1-2\sin^2\theta$

C. $\tan(2\theta)=\dfrac{2\tan\theta}{1-\tan^2\theta}$

Integration of Square Trigonometric Functions:

1- $\int \sin^2 x\,dx=\int \dfrac{(1-\cos 2x)}{2}dx=\dfrac{1}{2}\int(1-\cos 2x)dx$

$=\dfrac{1}{2}\left(x-\dfrac{1}{2}\sin 2x\right)+c$

2- $\int \cos^2 x\,dx=\int \dfrac{(1+\cos 2x)}{2}dx=\dfrac{1}{2}\int(1+\cos 2x)dx$

$=\dfrac{1}{2}\left(x+\dfrac{1}{2}\sin 2x\right)+c$

3- $\int \tan^2 x dx=\int(\sec^2 x-1)dx$

$=\tan x-x+c$

4- $\int \cot^2 x dx=\int(\csc^2 x-1)dx$

$=-\cot x-x+c$

EXAMPLE 10

b) $\int \tan^2 5x\,dx$ Find the integral $a)\int \sin^2 3x\,dx$

c) $\int(\sin 2x-\cos 2x)^2\,dx$

Solution

a) $\int \sin^2 3x\,dx = \frac{1}{2}\int (1-\cos 6x)\,dx$

$= \frac{1}{2}(x - \frac{1}{6}\sin 6x) + c$

b) $\int \tan^2 5x\ dx = \int \left(\sec^2 5x - 1\right) dx$

$= \frac{1}{5}\tan 5x - x + c$

c) $\int (\sin 2x - \cos 2x)^2\ dx$

$= \left(\sin^2 2x + \cos^2 2x - 2\sin 2x\cos 2x\right) dx$

$= \int (1 - \sin 4x)\,dx$

$= x + \dfrac{\cos 4x}{4} + c$

Exercises

1) Discuss whether the following integrals are right or wrong?

$a)\ \int \left(\sin^{-1} x\right)^2 dx = x\left(\sin^{-1} x\right)^2 - 2x + 2\sqrt{1-x^2}\ \sin^{-1} x + C$

$b)\ \int \left(\tan^{-1} x + 2x\right) dx = x\ \tan^{-1} x - 0.5\ \ln(x^2+1) + x^2 + C$

$c)\ \int x\cos x\,dx = x\sin x + \cos x + c$

2) Evaluate the following integrals:

$a)\ \int \left(x^2 - 2x + 5\right) dx$

b) $\int \left(\frac{1}{\sqrt{x^2 - 1}} + e^x \right) dx \ , \ 1 < x$

c) $\int \left(\sinh x + \frac{1}{\sqrt{1 + x^2}} \right) dx$

d) $\int \left(\sec^2 x - 6 \sec x \tan x \right) dx$

Techniques of Integrations

1-Integration by Substitution

Integration by substitution is in this case the “opposite” of the chain rule for differentiation

If u is a differentiable function of x and n is any number different from the Chain Rule tells us that

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right)=u^n\,\frac{du}{dx}.$$

From another point of view, this same equation says that $u^{n+1}/(n+1)$ is one of the antiderivatives of the function $u^n\,(du/dx)$ Therefore,

$$\int u^n\,\frac{du}{dx}=\frac{u^{n+1}}{n+1}+C.$$

Let’s have a look at an example: calculate $\int\frac{2x}{x^2+1}dx$.

We are looking for a function $F(x)$ such that $\frac{dF(x)}{dx}=\frac{2x}{x^2+1}$.

Now, if we set $u(x)=x^2+1$, we get

$$\frac{2x}{x^2+1}=\frac{u'(x)}{u(x)}$$

and we know that only one type of function F can have a derivative of this form: the logarithm

$$F(x)=\ln(u(x))\quad\Rightarrow\quad\frac{dF(x)}{dx}=\frac{u'(x)}{u(x)}.$$

Note that we used the chain rule to differentiate F .

Hence $\int \frac{2x}{x^2+1}\,dx = \ln(u(x)) + \mathrm{C} = \ln(x^2+1) + \mathrm{C}$.

EXAMPLE 1

Find the integral $\int \left(x^3+x\right)^3 \left(3x^2+1\right) dx$

Solution

We set $u = \left(x^3+x\right)^3$ Then $\frac{du}{dx} = \frac{d}{dx}\left(x^3+x\right) = 3x^2+1$

$\therefore du = \left(3x^2+1\right)dx$

so that by substitution we have

$$\int \left(x^3+x\right)^3 \left(3x^2+1\right)dx = \int u^3 \; du = \frac{u^4}{4} + C$$

$$= \frac{\left(x^3+x\right)^4}{4} + C$$

EXAMPLE 2

Find the integral $\int \sqrt{2x+1}\; dx$

Solution

We set $u = 2x+1$ Then $\frac{du}{dx} = \frac{d}{dx}(2x+1) = 2$

$\therefore du = (2)dx$

so that by substitution we have

$$\int \sqrt{2x+1}dx = \frac{1}{2}\int \sqrt{2x+1}\,(2)\,dx = \frac{1}{2}\int u^{1/2} \; du = \frac{1}{2}\,\frac{u^{3/2}}{3/2} + C$$

$$= \frac{(2x+1)^{3/2}}{3} + C$$

EXAMPLE 3

Find the integral $\int \frac{x}{x^2+1}\left(\ln(x^2+1)-2\right)^{-3} dx.$

Solution

We set $u=\left(\ln(x^2+1)-2\right)$ Then $\frac{du}{dx}=\frac{d}{dx}\left(\ln(x^2+1)-2\right)=\frac{2x}{x^2+1}$

$\therefore du=\left(\frac{2x}{x^2+1}\right)dx$

so that by substitution we have

$$\int \frac{x}{x^2+1}\left(\ln(x^2+1)-2\right)^{-3} dx=\int \mathrm{u}^{-3}du=\int \frac{u^{-2}}{-2}\,du$$
$$=-\frac{\left(\ln(x^2+1)-2\right)^{-2}}{2}+C$$

EXAMPLE 4

Find the integral $\int \sec^2(5t+1)\,dt$

Solution

We set $u=(5t+1)$ Then $\frac{du}{dt}=\frac{d}{dt}(5t+1)=5$

$\therefore du=(5)dt$

so that by substitution we have

$$\int \sec^2(5t+1)\,dt=\int \sec^2(u)\frac{du}{5}=\frac{1}{5}\int \sec^2(u)\,du$$
$$=\frac{1}{5}\tan u+C=\frac{1}{5}\tan(5t+1)+C$$

EXAMPLE 5

Find the integral $\int \sec^2(5t+1)\,dt$

Solution

We set $u=(5t+1)$ Then $\frac{du}{dt}=\frac{d}{dt}(5t+1)=5$

$\therefore du=(5)dt$

so that by substitution we have

$$\int \sec^2(5t+1)\,dt=\int \sec^2(u)\frac{du}{5}=\frac{1}{5}\int \sec^2(u)\,du$$

$$=\frac{1}{5}\tan u+C=\frac{1}{5}\tan(5t+1)+C$$

EXAMPLE 6

Find the integral $\int \cot x \ln(\sin x)\,dx$

Solution

We set $u=\ln(\sin x)$ Then $\frac{du}{dx}=\frac{d}{dx}(\ln(\sin x))=\frac{\cos x}{\sin x}=\cot x$

$\therefore du=(\cot x)dx$

so that by substitution we have

$$\int \cot x\ln(\sin x)\,dx=\int u\,du=\frac{u^2}{2}+C=\frac{1}{2}\left[\ln(\sin x)\right]^2+C$$

EXAMPLE 7

Find the integral $\int x\sqrt{2x+1}\,dx$

Solution

We set $u=(2x+1)$ Then $\frac{du}{dx}=\frac{d}{dx}(2x+1)=2$

$\therefore du = (2)dx$

so that by substitution we have

$$\int x\sqrt{2x+1}\,dx = \int\left(\frac{u-1}{2}\right)\sqrt{u}\,\frac{du}{2} = \frac{1}{4}\int(u-1)\sqrt{u}\,du$$

$$= \frac{1}{4}\int\left(u^{3/2} - u^{1/2}\right)du = \frac{1}{4}\left[\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right] + C$$

EXAMPLE 8

Find the integral $\int \frac{1}{e^x + e^{-x}}\,dx$

Solution

Multiply by e^x/e^x

$$\int \frac{1}{e^x + e^{-x}}\,dx = \int \frac{e^x}{e^x}\,\frac{1}{e^x + e^{-x}}\,dx = \int \frac{e^x}{e^{2x}+1}\,dx$$

We set $u = e^x$ Then $\frac{du}{dx} = \frac{d}{dx}\left(e^x\right) = e^x$

$\therefore du = e^x dx$

so that by substitution we have

$$\int \frac{e^x}{e^{2x}+1}\,dx = \int \frac{1}{u^2+1}\,du = \tan^{-1}u + C$$

$$= \tan^{-1}\left(e^x\right) + C$$

EXAMPLE 9

Find the integral $\int \sec x\,dx$

Solution

Multiply by $\dfrac{\sec x+\tan x}{\sec x+\tan x}$

$$\int \sec x\,dx=\int \frac{\sec x+\tan x}{\sec x+\tan x}\sec x\,dx=\int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}dx$$

We set $u=\sec x+\tan x$ Then $\dfrac{du}{dx}=\dfrac{d}{dx}(\sec x+\tan x)=\sec^2 x+\sec x\tan x$

$\therefore du=\left(\sec^2 x+\sec x\tan x\right)dx$

so that by substitution we have

$$\int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}dx=\int \frac{1}{u}du=\ln|u|+C$$
$$=\ln\left|\sec x+\tan x\right|+C$$

EXAMPLE 10

Find the integral $\displaystyle\int \frac{2^x+3^x}{e^x}dx$

Solution

$$\int \frac{2^x+3^x}{e^x}dx=\int\left[(2/e)^x+(3/e)^x\right]dx=\int (2/e)^x\,dx+\int (3/e)^x\,dx$$

We set $u=(2/e)^x$ Then $\dfrac{du}{dx}=\dfrac{d}{dx}\left((2/e)^x\right)=(2/e)^x\ln(2/e)$

$\therefore du=\left((2/e)^x\ln(2/e)\right)dx$

so that by substitution we have

$$\int (2/e)^x\,dx=\int \frac{du}{\ln(2/e)}=\frac{1}{\ln(2)-1}\int du=\frac{1}{\ln(2)-1}u+C$$
$$=\frac{1}{\ln(2)-1}(2/e)^x+C$$

So

$$\int \frac{2^x + 3^x}{e^x} dx = \int (2/e)^x dx + \int (3/e)^x dx$$

$$= \frac{1}{\ln(2) - 1}(2/e)^x + \frac{1}{\ln(3) - 1}(3/e)^x + C$$

EXAMPLE 11

Find the integral $\int \frac{\sinh(\ln x)}{x} dx$

Solution

We set $u = \ln x$ Then $\frac{du}{dx} = \frac{d}{dx}(\ln x) = \frac{1}{x}$

$$\therefore du = \left(\frac{1}{x}\right) dx$$

so that by substitution we have

$$\int \frac{\sinh(\ln x)}{x} dx = \int \sinh(u)\, du = \cosh(u) + C$$

$$= \cosh(\ln x) + C$$

EXAMPLE 12

Find the integral $\int \frac{\cos x}{\sqrt{\sin^2 x + 1}} dx$

Solution

We set $u = \sin x$ Then $\frac{du}{dx} = \frac{d}{dx}(\sin x) = \cos x$

$$\therefore du = (\cos x) dx$$

so that by substitution we have

$$\int \frac{\cos x}{\sqrt{\sin^2 x + 1}} dx = \int \frac{1}{\sqrt{u^2 + 1}} du = \sinh^{-1}(u) + C$$

$$= \sinh^{-1}(\sin x) + C$$

EXAMPLE 13

Find the integral $\int \frac{1}{\sqrt{1+e^x}}\,dx$

Solution

We set $u^2 = e^x$ Then $\frac{du}{dx} = \frac{d}{dx}\left(\sqrt{e^x}\right) = \frac{1}{2\sqrt{e^x}}e^x$

$$\therefore du = \left(\frac{1}{2\sqrt{e^x}}e^x\right)dx = \left(\frac{1}{2}\sqrt{e^x}\right)dx$$

so that by substitution we have

$$\int \frac{1}{\sqrt{1+e^x}}\,dx = \int \frac{1}{\sqrt{u^2+1}}\frac{2du}{u} = \int \frac{1}{\sqrt{u^2+1}}\frac{2du}{u} = -2\operatorname{csch}^{-1}(u) + C$$

$$= -\operatorname{csch}^{-1}\left(\sqrt{e^x}\right) + C$$

EXAMPLE 14

Find the integral $\int \frac{1}{\sqrt{x^2+6x+10}}\,dx$

Solution

By using complete square method, we have

$$\int \frac{1}{\sqrt{x^2+6x+10}}\,dx = \int \frac{1}{\sqrt{(x+3)^2+1}}\,dx$$

We set $u = x+3$ Then $\frac{du}{dx} = \frac{d}{dx}(x+3) = 1$

$$\therefore du = dx$$

so that by substitution we have

$$\int \frac{1}{\sqrt{(x+3)^2+1}}\,dx = \int \frac{1}{\sqrt{u^2+1}}du = \sinh^{-1}(u) + C$$

$$= \sinh^{-1}(x+3) + C$$

EXAMPLE 15

Find the integral $\int \frac{1}{\sqrt{8+2x-x^2}}\,dx$

Solution

By using complete square method, we have

$$\int \frac{1}{\sqrt{8+2x-x^2}}\,dx = \int \frac{1}{\sqrt{-\left(x^2-2x-8\right)}}\,dx = \int \frac{1}{\sqrt{9-\left(x-1\right)^2}}\,dx$$

$$= \int \frac{1}{3\sqrt{1-\left(\frac{x-1}{3}\right)^2}}\,dx$$

We set $u = \frac{x-1}{3}$ Then $\frac{du}{dx} = \frac{d}{dx}\left(\frac{x-1}{3}\right) = \frac{1}{3}$

$$\therefore du = \frac{1}{3}dx$$

so that by substitution we have

$$\int \frac{1}{3\sqrt{1-\left(\frac{x-1}{3}\right)^2}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}\left(u\right)+C$$

$$= \sin^{-1}\left(\frac{x-1}{3}\right)+C$$

EXAMPLE 16

Find the integral $\int \frac{x+16}{x^2+2x-8}\,dx$

Solution

$$\int \frac{x+16}{x^2+2x-8}\,dx = \int \frac{x+1}{x^2+2x-8}\,dx + \int \frac{15}{x^2+2x-8}\,dx$$

$$= \frac{1}{2}\int \frac{2x+2}{x^2+2x-8}\,dx + \int \frac{15}{\left(x+1\right)^2-9}\,dx$$

By using complete square method, we have

$$= \frac{1}{2}\ln\left| x^2 + 2x - 8 \right| dx + \frac{15}{9}\int \frac{1}{\left(\frac{x+1}{3}\right)^2 - 1} dx + C$$

We set $u = \frac{x+1}{3}$ Then $\frac{du}{dx} = \frac{d}{dx}\left(\frac{x+1}{3}\right) = \frac{1}{3}$

$\therefore du = \frac{1}{3}dx$

so that by substitution we have

$$= \frac{1}{2}\ln\left| x^2 + 2x - 8 \right| dx - 5\int \frac{1}{1-(u)^2} du + C$$

$$= \frac{1}{2}\ln\left| x^2 + 2x - 8 \right| dx - 5\tanh(u) + C$$

$$= \frac{1}{2}\ln\left| x^2 + 2x - 8 \right| dx - 5\tanh\left(\frac{x+1}{3}\right) + C$$

Exercise

By using substitution method find the following integrals

$1-\int \frac{\sin 2x+\sin x}{\sqrt{\sin^2 x-\cos x}}\,dx$ $\qquad$ $2-\int \frac{1}{1+e^x}\,dx$

$3-\int \frac{x}{(x+3)^6}\,dx$ $\qquad$ $4-\int \frac{(x+2)^2}{(x+1)^3}\,dx$

$5-\int \frac{x\sqrt{\ln(x^2+1)}}{(x^2+1)}\,dx$ $\qquad$ $6-\int \frac{1}{\sqrt{(1-x^2)\,\sin^{-1} x}}\,dx$

$7-\int \frac{x^2}{x+1}dx$ $\qquad$ $8-\int \frac{e^{-x}}{e^{-x}+2}dx$

$9-\int \frac{3\sqrt{t}+5}{t}\,dt$ $\qquad$ $10-\int \sin x\cos x\,dx$

$11-\int \cos x\sqrt{1-\cos^2 x}\,dx$ $\qquad$ $12-\int \sin^5 x\cos x\,dx$

$13-\int \csc x\,dx$ $\qquad$ $14-\int \frac{\sec(x^{2/3})\,\tan(x^{2/3})}{x^{1/3}}\,dx$

$15-\int \frac{\tan^2\left(2\sqrt[3]{x}\right)}{\sqrt[3]{x^2}}\,dx$ $\qquad$ $16-\int \frac{1}{x\sqrt{x^{10}+1}}\,dx$

$17-\int \frac{\sec h^2 x}{\sqrt{\tanh^2 x+1}}\,dx$ $\qquad$ $18-\int (\tan x+\sec x)^2\,dx$

$19-\int \sin^4 x\,dx$ $\qquad$ $20-\int \cos^6 x\,dx$

$21-\int \frac{1}{\sqrt{4x-x^2}}\,dx$ $\qquad$ $22-\int \frac{1}{\sqrt{4x^2+8x+20}}\,dx$

$23-\int \frac{1}{x^2+6x+12}\,dx$ $\qquad$ $24-\int \frac{3x}{x^2+6x+12}\,dx$

Techniques of Integrations

2-Integration by Parts:

Introduction

If the techniques/formulas we introduced earlier do not work then there is another technique that you can use: Integration by Parts. It is a way of simplifying integrals of the form $\int f(x)g(x)dx$ in which $f(x)$ can be differentiated repeatedly and $g(x)$ can be integrated repeatedly without difficulty.

If u and v are two functions of x, then using the rule of the derivative of the product of two functions we have

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Hence

$$d(uv) = udv + vdu$$

The Fundamental Theorem of Calculus tells us that if we take the derivative of the integral of a function, then we are left with the original function. The derivative and the integral "cancel" each other out.

Integrating both sides of this equation, we get

$$uv = \int udv + \int vdu$$

$\int udv = uv - \int vdu$ i.e

This formula is called the rule of integrating by parts, and it suits the purpose if $\int udv$ is evaluated more than $\int vdu$ or if one of these integrals can be expressed in terms of the other.

Strategies to use integration by parts

1) Use derivative to drive a polynomial function to zero

EXAMPLE 1

Find the integral $\int xe^x dx$

Solution

First, pick u and dv.

$$u = x \qquad dv = e^x dx$$
$$du = dx \qquad v = e^x$$

Use the formula for Integration by Parts.

$$\int udv = uv - \int vdu$$

$$\int xe^x dx = xe^x - \int e^x dx \quad = xe^x - e^x + C$$

EXAMPLE 2

Find the integral $\int x^2 \cos x \, dx$

Solution

First, pick u and dv.

$$u = x^2 \qquad dv = \cos x\, dx$$
$$du = 2x\, dx \qquad v = \sin x$$

Use the formula for Integration by Parts.

$$\int u dv = uv - \int v du$$

$$\int x^2 \cos x\, dx = x^2 \sin x - 2\int x \sin x\, dx + C$$

using the formula for integration by parts again

$$u = x \qquad dv = \sin x\, dx$$
$$du = dx \qquad v = -\cos x$$

$$\int x \sin x\, dx = -x\cos x + \int \cos x\, dx = -x\cos x + \sin x + C$$

$$\therefore \int x^2 \cos x\, dx = x^2 \sin x - 2\left[-x\cos x + \sin x\right] + C$$
$$= x^2 \sin x + 2x\cos x - 2\sin x + C$$

Tabular view of repeated Integration by Parts

If you have a polynomial as one of the two factors in a integration by parts problem, then the following is a short cut to solving the problem.

Steps:

1. Draw a 3 column table and label the first column "Dif" and the third column "Int"
2. Put $p(x)$ (the polynomial) in the first column and differentiate it until you obtain 0
3. Put $f(x)$ (the other function) in the third column and integrate repeatedly until you reach the 0 in the first column
4. Draw an arrow from the Dif column to the row below it in the Int column
5. Label the arrows, starting with " $+$ " and alternating with " $-$ "
6. From each arrow form the product of the expressions at the tail and the tip of the arrow and multiply that expression by $+1$ or -1 based on the sign on the arrow
7. Add the results together to obtain the value of the integral

EXAMPLE 3

Find the integral of $\int x^3 \sin x \, dx$

Solution

Dif		*Int*
x^3	$+$	$\sin x$
$3x^2$	$-$	$-\cos x$
$6x$	$+$	$-\sin x$
6	$-$	$\cos x$

0	+	$\sin x$

$$\int x^3 \sin x\,dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6\sin x + C$$

$$= \left(-x^3 + 6x\right)\cos x + \left(3x^2 - 6\right)\sin x + C$$

EXAMPLE 4

Find the integral of $\int \left(2x^3 - 7x^2 + 3x - 4\right)e^x\,dx$

Solution

Dif		***Int***
$2x^3 - 7x^2 + 3x - 4$	+	e^x
$6x^2 - 14x + 3$	−	e^x
$12x - 14$	+	e^x
12	−	e^x
0	+	e^x

$$\int \left(2x^3 - 7x^2 + 3x - 4\right)e^x\,dx = \left(2x^3 - 7x^2 + 3x - 4\right)e^x - \left(6x^2 - 14x + 3\right)e^x$$

$$+\left(12x - 14\right)e^x - 12e^x + C$$

$$= \left(2x^3 - 13x^2 + 29x - 33\right)e^x + C$$

How to pick u and dv

When deciding on your choice for u and dv, use the acronym: LIPET with the higher one having the priority for u.

L - logarithmic function

I - inverse trig function

P - polynomial function

E - exponential function

T - trigonometry function

EXAMPLE 5

Find the integral $\int \ln x\, dx$

Solution

First, pick u and dv.

$$u = \ln x \qquad dv = dx$$

$$du = \frac{1}{x}dx \qquad v = x$$

Use the formula for Integration by Parts.

$$\int u\,dv = uv - \int v\,du$$

$$\int \ln x\, dx = x \ln x - \int x\left(\frac{1}{x}\right)dx$$

$$= x \ln x - x + C$$

EXAMPLE 6

Find the integral $\int x \ln x\, dx$

Solution

First, pick u and dv.

$$u = \ln x \qquad dv = x\,dx$$

$$du = \frac{1}{x}dx \qquad v = \frac{x^2}{2}$$

Use the formula for Integration by Parts.

$$\int u dv = uv - \int v du$$

$$\int \ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\left(\frac{1}{x}\right)dx = \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$$

EXAMPLE 7

Find the integral $\int x \tan^{-1} x\ dx$

Solution

First, pick *u* and *dv*.

$$u = \tan^{-1} x \qquad dv = x\,dx$$

$$du = \frac{dx}{x^2+1} \qquad v = \frac{x^2}{2}$$

Use the formula for Integration by Parts.

$$\int u dv = uv - \int v du$$

$$\int x\tan^{-1}x\ dx=\frac{x^2}{2}\tan^{-1}x-\int\frac{x^2}{2}\left(\frac{1}{x^2+1}\right)dx+C$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\left(1-\frac{1}{x^2+1}\right)dx+C$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{x}{2}+\frac{1}{2}\tan^{-1}x+C$$

EXAMPLE 8

Find the integral $\int x^3\ \cos(2x^2-5)\ dx$

Solution

First, pick u and dv.

$u=x^2$ $\qquad$ $dv=x\cos\left(2x-5\right)dx$

$du=2xdx$ $\qquad$ $v=\dfrac{\sin\left(2x-5\right)}{2}$

Use the formula for Integration by Parts.

$$\int udv=uv-\int vdu$$

$$\int x^3\cos\left(2x-5\right)dx=\frac{1}{2}x^2\sin\left(2x-5\right)-\int x\sin\left(2x-5\right)dx+C$$

$$=\frac{1}{2}x^2\sin\left(2x-5\right)+\frac{1}{2}\cos\left(2x-5\right)+C$$

2) Reduce polynomials to get u-substitution

Perform integration by parts until the integral you began with appears on the right or you get to a point that you can use a basic integration rule. Then add or subtract accordingly and then multiply or divide.

EXAMPLE 9

Find the integral $\int e^x \cos x \, dx$

Solution

First, pick u and dv.

$$u = e^x \qquad dv = \cos x \, dx$$
$$du = e^x dx \qquad v = \sin x$$

Use the formula for Integration by Parts.

$$\int u dv = uv - \int v du$$

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx + C$$

Notice that the second integral looks the same as our original integral in form, except that it has a $\sin x$ instead of $\cos x$. To evaluate it, we again apply integration by parts to the second term with

$$u = e^x \qquad dv = \sin x \, dx$$
$$du = e^x dx \qquad v = -\cos x$$

Then

$$\int e^x \cos x\,dx = e^x \sin x - \left[-e^x \cos x - \int -e^x \cos x\,dx\right] + C$$
$$= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx + C$$

Notice that the unknown integral now appears on both sides of the equation. We can simply move the unknown integral on the right to the left side of the equation, thus adding it to our original integral:

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x + C$$

Dividing both sides by 2 we obtain

$$\int e^x \cos x\,dx = \frac{1}{2}\left(e^x \sin x + e^x \cos x\right) + C$$

EXAMPLE 10

Find the integral $\int \cos\left(\sqrt{x}\right) dx$

Solution

We set $z = \sqrt{x}$ Then $\frac{dz}{dx} = \frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$

$$\therefore dz = \left(\frac{1}{2\sqrt{x}}\right) dx$$

so that by substitution we have

$$\int \cos\sqrt{x}\;dx = \int 2z \cos z\,dz = 2\int z \cos z\,dz$$

Now, pick *u* and *dv*.

$u = z \qquad\qquad\qquad dv = \cos z\, dz$

$du = dz \qquad\qquad\qquad v = \sin z$

Use the formula for Integration by Parts.

$$\int u dv = uv - \int v du$$

$$\begin{aligned}\int \cos\sqrt{x}\ dx &= 2\int z\cos z\, dz \\ &= 2\left(z\sin z - \int \sin z\, dz\right) + C \\ &= 2\left(z\sin z + \cos z\right) + C \\ &= 2\left(\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right) + C\end{aligned}$$

Reduction Formulae

Integration by parts can be used to derive reduction formulas for integrals. are formulas that express an integral involving a power of a These integral that involves a lower power of that function in terms of an function

EXAMPLE 11

Find the reduction formulae of $\int \cos^n x dx$

Solution

$$u = \cos^{n-1} x \qquad\qquad dv = \cos x\, dx$$

$$du = (n-1)\cos^{n-2} \sin x \; dx \qquad\qquad v = \sin x$$

Use the formula for Integration by Parts.

$$\int \cos^n xdx = \sin x \cos^{n-1} x - (n-1)\int \sin^2 x \cos^{n-2} xdx + C$$

$$= \sin x \cos^{n-1} x - (n-1)\int \left(1 - \cos x^2\right)\cos^{n-2} xdx + C$$

$$= \sin x \cos^{n-1} x - (n-1)\int \cos^{n-2} xdx + (n-1)\int \cos^n xdx + C$$

Notice that the unknown integral now appears on both sides of the equation. We can simply move the unknown integral from the right to the left side of the equation, thus adding it to our original integral:

$$\int \cos^n x dx - (n-1)\int \cos^n x dx = \sin x \cos^{n-1} x - (n-1)\int \cos^{n-2} x dx + C$$

$$n\int \cos^n x dx = \sin x \cos^{n-1} x - (n-1)\int \cos^{n-2} x dx + C$$

$$\int \cos^n x dx = \frac{1}{n}\left[\sin x \cos^{n-1} x - (n-1)\int \cos^{n-2} x dx\right] + C$$

EXAMPLE 12

Find reduction formulae of $\int \sin^n x dx$

And then find $\int \sin^3 x\, dx$ & $\int \sin^4 x\, dx$ & $\int \sin^5 x\, dx$

Solution

$u = \sin^{n-1} x$ $\qquad\qquad$ $dv = \sin x\, dx$

$du = (n-1)\sin^{n-2} \cos x\ dx$ $\qquad\qquad$ $v = -\cos x$

Use the formula for Integration by Parts.

$$\therefore \int \sin^n x dx + (n-1)\int \sin^n x\, dx = -\cos x \sin^{n-1} x - (n-1)\int \sin^{n-2} x\, dx + C$$

$$\therefore \int \sin^n x dx = \frac{1}{n}\left[-\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\, dx\right] + C$$

So

$$= \frac{1}{3}\left[-\cos x \sin^2 x - 2\cos x\right] + C$$

$$\int \sin^4 x\,dx = \frac{1}{4}\left[-\cos x \sin^3 x + 3\int \sin^2 x\,dx\right] + C$$

$$= \frac{1}{4}\left[-\cos x \sin^3 x + 3\int\left(\frac{1-cox2x}{2}\right)dx\right] + C$$

$$= \frac{1}{4}\left[-\cos x \sin^3 x + \frac{3}{2}\left(x - \frac{\sin 2x}{2}\right)\right] + C$$

$$= \frac{1}{5}\left[-\cos x \sin^4 x + 4\left(\frac{1}{3}\left[-\cos x \sin^2 x - 2\cos x\right]\right)\right] + C$$

$$= \frac{1}{5}\left[-\cos x \sin^4 x + \frac{4}{3}\left(-\cos x \sin^2 x - 2\cos x\right)\right] + C$$

EXAMPLE 13

Find reduction formulae of $\int (\ln x)^n\,dx$ And then find $\int (\ln x)^4\,dx$

Solution

$u = (\ln x)^n \qquad\qquad dv = dx$

$du = n(\ln x)^{n-1}\,\frac{1}{x}\,dx \qquad\qquad v = x$

Use the formula for Integration by Parts.

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx + C$$

So

$$\int (\ln x)^4\,dx = x(\ln x)^4 - 4\int (\ln x)^3\,dx + C$$

$$= x(\ln x)^4 - 4\left[x(\ln x)^3 - 3\int (\ln x)^2\,dx\right] + C$$

$$= x(\ln x)^4 - 4x(\ln x)^3 + 12\int (\ln x)^2\,dx + C$$

$$= x(\ln x)^4 - 4x(\ln x)^3 + 12\left[x(\ln x)^2 - 2\int \ln x\,dx\right] + C$$
$$= x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24\int \ln x\,dx + C$$
$$= x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24\left[x\ln x - \int dx\right] + C$$
$$= x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24x\ln x + 24x + C$$

EXAMPLE 14

Find reduction formulae of $\int \sec^n x\,dx$ And then find $\int \sec^5 x\,dx$

Solution

$u = \sec^{n-2} x \qquad dv = \sec^2 x\,dx$

$du = (n-2)\sec^{n-3}(\sec x \tan x)\ dx \qquad v = \tan x$

Use the formula for Integration by Parts.

$$\int \sec^n x\,dx = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} \tan^2 x\,dx + C$$
$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x\left(\sec^2 x - 1\right)dx + C$$
$$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx + C$$
$$\int \sec^n x\,dx + (n-2)\int \sec^n x\,dx = \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} x\,dx + C$$
$$\int \sec^n x\,dx = \frac{1}{n-1}\left[\sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} x\,dx\right] + C$$

So

$$= \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\left[\frac{1}{2}\left(\sec x \tan x + \int \sec x\,dx\right)\right] + C$$

$$=\frac{1}{4}\sec^3 x\tan x+\frac{3}{8}\sec x\tan x+\frac{3}{8}\ln\left|\sec x+\tan x\right|+C$$

EXAMPLE 15

Find reduction formulae of $\int \tan^n x\,dx$ And then find $\int \tan^5 x\,dx$

Solution

$$\int \tan^n x\,dx=\int \tan^{n-2} x\tan^2 x\,dx=\int \tan^{n-2} x\left(\sec^2 x-1\right)dx$$

$$=\int \tan^{n-2} x\sec^2 x\,dx-\int \tan^{n-2} x\,dx$$

$$=\frac{\tan^{n-1} x}{n-1}-\int \tan^{n-2} x\,dx+C$$

So

$$=\frac{\tan^4 x}{4}-\int\left(\sec^2 x-1\right)dx+C\ \int \tan^5 x\,dx=\frac{\tan^4 x}{4}-\int \tan^2 x\,dx+C$$

$$=\frac{\tan^4 x}{4}-\tan x+x+C$$

Exercise

i) Evaluate the following integrals:

$1-\int x\sec^2 x\,dx$

$2-\int (x^2+x)\,e^x\,dx$

$3-\int \left(x\sec h^{-1}x+\dfrac{\ln x}{\sqrt{x}} \right) dx$

$4-\int x\tanh^{-1} x\,dx$

$5-\int xe^{3x}dx$

$6-\int \ln 2x dx$

$11-\int \dfrac{\ln x}{x^2}dx$

$12-\int x^3\ln x dx$

$13-\int t\ln(t+1)dt$

$14-\int x(\ln x)^2\,dx$

$15-\int x\sqrt{x-1}dx$

$16-\int (x^2-1)e^x dx$

$17-\int e^x\sin 3x\,dx$

$18-\int e^{-x}\cos 2x\,dx$

$19-\int \cos(\ln x)\,dx$

ii) Find reduction formulae of $\int \csc^n x\,dx$ and then evaluate $\int \csc^4 x\,dx$

iii) Find reduction formulae of $\int \cot^n x\,dx$ and then evaluate

$$\int \cot^5 x\,dx$$

Integration of Trigonometric Functions

In this section we want to look at various methods to integrate functions which are products and quotients of trigonometric functions. The starting point is the integrals of sine and cosine.

(1) $\int \sin x \; dx = -\cos x$

(2) $\int \cos x \, dx = \sin x$

Next come the integrals of tangent and cotangent

EXAMPLE 1

Find $\int \tan x \, dx$

Solution.

Since $\tan x = \frac{\sin x}{\cos x}$ one can substitute $u = \cos x$ with

$du = -\, sin\, x\, dx$ to get

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{1}{u} du = -\ln|u| = -\ln|\cos x| = \ln|\sec x|$$

So

(3) $\int \tan x \, dx = \ln|\sec x|$

If we are working in an interval where cos x is positive then we can just write $\int \tan x \, dx = \ln \sec x$.

Similarly

(4) $$\int \cot x \, dx = \ln|\sin x|$$

Finally, we come to the integrals of secant and cosecant. The easiest way to do these is to multiply top and bottom by $\sec x + \tan x$ or $\csc x + \cot x$.

EXAMPLE 2

Find $\int \sec x \, dx$

Solution

Multiply top and bottom by $\sec x + \tan x$

then substitute $u = \sec x + \tan x$ with $du = (\sec^2 x + \tan x \sec x)dx$ to get

$$\int \sec x \, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx = \int \frac{1}{u} du = \ln|u|$$

$$= \ln|\sec x + \tan x|$$

So

$$(5) \qquad \int \sec x\, dx = \ln|\sec x + \tan x|$$

If we are working in an interval where $\sec x + \tan x$ is positive then we can just write $\int \sec x\, dx = \ln(\sec x + \tan x)$

Similarly

$$(6) \qquad \int \csc x\, dx = -\ln|\csc x + \cot x||$$

Now we look at some techniques for integrating products of powers of sines and cosines, i.e. $\int \sin^n x \cos^m x\, dx$.

Case 1. One power is odd and positive.

The simplest case is when one of the powers is an odd positive integer. Suppose for definiteness that m is odd. In that case we split off a $\cos x$ and use it with the dx in the substitution $u = sin\, x$. We express the remaining cosine terms in terms of sine using $\sin^2 x + \cos^2 x = 1$ for the substitution. Let's look at an example.

$$\int \sin^m x \cos^{2k+1} x\, dx = \int \sin^m x (\cos^2 x)^k \cos x\, dx$$
$$= \int \sin^m x (1 - \sin^2 x)^k \cos x\, dx$$

EXAMPLE 3

Find $\int \frac{\cos^5 x}{\sin^6 x} dx$.

Solution

$$\int \frac{\cos^5 x}{\sin^6 x} dx = \int \frac{\cos^4 x}{\sin^6 x} \cos x \, dx = \int \frac{\left(\cos^2 x\right)^2}{\sin^6 x} \cos x \, dx$$

$$= \int \frac{\left(1-\sin^2 x\right)^2}{\sin^6 x} \cos x \, dx = \int \frac{\left(1-2\sin^2 x+\sin^4 x\right)}{\sin^6 x} \cos x \, dx$$

$$= \int \left(\frac{1}{\sin^6 x} - 2\frac{1}{\sin^4 x} + \frac{1}{\sin^2 x} \right) \cos x \, dx$$

Now let $u = \sin x$ with $du = \cos x \, dx$. We get

$$= \int \left(\frac{1}{u^6} - 2\frac{1}{u^4} + \frac{1}{u^2} \right) du = \frac{u^{-5}}{-5} - 2\frac{u^{-3}}{-3} + \frac{u^{-1}}{-1} + C$$

$$= -\frac{1}{5\sin^5 x} + \frac{2}{3\sin^3 x} - \frac{1}{\sin^{-1} x} + C$$

EXAMPLE 4

Find $\int \sin^5 x \, dx$.

Solution

$$\int \sin^5 x\,dx = \int \sin^4 x \sin x\,dx = \int \left(\sin^2 x\right)^2 \sin x\,dx$$

$$= \int \left(1-\cos^2 x\right)^2 \sin x\,dx = \int \left(1-2\cos^2 x+\cos^4 x\right)\sin x\,dx$$

Now let $u = \cos x$ with $du = -\sin x\,dx$. We get

$$= -\int \left(1-2u^2+u^4\right)du = -u+2\frac{u^3}{3}-\frac{u^5}{5}+C$$

$$= -\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x+C$$

We can also use reduction formulae

$$\int \sin^n x dx = \frac{-1}{n}\cos x \sin^{n-1} x+\frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\int \sin^5 x dx = \frac{-1}{5}\cos x \sin^4 x+\frac{4}{5}\int \sin^3 x\,dx$$

$$= \frac{-1}{5}\cos x \sin^4 x+\frac{4}{5}\left[\frac{-1}{3}\cos x \sin^2 x-\frac{2}{3}\cos x\right]+C$$

$$= \frac{-1}{5}\cos x \sin^4 x-\frac{4}{15}\cos x \sin^2 x-\frac{8}{15}\cos x+C$$

Case 2. Both powers are even.

In this case use the following identities to reduce the powers,

$$\sin 2x = 2\sin x\cos x \qquad (7)$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad (8)$$

$$\cos^2 x = \frac{1+\cos 2x}{2} \qquad (9)$$

EXAMPLE 5

$$\int \sin^2 x\cos^4 x dx$$

Solution

$$\int \sin^2 x\cos^4 x dx = \int (\sin x\cos x)^2 \cos^2 x dx = \int \left(\frac{\sin 2x}{2}\right)^2 \left(\frac{1+\cos 2x}{2}\right) dx$$

Try to solve using reduction formulae

We can do similar things for integrating products of powers of tangents and secants, i.e. $\int \tan^n x \sec^m x \, dx$.

Case 1. sec is to an even positive power.

In that case we split off a $\sec^2 x$ and use it with dx in the substitution $u = \tan x$. We express the remaining secant terms in terms of tangent using $\sec^2 x - \tan^2 x = 1$ for the substitution. Let's look at an example.

EXAMPLE 6

Find $\int \tan^5 x \sec^2 x \, dx$.

Solution

Let $u = \tan x$ with $du = \sec^2 x \, dx$. The integral becomes

$$= \int u^5 \, du = \frac{u^6}{6} + C = \frac{\tan^6 x}{6} + C$$

Try to solve using reduction formulae

EXAMPLE 7

Find $\int \tan^5 x \sec^6 x \, dx$.

Solution

$$\int \tan^5 x \sec^4 x\, dx = \int \tan^5 x \sec^4 x \sec^2 x\, dx = \int \tan^5 x\left(\tan^2 x + 1\right)^2 \sec^2 x\, dx$$

$$= \int \left(\tan^9 x + 2\tan^7 x + \tan^5 x\right)\sec^2 x\, dx$$

Let $u = \tan x$ with $du = \sec^2 x\, dx$. The integral becomes

$$= \int \left(u^9 + 2u^7 + u^5\right) du = \frac{u^{10}}{10} + 2\frac{u^8}{8} + \frac{u^6}{6} + C$$

$$= \frac{\tan^{10} x}{10} + \frac{\tan^8 x}{4} + \frac{\tan^6 x}{6} + C$$

Try to solve using reduction formulae

EXAMPLE 8

Find $\int \tan^5 x\, dx$.

Solution

$$\int \tan^5 x\, dx = \int \left(\tan^2 x\right)^2 \tan x\, dx = \int \left(\sec^2 x - 1\right)^2 \tan x\, dx$$

$$= \int \sec^3 x \sec x \tan x\, dx - 2\int \tan x \sec^2 x\, dx + \int \tan x\, dx$$

$$= \frac{\sec^4 x}{4} - 2\frac{\tan^2 x}{2} + \ln\left|\sec x\right| + C$$

Try to solve using reduction formulae

EXAMPLE 9

Find $\int \sec^3 x\,dx$.

Solution

$$\int \sec^3 x\,dx = \int \sec^2 x \sec x\ dx$$

pick u and dv.

$$u = \sec x \qquad dv = \sec^2 x\,dx$$
$$du = \sec x \tan x\,dx \qquad v = \tan x$$

Use the formula for Integration by Parts.

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx$$

$$= \sec x \tan x - \int \sec x\left(\sec^2 x - 1\right) dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln\left|\sec x + \tan x\right| + C$$

$$\int \sec^3 x\,dx = \frac{1}{2}\left[\sec x \tan x + \ln\left|\sec x + \tan x\right|\right] + C$$

Try to solve using reduction formulae

Case 2. tan is to a odd positive power.

In that case we split off a $\tan x$ and use it with $\sec x\, dx$ in the substitution $u = \sec x$. We express the remaining tangent terms in terms of secant using $\sec^2 x - \tan^2 x = 1$ for the substitution. Let's look at an example.

EXAMPLE 10

Find $\int \tan x \sec^3 x\, dx$.

Solution

$$\int \tan x \sec^3 x\, dx = \int \sec^2 x \left(\tan x \sec x\right) dx$$

Let $u = \sec x$ with $du = \tan x \sec x\, dx$. The integral becomes

$$= \int u^2 du = \frac{u^3}{3} + C$$

$$= \frac{\sec^3 x}{3} + C$$

EXAMPLE 11

Find $\int \tan^5 x \sec^3 x\, dx$.

Solution

$$\int \tan^5 x \sec^3 x\, dx = \int \tan^4 x \sec^2 x \left(\tan x \sec x\right) dx$$

$$= \int \left(\sec^2 x - 1\right)^2 \sec^2 x \left(\tan x \sec x\right) dx$$

$$= \int \left(\sec^6 x - 2\sec^4 x + \sec^2 x \right)\left(\tan x \sec x \right) dx$$

Let $u = \sec x$ with $du = \tan x \sec x\, dx$. The integral becomes

$$= \int \left(u^6 - 2u^4 + u^2 \right) du = \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + C$$

$$= \frac{\sec^7 x}{7} - \frac{2\sec^5 x}{5} + \frac{\sec^3 x}{3} + C$$

Integration of sin (mx) × cos (nx) where m≠n

we reverse the identities

$$\sin((m-n)x) = \sin(mx)\cos(nx) - \cos(mx)\sin(nx)$$

$$\sin((m+n)x) = \sin(mx)\cos(nx) + \cos(mx)\sin(nx)$$

$$\cos((m-n)x) = \cos(mx)\cos(nx) + \sin(nx)\sin(mx)$$

$$\cos((m+n)x) = \cos(mx)\cos(nx) - \sin(nx)\sin(mx)$$

to get

$$\sin(mx)\cos(nx) = \frac{1}{2}[\sin((m-n)x) + \sin((m+n)x]$$

$$\sin(mx)\sin(nx) = \frac{1}{2}[\cos((m-n)x) - \cos((m+n)x)]$$

$$\cos(mx)\cos(nx) = \frac{1}{2}[\cos((m-n)x) + \cos((m+n)x)]$$

EXAMPLE 12

Find the integral $\int \sin 3x \cos 2x dx$

Solution

$$\int \sin(mx)\cos(nx)\,dx = \frac{1}{2}\Big[\sin\big((m-n)x\big) + \sin\big((m+n)x\big)\Big]$$

$$\int \sin(3x)\cos(2x)\,dx = \frac{1}{2}\int\big[\sin(x) + \sin(5x)\big]dx$$

$$= \frac{1}{2}\left[-\cos(x) - \frac{\cos(5x)}{5}\right] + C$$

EXAMPLE 13

Find the integral $\int \sin 3x \sin 2x dx$

Solution

$$\int \sin(mx)\sin(nx)\,dx = \frac{1}{2}\left[\cos((m-n)x) - \cos((m+n)x)\right]$$

$$\int \sin(3x)\sin(2x)\,dx = \frac{1}{2}\int\left[\cos(x) - \cos(5x)\right]dx$$

$$= \frac{1}{2}\left[\sin(x) - \frac{\sin(5x)}{5}\right] + C$$

EXAMPLE 14

Find the integral $\int \cos 3x \cos 2x dx$

Solution

$$\int \cos(mx)\cos(nx)\,dx = \frac{1}{2}\left[\cos((m-n)x) + \cos((m+n)x)\right]$$

$$\int \sin(3x)\sin(2x)\,dx = \frac{1}{2}\int\left[\cos(x) + \cos(5x)\right]dx$$

$$= \frac{1}{2}\left[\sin(x) + \frac{\sin(5x)}{5}\right] + C$$

Exercise

i) Evaluate the following integrals**:**

$1-\int \sin^3 x \cos^5 x\, dx$

$2-\int \cos^4 x \ \sin^5 x \ dx$

$3-\int \ \sin^5 x \ dx$

$4-\int \sin^4 x \ \cos^4 x \ dx$

$5-\int \sin^4 x\, dx$

$6-\int \sec^8 x \ \ dx$

$7-\int \csc^3 x \cot^5 x \ \ dx$

$8-\int \tan^4 x \ \ dx$

$9-\int \tan x \sec^5 x\, dx$

$10-\int \tan x \sec^4 x\, dx$

$11-\int \tan^5 x \sec^9 x\, dx$

$12-\int \tan^{-3/2} x \sec^4 x\, dx$

$13-\int \tan^5 x \sec^8 x\, dx$

$14-\int \tan^5 x \ \cos^4 x \ dx$

$15-\int \dfrac{\left(\tan^3(\ln x)\right)\left(\sec^6(\ln x)\right)}{x}\, dx$

$16-\int \cos 3x \sin 5x\, dx$

$17-\int \cos 3x \cos 5x\, dx$

$18-\int \sin 3x \sin 5x\, dx$

Integration by Trigonometric Substitution

If the integrand of an integral contains an expression of the form $(a^2 - x^2)^{n/2}$, where a is a positive real number and n is an odd integer, then the domain of the integrand is either the closed interval $[-a, a]$ or is a subset of this interval.

Then try the trigonometric substitution: Let $x = a\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = a\cos\theta\, d\theta$.

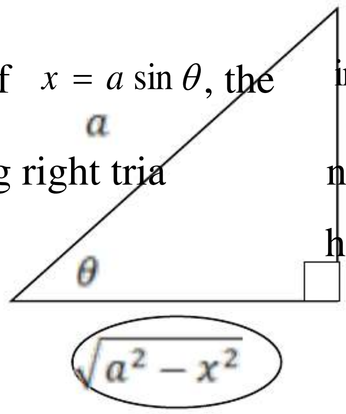
Note that the range of the function $x = a\sin\theta$ is the closed interval $[-a, a]$. Also, we have that $a^2 - x^2 = a^2 - (a\sin\theta)^2 = a^2 - a^2\sin^2\theta = a^2(1 - \sin^2\theta) = a^2\cos^2\theta$.

Thus, $(a^2 - x^2)^{n/2} = (a^2\cos^2\theta)^{n/2} = (\sqrt{a^2\cos^2\theta}\,)^n = (|a||\cos\theta|)^n = |a|^n\,|\cos\theta|^n$.

Since $a > 0$, then $|a| = a$. Since $\cos\theta \ge 0$ whenever $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then $|\cos\theta| = \cos\theta$. Thus, $(a^2 - x^2)^{n/2} = a^n\cos^n\theta$.

If $x = a\sin\theta$, the in $\theta = \frac{x}{a}$.

Using right tria nometry, have that



, $\sec\theta = \dfrac{a}{\sqrt{a^2 - x^2}}$ $\cos\theta = \dfrac{\sqrt{a^2 - x^2}}{a}$

$\tan\theta = \dfrac{x}{\sqrt{a^2 - x^2}}$, $\cot\theta = \dfrac{\sqrt{a^2 - x^2}}{x}$

If the integrand of an integral contains an expression of the form $(a^2 + x^2)^{n/2}$, where a is a positive real number and n is an integer, then the domain of the integrand is the set of all real numbers.

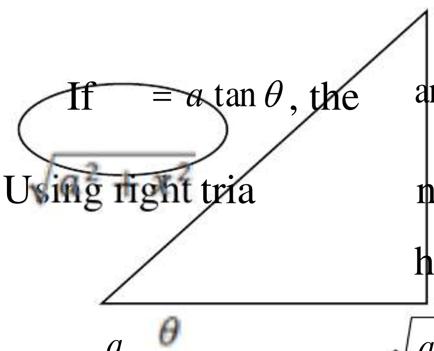
Then try the trigonometric substitution: Let $x = a\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = a\sec^2\theta\, d\theta$.

Note that the range of the function $x = a\tan\theta$ is the set of all real numbers. Also, we have that $a^2 + x^2 = a^2 + (a\tan\theta)^2 = a^2 + a^2\tan^2\theta$

$$= a^2(1 + \tan^2\theta) = a^2\sec^2\theta.$$

Thus, $(a^2 + x^2)^{n/2} = (a^2\sec^2\theta)^{n/2} = (\sqrt{a^2\sec^2\theta}\,)^n = (|a||\sec\theta|)^n = |a|^n|\sec\theta|^n$. Since $a > 0$, then $|a| = a$. Since $\sec\theta \geq 1 > 0$ whenever $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then $|\sec\theta| = \sec\theta$. Thus, $(a^2 + x^2)^{n/2} = a^n\sec^n\theta$.

If $x = a\tan\theta$, then $\tan\theta = \frac{x}{a}$.



Using right triangle trigonometry, we have that

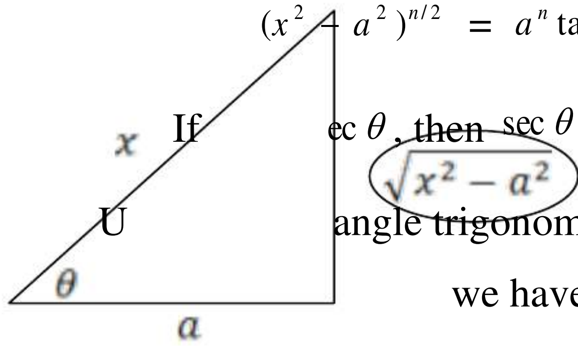
$$\cos\theta = \frac{a}{\sqrt{a^2 + x^2}}, \qquad \sec\theta = \frac{\sqrt{a^2 + x^2}}{a},$$

$$\sin\theta = \frac{x}{\sqrt{a^2 + x^2}}, \qquad \csc\theta = \frac{\sqrt{a^2 + x^2}}{x}$$

If the integrand of an integral contains an expression of the form $(x^2 - a^2)^{n/2}$, where a is a positive real number and n is an odd integer, then the domain of the integrand is the set $(-\infty, -a] \cup [a, \infty)$ or is a subset of this set.

Then try the trigonometric substitution: Let $x = a \sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = a \sec\theta \tan\theta \, d\theta$. Note that the range of the function $x = a\sec\theta$ is the set $(-\infty, -a] \cup [a, \infty)$. Also, we have that $x^2 - a^2 = (a\sec\theta)^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1)$ $= a^2\tan^2\theta$. Thus, $(x^2 - a^2)^{n/2} = (a^2\tan^2\theta)^{n/2} = (\sqrt{a^2\tan^2\theta}\,)^n =$ $(|a||\tan\theta|)^n = |a|^n\,|\tan\theta|^n$. Since $a > 0$, then $|a| = a$. Since $\tan\theta \ge 0$ whenever $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$, then $|\tan\theta| = \tan\theta$. Thus,

$(x^2 - a^2)^{n/2} = a^n \tan^n\theta$.



If ec θ, then $\sec\theta = \frac{x}{a}$.

angle trigonometry,

we have that

$$\sin\theta = \frac{\sqrt{x^2 - a^2}}{x}, \quad \csc\theta = \frac{x}{\sqrt{x^2 - a^2}}, \quad \tan\theta = \frac{\sqrt{x^2 - a^2}}{a} \text{ and}$$

$$\cot\theta = \frac{a}{\sqrt{x^2 - a^2}}$$

Table of Trigonometric Substitutions

Expression	Substitution	Identity used

$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \quad -\pi/2 \le \theta \le \pi/2$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \; -\pi/2 < \theta < \pi/2$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, \quad 0 \le \theta < \pi/2$ or $\pi \le \theta < 3\pi/2$	$\sec^2\theta - 1 = \tan^2\theta$

EXAMPLE 1

Solve the following integral $\int \frac{1}{x^2\sqrt{4-x^2}}\,dx$

Solution

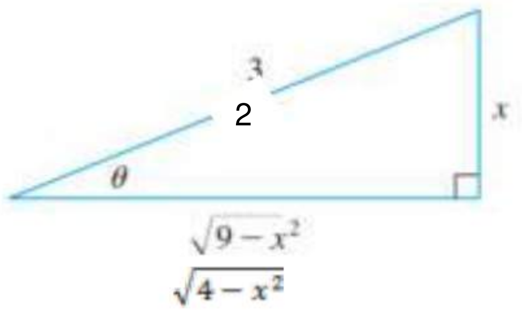
Note that we do not have the x in the integral to do the simple substitution to let $u = x^2 + 9$. We will use the technique of Trigonometric Substitution.

Let $x = 2sin\theta$ ➔ $dx = 2cos\theta\, d\theta$

$\boldsymbol{\sqrt{4 - x^2} = 2cos\theta}$

$$\int \frac{1}{4sin^2\theta(2cos\theta)} 2cos\theta\, d\theta$$

$$= \frac{1}{4}\int \frac{1}{sin^2\theta}\,d\theta = \frac{1}{4}\int csc^2\theta\, d\theta$$



$$= -\frac{1}{4}cot\theta + c$$

$$= -\frac{\sqrt{4 - x^2}}{4x} + c$$

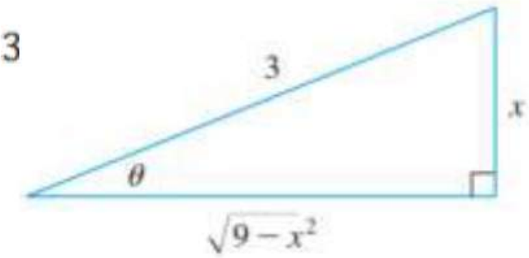
EXAMPLE 2

Solve the following integral $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

Solution

Let $x = 3sin\theta$ ➔ $dx = 3$

$= 3\ cos\theta\ \sqrt{9-x^2}$



$$\int \frac{3cos\theta}{9sin^2\theta} 3cos\theta\ d\theta = \int \frac{cos^2\theta}{sin^2\theta} d\theta$$

$$= \int cot^2\theta\, d\theta = \int csc^2\theta - 1\ \ d\theta = -cot\theta - \theta + C$$

$$= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\frac{x}{3} + c$$

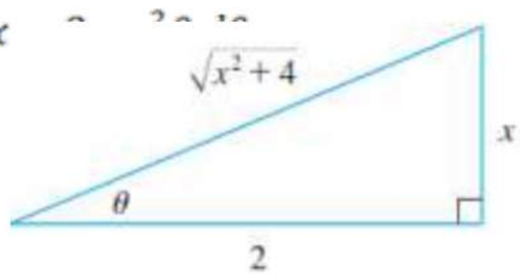
EXAMPLE 3

Solve the following integral $\int \frac{1}{x^2\sqrt{x^2+4}}dx$

Solution

Let $x = 2tan\theta$ ➔ dx

$= 2sec\theta\ \sqrt{4+x^2}$



$$\int \frac{1}{4tan^2\theta\ \ 2sec\theta} 2sec^2\theta\ d\theta$$

$$= \frac{1}{4}\int \frac{sec\theta}{tan^2\theta}\,d\theta = \frac{1}{4}\int \frac{cos\theta}{sin^2\theta}\,d\theta$$

Let $u = \sin\theta$ ➔ $du = \cos\theta\ d\theta$

$$= \frac{1}{4}\int \frac{1}{u^2}\,du = -\frac{1}{4u} + c$$

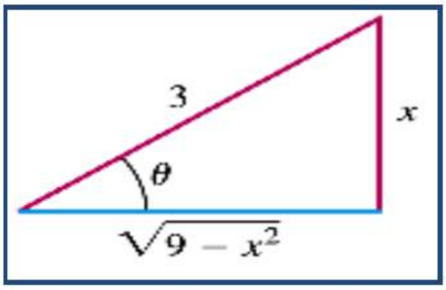
$$= -\frac{1}{4sin\theta} + C = -\frac{\sqrt{4+x^2}}{4x} + c$$

EXAMPLE 4

Solve the following integral $\int x^3\sqrt{9-x^2}\,dx$

Solution

Let $x = 3\sin\theta \qquad \Rightarrow \qquad dx = 3\cos\theta\, d\theta$



$\sqrt{9-x^2} = 3\cos\theta$

$$I = \int 27\sin^3\theta\, 3\cos\theta\, 3\cos\theta\, d\theta$$

$$= \int 243\cos^2\theta \sin^3\theta\, d\theta$$

$$= 243\int \cos^2\theta\,(1-\cos^2\theta)\sin\theta\, d\theta$$

Let $u = \cos\theta \Rightarrow \quad du = -\sin\theta\, d\theta$

$$\therefore I = -243\int u^2(1-u^2)\,du = -\frac{243}{3}u^3 + \frac{243}{5}u^5 + C$$

but $\cos\theta = \dfrac{\sqrt{9-x^2}}{3}$ $\qquad = -\dfrac{243}{3}\cos^3\theta + \dfrac{243}{5}\cos^5\theta + C$

$$\therefore I = -81\left(\frac{\sqrt{9-x^2}}{3}\right)^3 + \frac{243}{5}\left(\frac{\sqrt{9-x^2}}{3}\right)^5 + C$$

Try to solve using integration by substitution $u = 9 - x^2$

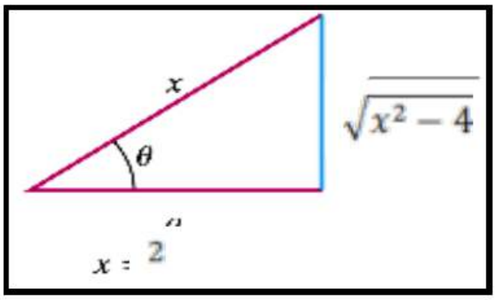
EXAMPLE 5

Solve the following integral $\int \frac{x^2}{\sqrt{x^2-4}} dx$

Solution

Let $x = 2\sec\theta \qquad \Rightarrow \qquad dx = 2\sec\theta\tan\theta \; d\theta$

$\Rightarrow \sqrt{x^2-4} = 2\tan\theta$



$$I = \int \frac{x^2}{\sqrt{x^2-4}} dx = \int \frac{2^3 \sec^3\theta\tan\theta}{2\tan\theta} d\theta$$

$$= 4\int \sec^3\theta \, d\theta$$

See example 9 Page 61

$$= \frac{4}{2}[\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta)] + C$$

$$= 2\left(\frac{\sqrt{x^2-4}}{2} . \frac{x}{2} + \ln\left(\frac{\sqrt{x^2-4}}{2} + \frac{x}{2}\right)\right) + C$$

EXAMPLE 6

Solve the following integral $\int \sqrt{1-x^2} \, dx$

Solution

Let $x = sin\theta \qquad \rightarrow \qquad dx = cos\theta \; d\theta$

$\sqrt{1-x^2} = cos\theta$

$$\int cos\theta\, cos\theta\, d\theta = \int \cos^2\theta\, d\theta$$

$$= \int \frac{1}{2}(1 + cos2\theta)d\theta$$

$$= \frac{\theta}{2} + \frac{sin2\theta}{4} + c$$

$$= \frac{\theta}{2} + \frac{2sin\theta cos\theta}{4} + c$$

$$= \frac{1}{2}\sin^{-1} x + \frac{x\sqrt{1-x^2}}{2} + c$$

EXAMPLE 7

Solve the following integral $\int \sqrt{2x - x^2}\, dx$

Solution

By using complete square

$2x - x^2 = -(x^2 - 2x) = -((x-1)^2 - 1)$

$= 1 - (x-1)^2$

Then let $z = x - 1$ ➔ $dz = dx$

$$\int \sqrt{2x - x^2}\, dx = \int \sqrt{1 - (x-1)^2}\, dx$$

$= \int \sqrt{1 - z^2}\, dz$

Which is identical to the previous example (6)

$$I = \frac{1}{2}\sin^{-1} z + \frac{z\sqrt{1-z^2}}{2} + c$$

Then the integral is

$$I = \frac{1}{2}\sin^{-1}(x-1) + \frac{(x-1)\sqrt{1-(x-1)^2}}{2} + c$$

Exercise

Evaluate the following integrals**:**

$1-\int \frac{x^3}{\sqrt{16-x^2}}dx$ $\qquad$ $2-\int \frac{dx}{x^4\sqrt{x^2+4}}$

$3-\int \frac{x^5}{(x^2-4)^{\frac{3}{2}}}dx$ $\qquad$ $4-\int \frac{dx}{x^2\sqrt{x^2-1}}$

$5-\int \frac{x^2}{\sqrt{9-x^2}}dx$ $\qquad$ $6-\int \frac{1}{\sqrt{x^2+9}}dx$

$8-\int y^5\,(9+7y^2)^{3/2}\ dy 7-\int \sqrt{4-(x-1)^2}dx$

$10-\int t^3(6-t^2)^{5/2}\ dt 9-\int 5x\,\sqrt{4x^2-11}\ dx$

$11-\int \frac{1}{(9y^2-25)^{3/2}}dy$

Method of integration by partial fractions

We know how to add/subtract functions:

EXAMPLE 1

Simplify the expression: $\frac{1}{x+3}-\frac{2}{x}+\frac{3}{x-2}$.

Solution

First, we need a common denominator. The Common denominator is

$x(x+3)(x-2)$ So we must

Simplifying we get: $\frac{1(x)(x-2)-2(x+3)(x-2)+3(x)(x+3)}{x(x+3)(x-2)}$

Simplifying again we get: $\frac{2x^2+5x+12}{x(x+3)(x-2)}$

Therefore, $\frac{1}{x+3}-\frac{2}{x}+\frac{3}{x-2}=\frac{2x^2+5x+12}{x(x+3)(x-2)}$

The method of Partial Fraction Decomposition is basically this process in reverse. We want to start with a rational expression and "Decompose it" into several pieces. Below are the steps necessary to complete the partial fraction decomposition. Let's see if we can start with $\frac{2x^2+5x+12}{x(x+3)(x-2)}$ and decompose it into the 3

pieces: $\frac{1}{x+3}-\frac{2}{x}+\frac{3}{x-2}$

Different cases of Partial Fractions

I) First order non-repeated brackets

STEP 1: Factor the denominator. In our example, the denominator is already factored.

STEP 2: If the rational function has the form $\frac{R(x)}{Q(x)}$ where

$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_nx + b_n)$, then

there exists constants $A_1, A_2, A_3 \ldots A_n$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_n}{a_nx + b_n}$$

So, $\frac{2x^2 + 5x + 12}{x(x+3)(x-2)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-2}$

STEP 3: Multiply both sides of the equation above by the denominator, and re-write the right-hand side in standard form:

$$\frac{2x^2 + 5x + 12}{x(x+3)(x-2)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-2}$$

$$2x^2 + 5x + 12 = A(x+3)(x-2) + Bx(x-2) + Cx(x+3)$$

$$2x^2 + 5x + 12 = Ax^2 + Ax - 6A + Bx^2 - 2Bx + Cx^2 + 3Cx$$

$$2x^2 + 5x + 12 = \left(A + B + C\right)x^2 + (A - 2B + 3C)x - 6A$$

STEP 4: Since the polynomial on the left is equal to the polynomial on the right implies that each corresponding coefficient must be equal. Therefore, we can set up a system of equations by comparing each coefficient.

$$\text{So,}\quad \begin{aligned} &A+B+C=2\\ &A-2B+3C=5\\ &-6A=12 \end{aligned}$$

STEP 5: Solve the system of equations:

Since $-6A=12 \Rightarrow \; A=-2$

$-2+B+C=2$ and $-2-2B+3C=5$

$B+C=4$ and $-2B+3C=7$

So, $B=4-C$

Therefore, $-2(4-C)+3C=7$, so $-8+2C+3C=7$ so $C=3$. If follows that $B=1$

Hence, Since $A=-2,\ B=1$ and $C=3$ means that =

$$\frac{2x^2+5x+12}{x(x+3)(x-2)}=-\frac{2}{x}+\frac{1}{x+3}+\frac{3}{x-2}$$

and our Partial Fraction Decomposition is complete.

SHORT CUT METHOD

In the example above, start the process exactly the same getting your expression into the form:

$$2x^2+5x+12=A(x+3)(x-2)+Bx(x-2)+Cx(x+3)$$

Now, instead of equating the coefficients, lets let $x=0, x=-3$ and $x=2$.

If $x=0$ we get: $12=A(0+3)(0-2)$ or $12=-6A$ so $A=-2$

If $x=-3$ we get: $15 = B(-3)(-5)$ or $15=15B$ so $B=1$

If $x=2$ we get: $30 = C(2)(5)$ or $30=10C$ so $C=3$

As you can see, this method is much quicker. I will use this method if at all possible. Sometimes we can find one or two coefficients this way and find the others by solving systems.

II) First order repeated brackets

Suppose that $(ax+b)^n$ is a factor of the denominator. This means that the linear factor $ax+b$ is repeated n times. When this occurs, the partial fraction decomposition will contain a sum of n fractions for this factor of the denominator

The Partial Fraction Decomposition of $\frac{P(x)}{Q(x)}$: $Q(x)$ Has Repeated Linear Factors

The form of the partial fraction decomposition for a rational expression containing the linear factor $ax+b$ occurring n times as its denominator is

$$\frac{P(x)}{(ax+b)^n}=\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(ax+b)^3}+\ldots+\frac{A_n}{(ax+b)^n}$$

EXAMPLE 2

Find the partial fraction decomposition of $\dfrac{x-18}{x(x-3)^2}$

Solution

Step 1 Set up the partial fraction decomposition with the unknown constants. Because the linear factor $x-3$ occurs twice, we must include one fraction with a constant numerator for each power of $x-3$.

$$\frac{x-18}{x(x-3)^2}=\frac{A}{x}+\frac{B}{x-3}+\frac{C}{(x-3)^2}$$

Step 2 Multiply both sides of the resulting equation by the least common denominator. We clear fractions, multiplying both sides by $x(x-3)^2$, the least common denominator.

$$x(x-3)^2\left[\frac{x-18}{x(x-3)^2}\right]=x(x-3)^2\left[\frac{A}{x}+\frac{B}{x-3}+\frac{C}{(x-3)^2}\right]$$

We use the distributive property on the right side. Then dividing out common factors in numerators and denominators, we obtain

$$\begin{aligned}x-18&=A(x-3)^2+Bx(x-3)+Cx\\&=(A+B)x^2+(-6A-3B+C)x+9A\end{aligned}$$

Step 3 Equating coefficients of like powers of and equating constant terms results in the following system of linear equations:

$$\begin{cases}A+B=0\\-6A-3B+C=1\\9A=-18\end{cases}$$

Step 4 Solve the resulting system for A, B and C. We obtain $A=-2,\ B=2$ and $C=-5$. The required partial fraction decomposition is

$$\frac{x-18}{x(x-3)^2}=-\frac{2}{x}+\frac{2}{x-3}-\frac{5}{(x-3)^2}$$

III) Second order non-repeated brackets

Our final two cases of partial fraction decomposition involve prime quadratic factors of the form ax^2+bx+c. Based on our work with the discriminant, we know that ax^2+bx+c is prime and cannot be factored over the integers if $b^2-4ac<0$ or if b^2-4ac is not a perfect square.

The Partial Fraction Decomposition of $\frac{P(x)}{Q(x)}: Q(x)$ Has a Nonrepeated, Prime Quadratic Factor

If $ax^2 + bx + c$ is a prime quadratic factor of $Q(x)$, the partial fraction decomposition will contain a term of the form $\frac{Ax+B}{ax^2+bx+c}$.

The voice balloons in the box show that each distinct prime quadratic factor in the denominator produces a partial fraction of the form *linear numerator over quadratic factor*. For example,

EXAMPLE 3

Find the partial fraction decomposition of $\frac{3x^2+17x+14}{(x-2)(x^2+2x+4)}$

Solution

Step 1 Set up the partial fraction decomposition with the unknown constants. We put a constant (A) over the linear factor and a linear expression $(Bx+C)$ over the prime quadratic factor.

$$\frac{3x^2+17x+14}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}$$

Step 2 Multiply both sides of the resulting equation by the least common denominator. We clear fractions, multiplying both sides by $(x-2)(x^2+2x+4)$, the least common denominator. Then We use the distributive property on the right side.

$$3x^2+17x+14 = A(x^2+2x+4) + (Bx+C)(x-2)$$

Step 3 Write both sides in descending powers, equate coefficients of like powers of and equate constant terms. The left side, $3x^2+17x+14$ is in descending powers of x. We write the right side in descending powers of x. And express both sides in the same form.

$$3x^2+17x+14=(A+B)x^2+(2A-2B+C)x+(4A-2C)$$

Equating coefficients of like powers of and equating constant terms results in the following system of linear equations:

$$\begin{cases} A+B=3 \\ 2A-2B+C=17 \\ 4A-2C=14 \end{cases}$$

Step 4 Solve the resulting system for $A,\ B$ and C. We obtain $A=5,\ B=-2$ and $C=3$. The required partial fraction decomposition is

$$\frac{3x^2+17x+14}{(x-2)(x^2+2x+4)}=\frac{5}{x-2}+\frac{-2x+3}{x^2+2x+4}$$

IV) Second order repeated brackets

Suppose that $(ax^2+bx+c)^n$ is a factor of the denominator and that ax^2+bx+c cannot be factored further. This means that the quadratic factor ax^2+bx+c occurs n times. When this occurs, the partial fraction decomposition will contain a linear numerator for each power of ax^2+bx+c

The Partial Fraction Decomposition of $\frac{P(x)}{Q(x)}: Q(x)$ Has a Prime Repeated Quadratic Factor

The form of the partial fraction decomposition for a rational expression containing the prime factor $ax^2 + bx + c$ occurring n times as its denominator is

$$\frac{P(x)}{(ax^2+bx+c)^n} = \frac{A_1x+b_1}{ax^2+bx+c} + \frac{A_2x+b_2}{(ax^2+bx+c)^2} + \ldots + \frac{A_nx+b_n}{(ax^2+bx+c)^n}$$

EXAMPLE 4

Find the partial fraction decomposition of $\dfrac{5x^3-3x^2+7x-3}{(x^2+1)^2}$

Solution

Step 1 Set up the partial fraction decomposition with the unknown constants. Because the quadratic factor x^2+1 occurs twice, we must include one fraction with a linear numerator for each power of x^2+1

$$\frac{5x^3-3x^2+7x-3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

Step 2 Multiply both sides of the resulting equation by the least common denominator. We clear fractions, multiplying both sides by $(x^2+1)^2$ the least common denominator and simplify.

$$5x^3-3x^2+7x-3 = (Ax+B)(x^2+1) + (Cx+D)$$

Step 3 Write both sides in descending powers, equate coefficients of like powers of and equate constant terms.

$$5x^3-3x^2+7x-3 = Ax^3+Bx^2+(A+C)x+(B+D)$$

Equating coefficients of like powers of x and equating constant terms results in the following system of linear equations

$$\begin{cases} A = 5 \\ B = -3 \\ A + C = 7 \\ B + D = -3 \end{cases}$$

Step 4 Solve the resulting system for A, B, C and D Based on our observations in step 3, $A = 5,\ B = -3,\ C = 2$ and $D = 0$

Step 5 Substitute the values of A, B, C and D, and write the partial fraction decomposition.

$$\frac{5x^3 - 3x^2 + 7x - 3}{(x^2+1)^2} = \frac{5x-3}{x^2+1} + \frac{2x}{(x^2+1)^2}$$

EXAMPLE 5

Evaluate $\int \frac{1}{x^2 + x - 2} dx$

Solution

$\frac{1}{x^2+x-2}$ can be rewritten as $\frac{1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$

$$1 = A(x-1) + B(x+2)$$

By using short cut method

$$at\ x = 1 \quad \Rightarrow \quad 1 = 3B \quad \Rightarrow \quad 1/3 = B$$

$$at\ x = -2 \quad \Rightarrow \quad 1 = -3A \quad \Rightarrow \quad -1/3 = A$$

$$\int \frac{1}{(x+2)(x-1)} dx = \int \frac{-1/3}{x+2} + \frac{1/3}{x-1} dx$$

$$=-\frac{1}{3}\ln|x+2|+\frac{1}{3}\ln|x-1|+C$$

EXAMPLE 6

Evaluate $\displaystyle\int\frac{x^2+2x-1}{x^3-3x^2+2x}\,dx$

Solution

can be rewritten as $\displaystyle\frac{x^2+2x-1}{x(x-2)(x-1)}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x-1}\ \frac{x^2+2x-1}{x^3-3x^2+2x}$

$$x^2+2x-1=(x-2)(x-1)A+x(x-1)B+x(x-2)C$$

By using short cut method

$$at\ x=0 \quad\Rightarrow\quad -1=2A \quad\Rightarrow\quad A=-\frac{1}{2}$$

$$at\ x=2 \quad\Rightarrow\quad 7=2B \quad\Rightarrow\quad B=\frac{7}{2}$$

$$at\ x=1 \quad\Rightarrow\quad 2=-C \quad\Rightarrow\quad C=-2$$

$$\int\frac{x^2+2x-1}{x^3-3x^2+2x}dx=\int\frac{-1/2}{x}\,dx+\int\frac{7/2}{x-2}\,dx+\int\frac{-2}{x-1}\,dx$$

$$=-\frac{1}{2}\ln|x|+\frac{7}{2}\ln|x-2|-2\ln|x-1|+C$$

EXAMPLE 7

Evaluate $\displaystyle\int\frac{1}{x\left((\ln x)^2-6\ln x+8\right)}\,dx$

Solution

$$u = \ln x \qquad \Rightarrow \qquad du = \frac{1}{x}dx$$

can be rewritten as $\displaystyle\int \frac{1}{x\left((\ln x)^2 - 6\ln x + 8\right)}\,dx = \int \frac{du}{u^2 - 6u + 8}$

$$\frac{1}{(u-2)(u-4)} == \frac{A}{u-2} + \frac{B}{u-4}$$

$$1 = (u-4)A + (u-2)B$$

By using short cut method

$$at\ x = 4 \qquad \Rightarrow \qquad 1 = 2B \qquad \Rightarrow \qquad B = \frac{1}{2}$$

$$at\ x = 2 \qquad \Rightarrow \qquad 1 = -2A \qquad \Rightarrow \qquad A = -\frac{1}{2}$$

$$\int \frac{du}{u^2 - 6u + 8} = -\frac{1}{2}\int \frac{1}{u-2}du + \frac{1}{2}\int \frac{1}{u-4}\,du$$

$$= -\frac{1}{2}\ln|u-2| + \frac{1}{2}\ln|u-4| + C$$

$$= -\frac{1}{2}\ln|\ln x - 2| + \frac{1}{2}\ln|\ln x - 4| + C$$

Important note:

If **the numerator degree ≥ the denominator degree,** use long division first, then use partial fractions.

EXAMPLE 8

Evaluate $\displaystyle\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}dx$

Solution

the degree of the numerator is higher than the degree of the denominator, use long division first.

can be rewritten as $\dfrac{2x^3-4x^2-x-3}{x^2-2x-3}$

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad 2x \\ \hline x^2-2x-3 & 2x^3-4x^2-x-3 \\ & 2x^3-4x^2-6x \\ \hline & \quad\quad\quad\quad\quad 5x-3 \end{array}$$

$$2x+\frac{5x-3}{x^2-2x-3}$$

$$=2x+\frac{A}{(x-3)}+\frac{B}{(x+1)}$$

$$5x-3=(x+1)A+(x-3)B$$

By using short cut method

$$at\ \ x=-1 \quad\Rightarrow\quad -8=-4B \quad\Rightarrow\quad B=2$$

$$at\ \ x=3 \quad\Rightarrow\quad 12=4A \quad\Rightarrow\quad A=3$$

$$\int\frac{2x^3-4x^2-x-3}{x^2-2x-3}dx=\int 2x\ dx+\int\frac{3}{x-3}\ dx+\int\frac{2}{x+1}\ dx$$

$$=x^2+3\ln|x-3|+2\ln|x+1|+C$$

EXAMPLE 9

Evaluate $\displaystyle\int\frac{x^2+2}{(x-1)(x+2)^2}dx$

Solution

can be rewritten as $\dfrac{A}{x-1}+\dfrac{B}{x+2}+\dfrac{C}{(x+2)^2}$ $\dfrac{x^2+2}{(x-1)(x+2)^2}$

$$x^2+2=(x+2)^2A+(x-1)(x+2)B+(x-1)C$$

By using short cut method

$$at\ \ x=1 \quad \Rightarrow \quad 3=9A \quad \Rightarrow \quad A=\frac{1}{3}$$

$$at\ \ x=-2 \quad \Rightarrow \quad 6=-3C \quad \Rightarrow \quad C=-2$$

To find B: substitute with any number x ($x \neq \{1/3, -2\}$).

$$Let\ \ \ x=0 \quad \Rightarrow \quad 2=4A-2B-C$$

$$2=4\frac{1}{3}-2B+2 \quad \Rightarrow \quad B=\frac{2}{3}$$

$$\int \frac{x^2+2}{(x-1)(x+2)^2}dx=\int \frac{1/3}{x-1}+\frac{2/3}{x+2}+\frac{-2}{(x+2)^2}dx$$

$$=\frac{1}{3}\ln|x-1|+\frac{2}{3}\ln|x+2|+\frac{2}{x+2}+C$$

EXAMPLE 10

Evaluate $\int \frac{2x+4}{x^3-2x^2}dx$

Solution

can be rewritten as $\frac{2x+4}{x^2(x-2)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}\ \frac{2x+4}{x^3-2x^2}$

$$2x+4=x(x-2)A+(x-2)B+x^2C$$

By using short cut method

$$at\ \ x=0 \quad \Rightarrow \quad 4=-2B \quad \Rightarrow \quad B=-2$$

$$at\ \ x=2 \quad \Rightarrow \quad 8=4C \quad \Rightarrow \quad C=2$$

To find A: substitute with any number x ($x \neq \{0, 2\}$).

$$Let\ \ \ x=-1 \quad \Rightarrow \quad 2=3A-3B+C$$

$$2=3A-3(-2)+2 \quad \Rightarrow \quad A=-2$$

$$\int \frac{2x+4}{x^2(x-2)}dx = -2\int \frac{dx}{x} - 2\int \frac{dx}{x^2} + 2\int \frac{dx}{x-2}$$

$$-2\ln|x| + \frac{2}{x} + 2\ln|x-2| + C$$

EXAMPLE 11

Evaluate $\int \frac{5x^2+20x+6}{x^4+2x^3+x^2}dx$

Solution

can be rewritten as $\frac{5x^2+20x+6}{x^2(x+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$ $\frac{5x^2+20x+6}{x^4+2x^3+x^2}$

$$5x^2+20x+6 = x(x+1)^2 A + (x+1)^2 B + x^2(x+1)C + x^2 D$$

By using short cut method

$at\ x=0 \qquad \Rightarrow \qquad 6=B$

$at\ x=-1 \qquad \Rightarrow \qquad -9=D$

To find A & C substitute with any number x ($x \neq \{0,-1\}$).

$Let\ \ x=1 \qquad \Rightarrow \qquad 31=4A+4B+2C+D$

$$24=18A+12C$$

Solve simultaneously

$A=8,\ C=-8.$

$$\int \frac{5x^2+20x+6}{x^2(x+1)^2}dx = \int\left(\frac{8}{x}+\frac{6}{x^2}+\frac{-8}{x+1}+\frac{-6}{(x+1)^2}\right)dx$$

$$= 8\ln|x| - \frac{6}{x} - 8\ln|x+1| + \frac{6}{x+1} + C$$

EXAMPLE 12

Evaluate $\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}dx$

Solution

$\frac{2x^3-4x-8}{(x^2-x)(x^2+4)}$ can be rewritten as $\frac{2x^3-4x-8}{x(x-1)(x^2+4)} = \frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+1}$

$$2x^3-4x-8 = (x-1)(x^2+4)A + x(x^2+4)B + x(x-1)(Cx+D)$$

By using short cut method

$at\ x=0 \quad \Rightarrow \quad 2=A$

$at\ x=1 \quad \Rightarrow \quad B=-2$

To find C & D: substitute with any number x ($x \neq \{0,1\}$).

$Let\ \ x=-1 \quad \Rightarrow \quad -6=-10A-5B-2C+2D$

$-6=-20+10-2C+2D$

$2=-C+D$

$Let\ \ x=2 \quad \Rightarrow \quad 0=8A+16B+4C+2D$

$0=16-32+4C+2D$

$8=2C+D$

Solve simultaneously

$C=2,\ D=4.$

$$\int \frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}dx = \int\left(\frac{2}{x}-\frac{2}{x-1}+\frac{2x+4}{x^2+4}\right)dx$$

$$= 2\ln|x| - 2\ln|x-1| + \ln\left|x^2+4\right| + 2\tanh^{-1}\left(\frac{x}{2}\right) + C$$

EXAMPLE 13

Evaluate $\int \frac{1}{\sqrt{x}\left(1-\sqrt[3]{x}\right)}dx$

Solution

$$u=\sqrt[6]{x} \qquad \Rightarrow \qquad u^6 = x \qquad\qquad 6u^5du = dx$$

$$\int \frac{1}{\sqrt{x}\left(1-\sqrt[3]{x}\right)}dx = \int \frac{1}{u^3\left(1-u^2\right)}6u^5du = 6\int \frac{u^2}{1-u^2}du$$

The degree of the numerator is equal to the degree of the denominator, use long division first.

can be rewritten as $\frac{u^2}{1-u^2}$

$$\begin{array}{r|l} & -1 \\ \hline -u^2+1 & u^2 \\ & u^2-1 \\ \hline & 1 \end{array}$$

$$= -1 + \frac{A}{1-u} + \frac{B}{1+u} \qquad -1 + \frac{1}{1-u^2}$$

$$1 = (1+u)A + (1-u)B$$

By using short cut method

$$at\ u=-1 \quad \Rightarrow \quad 1 = 2B \quad \Rightarrow \quad B = 1/2$$

$$at\ x=1 \quad \Rightarrow \quad 1 = 2A \quad \Rightarrow \quad A = 1/2$$

$$6\int \frac{u^2}{1-u^2}du = -6\int du + 6\int \frac{1/2}{1-u}du + 6\int \frac{1/2}{1+u}du$$

$$= -6u + 3\ln\left|1-u\right| + 3\ln\left|1+u\right| + C$$

$$= -6\sqrt[6]{x} + 3\ln\left|1-\sqrt[6]{x}\right| + 3\ln\left|1+\sqrt[6]{x}\right| + C$$

Exercise

Evaluate the following integrals:

$$1-\int \frac{x-6}{x^2-2x}dx \qquad 2-\int \frac{1}{x^2\left(x^2+1\right)}dx$$

$$3-\int \frac{-x^2+3x+4}{x(x+2)^2}dx \qquad 4-\int \frac{\sqrt{x}}{\left(1+\sqrt[3]{x}\right)}dx$$

$$5-\int \frac{e^x}{e^{2x}+3e^x+2}dx \qquad 6-\int \frac{1}{x\sqrt{x+1}}dx$$

$$7-\int \frac{3}{x^3-1}dx. \qquad 8-\int \frac{x^3+1}{x^2+1}dx$$

$$9-\int \frac{e^{3x}}{e^{2x}-e^x-6}\,dx \qquad 10-\int \frac{2x+1}{x^2+2x+1}dx$$

$$11-\int \frac{1}{x^3+x^2+x}\,dx \qquad 12-\int \frac{x+3}{x^3-4x}dx$$

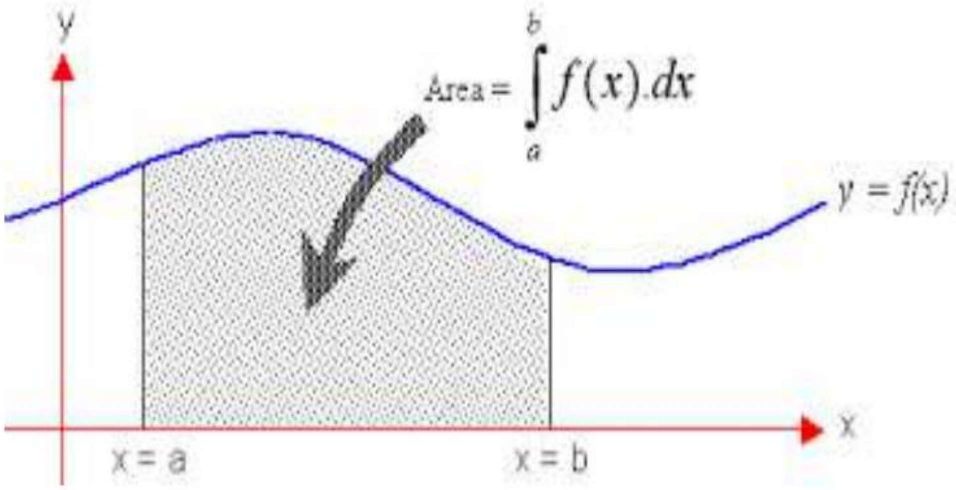
$$13-\int \frac{2x^3}{\left(x^2+1\right)^2}\,dx \qquad 14-\int \frac{x^4-x^3-x-1}{x^3-x^2}dx$$

$$15-\int \frac{x+6}{x^3-3x^2-4x+12}dx \qquad 16-\int \frac{x^2-4x+7}{(x+1)\left(x^2-2x+3\right)}dx$$

$$17-\int \frac{x^3+2x^2-x+1}{x^2+3x-4}dx \qquad 18-\int \frac{x^2+5}{(x+1)\left(x^2-2x+3\right)}dx$$

Definite Integral

In the previous chapter we found how to take an antiderivative and investigated the indefinite integral. In this chapter the connection between antiderivatives and definite integrals is established as we try to solve one of the most famous problems in mathematics, finding the area under a given curve.



Definition

If f is a continuous function on the closed interval $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

EXAMPLE 1

Evaluate $\int_0^3 x^2 dx$

Solution

$$\int_0^1 x^2 dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

Properties of Definite Integrals

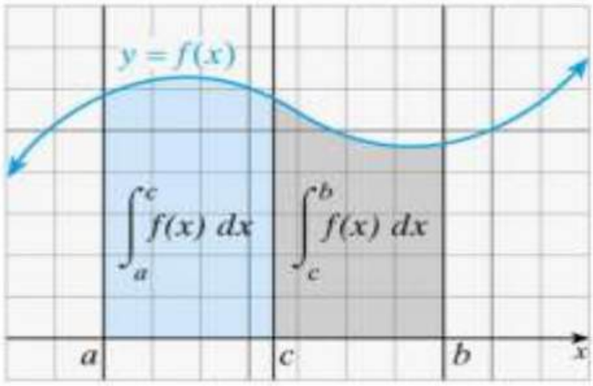
Zero Width Interval

Multiple by Constant $$1)\ \int_a^b k \cdot f(x)\, dx = k\int_a^b f(x)\, dx$$

$$2)\ \int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$$

Sum and Difference between two functions

Additivity $$3)\ \int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx$$



Zero Width Interval $$4)\ \int_a^a f(x)\, dx = 0$$

Order of Integration $$5)\ \int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

EXAMPLE 2

Evaluate the following definite integral

$$\int_{-1}^{1} f(x)\,dx, \quad f(x) = \begin{cases} e^x & , \quad x < 0 \\ \dfrac{1}{1+x^2} & , 0 \le x \le 3 \end{cases}$$

Solution

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} e^x\,dx + \int_{0}^{1} \frac{1}{1+x^2}\,dx$$

$$= e^x|_{-1}^{0} + \tan^{-1} x|_0^1$$

$$= (e^0 - e^{-1}) + (\tan^{-1} 1 - \tan^{-1} 0)$$

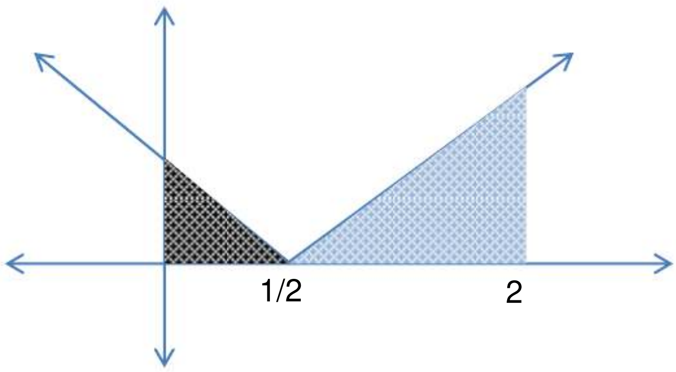
$$= 1 - 0.368 + \frac{\pi}{4} - 0 = 1.4174$$

EXAMPLE 3

Evaluate $\int_0^2 |2x-1|\,dx$

Solution

$$|2x-1| = \begin{cases} 2x-1 & x \ge 0.5 \\ -(2x-1) & x < 0.5 \end{cases}$$



$$\int_0^{1/2} -(2x-1)\,dx + \int_{1/2}^{2} (2x-1)\,dx$$

$$= \left[-x^2 + x\right]_0^{1/2} + \left[x^2 - x\right]_{1/2}^{2}$$

$$= \left(-\frac{1}{4} + \frac{1}{2}\right) - (0+0) + (4-2) - \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{5}{2}$$

Wallias's Formula

In a previous section we have seen the reduction formulas for the case where integrands were powers of a single trigonometric function. Here we shall consider some integrands involving products of powers of trigonometric functions. The technique of finding a reduction formula basically involves integration by prqts. Since there can be more than one way of writing the integrand as a product of two functions, you will see that we can have many reduction formulas for the same integral. We start with the first one of the two types of integrands which we shall study in this section.

EXAMPLE 4

Find the reduction formula of the integral $\int_0^{\pi/2} cos^n x\, dx$

Solution

$= \int_0^{\pi/2} \cos^{n-1} x \quad \cos x\, dx I_n = \int_0^{\pi/2} cos^n x\, dx$

By using By-Parts as follows: -

$u = cos^{n-1} x \qquad\qquad dv = \cos x\, dx$

$du = -(n-1)\cos^{n-2} x \sin x\, dx \qquad\qquad v = \sin x$

$$I_n = (\cos^{n-1} x \sin x)|_0^{\pi/2} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \ \ \sin^2 x\, dx$$

Since $\sin 0 = 0, and \cos\left(\frac{\pi}{.2}\right) = 0$ the first part will be zero. Then

$$I_n = (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \ \ (1-\cos^2 x)dx = (n-1)(I_{n-2} - I_n)$$

$$I_n\left(1 + (n-1)\right) = (n-1)I_{n-2} \qquad \rightarrow \qquad I_n = \frac{n-1}{n} I_{n-2}$$

$$I_n = \frac{n-1}{n} \times \frac{n-3}{n-2} I_{n-4}$$

$$I_n = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} I_{n-6}$$

$$I_n = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \times \ldots \times I_i$$

if n is even $i = 0$
if n is odd $i = 1$

Note that

$$I_0 = \int_0^{\pi/2} \cos^0 x \, dx = \frac{\pi}{2}$$

$$I_1 = \int_0^{\pi/2} \cos x \, dx = \sin x \Big|_0^{\pi/2} = 1$$

Wallis's Formulas

If ***n* is even** and (n ≥ 2) then

$$\int_0^{\pi/2} \cos^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \cdots \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

If ***n* is odd** and (n ≥ 3) then

$$\int_0^{\pi/2} \cos^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \cdots \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}$$

These formulas are also valid if $cos^n x$ is replaced by $sin^n x$

EXAMPLE 5:- Evaluate

$1- \int_0^{\pi/2} \cos^8 x \, dx$ $\qquad$ $2- \int_0^{\pi/2} \sin^7 x \, dx$

$3- \int_0^{\pi/2} \sin^{10} x \, dx$ $\qquad$ $4- \int_0^{\pi/2} \cos^{11} x \, dx$

Solution

Using Wallis's Formulas

$$\int_0^{\pi/2} \cos^n x\, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots$$

$$3 - \int_0^{\pi/2} \sin^{10} x\, dx = \frac{9}{10} \times \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$4 - \int_0^{\pi/2} \cos^{11} x\, dx = \frac{10}{11} \times \frac{8}{9} \times \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3}$$

EXAMPLE 6

Evaluate $\int_0^2 \left(4 - x^2\right)^{7/2} dx$

Solution

$let \quad x = 2\sin\theta \quad dx = 2\cos\theta d\theta \qquad \sqrt{4 - x^2} = 2\cos\theta$

at $x = 0 \Rightarrow \theta = 0$ $\qquad$ at $x = 2 \Rightarrow \theta = \pi/2$

$$\int_0^2 \left(4 - x^2\right)^{7/2} dx = \int_0^{\pi/2} \left(2\cos\theta\right)^7 2\cos\theta d\theta \quad = 2^8 \int_0^{\pi/2} \cos^8 \theta d\theta$$

Using Wallis's Formulas

$$= 2^8 \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

Wallis's Formula for the integral $\int_0^{\pi/2} \sin^n x \cos^m x \, dx$, $n \,\&\, m \neq 1$

Hence we assume that $n > 1$.. Now,

$$I_{m,n} = \int \sin^n x \cos^m x \, dx = \int \cos^{m-1} x \left(\sin^n x \cos x\right) dx$$

Integrating by parts we get

u	dv
$\cos^{m-1} x$	$\sin^n x \cos x dx$
du	v
$-(m-1)\cos^{m-2} x \sin x \, dx$	$\dfrac{\sin^{n+1} x}{n+1}$

Therefor,

$$I_{m,n} = \frac{1}{n+1}\cos^{m-1} x \sin^{n+1} x + \frac{m-1}{n+1}\int \cos^{m-2} x \sin^{n+2} x \, dx$$

$$= \frac{1}{n+1}\cos^{m-1} x \sin^{n+1} x + \frac{m-1}{n+1}\int \cos^{m-2} x \sin^n x\left(\sin^2 x\right) dx$$

$$= \frac{1}{n+1}\cos^{m-1} x \sin^{n+1} x + \frac{m-1}{n+1}\int \cos^{m-2} x \sin^n x\left(1-\cos^2 x\right) dx$$

$$= \frac{1}{n+1}\cos^{m-1} x \sin^{n+1} x + \frac{m-1}{n+1}\int \cos^{m-2} x \sin^n x \, dx - \frac{m-1}{n+1}\int \cos^m x \sin^n x \, dx$$

$$\left(1+\frac{m-1}{n+1}\right) I_{m,n} = \frac{1}{n+1}\cos^{m-1} x \sin^{n+1} x + \frac{m-1}{n+1} I_{m-2,n}$$

$$\left(\frac{m+n}{n+1}\right) I_{m,n} = \frac{1}{n+1}\cos^{m-1} x \sin^{n+1} x + \frac{m-1}{n+1} I_{m-2,n}$$

This gives us,

$$I_{m,n} = \frac{\cos^{m-1} x \sin^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}$$

But surely this formula will not work if $m + n = 0$. So, what do we do if $m + n = 0$?

Actually we have a simple way out. If $m + n = 0$, then since, n is positive, we write $m = -n$.

$$I_{-n,n} = \int \sin^n x \cos^{-n} x \, dx = \int \cot^n x \, dx$$

which is easy to evaluate using the reduction formula

Hence we assume that $m > 1$.. Now,

$$I_{m,n} = \int \sin^n x \cos^m x \, dx = \int \sin^{n-1} x \left(\cos^m x \sin x\right) dx$$

Integrating this by parts we get

$$I_{m,n} = \frac{-1}{m+1} \cos^{m+1} x \sin^{n-1} x + \frac{n-1}{m+1} \int \cos^{m+2} x \sin^{n-2} x \, dx$$

$$\left(1 + \frac{n-1}{m+1}\right) I_{m,n} = \frac{-1}{m+1} \cos^{m+1} x \sin^{n-1} x + \frac{n-1}{m+1} I_{m,n-2}$$

$$\left(\frac{m+n}{m+1}\right) I_{m,n} = \frac{1}{m+1} \cos^{m+1} x \sin^{n-1} x + \frac{n-1}{m+1} I_{m,n-2}$$

This gives us,

$$I_{m,n} = \frac{-\cos^{m+1} x \sin^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$$

Now we want to evaluate

$$\int_0^{\pi/2} \cos^m x \sin^n x \, dx = \left. \frac{-\cos^{m+1} x \sin^{n-1} x}{m+n} \right|_0^{\pi/2} + \frac{n-1}{m+n} I_{m,n-2}$$

$$= 0 + \frac{n-1}{m+n} \left[\left. \frac{\cos^{m-1} x \sin^{n-1} x}{m+n-2} \right|_0^{\pi/2} + \frac{m-1}{m+n-2} I_{m-2,n-2} \right]$$

$$= 0 + \frac{n-1}{m+n} \frac{m-1}{m+n-2} I_{m-2,n-2}$$

Therefor,

$$\int_0^{\pi/2} \sin^n x \cos^m x \, dx = \frac{(n-1)(n-3)........(m-1)(m-3).....}{(n+m)(n+m-2)......} \cdot I_{i,j}$$

Case I) n and m are both even then $I_{i,j} = \frac{\pi}{2}$

Case II) n and m are both odd then $I_{i,j} = \frac{1}{2}$

Case III) one of n or m is odd and the other is even then $I_{i,j} = 1$

EXAMPLE 7

Evaluate each of the following integrals

$1- \int_0^{\pi/2} \sin^6 x \cos^8 x \, dx$ $\qquad$ $2- \int_0^{\pi/2} \sin^6 x \cos^5 x \, dx$

$3- \int_0^{\pi/2} \sin^7 x \cos^5 x \, dx$

Solution

By using Wallis's Formula

$$\int_0^{\pi/2} \sin^n x \cos^m x \, dx = \frac{(n-1)(n-3)........(m-1)(m-3).....}{(n+m)(n+m-2)......} \cdot I_{i,j}$$

$$1- \int_0^{\pi/2} \sin^6 x \cos^8 x \, dx = \frac{(5)\ \ (3)\ \ (1)(7)(5)(3)(1)}{(14)(12)(10)(8)(6)(4)(2)} \cdot \frac{\pi}{2}$$

$$2- \int_0^{\pi/2} \sin^6 x \cos^5 x \, dx = \frac{(5)(3)(1)(4)(2)}{(11)(9)(7)(5)(3)} \cdot 1$$

$$3- \int_0^{\pi/2} \sin^7 x \cos^5 x \, dx = \frac{(6)\ \ (4)(2)(4)(2)}{(12)(10)(8)(6)(4)(2)} \cdot \frac{1}{2}$$

EXAMPLE 8

Evaluate $\int_0^2 x^5 \left(4-x^2\right)^{5/2} dx$

Solution

Let $x = 2\sin\theta$ $\quad dx = 2\cos\theta \mathrm{d}\theta$ $\quad \sqrt{4-x^2} = 2\cos\theta$

$at\ x = 0 \Rightarrow \theta = 0$

$at\ x = 2 \Rightarrow \theta = \pi/2$

$$\int_0^2 x^5 \left(4-x^2\right)^{5/2} dx = \int_0^{\pi/2} \left(2\sin\theta\right)^5 \left(2\cos\theta\right)^5 2\cos\theta d\theta$$

$$= 2^{11} \int_0^{\pi/2} \sin^5\theta \cos^6\theta d\theta$$

By using wallis's formula

$$\int_0^2 x^5 \left(4-x^2\right)^{5/2} dx = 2^{11} \int_0^{\pi/2} \sin^5\theta \cos^6\theta d\theta = 2^{11} \frac{4\times 2\times 5\times 3\times 1}{11\times 9\times 7\times 5\times 3\times 1} \times 1$$

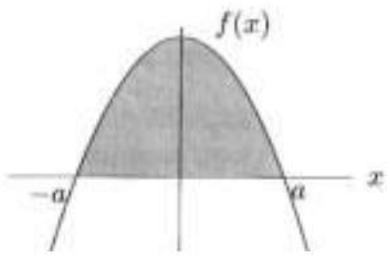
Definite Integrals of Symmetric Functions

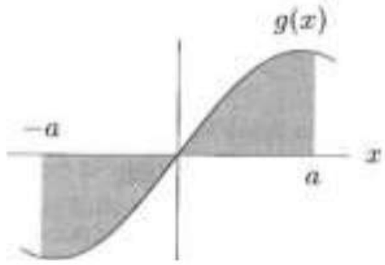
Function f is called odd if $f(-x) = -f(x)$ and even if $f(-x) = f(x)$. Most functions are neither odd nor even. But from any given function defined for both x and $-x$ it is possible to create odd and even combination:

Even functions are symmetric about the y-axis.
An odd function is symmetric about the origin.
So, when evaluating integrals:

a) If f is even, then b) If f is odd, then

$$\int_{-a}^{a} g(x)dx = 0 \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$





EXAMPLE 9: - Evaluate

$2 - \int_{-1}^{1}(xsinx)dx$ $1 - \int_{-2}^{2}(|x| + x^2 - cosx)dx$

$3 - \int_{-2}^{2}(x^3 + sinx)dx$ $4 - \int_{-2}^{2}(x^5 + xcosx)dx$

Solution

$$1 - \int_{-2}^{2}\left(|x| + x^2 - \cos x\right)dx = 2\int_{0}^{2}\left(x + x^2 - \cos x\right)dx$$

$$= 2\left[\frac{x^2}{2} + \frac{x^3}{3} - \sin x\right]_0^2 = 2\left[2 + \frac{8}{3} - \sin(2)\right] = 2.514738$$

$$2-\int_{-1}^{1}(x\sin x)\,dx=2\int_{0}^{1}(x\sin x)\,dx$$

By using integration by parts

$$=2\left[-x\cos x\right]_0^1+2\int_0^1\cos x\,dx=2\left[-x\cos x\right]_0^1+2\left[\sin x\right]_0^1=0.301169$$

$$3-\int_{-2}^{2}\left(x^3+\sin x\right)dx=0$$

$$4-\int_{-2}^{2}\left(x^5+x\cos x\right)dx=0$$

EXAMPLE 10:- Evaluate $\int_{-\pi/2}^{\pi/2}\sin^8 x\,dx$

Solution

$$f(-x)=\sin^8(-x)=\left[-\sin(x)\right]^8$$
$$=\sin^8(x)=f(x)$$

So, $f(x)$ is even function

$$\int_{-\pi/2}^{\pi/2}\sin^8 x\,dx=2\int_{0}^{\pi/2}\sin^8 x\,dx$$
$$=2\,\frac{7\times5\times3\times1}{8\times6\times4\times2}\,\frac{\pi}{2}$$

Exercise

$$1-\int_{0}^{\pi/2} \cos^7 x\,dx$$

$$2-\int_{0}^{\pi/2} \sin^4 x\cos^6 x\,dx$$

$$3-\int_{0}^{\pi/2} \cos^7 x\sin^6 x\,dx$$

$$4-\int_{0}^{\pi} \tan^4\left(\frac{x}{2}\right)\cos^9\left(\frac{x}{2}\right)dx$$

$$5-\int_{-1}^{1}\left(x^3-\tan^{-1} x\right)dx$$

$$6-\int_{-\pi/2}^{\pi/2} \frac{x^3\cos x}{\cosh^2(\sin x)}\,dx$$

$$7-\int_{-2/3}^{2/3} \ln\left(\frac{1+x}{1-x}\right)dx$$

Improper Integrals

Overview: In this section, we will continue to use prior knowledge of limits in order to investigate how to solve improper integrals. we will learn the different types of improper integrals and how to solve each of them. we will also investigate and find guidelines for dealing with integrals involving $(1/x^p)$.

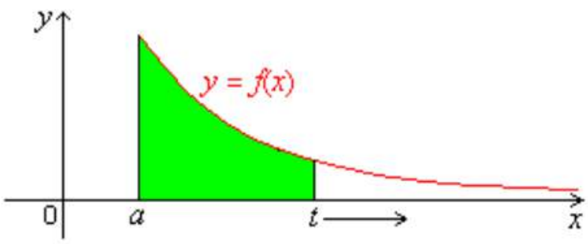
Improper Integrals (Type I)

Integrals over infinite intervals

First, we consider integrals of functions defined over infinite integrals of the form $[a,\infty), (-\infty,b]$ and $(-\infty,\infty)$. Recall that Rieman integral was defined over intervals of the form $[a,b]$.

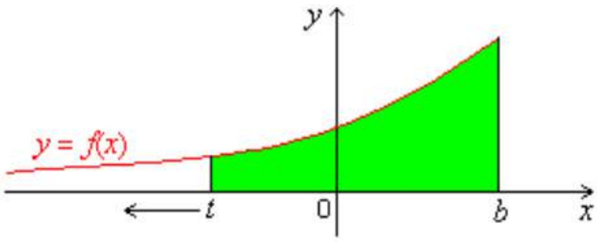
Definition

(i) Suppose f is defined on $[a,\ \infty)$ and integrable on $[a,\ t]$ for all $t > a$. If $\lim_{t\to\infty}\int_a^t f(x)dx$ exists, then we define the improper integral of f over $[a,\infty)$ as



$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$$

(ii) Suppose f is defined on $(-\infty, b]$ and integrable on $[t;\ b]$ for all $t < b$. If $\lim_{t\to\infty} \int_t^b f(x)dx$ exists, then we define the improper integral of f over $(-\infty, b]$ as



$$\int_{-\infty}^{b} f(x)dx = \lim_{t\to\infty} \int_{t}^{b} f(x)dx$$

(iii) Suppose f is defined on $R := (-\infty,\ \infty)$ and integrable on $[a, b]$ for every closed and bounded interval $[a, b] \subseteq R$. If $\int_{-\infty}^{c} f(x)dx$ and $\int_{c}^{\infty} f(x)dx$ exist for some $c \in R$, then we define the improper integral of f over $(-\infty,\ \infty)$ as

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$$

EXAMPLE 1

Evaluate the following integrals

$2-\int_{1}^{\infty} \frac{1}{x}dx$ $\qquad$ $1-\int_{1}^{\infty} \frac{1}{x^2}dx$

$4-\int_{0}^{\infty} xe^{-x^2}dx$ $\qquad$ $3-\int_{-\infty}^{0} e^{x+1}dx$

Solution

$$1-\int_{1}^{\infty} \frac{1}{x^2}dx = \lim_{b\to\infty} \int_{1}^{b} \frac{1}{x^2}dx$$

Convergent integration $$=\lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b=\lim_{b\to\infty}\left[-\frac{1}{b}+1\right]=1$$

$$2-\int_1^\infty \frac{1}{x}dx=\lim_{b\to\infty}\int_1^b \frac{1}{x}dx=\lim_{b\to\infty}\big[\ln|x|\big]_1^b$$

(Divergent integration) $$=\lim_{b\to\infty}\big[\ln(b)+\ln(1)\big]=\lim_{b\to\infty}\big[\ln(b)\big]=\infty$$

$$3-\int_{-\infty}^0 e^{x+1}dx=\lim_{a\to-\infty}\int_a^0 e^{x+1}dx$$

Convergent integration $$=\lim_{a\to-\infty}\left[e^{x+1}\right]_a^0=\lim_{a\to-\infty}\left[e^1-e^{a+1}\right]=e^1$$

$$4-\int_0^\infty xe^{-x^2}dx=\lim_{b\to\infty}\int_0^b xe^{-x^2}dx$$

(Convergent) $$=-\frac{1}{2}\lim_{b\to\infty}\left[e^{-x^2}\right]_0^b=-\frac{1}{2}\lim_{b\to\infty}\left[e^{-b^2}-e^0\right]=\frac{1}{2}$$

EXAMPLE 2

Evaluate the following integrals

$$2-\int_{-\infty}^{\infty}\frac{1}{x^2+1}dx \qquad\qquad 1-\int_{1}^{\infty}\frac{1}{\sqrt{x}}dx$$

$$3-\int_{1}^{\infty}\frac{\ln x}{x^2}dx$$

Solution

$$1-\int_{1}^{\infty}\frac{1}{\sqrt{x}}dx=\lim_{b\to\infty}\int_{1}^{b}\frac{1}{\sqrt{x}}dx$$

Diverges integration $\qquad =\lim_{b\to\infty}[2\sqrt{x}\,]_1^b=2\sqrt{b}-2=\infty$

is even $f(x)$ $\qquad 2-\int_{-\infty}^{\infty}\frac{1}{x^2+1}dx$

$$=2\int_{0}^{\infty}\frac{1}{x^2+1}dx=2\lim_{b\to\infty}\int_{0}^{b}\frac{1}{x^2+1}dx$$

$$=2\lim_{b\to\infty}\left[\tan^{-1}x\right]_0^b+2\lim_{b\to\infty}\left[\tan^{-1}b-\tan^{-1}0\right]_0^b$$

Convergent $\qquad =2\left[\frac{\pi}{2}-0\right]=\pi$

by using integration by parts $3-\int_{1}^{\infty}\frac{\ln x}{x^2}dx$

$$=\lim_{a\to\infty}\int_{1}^{a}\frac{\ln x}{x^2}dx=\lim_{a\to\infty}\left[-\frac{1}{x}\ln x\right]_1^a+\lim_{a\to\infty}\int_{1}^{a}\frac{1}{x^2}dx$$

$$=\lim_{a\to\infty}\left[-\frac{1}{x}\ln x\right]_1^a+\lim_{a\to\infty}\left[-\frac{1}{x}\right]_1^a$$

By using L' Hopital's Rule

Convergent $$= \lim_{a\to\infty}\left[-\frac{1/a}{1}\right] + \left[-0+\frac{1}{1}\right] = 0+1 = 1$$

EXAMPLE 3

Find the following integral $\int_2^{\infty} \frac{2}{x^2-1}dx$

Using partial fractions technique

$$\frac{2}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} \quad \Rightarrow \quad 2 = (x+1)A + (x-1)B$$

At $x=1 \quad \Rightarrow \quad 2=2A \quad \Rightarrow \quad A=1$

At $x=-1 \quad \Rightarrow \quad 2=-2B \quad \Rightarrow \quad B=-1$

$$\int_2^{\infty} \frac{2}{x^2-1}dx = \int_2^{\infty}\left[\frac{1}{x-1} + \frac{-1}{x+1}\right]dx = \int_2^{\infty}\frac{1}{x-1}dx - \int_2^{\infty}\frac{1}{x+1}dx$$

$$= \lim_{a\to\infty}\left[\int_2^{a}\frac{1}{x-1}dx - \int_2^{a}\frac{1}{x+1}dx\right] = \lim_{a\to\infty}\left[\ln|x-1|_2^a - \ln|x+1|_2^a\right]$$

$$= \lim_{a\to\infty}\left[\ln\left|\frac{x-1}{x+1}\right|_2^a\right] = \lim_{a\to\infty}\left[\ln\left|\frac{a-1}{a+1}\right|\right] - \ln\left|\frac{1}{3}\right| = \ln\left[\lim_{a\to\infty}\left|\frac{a-1}{a+1}\right|\right] - \ln\left|\frac{1}{3}\right|$$

By using L' Hopital's Rule

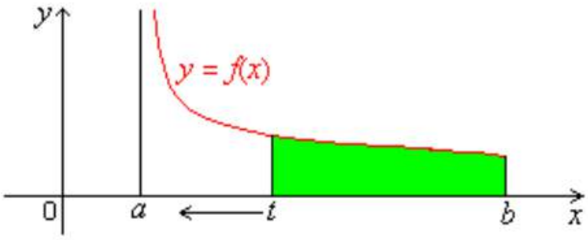
Convergent integration $$= \ln\left[\lim_{a\to\infty}|1|\right] - \ln\left|\frac{1}{3}\right| = -\ln\left|\frac{1}{3}\right|$$

Improper Integrals (Type II)

Integrands with Vertical Asymptotes

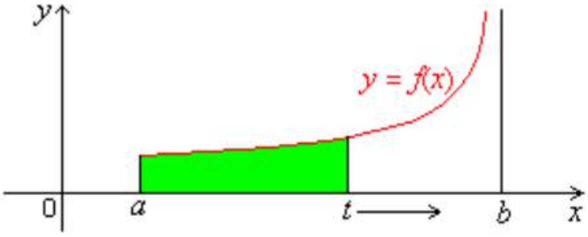
Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a then



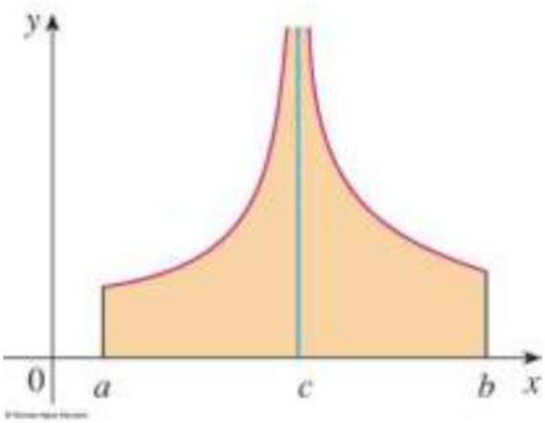
$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$$

2. If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then



$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$. Then



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

In each case, if the limit is finite, we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

EXAMPLE 4

Investigate the convergence of $\int_0^1 \frac{1}{1-x}dx$

Solution

The integrand $f(x) = 1/(1-x)$ is continuous on [0, 1) but is discontinuous at $x = 1$ and becomes infinite. We evaluate the integral as

$$\int_0^1 \frac{1}{1-x}dx = \lim_{b\to 1^-}\int_0^b \frac{1}{1-x}dx$$

$$= \lim_{b\to 1^-}\Big[-\ln|1-x|\Big]_0^b = \lim_{b\to 1^-}\Big[-\ln|1-b|+0\Big] = \infty$$

The limit is infinite, so the integral diverges.

EXAMPLE 5

Find the following integral $\int_0^1 \frac{1}{\sqrt{x}}dx$

Solution

The integrand $f(x) = 1/\sqrt{x}$ is continuous on (0, 1] but is discontinuous at $x = 0$ and becomes infinite. We evaluate the integral as

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{a\to 0^+}\left[2\sqrt{x}\right]_a^1 = \lim_{a\to 0^+}\left[2\sqrt{1}-2\sqrt{a}\right] = 2$$

The limit is finite, so the integral converges.

EXAMPLE 6

Find the following integral $\int_0^{\pi/2} \sec x\,dx$

Solution

So, the integrand $f(x) = \sec x$ is continuous on $[0, \pi/2)$ but is discontinuous at $x = \pi/2$ and becomes infinite. We evaluate the integral as

$$\int_0^{\pi/2} \sec x\,dx = \lim_{x\to(\pi/2)^-}\int_0^b \sec x\,dx$$

$$= \lim_{x\to(\pi/2)^-} \ln\left|\sec x + \tan x\right|\Big]_0^b$$

$$= \lim_{x\to(\pi/2)^-}\left[\ln(\sec b + \tan b) - \ln 1\right] = \infty$$

This is because sec $b \to \infty$ and tan $b \to \infty$ as $b \to (\pi/2)^-$.

The limit is infinite, so the integral divergence.

EXAMPLE 7

Evaluate $\int_0^3 \frac{dx}{x-1}$ if possible.

Solution

Observe that the line $x = 1$ is a vertical asymptote of the integrand. As it occurs in the middle of the interval [0, 3],

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

Where

$$\begin{aligned}\int_0^1 \frac{dx}{x-1} = \lim_{c\to 1^-} \int_0^c \frac{dx}{x-1} &= \lim_{c\to 1^-} |x-1|\Big]_0^c \\ &= \lim_{c\to 1^-} (\ln|c-1| - \ln|-1|) \\ &= \lim_{c\to 1^-} \ln(1-c) = -\infty\end{aligned}$$

Exercise

$1-\int_{0}^{\infty}\sin x\,dx$

$2-\int_{-\infty}^{-5}\frac{dx}{x^{4}}$

$3-\int_{2}^{\infty}\frac{\ln x\,dx}{x^{2}}$

$4-\int_{-\infty}^{1}\frac{dx}{2x-3}$

$5-\int_{-\infty}^{\infty}\frac{x}{\sqrt{x^{2}+9}}dx$

$6-\int_{e}^{\infty}\frac{1}{x\ln x}dx$

$7-\int_{1}^{\infty}2xe^{-x^{2}}\,dx$

$8-\int_{-\infty}^{\infty}\frac{1}{x^{2}+2x+10}$

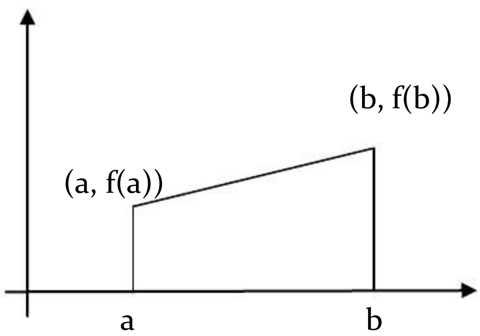
$9-\int_{8}^{\infty}\frac{1}{x^{1/3}}dx$

$10-\int_{1}^{2}\frac{x}{x^{2}-1}dx$

Numerical integration

Trapezoidal Rule of Integration

The *trapezoidal rule* is the first of the Newton-Cotes closed integration formulas. It corresponds to the case where the polynomial in $I = \int_a^b f(x)dx \cong \int_a^b f_n(x)dx$ is first-order:



$$I = \int_a^b f(x)dx \cong \int_a^b f_n(x)dx$$

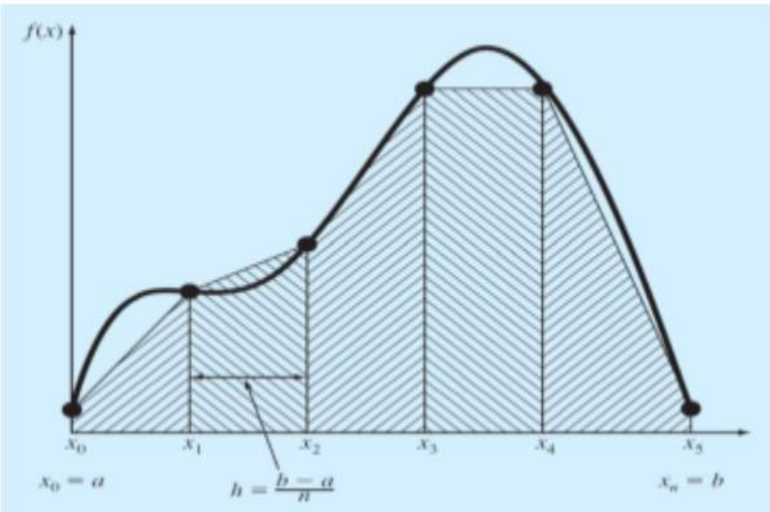
$$I = \left[f(a)x + \left(\frac{f(b) - f(a)}{b - a} \right) \frac{(x-a)^2}{2} \right]_{x=a}^{x=b}$$

$$I = \left[\frac{\left(f\left(b\right) + f\left(a\right) \right)}{2} \left(b-a\right) \right]$$

The Composite Trapezoidal Rule

One way to improve the accuracy of the trapezoidal rule is to divide the integration interval from a to b into a number of segments and apply the method to each segment. The areas of individual segments can then be added to yield the integral for the entire interval. The resulting equations are called *composite,* or *multiple-segment, integration formulas.*

If a and b are designated as x_0 and x_n, respectively, the total integral can be represented as



$$I = \int_{x_0}^{x_1} f\left(x\right) dx + \int_{x_1}^{x_2} f\left(x\right) dx + \ldots + \int_{x_{n-1}}^{x_n} f\left(x\right) dx$$

Substituting the trapezoidal rule for each integral yields

$$I = h \frac{f\left(x_0\right) + f\left(x_1\right)}{2} + h \frac{f\left(x_1\right) + f\left(x_2\right)}{2} + \ldots + h \frac{f\left(x_{n-1}\right) + f\left(x_n\right)}{2}$$

$$I=\frac{h}{2}\left[f\left(x_0\right)+2f\left(x_1\right)+2f\left(x_2\right)+2f\left(x_3\right)+\ldots+f\left(x_n\right)\right]$$

$$=\frac{h}{2}\left[f\left(x_0\right)+2\sum_{i=1}^{i=n-1}f\left(x_i\right)+f\left(x_n\right)\right]$$

EXAMPLE 1

Use the two-segment trapezoidal rule to estimate the integral of

$f(x) = 0.2 + 25\,x - 200\,x^2 + 675\,x^3 - 900\,x^4 + 400\,x^5$

from $a = 0$ to $b = 0.8$. estimate the approximated error. And true error

recall that the exact value of the integral is 1.640533.

Solution

For n = 2

$$h=\frac{b-a}{n}=\frac{0.8-0}{2}=0.4$$

x	0	0.4	0.8
$f(x)$	0.2	2.456	0.232

$$I=\frac{h}{2}\left[f\left(x_0\right)+2\sum_{i=1}^{i=n-1}f\left(x_i\right)+f\left(x_n\right)\right]$$

$$=\frac{0.4}{2}\left[0.2+2\times 2.456+0.232\right]=1.0688$$

$E_t=1.640533-1.0688=0.57173$ $\qquad\qquad \varepsilon_t=34.9\%$

EXAMPLE 2

in the previous example Use the four-segment

Solution

n = 4 For

$$h=\frac{b-a}{n}=\frac{0.8-0}{4}=0.2$$

x	0	0.2	0.4	0.6	0.8
$f(x)$	0.2	1.288	2.456	3.464	0.232

$$I=\frac{h}{2}\left[f\left(x_0\right)+2\sum_{i=1}^{i=n-1}f\left(x_i\right)+f\left(x_n\right)\right]$$

$$=\frac{0.2}{2}\left[0.2+2\times\left(1.288+2.456+3.464\right)+0.232\right]=1.4848$$

$E_t=1.640533-1.4848=0.155733$ $\qquad\qquad \varepsilon_t=9.5\%$

EXAMPLE 3

Use 4 – segment multiple trapezoidal rule to evaluate

$\int_{-1}^{1}\frac{1}{\sqrt{4-x^2}}\,dx$ Then find the true error.

Solution

$$h=\frac{b-a}{n}=\frac{1+1}{4}=0.5$$

x	-1	-0.5	0	0.5	1
$f(x)$	0.57735	0.516398	0.5	0.516398	0.57735

$$I=\frac{h}{2}\left[f(a)+2\sum_{i=1}^{n-1}f(x_i)+f(b)\right]=1.05507$$

$$\int_{-1}^{1}\frac{1}{\sqrt{4-x^2}}dx=\int_{-1}^{1}\frac{1}{2\sqrt{1-\left(\frac{x}{2}\right)^2}}dx=\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-1}^{1}=1.0472$$

$$E_t=\left|1.0472-1.05507\right|=0.00787$$

Simpson's 1/3 Rule

Simpson's rule is the most accurate method of finding the area under a curve.

It is better than the trapezoidal rule because instead of using straight lines to model the curve, it uses parabolic arches to approximate each part of the curve.

Trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial in the interval of integration. Simpson's 1/3rd rule is an extension of Trapezoidal rule where the integrand is approximated by a second order polynomial.

Hence

$$I = \int_a^b f(x)dx \approx \int_a^b f_2(x)dx$$

Where $f_2(x)$ is a second order polynomial.

$$f_2(x) = a_0 + a_1 x + a_2 x^2$$

Choose

$$(a, f(a)), \left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right), \& (b, f(b))$$

as the three points of the function to evaluate a_0, a_1 and a_2.

$$f(a) = f_2(a) = a_0 + a_1 a + a_2 a^2$$

$$f\left(\frac{a+b}{2}\right) = f_2\left(\frac{a+b}{2}\right) = a_0 + a_1\left(\frac{a+b}{2}\right) + a_2\left(\frac{a+b}{2}\right)^2$$

$$f(b) = f_2(b) = a_0 + a_1 b + a_2 b^2$$

Solving the previous equations for a_0, a_1 and a_2 give

$$a_0 = \frac{a^2 f(b) + abf(b) - 4abf\left(\frac{a+b}{2}\right) + abf(a) + b^2 f(a)}{a^2 - 2ab + b^2}$$

$$a_1 = -\frac{af(a) - 4af\left(\frac{a+b}{2}\right) + 3af(b) + 3bf(a) - 4bf\left(\frac{a+b}{2}\right) + bf(b)}{a^2 - 2ab + b^2}$$

$$a_2 = \frac{2\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)}{a^2 - 2ab + b^2}$$

Then

$$I \approx \int_a^b f_2(x)dx$$

$$= \int_a^b \left(a_0 + a_1 x + a_2 x^2\right) dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3}\right]_a^b$$

$$= a_0 (b-a) + a_1 \frac{b^2 - a^2}{2} + a_2 \frac{b^3 - a^3}{3}$$

Substituting values of a_0, a_1 and a_2 give

$$\int_a^b f_2(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

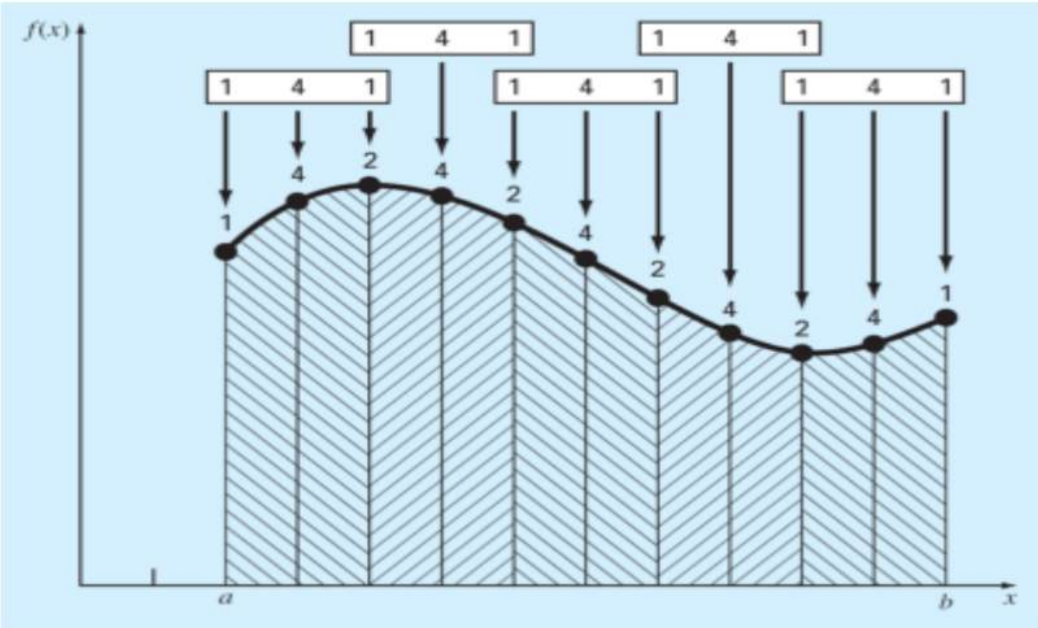
Since for Simpson's 1/3rd Rule, the interval [a, b] is broken into 2 segments, the segment width $h = \frac{b-a}{2}$

Hence

$$\int_a^b f_2(x)dx = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

Because the above form has 1/3 in its formula, it is called Simpson's 1/3rd Rule.

Multiple Simpson's Rule



$$I = \int_a^b f\left(x\right)dx$$

$$I = \int_{x_0}^{x_2} f\left(x\right)dx + \int_{x_2}^{x_4} f\left(x\right)dx + \int_{x_4}^{x_6} f\left(x\right)dx + \ldots + \int_{x_{n-2}}^{x_n} f\left(x\right)dx$$

$$= \frac{h}{3}\left[f\left(x_0\right) + 4f\left(x_1\right) + f\left(x_2\right)\right] + \frac{h}{3}\left[f\left(x_2\right) + 4f\left(x_3\right) + f\left(x_4\right)\right] + \ldots$$
$$+ \frac{h}{3}\left[f\left(x_{n-2}\right) + 4f\left(x_{n-1}\right) + f\left(x_n\right)\right]$$

$$I = \frac{h}{3}\left[f\left(x_0\right) + 4\sum_{i=1,3,5}^{n-1} f\left(x_i\right) + 2\sum_{i=2,4,6}^{n-2} f\left(x_i\right) + f\left(x_n\right) \right]$$

EXAMPLE 4

Estimate using Simpson's rule $\int_0^{\pi} \sin x \; dx$ with $n = 4$

Solution

$$h = \frac{b-a}{n} = \frac{\pi - 0}{4} = \frac{\pi}{4}$$

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
$f(x)$	$sin0$	$sin(\pi/4)$	$sin(\pi/2)$	$sin(3\pi/4)$	$sin(\pi)$

$$\int_0^{\pi} \sin x \; dx = \frac{\pi}{3(4)}\left[\sin 0 + 4\sin\frac{\pi}{4} + 2\sin\frac{\pi}{2} + 4\sin\frac{3\pi}{4} + \sin\pi\right]$$

$$= \frac{\pi}{12}\left[0 + 2\sqrt{2} + 2 + 2\sqrt{2} + 0\right] \approx 2.005$$

if n = 8 $\quad Ans. \approx 2.0003$

$$\int_0^{\pi} \sin x dx = -\cos x\Big]_0^{\pi} = 2$$

EXAMPLE 5

Use 4-segment Simpson's 1/3 rule to approximate the distance covered by a rocket in meters from $t = 8$ s to $t = 30$ s as given by

$$x = \int_8^{30}\left(2000\ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t\right)dt$$

Solution

Using 4 segment Simpson's 1/3 rule,

$$x \approx \frac{b-a}{3n}\left[f(t_0) + 4\sum_{\substack{i=1\\ i=odd}}^{n-1} f(t_i) + 2\sum_{\substack{i=2\\ i=even}}^{n-2} f(t_i) + f(t_n) \right]$$

$$h = \frac{b-a}{n} = \frac{30-8}{4} = 5.5$$

$$f(t) = 2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t$$

So

$$f(8) = 2000\ln\left[\frac{140000}{140000-2100(8)}\right] - 9.8(8) = 177.27 m/s$$

$$f(13.5) = 2000\ln\left[\frac{140000}{140000-2100(13.5)}\right] - 9.8(13.5) = 320.25 m/s$$

$$f(19) = 2000\ln\left(\frac{140000}{140000-2100(19)}\right) - 9.8(19) = 484.75 m/s$$

$$f(24.5) = 2000\ln\left[\frac{140000}{140000-2100(24.5)}\right] - 9.8(24.5) = 676.05 m/s$$

$$f(30) = 2000\ln\left[\frac{140000}{140000-2100(30)}\right] - 9.8(30) = 901.67 m/s$$

$$x = \frac{b-a}{3n}\left[f(t_0) + 4\sum_{\substack{i=1\\ i=odd}}^{n-1} f(t_i) + 2\sum_{\substack{i=2\\ i=even}}^{n-2} f(t_i) + f(t_n) \right]$$

$$= \frac{30-8}{3(4)}\left[f(8) + 4\sum_{\substack{i=1\\ i=odd}}^{3} f(t_i) + 2\sum_{\substack{i=2\\ i=even}}^{2} f(t_i) + f(30) \right]$$

$$= \frac{22}{12}\left[f(8) + 4f(t_1) + 4f(t_3) + 2f(t_2) + f(30) \right]$$

$$= \frac{11}{6}\left[f(8) + 4f(13.5) + 4f(24.5) + 2f(19) + f(30) \right]$$

$$= \frac{11}{6}\left[177.27 + 4(320.25) + 4(676.05) + 2(484.75) + 901.67 \right]$$

$$= 11061.64\ m$$

n	Approximate Value	E_t	$\lvert \epsilon_t \rvert$
2	11065.72	-4.38	0.0396%
4	11061.64	-0.30	0.0027%
6	11061.40	-0.06	0.0005%
8	11061.35	-0.02	0.0002%
10	11061.34	-0.01	0.0001%

We can see that as n increase errors decrease

Exercise

Evaluate the following integral

1– $\int_0^4 \left(1-e^{-x}\right) dx$

2– $\int_0^{\pi/2} \left(8+4\cos x\right) dx$

3– $\int_{-2}^{4} \left(1-x-4x^3+2x^5\right) dx$

a. Analytically,

b. Using the trapezoidal rule,

c. By using composite trapezoidal rule with $n = 2$ and 4,

d. Using Simpson's 1/3 rule,

e. Composite Simpson's 1/3 rule with $n = 4$,

Chapter 7
Applications of Integrals

In this section we're going to take a look at some of applications of integration. It should be noted as well that these applications are presented here, as opposed to Calculus I, simply because many of the integrals that arise from these applications tend to require techniques that we discussed in the previous chapter.

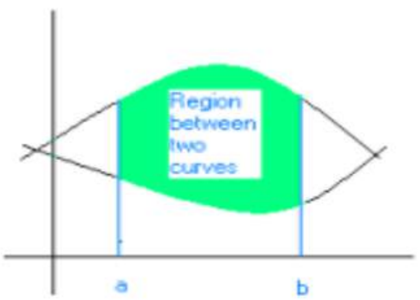
Here is a list of applications that we'll be taking a look at in this chapter.

Area Between Two Curves – In this section we'll take a look at determining the area between two curves.

Volumes of Solids of Revolution / Method of Rings – This is the first of two sections devoted to find the volume of a solid of revolution. In this section we look at the method of rings/disks.

1) *Area Between Curves*

In this section we are going to look at finding the area between two curves. we are want to determine the area between $y = f(x)$ and $y = g(x)$ on the interval $[a,b]$. We are also going to assume that $f(x) \geq g(x)$. Take a look at the following sketch to get an idea of what we're initially going to look at.



$$\begin{aligned} &\textit{Area of region between } f \textit{ and } g \\ &\qquad = \textit{Area of region under } f(x) - \textit{Area of region under } g(x) \end{aligned}$$

$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx =$$

Steps to calculate the Area between curves

1. Sketch the given graphs.
2. Find the points of intersection.
3. Determine the upper and lower curves.

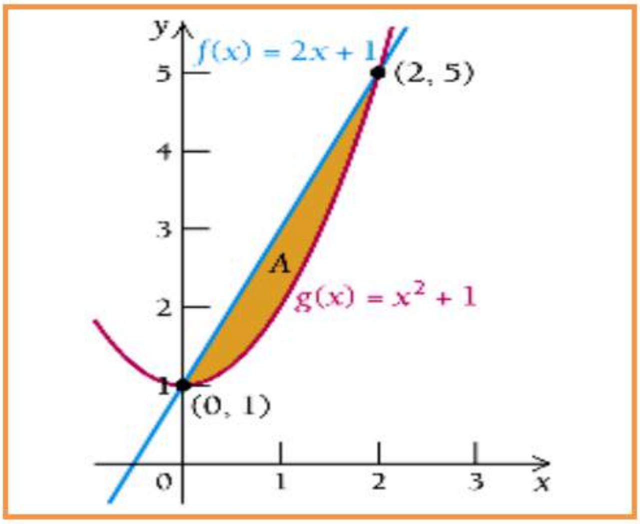
y = f(x) "upper" *y = g(x) "lower"*

4. Calculate the area using the following formula

$$Area = \int_{min\_x}^{max\_x} [upper - lower]dx$$

Example1: Find the area of the region that is bounded by the graphs of $f(x) = 2x + 1$ and $g(x) = x^2 + 1$.

Solution



First step, Sketch

Second step, points of intersection

by setting $f(x) = g(x)$

$2x + 1 = x^2 + 1$

$x^2 - 2x = 0$

$x(x - 2) = 0$

$x = 0$ or $x = 2$

Step 3,

Upper $y = 2x + 1$

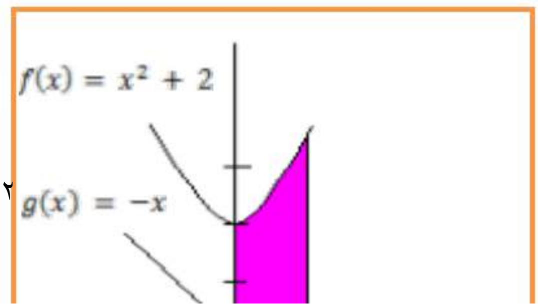
Lower $y = x^2 + 1$

Step 4

$$= x^2 - \frac{x^3}{3}\Bigg|_0^2 = 4 - \frac{8}{3} = 1.3333 \quad A = \int_0^2 \left[(2x+1) - (x^2+1)\right] dx$$

Example2: Find the area of the region bounded by the graphs of *f(x)* = x^2 + *2*, *g(x)* = *- x*, *x* = *0*, and *x* = *1* .

Solution



$$A=\int_a^b [f(x)-g(x)]dx =$$

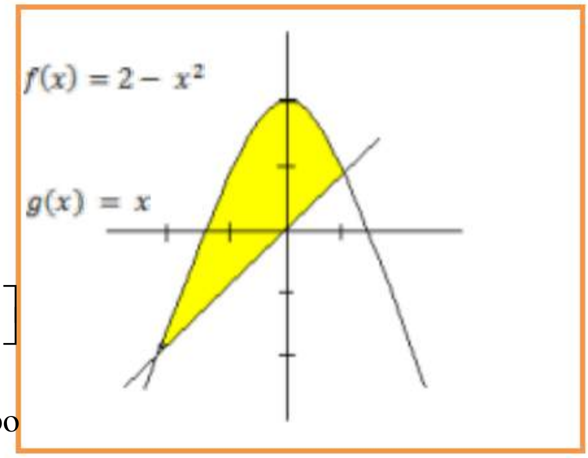
$$\int_0^1 \left[(x^2+2)-(-x)\right]dx=$$

$$\left[\frac{x^3}{3}+\frac{x^2}{2}+2x\right]_0^1 = \frac{1}{3}+\frac{1}{2}+2=\frac{17}{6}$$

Example3: Find the area of the region bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$

Solution

First, set $f(x) = g(x)$ to find their points of intersection.



$2 - x^2 = x \quad \rightarrow \quad 0 = x^2 + x - 2$

$0 = (x + 2)(x - 1) \quad \rightarrow \quad x = -2 \text{ and } x = 1$

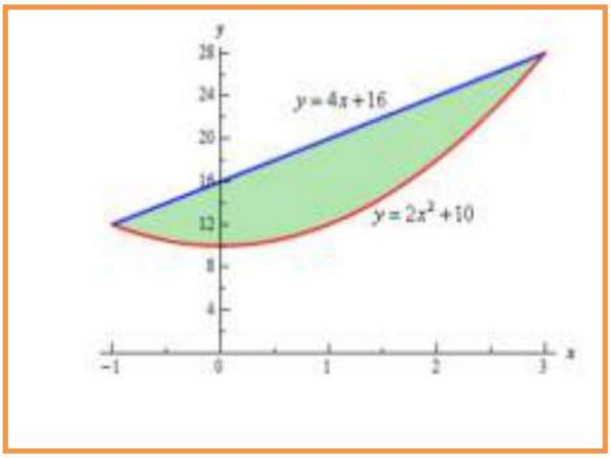
$$A=\int_a^b [f(x)-g(x)]dx = \quad \int_{-2}^1 \left[\left(2-x^2\right)-x\right]$$

Example4: Determine the area of the region bo

$y = 4x + 16.$

Solution

First, set $f(x) = g(x)$ to find their points of intersection.



$2x^2 + 10 = 4x + 16 \quad \rightarrow \quad 0 = x^2 - 2x - 6$

$0 = (x + 1)(x - 3) \quad \rightarrow \quad x = -1 \text{ and } x = 3$

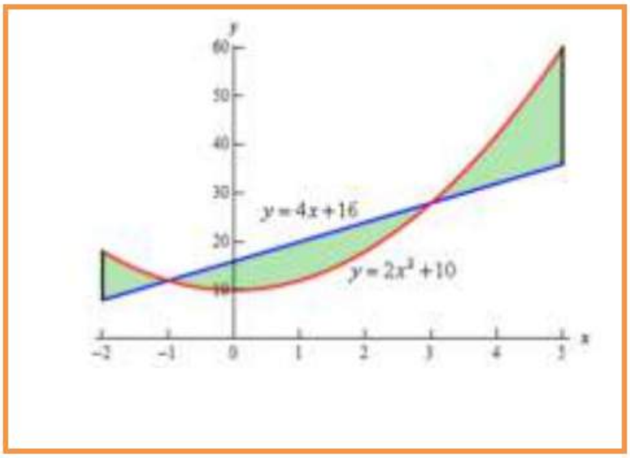
$$A=\int_a^b [f(x)-g(x)]dx =$$

$$\int_{-1}^3 \left[\left(4x+16\right)-\left(2x^2+10\right)\right]dx = \frac{64}{3}$$

Example5: Determine the area of the region bounded by $= 2x^2 + 10$,
$y = 4x + 16, x = -2$ and $x = 5$

Solution

There are three regions in which one function is always the upper function and the other is always the lower function. So, all that we need to do is find the area of each of the three regions,



$$A=\int_a^b[f(x)-g(x)]dx = \int_{-2}^{-1}\left[\left(2x^2+10\right)-\left(4x+16\right)\right]dx$$

$$+\int_{-1}^{3}\left[\left(4x+16\right)-\left(2x^2+10\right)\right]dx \quad +\int_{3}^{5}\left[\left(2x^2+10\right)-\left(4x+16\right)\right]dx$$

$$=\frac{14}{3}+\frac{64}{3}+\frac{64}{3}=\frac{142}{3}$$

Example6: The sine and cosine curves intersect infinitely many times, bounding regions of equal areas. Find the area of one of these regions.

Solution

points of intersection?

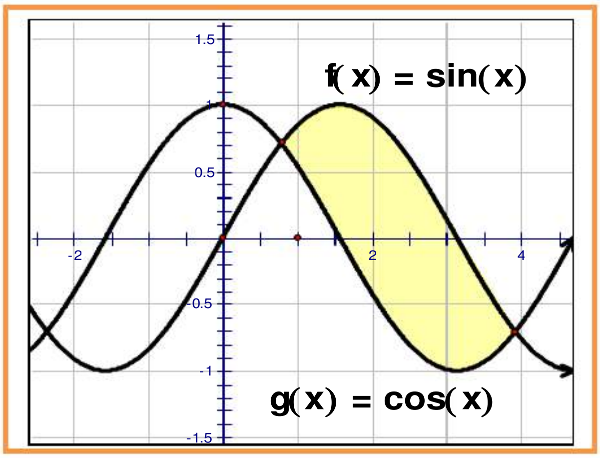
$$\sin x=\cos x$$

$$\tan x=\frac{\sin x}{\cos x}=\frac{\cos x}{\cos x}=1$$

$$x=\frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$

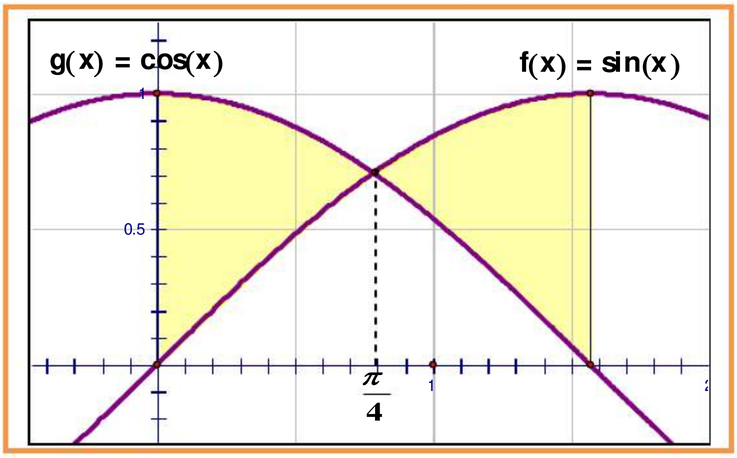
$$A=\int_{\pi/4}^{5\pi/4}[\sin x-\cos x]dx=$$

$$[-\cos x-\sin x]_{\pi/4}^{5\pi/4}=2\sqrt{2}$$



Example7: Find the area of the region between the curve $y = \sin x$ and the curve $y = \cos x$ from 0 to $\pi/2$.

First let's get a graph of the region. point of intersection? $\pi/4$



$$A=\int_a^b [f(x)-g(x)]dx$$

$$=\int_0^{\pi/4}[\cos x-\sin x]dx+\int_{\pi/4}^{\pi/2}[\sin x-\cos x]dx \qquad =2\sqrt{2}-2$$

Horizontal Representative Rectangles

If a region is bounded by *f*(*y*) on the right and *g*(*y*) on the left at all points of the interval [*c*, *d*], then the area of the region is given by

$$A=\int_{y_1}^{y_2}[f(y)-g(y)]dy$$

Example8: Find the area of the region bounded by the graphs of $x = 3 - y^2$ and $x = y + 1.$

Solution

Points of intersection

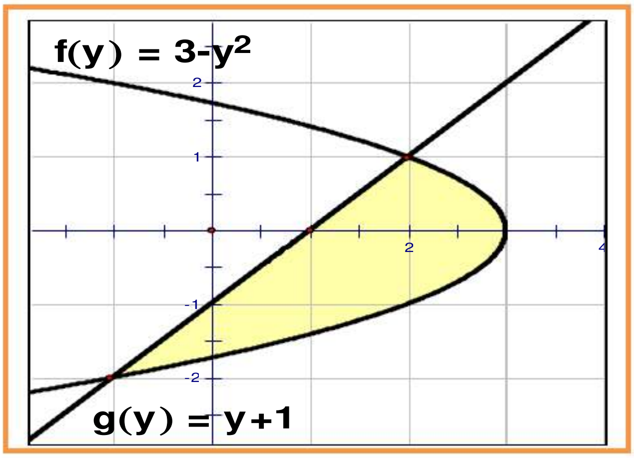
$3 - y^2 = y + 1$

$y = -2 \text{ or } y = 1$

$$A=\int_{-2}^{1}\left[\left(3-y^2\right)-\left(y+1\right)\right]dy$$

$$=\int_{-2}^{1}\left[-y^2-y+2\right]dy$$

$$=\left[-\frac{y^3}{3}-\frac{y^2}{2}+2y\right]_{-2}^{1}=\frac{9}{2}$$



Example9: Determine the area of the region enclosed by $x = \frac{1}{2}y^2 - 3$ and $y = x - 1.$

Solution

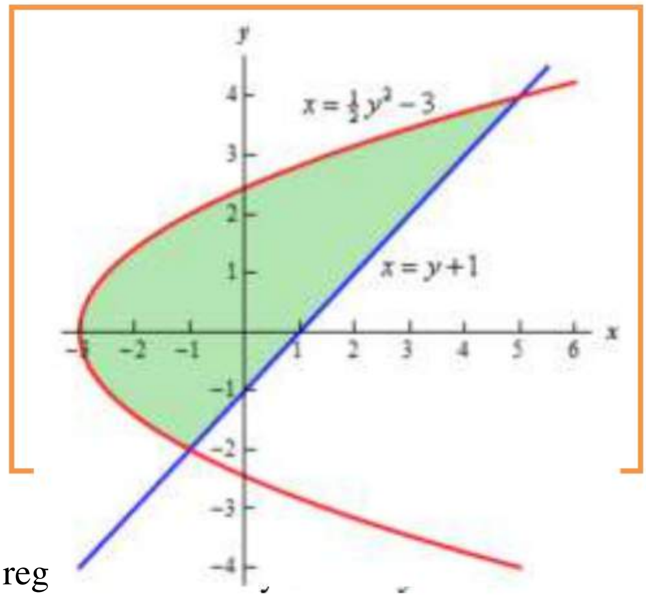
Points of intersection

$$\frac{1}{2}y^2 - 3 = y + 1$$

$$y = -2 \text{ or } y = 4$$

$$A = \int_{-2}^{4}\left[(y+1) - \left(\frac{1}{2}y^2 - 3\right)\right]dy$$

$$= \left[-\frac{y^3}{6} + \frac{y^2}{2} + 4y\right]_{-2}^{4} = 18$$



Example10*:* Determine the area of the reg

$x = (y-2)^2$.

Solution

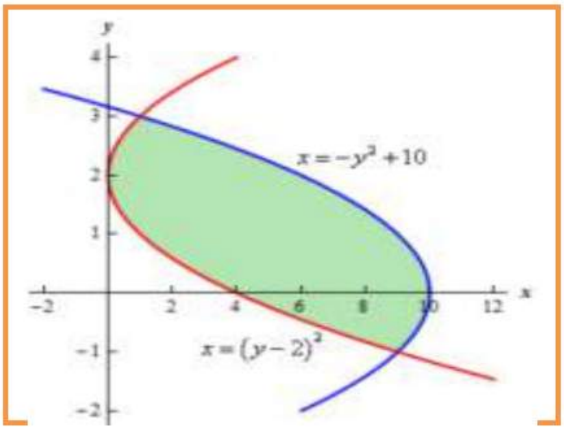
Points of intersection

$$-y^2 + 10 = (y-2)^2$$

$$y = -1 \text{ or } y = 3$$

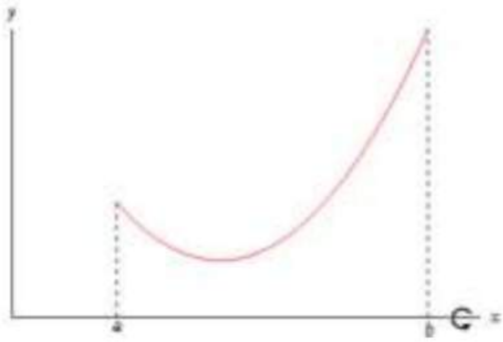
$$A = \int_{-1}^{3}\left[\left(-y^2 + 10\right) - (y-2)^2\right]dy$$

$$= \left[-\frac{2y^3}{3} + 2y^2 + 6y\right]_{-1}^{3} = \frac{64}{3}$$

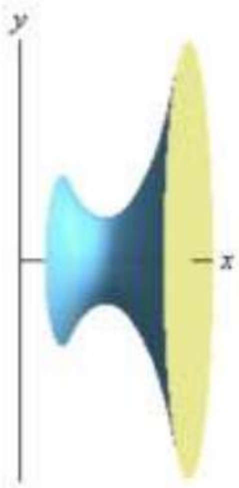


Volumes of Revolution

In this section we will start looking at the volume of a solid of revolution. We should first define just what a solid of revolution is. To get a solid of revolution we start out with a function, $y = f(x)$, on an interval $[a,b]$.

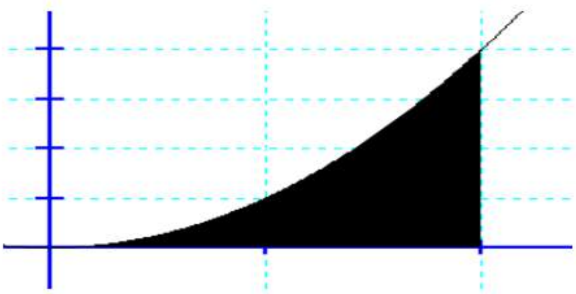


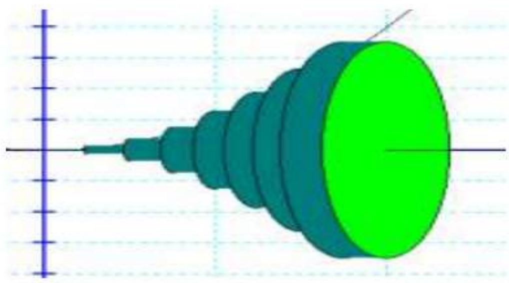
We then rotate this curve about a given axis to get the surface of the solid of revolution. For purposes of this discussion let's rotate the curve about the x-axis, although it could be any vertical or horizontal axis. Doing this for the curve above gives the following three dimensional region.



How is it calculated

be approximated by a series of narrow Just like the area under a continuous curve can rectangles, the volume of a solid of revolution can be approximated by a series of thin circular discs:





we

coul

d improve our accuracy by using a larger and larger number of circular discs, making them thinner and thinner

- Volume of the i-th circular disc $= \pi[f(x_i)]^2 \Delta x$
- Sum of the volumes of all the discs $= \sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x$

As n tends to infinity It means the discs get thinner and thinner. And it becomes a better and better approximation.

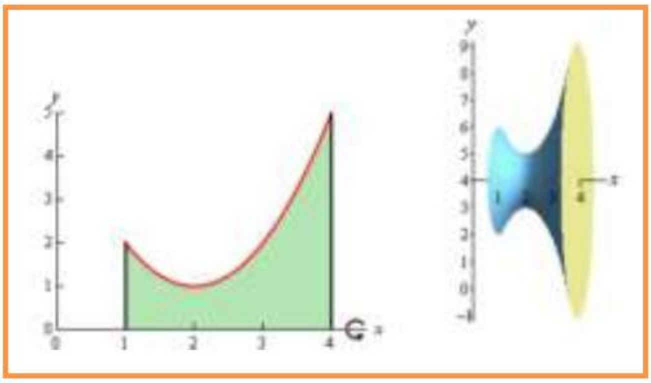
It can be replaced by an integral

- Sum of the volumes of all the discs $= \sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x = \int_a^b \pi[f(x_i)]^2 dx$

Example1: Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 4x + 5$, $x = 1$, $x = 4$, and the x-axis about the x-axis.

Solution

The first thing to do is get a sketch of the bounding region and the solid obtained by rotating the region about the x-axis. Here are both of these sketches.



The cross-sectional area is then: $A = \pi r^2 = \pi\left(x^2 - 4x + 5\right)^2$.

The volume of this solid is then,

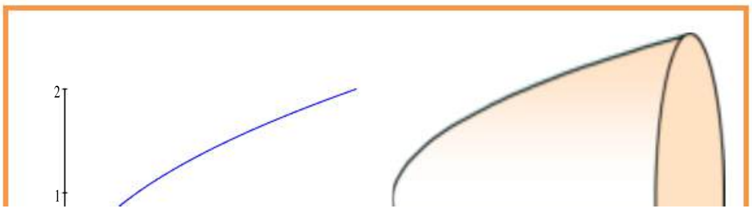
$$V = \int_a^b A(x)\,dx = \int_a^b \pi r^2 dx = \int_1^4 \pi\left(x^2 - 4x + 5\right)^2 dx$$

$$= \pi\left(\frac{x^5}{5} - 2x^4 + \frac{26}{3}x^3 - 20x^2 + 25x\right)\Bigg|_1^4 = \frac{78\pi}{5}$$

Example2: Consider the area under the graph of $y = \sqrt{x}$ and above $x -$ axis from $x = 0$ *to* $x =$ What is the volume of revolution about the $x -$axis

Solution

In this case: r = the y value of



the function

thickness = dx

The cross-sectional area is then:

$$A = \pi r^2 = \pi\left(\sqrt{x}\right)^2 \cdot$$

The volume of this solid is then,

$$V = \int_a^b A(x)\,dx = \int_a^b \pi r^2 dx = \int_0^4 \pi\left(\sqrt{x}\right)^2 dx$$

$$= \int_0^4 \pi x\, dx = \frac{\pi}{2} x^2 \Big|_0^4 = 8\pi$$

Example3: Calculate the volume of the solid obtained by rotating the region bounded by $y = x^2$ and $y = 0$ about the $x-$ axis for $0 \le x \le 2.$

Solution

In this case: r = the y value of the function and thickness = dx

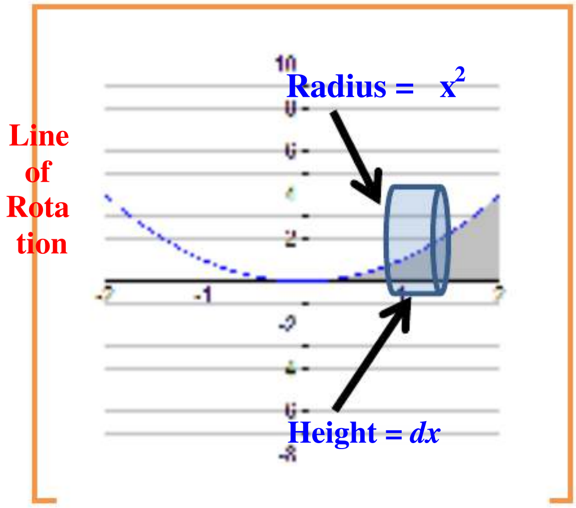
The cross-sectional area is then:

$$A = \pi r^2 = \pi \left(x^2\right)^2 .$$

The volume of this solid is then,

$$V = \int_a^b A(x)\,dx = \int_a^b \pi r^2 dx = \int_0^2 \pi \left(x^2\right)^2 dx$$

$$= \int_0^2 \pi x^4 \ dx = \frac{\pi}{5} x^5 \Big|_0^2 = \frac{32\pi}{5}$$



Example4: Consider the area under the graph of $y = 0.5\,x$ from $x = 0$ to $x = 1$.

Solution

In this case: r = the y value of the function and thickness = dx

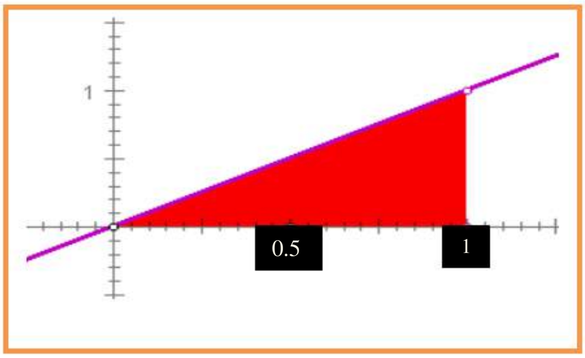
The cross-sectional area is then:

$$A = \pi r^2 = \pi \left(0.5\,x\right)^2 .$$

The volume of this solid is then,

$$V = \int_a^b A(x)\,dx = \int_a^b \pi r^2 dx = \int_0^1 \pi \left(0.5x\right)^2 dx$$

$$= 0.25\int_0^1 \pi x^2 \ dx = 0.25\frac{\pi}{3} x^3 \Big|_0^1 = \frac{\pi}{12}$$



Example of rotating the region about y-axis

Example5: Calculate the volume of the solid obtained by rotating the region bounded by $y = x^2,\ x > 0,\ x = 0,$ and $y = 4$ about the $y -$ axis.

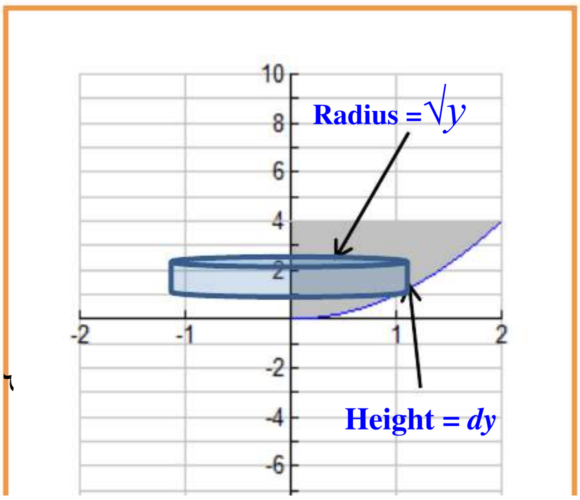
Solution

In this case: r = the x value of the function and thickness $= dy$

The cross-sectional area is then:

$$A = \pi r^2 = \pi \left(\sqrt{y}\right)^2 .$$

The volume of this solid is then,



$$V=\int_a^b A(y)dy=\int_a^b \pi r^2 dy=\int_0^4 \pi\left(\sqrt{y}\right)^2 dy$$

$$=\int_0^4 \pi y\,dy=\frac{\pi}{2}y^2\Big|_0^4=8\pi$$

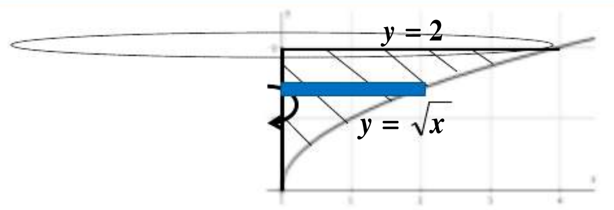
Example6: The area bounded by the curve $y=\sqrt{x}$ the y-axis and the line $y=2$ is rotated about the y-axis. Find the volume of the solid formed.

Solution

In this case: r = the x value of the function and thickness $= dy$

The cross-sectional area is then:

$$A=\pi r^2=\pi\left(y^2\right)^2.$$



Line of Rotation

The volume of this solid is then,

$$V=\int_a^b A(y)dy=\int_a^b \pi r^2 dy=\int_0^2 \pi\left(y^2\right)^2 dy$$

$$=\int_0^2 \pi y^4\,dy=\frac{\pi}{5}y^5\Big|_0^2=\frac{32\pi}{5}$$

<u>**Example7:**</u> Find the volume of the solid generated by revolving the region defined by y = 8, and $x=0$ about the y-axis. $=x^3$,

Solution

In this case: r = the x value of the function and thickness $= dy$

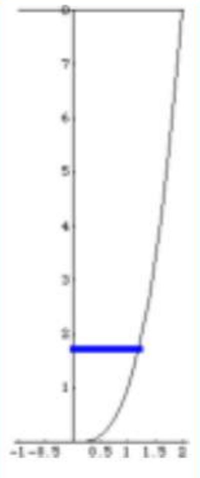
The cross-sectional area is then:

$$A=\pi r^2=\pi\left(\sqrt[3]{y}\right)^2.$$

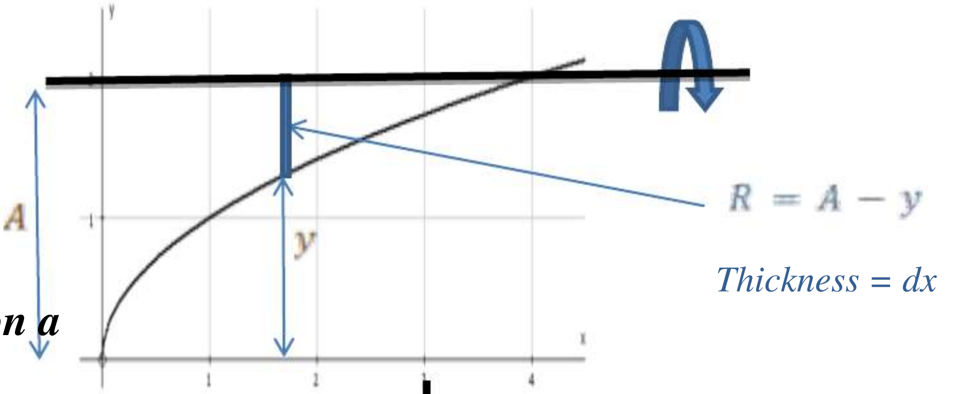
The volume of this solid is then,

$$V=\int_a^b A(y)dy=\int_a^b \pi r^2 dy=\int_0^8 \pi\left(\sqrt[3]{y}\right)^2 dy$$

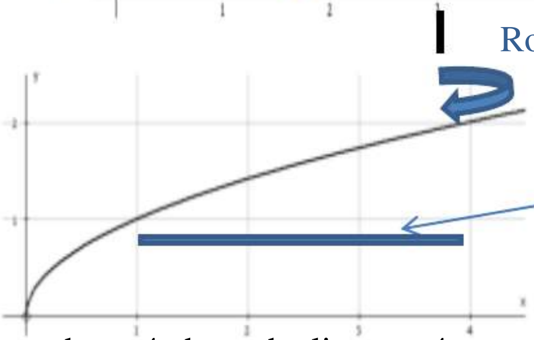
$$=\int_0^8 \pi y^{2/3}\,dy=\frac{3\pi}{5}y^{5/3}\Big|_0^8=\frac{96\pi}{5}$$


Line of Rotation

Rotation about line parallel to x − axis



Rotation a



Examp obtained by rotating the region bounded by $y = x$ and $y = 4$ about the line $y = 4$.

Solution

In this case: r = the $4 - x^2$ value of the function and thickness $= dx$

The cross-sectional area is then:

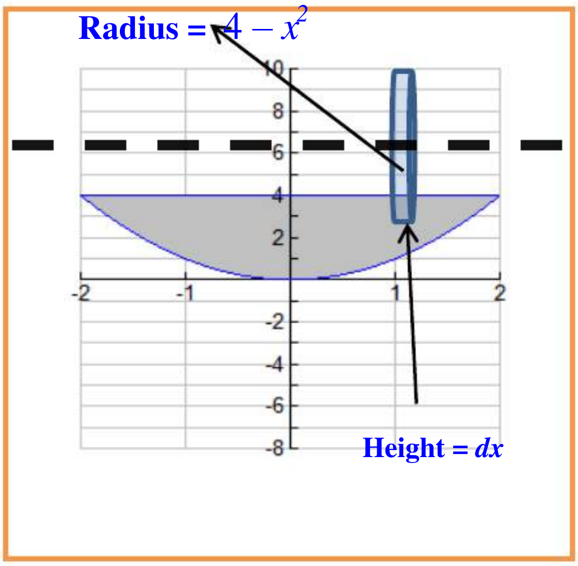
$$A = \pi r^2 = \pi\left(4 - x^2\right)^2 .$$

The volume of this solid is then,

$$V = \int_a^b A(x)\,dx = \int_a^b \pi r^2 dx = \int_{-2}^{2} \pi\left(4 - x^2\right)^2 dx$$

$$= \int_{-2}^{2} \pi\left(16 - 8x^2 + x^4\right) dx$$

$$= \pi\left(16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right)\Bigg|_{-2}^{2} = \frac{512\pi}{15}$$



Example9: Calculate the volume of the solid obtained by rotating the region bounded and $y = 0$, *and* $x = 4$ about the line $x = 4$. $y = \sqrt{x}$ by

Solution

In this case: r = the $4 - y^2$ value of the function and thickness $= dy$

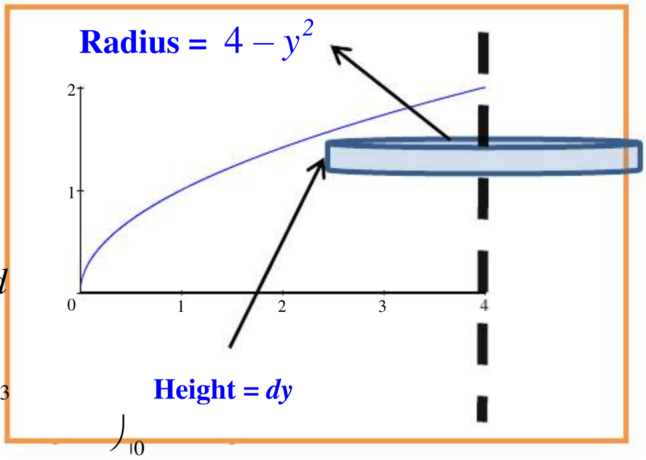
The cross-sectional area is then:

$$A = \pi r^2 = \pi\left(4 - y^2\right)^2.$$

The volume of this solid is then,

$$V = \int_a^b A(y)\,dy = \int_a^b \pi r^2 dy = \int_0^4 \pi\left(4 - y^2\right)^2 d$$

$$= \int_0^4 \pi\left(16 - 8y^2 + y^4\right)dy = \pi\left(16x - \frac{8}{3}y^3\right.\Big|_0$$



Washer method (hole in the solid)

In the case that we get a ring the area is,

$$A = \pi r^2 = \pi\left(\begin{pmatrix} outer \\ radius \end{pmatrix}^2 - \begin{pmatrix} inner \\ radius \end{pmatrix}^2\right).$$

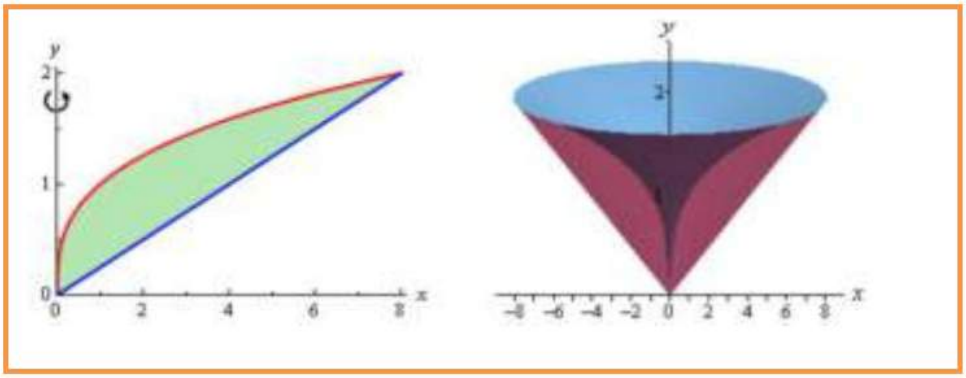
where again both of the radii will depend on the functions given and the axis of rotation. Note as well that in the case of a solid disk we can think of the inner radius as zero and we'll arrive at the correct formula for a solid disk and so this is a much more general formula to use.

Also, in both cases, whether the area is a function of x or a function of y will depend upon the axis of rotation as we will see. This method is often called the **method of disks** or the **method of rings**.

Example10: Determine the volume of the solid obtained by rotating the portion of the region bounded by $y = \sqrt[3]{x}$ and $y = \frac{x}{4}$ that lies in the first quadrant about the $y-$axis.

Solution

First, let’s get a graph of the bounding region and a graph of the object.



The cross section will be a ring for this example and it will be horizontal at some *y*. This means that the inner and outer radius for the ring will be *x* values and so we will need to rewrite our functions into the form $x = f(y)$.

$$y = \sqrt[3]{x} \quad \Rightarrow \quad x = y^3 \quad \& \quad y = \frac{x}{4} \quad \Rightarrow \quad x = 4y$$

The cross-sectional area is then:

$$A(y) = \pi\left((4y)^2 - \left(y^3\right)^2\right) = \pi\left(16y^2 - y^6\right) \cdot$$

Working from the bottom of the solid to the top we can see that the first cross-section will occur at $y = 0$ and the last cross-section will occur at $y = 2$.

The volume of this solid is then,

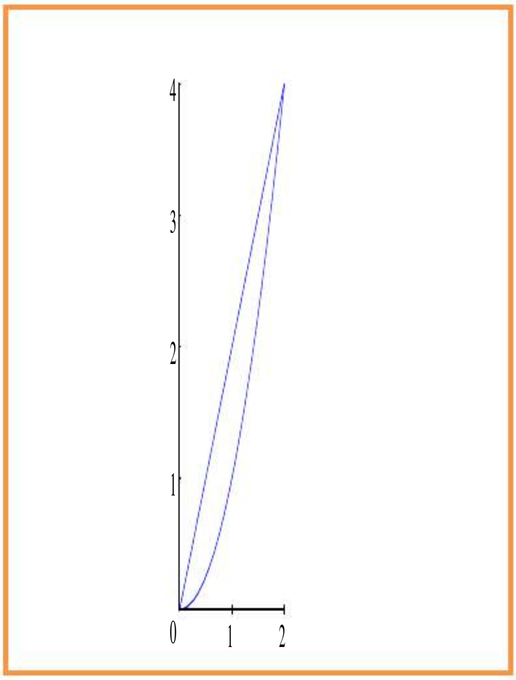
$$V = \int_a^b A(y)\,dy = \int_0^2 \pi\left(16y^2 - y^6\right)dy \cdot$$

$$= \pi\left(\frac{16}{3}y^3 - \frac{1}{7}y^7\right)\Bigg|_0^2 = \frac{512\pi}{21}$$

Example11: The region bounded by $y = x^2$ and $y = 2x$ is revolved about the y-axis. Find the volume.

Solution

First, let's get a graph of the bounding region and a graph of the object.



The inner and outer radius for the ring will be x values and so we will need to rewrite our functions into the form $x = f(y)$.

$$y = x^2 \quad \Rightarrow \quad x = \sqrt{y} \quad \&$$

$$y = 2x \quad \Rightarrow \quad x = \frac{y}{2}$$

The cross-sectional area is then:

$$A(y) = \pi\left(\left(\sqrt{y}\right)^2 - \left(\frac{y}{2}\right)^2\right) = \pi\left(y - y^2/4\right).$$

The volume of this solid is,

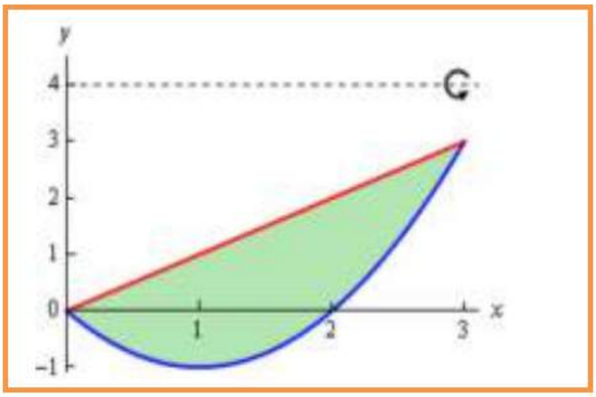
$$V = \int_0^4 \pi\left(y - \frac{1}{4}y^2\right)dy = \pi\int_0^4 y - \frac{1}{4}y^2 \; dy$$

$$= \pi\left[\frac{1}{2}y^2 - \frac{1}{12}y^3\right]_0^4 = \pi\left[8 - \frac{16}{3}\right] = \frac{8\pi}{3}$$

Example12: Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2x$ and $y = x$ about the line $= 4$.

Solution

First let's get the bounding region and the solid graphed. The inner and outer radius for the ring will be y values and so our functions on the form $y = f(x)$.



Inner radius $= 4 - x$

Outer radius $= 4 - (x^2 - 2x)$

$$= -x^2 + 2x + 4$$

The cross-sectional area is then:

$$A(x) = \pi\left(\left(-x^2 + 2x + 4\right)^2 - (4 - x)^2\right) = \pi\left[x^4 - 4x^3 - 5x^2 + 24x\right]$$

The volume of this solid is,

$$V = \int_0^3 \pi \left[x^4 - 4x^3 - 5x^2 + 24x \right] dx$$

$$= \pi \left[\frac{1}{5}x^5 - x^4 - \frac{5}{3}x^3 + 12x^2 \right]_0^3 = \frac{153\pi}{5}$$

Part II

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CHAPTER 1
The binomial theorem

Chapter 1

The binomial theorem

When you look at the following expansions you can see the symmetry and the emerging patterns.

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a^2+2ab+b^2.\end{aligned}$$

$$\begin{aligned}(a+b)^3 &= (a+b)^2(a+b)\\ &= (a^2+2ab+b^2)(a+b)\\ &= (a^3+2a^2b+ab^2)+(a^2b+2ab^2+b^3)\\ &= a^3+3a^2b+3ab^2+b^3.\end{aligned}$$

$$\begin{aligned}(a+b)^4 &= (a+b)^3(a+b)\\ &= (a^3+3a^2b+3ab^2+b^3)(a+b)\\ &= (a^4+3a^3b+3a^2b^2+ab^3)+(a^3b+3a^2b^2+3ab^3+b^4)\\ &= a^4+4a^3b+6a^2b^2+4ab^3+b^4.\end{aligned}$$

Notice that:

- the expansion of $(a+b)^2$ has three terms and in each term the sum of the indices is 2.

- the expansion of $(a+b)^3$ has four terms and in each term the sum of the indices is 3.
- the expansion of $(a+b)^4$ has five terms and in each term the sum of the indices is 4.

We conjecture that the expansion of $(a+b)^n$ has $n+1$ terms and in each term the sum of the indices is n.

1.1 Pascal's triangle

The following pattern show how can we predict the coefficients of the expansion of $(a+b)^n$?

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}
$$

The triangular numbers in Pascal's triangle.

We begin by looking at the expansions of $(1+x)^n$ for $n = 0, 1, 2, 3, 4, 5$.

$$(1+x)^0 = 1$$

$$(1+x)^1 = 1 + x$$

$$(1+x)^2 = 1+2x+x^2$$

$$(1+x)^3 = 1+3x+3x^2+x^3$$

$$(1+x)^4 = 1+4x+6x^2+4x^3+x^4$$

$$(1+x)^5 = 1+5x+10x^2+10x^3+5x^4+x^5$$

By examining Pascal's triangle, we can make the following observations,.

(1) Each number is the sum of the two numbers diagonally above it (with the exception of the 1Šs).

(2) Each row is symmetric (i.e., the same backwards as forwards).

(3) The sum of the numbers in each row is a power of 2.

(4) In any row, the sum of the first, third, fifth, . . . numbers is equal to the sum of the second, fourth, sixth, . . . numbers. (This is not a totally obvious result.)

Example 1.1.1. *Expand*

1. $(1+x)^6$

2. $(1-2x)^6$

Answer:

1. The coefficients of $(1+x)^6$ *are given in the sixth row of PascalŠs triangle:*

$$(1+x)^6 = 1+6x+15x^2+20x^3+15x^4+6x^5+x^6.$$

2. The expansion of $(1-2x)^6$ *can be obtained by replacing* $(-2x)$ *for* x *in the expansion of* $(1+x)^6$:

$$\begin{aligned}(1-2x)^6 &= 1+6(-2x)+15(-2x)^2+20(-2x)^3+15(-2x)^4+6(-2x)^5+(-2x)^6\\ &= 1-12x+60x^2-160x^3+240x^4-192x^5+64x^6.\end{aligned}$$

1.2 Expansions

We start by looking at the results of multiplying several binomials. With two binomials, we have

$$(a+b)(c+d) = a(c+d) + b(c+d) = ac + ad + bc + bd.$$

The expansion is obtained by multiplying each letter in the first bracket by each letter in the second and adding them. There are $2 \times 2 = 2^2 = 4$ terms. Similarly, with three binomials, we have

$$(a+b)(c+d)(e+f) = ace + acf + ade + adf + bce + bcf + bde + bdf.$$

There are $2 \times 2 \times 2 = 2^3 = 8$ terms.

Exercise 1.2.1. *How many terms are there in the expansion of*

(1) $(a+b+c)(e+f)$*?*

(2) $(a+b)(c+d)(e+f)(g+h)(i+j)$*?*

In general, the product of any number of polynomials is equal to the sum of all the products which can be formed by choosing one term from each polynomial and multiplying these terms together.

Example 1.2.1. *Find the coefficient of* x^2 *in the expansion of* $(2x-1)(3x+4)(5x-6)$.

Answer:

If we take the terms containing x *from any two of the factors and the constant from the remaining factor and multiply these terms together, we will obtain a term containing* x^2 *in the expansion. If we do this in all possible ways and add, we will find the required coefficient.*

The required coefficient is

$$\begin{aligned} 2 \times 3 \times (-6) + 3 \times 5 \times (-1) + 2 \times 5 \times 4 &= -36 - 15 + 40 \\ &= -11. \end{aligned}$$

1.3 Permutations and factorial notation

The number of ways to arrange n different objects in a row is **n!**. It is important to note that $n! = n(n-1)!$. For this statement to be true when $n = 1$, we define $0! = 1$.

If there are eight competitors in a race, in how many ways can the first four places be filled? We can draw a box diagram for this situation, with each box indicating the number of choices for each position.

8	7	6	5

So the answer is $8 \times 7 \times 6 \times 5 = 1680$ ways.

Definition 1.3.1. A permutation is an arrangement of elements chosen from a certain set.

Example 1.3.1. *List all the permutations of the letters in the word CAT.*

Answer:

There are $3 \times 2 \times 1 = 6$ *such permutations: CAT, CTA, TAC, TCA, ACT, ATC.*

The symbol ${}^nP_r$ is used to denote the number of permutations of r distinct objects chosen from n objects. We see that

$$ {}^nP_r = n(n-1)(n-2)\dots(n-r+1) = \frac{n!}{(n-r)!}. $$

Example 1.3.2. *Seven runners are competing in a race. In how many ways can the gold, silver and bronze medals be awarded?*

Answer:

The total number of ways of awarding the medals is

$$ {}^7P_3 = \frac{7!}{4!} = 7 \times 6 \times 5 = 210 \textit{ ways.} $$

We denote the number of ways of choosing r objects from n objects by $\begin{pmatrix} n \\ r \end{pmatrix}$, which is read " n **choose** r". We see that

$$\begin{pmatrix} n \\ r \end{pmatrix} = \frac{{}^nP_r}{r!} = \frac{n!}{(n-r)!\,r!}.$$

Example 1.3.3. *Consider the set* $\{a, b, c, d, e\}$.
The number of permutations of three letters chosen from five letters is

$${}^5P_3 = \frac{5!}{2!} = 5 \times 4 \times 3 = 60.$$

Therefore the number of ways of choosing three letters from five letters is

$$\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \frac{{}^5P_3}{3!} = \frac{60}{6} = 10.$$

Example 1.3.4. *In how many ways can you choose two people from group of seven people ?*

Answer:

There are $\begin{pmatrix} 7 \\ 2 \end{pmatrix} = \frac{{}^7P_2}{2!} = 21.$ *ways of choosing two people from seven.*

Exercise 1.3.1. *There are ten people in a basketball squad. Find how many ways :*

(1) the starting five can be chosen from the squad

(2) the squad can be split into two teams of five.

Exercise 1.3.2. *Evaluate :*

(1) $\begin{pmatrix} 1000 \\ 998 \end{pmatrix}$

(2) $\begin{pmatrix} 100 \\ 2 \end{pmatrix}$

1.4 The Binomial Theorem

Theorem 1.4.1. ***(Binomial theorem)***
For each positive integer n,

$$\begin{aligned}(a+b)^n &= a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \cdots + \binom{n}{r} a^{n-r}b^r + \\ &\quad + \ldots + \binom{n}{n-1} ab^{n-1} + b^n \\ &= \sum_{r=0}^{n} \binom{n}{r} a^{n-r}b^r. \end{aligned} \tag{1.4.1}$$

We will give the proof later by using mathematical induction.
Substituting with $a = 1$ and $b = x$ in (1.4.1) gives

$$\begin{aligned}(1+x)^n &= \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{r} x^r + \ldots + \\ &\quad + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n.\end{aligned}$$

We can now display Pascal's triangle with the notation of the binomial theorem. The nth row of the following table gives the coefficients of $(1+x)^n$, where $\binom{n}{r}$ is the coefficient of x^r in this expansion. Numbers of the form $\binom{n}{r}$ are called **binomial coefficients**.

Notice that:

$$\binom{5}{0} = 1, \quad \binom{5}{1} = 5, \quad \binom{5}{2} = 10, \quad \binom{5}{3} = 10,$$

$$\binom{5}{4} = 5, \quad \binom{5}{5} = 1.$$

This is the fifth row of Pascal's triangle.

Pascal's triangle using the binomial theorem

n	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7
0	$\binom{0}{0}$							
1	$\binom{1}{0}$	$\binom{1}{1}$						
2	$\binom{2}{0}$	$\binom{2}{1}$	$\binom{2}{2}$					
3	$\binom{3}{0}$	$\binom{3}{1}$	$\binom{3}{2}$	$\binom{3}{3}$				
4	$\binom{4}{0}$	$\binom{4}{1}$	$\binom{4}{2}$	$\binom{4}{3}$	$\binom{4}{4}$			
5	$\binom{5}{0}$	$\binom{5}{1}$	$\binom{5}{2}$	$\binom{5}{3}$	$\binom{5}{4}$	$\binom{5}{5}$		
6	$\binom{6}{0}$	$\binom{6}{1}$	$\binom{6}{2}$	$\binom{6}{3}$	$\binom{6}{4}$	$\binom{6}{5}$	$\binom{6}{6}$	
7	$\binom{7}{0}$	$\binom{7}{1}$	$\binom{7}{2}$	$\binom{7}{3}$	$\binom{7}{4}$	$\binom{7}{5}$	$\binom{7}{6}$	$\binom{7}{7}$

We now describe the general pattern.

Theorem 1.4.2. ***(Pascal's identity)***

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, \quad \textit{for } 0 < r < n. \tag{1.4.2}$$

Proof.

$$\begin{aligned}
\binom{n-1}{r-1} + \binom{n-1}{r} &= \frac{(n-1)!}{(n-r)!(n-1)!} + \frac{(n-1)!}{(n-1-r)!\,r!} \\
&= \frac{(n-1)!}{(n-1-r)!(r-1)!}\Big(\frac{1}{n-r} + \frac{1}{r}\Big) \\
&= \frac{(n-1)!}{(n-1-r)!(r-1)!}\Big(\frac{n}{(n-r)r}\Big) = \frac{n!}{(n-r)!\,r!}\,.
\end{aligned}$$

□

Now we give the proof of the binomial theorem using mathematical induction.

Proof. **of binomial theorem**
We aim to prove that (1.4.1). We first note that the result is true for $n = 1$ and $n = 2$.
Let k be a positive integer with $k \geq 2$ for which (1.4.1) is true. Then

$$(a+b)^k = a^k + \binom{k}{1} a^{k-1}b + \binom{k}{2} a^{k-2}b^2 + \ldots + \binom{k}{r} a^{k-r}b^r + \\ + \ldots + \binom{k}{k-1} ab^{k-1} + b^k.$$

Now consider the expansion

$$\begin{aligned}(a+b)^{k+1} &= (a+b)\Big(a^k + \binom{k}{1} a^{k-1}b + \binom{k}{2} a^{k-2}b^2 + \ldots + \binom{k}{r} a^{k-r}b^r \\ &\quad + \ldots + \binom{k}{k-1} ab^{k-1} + b^k\Big) \\ &= a^{k+1} + \Big[1 + \binom{k}{1}\Big]a^k b + \Big[\binom{k}{1} + \binom{k}{2}\Big]a^{k-1}b^2 + \cdots + \\ &\quad + \Big[\binom{k}{r-1} + \binom{k}{r}\Big]a^{k-r+1}b^r + \ldots + \Big[\binom{k}{k-1} + 1\Big]ab^k + \\ &\quad + b^{k+1}.\end{aligned}$$

From Pascal's identity, it follows that

$$(a+b)^{k+1} = a^{k+1} + \binom{k+1}{1} a^k b + \ldots + \binom{k+1}{r} a^{k-r+1}b^r + \\ + \ldots + \binom{k+1}{k} ab^k + b^{k+1}.$$

Hence the result is true for $k+1$. By induction, the result is true for all positive integers n. $\square$

1.5 Applying the binomial theorem

In this section, we give some examples of applying the binomial theorem.

Example 1.5.1. *Expand* $(2x+3)^4$.

Answer:

Using the binomial theorem :

Let $a = 2x$ *and* $b = 3$. *Then*

$$\begin{aligned}(2x+3)^4 &= (2x)^4 + 4(2x)^3 \times 3 + 6(2x)^2 \times 3^2 + 4(2x) \times 3^3 + 3^4 \\ &= 16x^4 + 96x^3 + 216x^2 + 216x + 81.\end{aligned}$$

Exercise 1.5.1. *Expand*

(a) $(2x-3y)^4$.

(b) $(1-x)^{10}$.

(c) $\left(x - \frac{2}{x}\right)^4$.

The general term in the binomial expansion

The general term in the expansion of $(a+b)^n$ is

$$\binom{n}{r} a^{n-r} b^r, \quad \text{where } 0 \leq r \leq n. \tag{1.5.1}$$

Example 1.5.2. *Find the general term and find the coefficient of* $x^3 z^4$, *for the expansion of* $(x - 3z^2)^5$.

Answer:

The general term is

$$\binom{5}{r} x^{5-r} (-3z^2)^r, \quad \textit{where } 0 \leq r \leq 5.$$

To find the coefficient of x^3z^4*, we have* $5-r=3$ *and* $r=2$.
Then the coefficient of x^3z^4 *is* $\begin{pmatrix} 5 \\ 2 \end{pmatrix}(-3)^2 = 90$.

Example 1.5.3. *Find the constant term in the expansion of* $(x^2+\frac{1}{x^2})^6$.
Answer:
The general term is

$$\begin{aligned}\begin{pmatrix} 6 \\ r \end{pmatrix}(x^2)^{6-r}(\frac{1}{x^2})^r &= \begin{pmatrix} 6 \\ r \end{pmatrix}x^{12-2r}(\frac{1}{x^2r}) \\ &= \begin{pmatrix} 6 \\ r \end{pmatrix}x^{12-4r}.\end{aligned}$$

This term will be a constant when $12-4r=0$*, that is, when* $r=3$.
Hence the constant term in the expansion of $(x^2+\frac{1}{x^2})^6$ *is* $\begin{pmatrix} 6 \\ 3 \end{pmatrix} = 20$.

Exercise 1.5.2. *Find the general term and the constant term in the expansion of*

(a) $(x+\frac{1}{x^2})^8$.

(b) $(x-2z^3)^7$.

Exercise 1.5.3. *Find the coefficient of* x^5 *in the expansion of* $(1-2x+3x^2)^5$.

The middle term in the binomial expansion

When n is even, there will be an odd number of terms in the expansion of $(a+b)^n$, and hence there will be a middle term. Let $n=2m$, for some positive integer m. Then, when the expansion of $(a+b)^n$ is arranged with terms in descending or ascending order, the middle term is

$$\begin{pmatrix} 2m \\ m \end{pmatrix}a^mb^m. \tag{1.5.2}$$

Example 1.5.4. *Find the middle term of* $(a+b)^8$.
Answer:

The middle term of $(a+b)^8$ *is*

$$\binom{8}{4} a^4 b^4 = 70a^4 b^4.$$

When n is odd, there will be two middle terms of $(a+b)^n$. Let $n = 2m+1$, for some positive integer m. Then the middle terms are

$$\binom{2m+1}{m} a^{m+1} b^m \quad \text{and} \quad \binom{2m+1}{m+1} a^m b^{m+1}. \tag{1.5.3}$$

Exercise 1.5.4. *Find the middle term of* $(2x-3yb)^9$.

Greatest coefficients

When n is even, we can show that the greatest coefficient of $(1+x)^n$ is the coefficient of the middle term, which is

$$\binom{n}{\frac{n}{2}}. \tag{1.5.4}$$

When n is odd, there are two greatest coefficients, which are the coefficients of the two middle terms

$$\binom{n}{\frac{1}{2}(n-1)} \quad \text{and} \quad \binom{n}{\frac{1}{2}(n+1)}. \tag{1.5.5}$$

Example 1.5.5. *Find the greatest coefficient of* $(1+3x)^{21}$.

Answer:

The general term is

$$\binom{21}{r} (3x)^r,$$

and its coefficient is

$$c_r = \binom{21}{r} (3)^r,$$

The next coefficient is

$$c_{r+1} = \left(\begin{array}{c} 21 \\ r+1 \end{array} \right) (3)^{r+1},$$

We have

$$\begin{aligned}
\frac{c_{r+1}}{c_r} &= \frac{\left(\begin{array}{c} 21 \\ r+1 \end{array} \right) (3)^{r+1}}{\left(\begin{array}{c} 21 \\ r \end{array} \right) (3)^{r}} \\
&= \frac{21!}{(20-r)!(r+1)!} \times \frac{(21-r)!\, r!}{21!} \times 3 \\
&= \frac{63-3r}{r+1}.
\end{aligned}$$

To find where the coefficients are increasing, we solve

$$c_{r+1} > c_r,$$

that is

$$\frac{c_{r+1}}{c_r} > 1.$$

From above, we have

$$\frac{63-3r}{r+1} > 1,$$

which is equivalent to

$$r < 15\frac{1}{2}.$$

Since k is an integer. Then $c_{r+1} > c_r$, for $k = 0, 1, \ldots, 14, 15$.
The sequence of coefficients is increasing from c_0 to c_{16} and decreasing from c_{16} to c_{21}. Hence c_{16} is the largest coefficient:

$$c_{16} = \left(\begin{array}{c} 21 \\ 16 \end{array} \right) (3)^{16} = 875957725629.$$

Exercise 1.5.5. *Find the greatest coefficient of $(2x+3y)^{15}$.*

CHAPTER 2
Partial Fractions

Chapter 2

Partial Fractions

In this chapter, we split a fraction up into its component parts which are called **partial fractions**. We discuss how to do this in the following sections. While using the method of partial fractions, we assume that:

(i) The numerator and the denominator have no common polynomials factors.

(ii) The degree of the numerator is less than that of the denominator.

(iii) If the degree of the numerator is not less than that of the denominator, we can divided the numerator by the denominator, till the reminder is of degree less than that of the denominator.

2.1 Fractions where the denominator is a product of distinct linear factors

Suppose we want to express $\frac{3x}{(x-1)(x+2)}$ as the sum of its partial fractions. Observe that the factors in the denominator are $x-1$ and $x+2$ so we write

$$\frac{3x}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)},$$

where A and B are numbers. We multiply both sides by the common denominator $(x-1)(x+2)$:

$$3x = A(x+2) + B(x-1).$$

This time the special values that we shall choose are $x = -2$ because then the first term on the right will become zero and A will disappear, and $x = 1$ because then the second term on the right will become zero and B will disappear.
If we put $x = -2$, we get

$$-6 = -3B \Rightarrow B = 2.$$

If we put $x = 1$, we get

$$3 = 3A \Rightarrow A = 1.$$

Putting these results together, we have

$$\frac{3x}{(x-1)(x+2)} = \frac{1}{(x-1)} + \frac{2}{(x+2)}.$$

Example 2.1.1. *Express the following fraction as a sum of partial fractions*

$$\frac{x^2 + 2x - 1}{2x^3 - 3x^2 + 2x}.$$

Answer:

$$\begin{aligned} 2x^3 - 3x^2 + 2x &= x(2x^2 - 3x + 2x) \\ &= x(2x+1)(x-2). \end{aligned}$$

Then

$$\frac{x^2 + 2x - 1}{2x^3 - 3x^2 + 2x} = \frac{A}{x} + \frac{B}{2x+1} + \frac{C}{(x-2)}. \qquad (2.1.1)$$

To determine the values of constants A, B and C, we multiply both sides of Eq. (2.1.1) by the product of the denominator $x(2x+1)(x-2)$, we get:

$$x^2 + 2x - 1 = A(2x+1)(x-1) + Bx(x-2) + Cx(2x+1). \qquad (2.1.2)$$

We choose values of x that simplify the equation (2.1.2).
If we put $x = 0$, we get

$$\begin{aligned} -1 &= A(1)(-2) \\ \Rightarrow A &= \frac{1}{2}. \end{aligned}$$

If we put $x = 2$, *we get*

$$4 + 4 - 1 = C(2)(5)$$
$$\Rightarrow\ 7 = 10C$$
$$\Rightarrow\ C = \frac{7}{10}.$$

If we put $x = -\frac{1}{2}$, *we get*

$$\frac{1}{4} - 1 - 1 = B(\frac{-1}{2})(\frac{-1}{2} - 2)$$
$$\Rightarrow\ -\frac{7}{4} = B(\frac{-1}{2})(\frac{-5}{2})$$
$$\Rightarrow\ B = \frac{-7}{5}.$$

Hence,

$$\frac{x^2 + 2x - 1}{2x^3 - 3x^2 + 2x} = \frac{\frac{1}{2}}{x} - \frac{\frac{7}{5}}{2x+1} + \frac{\frac{7}{10}}{(x-2)}.$$

Exercise 2.1.1. *Express the following as a sum of partial fractions*

(a) $\dfrac{2x-1}{(x+2)(x-3)}$

(b) $\dfrac{2x+5}{(x-2)(x+1)}$

(c) $\dfrac{x+6}{x^2-5x+6}$

(d) $\dfrac{1-3x^2}{x-x^3}$

(e) $\dfrac{1}{x^2-3x+2}$

2.2 Fractions where the denominator has a repeated factor

Consider the following example in which the denominator is the product of linear factors, some of which is repeated.

Example 2.2.1. *Suppose we want to express* $\frac{x^2}{(x-3)(x+1)(x+2)^2}$ *as the sum of its partial fractions.*
There are actually four possibilities for a denominator in the partial fractions: $x-3$, $x+1$, $x+2$ *and also the possibility of* $(x+2)^2$, *so in this case we write*

$$\frac{x^2}{(x-3)(x+1)(x+2)^2} = \frac{A}{x-3} + \frac{B}{x+1} + \frac{C}{x+2} + \frac{D}{(x+2)^2}.$$

where A, B, C *and* D *are numbers.*
As before we multiply both sides by the denominator $(x-3)(x+1)(x+2)^2$ *to give*

$$x^2 = A(x+1)(x+2)^2 + B(x-3)(x+2)^2 + C(x-3)(x+2) + D(x-3)(x+1). \quad (2.2.1)$$

Again we look for special values to substitute into this identity.
If we put $x = -1$, *we get*

$$1 = B(-1-3)(-1+2)^2$$

$$\Rightarrow 1 = -4B$$

$$\Rightarrow B = -\frac{1}{4}.$$

If we put $x = 3$, *we get*

$$9 = A(3+1)(3+2)^2$$

$$\Rightarrow 9 = 100A$$

$$\Rightarrow A = \frac{9}{100}.$$

If we put $x = -2$, *we get*

$$4 = D(-2-3)(-2+1)$$

$$\Rightarrow 4 = 5D$$

$$\Rightarrow D = \frac{4}{5}.$$

If we put $x = 0$, we get

$$0 = A(1)(2)^2 + B(-3)(2)^2 + C(-3)(1)(2) + D(-3)(1)$$

$$\begin{aligned}
\Rightarrow 0 &= 4A - 12B - 6C - 3D \\
\Rightarrow 0 &= \frac{9}{25} + 3 - 6C - \frac{12}{5} \\
\Rightarrow 6C &= \frac{9 + 75 - 60}{25} = \frac{24}{25} \\
\Rightarrow C &= \frac{4}{25}.
\end{aligned}$$

Hence,

$$\frac{x^2}{(x-3)(x+1)(x+2)^2} = \frac{\frac{9}{100}}{x-3} - \frac{\frac{1}{4}}{x+1} + \frac{\frac{4}{25}}{x+2} + \frac{\frac{4}{5}}{(x+2)^2}.$$

Example 2.2.2. *Express the following fraction as a sum of partial fractions*

$$\frac{3x+1}{(x-1)^2(x+2)}.$$

Answer:

$$\frac{3x+1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}.$$

where A, B and C are numbers.
Multiply both sides by the denominator $(x-1)^2(x+2)$ to give

$$3x + 1 = A(x-1)(x+2) + B(x+2) + C(x-1)^2. \tag{2.2.2}$$

If we put $x = 1$, we get

$$\begin{aligned}
4 &= 3B \\
\Rightarrow B &= \frac{4}{3}.
\end{aligned}$$

If we put $x = -2$, we get

$$\begin{aligned}
-5 &= 9C \\
\Rightarrow C &= \frac{-5}{9}.
\end{aligned}$$

We now need to find A. There is no special value of x that will eliminate B and C to give us A. We could use any value. We could use $x = 0$. This will give us an equation in A, B and C. Since we already know B and C, this would give us A. But here we shall demonstrate a different technique - one called equating coefficients. We take equation (2.2.2) and multiply-out the right-hand side, and then collect up like terms.

$$\begin{aligned} 3x + 1 &= A(x^2 + x - 2) + B(x + 2) + C(x^2 - 2x + 1) \\ &= (A + C)x^2 + (A + B - 2C)x + (-2A + 2B + C). \end{aligned}$$

By equating coefficients of x^2, we get

$$\texttt{Coeff.} \quad \texttt{of } x^2 : \Rightarrow 0 = A + C.$$

Since $C = \frac{-5}{9}$. Then $A = \frac{5}{9}$. Therefore,

$$\frac{3x + 1}{(x - 1)^2 (x + 2)} = \frac{\frac{5}{9}}{x - 1} + \frac{\frac{4}{3}}{(x - 1)^2} - \frac{\frac{5}{9}}{x + 2}.$$

Exercise 2.2.1. *Express the following as a sum of partial fractions*

(a) $\dfrac{5x^2 + 17x + 15}{(x + 2)^2(x + 1)}$

(b) $\dfrac{x}{(x - 3)^2(2x + 1)}$

(c) $\dfrac{x^2 + 1}{(x - 1)^2(x + 1)}$

(d) $\dfrac{x^3}{x(x - 1)^3}$

(e) $\dfrac{x}{(x^2 - 1)(x + 1)}$

2.3 Fractions in which the denominator has a quadratic term

If the denominator has a quadratic term which cannot be factorized. We will now learn how to deal with cases like this.

Example 2.3.1. *Express* $\frac{5x}{(x^2+x+1)(x-2)}$ *as a sum of its partial fractions.* ***Answer:*** *When the denominator contains a quadratic factor we have to consider the possibility that the numerator can contain a term in* x. *This is because if it did, the numerator would still be of lower degree than the denominator - this would still be a proper fraction. So we write*

$$\frac{5x}{(x^2+x+1)(x-2)} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-2}.$$

$$\Rightarrow 5x = (Ax+B)(x-2) + C(x^2+x+1).$$

If we put $x = 2$, *we get*

$$10 = C(4+2+1)$$

$$\Rightarrow 10 = 7C$$

$$\Rightarrow C = \frac{10}{7}.$$

Now, we use the technique of equating coefficients. We have

$$\begin{aligned} 5x &= Ax^2 - 2Ax + Bx - 2B + Cx^2 + Cx + C \\ &= (A+C)x^2 + (-2A+B+C)x + (-2B+C). \end{aligned}$$

Coefficient of x^2*:*

$$\Rightarrow 0 = A + C$$

$$\Rightarrow A = -\frac{10}{7}.$$

The left-hand side has no constant term and so

$$0 = -2B + C$$

$$\Rightarrow 2B = C = \frac{10}{7}$$

$$\Rightarrow B = \frac{5}{7}.$$

Hence,

$$\frac{5x}{(x^2+x+1)(x-2)} = \frac{-\frac{10}{7}x + \frac{5}{7}}{x^2+x+1} + \frac{\frac{10}{7}}{x-2}.$$

Exercise 2.3.1. *Express the following as a sum of partial fractions*

(a) $\dfrac{x^2-2x-1}{(x-1)^2(x^2+1)}$

(b) $\dfrac{1}{x^2+2x^3}$

(c) $\dfrac{2x^2+1}{(x^2-4)(x^2+5)}$

(d) $\dfrac{2x^2-3}{(x^2+1)(x^2-4)}$

(e) $\dfrac{2x^2+13}{(x^2+4)(x^2+9)}$

2.4 Fractions in which the denominator has a repeated quadratic term

If the denominator has a repeated quadratic term which cannot be factorized.

Example 2.4.1. *Express* $\dfrac{x^4+1}{x(x^2+1)^2}$ *as a sum of its partial fractions.*
Answer: *We have*

$$\frac{x^4+1}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}.$$

Then

$$x^4+1=A(x^2+1)^2+(Bx+C)x(x^2+1)+(Dx+E)x.$$

If we put $x=0$*, we get*

$$1=A.$$

Now, we use the technique of equating coefficients. We have

$$x^4+1=(A+B)x^4+Cx^3+(2A+B+D)x^2+(C+E)x+A.$$

$$\textit{Coefficient of } x^4 :\Rightarrow 1=A+B$$

$$\Rightarrow B=0.$$

$$\textit{Coefficient of } x^3 :\Rightarrow 0=C$$

$$\textit{Coefficient of } x^2 :\Rightarrow 0 = 2A + B + D$$

$$\Rightarrow D = -2.$$

$$\textit{Coefficient of } x :\Rightarrow 0 = C + E$$

$$\Rightarrow E = 0.$$

Hence,

$$\frac{x^4+1}{x(x^2+1)^2} = \frac{1}{x} + \frac{-2x}{(x^2+1)^2}.$$

Exercise 2.4.1. *Express the following as a sum of partial fractions*

(a) $\dfrac{x^2-2x-1}{(x-1)(x^2+1)^2}$

(b) $\dfrac{2x+1}{(2x+x^3)^2}$

(c) $\dfrac{2x^2+1}{(1-x^3)^2}$

2.5 Dealing with improper fractions

We now look at how to deal with improper fractions, for which the numerator is of degree not less than that of the denominator. We can divided the numerator by the denominator, till the reminder is of degree less than that of the denominator.

Example 2.5.1. *Express* $\dfrac{x^3+2x^2+6}{x^2+x+2}$ *as a sum of partial fractions.*
Answer: *Here the degree of the numerator is 3 and that of the denominator is 2. Hence we divide the numerator by the denominator as follows:*

x^2+x-2	x^3+2x^2+6
$x{+}1$	x^3+x^2-2x
	x^2+2x+6
	x^2+x-2
	$x{+}8$

$$\therefore \frac{x^3+2x^2+6}{x^2+x+2} = x+1+\frac{x+8}{x^2+x+2}.$$

Now, let

$$\frac{x+8}{x^2+x+2} = \frac{x+8}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}.$$

$$\Rightarrow x+8 = A(x+2) + B(x-1).$$

If we put $x = 1$, *we get*

$$9 = 3A$$

$$\Rightarrow A = 3.$$

If we put $x = -2$, *we get*

$$6 = -3B$$

$$\Rightarrow B = -2.$$

Hence,

$$\frac{x+8}{x^2+x+2} = \frac{3}{x-1} - \frac{2}{x+2}.$$

Therefore,

$$\frac{x^3+2x^2+6}{x^2+x+2} = x+1+\frac{3}{x-1} - \frac{2}{x+2}.$$

Example 2.5.2. *Express* $\frac{4x^3+10x+4}{x(2x+1)}$ *as a sum of partial fractions.*
Answer: *The degree of the numerator is 3 and that of the denominator is 2. Then, we have*

$$\frac{4x^3+10x+4}{x(2x+1)} = Ax+B+\frac{C}{x}+\frac{D}{2x+1}.$$

Multiplying both sides by the denominator $x(2x+1)$ *gives*

$$4x^3+10x+4 = Ax^2(2x+1) + Bx(2x+1) + C(2x+1) + Dx.$$

If we put $x = 0$, *we get*

$$4 = C.$$

If we put $x = -\frac{1}{2}$, *we get*

$$\frac{-4}{8} - \frac{10}{2} + 4 = -\frac{1}{2}D$$

$$\begin{aligned}
&\Rightarrow -\frac{1}{2} - 5 + 4 = -\frac{1}{2}D \\
&\Rightarrow -\frac{3}{2} = -\frac{1}{2}D \\
&\Rightarrow D = 3.
\end{aligned}$$

Special values will not give A or B so we shall have to equate coefficients.

$$\begin{aligned}
4x^3 + 10x + 4 &= Ax^2(2x+1) + Bx(2x+1) + C(2x+1) + Dx \\
&= 2Ax^3 + Ax^2 + 2Bx^2 + Bx + 2Cx + C + Dx \\
&= 2Ax^3 + (A+2B)x^2 + (B+2C+D)x + C.
\end{aligned}$$

$$\begin{aligned}
\textit{Coefficient of } x^3 &:\Rightarrow 2A = 4 \\
&\Rightarrow A = 2. \\
\textit{Coefficient of } x^2 &:\Rightarrow A + 2B = 0 \\
&\Rightarrow B = -1.
\end{aligned}$$

Hence,

$$\frac{4x^3 + 10x + 4}{x(2x+1)} = 2x - 1 + \frac{4}{x} + \frac{3}{2x+1}.$$

Exercise 2.5.1. *Express the following as a sum of partial fractions*

(a) $\dfrac{x^3 - 4x^2 + 3x + 1}{x^2 + 5x + 6}$

(b) $\dfrac{x^3 + 1}{x^2 + 1}$

(c) $\dfrac{2x^4 + 3x^2 + 1}{x^2 + 3x + 2}$

(d) $\dfrac{7x^2 - 1}{x + 3}$

CHAPTER 3
Sequences and Series

Chapter 3

Sequences and Series

In this chapter, we define just what we mean by sequence in a math class and give the basic notation we will use with them. We will focus on the basic terminology, limits of sequences and convergence of sequences in Section 3.1. We will also give many of the basic facts and properties we'll need as we work with sequences. In Section 3.2, we will formally define an infinite series. We will also give many of the basic facts, properties and ways we can use to manipulate a series. We will also briefly discuss how to determine if an infinite series will converge or diverge (a more in depth discussion of this topic will occur in the following sections).

3.1 Sequences

Definition 3.1.1. A sequence is an ordered list of real numbers or is a function represented by $\{a_n\} = \{a_1, a_2, \ldots\}$ where the domain is the variable n with positive integers.
The function values: a_1, a_2, a_3, $\ldots$, a_n, $\ldots$ are the terms of the sequence.

A finite sequence has a defined number of terms.
For example, 2, 6, 10, 14 this sequence is finite with 4 terms.
An infinite sequence has an undefined number of terms.
For example, 2, 6, 10, 14, $\ldots$ this sequence is infinite.
Because we will be dealing with infinite sequences each term in the sequence

will be followed by another term as:

$$a_1 \longrightarrow \text{ first term}$$

$$a_2 \longrightarrow \text{ second term}$$

$$a_3 \longrightarrow \text{ third term}$$

$$\vdots$$

$$a_n \longrightarrow \; n^{th} \text{ term or (the general term of the sequence)}$$

$$\vdots$$

Example 3.1.1. *Given the n^{th} term of a sequence, write the first 4 terms.*

(i) $a_n = \frac{n}{n+1}$.

(ii) $a_n = 3 + (0.1)^n$.

Answer:

(i)

$$a_n = \frac{n}{n+1} \longrightarrow a_1 = \frac{1}{2},\ a_2 = \frac{2}{3},\ a_3 = \frac{3}{4},\ a_4 = \frac{4}{5}.$$

the first 4 terms are $\{\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ \ldots\}$.

(ii)

$$a_n = 3 + (0.1)^n \longrightarrow a_1 = 3.1,\ a_2 = 3.01,\ a_3 = 3.001,\ a_4 = 3.0001.$$

the first 4 terms are $\{3.1,\ 3.01,\ 3.001,\ 3.0001,\ \ldots\}$.

Exercise 3.1.1. *Write down the first few terms of each of the following sequences*

(i) $a_n = \frac{(-1)^{n+1}}{2^n}$.

(ii) $a_n = 1 - \frac{1}{n^2}$.

(iii) $a_n = \frac{n^2+n}{2}$.

A better method of describing a sequence is to state how to determine the n^{th} term with an explicit formula.
For example, the sequence 1, 2, 3, 4, ... is easily specified by saying $a_n = n$.

Example 3.1.2. *Given the terms of 5 sequences, find the general term of each sequence:*

(i) $\frac{1}{3}, \frac{3}{6}, \frac{5}{9}, \frac{7}{12}, \frac{9}{15}, \ldots$.

(ii) $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots$.

Answer:

(i) $a_n = \frac{2n-1}{3n}$.

(ii) $a_n = \frac{\cos(\pi n)}{n}$.

Definition 3.1.2. An Arithmetic Sequence is a sequence such that the difference d between two consecutive terms is constant. We write this sequence as follows

$$a_1,\, a_1 + d,\, a_1 + 2d,\, \ldots,\, a_1 + (n-1)d,\, \ldots.$$

Therefore, The n^{th} term of an arithmetic sequence is

$$a_n = a_1 + (n-1)d, \quad n = 1, 2, \ldots,$$

and the base d is equal to

$$d = a_n - a_{n-1}, \;\; n = 1, 2, \ldots.$$

Example 3.1.3. *Given the arithmetic sequence:* 6, 4, 2, 0, -2, ..., *find* d, a_n, a_{50} *and which term equals* -40 *?*

Answer:

$d = a_2 - a_1 = 4 - 6 = -2$. *Hence*

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= 6 + (n-1)(-2) \\ &= 8 - 2n. \end{aligned}$$

Therefore, $a_{50} = 8 - 2(50) = -92$. *Also,*

$$a_n = -40, \Longrightarrow 8 - 2n = -40 \Longrightarrow n = 24.$$

Then the twenty-fourth term equals to -40.

Exercise 3.1.2. *Given the arithmetic sequence:* $\frac{2}{5}, \frac{1}{5}, 0, -\frac{1}{5}, -\frac{2}{5}, \ldots,$ *find* d, a_n, a_{43} *and which term equals* -9 *?*

Definition 3.1.3. A Geometric Sequence is a sequence such that the ratio r among two consecutive terms is constant. We write this sequence as follows:

$$a_1, \, a_1 r, \, a_1 r^2, \, a_1 r^3, \, \ldots, \, a_1 r^{n-1}, \, \ldots.$$

Therefore, The n^{th} term of a geometric sequence is

$$a_n = a_1 (r)^{n-1},$$

and the ratio $r = \frac{a_{n+1}}{a_n}, \;\; n = 1, 2, 3, \ldots.$

Example 3.1.4. *Given the geometric sequence:* 120, 60, 30, , 15, $\frac{15}{2}, \ldots$ *find* r, a_n, a_{10} *and which term equals* $\frac{15}{32}$ *?*

Answer:

The ratio $r = \frac{a_2}{a_1} = \frac{60}{120} = \frac{1}{2}$. *Then*

$$a_n = a_1 (r)^{n-1} = 120 \cdot (\frac{1}{2})^{n-1}.$$

Therefore,

$$a_{10} = 120(\frac{1}{2})^9 = \frac{15}{64}.$$

The tenth term equals to $\frac{15}{64}$. *If* $a_n = \frac{15}{32}$, *then*

$$120(\frac{1}{2})^{n-1} = \frac{15}{32}$$

$$\Longrightarrow (\frac{1}{2})^{n-1} = \frac{1}{256} = (\frac{1}{2})^{9-1}$$

$$\Longrightarrow n = 8, \quad \textit{the ninth term equals to } \frac{15}{32} \; .$$

Exercise 3.1.3. *Given the geometric sequence:* $-8, 4, -2, 1, -1, 2, \ldots$ *find the base, the general term and then find the eleventh term.*

Exercise 3.1.4. *Given 3 sequences, identify which sequence is arithmetic, geometric or neither:*

(i) $18, 16, 14, 12, \ldots.$

(ii) $18, 10, 22, 6, 30, \ldots.$

(iii) $18, 9, \frac{9}{2}, \frac{9}{4}, \frac{9}{8}, \ldots.$

3.1.1 Convergence and divergence of sequences

Definition 3.1.4. A sequence $\{a_n\}$ is said to be convergent if it has a unique limit point. That is

$$\lim_{n\to\infty} a_n = a,$$

for some real numbers a.

Definition 3.1.5. A sequence $\{a_n\}$ is said to be convergent to a number a if for every $\epsilon > 0$ there exists a natural number N such that

$$|a_n - a| < \epsilon, \ \forall\, n > N.$$

In this case we write,

$$\lim_{n\to\infty} a_n = a.$$

Definition 3.1.6. A sequence $\{a_n\}$ is said to be divergent if it is not convergent.

Definition 3.1.7. We say that $\lim_{n\to\infty} a_n = \infty$ if for every number $M > 0$ there is an integer N such that

$$a_n > M, \quad \forall\, n > N.$$

In this case we say that the sequence $\{a_n\}$ is a divergent sequence.

Example 3.1.5. *Given sequences with the* n^{th} *term , find if the sequences converges or diverges:*

(i) $\{\frac{1}{2}n^3 - 1\}$

(ii) $\{(-1)^{n+1}\}$

(iii) $\{\frac{4n}{e^{3n}}\}$

Answer:

(i) $\because a_n = \frac{1}{2}n^3 - 1$, *then* $\lim_{n\to\infty} a_n = \lim_{n\to\infty}[\frac{1}{2}n^3 - 1] = \infty$. *Then, the sequence* $\{\frac{1}{2}n^3 - 1\}$ *is divergent.*

(ii) $\because a_n = (-1)^{n+1}$, *then this sequence has two limit points* 1 *and* -1. *Hence the sequence* $\{(-1)^{n+1}\}$ *is divergent.*

(iii) $\because a_n = \frac{4n}{e^{3n}}$, *then*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4n}{e^{3n}} = \frac{\infty}{\infty}.$$

By using L'Hôpital's rule, we get

$$\lim_{n\to\infty} \frac{4n}{e^{3n}} = \lim_{n\to\infty} \frac{4}{3e^{3n}} = 0.$$

Therefore, the sequence $\{\frac{4n}{e^{3n}}\}$ *is convergent.*

Exercise 3.1.5. *Given sequences with the* n^{th} *term , find if the sequences converges or diverges:*

(i) $\{\frac{n-2}{n^2}\}$

(ii) $\{\frac{n^2+1}{2n}\}$

(iii) $\{1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots\}$

Theorem 3.1.1. *Let* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *be three sequences of real numbers such that* $a_n \leq b_n \leq c_n$. *If*

$$\lim_{n\to\infty} a_n = a = \lim_{n\to\infty} c_n,$$

then

$$\lim_{n\to\infty} b_n = a.$$

Example 3.1.6. *Find the limit of the sequence* $\{\frac{\sin^2 n}{2^n}\}$ *as* $n \to \infty$.

Answer:

Since $0 \leq \sin^2 n \leq 1$. *then*

$$0 \leq \frac{\sin^2 n}{2^n} \leq \frac{1}{2^n}.$$

But

$$\lim_{n\to\infty} 0 = 0 \quad \textit{and} \quad \lim_{n\to\infty} \frac{1}{2^n} = 0.$$

Therefore, by Theorem 3.1.1 we get

$$\lim_{n\to\infty} \frac{\sin^2 n}{2^n} = 0.$$

Theorem 3.1.2. *Let* r *be real number. If*

(1) $|r| < 1$, *then* $\lim_{n\to\infty} r^n = 0$.

(2) $|r| > 1$, *then* $\lim_{n\to\infty} r^n = \infty$.

Example 3.1.7. *Find* $\lim_{n\to\infty}(\frac{-2}{5})^n$.

Answer:

$$\because\ |r| = |\frac{-2}{5}| = \frac{2}{5} < 1.$$

Then

$$\lim_{n\to\infty} (\frac{-2}{5})^n = 0.$$

3.2 Infinite series

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of numbers and define the following:

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$S_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i=1}^{n} a_i$

The S_n are called **partial sums** and notice that they will form a sequence

$\{S_n\}_{n=1}^{\infty}$ and $a_n = S_n - S_{n-1}$. Also recall that the $\sum$ is used to represent this summation and called a variety of names. The most common names are : series notation, summation notation, and sigma notation.

3.2.1 Convergence and divergence of an infinite series

Definition 3.2.1. An infinite series $\sum_{n=1}^{\infty} a_n$ is convergent if :

$$\lim_{n\to\infty} S_n = S = \sum_{n=1}^{\infty} a_n, \quad \text{for some real number } S.$$

If $\lim_{n\to\infty} S_n$ does not exist, then the series diverges.

Remark 3.2.1. We know that:

(i)
$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + L \tag{3.2.1}$$

is arithmetic series, with a_1 in a first term, d is the base and L is the last term, then

$$S_n = L + (L - d) + (L - 2d) + \ldots + a_1. \tag{3.2.2}$$

Adding equations (3.2.1) and (3.2.2), we get

$$\begin{aligned} 2S_n &= (a_1 + L) + (a_1 + L) + (a_1 + L) + \ldots + (a_1 + L) \text{ to } n \text{ terms} \\ &= n(a_1 + L). \end{aligned}$$

Then

$$\boxed{S_n = \frac{n(a_1 + L)}{2}} \tag{3.2.3}$$

(ii)
$$S_n = a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-1} \tag{3.2.4}$$

is Geometric series, with a_1 in a first term, r is the base and $a_1 r^{n-1}$ is the last term, then

$$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^n. \tag{3.2.5}$$

Subtract equation (3.2.5) from (3.2.4), we obtain

$$S_n - rS_n = a_1 r^n.$$

Then,

$$\boxed{S_n = \frac{a_1(1-r^n)}{1-r}} \tag{3.2.6}$$

Example 3.2.1. *Discuss the convergence and divergence of the series*

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \ldots + n + \ldots.$$

Answer:
Since

$$S_n = 1 + 2 + 3 + \ldots + n$$

which is arithmetic series with $a_1 = 1$, $L = n$ *and the number of terms is* n. *Then*

$$S_n = \frac{n(1+n)}{2},$$

and

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(1+n)}{2} = \infty.$$

Therefore, the series $\sum_{n=1}^{\infty} n$ *is divergent.*

Example 3.2.2. *Discuss the convergence and divergence of the series*

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n} + \ldots.$$

Answer:

Since

$$S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n}$$

which is geometric series with $a_1 = 1$, $r = \frac{1}{2}$. *Then*

$$S_n = \frac{1(1 - (\frac{1}{2})^n)}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n-1}},$$

and

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2 - \frac{1}{2^{n-1}} = 2.$$

Therefore, the series $\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$, *and it is convergent.*

Example 3.2.3. *Discuss the convergence and divergence of the series*

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \ldots + \frac{1}{n\times(n+1)} + \ldots.$$

Answer:

Since

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Then

$$\begin{aligned} S_n &= (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \ldots + (\frac{1}{n-1} - \frac{1}{n}) + (\frac{1}{n} - \frac{1}{n+1}) \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (1 - \frac{1}{n+1}) = 1.$$

Consequently, the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ *is convergent and the sum equal to 1.*

Exercise 3.2.1. *Discuss the convergence and divergence of the following series*

(i) $\sum_{n=1}^{\infty}(-1)^{n-1}$.

(ii) $\sum_{n=0}^{\infty} x^n$.

3.2.2 Special series

Geometric series

Theorem 3.2.1. *The series* $\sum_{n=0}^{\infty} ar^n$ *is convergent to* $\frac{a}{1-r}$ *if* $|r| < 1$ *and its divergent if* $|r| \geq 1$.

Proof.

$$\because S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

and

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (\frac{a}{1-r} - \frac{ar^n}{1-r})$$

Then

(1) If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$, and $\lim_{n\to\infty} S_n = \frac{a}{1-r}$.

(2) If $|r| > 1$, then $\lim_{n\to\infty} r^n = \infty$, and $\lim_{n\to\infty} S_n = \infty$. Therefore, the series is divergent.

(3) If $r = 1$, then $S_n = \frac{0}{0}$ and $\lim_{n\to\infty} S_n$ is not defined, the series is divergent.

(4) If $r = -1$, then $S_n = \frac{a(1-(-1)^n)}{1-(-1)} = \frac{1}{2}(1-(-1)^n)$. Hence,

   (i) if n is even, then $S_n = 0$.

   (ii) if n is odd, then $S_n = a$.

   Therefore, $\lim_{n\to\infty} S_n$ does not exist, the series is divergent.

□

Example 3.2.4. *Determine if the series* $0.6 + .06 + .006 + \ldots + \frac{6}{10^n} + \ldots$ *is convergent or divergent.*

Answer:

Since $r = 0.1$, *then by the pervious theorem the series is convergent and its sum is* $\frac{2}{3}$.

Harmonic series

Definition 3.2.2. The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots .$$

is called harmonic series and it is divergent.

Proof. we rewrite the series as:

$$1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + (\frac{1}{9} + \ldots + \frac{1}{16}) + \ldots \qquad (3.2.7)$$

Note that

$$\frac{1}{3} > \frac{1}{4}.$$

Also,

$$\frac{1}{5} > \frac{1}{6} > \frac{1}{7} > \frac{1}{8}.$$

Then the harmonic series in (3.2.7) is greater than the series

$$\begin{aligned} 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) + (\frac{1}{16} + \ldots + \frac{1}{16}) + \ldots \\ = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots \qquad (3.2.8) \end{aligned}$$

Hence, $S_n = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2}$ n term.It follows that

$$S_n = 1 + \frac{n-1}{2},$$

and

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (1 + \frac{n-1}{2}) = \infty.$$

Therefore, the series (3.2.8) is divergent. Consequently, the series in (3.2.7) must be divergent. □

Remark 3.2.2. Removal (or additional) of a finite number of terms from (or to) the series does not affect the convergence or divergence.

Example 3.2.5. *Determine if the following series is convergent or divergent*

$$\frac{1}{3\times 4}+\frac{1}{4\times 5}+\ldots+\frac{1}{(n+2)(n+3)}+\ldots.$$

Answer:

The series is convergent because we can get it by removal the first term and the second term from the series $\sum\limits_{n=1}^{\infty}\frac{1}{n(n+1)}$.

The P-series

The series

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=1+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}+\ldots,$$

where p is constant, is convergent if $p>1$ and it is divergent if $p\leq 1$.

3.2.3 Properties of an infinite series

Theorem 3.2.2. *Let* $\sum\limits_{n=1}^{\infty}a_n$ *and* $\sum\limits_{n=1}^{\infty}b_n$ *be two convergent series to* a *and* b *respectively and* c *be a real number. Then the following properties are valid:*

(i) $\sum\limits_{n=1}^{\infty}ca_n=c\sum\limits_{n=1}^{\infty}a_n=ca.$

(ii) $\sum\limits_{n=1}^{\infty}(a_n\pm b_n)=\sum\limits_{n=1}^{\infty}a_n\pm\sum\limits_{n=1}^{\infty}b_n=a\pm b.$

Theorem 3.2.3. *If the series* $\sum\limits_{n=1}^{\infty}a_n$ *is convergent, then*

$$\lim_{n\to\infty}a_n=0.$$

The converse is not necessarily true.

Example 3.2.6. *For the harmonic series* $\sum\limits_{n=1}^{\infty}\frac{1}{n}$, *we obtain*

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{1}{n}=0,$$

however it is divergent.

Example 3.2.7. *Prove that the series* $\sum_{n=1}^{\infty}\Big[\frac{5}{n(n+1)}+\frac{4}{5^{n-1}}\Big]$ *is convergent and find its sum.*

Answer:

By Example 3.2.3 and Theorem 3.2.2 (i), we get

$$\sum_{n=1}^{\infty}\frac{5}{n(n+1)}=5\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=5(1)=5.$$

Also,

$$\sum_{n=1}^{\infty}\frac{4}{5^{n-1}}=4\sum_{n=1}^{\infty}\frac{1}{5^{n-1}}.$$

Since the series $\sum_{n=1}^{\infty}\frac{1}{5^{n-1}}$ *is geometric series with* $a=1$ *and* $r=\frac{1}{5}$*, then*

$$\sum_{n=1}^{\infty}\frac{1}{5^{n-1}}=\frac{1}{1-\frac{1}{5}}=\frac{5}{4}.$$

Therefor,Theorem 3.2.2 (ii), we get

$$\sum_{n=1}^{\infty}\Big[\frac{5}{n(n+1)}+\frac{4}{5^{n-1}}\Big]=5+4(\frac{5}{4})=5+5=10.$$

Example 3.2.8. *Discuss the convergence and divergence of the following series*

$$\sum_{n=1}^{\infty}\frac{n}{n+1}.$$

Answer:

By using the necessary condition for convergence, we obtain

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{n}{n+1}=1\neq 0.,$$

Then from Theorem 3.2.3, the series is divergent.

Exercise 3.2.2. *Discuss the convergence and divergence of the following series*

$$\sum_{n=1}^{\infty}\Big[\frac{1}{n^5}+\frac{1}{n^2}\Big].$$

3.2.4 Tests of divergence

n^{th} term test for Divergence

If an infinite series $\sum_{n=1}^{\infty} a_n$ has $\lim_{n\to\infty} a_n \neq 0$, then the series is divergent.

Example 3.2.9. *Given an infinite series* $\sum_{n=1}^{\infty} \frac{n}{3n+2}$*; test the series for divergence.*

Answer:

$$\because \lim_{n\to\infty} \frac{n}{3n+2} = \frac{1}{3} \neq 0.$$

Use the n^{th} term test for Divergence, the series is divergent.

Example 3.2.10. *Given an infinite series* $\sum_{n=1}^{\infty} \frac{6}{5^n}$*; test the series for divergence.*

Answer:

$$\because \lim_{n\to\infty} \frac{6}{5^n} = 0.$$

Then, n^{th} term test for Divergence fails.

Exercise 3.2.3. *Discuss the convergence and divergence of the following series*

(i) $\sum_{n=1}^{\infty} \frac{3-n}{2n+5}$

(ii) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

(iii) $\sum_{n=1}^{\infty} \frac{e^n}{n}$

3.2.5 Positive series and tests of convergence

Definition 3.2.3. We say that the series $\sum_{n=1}^{\infty} a_n$ is positive series if all terms of it is positive. Thst is $a_n > 0$ for all values of n.

Comparison test

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two positive series and $a_n \leq b_n$ for all values of n, then

(1) if the series $\sum_{n=1}^{\infty} b_n$ is convergent, then the series $\sum_{n=1}^{\infty} a_n$ is also convergent.

(2) if the series $\sum_{n=1}^{\infty} a_n$ is divergent, then the series $\sum_{n=1}^{\infty} a_n$ is also divergent.

Remark 3.2.3. In this part to determine the series $\sum_{n=1}^{\infty} b_n$, which will be used for comparison with $\sum_{n=1}^{\infty} a_n$ we take the term in the numerator and the dominator that have the most effect on the magnitude.

Example 3.2.11. *Discuss the convergence and divergence of the following series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}; \quad p < 1.$$

Answer:

Since $n^p < n; \ \ p < 1$. *Then*

$$\frac{1}{n^p} > \frac{1}{n}.$$

But the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent, then the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}; \quad p < 1.$$

is also divergent by the comparison test.

Example 3.2.12. *Discuss the convergence and divergence of the following series*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n^2+1)}}.$$

Answer:

Since

$$\frac{1}{\sqrt{n(n^2+1)}} < \frac{1}{\sqrt{n^3}}$$

and the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$$

is convergent. Then by the comparison test the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n^2+1)}}.$$

is convergent.

Example 3.2.13. *Discuss the convergence and divergence of the following series*

$$\sum_{n=1}^{\infty} \frac{\log n}{2n^5-1}.$$

Answer:

Since

$$\log n < n.$$

Then, we have

$$\frac{\log n}{2n^5-1} < \frac{n}{n^5} = \frac{1}{n^4}.$$

But the series

$$\sum_{n=1}^{\infty} \frac{1}{n^4}$$

is convergent. Then by the comparison test the series

$$\sum_{n=1}^{\infty} \frac{\log n}{2n^5-1}$$

is convergent.

Exercise 3.2.4. *Discuss the convergence and divergence of the following series*

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3+7^n}.$

(ii) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

(iii) $\sum_{n=1}^{\infty} \frac{n+2}{n^2}$.

Comparison test by using limit

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two positive series and if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0.$$

Then, either both series converge or both series diverge.

Example 3.2.14. *Discuss the convergence and divergence of the following series*

$$\sum_{n=1}^{\infty} \frac{5n^2 - 2n + 1}{n^3 + 3n - 1}.$$

Answer:

Let us consider

$$a_n = \frac{5n^2 - 2n + 1}{n^3 + 3n - 1}.$$

Then, we have

$$b_n = \frac{5n^2}{n^3}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{5n^2 - 2n + 1}{n^3 + 3n - 1} \times \frac{n}{5} = 1 > 0.$$

Since, the series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{5}{n}$$

is divergent. Then by the comparison test by using limit the series

$$\sum_{n=1}^{\infty} \frac{5n^2 - 2n + 1}{n^3 + 3n - 1}$$

is divergent.

Example 3.2.15. *Determine the convergence and divergence of the following series*

$$\sum_{n=1}^{\infty} \frac{3n^2+5}{2^n(n^2+1)}.$$

Answer:

Let

$$a_n = \frac{3n^2+5}{2^n(n^2+1)}.$$

Then, we have

$$b_n = \frac{3n^2}{2^n n^2} = \frac{3}{2^n},$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{3n^2+5}{2^n(n^2+1)} \times \frac{2^n}{3} = 1 > 0.$$

Since, the series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{3}{2^n}$$

is convergent geometric series with $a = 3$ and $r = \frac{1}{2}$. Then by the comparison test by using limit the series

$$\sum_{n=1}^{\infty} \frac{3n^2+5}{2^n(n^2+1)}$$

is convergent.

Exercise 3.2.5. *Determine the convergence and divergence of the following series*

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(n^2+1)(n+2)}$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n+\sqrt{n}}{2n^3-1}$

(iii) $\displaystyle\sum_{n=1}^{\infty} \sin^3(\frac{1}{n})$

(iv) $\displaystyle\sum_{n=1}^{\infty} \log(1+\frac{x}{n}),\ x > 0$

D'Alembert's test(Ratio test)

Consider the series $\sum_{n=1}^{\infty} a_n$. Let $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$. Then,

(i) the series $\sum_{n=1}^{\infty} a_n$ is convergent if $L < 1$

(ii) the series $\sum_{n=1}^{\infty} a_n$ is divergent if $L > 1$

(iii) If $L = 1$, then the test fails.

Example 3.2.16. *Determine the convergence and divergence of the following series*

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

Answer:

$$\because a_n = \frac{2^n}{n!}$$

Then

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \times \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{(n+1)} = 0 < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ *is convergent.*

Example 3.2.17. *Determine the convergence and divergence of the following series*

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2}$$

Answer:

$$\because a_n = \frac{2^n}{n^2}$$

Then

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)^2} \times \frac{n^2}{2^n} = \lim_{n\to\infty} \frac{2n^2}{n^2 + 2n + 1} = 2 > 1.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ *is divergent.*

Exercise 3.2.6. *Determine the convergence and divergence of the following series*

(i) $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

(ii) $\sum_{n=1}^{\infty} n^4 e^{-n^2}$

(iii) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^n}{n^2}$

(iv) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^2+1}$

(v) $\sum_{n=1}^{\infty} \frac{(n-1)(n+3)}{2 \times 4 \times 6 \times \ldots (2n)}$

Cauchy's test(The n^{th} root test)

Let $\sum_{n=1}^{\infty} a_n$ be a positive series and suppose that $\lim_{n \to \infty} \sqrt[n]{a_n} = L$. Then,

(i) if $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent

(ii) if $L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent

(iii) If $L = 1$, then the test fails.

Example 3.2.18. *Determine the convergence and divergence of the following series*

$$\sum_{n=1}^{\infty} \frac{2^{3n+1}}{n^n}$$

Answer:

Applying Cauchy's test,

$$\because a_n = \frac{2^{3n+1}}{n^n}$$

Then

$$\lim_{n\to\infty} |\sqrt[n]{a_n} = \lim_{n\to\infty} \frac{2^{3+\frac{1}{n}}}{n} = \frac{8}{\infty} = 0 < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{2^{3n+1}}{n^n}$ *is convergent.*

Example 3.2.19. *Determine the convergence and divergence of the following series*

$$\sum_{n=1}^{\infty} (1 - \frac{1}{n})^{n^2}$$

Answer:

Applying Cauchy's test,

$$\because\ a_n = (1 - \frac{1}{n})^{n^2}$$

Then

$$\lim_{n\to\infty} |\sqrt[n]{a_n} = \lim_{n\to\infty} (1 - \frac{1}{n})^n = e^{-1} < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} (1 - \frac{1}{n})^{n^2}$ *is convergent.*

Exercise 3.2.7. *Determine the convergence and divergence of the following series*

(i) $\sum_{n=1}^{\infty} (\frac{n}{4n+1})^{4n}$

(ii) $\sum_{n=1}^{\infty} \frac{n^{n^2}}{(1+n)^{n^2}}$

(iii) $\sum_{n=1}^{\infty} \frac{4^{5n+1}}{n^n}$

Integral test(The n^{th} root test)

Suppose that $f(n)$ is a positive, continuous and decreasing function on the interval $[k, \infty)$ and that $f(n) = a_n$. Then,

(i) if $\int_k^{\infty} f(x)dx < \infty$, then $\sum_{n=k}^{\infty} a_n$ is convergent

(ii) if $\int_k^{\infty} f(x)dx = \infty$, then $\sum_{n=k}^{\infty} a_n$ is divergent

Example 3.2.20. *Determine the convergence and divergence of the following series*

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}.$$

Answer:

Applying integral test,

$$\because \ f(n) = a_{\,n} = \frac{1}{n \ln n}$$

Then

$$\int_2^x f(n)dn = \int_2^{\infty} \frac{1}{n \ln n} dn = \ln(\ln x) - \ln(\ln 2).$$

Hence,

$$\lim_{x \to \infty} [\ln(\ln x) - \ln(\ln 2)] = \infty.$$

Therefore, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ *is divergent.*

Example 3.2.21. *Determine the convergence and divergence of the following series*

$$\sum_{n=1}^{\infty} \frac{n^2}{e^{n^3}}.$$

Answer:

Applying integral test,

$$\because \ f(n) = a_n = \frac{n^2}{e^{\,-n^3}}$$

Then

$$\int_1^x f(n)dn = \int_1^{\infty} \frac{n^2}{e^{n^3}} dn = \int_1^{\infty} n^2 e^{-n^3} \, dn = \frac{1}{3}[e^{-x^3} - e^{-1}).$$

Hence,

$$\lim_{x \to \infty} \frac{1}{3}[e^{-x^3} - e^{-1}) = \frac{1}{3e}.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{e^{n^3}}$ *is convergent.*

Exercise 3.2.8. *Determine the convergence and divergence of the following series*

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \ln n \cdot (\ln(\ln n))}$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Exercise 3.2.9. *Prove that the harmonic series* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ *is convergent if* $p > 1$ *and it is divergent if* $p \leq 1$.

Alternating series

Definition 3.2.4. The alternating series is any series $\displaystyle\sum_{n=1}^{\infty} a_n$, for which the series terms are alternately positive and negative and can be written in one of the following two forms

$$a_n = (-1)^n b_n, \;\; b_n > 0$$

Or

$$a_n = (-1)^{n+1} b_n, \;\; b_n > 0,$$

for all n.

Alternating series test

The alternating series $\displaystyle\sum_{n=1}^{\infty}(-1)^n b_n$ or $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1} b_n$, where $b_n > 0$ for all n, is convergent if the following conditions hold

(i) $\lim_{n\to\infty} b_n = 0$,

(ii) $\{b_n\}$ is a decreasing sequence, i.e, $b_{n+1} \leq b_n$.

Example 3.2.22. *Determine the convergence and divergence of the following series*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + (-1)^{n-1}\frac{1}{n} + \ldots.$$

Answer:

Let $a_n = \frac{1}{n}$. *We can see that* *(i)* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0,$

(ii) $a_{n+1} \leq a_n.$

Then, the series is convergent.

Exercise 3.2.10. *Determine if each alternating series is convergent or divergent*

(i) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{4n-3}.$

(ii) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{4n^2-3}.$

(iii) $\sum_{n=1}^{\infty}(-1)^{n-3}\frac{\sqrt{n}}{n+4}.$

Absolute and conditional convergence

The following concept is useful when investigating a series that contains both positive and negative terms but is not alternating.

Definition 3.2.5. An infinite series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Definition 3.2.6. An infinite series $\sum_{n=1}^{\infty} a_n$ in conditionally convergent if the series $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty}|a_n|$ divergent.

Example 3.2.23. *Examine for absolute convergent and conditionally convergent*

$$\sum_{n=2}^{\infty}(-1)^{n-1}\frac{1}{n\ln^2 n}. \tag{3.2.9}$$

Answer:

Let $a_n = (-1)^{n-1}\frac{1}{n\ln^2 n}$. *Then*

$$\sum_{n=2}^{\infty}|a_n| = \sum_{n=2}^{\infty}\frac{1}{n\ln^2 n} \tag{3.2.10}$$

We will discuss the convergence of the series (3.2.10) *by the integral test. Since* $f(n) = \dfrac{1}{n\ln^2 n}$ *is positive, continuous and decreasing, then*

$$\int_2^x \frac{1}{n\ln^2 n}dn = \int_2^x \frac{1}{n}[\ln n]^{-2}dn = -\frac{1}{\ln x} + \frac{1}{\ln 2}.$$

It follows that

$$\lim_{x\to\infty}[-\frac{1}{\ln x} + \frac{1}{\ln 2}] = \frac{1}{\ln 2} < \infty.$$

Hence, the series (3.2.10) *is convergent. Therefore, the series* (3.2.9) *is absolutely convergent.*

Exercise 3.2.11. *Examine for absolute convergent and conditionally convergent*

(i) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$.

(ii) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n^2}$.

(iii) $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{n^3}$.

3.2.6 Power series

Definition 3.2.7. A power series is any series that can be written in the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots,$$

where a and c_n are real numbers. The c_n's are often called the coefficients of the series.

The first thing to notice about a power series is that it is a function of x. In general a power series converges for some values of x and diverges for other values of x.

Definition 3.2.8. If there is a number R, then we say that the power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ will converge for $|x-a| < R$ and will diverge for $|x-a| > R$. If

$|x-a| = R$, then the series may or may not converge. The number R is called the radius of convergence for the series and the interval of all x's, including the endpoints if need be, for which the power series converges is called the interval of convergence of the series.

Example 3.2.24. *Find the interval of the convergence of the series*

$$\sum_{n=0}^{\infty} \frac{nx^n}{4^n}.$$

Answer:
Let $a_n = \dfrac{nx^n}{4^n}$. *Applying the ratio test*

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(n+1)x^{n+1}}{4^{n+1}} \times \frac{4^n}{nx^n}\right| \\ &= \frac{|x|}{4} \lim_{n\to\infty} \frac{n+1}{n} \\ &= \frac{|x|}{4}.\end{aligned}$$

So, the ratio test tells us that if $\frac{|x|}{4} < 1$, *then the series will converge, if* $\frac{|x|}{4} > 1$, *then the series will diverge and if* $L = 1$, *we don't know what will happen. So, we have,*

$$|x| < 4 \quad \textit{the series converges.}$$

$$|x| > 4 \quad \textit{the series diverges.}$$

Then, the series converges for $-4 < x < 4$. *Also the series diverges for* $x > 4$ *and* $x < -4$. *In this case the the radius of convergence for the series is* $R = 4$. *At* $x = 4$, *we have*

$$\sum_{n=0}^{\infty} \frac{n4^n}{4^n} = \sum_{n=0}^{\infty} n$$

is divergent series, why?
At $x = -4$, *we have*

$$\sum_{n=0}^{\infty} \frac{n(-4)^n}{4^n} = \sum_{n=0}^{\infty} (-1)^n n$$

is divergent series, why?

Therefore, the series $\sum_{n=0}^{\infty} \frac{nx^n}{4^n}$ *converges on* $(-4,4)$ *and diverges for* $x \leq -4$ *and* $x \geq 4$.

Example 3.2.25. *Find the interval of the convergence of the series*

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n+1}}{(2n-1)!}.$$

Answer:
Let $a_n = (-1)^{n-1}\frac{x^{2n+1}}{(2n-1)!}$. *Applying the ratio test*

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(2n+1)!}\times\frac{(2n-1)!}{x^{2n+1}}\right| \\ &= |x^2|\lim_{n\to\infty}\frac{1}{(2n)(2n+1)} \\ &= 0 < 1.\end{aligned}$$

Then, the given series converges for all x *or the interval of convergence is* $-\infty < x < \infty$.

Exercise 3.2.12. *Find the interval of the convergence of the series*

(i) $\sum_{n=1}^{\infty}\frac{n(x-1)^n}{2^n(3n-1)}$.

(ii) $\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)x^n}$.

(iii) $\sum_{n=1}^{\infty} n!\cdot(x-a)^n$.

CHAPTER 4
Theory of Equations

Chapter 4

Theory of Equations

In this chapter, we we will study about polynomial functions and various methods to find out the roots of polynomial equations. "Solving equations" was an important problem from the beginning of study of Mathematics itself.

4.1 Polynomial Functions

Definition 4.1.1. A function defined by

$$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n,$$

where $a_0 \neq 0$, n is a non negative integer and a_i, $i = 1, 2, \ldots, n$ are fixed real or complex numbers, is called a polynomial of degree n in x. Then the numbers $a_0, a_1, a_2, \ldots, a_n$ are called the coefficients of P.

If α is a real or complex number such that $P(\alpha) = 0$, then α is called **zero** of the polynomial.

Theorem (Fundamental Theorem of Algebra)

Theorem 4.1.1. *Every polynomial function of degree $n > 1$ has at least one zero.*

Remark 4.1.1. Fundamental theorem of algebra says that, if

$$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n,$$

where $a_0 \neq 0$, is the given polynomial of degree $n \geq 1$, then there exists a complex number α such that

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0.$$

We use the Fundamental Theorem of Algebra, to prove the following result.

Theorem 4.1.2. *Every polynomial of degree n has n and only n zeroes.*

Proof. Let $P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n$, where $a_0 \neq 0$, be a polynomial of degree $n \geq 1$. By the fundamental theorem of algebra, $P(x)$ has at least one zero,let α_1 be that zero. Then $(x - \alpha_1)$ is a factor of $P(x)$. Therefore, we can write:

$$P(x) = (x - \alpha_1)Q_1(x),$$

where $Q_1(x)$ is a polynomial function of degree $n - 1$. If $n - 1 \geq 1$, again by Fundamental Theorem of Algebra, $Q_1(x)$ has at least one zero, say α_2.
Therefore,

$$P(x) = (x - \alpha_1)(x - \alpha_2)Q_2(x),$$

where $Q_2(x)$ is a polynomial function of degree $n - 2$. Repeating the above arguments, we get

$$P(x) = (x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n)Q_n(x),$$

where $Q_n(x)$ is a polynomial function of degree $n - n = 0$. i.e., $Q_n(x)$ is a constant.
Equating the coefficient of x^n on both sides of the above equation, we get

$$Q_n(x) = a_0 .$$

Therefore,

$$P(x) = a_0(x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n).$$

If α is any number other than $\alpha_1, \alpha_2, \ldots, \alpha_n$, then $P(\alpha) \neq 0$. This implies that α is not a zero of $P(x)$.

Hence $P(x)$ has n and only n zeros, namely $\alpha_1, \alpha_2, \ldots, \alpha_n$. $\square$

Definition 4.1.2. Let $P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n$; $a_0 \neq 0$ be an n^{th} degree polynomial in x. Then

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0 \tag{4.1.1}$$

is called a polynomial equation in x of degree n.
A number α is called a root of the equation (4.1.1) if α is a zero of the polynomial $P(x)$.

Example 4.1.1. *Find the polynomial of degree 3 that has roots -3, $\frac{1}{2}$ and $\frac{1}{3}$.*
Answer:
Since $\alpha_1 = -3$, $\alpha_2 = \frac{1}{2}$ and $\alpha_3 = \frac{1}{3}$ are the roots, then

$$a_0(x+3)(x-\frac{1}{2})(x-\frac{1}{3}) = 0$$

$$\Longrightarrow a_0(x+3)(\frac{2x-1}{2})(\frac{3x-1}{3}) = 0$$

$$\Longrightarrow (x+3)(6x^2-5x+1) = 0$$

$$\Longrightarrow 16x^3 + 13x^2 - 14x + 3 = 0.$$

Theorem 4.1.3. *If the equation $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$, where $a_0, a_1, \ldots, a_n$ are real numbers ($a_0 \neq 0$), has a complex root $\alpha + i\beta$, then it also has a complex root $\alpha - i\beta$ (i.e., complex roots occur in conjugate pairs for a polynomial equation with real coefficients).*

Proof. Let $P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n \neq 0$. Given that $\alpha + i\beta$ is a root of $P(x) = 0$.
Consider

$$(x - (\alpha + i\beta))(x - (\alpha - i\beta)) = (x-\alpha)^2 + \beta^2.$$

Divide $P(x)$ by $((x-\alpha)^2 + \beta^2)$. Let $Q(x)$ be the quotient and $Ax + B$ be the remainder. Then,

$$\begin{aligned} P(x) &= ((x-\alpha)^2+\beta^2)Q(x)+Ax+B \\ &= [(x-(\alpha+i\beta))(x-(\alpha-i\beta))]Q(x)+Ax+B. \end{aligned}$$

$$\Longrightarrow P(\alpha+i\beta)=0+A(\alpha+i\beta)+B=A\alpha+B+iA\beta.$$

But $P(\alpha+i\beta)=0$. Equating real and imaginary parts, we see that

$$A\alpha+B=0,$$

and

$$A\beta=0.$$

But $\beta\neq 0 \implies A=0$ and so $B=0$. Therefore, The remainder $Ax+B$ is zero, i.e., $[(x-(\alpha+i\beta))(x-(\alpha-i\beta))]$ is a factor of $P(x)$. Then $\alpha-i\beta$ is a root of $P(x)=0$. $\square$

Example 4.1.2. *If the equation*

$$x^4-4x^2+8x+35=0 \tag{4.1.2}$$

has a root $2+i\sqrt{3}$, *then find all remaining roots.*

Answer:

Since $2+i\sqrt{3}$ *is a root of equation* (4.1.2), *then* $2-i\sqrt{3}$ *is also root. This implies that*

$$(x-(2+i\sqrt{3}))(x-(2+i\sqrt{3}))=(x-2)^2+3=x^2-4x+7$$

is a factor of the given polynomial.

Dividing the given polynomial by this factor, we obtain the other factor as x^2+4x+5.

The roots of $x^2+4x+5=0$ *are given by*

$$\frac{-4\pm\sqrt{16-20}}{2}=-2\pm i.$$

Hence the roots of the given polynomial are

$$2 + i\sqrt{3},\ \ 2 - i\sqrt{3},\ \ 2 + i,\ \ \textit{and}\ \ -2 - i.$$

Theorem 4.1.4. *Let an equation $P(x) = 0$ with rational coefficients and the number $a + \sqrt{b}$, where $\sqrt{b}$ is irrational number, is a root of it. Then the number $a - \sqrt{b}$ is also root of it.*

Proof. Let $P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n \neq 0$, $a_0 \neq 0$ be an n^{th} degree polynomial with rational coefficients. Given that $\alpha + \sqrt{\beta}$ is a root of $P(x) = 0$. Divide $P(x)$ by

$$(x - (\alpha + \sqrt{\beta}))(x - (\alpha - \sqrt{\beta})) = (x - \alpha)^2 - \beta.$$

. Let $Q(x)$ be the quotient and $Ax + B$ be the remainder. Then, Proceeding exactly as in the previous theorem, we get

$$Ax + B = 0.$$

Thus we conclude that $\alpha - \sqrt{\beta}$ is a root of $P(x) = 0$. □

Example 4.1.3. *If the equation*

$$x^4 - 5x^3 + 4x^2 + 8x - 8 = 0 \tag{4.1.3}$$

has a root $1 - \sqrt{5}$, then find all remaining roots.
Answer:
Since $1 - \sqrt{5}$ is a root of equation (4.1.3), then $1 + \sqrt{5}$ is also root. This implies that

$$(x - (1 - \sqrt{5}))(x - (1 + \sqrt{5})) = (x - 1)^2 - 5 = x^2 - 2x - 4$$

is a factor of the given polynomial.
Dividing the given polynomial by this factor, we obtain the other factor as $x^2 - 3x + 2 = (x - 2)(x - 1)$. Thus the roots of the given polynomial are

$$1 - \sqrt{5},\ \ 1 + \sqrt{5},\ \ 1,\ \ \textit{and}\ 2.$$

Theorem 4.1.5. *If the rational number $\frac{p}{q}$, a fraction in its lowest terms (so that p, q are integers prime to each other, $q \neq 0$) is a root of the equation $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$, where $a_0, a_1, \ldots, a_n$ are integers and $a_0 \neq 0$, then p is a divisor of a_n and q is a divisor of a_0.*

Proof. Since $\frac{p}{q}$ is a root of the given polynomial equation, we have

$$a_0(\frac{p}{q})^n + a_1(\frac{p}{q})^{n-1} + a_2(\frac{p}{q})^{n-2} + \ldots + a_n = 0.$$

Multiplying by q^n, we get

$$a_0p^n + a_1p^{n-1}q + a_2p^{n-2}q^2 + \ldots + a_nq^n = 0. \tag{4.1.4}$$

Dividing by p, we have

$$a_0p^{n-1} + a_1p^{n-2}q + a_2p^{n-3}q^2 + \ldots + a_{n-1}q^{n-1} = -\frac{a_nq^n}{p}.$$

Now, the left side of the above equation is an integer and therefore $-\frac{a_nq^n}{p}$ is also must be an integer. Since p and q have no common factor, p must be a divisor of a_n.
Also, from (**??**),

$$a_1p^{n-1}q + a_2p^{n-2}q^2 + \ldots + a_nq^n = -a_0p^n.$$

Dividing this expression by q, we get

$$a_1p^{n-1} + a_2p^{n-2}q + \ldots + a_nq^{n-1} = -\frac{a_0p^n}{q}.$$

Since the left side is an integer and since q does not divide p, q must be a divisor of a_0. □

Corollary 4.1.6. *Every rational root of the equation $x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$, where each a_i is an integer must be an integer. Moreover, every such root must be a divisor of the constant a_n.*

Example 4.1.4. *Find a polynomial equation of the lowest degree with rational coefficients having* $\sqrt{3}$ *and* $1-2i$ *as two of its roots.*

Answer:

Since $\sqrt{3}$ *and* $1-2i$ *are two roots, then* $-\sqrt{3}$ *and* $1+2i$ *are also roots of the required polynomial equation. Therefore the desired equation is given by*

$$(x-\sqrt{3})(x+\sqrt{3})(x-(1-2i))(x-(1+2i))=0$$

i.e.,

$$x^4-2x^3+2x^2+6x-15=0.$$

Example 4.1.5. *Solve* $4x^5+x^3+x^2-3x+1=0$ *given that it has rational roots.*

Answer:

By theorem 4.1.5, any rational root $\dfrac{p}{q}$ *(in its lowest terms) must satisfy the condition that,* p *is divisor of* 1 *and* q *is positive divisor of* 4*. So the possible rational roots are* ± 1, $\pm\frac{1}{2}$ *and* $\pm\frac{1}{4}$*. Note that* $P(-1)=0$ *and* $P(\frac{1}{2})=0$*. But* $P(1)\neq 0$, $P(\frac{1}{4}\neq 0$, $P(-\frac{1}{4}\neq 0$ *and* $P(-\frac{1}{2})\neq 0$*, we see that* $(x+1)$ *and* $(x-\frac{1}{2})$ *are factors of the given polynomial. Also by factorizing , we find that*

$$P(x)=(x-\frac{1}{2})(x+1)(4x^3-2x^2+4x-2).$$

Note that $x=\frac{1}{2}$ *is a root of the third factor, if we divide* $4x^3-2x^2+4x-2$ *by* $x-\frac{1}{2}$*, we obtain*

$$\begin{aligned}P(x)&=(x-\frac{1}{2})^2(x+1)(4x^2+4)\\&=4(x-\frac{1}{2})^2(x+1)(x^2+1).\end{aligned}$$

Hence the roots of $P(x)=0$*, are* -1, $\frac{1}{2}$, $\frac{1}{2}$ *and* $\pm i$*.*

Multiple Roots

If a root α of $P(x)=0$ repeats r times, then α is called an r-multiple root. A 2- multiple root is usually called a double root.

For example, consider $P(x)=(x-2)^3(x-5)^2(x+1)$. Here 2 is a 3-multiple root, 5 is a double root, and -1 is a single root of the equation $P(x)=0$.

Theorem 4.1.7. *If α is r-multiple root of $P(x) = 0$ then α is an $(r-1)$-multiple root of $P'(x) = 0$, where $P'(x)$ is the derivative of $P(x)$.*

Proof. Given that α is an r-multiple root of $P(x) = 0$. Then

$$P(x) = (x - \alpha)^r \varphi(x),$$

where $\varphi(\alpha) \neq 0$. Now, by applying product rule of differentiation, we obtain:

$$\begin{aligned} P'(x) &= (x-\alpha)^r \varphi'(x) + r(x-\alpha)^{r-1}\varphi(x) \\ &= (x-\alpha)^{r-1}[(x-\alpha)\varphi'(x) + r\varphi(x)]. \end{aligned}$$

When $x = \alpha$, then

$$(x-\alpha)\varphi'(x) + r\varphi(x) = r\varphi(\alpha) \neq 0.$$

Then α is an $(r-1)$-multiple root of $P'(x) = 0$. □

Remark 4.1.2. If α is $(r-1)$-multiple root of $P'(x) = 0$ then α is an $(r-2)$-multiple root of $P''(x) = 0$ and so on.

Example 4.1.6. *Solve $x^3 - x^2 - 8x + 12 = 0$, given that has a double root.*
Answer:
Let $P(x) = x^3 - x^2 - 8x + 12 = 0$. Differentiating, we obtain:

$$P'(x) = 3x^2 - 2x - 8.$$

Since the multiple roots of $P(x) = 0$ are also the roots of $P'(x) = 0$, the product of the factors corresponding to these roots will be the g.c.d of $P(x)$ and $P'(x)$. Let us find the g.c.d of $P(x)$ and $P'(x)$.

$3x^2-2x-8$	$x^3-x^2-8x+12$
$3x^2-6x$	3
$4x-8$	$3x^3-3x^2-24x+36$
$4x-8$	$3x^2-2x-8$
0	$-x^2-16x+36$
	3
	$-3x^2-48x+108$
	$-3x^2+2x+8$
	$-50x+100$
	$-50(x-2)$

Therefore, g.c.d $=(x-2)$*. This implies that* $P(x)$ *has a factor* $(x-2)^2$*. Also,* $P(x)=(x-2)^2(x+3)$*. Thus the roots are* $2,\ 2,\ -3$*.*

Exercise 4.1.1. *(1) If the equation*

$$2x^4-8x^3+27x^2-4x+13=0$$

has a root $2-3i$*, the find all remaining roots.*

(2) Find all roots of the equation

$$x^4-6x^3+11x^2-2x-10=0.$$

Given that $1+\sqrt{3}$ *is a root of it.*

4.2 Relation between the Roots and Coefficients of a Polynomial Equation

Consider the polynomial function $P(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n\neq 0$, $a_0\neq 0$. Let $\alpha_1,\alpha_2,\ldots,\alpha_n$ be the roots of $P(x)=0$. Then we can write

$$P(x)=a_0(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n).$$

Equating the two expressions for $P(x)$, we obtain:

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = a_0(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n).$$

Dividing both sides by a_0,

$$\begin{aligned} x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + \ldots + \frac{a_n}{a_0} &= (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n) \\ &= x^n - S_1x^{n-1} + S_2x^{n-2} - \ldots + (-1)^nS_n, \end{aligned}$$

where S_r stands for the sum of the products of the roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ taken r at a time.
Comparing the coefficients on both sides, we see that

$$S_1 = -\frac{a_1}{a_0},\ S_2 = \frac{a_2}{a_0},\ \ldots,\ S_n = (-1)^n\frac{a_n}{a_0}.$$

Special Cases
If α and β are the roots of $ax^2 + bx + c = 0$, $(a \neq 0)$, then $\alpha + \beta = \frac{-b}{a}$ and $\alpha\beta = \frac{c}{a}$.
If α, β and γ are the roots of $ax^3+bx^2+cx+d = 0$, $(a \neq 0)$, then $\alpha+\beta+\gamma = \frac{-b}{a}$, $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$ and $\alpha\beta\gamma = \frac{-d}{a}$.

Example 4.2.1. *If the roots of the equation $x^3 + px^2 + qx + r = 0$ are in arithmetic progression, show that*

$$2p^3 - 9pq + 27r = 0.$$

Answer:
Let the roots of the given equation be ad, a and $a + d$. Then

$$S_1 = ad + a + a + d = 3a = -p \implies a = \frac{-p}{3}.$$

Since a is a root, it satisfies the given polynomial

$$(\frac{-p}{3})^3 + p(\frac{-p}{3})^2 + q(\frac{-p}{3}) + r = 0.$$

On simplification, we obtain

$$2p^3 - 9pq + 27r = 0.$$

Example 4.2.2. *Solve* $27x^3 + 42x^2 - 28x - 8 = 0$, *given that its roots are in geometric progression.*

Answer:

Let the roots be $\frac{a}{r}$, a, ar. *Then,*

$$(\frac{a}{r})(a)(ar) = a^3 = \frac{8}{27} \implies a = \frac{2}{3}.$$

Since $a = \frac{2}{3}$ *is a root, then* $(x - \frac{2}{3})$ *is a factor. On division, the other factor of the polynomial is* $27x^2 + 60x + 12$.

Its roots are

$$\frac{-60 \pm \sqrt{(60)^2 - (4)(27)(12)}}{2 \times 27} = \frac{-2}{9} \ or \ -2.$$

Hence the roots of the given polynomial equation are $\frac{-2}{9}$, -2 *and* $\frac{2}{3}$.

Exercise 4.2.1. *Answer to the following*

(1) *Show that the roots of the equation* $ax^3 + bx^2 + cx + d = 0$ *are in geometric progression, then* $c^3 a = b^3 d$.

(2) *Solve the equation* $x^3 - 3x^2 - 6x + 8 = 0$ *if its root are arithmetic progression.*

(3) *Solve the equation* $x^3 - 3x^2 - 4x + 12 = 0$ *if the sum of two roots are equal to zero.*

(4) *Solve the equation* $x^4 + 3x^3 - 6x^2 - 28x - 24 = 0$ *if it has a root repeated 3-times.*

4.3 Cardan's Method of Solving a Standard Cubic Equation

In this section we describe some methods to find one root of the cubic equation

$$ax^3 + bx^2 + cx + d = 0. \tag{4.3.1}$$

so that other two roots (real or complex) can then be found by polynomial division and the quadratic formula. The solution proceeds in two steps. First, the cubic equation is depressed; then one solves the depressed cubic.

Depressing the cubic equation

To remove the second term of Eq. (4.3.1), we diminish the roots of (4.3.1) by $h = -\frac{a_1}{na_0}$ with $n = 3$ (i.e., the degree of the polynomial equation), $a_0 = a$, $a_1 = b$, so that $h = -\frac{b}{3a}$. Set $x = y + h = y - \frac{b}{3a}$.

We apply the substitution $x = y - \frac{b}{3a}$ to the cubic equation (4.3.1), and obtain

$$a(y - \frac{b}{3a})^3 + b(y - \frac{b}{3a})^2 + c(y - \frac{b}{3a}) + d = 0.$$

Multiplying out and simplifying, we obtain

$$ay^3 + (c - \frac{b^2}{3a})^2 + (d + \frac{2b^3}{27a^2} - \frac{bc}{3a}) = 0$$

a cubic equation in which y^2-term is absent.

Solving the Depressed Cubic

Method to solve a depressed cubic equation of the form

$$y^3 + Ay = B. \tag{4.3.2}$$

The procedure is as follows:

First find s and t so that

$$3st = A, \tag{4.3.3}$$

and

$$s^3 - t^3 = B. \tag{4.3.4}$$

Then $y = s - t$ will be a solution of the depressed cubic. This can be verified as follows: Substituting for A, B and y, Eq. (4.3.2) gives

$$(s - t)^3 + 3st(s - t) = s^3 - t^3.$$

This is true since we can simplify the left side using the binomial formula to obtain $s^3 - t^3$.
Now to find s and t satisfying (4.3.3) and (4.3.4), we proceed as follows: From Eq.(4.3.3) , we have

$$s = \frac{A}{3t},$$

and substituting this into Eq.(4.3.4), we obtain,

$$(\frac{A}{3t})^3 - t^3 = B.$$

Simplifying, this turns into the tri-quadratic equation,

$$t^6 + Bt^3 - \frac{A^3}{27} = 0,$$

which using the substitution $u = t^3$ becomes the quadratic equation,

$$u^2 + Bu - \frac{A^3}{27} = 0.$$

From this, we can find a value for u by the quadratic formula, then obtain t, afterwards s. Hence the root $s - t$ can be obtained.

Example 4.3.1. *Using Cardan's Method to solve the equation*

$$2x^3 - 30x^2 + 162x - 350 = 0. \tag{4.3.5}$$

Answer:
Comparing Eq. (4.3.5) *with* (4.3.1)*, we have*

$$a = 2,\ b = -30,\ c = 162, \ \ and \ \ d = -350.$$

Hence substituting $x = y - \frac{b}{3a} = y + 5$ in (4.3.5), expanding and simplifying, we obtain the depressed cubic equation

$$y^3 + 6y - 20 = 0.$$

Now to find the solution of depressed equation $y^3 + 6y = 20$, we proceed as follows:
We need s and t to satisfy

$$3st = 6, \tag{4.3.6}$$

and

$$s^3 - t^3 = 20. \tag{4.3.7}$$

Solving for s in (4.3.6) and substituting the result into (4.3.7) yields:

$$\frac{8}{t^3} - t^3 = 20,$$

which multiplied by t^3 becomes,

$$t^6 + 20t^3 - 8 = 0.$$

Using the substitution $u = t^3$ the above becomes the quadratic equation

$$u^2 + 20u - 8 = 0.$$

Using the quadratic formula, we obtain that

$$u = -10 \pm \sqrt{108}.$$

We take the cube root of the positive value of u and obtain

$$t = \sqrt[3]{-10 + \sqrt{108}}.$$

By Eq. (4.3.7)we obtain

$$s^3 = t^3 + 20 = 10 + \sqrt{108}$$

and hence

$$s = \sqrt[3]{10 + \sqrt{108}}.$$

Hence a root y for the depressed cubic equation is

$$y = s - t = \sqrt[3]{10 + \sqrt{108}} - \sqrt[3]{-10 + \sqrt{108}}.$$

Hence a root of the original cubic Eq. (4.3.5) is s given by

$$x = y + 5 = \sqrt[3]{10 + \sqrt{108}} - \sqrt[3]{-10 + \sqrt{108}} + 5.$$

Exercise 4.3.1. *Solve the following equations by Cardan's Method*

(i) $x^3 + 6x^2 + 3x + 18 = 0$

(ii) $x^3 + 9x^2 + 18x + 28 = 0$

(iii) $x^3 - 2x - 5 = 0$

(iv) $x^3 - 6x^2 + 3x - 2 = 0$

4.4 Ferrari's Method of Solving a Quartic Equations

Shortly after the discovery of a method to solve the cubic equation, Lodovico Ferrari, a student of Cardano, found a similar method to solve the quartic

$$x^4 + ax^3 + bx^2 + cx + d = 0. \tag{4.4.1}$$

In this method the solution of the quartic depends on the solution of a cubic. We now describe the Ferrari's method.
Writing the quartic equation

$$u = x^4 + ax^3 + bx^2 + cx + d = 0.$$

We assume that

$$u = (x^2 + \frac{a}{2}x + r)^2 - (Ax + B)^2$$

Equating coefficients of like powers of x and then eliminating A and B we will obtain a cubic equation in r. Then corresponding values of A, B and will be obtained. In this case, we have

$$x^2 + \frac{a}{2}x + r = \pm(Ax + B).$$

That is

$$x^2 + \frac{a}{2}x + r = Ax + B \tag{4.4.2}$$

and

$$x^2 + \frac{a}{2}x + r = -Ax + B. \tag{4.4.3}$$

If the roots of (4.4.2) are x_1 and x_2, and the roots of (4.4.3) are x_3 and x_4, then the roots of (4.4.1) are x_1, x_2, x_3 and x_4.

Example 4.4.1. *Solve* $x^4 - 2x^3 - 12x^2 + 10x + 3 = 0.$

Answer:

Let $u = x^4 - 2x^3 - 12x^2 + 10x + 3 = 0$ *and assume that*

$$u = (x^2 - x + r)^2 - (Ax + B)^2.$$

Then,

$$x^4 - 2x^3 - 12x^2 + 10x + 3 = (x^2 - x + r)^2 - (Ax + B)^2.$$

$$\Longrightarrow x^4 - 2x^3 - 12x^2 + 10x + 3 = x^4 - 2x^3 + (-A^2 + 2r + 1)x^2 - (2AB + 2r)x + (r^2 - B^2).$$

Equating coefficients of like powers of x*, we obtain*

$$-12 = -A^2 + 2r + 1,$$

$$10 = -2AB - 2r,$$

and

$$3 = r^2 - B^2$$

$$\Longrightarrow A^2 = 12 + 2r + 1,\ AB = -r - 5,\ B^2 = r^2 - 3.$$

Since $A^2B^2 = (AB)^2$, *then*

$$(12 + 2r + 1)(r^2 - 3) = -r - 5$$

It follows that

$$r^3 + 6r^2 - 8r - 32 = 0. \tag{4.4.4}$$

*(**Remark:***

- *Equation* (4.4.4), *which is a cubic in* r *, is known as the reducing cubic.*
- *The reducing cubic gives three values of* r *. These do not however lead to three different sets of roots for the quartic equation. They only give three different methods of factorizing the left hand side of the quartic. Hence it is enough to find any one root of the reducing cubic.)*

By inspection, $r = -2$ *is a root of the cubic equation* (4.4.4)*. Hence,*

$$A^2 = 12 + 2r + 1 = 12 - 4 + 1 = 9,$$

$$B^2 = r^2 - 3 = 4 - 3 = 1$$

and

$$AB = -r - 5 = 2 - 5 = -3.$$

We take $A = 3$*, then* $B = -1$*. Therefore,*

$$x^2 - x - 2 = 3x - 1$$

$$\Longrightarrow x^2 - 4x + 1 = 0$$

$$\Longrightarrow x^2 - 4x - 1 = 0$$

$$\Longrightarrow x = \frac{4 \pm \sqrt{16 + 4}}{2} = 2 \pm \sqrt{5}.$$

Also,

$$x^2 - x - 2 = -3x + 1$$

$$\Longrightarrow x^2 + 2x - 3 = 0$$

$$\Longrightarrow (x-1)(x+3) = 0$$

$$\Longrightarrow x = 1 \; or \; 3.$$

Consequently, the roots of quartic equation $1,\ 3,\ 2+\sqrt{5}$ *and* $2-\sqrt{5}$.

Exercise 4.4.1. *Solve the following equations by Ferrari's Method*

(i) $x^4 - 2x^3 - 11x^2 + 4x + 3 = 0$

(ii) $x^4 - 2x^3 - x^2 - 10x - 3 = 0$

(iii) $x^4 + 3x^3 + x^2 - 2 = 0$

CHAPTER 5
Matrices and solving system of linear equations

Chapter 5

Matrices and solving system of linear equations

In this chapter, We introduce matrix notation to describe the original system.

5.1 Matrices and matrix algebra

Definition 5.1.1. A matrix is a rectangular array of numbers and/or variables. An $m \times n$ matrix is a rectangular array of mn real numbers or complex arranged in m horizontal rows and n vertical column.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = (a_{ij}),$$

where $i = 1, 2, \dots, m$ and $j = i = 1, 2, \dots, n$.

For instance,

$$A = \begin{pmatrix} 1 & -1 & 4 & 0 & 2 \\ 3 & 1 & 2 & -1 & 5 \\ -2 & 3 & -1 & 2 & 3 \end{pmatrix}$$

is a matrix with 3 rows and 5 columns (a 3×5 matrix). We indicate the fact that A is 3×5 (this is read as "three by five") by writing $A_{3\times 5}$. Matrices can also be enclosed in square brackets as

$$A = \begin{bmatrix} 1 & -1 & 4 & 0 & 2 \\ 3 & 1 & 2 & -1 & 5 \\ -2 & 3 & -1 & 2 & 3 \end{bmatrix}.$$

Real numbers are 1×1 matrices. A vector such as

$$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

is a 3×1 matrix. We will generally use upper case Latin letters such as A, B, C, ... as symbols for general matrices, boldface lower case letters for the special case of vectors, and ordinary lower case letters for real numbers.

Example 5.1.1. *construct the matrix $A = (a_{ij})_{4\times3}$ such that $a_{ij} = i + j$.*
Answer:

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{pmatrix}.$$

Example 5.1.2. *construct the matrix $A = (a_{ij})_{4\times4}$ such that*

$$a_{ij} = \{ \begin{array}{ll} 0 & if\, i < j \\ 2 & if\, i \geq j \end{array}.$$

Answer:

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 2 \end{pmatrix}.$$

Exercise 5.1.1. *construct the matrix A such that* *(i)* $A = (a_{ij})_{4\times 5}$ *such that*

$$a_{ij} = \begin{cases} i+j & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases}$$

(ii) $A = (a_{ij})_{4\times 4}$ *such that*

$$a_{ij} = \begin{cases} i+j & \text{if } i > j \\ i-j & \text{if } i \leq j \end{cases}$$

5.1.1 Algebra of matrices

Definition 5.1.2. The $m \times n$ matrix whose entries are all 0 is denoted $0_{m\times n}$ (or, more often, just by O if the dimensions are obvious from context). It's called the **zero matrix**.

Definition 5.1.3. Two matrices A and B are **equal** if all their corresponding entries are equal:

$$A \Longleftrightarrow B \;\; a_{ij} = b_{ij}, \text{ for all } i,\, j.$$

Example 5.1.3. *If* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ *find the values of* a, b, c *and* d.

Answer:

$a = 1$, $b = 2$, $c = 3$ *and* $d = 4$.

Definition 5.1.4. If the matrices A and B have the same size, then their sum is the matrix C defined by

$$C = A + B = (c_{ij}),\;\; C_{ij} = a_{ij} + b_{ij}.$$

Their difference is the matrix $A - B$ defined by

$$(A - B)_{ij} = a_{ij} - b_{ij}.$$

Example 5.1.4. *Let* $A = \begin{pmatrix} 1 & -1 & 3 \\ 4 & 0 & -2 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 2 \end{pmatrix}$. *Find* $A + B$ *and* $A - B$.

Answer:

$$
\begin{aligned}
A+B &= \begin{pmatrix} 1 & -1 & 3 \\ 4 & 0 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 1+1 & -1+0 & 3+(-2) \\ 4+1 & 0+1 & -2+2 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -1 & 1 \\ 5 & 1 & 0 \end{pmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
A-B &= \begin{pmatrix} 1 & -1 & 3 \\ 4 & 0 & -2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 1-1 & -1-0 & 3-(-2) \\ 4-1 & 0-1 & -2-2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 & 5 \\ 3 & -1 & -4 \end{pmatrix}.
\end{aligned}
$$

The fundamental properties of matrix addition:

(1) $A+B=B+A$ (Commutative)

(2) $A+(B+C)=(A+B)+C$ (Associative)

(3) $A+O=O+A$ (Identity property)

where A, B, C and O are matrices, which have the same size.

Definition 5.1.5. A matrix A can be multiplied by a scalar c to obtain the matrix cA, where

$$cA=(ca_{ij}).$$

This is called scalar multiplication. We just multiply each entry of A by c.

Example 5.1.5. *If* $A = \begin{pmatrix} 1 & -1 & 3 \\ 4 & 0 & -2 \\ 1 & 1 & 2 \end{pmatrix}$, *then find* $2A$

$$2A = 2\begin{pmatrix} 1 & -1 & 3 \\ 4 & 0 & -2 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 6 \\ 8 & 0 & -4 \\ 2 & 2 & 4 \end{pmatrix}.$$

The fundamental properties of matrix multiplication:

(1) If AB and AC are defined, then $A(B \pm C) = AB \pm AC$.

(2) If AB is defined, and c is a scalar, then $A(cB) = c(AB)$.

(3) $(AB)C = (AB)C$.

(4) $(B \pm C)A = BA \pm CA$.

(5) $(a \pm b)C = aC \pm bC$.

(6) $(ab)C = a(bC)$.

Definition 5.1.6. Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then the product AB is defined by

$$AB = (c_{ij}, \; c_{ij} = \sum_{s=1}^{k} a_{is} b_{sj}.$$

Here k is the number of columns of A or rows of B.
If AB is defined, then the number of rows of AB is the same as the number of rows of A, and the number of columns is the same as the number of columns of B:

$$A_{m\times n} B_{n\times p} = (AB)_{m\times p}.$$

Example 5.1.6. *Find the product* AB *if* $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{pmatrix}$ *and* $B = \begin{pmatrix} 3 & 5 \\ 4 & 2 \\ 5 & 7 \end{pmatrix}$.

Answer:

$$\begin{aligned} AB &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 4 & 2 \\ 5 & 7 \end{pmatrix} \\ &= \begin{pmatrix} (1)(3)+(2)(2)+(3)(5) & (1)(5)+(2)(4)+(3)(7) \\ (2)(3)+(4)(2)+(5)(5) & (2)(5)+(4)(4)+(5)(7) \end{pmatrix} \\ &= \begin{pmatrix} 22 & 34 \\ 39 & 61 \end{pmatrix}. \end{aligned}$$

Example 5.1.7. *Find the product AB if $A = \begin{pmatrix} 2 & 1 & -6 \\ 1 & -3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 4 & 2 \\ -2 & 1 & 1 \end{pmatrix}$.*

Answer:

$$\begin{aligned} AB &= \begin{pmatrix} 2 & 1 & -6 \\ 1 & -3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 4 & 2 \\ -2 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 14 & -2 & -10 \\ -3 & -10 & -7 \end{pmatrix}. \end{aligned}$$

Remark 5.1.1. The product of matrices are not commutative. i.e., $AB \neq BA$.

Exercise 5.1.2. *Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{pmatrix}$, $D = \begin{pmatrix} 3 & -2 \\ 2 & 4 \end{pmatrix}$, $E = \begin{pmatrix} 2 & -4 & 5 \\ 0 & 1 & 4 \\ 2 & 1 & 3 \end{pmatrix}$, and $F = \begin{pmatrix} 3 & -1 \\ 4 & 1 \end{pmatrix}$, If possible compute*

(i) $C + E$

(ii) AB *and* BA

(iii) $2D - 3F$

(iv) $AB + DF$

(v) $CB + D$

(vi) $A(BD)$ *and* $(AB)D$

(vii) $EF + 2A$

5.1.2 Properties and types of matrices

Definition 5.1.7. The transpose of the matrix $A_{m\times n}$, denoted by $A^t_{n\times m}$, is obtained from A by making the first row of A into the first column of A^t, the second row of A into the second column of A^t, and so on. Formally,

$$a^t_{ij} = a_{ji}.$$

Example 5.1.8. *If* $A = \begin{pmatrix} 1 & 5 \\ 0 & 4 \\ -2 & 1 \end{pmatrix}$, *find* A^t*?*

Answer:
Since A *is a* 3×2 *matrix, then* A^t *is a* 2×3 *matrix and it is given by*

$$A^t = \begin{pmatrix} 1 & 0 & -2 \\ 5 & 4 & 1 \end{pmatrix}.$$

Properties of transpose of the matrix:

(1) $(A^t)^t = A$.

(2) $(A + B)^t = A^t + B^t$.

(3) $(kA)^t = kA^t$, where k is scalar number.

(4) $(AB)^t = B^tA^t$.

Definition 5.1.8. We say that A is a **square matrix** if it has the same number of rows and columns.

An important instance is the identity matrix I_n, which has ones on the main diagonal and zeros elsewhere:

$$I = I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Definition 5.1.9. The square matrix $A = (a_{ij})$ is called **upper triangle matrix** if

$$a_{ij} = 0, \text{ for } \ i > j.$$

That is

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}.$$

Definition 5.1.10. The square matrix $A = (a_{ij})$ is called **lower triangle matrix** if

$$a_{ij} = 0, \text{ for } \ i < j.$$

That is

$$A = \begin{pmatrix} a_{11} & 0 & \dots & & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ a_{n1} & a_{n2} & \dots & & a_{nn} \end{pmatrix}.$$

Definition 5.1.11. The square matrix $A = (a_{ij})$ is called **diagonal matrix** if

$$a_{ij} = 0, \text{ for } \ i \neq j.$$

That is

$$A = \begin{pmatrix} a_{11} & 0 & \dots & & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \dots & & a_{nn} \end{pmatrix}.$$

Definition 5.1.12. The square matrix $A = (a_{ij})$ is called **scalar matrix** if

$$a_{ij} = \begin{cases} 0 & \text{for} \quad i \neq j \\ k & \text{for} \quad i = j. \end{cases}$$

That is

$$A = \begin{pmatrix} k & 0 & \dots & & 0 \\ 0 & k & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \dots & & k \end{pmatrix},$$

where k is scalsr.

Example 5.1.9. *construct the matrix $A = (a_{ij})$ of type 2×2, if*

$$a_{ij} = \begin{cases} 0 & \textit{for} \quad i \neq j \\ -2 & \textit{for} \quad i = j. \end{cases}$$

Answer:

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Definition 5.1.13. The square matrix $A = (a_{ij})$ is called **symmetric matrix** if $A = A^t$. That is A is symmetric if $a_{ij} = a_{ji}$, and the elements of A are symmetric with respect to the main diagonal of A

Definition 5.1.14. The square matrix $A = (a_{ij})$ is called **skew-symmetric matrix** if $A = -A^t$.

Example 5.1.10. *Let* $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & -5 \\ -3 & 5 & 0 \end{pmatrix}$ *. Then,*

$$A^t = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{pmatrix},$$

and

$$B^t = \begin{pmatrix} 0 & -2 & -3 \\ 2 & 0 & 5 \\ 3 & -5 & 0 \end{pmatrix}.$$

Hence $A = A^t$ and $B = -B^t$. Therefore, A is symmetric matrix and B is skew-symmetric matrix.

Exercise 5.1.3. *Put the following matrices as sum of symmetric and skew-symmetric matrices*

(i) $\begin{pmatrix} 3 & 5 \\ 4 & -7 \end{pmatrix}$

(ii) $\begin{pmatrix} 10 & 1 & 8 \\ 9 & 3 & 11 \\ -6 & 7 & 12 \end{pmatrix}$

Exercise 5.1.4. *Show that* $A = \begin{pmatrix} 0 & \sqrt{2} & 2 \\ \sqrt{3} & \sqrt{2} & -1 \\ -\sqrt{3} & \sqrt{2} & 1 \end{pmatrix}$ *commutative with its transpose.*

Exercise 5.1.5. *Prove that* $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ *satisfies the equation*

$$A^2 - 4A - 5I = 0.$$

5.2 Determinants

In this section, we define the notion of determinant and study some of its properties.

Definition 5.2.1. Let $A = (a_{ij}$ be an $n \times n$ matrix, we define the determinant of A and denoted by $|A|$ or $\det(A)$,

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{vmatrix}.$$

Example 5.2.1. *If* $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, *then*

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}\,.$$

e.g

$$\begin{vmatrix} -1 & 2 \\ 4 & 3 \end{vmatrix} = (-1)(3) - (2)(4) = -3 - 8 = -11.$$

Exercise 5.2.1. *Find the determinants of the following matrices*

(i) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

(ii) $\begin{pmatrix} \sin x & \cos x \\ -\cos x & \sin x \end{pmatrix}$

Definition 5.2.2. Let $A = (a_{ij}$ be an $n \times n$ matrix and let $M = (m_{ij})$ be the $(n-1)\times(n-1)$ submatrix of A obtained by deleting the i^{th}-row and j^{th}-column of A. The cofactor A_{ij} of a_{ij} is defined by

$$A_{ij} = (-1)^{i+j}|M_{ij}|.$$

Example 5.2.2. *Let* $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ *be* 3×3 *matrix, then*

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

e.g.,

$$\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = (0) \begin{vmatrix} -6 & 9 \\ 6 & 1 \end{vmatrix} - (1) \begin{vmatrix} 3 & 9 \\ 2 & 1 \end{vmatrix} + 5 \begin{vmatrix} 3 & -6 \\ 2 & 6 \end{vmatrix}$$
$$= 0 - (3 - 18) + 5(18 + 12) = 15 + 150 = 165.$$

5.2.1 Properties of determinants

Let A be $n \times n$ matrix.

(1) The determinants of a matrix A and its transpose are equal

$$i.e., \ |A| = |A^t|.$$

(2) If matrix B results from matrix A by interchanging two rows (or two columns) of A, then
$$|B| = -|A|.$$

(3) If two rows (or two column) are equal, then

$$|A| = 0.$$

(4) If a row (or column) of A consists of zeros, then

$$|A| = 0.$$

(5) If $B = (b_{ij}$ is obtained from $A = (a_{ij}$by adding to each element of the r^{th} row (column) of A, a constant c times the corresponding element of its s^{th} row (column) $r \neq s$, then

$$|B| = |A|.$$

e.g.,

$$\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 13 & 7 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = 165.$$

(6) If a matrix $A = (a_{ij}$ is upper (lower) triangle matrix, then

$$|A| = a_{11}a_{22}a_{33}.$$

(7) The determinants of a product two matrices is the product of their determinants, that is

$$|AB| = |A||B|.$$

e.g., Let $A = \begin{pmatrix} -1 & 2 \\ 6 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$, Then

$$|A| = -13, \ |B| = -5.$$

Also,

$$AB = \begin{pmatrix} -1 & 2 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 9 & 13 \end{pmatrix}.$$

Therefore,

$$|AB| = 165 = |A||B|.$$

Exercise 5.2.2. *Find the determinants of the following matrices*

(i) $\begin{pmatrix} 2 & 7 & 8 \\ 3 & 2 & 4 \\ 2 & 7 & 8 \end{pmatrix}$

(ii) $\begin{pmatrix} 2 & -4 & 11 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}$

(iii) $\begin{pmatrix} 3 & -1 & 2 \\ 6 & -2 & 4 \\ 1 & 7 & 3 \end{pmatrix}$

5.3 solving system of linear equations

In this section, we shall solve the system of linear equations.

5.3.1 Non-homogenous linear equations

Consider m of non-homogenous linear equations in variables $x_1, x_2, \ldots, x_n$ such as

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

The system of given equations is equivalent to a matrix equation

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Or

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Denoting by,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Here, A is called **the coefficient matrix** of type $m \times n$, X is the **matrix of unknown** of type $n \times 1$ and B is the **the matrix of absolute terms** of type $m \times 1$. Then, we have

$$AX = B. \tag{5.3.1}$$

Definition 5.3.1. The square matrix A is called singular if $|A| = 0$, and it is called non-singular if $|A| \neq 0$.

First if the number of equations in the system (5.3.1) are equal to the number of unknowns i.e., $m = n$, then equation (5.3.1) can be solved by two different methods.

First method: Cramer's rule

If $AX = B$ is a system of n linear equations in n variables such that $\det(A) \neq 0$, then there is a unique solution of this system and it is given by

$$x_1 = \frac{\Delta_1}{\Delta},\ x_2 = \frac{\Delta_2}{\Delta},\ \dots,\ x_n = \frac{\Delta_n}{\Delta},$$

where Δ is the determinant $|A|$ of the non-singular coefficient matrix A and Δ_i, $i = 1, 2, \dots, n$ is a determinant obtained from $|A|$ by substituting column vector B for the i^{th} column of $|A|$.

Example 5.3.1. *Solve by Cramer's rule the system of the following equations*

$$x + y + z = 6,$$

$$x - y + z = 2,$$

$$2x + y - z = 1.$$

Answer:

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = (1)(0) - 1(-3) + (1)(3) = 6 \neq 0,$$

$$\Delta_1 = \begin{vmatrix} 6 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = (6)(0) - 1(-3) + (1)(3) = 6,$$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = (1)(-3) - 6(-3) + (1)(-3) = 12$$

and

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{vmatrix} = (1)(-3) - 1(-3) + (6)(3) = 18.$$

Then

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{6}{6} = 1,$$
$$x_2 = \frac{\Delta_2}{\Delta} = \frac{12}{6} = 2,$$
$$x_3 = \frac{\Delta_3}{\Delta} = \frac{18}{6} = 3.$$

Second method by using the inverse matrix of matrix A

Definition 5.3.2. Suppose A and B are square matrices of the same dimension, and suppose that $AB = I = BA$. Then B is said to be the inverse of A, and we write this as $B = A^{-1}$ (In this case A is said to be invertible). Similarly, $B^{-1} = A$.

For instance, you can easily check that

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and so these two matrices are inverses of one another:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Theorem 5.3.1. *The square matrix A is invertible if and only if* $\det(A) \neq 0$.

The inverse of 2×2 matrix

To find the inverse of the non-singular matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we assume that $A^{-1} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Then,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\Longrightarrow ax + bz = 1, \;\; ay + bw = 0, \;\; cx + dz = 0, \;\; cy + dw = 1$$

By solving the above equations, we get

$$x = \frac{d}{ad - bc}, \; y = \frac{-b}{ad - bc}, \; z = \frac{-c}{ad - bc}, \; w = \frac{a}{ad - bc},$$

where $|A| = ad - bc \neq 0$. Therefore,

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Example 5.3.2. *Find the inverse of* $A = \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}$.

Answer:

Since $|A| = (1)(2) - (-1)(3) = 2 + 3 = 5 \neq 0$. *Then A is invertible and*

$$A^{-1} = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}$$

.

The inverse of $n \times n$ matrix

To calculate the inverse of $n \times n$ matrix, we use two different methods.

The first method: Inverse of a matrix in terms of its adjoint.

Consider a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

If $|A| \neq 0$, then the inverse of A is given by

$$A^{-1} = \frac{1}{|A|}\,\mathrm{adj}(A),$$

Where

$$\mathrm{adj}(A) = \begin{pmatrix} A_{11} & -A_{21} & \dots & (-1)^{n+1}A_{n1} \\ -A_{12} & A_{22} & \dots & (-1)^{n+1}A_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ (-1)^{n+1}A_{1n} & (-1)^{n+2}A_{2n} & \dots & A_{nn} \end{pmatrix}.$$

Example 5.3.3. *Find the inverse of* $A = \begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 1 \\ -2 & 1 & 5 \end{pmatrix}$.

Answer:

$$\because |A| = \begin{vmatrix} -1 & 1 & 2 \\ 0 & 0 & 1 \\ -2 & 1 & 5 \end{vmatrix} = -1 \neq 0.$$

Then,

$$A_{11} = \begin{vmatrix} 0 & 1 \\ 1 & 5 \end{vmatrix} = -1, \ A_{12} = \begin{vmatrix} 0 & 1 \\ -2 & 5 \end{vmatrix} = 2, \ A_{13} = \begin{vmatrix} 0 & 0 \\ -2 & 1 \end{vmatrix} = 0,$$

$$A_{21} = \begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} = 3, \ A_{22} = \begin{vmatrix} -1 & 2 \\ -2 & 5 \end{vmatrix} = -1, \ A_{23} = \begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix} = 1,$$

$$A_{31} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1, \ A_{32} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1, \ A_{33} = \begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} = 0.$$

Therefore,

$$\begin{aligned} A^{-1} &= (-1)\begin{pmatrix} -1 & -3 & 1 \\ -2 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 3 & -1 \\ 2 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

Exercise 5.3.1. *Find the inverse of the following materices*

(i) $\begin{pmatrix} -2 & -2 & 7 \\ 4 & 3 & -12 \\ -1 & 0 & 2 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

(ii) $\begin{pmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{pmatrix}$

Exercise 5.3.2. *Let* $A = \begin{pmatrix} 3 & 3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$. *Prove that*

$$A^3 = A^{-1}.$$

The second method: Elementary row operations

Let's write down the different "operations" we've been using on the systems of equations. These three operations are called **elementary row operations**

(1) Interchange two rows

(2) Multiplying any rows by a scalar $c \neq 0$

(3) Replace any row of a matrix by that row plus a multiple of another row.

Definition 5.3.3. The leading entry of a matrix row is the first non-zero entry in the row, starting from the left. A row without a leading entry is a row of zeros.

Now, we write the matrix $(A|I)$ and by using elementary row operations, we obtain it in the form $(I|A^{-1})$.

Example 5.3.4. *Applying elementary row operations to find the inverse of*

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{pmatrix}.$$

Answer:

Let

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right)$$

The computation can be carried out as follows:

The leading entry of every non-zero row is a 1 *and then eliminate the elements above and below it.*

$$(-5)R_1 + R_3 \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right)$$

$$\frac{1}{2}R_2 \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right)$$

$$(-1)R_2 + R_1 \implies \left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right)$$

$$-\frac{1}{4}R_3 \implies \left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right)$$

$$-\frac{3}{2}R_3 + R_2 \implies \left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right)$$

$$-\frac{1}{2}R_3 + R_1 \implies \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ 0 & 1 & 0 & -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right).$$

Hence,

$$A^{-1} = \begin{pmatrix} \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{pmatrix}.$$

Verify that $A^{-1}A = AA^{-1} = I$?

Example 5.3.5. *Applying elementary row operations to find the inverse of*

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{pmatrix}.$$

Answer:

Let

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 5 & -2 & -3 & 0 & 0 & 1 \end{array}\right)$$

Then,

$$-R_1 + R_2 \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 4 & -1 & 1 & 0 \\ 5 & -2 & -3 & 0 & 0 & 1 \end{array}\right)$$

$$(-5)R_1 + R_3 \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 4 & -1 & 1 & 0 \\ 0 & -12 & 12 & -5 & 0 & 1 \end{array}\right)$$

$$(-3)R_2 + R_3 \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 4 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & -3 & 1 \end{array}\right)$$

Note that, the matrix in the left contains the row of zeros, so we must stop and in this case, we say that the matrix A is singular matrix.

Exercise 5.3.3. *Find the inverse of the following matrices (if possible)*

(i) $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$.

(ii) $A = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$.

(iii) $A = \begin{pmatrix} 2 & 1 & -2 \\ 3 & 4 & 6 \\ 7 & 6 & 2 \end{pmatrix}$.

Properties of the inverse product

Theorem 5.3.2. *The following statement are true*

(i) If B and C are two inverse of matrix A, then $B = C$.

(ii) Let A and B be two invertible matrices. Then

(a) AB is invertible,

(b) $(AB)^{-1} = B^{-1}A^{-1}$

(c) $(A^{-1})^{-1} = A$

(d) *For any non-zero number k, kA is invertible and* $(kA)^{-1} = \frac{1}{k}A^{-1}$.

Theorem 5.3.3. *Let A be an invertible matrix of type $n \times n$. Then for any matrix B of type $n \times 1$, the system of linear equations $AX = B$ has a unique solution, which is given by*

$$X = A^{-1}B.$$

Proof.

$$\because\ A(A^{-1}B) = B.$$

Then $X = A^{-1}B$ is a solution of $AX = B$. Assume that X_0 is another solution of $AX = B$. Then,

$$AX_0 = B.$$

Multiplying both sides by A^{-1}, we get

$$A^{-1}(AX_0) = A^{-1}B \implies IX_0 = A^{-1}B \implies X_0 = A^{-1}B.$$

$\square$

Example 5.3.6. *Solve the following system of non-homogenous linear equations*

$$x + y + z = 2,$$

$$2y + 3z = 3,$$

$$5x + 5y + z = 4.$$

Answer:

The given system can be written as

$$AX = B,$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \; X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

In Example 5.3.4*, we found the inverse of* A *and it is given by*

$$A^{-1} = \begin{pmatrix} \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{pmatrix}.$$

Then,

$$X = A^{-1}B = \begin{pmatrix} \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{5}{4} \\ -\frac{3}{4} \\ \frac{3}{2} \end{pmatrix}.$$

Example 5.3.7. *Solve the following system of linear equations*

$$x + 2y + 2z = 5,$$

$$2x + y - 2z = 0,$$

$$2x - 2y + z = 2.$$

Answer:

The given system can be written as

$$AX = B,$$

where

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}, \; X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

First we calculate the inverse of A *by elementary row operations Let*

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right)$$

Then,

$$R_2 - 2R_1,\ R_3 - 2R_1 \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right)$$

$$(-\frac{1}{3})R_2,\ R_1 - 2R_2,\ R_3 + 6R_2 \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 9 & 2 & -2 & 1 \end{array}\right)$$

$$\frac{1}{9}R_3 \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right)$$

$$(-2)R_3 + R_2,\ 2R_3 + R_1 \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & -\frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right)$$

Hence,

$$A^{-1} = \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & -1 & -2 \\ 2 & -2 & 1 \end{pmatrix}.$$

Therefore,

$$X = A^{-1}B = \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & -1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{2}{3} \\ \frac{4}{3} \end{pmatrix}.$$

Exercise 5.3.4. *Solve the following the systems of linear equations*

(i) $x + 2y + 3z = 10,\ 2x - 3y + z = 1,\ 3x + y - 2z = 9.$

(ii) $2x + 2y + z = 5,\ 2x + 3y + z = 1,\ 2x + y + 3z = 11.$

(iii) $x + y + 2z = -1,\ 2x - y + 2z = -4,\ 4x + y + 4z = -2.$

Second if the number of equations in the system (5.3.1) don't equal the number of unknowns i.e., $m \neq n$ and if $m = n$, $|A| = 0$, then equation (5.3.1) can be solved by using the rank of matrix.

Definition 5.3.4. The rank of the matrix A of type $m \times n$ is the largest number of non-singular rows in it.

We shall determine the number of non-singular rows in a matrix A by series of elementary row operations.

Example 5.3.8. *Find the rank of the matrix*

$$A = \begin{pmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ -1 & -3 & -4 & -3 \end{pmatrix}.$$

Answer:

$$R_2 - 3R_1 \Longrightarrow \begin{pmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ -1 & -3 & -4 & -3 \end{pmatrix}.$$

$$R_3 + R_1 \Longrightarrow \begin{pmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the rank of A is one.

Example 5.3.9. *Find the rank of the matrix*

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & -2 & -3 & -3 \\ 1 & 1 & 2 & 3 \end{pmatrix}.$$

Answer:

$$R_2 - R_1,\ R_3 - R_1,\ R_4 - R_1 \Longrightarrow \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 2 & -2 & 0 \\ 0 & -3 & -5 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{1}{2}R_2 \Longrightarrow \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & -5 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R_1 - R_2,\ R_3 + 3R_2 \Longrightarrow \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -8 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$-\frac{1}{8}R_3 \Longrightarrow \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{3}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$R_3 + R_2,\ (-3)R_3 + R_1 \Longrightarrow \begin{pmatrix} 1 & 0 & 0 & \frac{3}{4} \\ 0 & 1 & 0 & \frac{3}{4} \\ 0 & 0 & 1 & \frac{3}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the rank of A is 3.

Exercise 5.3.5. *Find the rank of the following matrices*

(i) $\begin{pmatrix} 1 & 3 & 4 \\ -2 & -6 & -8 \\ 3 & 9 & 12 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 3 & 4 \\ 2 & 4 & 5 \\ 3 & 1 & 2 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$

Definition 5.3.5. Let the system of non-homogenous linear equations be equivalent to the matrix

$$AX = B, \tag{5.3.2}$$

where A is of type $m \times n$, X be of type $n \times 1$ and B be of type $m \times 1$. The matrix $\mathbf{A}^* = (\mathbf{A}|\mathbf{B})$ is called the **augmented matrix** of the system.

Theorem 5.3.4. *Consider the system* (5.3.2). *The following statements are true*

(i) If rank(A) =*rank*(A^*) = *number of unknown variables, then Eq.* (5.3.2) *has a unique solution.*

(ii) If rank(A) =*rank*(A^*) < *number of unknown variables, then Eq.* (5.3.2) *has an infinite number of solutions.*

(iii) If rank(A) <*rank*(A^*)*, then Eq.* (5.3.2) *has no solution.*

Example 5.3.10. *Investigate the system of equations*

$$x_1 + 2x_2 + 3x_3 = 9,$$

$$2x_1 - x_2 + x_3 = 8,$$

$$3x_1 - \ x_3 = 3.$$

Answer:
The given system can be written as

$$AX = B,$$

where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{pmatrix}, \ B = \begin{pmatrix} 9 \\ 8 \\ 3 \end{pmatrix} \ and\, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

The augmented matrix A^* of this system is given by

$$A^* = (A|B) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right).$$

$$-2R_1+R_2,\ -3R_1+R_3 \Longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array}\right).$$

$$-\frac{1}{5}R_2 \Longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & -6 & -10 & -24 \end{array}\right).$$

$$6R_2+R_3,\ -2R_2+R_1 \Longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right).$$

$$-\frac{1}{4}R_3 \Longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right).$$

$$-R_3+R_2,\ -R_3+R_1 \Longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right).$$

Then, the rank(A) =rank$(A^*)=3$. Therefor, the system has the unique solution

$$x_1=2,\ x_2=-1 \text{ and } x_3=3.$$

Example 5.3.11. *Investigate the system of equations*

$$x_1+5x_2+4x_3=1,$$

$$2x_1+10x_2+8x_3=3,$$

$$3x_1+15x_2+12x_3=5.$$

Answer:

The given system can be written as

$$AX=B,$$

where

$$A = \begin{pmatrix} 1 & 5 & 4 \\ 2 & 10 & 8 \\ 3 & 15 & 12 \end{pmatrix}, \ B = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \ and\, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

The augmented matrix A^* of this system is given by

$$A^* = (A|B) = \left(\begin{array}{ccc|c} 1 & 5 & 4 & 1 \\ 2 & 10 & 8 & 3 \\ 3 & 15 & 12 & 5 \end{array}\right).$$

$$-2R_1 + R_2, \ -3R_1 + R_3 \Longrightarrow \left(\begin{array}{ccc|c} 1 & 5 & 4 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right).$$

Then, the rank$(A) = 1 <$rank$(A^*) = 3$. Therefor, the system has no solution.

Example 5.3.12. *Investigate the system of equations*

$$x_1 - 3x_2 + 2x_3 = -1,$$

$$x_1 + 9x_2 + 6x_3 = 3,$$

$$x_1 + 3x_2 + 4x_3 = 1.$$

Answer:

The given system can be written as

$$AX = B,$$

where

$$A = \begin{pmatrix} 1 & -3 & 2 \\ 1 & 9 & 6 \\ 1 & 3 & 4 \end{pmatrix}, \ B = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \ and\, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

The augmented matrix A^* of this system is given by

$$A^* = (A|B) = \left(\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 1 & 9 & 6 & 3 \\ 1 & 3 & 4 & 1 \end{array}\right).$$

$$-R_1 + R_2,\ -R_1 + R_3 \Longrightarrow \left(\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 12 & 4 & 4 \\ 0 & 6 & 2 & 2 \end{array}\right).$$

$$\frac{1}{12}R_2,\ \frac{1}{6}R_3 \Longrightarrow \left(\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} \end{array}\right).$$

$$-R_2 + R_3,\ 3R_2 + R_1 \Longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Then, the rank(A) =rank(A^*) $= 2 <$ number of unknown variables. Therefor, the system has an infinite number of solutions, and it is equivalent to the system

$$x_1 + 3x_3 = 0,\ 3x_2 + x_3 = 1$$

Let $x_3 = a$, then

$$x_1 = -3a,\ x_2 = \frac{1}{3}(1 - a).$$

Exercise 5.3.6. *Investigate the following systems of linear equations*

(i) $3x - y + z = 4,\ 2x + y - 2z = -1$ *and* $x - 2y + 3z = 5.$

(ii) $2x + 2y + 3z = 7,\ x - y = -1$ *and* $-x + 2y + z = 3.$

(iii) $x + 2y - z = 3,\ 3x - y + 2z = 1,\ 2x - 2y + 3z = 2$ *and* $x - y + z = -1.$

5.3.2 Homogeneous systems of linear equations

Definition 5.3.6. A homogeneous system of linear algebraic equations is one in which all the numbers on the right hand side are equal to 0:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0.$$

The system of given equations is equivalent to a matrix equation $AX = 0$.

Definition 5.3.7. Any non-zero solutions to $AX = 0$, if they exist, are called non-trivial solutions.

Theorem 5.3.5. *If A is of type $n \times n$, then $AX = 0$ has non-trivial solution if and only if A is singular.*

Example 5.3.13. *Solve the homogenous system of equations*

$$2x_1 + x_2 + 3x_3 = 0,$$

$$x_1 + 2x_2 \ = 0,$$

$$x_2 + x_3 = 0.$$

Answer:
The given system can be written as $AX = 0$, where

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Then,

$$A^* = (A|0) = \left(\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

$$R_1 \longleftrightarrow R_2 \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

$$-2R_1 + R_2 \implies \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

$$R_2 + R_3 \implies \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right)$$

$$\frac{1}{5}R_3 \implies \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$-3R_3 + R_2 \implies \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Then rank(A) =*rank*$(A^*) = 3$. *Therefor, the given system has a unique solution* $X = 0$.

Exercise 5.3.7. *Investigate the following systems of linear equations*

(i) $x + 2y + 3z = 0,\ -x + 3y + 2z = 0$ *and* $2x + y - 2z = 0$.

(ii) $x_1 + x_2 + x_3 + x_4 = 0,\ x_1 \quad + x_4 = 0$ *and* $x_1 + 2x_2 + x_3 = 0$.

(iii) $x + 2y - z = 0,\ x + y + z = 0$ *and* $5x + 7y + 4z = 0$.

General Exercises

General Exercises

(1) Find the general term, the greatest coefficient and the middle term in the expansion $(3+2y)^5$.

(2) Express the following as a sum of partial fractions

(i) $\dfrac{x}{x^2 - 1}$.

(ii) $\dfrac{x^6+2x-1}{(x^2+1)^2(x+1)^2}$

(iii) $\dfrac{x^4}{x^3 + 4x^2 + x - 2}$

(3) Discuss for convergence and divergence of the following series

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+5^n}$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{8n+\sqrt{n}}{n^{7/2}+n^2+5}$

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 2^{3n}}{3^{2n}}$

(4) Find an interval of convergence for the following series

$$\sum_{n=0}^{\infty} \frac{(n+1)(x-4)^n}{10^n}.$$

(5) Solve the equation

$$2x^4 - 8x^3 + 27x^2 - 4x + 13 = 0,$$

if it has $2 + 3i$ is a root.

(6) Solve the equation

$$2x^5 - 11x^4 + 19x^3 - 10x^2 + 4x - 8 = 0,$$

if it has a root 3 times.

(7) Solve the following systems of linear equation by two different methods

$$x + 2y + 4z = 31,$$

$$5x + y + 2z = 29,$$

$$3x - y + z = 10.$$

(8) Find the values of k such that the system of equations

$$x + 2y - 3z = 4,$$

$$3x - y + 5z = 2,$$

$$4x + y + (k^2 - 14)z = k + 2.$$

has

(i) unique solution and find it

(ii) an infinite solutions and then find them

(iii) no solution.

Part III

Part 3
Function of Several variabels

Introduction

Through our previous study of the function in one variable x

Which is in the form $y = f(x)$ Where x is the independent variable and y is the dependent variables varies depending on the variable x

In this section we define functions of more than one independent variable and discuss ways to graph them. Real-valued functions of several independent real variables are defined similarly to functions in the single-variable case. Points in the domain are ordered pairs (triples, quadruples, n-tuples) of real numbers, and values in the range are real numbers as we have worked with all along.

Definition: Suppose *D* is a set of n-tuples of real numbers

$(x_1, x_2, ..., x_n)$. **A real-valued function** f on D is a rule that assigns a unique (single) real number

$$w = f\,(x_1, x_2, ..., x_n)$$

to each element in *D.* The set *D* is the function's **domain**. The set of w-values taken on by *f* is the function's **range**. The symbol w is the **dependent variable** of *f,* and *f* is said to be a function of the *n* **independent variables** x_1 to x_n. We also call the x_j's the function's **input variables** and call w the function's **output variable**.

Functions of Two Variables

If f is a function of two independent variables, we usually call the independent variables x and y and the dependent variable z and we picture the domain of f as a region in the xy-plane (Figure .1).

If f is a function of three independent variables, we call the independent variables x , y , and z , and the dependent variable w , and we picture the domain as a region in space.

In applications, we tend to use letters that remind us of what the variables stand for. To say that the volume of a right circular cylinder is a function of its radius and height, we might write $V = f(r,h)$. To be more specific, we might replace the notation $f(r,h)$ by the formula that calculates the value of V from the values of rand h, and write $V = \pi r^2 h$. In either case, r and h would be the independent variables and V the dependent variable of the function.

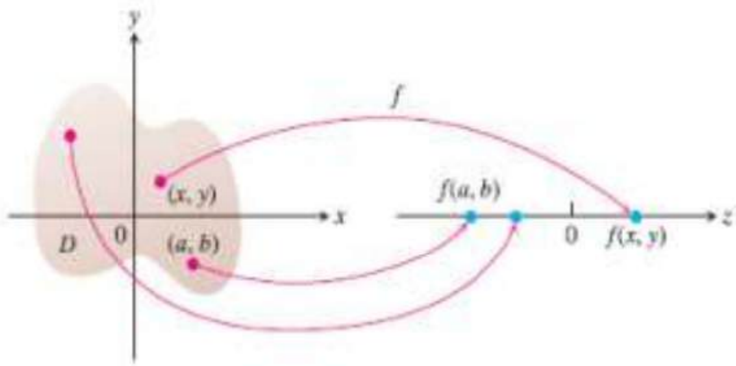

Figure .1.

Example (1)

The equation of the sphere in the space which have unit radius and its center is the origin has the form

$$x^2 + y^2 + z^2 = 1$$

So

$$z = \pm\sqrt{1 - x^2 - y^2}$$

If we take the positive sigin , then

$$z = \sqrt{1 - x^2 - y^2}$$

Represented the distance between any point on the upper surface of the ball and the horizontal plane xoy

Example (2):

These are functions of two variabels. Note the restricition that may apply to their domain in order to obtain a real value for the dpendent variable z

Function	Domain	Range
$z = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$z = \dfrac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$z = \sin xy$	Entire plane	$[-1, 1]$

(b) These are functions of three variables with restrictions on some of their domains.

Function	Domain	Range
$w = \sqrt{x^2 + y^2 + z^2}$	Entire space	$[0, \infty)$
$w = \dfrac{1}{x^2 + y^2 + z^2}$	$(x, y, z) \neq (0, 0, 0)$	$(0, \infty)$
$w = xy \ln z$	Half-space $z > 0$	$(-\infty, \infty)$

Example (3): Describe the domain of the function $z = \sqrt{y - x^2}$

Solution: Since f is defined only where $y - x^2 \geq 0$, the domain is the closed, unbounded region shown in Figure 2. The parabola

$y = x^2$ is the boundary of the domain. The points above the parabola

make up the domain's interior

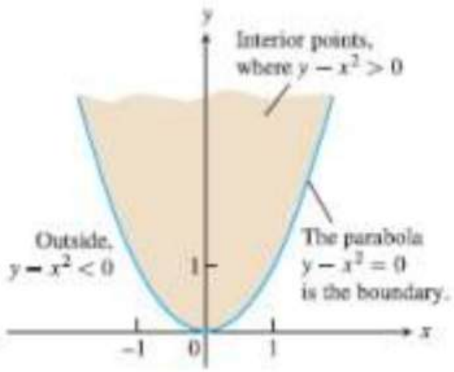

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Figure .2

Example (4): Find the domain and range of the function

$$z = f(x, y) = \sqrt{9 - y^2 - x^2}$$

Solution: **The domain of f is**

$$D = \{(x, y) : 9 - y^2 - x^2 \geq 0\} = \{(x, y) : x^2 + y^2 \leq 9\}$$

which is the disk with center $(0,0)$ and radius 3 (see Figure 3).

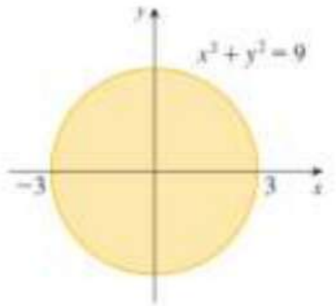

Figure .3

The range of f is

$$\{Z : Z = \sqrt{9 - y^2 - x^2} \ , \ (x, y) \in D\}$$

Since z is a positive square root, $z \geq 0$. Also

$$9 - y^2 - x^2 \leq 9 \ \Rightarrow \ \sqrt{9 - y^2 - x^2} \leq 3$$

So the range is

$$\{Z : 0 \leq Z \leq 3\} = [0, 3]$$

(2) The Limit of function in two variables

If the values of f(x, y) lie arbitrarily close to a fixed real number M for all points (x, y) sufficiently close to a point (a,b), we say that f approaches the limit M as (x, y) approaches (a,b). This is similar to the informal definition for the limit of a function of a single variable. Notice, however, that if (a,b)lies in the interior of j's domain, (x, y) can approach (a,b) from any direction. For the limit to exist, the same limiting value must be obtained whatever direction of approach is taken. We illustrate this issue in several examples following the definition

Definition:

We say that the function $z = f(x, y)$ approaches the limit M as (x, y) approaches (a, b) and write $\lim\limits_{(x,y)\to(a,\mathbf{b})} f(x, y) = M$

If for every $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ for all (x, y) in the domain of f such that :

$$|f(x, y) - M| < \varepsilon$$

i.e.,

$$\forall\ \varepsilon > 0\ \exists\ \ \delta(\varepsilon) > 0$$

Such that

$$|f(x, y) - M| < \varepsilon$$

Whenever

$|x - a| < \delta$, $|y - b| < \delta$ i.e.,

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

In this case we write

$$\lim_{(x,y)\to(a,\mathbf{b})} f(x, y) = M$$

Or

$$f(x, y) \to M \quad \text{as } (x, y) \to (a, b)$$

Remark(1):

From the above definition of the limit of the function, it is clear that there is no requirement on how the independent variables

(x, y) approach the to any point (a,b) that there is no requirement on the track for the pair (x, y) could be to get to the point (a,b) , so the path can be straight lines like

$$y-b=m(x-a)$$

Or any curve , like $y=\phi(x)$ such that , when $x \to a$ then $y \to b$, for example the parabola

$$(y-b)^2 = m(x-a)$$

This means that ,

1- when we take any path , if the results of the limit depends on m , then the limit does not exists

2- If the results of the limit is the same for any number of paths , then the limit exists

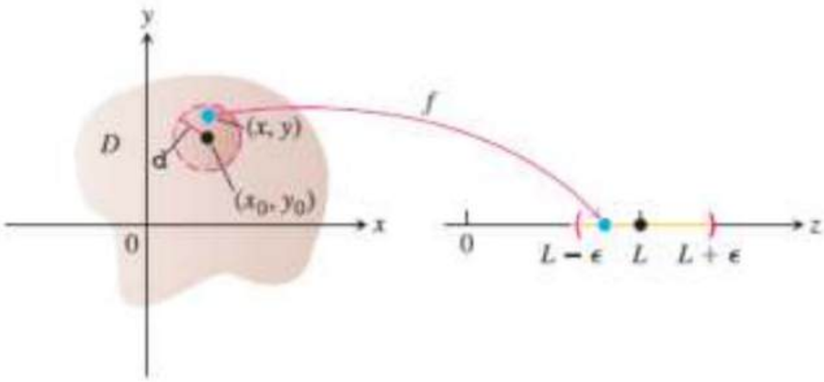

Figure .4

Remark(2):

The limit does not exists if :

$$\lim_{x \to a} \left(\lim_{y - b} f(x, y) \right) \neq \lim_{y \to b} \left(\lim_{x \to a} f(x, y) \right)$$

But if we have

$$\lim_{x \to a} \left(\lim_{y - b} f(x, y) \right) = \lim_{y \to b} \left(\lim_{x \to a} f(x, y) \right)$$

Then the limit

$$\lim_{(x \to y) \to (a,b)} f(x,y)$$

It can exist (not a requirement) and can not exist, and these are called successive limits.

As with single-variable functions, the limit of the sum of two functions is the sum of their limits (when they both exist), with similar results for the limits of the differences, constant multiples, products, quotients, powers, and roots.

Theorem [1]: **Properties of Limits of Functions of Two variables**

The following rules hold if M_1, M_2 and K are real numbers and

$$\lim_{(x,y)\to(a,b)} f(x,y) = M_1, \lim_{(x,y)\to(a,b)} g(x,y) = M_2 \ .$$

Then:

(I) $\lim\limits_{(x,y)\to(a,b)} \{f(x,y) \pm g(x,y)\} = M_1 \pm M_2$

(II) $\lim\limits_{(x,y)\to(a,b)} \{f(x,y).g(x,y)\} = M_1 \cdot M_2$

(III) $\lim\limits_{(x,y)\to(a,b)} \left\{\dfrac{f(x,y)}{g(x\ y)}\right\} = \dfrac{M_1}{M_2} \qquad ;M_2 \neq 0$

Example (4)

Determine the limit of the following function exixts or not ?

$$f(x,y) = \begin{cases} \dfrac{x\ y}{x^2+y^2} & ;(x,y) \neq (0,0) \\ 0 & ;(x,y) = (0,0) \end{cases}$$

Solution:

$$\because \lim_{x\to 0}\left(\lim_{y\to 0}\ \frac{x\,y}{x^2+y^2}\right)=0$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\ \frac{x\,y}{x^2+y^2}\right)=0$$

Since the two limits equal , so we still can not rule on the presence of the limit or not

Then , by using the path $y=mx$:

$$\therefore\ \lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{xy}{x^2+y^2}\right)=\lim_{x\to 0}\left(\frac{x(m\,x)}{x^2+m^2\,x^2}\right)$$

$$=\lim_{x\to 0}\ \frac{m\,x^2}{x^2\,(1+m^2)}$$

$$=\frac{m}{1+m^2}$$

Since the limit depends on m , then the limit does not exists

Example (5): Prove that the following limits does not exist

(I) $\displaystyle\lim_{(x,\,y)\to(0,0)}\left(\frac{x-3y}{3x-y}\right)$

(II) $\displaystyle\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{x^3-y^3}{x^3+y^3}\right)$

(III) $\displaystyle\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{2x-y}{x^2+y^2}\right)$

Solution:

(I)

$$\lim_{x\to 0}\left(\lim_{y\to 0}\ \frac{x-3y}{3x-y}\right)=\lim_{x\to 0}\left(\frac{x}{3\,x}\right)=\frac{1}{3}$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\ \frac{x-3y}{3x-y}\right)=\lim_{y\to 0}\left(\frac{-3\,y}{-\,y}\right)=3$$

$\therefore$ Then it is not exists

(II)

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^3-y^3}{x^3+y^3}\right)=\lim_{x\to 0}\left(\frac{x^3}{x^3}\right)=1$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^3-y^3}{x^3+y^3}\right)=\lim_{y\to 0}\left(\frac{-y^3}{y^3}\right)=-1$$

$\therefore$ Then it is not exists

(III)

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{2x-y}{x^2+y^2}\right)=\lim_{x\to 0}\frac{2x}{x^2}=\infty$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{2x-y}{x^2+y^2}\right)=\lim_{y\to 0}\left(\frac{-y}{y^2}\right)=-\infty$$

$\therefore$ Then it is not exists

Example (6): Show that the limit $\lim_{(x,y)\to(0,0)}\left(\frac{2x\,y^2}{x^2+y^4}\right)$ is not exists .

Solution: By using the path $x=my^2$

$$\lim_{(x,y)\to(0,0)}\left(\frac{2x\,y^2}{x^2+y^4}\right)=\lim_{y\to 0}\left(\frac{2my^4}{(m^2+1)\,y^4}\right)$$

$$=\frac{2m}{1+m^2}$$

$\therefore$The limit depends on m $\therefore$ Then it is not exists

Example (7):

Show that the limit $\lim_{(x,y)\to(0,0}f(x,y)$ is not exists , where

$$f(x,y)=\begin{cases}\dfrac{x^2\,y}{x^4+y^2} & if\quad x^2+y^2\neq 0\\ 0 & if\quad x=y=0\end{cases}$$

Solution:

By using the path $y=mx^2$

$$\therefore = \lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{x^2 y}{x^4 + y^2} \right) = \lim_{x \to 0} \left(\frac{x^2 (m x^2)}{x^4 + m^2 x^4} \right)$$

$$= \lim_{x \to 0} \left(\frac{m x^4}{x^4 (1+m^2)} \right) = \frac{m}{1+m^2}$$

$\therefore$ The limit depends on m

$\therefore$ Then it is not exists

Example (8):

Discuss the following limit

$$\lim_{(x,y) \to (0,0)} \left(\frac{y-x}{y+x} \cdot \frac{1+x}{1+y} \right)$$

Solution:

$$\therefore \lim_{x \to 0} \left(\lim_{y \to 0} f(x,y) \right) = \lim_{x \to 0} \left(\lim_{y \to 0} \left(\frac{y-x}{y+x} \cdot \frac{1+x}{1+y} \right) \right)$$

$$= \lim_{x \to 0} \left(\frac{-x}{x} \cdot \frac{1+x}{1} \right) = -\lim_{x \to 0} (1+x) = -1$$

but

$$\therefore \lim_{y \to 0} \left(\lim_{x \to 0} f(x,y) \right) = \lim_{y \to 0} \left(\lim_{x \to 0} \left(\frac{y-x}{y+x} \cdot \frac{1+x}{1+y} \right) \right)$$

$$= \lim_{y \to 0} \left(\frac{y}{y} \cdot \frac{1}{1+0} \right) = 1$$

$$\therefore \lim_{x \to 0} \left(\lim_{y \to 0} f(x,y) \right) \neq \lim_{y \to 0} \left(\lim_{x \to 0} f(x,y) \right)$$

$\therefore$ The limit does not exists.

Example (9): Evaluate the following limit

(I) $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \left(\dfrac{x \sin(x^2+y^2)}{(x^2+y^2)} \right)$ (ii) $\lim\limits_{(x,y) \to (2,1)} \left(\dfrac{\tan^{-1}(3xy+1)}{\sin^{-1}(xy-1)} \right)$

Solution:

By using the path $y = mx$

(I) $$\therefore \lim_{x\to 0}\left(\frac{x\sin(x^2+m^2x^2)}{x^2+m^2x^2}\right)=\lim_{x\to 0}\left(x\cdot\frac{\sin\left(x^2(1+m^2)\right)}{x^2(1+m^2)}\right)$$

By using the properties of limits

$$=\lim_{x\to 0}(x)\cdot\lim_{x\to 0}\left(\frac{\sin\left(x^2(1+m^2)\right)}{x^2(1+m^2)}\right)=0\cdot(1)=0$$

(II) By direct subsitiution, we find

$$\frac{\tan^{-1}3(2)(1)+1}{\sin^{-1}(2(1)-1)}=\frac{\tan^{-1}7}{\sin^{-1}1}=\frac{\tan^{-1}7}{\frac{\pi}{2}}=\frac{\pi}{2}\tan^{-1}7$$

Example (10): **Prove that** $\lim\limits_{(x,y)\to(1,2)}(x^2+2y)=5$

Solution: We have two methods

The first: From definition

$$\left|f(x,y)-5\right|=\left|x^2+2y-5\right|$$

Assuming the existence of a small number $\sigma\le 1$ so that

$$|x-1|<\delta \qquad ;and \quad |y-2|<\delta$$

then

$$-\delta<x-1<\delta \quad ;and \quad -\delta<y-2<\delta$$

$$1-\delta<x<\delta+1 \quad ;and \quad 2-\delta<y<\delta+2$$

$$1-2\delta+\delta^2<x^2<\delta^2+2\delta+1 \quad ;and \quad 4-2\delta<2y<4+2\delta$$

So $$-5\delta<-4\delta+\delta^2<x^2+2y-5<4\delta+\delta^2<5\delta$$

$$|f(x,y)-5|<5\delta$$

if we take , so $\varepsilon=5\delta$

$$\because \forall\varepsilon>0\ \exists\ \delta(\varepsilon)=\frac{\varepsilon}{5}$$

Where $\quad |f(x,y)-5| < \varepsilon$

At $\quad |x-1|<\delta \quad and \quad |y-2|<\delta$

then $\quad \lim\limits_{(x,y)\to(1,2)} f(x,y)=5$

<u>The second method</u>: By using Theorem 1

$$\lim_{(x,y)\to(1,2)}\left(x^2+2y\right)=\lim_{(x,y)\to(1,2)}\left(x^2\right)+\lim_{(x,y)\to(1,2)}\left(2y\right)$$

$$=\lim_{x\to1}\left(x^2\right)+\lim_{y\to2}\left(2y\right)=1+4=5$$

<u>Exercises on limits</u>

1- Prove that by using the definition of limit

(I) $\lim\limits_{(x,y)\to(1,2)}\left(3xy\right)=6$

(II) $\lim\limits_{(x,y)\to(2,1)}\left(x^2+y^2\right)=5$

(III) $\lim\limits_{(x,y)\to(1,-2)}\left(3x-4y\right)=11$

2- Prove that

(I) $\lim\limits_{(x,y)\to(0,0)}\left(\dfrac{\sin^{-1}(xy-2)}{\tan^{-1}(3xy-6)}\right)=\dfrac{1}{3}$

(II) $\lim\limits_{(x,y)\to(0,1)}\left(\dfrac{x+y-1}{\sqrt{x}-\sqrt{1-y}}\right)=0 \quad ,x>0 \quad , y<1$

3- Consider the function

$$f(x,y)=\begin{cases} y+x\sin\left(\dfrac{1}{y}\right) & ;\ y\neq 0 \\ 0 & ; y=0 \end{cases}$$

Find the following limits

(I) $\lim\limits_{(x,y)\to(0,0)} f(x,y)$

(II) $\lim\limits_{y\to0}\left(\lim\limits_{x\to0} f(x,y)\right)$

(III) $\lim\limits_{x\to0}\left(\lim\limits_{y\to0} f(x,y)\right)$

3- 4- Prove that the following limit does not exists

$$\lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{2x - 5y}{5x - 2y} \right)$$

5- Prove that the following limit $\lim_{(x,y) \to (0,0)} f(x,y)$

and the successive limit exists for the function

$$f(x,y) = \begin{cases} xy \dfrac{x^2 - y^2}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

6- Prove that the successive limit exists for the function $f(x,y)$ but tle limit $\lim_{(x,y) \to (0,0)} f(x,y)$ not exists

(I) $f(x,y) = \left(\dfrac{x - y}{x + y} \right)$

(II) $f(x,y) = \left(\dfrac{x^2 y^2}{x^4 + y^4 - x^2 y^2} \right)$

(III) $f(x,y) = \begin{cases} \dfrac{x^3 + y^3}{x - y} & , \quad x \neq y \\ 0 & , \quad x = y \end{cases}$

(IV) $f(x,y) = \begin{cases} \dfrac{x^2 + y^2}{x^2 - y^2} & , \quad x \neq y \\ 0 & , \quad x = y \end{cases}$

7- Find each of the following limits at the corresponding point

(I) $f(x,y) = \sqrt{x^2 + y^2} - 2$ at (1 , 2)

(II) $f(x,y) = \left(\dfrac{\tan^{-1} x}{x + y} \right)$ at (1 , 1)

(III) $f(x,y) = e^{-\frac{1}{x}} \log y$ at (1 , 1)

(IV) $f\ (x\,,y)=\left(\dfrac{1}{x-y}\right)$ at (9 , 9)

8- Find each of the following limits , or explain that the limit does not exists

(I) $\lim\limits_{(x,y)\to(-1,3)}\left(\dfrac{\sqrt{2xy+y^2}}{x^2}\right)$

(II) $\lim\limits_{(x,y)\to(0\,,\,0)}\left(\dfrac{\sin\ (x+y)}{x+y}\right)$

(III) $\lim\limits_{(x,y)\to(0\,,\,0)}\left(\dfrac{|x\,y|}{x^2+y^2}\right)$

(IV) $\lim\limits_{(x,y)\to(0\,,\,0)}\left(y^2\ \log\left(x^2+y^2\right)\right)$

(V) $\lim\limits_{(x,y)\to(3\,,\,1)}\left(\dfrac{3x^2-2xy}{x^2+xy+y^2}\right)$

Continuity

As with functions of a single variable, continuity is defined in terms of limits.

Continuity of function of two variabels

Definition: A function $z=f(x,y)$ is continuous at the point (a,b) in its domain if $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$,

(1) - $z=f(x,y)$ is defined at (a,b)

(2) - $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists

(3) - $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

A function is continuous if it is continuous at every point of its domain.

As with the definition of limit, the definition of continuity applies at boundary points. As well as interior points of the domain of f. The only requirement is that each point (x, y) near (a,b) be in the domain of $z = f(x, y)$.

A consequence of Theorem 1 is that algebraic combinations of continuous functions are continuous at every point at which all the functions involved are defined . This means that sums, differences, constant multiples, products, quotients, and powers of continuous functions are continuous where defined In particular, polynomials and rational functions of two variables are continuous at every point at which they are defined .

Remark: If the function $z = f(x, y)$ is continuous at the point (a,b) , then the function $f(x,b)$ is continuous at the point $x = a$. Also the $f(a, y)$ nis continuous at the point $y = b$. But The converse is not true i.e., the function $z = f(x, y)$ may be continuous at every variables separately but not continuous at the point (a,b)

Exampels on continuity

Example(11): Let

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2} & (x, y) \neq (0,0) \\ 0 & (x, y) = (0,0) \end{cases}$$

We have

$$\lim_{x \to 0} f(x,0) = 0 = f(0,0)$$

$$\lim_{y \to 0} f(0, y) = 0 = f(0,0)$$

the function $z = f(x, y)$ is continuous at every variables separately but not continuous at the point $(0,0)$ since the limit not exists

Exampels on continuity

Example (12):

Test the continuity of the following function at the origin

$$f\ (x,y)=\begin{cases}\dfrac{x^2\ y}{x^2+y^4} & ;\ x^2y^4\neq 0\\ 0 & ;\ x=y\ =0\end{cases}$$

Solution

$$\because\ f\ (0,\ 0)=\ 0\qquad \rightarrow (1)$$

The limits

$$\lim_{\substack{x\to 0\\ y\to 0}} f\,(x,y)\ =\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{x^2\ y}{x^2\ +y^4}\right)$$

By using the path $y=mx$

$$=\lim_{x\to 0}\left(\frac{x^2\ (m\,x)}{x^2\ +\ m^4\ x^4}\right)=\lim_{x\to 0}\left(\frac{mx^3}{x^2\ (1\ +m^4\ x^2)}\right)$$

$$=m\lim_{x\to 0}\left(\frac{x}{1\ +\ m^4\ x^2}\right)=\ m\ (0)=0\qquad \rightarrow (2)$$

From (1) and (2) we find that

$\therefore$ The function is continuous at $(0,0)$.

Example (13):

Discus the continuity of the function at origin

$$g\ (x,y)=\begin{cases}\dfrac{\sin xy}{x\,y} & (x,\ y)\neq(0,0)\\ 2 & (x,\ y)=(0,0)\end{cases}$$

Solution:

$$\because\ g\ (0,0)=2\qquad \rightarrow (1)$$

The limit

$$\lim_{\substack{x \to 0 \\ y \to 0}} g(x, y) = \lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{\sin(xy)}{xy} \right)$$

By using the path $y = mx$

$$= \lim_{x \to 0} \left(\frac{\sin(mx^2)}{mx^2} \right) = 1 \qquad \to (2)$$

From (2) ، (1) we find that

$$\lim_{\substack{x \to 0 \\ y \to 0}} g(x, y) \neq g(0, 0)$$

$\therefore$ The function is not continuous.

Example (14) :

Discus the continuity of the function at origin

$$f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^3 + y^3} & ; (x, y) \neq (0,0) \\ 0 & ; (x, y) = (0,0) \end{cases}$$

Solution:

$\because f(0,0) = 0 \qquad \to (1)$

The limit

$$\lim_{\substack{x \to 0 \\ y \to 0}} f(x, y) = \lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{x^3 - y^3}{x^3 + y^3} \right)$$

By using the path $y = mx$

$$\lim_{\substack{x \to 0 \\ y \to 0}} f(x, y) = \lim_{x \to 0} \left(\frac{x^3 - m^3 x^3}{x^3 + m^3 x^3} \right) = \lim_{x \to 0} \left(\frac{x^3 (1 - m^3)}{x^3 (1 + m^3)} \right)$$

$$= \left(\frac{1 - m^3}{1 + m^3} \right) \qquad \to (2)$$

From (1) and (2) we find that the limit depends on m

$\therefore$ the function is discontinuous.

Example (15):

Prove that the following function is gontinuous at the point $(0,0)$

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & ;\ (x,y)\neq(0,0)\\ 0 & ;\ (x,y)=(0,0)\end{cases}$$

Solution

$$\because f(0,0)=0 \qquad \rightarrow(1)$$

The limit

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)=\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{x\ y}{\sqrt{x^2+y^2}}\right)$$

By using the path $y=mx$

$$\therefore \lim_{\substack{x\to 0\\ y\to 0}} f(x,y)=\lim_{x\to 0}\left(\frac{m\ x^2}{\sqrt{x^2+m^2x^2}}\right)=\lim_{x\to 0}\left(\frac{m\,x^2}{x\sqrt{1+m^2}}\right)$$

$$=\frac{m}{\sqrt{1+m^2}}\lim_{x\to 0}(x)=0 \qquad \rightarrow(2)$$

From (1) and (2) we find that

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)=f(0,0)$$

$\therefore$ The function is continuous at $(0,0)$.

Exercises on continuity

1- Discuss the continuity of the function at the origin

$$f(x,y)=\begin{cases}\dfrac{x^2-y^2}{x^2+y^2} & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$$

2- Discuss the continuity of the function at $(1,2)$

$$f(x,y)=\begin{cases}x^2+2y & ;(x,y)\neq(1,2)\\ 0 & ;(x,y)=(1,2)\end{cases}$$

3- Discuss the continuity of the function at the origin

$$f(x,y)=\begin{cases}\dfrac{x^2}{x^2+y^2} & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$$

4- Discuss the continuity of the function at the origin

(I) $$f(x,y)=\begin{cases}\dfrac{5\sin xy}{xy} & ;(x,y)\neq(0,0)\\ 5 & ;(x,y)=(0,0)\end{cases}$$

(II) $$f(x,y)=\begin{cases}\dfrac{2\tan xy}{xy} & ;(x,y)\neq(0,0)\\ 2 & ;(x,y)=(0,0)\end{cases}$$

(III) $$f(x,y)=\begin{cases}\dfrac{1}{x^2+y^2} & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$$

(IV) $$f(x,y)=\begin{cases}\dfrac{x^4-y^4}{x^4+y^4} & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$$

(V) $$f(x,y)=\begin{cases}\dfrac{x^3\,y^3}{x^2+y^2} & ;(x,\ y)\neq(0\ ,\ 0)\\ 0 & ;(x\ ,\ y)=(0\ ,\ 0)\end{cases}$$

5- Discuss the continuity of the function at $(1{,}0)$

$$f(x,y)=\begin{cases}\dfrac{(x-1)\,y}{(x-1)^2+y^2} & ;(x\ ,\ y)\neq(1\ ,\ 0)\\ 0 & ;(x\ ,\ y)=(1\ ,\ 0)\end{cases}$$

6- Discuss the continuity of the function at $(0{,}2)$

$$h(x,y)=\begin{cases}\dfrac{x\ (y-2)}{x^2+(y-2)^2} & ;(x,\ y)\neq(0\ ,\ 2)\\ 0 & ;(x,\ y)=(0\ ,\ 2)\end{cases}$$

Partial Derivatives of a Function of Two Variables

We define the partial derivative of $z=f(x,y)$ with respect to *x* at the point *(xo,* YO) as the ordinary

derivative of *f(x,* YO) with respect to *x* at the point *x* = Xc. To distinguish partial derivatives

from ordinary derivatives we use the symbol a rather than the *d* previously used.

In the definition, *h* represents a real number, positive or negative.

If we have the function $z=f(x,y)$, then the partial differentiation for this function with respect to the variable (for example) x is defined as the normal differential of function, as it is a function of the independent variable x and the rest of the variables is fixed amounts " y fixed ". And it symbolized to the partial derivative of $z=f(x,y)$ with respect to x

$$f_x=\frac{\partial z}{\partial x}=\frac{\partial f}{\partial x}=f_x\ (x\ ,\ y)$$

The function $z = f(x, y)$ is differentiable partially with respect to x if the limit $\lim\limits_{h \to 0} \dfrac{f(x+h, y) - f(x, y)}{h}$ exists .

Similarly the function $z = f(x, y)$ is differentiable partially with respect to y if the limit $\lim\limits_{k \to 0} \dfrac{f(x, y+k) - f(x, y)}{k}$ exists .

Also the partial derivatives for $z = f(x, y)$ at the point (a, b) is defined by

$$\left(\frac{\partial f}{\partial x}\right)_{(a,\, b)} = \frac{\partial f(a, b)}{\partial x} = f_x(a, b)$$

$$\left(\frac{\partial f}{\partial y}\right)_{(a,\, b)} = \frac{\partial f(a, b)}{\partial y} = f_y(a, b)$$

i.e.,

$$\left(\frac{\partial f}{\partial x}\right)_{(a,\, b)} = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

$$\left(\frac{\partial f}{\partial y}\right)_{(a,\, b)} = \lim_{k \to 0} \frac{f(a, b+k) - f(a, b)}{k}$$

Example (16):

Let $\qquad f(x, y) = e^x \sin y$

By using the definition, find $\quad f_x(1,1) \quad$ and $\quad f_y(1,1)$

Solution:

$$\because f_x(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

$$\therefore f_x(1, 1) = \lim_{h \to 0} \frac{f(1+h, 1) - f(1, 1)}{h} = \lim_{h \to 0} \frac{e^{1+h} \sin 1 - e \sin 1}{h}$$

$$= e \sin 1 \lim_{h \to 0} \left(\frac{e^h - 1}{h}\right)$$

By using L'Hopital Theorem

$$= e \sin 1 \lim_{h \to 0} e^h = e \sin 1$$

similarly

$$\because f_x\left(a, b\right) = \lim_{k \to 0} \frac{f\left(a, b+k\right) - f\left(a, b\right)}{k}$$

$$\therefore f_x\ (1, 1) = \lim_{k \to 0} \frac{f\left(1, 1+k\right) - f\left(1, 1\right)}{k} = \lim_{k \to 0} \frac{e \sin\left(1+k\right) e \sin 1}{k}$$

$$= e \lim_{k \to 0} \frac{\sin\left(1+k\right) - \sin 1}{k}$$

By using L'Hopital Theorem

$$= e \lim_{h \to 0}\left(\cos\left(1+k\right)\right) = e \cos 1$$

Example (17): **Let**

$$f(x, y) = \begin{cases} \dfrac{x^2 - xy}{x+y} & ; (x, y) \neq (0, 0) \\ 0 & ; (x, y) = (0, 0) \end{cases}$$

Find $f_x(0,0)$ *and* $f_y(0,0)$

Solution:

$$\because f_x\left(a, b\right) = \lim_{h \to 0} \frac{f\left(a+h, b\right) - f\left(a, b\right)}{h}$$

$$\therefore f_x\ (0, 0) = \lim_{h \to 0} \frac{f\left(h, 0\right) - f\left(0, 0\right)}{h}$$

$$= \lim_{h \to 0}\left(\frac{\dfrac{h2 - h\,(0)}{h+0} - 0}{h}\right) = \lim_{h \to 0}\left(\frac{h}{h}\right) = 1$$

$$\therefore f_x\ (0, 0) = 1$$

similarly

$$\because f_y(a,b) = \lim_{k\to 0} \frac{f(a, b+k) - f(a,b)}{k}$$

$$\therefore f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \left(\frac{\frac{0-0(k)-0}{0+k}}{k} \right)$$

$$= \lim_{k\to 0} \frac{0}{k} = 0 \quad \Rightarrow f_y(0,0) = 0$$

Example (18): Show that the function

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & ; x^2 + y^2 \neq 0 \\ 0 & ; (x,y) = (0,0) \end{cases}$$

Has derivative of the first order at the point $(0,0)$ while it is not connected at the point. $(0,0)$

Solution;

$$\because f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

Similarly

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0, 0+h) - f(0,0)}{h} = \lim_{h\to 0} \left(\frac{0-0}{h} \right) = 0$$

$\therefore$ The partial derivatives of the first order exist at the point $(0,0)$.

But by studying the continuity , we find . $f(0,0) = 0 \qquad \rightarrow (1)$

The limit : $\lim\limits_{\substack{x\to 0 \\ y\to 0}} f(x,y) = \lim\limits_{\substack{x\to 0 \\ y\to 0}} \left(\dfrac{x^2 y}{x^4 + y^2} \right)$

by using the path $y = mx^2$

$$\therefore \lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{x\to 0} \frac{x^2\ (m x^2)}{x^4 + m^2 x^4} = \lim_{x\to 0} \frac{m x^4}{x^4\ (1\ m^2)} = \left(\frac{m}{1+m^2}\right) \quad (2)$$

From (1) and (2) we find that the limit depends on m

$\therefore$ the function is discontinuous at $(0,0)$

From the above example it is clear that there is a difference between ordinary derivatives and partial derivatives in its relationship with the continuity . In a single-variable function, we see that the continuity is condition of the presence of the first derivative and if the first derivative present then the function must be connected to the function.

As in the case of multivariate functions relationship between communication and partial derivatives will become clear from the following theories:

Theorem: If the partial derivative f_x exists in the neighborhood of the point (a,b) and the derivative $f_y(a,b)$ exists . Then for any point $(a+h,b+k)$ in the neighborhood of the point (a,b) we have

$$f(a+h,b+k) - f(a,b) = hf_x(a+\theta h, b+k) + kf_y(a, b+g)$$

Where $0<\theta<1$, and g is a function on k tends to zero with k

Proof:

But $\because f(a+h,b+k)-f(a,b) = f(a+h,b+k)-f(a,b+k)+f(a,b+k)-f(a,b)$

$$f(a+h,b+k)-f(a,b+k) = hf_x(a+\theta h,b+k) \qquad \to (1)$$

From Lagrangian mean theorem $0<\theta<1$ and since $f_y(a,b)$ exist , we have

$$\lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k} = f_y\ (a\ ,b)$$

Then $\quad f(a,b+k)-f(a,b) = k(f_y(a,b)+g) \qquad \to (2)$

Where $k\to 0$ when $g\to 0$

From (1) and (2), the teorem is proved

Differentiability

If $y = f(x)$ is differentiable at $x = x_0$, then the change in the value of f that results from changing x from x_0 to $x_0 + \Delta x$ is given by an equation of the form

$$\Delta y = f'(x_0)\Delta x + \varepsilon \Delta x$$

in which $\varepsilon_1 \to 0$ as $\Delta x \to 0$.

For functions of two variables, the analogous property becomes the definition of differentiability. The following Theorem tells us when to expect the property to hold.

Theorem3: Suppose that the first partial derivatives of $z = f(x, y)$ are defined throughout an open region *R* containing the point (x_0 , y_0) and that f_x and f_y are continuous at (x_0 , y_0).Then the change

$$\Delta Z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

in the value of f that results from moving from (x_0 , y_0) to another point $(x_0 + \Delta x,\ y_0 + \Delta y)$ in *R* satisfies an equation of the form

$$\Delta Z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

in which each of $\varepsilon_1 , \varepsilon_2 \to 0$ as both $(\Delta x, \Delta y) \to (0,0)$

Definition: A function $z = f(x, y)$ is differentiable at (x_0 , y_0) if

$f_x(x_0 , y_0)$ and $f_y(x_0 , y_0)$ exist and ΔZ satisfies an equation of the form

$$\Delta Z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

in which each of $\varepsilon_1 , \varepsilon_2 \to 0$ as both $(\Delta x, \Delta y) \to (0,0)$. We call f **differentiable** if it is differentiable at every point in its domain, and say that its graph is a **smooth surface**.

Because of this definition, an immediate corollary of Theorem 3 is that a function is differentiable at (x_0 , y_0) if its first partial derivatives are ***continuous*** there.

Corollary of Theorem 3: If the partial derivatives f_x and f_y of the function $z = f(x, y)$ are continuous throughout an open region *R,* then f is differentiable at every point of *R.*

If $z = f(x, y)$ is differentiable, then the definition of differentiability assures that $\Delta Z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$ approaches 0 as $(\Delta x, \Delta y) \to (0,0)$. This tells us that a function of two variables is continuous at every point where it is differentiable.

Theorem 4 (DIfferentiabillty Implies Continuity)
If a function $z = f(x, y)$ is differentiable at (x_0 , y_0), then I is continuous at (x_0 , y_0).

Second-Order Partial Derivatives

When we differentiate a function $z = f(x, y)$ twice, we produce its second-order derivatives. These derivatives are usually denoted by

(I) $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial x^2} = f_{xx}$

(II) $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial y^2} = f_{xyy}$

(III) $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y \partial x} = f_{yx}$

(IV) $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x \partial y} = f_{xy}$

The second – order derivatives can be defined at any point (a,b) in the form :

(V) $f_{xx}(a,b) = \lim\limits_{h\to 0} \dfrac{f_x(a+h,\ b) - f_x(a,\ b)}{h}$

(VI) $f_{xy}(a,b) = \lim\limits_{h\to 0} \dfrac{f_y(a+h,\ b) - f_y(a,\ b)}{h}$

(VII) $f_{yy}(a,b) = \lim\limits_{k\to 0} \dfrac{f_y(a,b+k) - f_y(a,\ b)}{k}$

(VIII) $f_{yx}(a,b) = \lim\limits_{k\to 0} \dfrac{f_x(a,b+k) - f_x(a,\ b)}{k}$

If this limits exist.

Similarly , the derivative of any order can be defined , for example:

$$\frac{\partial^3 f}{\partial x\,\partial x\,\partial y} = \frac{\partial^3 f}{\partial x^2\,\partial y} = f_{xxy}$$

Example (19): Let

$$f(x,y) = \begin{cases} xy\,\dfrac{x^2 - y^2}{x^2 + y^2} & ;\ (x,y) \neq (0,0) \\ 0 & ;(x,\ y) = (0,0) \end{cases}$$

By using the definition of derivatives, find the first and second order derivatives at $(0,0)$

Solution:

$$\because f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$\therefore f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

-

$$= \lim_{h\to 0} \frac{h(0)\left(\dfrac{h^2 - 0}{h^2 + 0}\right) - 0}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

$$\therefore f_x(0,0) = 0$$

Similarly

$$\because f_y(a,b) = \lim_{k \to 0} \frac{f(a, b+k) - f(a,b)}{k}$$

$$\therefore f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{h\to 0} \frac{(0)(k)\left(\frac{0-k^2}{0+k^2}\right) - 0}{h}$$

$$\therefore f_y(0,0) = 0$$

The second order derivatives

$$\because f_{xx}(a,b) = \lim_{h\to 0} \frac{f_x(a+h, b) - f_x(a,b)}{h}$$

$$\therefore f_{xx}(0,0) = \lim_{h\to 0} \frac{f_x(h,0) - f_x(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f_x(h,0)}{h} \qquad \to (1)$$

Since $\qquad f(x,y) = \dfrac{x^3 y - xy^3}{x^2 + y^2}$

$$\therefore f_x = \frac{(x^2+y^2)(3x^2 y - y^3) - (x^3 y - xy^3)(2x)}{(x^2+y^2)^2}$$

$$\therefore f_x(h,0) = \frac{(h^2+0)\left[3h^2(0) - 0\right] - 2h\left[h_3(0) - h(0)\right]}{(h^2+0)^2}$$

$$\therefore f_x(h,0) = 0$$

Substitution in (1)

$$\therefore f_{xx}(0,0) = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0}(0) = 0$$

$$\therefore f_{xx}(0,0) = 0$$

$$\because f_{xy}\ (a\,,b) = \lim_{k\to 0} \frac{f_x\ (a\,,b+k)\ - f_x\ (0\,,\,0)}{k}$$

$$\therefore f_{xy}\ (0\,,0)\ = \lim_{k\to 0} \frac{f_x\,(0\,,\,k) - f_x\ (0\,,\,0)}{k}$$

$$= \lim_{k\to 0} \frac{f_x\ (0\,,\,k)}{k}$$

$$\lim_{k\to 0} = \frac{\dfrac{(0+k^2)\left\lfloor 3(0)\ -k^3\right\rfloor - (0-0)\,(0)}{(0+k^2)^2}}{k}$$

$$\lim_{k\to 0} = \frac{\dfrac{-k^5}{k^4}}{k} = \lim_{k\to 0} \frac{-k}{k} = \lim_{k\to 0}\ (-1)$$

$$\therefore f_x y\,(0\,,\,0) = -1$$

Similarly , we can find $f_{xy}(0{,}0), f_{yy}(0{,}0)$

Example (20)

I**forder derivatives–find the second** , $f(x,y) = x^3 y + e^{xy^2}$

Solution: The first step is to calculate both first partial derivatives.

$$\because f(x,y) = x^3 y + e^{xy^2}$$

By differentiate with respect to x

$$\therefore f_x = 3x^2 y + y^2 e^{xy^2} \qquad \rightarrow (1)$$

By differentiate with respect to y

$$\therefore f_y = x^3\,(1) + 2\,x\,y\,e^{xy^2} \qquad \rightarrow (2)$$

Now we find both partial derivatives of each first partial:

By differentiate (1) with respect to x

$$f_{xx} = 6xy + y^2 y^2 e^{xy^2} = 6xy + y^4 e^{xy^2}$$

By differentiate (1) with respect to y

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\partial x}=f_{yx}$$

$$=3x^2\,(1)+y^2\,2xye^{xy^2}+2ye^{xy^2}$$

$$=3x^2+2xy^3e^{xy^2}+2ye^{xy^2}$$

By differentiate (2) with respect to x

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\partial y}=f_{yx}$$

$$=3x^2+2xy^3e^{xy^2}+2ye^{xy^2}$$

By differentiate (2) with respect to y

$$\therefore f_{yy}=(0)+\;2x\left[y\,2xy\,e^{xy^2}+(1)e^{xy^2}\right]$$

$$=2xe^{xy^2}\cdot\left[2xy^2+1\right]$$

Example (21) Let $u(x,y,z)=\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$

Prove that $u_{xx}+u_{yy}+u_{zz}=0$

Solution:

$$\because u(x,y,z)=\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$$

By differentiate with respect to x

$$u_x=-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2x)$$

$$=-x\left(x^2+y^2+z^2\right)^{-3/2}$$

By differentiate again with respect to x

$$u_{xx}=-x\left\{\left(\frac{-3}{2}\right)\left(x^2+y^2+z^2\right)^{-5/2}(2x)\right\}+\left(x^2+y^2+z^2\right)^{-3/2}(-1)$$

$$=3x^2\left(x^2+y^2+z^2\right)^{-5/2}-\left(x^2+y^2+z^2\right)^{-3/2}\qquad\rightarrow(1)$$

Similarly , by differentiate twice with respect to y

$$u_{yy} = 3y^2\left(x^2+y^2+z^2\right)^{-5/2} - \left(x^2+y^2+z^2\right)^{-3/2} \quad \rightarrow (2)$$

Also , $u_{zz} = 3y^2\left(x^2+y^2+z^2\right)^{-5/2} - \left(x^2+y^2+z^2\right)^{-3/2} \quad \rightarrow (3)$

From (1) , (2) and (3) by collection we get

$$u_{xx} + u_{yy} + u_{zz} = 0$$

Example (22): Find the first and second derivatives of

$$z = x\tan(xy)$$

Solution:

$$z_x = \frac{\partial z}{\partial x} = xy\sec^2(xy) + \tan(xy) \quad \rightarrow (1)$$

$$z_y = \frac{\partial z}{\partial y} = x^2\sec^2(xy) \quad \rightarrow (2)$$

By differentiate (1) with respect to x

$$\therefore z_{xx} = 2xy^2\sec^2(xy)\tan(xy) + 2y\sec^2(xy)$$

By differentiate (1) with respect to y

$$z_{yx} = 2x^2y\sec^2(xy)\tan(xy) + 2x\sec^2(xy)$$

By differentiate (2) with respect to x

$$\therefore z_{xy} = 2xy^2\sec^2(xy)\tan(xy) + 2y\sec^2(xy)$$

By differentiate (2) with respect to y

$$z_{yy} = 2x^3\sec^2(xy)\tan(xy)$$

Remark: We notice that in the above example , $z_{xy} = z_{yx}$.
Does this exists everywhere or there exist a condition to have $f_{xy} = f_{yx}$, the following theorem gives the conditions

Theorem (1):

If both f_x, f_y are differentiable at the point (a,b) in the domain D of the function f . Then $f_{xy}(a,b) = f_{yx}(a,b)$

Theorem (2)

If the function $f(x, y)$ is defined in the domain D , and f_{xy}, f_{yx} exists and continuous at the point (a,b) in D . Then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Example (23) : **Let**

$$f(x, y) = \begin{cases} \dfrac{x^2 y^2}{x^2 + y^2} & ;(x, y) \neq (0,0) \\ 0 & ;(x, y) = (0,0) \end{cases}$$

Prove that : $f_{xy}(0,0) = f_{yx}(0,0)$

Solution:

Since , $f_x(0,0) = \lim\limits_{h \to 0}\left(\dfrac{f(h,0) - f(0,0)}{h} \right) = 0$

Also , $f_y(0,0) = 0$

$$f_x(x, y) = \frac{(x^2 + y^2) \cdot 2xy^2 - x^2 y^2 \cdot 2x}{(x^2 + y^2)^2} = \frac{2xy^4}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{2x^4 y}{(x^2 + y^2)^2}$$

$$f_{yx}(0,0) = \lim_{k \to 0}\left(\frac{f_x(0,k) - f_x(0,0)}{k} \right) = 0$$

$$f_{xy}(0,0) = \lim_{h \to 0}\left(\frac{f_y(h,0) - f_y(0,0)}{h} \right) = 0$$

$$\therefore f_{xy}(0,0) = f_{yx}(0,0)$$

Exercises on partial derivatives

1. Find for the following function $f_x(0,0), f_y(0,0)$

$$f(x,y)=\begin{cases}\dfrac{x^2-xy}{x+y} & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$$

2- proved that the following functions are not continuous at the origin, despite of the exists of partial derivatives of the first order in the domain of the function and at the original point .

(I) $$f(x,y)=\begin{cases}\dfrac{x^3+y^3}{x-y} & ;x\neq y\\ 0 & ;x=y\end{cases}$$

(II) $$f(x,y)=\begin{cases}\dfrac{x^2y}{x^4+y^2} & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$$

3-Let

$$f(x,y)=\begin{cases}xy\tan(y,x) & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$$

Prove that : $x f_x + y f_y = 2f$

4- Discuss the continuity and differentiability of the functions

(I) $$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & if\quad x^2+y^2\neq 0\\ 0 & if\quad x=y=0\end{cases}$$

(II) $$f(x,y)=\begin{cases}\dfrac{xy^2}{x^2+y^2} & ;(x,y)\neq 0\\ 0 & ;(x,y)=(0,0)\end{cases}$$

(III) $f(x,y)=y\sin\left(\dfrac{1}{x}\right) \qquad if\ \ x\neq 0$

5- Find the partial derivative of first and second order of the following functions:

(I) $f(x,y) = x\tan^{-1}(xy)$

(II) $f(x,y) = e^{xy} + \log(x+y)$

(III) $f(x,y) = \log\left(\dfrac{x}{x+y}\right)$

(IV) $f(x,y,z) = \tan^{-1}(xyz)$

(V) $f(x,y) = x^{y}$

(VI) $f(x,y,z) = \sin(x + z^{y})$

(VII) $h(x,y) = x^2\cos(y) + y^2\sin(x)$

(VIII) $h(x,y) = y\cos^{-1}(x+y)$

(IX) $k(x,y) = x\tan^{-1}(y\cos(x))$

(X) $f(x,y,z) = x^{yz^2}$

(XI) $f(x,y,z) = e^{x}\log(x^2 + y^2 + z^2)$

6- Prove that the function : $z = \log\left[(x-a)^2 + (y-b)^2\right]$ satisfy the relation $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$ except at $(x,y) = (a,b)$

7- Prove that the function : $z = x\cos\left(\dfrac{y}{x}\right) + \tan\left(\dfrac{y}{x}\right)$

satisfy the relation $x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} = 0$

8- Prove that the function $u = e^{x}\cos(y) + e^{y}\sin(z)$

Prove that : $u_{xy} = u_{yx}, u_{xz} = u_{zx}, u_{yz} = u_{zy}$

9- Let the function $z = f(x^2 y)$ be differentiable function . Prove that : $xz_x = 2yz_y$

10- Find the value of the first partial derivatives of each function at the corresponding point :

(I) $f(x,y) = e^x \log(y)$; $(0,e)$

(II) $g(x,y) = \dfrac{x}{x+y}$; $(1,2)$

(III) $h(x,y) = e^{-x}\sin(x+2y)$; $\left(0,\dfrac{\pi}{4}\right)$

The Chain Rule

For functions of more than one variable, the Chain Rule has several versions, each of them giving a rule for differentiating a composite function. The first version

(Theorem 2) deals with the case where $z = f(x,y)$ and each of the variables x and y is, in turn, a function of a variable t. This means that $z = f(x,y)$ is indirectly a function of t, $z = f(g(t),h(t))$, and the Chain Rule gives a formula for differentiating as a function of t. We assume that f is differentiable. Recall that this is the case when f_x and f_y are continuous.

Theorem 2: Suppose that $z = f(x,y)$ is a differentiable function of x and y, where $x = g(t)$; $y = h(t)$ are both differentiable functions of t.

Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Since

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \qquad \to (1)$$

And $dx = \dfrac{dx}{dt}dt$, $dy = \dfrac{dy}{dt}dt$

but z is a differentiable function

$$dz = \frac{\partial z}{\partial x}\frac{dx}{dt}dt + \frac{\partial z}{\partial y}\frac{dy}{dt}dt \qquad \to (2)$$

Also

$$dz = \frac{dz}{dt}dt \qquad \rightarrow (3)$$

Then we have $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

Example (24): If $z = e^{xy^2}$; $x = t\cos(t)$; $y = t\sin(t)$

Find $\frac{dz}{dt}$ at $t = \frac{\pi}{2}$

Solution:

$$\because z = e^{xy^2}$$

$$\therefore \frac{\partial z}{\partial x} = y^2 e^{xy^2} \qquad , \frac{\partial z}{\partial y} = 2xye^{xy^2} \qquad \rightarrow (1)$$

$$\because x = t\cos(t) \Rightarrow \frac{dx}{dt} = -t\sin(t) + \cos(t) \qquad \rightarrow (2)$$

$$\because y = t\sin(t) \Rightarrow \frac{dy}{dt} = t\cos(t) + \sin(t) \qquad \rightarrow (3)$$

$$\because \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

From (1),(2) and (3) we have

$$\therefore \frac{dz}{dt} = \left(y^2 e^{xy^2}\right)\left(-t\sin(t) + \cos(t)\right) + \left(xe^{xy^2}\right)\left(t\cos(t) + \sin(t)\right)$$

$$= e^{xy^2}\left\{\left(y^2[\cos(t) - t\sin(t)]\right) + x[t\cos(t) + \sin(t)]\right\}$$

$$at \quad t = \frac{\pi}{2} \Rightarrow x = 0 \qquad ; y = \frac{\pi}{2}$$

$$\therefore \frac{dz}{dt} = e^{0\left(\frac{\pi}{2}\right)^2}\left[\frac{\pi^2}{4}\left(0 - \frac{\pi}{2}\right) + 0\right] = -\frac{\pi^3}{8}$$

We now consider the situation where $z = f(x, y)$ but each of x and y is a function of two variables u and v : where $x = g(u, v)$; $y = h(u, v)$. Then $z = f(x, y)$ is indirectly a function of u ,and

Suppose that $z = f(x, y)$ is a differentiable function

of x and y , where $x = g(u,v)$; $y = h(u,v)$ are differentiable functions of u and v Then

Theorem 3: Suppose that $z = f(x, y)$ is a differentiable function of x and y , where $x = y(u,v)$, $y = h(u,v)$ are both differentiable functions of u and v .

Then z is a differentiable function of of u and v , then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Since

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$$
$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv \qquad \rightarrow (4)$$

But $z = f(x, y)$ is a differentiable function of x and y ,so z is a differentiable function of of u and v

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv \qquad \rightarrow (5)$$

But $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$. Then

$$dz = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)$$
$$= \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\right)du + \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\right)dv$$

Comparing with (5) , we have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Example (25): If $z = x^3 - xy + y^3$ where

$$x = r\cos(\theta) \quad ; \quad y = r\sin(\theta)$$

Find : $\dfrac{\partial z}{\partial r}, \dfrac{\partial z}{\partial \theta}$

Solution: $\dfrac{\partial z}{\partial x} = 3x^2 - y \quad , \dfrac{\partial z}{\partial y} = -x + 3y^2 \qquad \to (1)$

$$\because x = r\cos(\theta) \Rightarrow \begin{cases} \dfrac{\partial x}{\partial r} = \cos(\theta) \\ \dfrac{\partial x}{\partial \theta} = -r\sin(\theta) \end{cases} \qquad \to (2)$$

$$\because y = r\sin(\theta) \Rightarrow \begin{cases} \dfrac{\partial y}{\partial r} = \sin(\theta) \\ \dfrac{\partial y}{\partial \theta} = r\cos(\theta) \end{cases} \qquad \to (3)$$

$$\because \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

From (1),(2) and (3) we have

$$\therefore \frac{\partial z}{\partial r} = (3x^2 - y)(\cos\theta) + (3y^2 - x)(\sin\theta)$$
$$= 3r^3(\cos^3\theta + \sin^3\theta) - 2r\sin\theta\cos\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

From (1),(2) and (3) we have

$$\therefore \frac{\partial z}{\partial \theta} = (3x^2 - y)(-r\sin\theta) + (3y^2 - x)(r\cos\theta)$$
$$= 3r^3(\sin\theta\cos^2\theta + \cos\theta\sin^2\theta) + r^2(\sin^2\theta - \cos^2\theta)$$

Example (26): If $v = v(x, y)$ where $x = r\cos(\theta)$; $y = r\sin(\theta)$

Prove that: $$\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial v}{\partial \theta}\right)^2$$

Solution:

$$\because x = r\cos(\theta) \Rightarrow \begin{cases} \frac{\partial x}{\partial r} = (1)\cos(\theta) = \frac{x}{r} & \Rightarrow \frac{\partial x}{\partial r} = \frac{x}{r} \\ \frac{\partial x}{\partial \theta} = -r\sin(\theta) = -y & \Rightarrow \frac{\partial x}{\partial \theta} = -y \end{cases} \quad \rightarrow (1)$$

$$\because y = r\sin(\theta) \Rightarrow \begin{cases} \frac{\partial y}{\partial r} = (1)\sin(\theta) = \frac{y}{r} & \Rightarrow \frac{\partial y}{\partial r} = \frac{y}{r} \\ \frac{\partial y}{\partial \theta} = r\cos(\theta) = x & \Rightarrow \frac{\partial y}{\partial \theta} = x \end{cases} \quad \rightarrow (2)$$

$$\because \frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}$$

By (1) and (2)

$$\therefore \frac{\partial v}{\partial r} = \frac{x}{r}\frac{\partial v}{\partial x} + \frac{y}{r}\frac{\partial v}{\partial y}$$

Multiply by r

$$\therefore r\frac{\partial v}{\partial r} = x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}$$

$$\therefore r^2\left(\frac{\partial v}{\partial r}\right)^2 = x^2\left(\frac{\partial v}{\partial x}\right)^2 + y^2\left(\frac{\partial v}{\partial y}\right)^2 + 2xy\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} \quad \rightarrow (3)$$

Similarly

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}$$

By (1) and (2)

$$\therefore \frac{\partial v}{\partial \theta} = -y\frac{\partial v}{\partial x} + x\frac{\partial v}{\partial y}$$

$$\therefore \left(\frac{\partial v}{\partial \theta}\right)^2 = y^2\left(\frac{\partial v}{\partial x}\right)^2 + x^2\left(\frac{\partial v}{\partial y}\right)^2 - 2xy\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} \quad \rightarrow (4)$$

From (3) and (4) by addition

$$\therefore r^2\left(\frac{\partial v}{\partial r}\right)^2+\left(\frac{\partial v}{\partial \theta}\right)^2=\left(x^2+y^2\right)\left(\frac{\partial v}{\partial x}\right)^2+\left(x^2+y^2\right)\left(\frac{\partial v}{\partial y}\right)^2$$

$$=\left(x^2+y^2\right)\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right] \qquad \rightarrow (5)$$

$$\because x^2+y^2=r^2\cos^2(\theta)+r^2\sin^2(\theta)$$
$$=r^2(1)=r^2$$

From (5) by dividing on r^2 we have

$$\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2=\left(\frac{\partial v}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial v}{\partial \theta}\right)^2$$

Example (27): If $z=f(x,y)$ where $x=ue^{v}$; $y=ue^{-v}$

Prove that: $u^2 z_{uu}-uz_u+z_{vv}=2\left(x^2 z_{xx}+y^2 z_{yy}\right)$

Solution:

$$\because x=ue^{v} \Rightarrow \begin{cases} \dfrac{\partial x}{\partial u}=e^{v}=\dfrac{x}{u} & \Rightarrow \dfrac{\partial x}{\partial u}=\dfrac{x}{u} \\ \dfrac{\partial x}{\partial v}=ue^{v}=x & \Rightarrow \dfrac{\partial x}{\partial v}=x \end{cases} \qquad \rightarrow (1)$$

$$\because y=ue^{-v} \Rightarrow \begin{cases} \dfrac{\partial y}{\partial u}=e^{-v}=\dfrac{y}{u} & \Rightarrow \dfrac{\partial y}{\partial u}=\dfrac{y}{u} \\ \dfrac{\partial y}{\partial v}=-ue^{-v}=-y & \Rightarrow \dfrac{\partial y}{\partial v}=-y \end{cases} \qquad \rightarrow (2)$$

$$\because z_u=\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

By (1) and (2)

$$\therefore z_u=z_x\left(\frac{x}{u}\right)+z_y\left(\frac{y}{u}\right)$$

Multiply by u

$$\therefore uz_u=xz_x+yz_y \qquad \rightarrow (3)$$

By differentiate (3) with respect to u

$$\therefore uz_{uu} + z_u(1) = \frac{\partial}{\partial u}\left(xz_x + yz_y\right)$$

$$\therefore uz_{uu} + z_u = \frac{\partial}{\partial x}\left(xz_x + yz_y\right)\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left(xz_x + yz_y\right)\frac{\partial y}{\partial u}$$

from (1), (2)

$$\therefore uz_{uu} + z_u = \left(xz_{xx} + z_x + yz_{yx}\right)\left(\frac{x}{u}\right) + \left(xz_{yx} + yz_{yy} + z_y\right)\left(\frac{y}{u}\right)$$

Multiply by u

$$\therefore u^2 z_{uu} + uz_u = x^2 z_{xx} + xz_x + xyz_{yx} + xyz_{xy} + y^2 z_{yy} + yz_y \qquad \rightarrow (4)$$

Similarly

$$z_v = \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

From (1) and (2)

$$\therefore z_v = xz_x - yz_y \qquad \rightarrow (5)$$

By differentiate (5) with respect to v

$$\therefore z_{vv} = \frac{\partial}{\partial v}\left(xz_x - yz_y\right)$$

$$\therefore z_{vv} = \frac{\partial}{\partial x}\left(xz_x - yz_y\right)\frac{\partial x}{\partial v} + \frac{\partial}{\partial y}\left(xz_x - yz_y\right)\frac{\partial y}{\partial v}$$

$$\therefore z_{vv} = \left(xz_{xx} + z_x - yz_{yx}\right)(x) + \left(xz_{xy} - yz_{yy} - z_y\right)(-y)$$

$$\therefore z_{vv} = x^2 z_{xx} + xz_x - xyz_{yx} - xyz_{xy} + y^2 z_{yy} + yz_y \qquad \rightarrow (6)$$

By adding (4) and (6), we have

$$\therefore u^2 z_{uu} + uz_u + z_{vv} = 2x^2 z_{xx} + 2xz_x + 2y^2 z_{yy} + 2yz_y$$

$$\therefore u^2 z_{uu} + uz_u + z_{vv} = 2\left(x^2 z_{xx} + y^2 z_{yy}\right) + 2\left(xz_x + yz_y\right)$$

By using (3) $\therefore u^2 z_{uu} - uz_u + z_{vv} = 2\left(x^2 z_{xx} + y^2 z_{yy}\right)$

Example (28): If $z = f(x, y)$ where $x = \frac{1}{2}\log(u^2 + v^2)$; $y = \tan^{-1}\left(\frac{v}{u}\right)$

Prove that : (I) $(z_x)^2 + (z_y)^2 = (u^2 + v^2)\left[(z_u)^2 + (z_v)^2\right]$

(II) $z_{xx} + z_{yy} = (u^2 + v^2)\left[z_{uu} + z_{vv}\right]$

Solution:

$$\because y = \tan^{-1}\left(\frac{v}{u}\right) \Rightarrow \tan(y) = \frac{v}{u}$$

$$\therefore \sin(y) = \frac{v}{\sqrt{u^2 + v^2}}, \cos(y) = \frac{u}{\sqrt{u^2 + v^2}}$$

$$\therefore x = \log(u^2 + v^2)^{\frac{1}{2}} = \log\sqrt{u^2 + v^2}$$

$$\therefore x = \begin{cases} x = \log\left(\dfrac{v}{\sin(y)}\right) \Rightarrow e^x = \dfrac{v}{\sin(y)} \\ x = \log\left(\dfrac{u}{\cos(y)}\right) \Rightarrow e^x = \dfrac{u}{\cos(y)} \end{cases}$$

$$\therefore u = e^x \cos(y) \qquad v = e^x \sin(y)$$

$$\left.\begin{array}{l} \therefore \dfrac{\partial u}{\partial x} = e^x \cos(y) = u \,, \quad \dfrac{\partial u}{\partial y} = -e^x \sin(y) = -v \\ \dfrac{\partial v}{\partial x} = e^x \sin(y) = v \,, \quad \dfrac{\partial v}{\partial y} = e^x \cos(y) = u \end{array}\right\} \rightarrow (1)$$

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$\therefore \frac{\partial z}{\partial x} = u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} \qquad \rightarrow (2)$$

Squaring the equation (2) $\therefore \left(\frac{\partial z}{\partial x}\right)^2 = u^2\left(\frac{\partial z}{\partial u}\right)^2 + v^2\left(\frac{\partial z}{\partial v}\right)^2 + 2uv\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} \qquad \rightarrow (3)$

similarly

$$\therefore \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$\therefore \frac{\partial z}{\partial y} = -v\frac{\partial z}{\partial u} + u\frac{\partial z}{\partial v} \qquad \rightarrow (4)$$

Squaring the equation (4)

$$\therefore \left(\frac{\partial z}{\partial y}\right)^2 = v^2\left(\frac{\partial z}{\partial u}\right)^2 + u^2\left(\frac{\partial z}{\partial v}\right)^2 - 2uv\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} \qquad \rightarrow (5)$$

By adding (3) and (5) we have

$$\therefore \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(u^2+v^2\right)\left(\frac{\partial z}{\partial u}\right)^2 + \left(u^2+v^2\right)\left(\frac{\partial z}{\partial v}\right)^2$$

$$= \left(u^2+v^2\right)\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

From (2) by differentiate with respect to x

$$\therefore z_{xx} = \frac{\partial}{\partial x}\left(u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(uz_u + vz_v\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(uz_u + vz_v\right)\frac{\partial v}{\partial x}$$

$$\therefore z_{xx} = \left(uz_{uu} + z_u + vz_{vu}\right)(u) + \left(uz_{uv} + vz_{vv} + z_v\right)(v)$$

$$\therefore z_{xx} = u^2 z_{uu} + uz_u + uvz_{vu} + uvz_{uv} + v^2 z_{vv} + vz_v \qquad \rightarrow (6)$$

From (4) by differentiate with respect to y

$$\therefore z_{yy} = \frac{\partial}{\partial y}\left(-vz_u + uz_v\right)$$

$$= \frac{\partial}{\partial u}\left(-vz_u + uz_v\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(-vz_u + uz_v\right)\frac{\partial v}{\partial y}$$

$$\therefore z_{yy} = \left(-vz_{uu} + uz_{vu} + z_v\right)(-v) + \left(-vz_{uv} - z_u + uz_{vv}\right)(u)$$

$$\therefore z_{yy} = v^2 z_{uu} - uvz_{vu} - vz_v - uvz_{uv} - uz_u + u^2 z_{vv} \qquad \rightarrow (7)$$

By adding (6) and (7) we have

$$\therefore z_{xx} + z_{yy} = \left(u^2+v^2\right)z_{uu} + \left(u^2+v^2\right)z_{vv}$$

$$\therefore z_{xx} + z_{yy} = \left(u^2+v^2\right)\left(z_{uu} + z_{vv}\right)$$

<u>Exercise on The Chain Rule</u>

(1) Prove that **Laplace's equation** $u_{xx} + u_{yy} = 0$ Transformed by the substitution $x = r\cos(\theta) \quad ; y = r\sin(\theta)$

To $u_{rr} + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r = 0$

(2) Prove that the substitution $x = r\cos(\theta)$; $y = r\sin(\theta)$

Transform Cauchy –Riemann equation $u_y = -v_x, u_x = v_y$ to

$$v_r = -\frac{1}{r}u_\theta, u_r = \frac{1}{r}v_\theta$$

(3) If $f = f(u,v)$ where $u = xe^y, v = xe^{-y}$, Prove that :

$$x^2 f_{xx} - xf_x + f_{yy} = 2\left(u^2 f_{uu} + v^2 f_{vv}\right)$$

(4) Let $x = u\cos\theta - v\sin\theta, y = u\sin\theta + v\cos\theta$

Prove that , if $V = v(x,y)$ then

$$\left(\frac{\partial V}{\partial u}\right)^2 + \left(\frac{\partial V}{\partial v}\right)^2 = \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2$$

(5) If $z = f(x^2 y)$ where f is differentiable function .

Prove that : $x\left(\frac{\partial z}{\partial x}\right) = 2y\left(\frac{\partial z}{\partial y}\right)$

(6) Let $z = f(x,y)$ and $x = e^u \cos t, y = e^u \sin t$.

Prove that : $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^{-2u}\left[\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial t^2}\right]$

(7) Let $z = f(x,y)$ and $x = e^u + e^{-v}, y = e^v + e^{-u}$

Prove that : $z_{uu} - 2z_{uv} + z_{vv} = x^2 z_{xx} - 2xyz_{xy} + y^2 z_{yy} + xz_x + yz_y$

(8) If $u = \Phi(x + at) + \Psi(x - at)$

Prove that : $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$

(9) proved that the amount $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}$ does not change as a result of the transfer of the original axes x, y Parallel to their position the point (a, b)

(10) If $z=f(x, y)+g(u)$ where $u=xy$.

Prove that : $w=xz_x-yz_y$ does not depend on the function g , then find w when $f=xye^{x-y}$

(11) Let V be a function on two variables x, y , and both x, y is a function on two variables u, v satisfy $x_u=y_v, x_v=-y_u$. Prove that : $V_{uu}+V_{vv}=\left(x_u{}^2+x_v{}^2\right)\left(V_{xx}+V_{yy}\right)$

(12) Let $u=y\log(y+v)-r$ where $r^2=x^2+y^2$.

Prove that : $u_{xx}+u_{yy}=\frac{1}{y+r}$

(13) Prove that : If we change the variables x, y in the equation

$$z_{xx}+2xy^2z_x+2\left(y-y^3\right)z_y+x^2y^2=0$$

By the variables u, v , where $x=uv, y=\frac{1}{v}$, then we can get the same equation by replacing u by x and v by y

(14) If the function $f(x, y)$ changed to the function $\Phi(u, v)$ by the substitutions $x=u\cosh v, y=u\sinh v$, then prove that

$$f_{xx}-f_{yy}=\Phi_{uu}-\frac{1}{u^2}\Phi_{vv}+\frac{1}{u}\Phi_u$$

(15) Change the variables x, y, z in the equation

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu$$

By the variables ξ, η, ζ , where $\xi = \dfrac{x}{z}, \eta = \dfrac{y}{z}, \zeta = z$

Then prove that the equation changes to the form $\xi u_{\xi} = nu$

Directional Derivatives in the Plane

We know that if $f(x,y)$ is differentiable, then the rate at which f changes with respect to t along a differentiable curve $x = g(t), y = h(t)$ is

$$\frac{d\,f}{d\,t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

At any point $p_0(x_0, y_0) = p_0(g(t_0), h(t_0))$, h(to)), this equation gives the rate of change of f with respect to increasing t and therefore depends, among other things, on the direction of motion along the curve. If the curve is a straight line and t is the arc length parameter along the line measured from p_0 in the direction of a given unit vector u , then df/dt is the rate of change of f with respect to distance in its domain in the direction of u . By varying u , we find the rates at which f changes with respect to distance as we move through p_0 in different directions. We now define this idea more precisely. Suppose that the function $f(x,y)$ is defined throughout a region R in the xy-plane, that $p_0(x_0, y_0)$ is a point in R , and that $u = u_1 i + u_2 j$ is a unit vector. Then the equations

$$x = x_0 + su_1, \quad y = y_0 + su_2$$

parameterize the line through p_0 parallel to u . If the parameter s measures arc length from p_0 in the direction of u , we find the rate of change of f at p_0 in the direction of u by calculating df/ds at p_0 Figure (5).

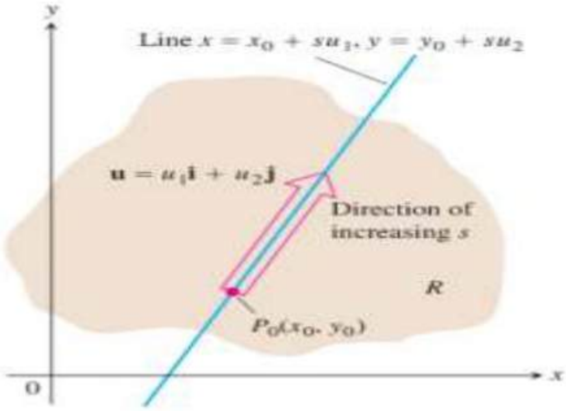

Figure .5 The rate of change of f in the direction of u at a point p_0 is the rate at which f changes along this line at p_0

Definition: The derivative of f at $p_0(x_0, y_0)$ in the direction of the unit vector $u = u_1 i + u_2 j$ is the number

$$\left(\frac{d\,f}{d\,s}\right)_{u,\,p_0} = \lim_{s\to 0}\frac{f(x_0+su_1, y_0+su_2)-f(x_0\,,y_0)}{s} \qquad (1)$$

Provided the limit exists.

The **directional derivative** defined by Equation (1) is also denoted by

$(D_u f)_{p_0}$ "The derivative of f at p_0 in the direction of u "

The partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ are the directional derivatives of f at p_0 in the i and j directions. This observation can be seen by comparing Equation (1) to the definitions of the two partial derivatives given before.

Example(29): Using the definition, find the derivative of $f(x,y) = x^2 + xy$ at $p_0\ (1, 2)$ in the direction of the unit vector $u = (1/\sqrt{2})i + (1/\sqrt{2})\,j$

Solution:

$$\left(\frac{d\,f}{d\,s}\right)_{u,\,p_0} = \lim_{s\to 0}\frac{f(x_0+su_1, y_0+su_2)-f(x_0\,,y_0)}{s}$$

$$= \lim_{s\to 0}\frac{f(1+s\frac{1}{\sqrt{2}}, 2+s\frac{1}{\sqrt{2}})-f(1,2)}{s}$$

$$= \lim_{s\to 0}\frac{(1+\frac{2s}{\sqrt{2}}+\frac{s^2}{2})+(2+\frac{3s}{\sqrt{2}}+\frac{s^2}{2})-3}{s}$$

$$= \lim_{s \to 0} \frac{\frac{5s}{\sqrt{2}} + s^2}{s} = \lim_{s \to 0} \left(\frac{5}{\sqrt{2}} + s \right) = \frac{5}{\sqrt{2}}$$

The rate of change of $f(x, y) = x^2 + xy$ at $p_0\ (1, 2)$ in the direction u is $5/\sqrt{2}$

For a physical interpretation of the directional derivative, suppose that $T = f(x, y)$ is the temperature at each point (x, y) over a region in the plane. Then $f(x_0, y_0)$ is the temperature at the point $p_0(x_0, y_0)$ and $(D_u f)_{p_0}$ is the instantaneous rate of change of the temperature at p_0 stepping of f in the direction u.

Calculation and Gradients

We now develop an efficient formula to calculate the directional derivative for a differentiable function f. We begin with the line

$$x = x_0 + su_1, \quad y = y_0 + su_2$$

through $p_0(x_0, y_0)$, parameterized with the arc length parameter s increasing in the direction of the unit vector $u = u_1 i + u_2 j$ .Then by the Chain Rule we find

$$\begin{aligned}
\left(\frac{d f}{d s} \right)_{u, p_0} &= \left(\frac{\partial f}{\partial x} \right)_{p_0} \frac{dx}{ds} + \left(\frac{\partial f}{\partial y} \right)_{p_0} \frac{dy}{ds} \\
&= \left(\frac{\partial f}{\partial x} \right)_{p_0} u_1 + \left(\frac{\partial f}{\partial y} \right)_{p_0} u_2 \\
&= \left[\left(\frac{\partial f}{\partial x} \right)_{p_0} i + \left(\frac{\partial f}{\partial y} \right)_{p_0} j \right] \bullet [u_1\, i + u_2\, j] \qquad (3)
\end{aligned}$$

Equation (3) says that the derivative of a differentiable function f in the direction of u at p_0 is the dot product of u with the special vector called the ***gradient*** of f at p_0 .

Definition: The gradient vector (gradient) of $f(x, y)$ at a point $p_0(x_0, y_0)$ is the vector $\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$

obtained by evaluating the partial derivatives of f at p_0 .

Theorem: The Directional Derivative Is a Dot Product

If $f(x, y)$ differentiable in an open region containing $p_0(x_0, y_0)$, then

$$\left(\frac{d f}{d s} \right)_{u, p_0} = (\nabla f)_{p_0} \bullet u,$$

the dot product of the gradient ∇f at p_0 and u .

Example(29): Find the derivative of $f(x,y) = xe^y + \cos(xy)$ at the point $(2,0)$ in the direction of the unit vector $v = 3i - 4j$

Solution: The direction of v is the unit vector obtained by dividing v by its length:

$$u = \frac{v}{|v|} = \frac{v}{5} = \frac{3}{5}i - \frac{4}{5}j$$

The partial derivatives of f are everywhere continuous and at (2, 0) are given by

$$f_x(2,0) = (e^y - y\sin(xy))_{(2,0)} = e^0 - 0 = 1$$

$$f_y(2,0) = (xe^y - x\sin(xy))_{(2,0)} = 2e^0 - 2\cdot 0 = 2$$

The gradient of f at (2, 0) is

$$\nabla f\,|_{(2,0)} = f_x(2,0)i + f_y(2,0)j = i + 2j$$

Figure ( ). The derivative of f at (2, 0) in the direction of v is therefore

$$(D_u f)|_{(2,0)} = \nabla f\,|_{(2,0)} \cdot u = (i+2j)\cdot(\frac{3}{5}i - \frac{4}{5}j) = \frac{3}{5} - \frac{8}{5} = -1$$

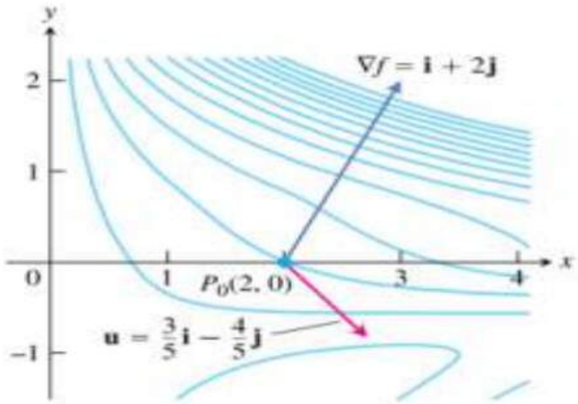

Figure .6 Picture ∇f as a vector in the domain of f .

Tangent Planes and Differentials

In this section we define the tangent plane at a point on a smooth surface in space. Then we show how to calculate an equation of the tangent plane from the partial derivatives of the function defining the surface. This idea is similar to the definition of the tangent line at a point on a curve in the coordinate plane for single-variable functions . We then study the total differential and linearization of functions of several variables.

Tangent Planes and Normal Lines

If $r=g(t)i+h(t)j+k(t)k$ is a smooth curve on the level surface $f(x,y,z)=c$ of a differentiable function f, then $f(g(t),h(t),k(t))=c$. Differentiating both sides of this equation with respect to t leads to

$$\frac{d}{dt}f(g(t),h(t),k(t))=\frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt}+\frac{\partial f}{\partial y}\frac{dh}{dt}+\frac{\partial f}{\partial z}\frac{dk}{dt}=0$$

$$\underbrace{\left(\frac{\partial f}{\partial x}i+\frac{\partial f}{\partial y}j+\frac{\partial f}{\partial z}k\right)}_{\nabla f}\cdot\underbrace{\left(\frac{dg}{dt}i+\frac{dh}{dt}j+\frac{dk}{dt}k\right)}_{\frac{dr}{dt}}=0$$

At every point along the curve, ∇f is orthogonal to the curve's velocity vector.
Now let us restrict our attention to the curves that pass through p_0 Figure ( ). All the velocity vectors at p_0 are orthogonal to ∇f at Po, so the curves' tangent lines all lie in the plane through p_0 normal to ∇f. We now define this plane.

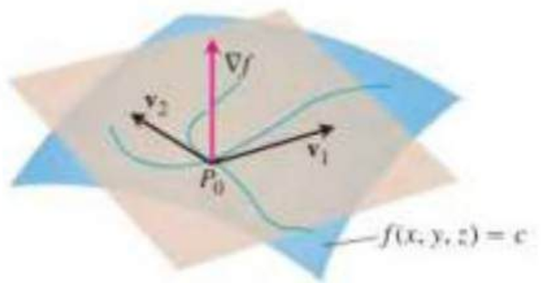

Figure .7 The gradient ∇f is orthogonal to the velocity vector of every smooth curve in the surface through p_0. The velocity vectors at Po therefore lie in a common plane, which we call the tangent plane at p_0.

Definition: The **tangent plane** at the point $p_0(x_0, y_0, z_0)$ on the level surfaces $f(x,y,z)=c$ of the differentiable function f is the plain through p_0 normal to t a point is the vector $\nabla f|_{p_0}$
The normal line of the surfaces at p_0 is the line through p_0 parallel to $\nabla f|_{p_0}$.

the tangent plane and normal line have the following equations:

Tangent plane to $f(x,y,z)=c$ at $p_0(x_0, y_0, z_0)$

$$f_x(p_0)(x-x_0)+f_y(p_0)(y-y_0)+f_z(p_0)(z-z_0)=0$$

Normal line to $f(x,y,z)=c$ at $p_0(x_0, y_0, z_0)$

$$x = x_0 + f_x(p_0)t \quad , \quad y = y_0 + f_y(p_0)t \, , \quad z = z_0 + f_z(p_0)t$$

Example(30): Find the tangent plane and normal line of the surface

$$f(x, y, z) = x^2 + y^2 + z - 9 = 0$$

at the point $p_0(1,2,4)$.

Solution: The surface is shown in Figure 8. The tangent plane is the plane through p_0 perpendicular to the gradient of f at p_0. The gradient is

$$\nabla f|_{p_0} = (2xi + 2yj + k)_{(1,2,4)} = 2i + 4j + k$$

The tangent plane is therefore the plane

$$2(x-1) + 4(y-2) + (z-4) = 0 \qquad or \qquad 2x + 4y + z = 14$$

The line normal to the surface at p_0 is

$x = 1 + 2t \; , \quad y = 2 + 4t, \quad z = 4 + t$

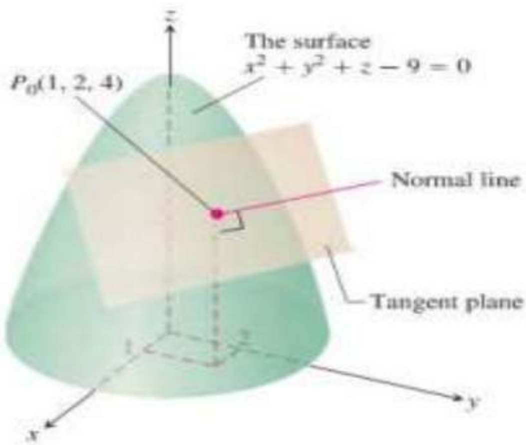

Figure .8 The tangent plane and normal line to this surface at p_0

To find an equation for the plane tangent to a smooth surface $z = f(x, y)$ at a point $p_0(x_0, y_0, z_0)$ where $z_0 = p_0(x_0, y_0)$ we first observe that the equation $z = f(x, y)$ is equivalent to $f(x, y) - z = 0$. The surface $z = f(x, y)$ is therefore the zero level surface of the function $F(x, y, z) = f(x, y) - z$. The partial derivatives of F are

$$F_x = \frac{\partial}{\partial x}(f(x, y) - z) = f_x - 0 = f_x$$

$$F_y = \frac{\partial}{\partial y}(f(x, y) - z) = f_y - 0 = f_y$$

$$F_z = \frac{\partial}{\partial z}(f(x, y) - z) = 0 - 1 = -1$$

The formula

$$F_x(p_0)(x - x_0) + F_y(p_0)(y - y_0) + F_z(p_0)(z - z_0) = 0$$

for the plane tangent to the level surface at Po therefore reduces to

$$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)+(z-z_0)=0 \qquad (4)$$

Example(31): Find the plane tangent to the surface $z = x\cos y - y e^x$ at the point $p_0(0,0,0)$.

Solution: We calculate the partial derivatives of $f(x,y) = x\cos y + y e^x$ and use Equation (4):

$$f_x(0,0) = (\cos y - y e^x)_{(0,0)} = 1 - 0\cdot 1 = 1$$

$$f_y(0,0) = (-x\sin y - e^x)_{(0,0)} = 0 - 1 = -1$$

The tangent plane is therefore the plane

$$1(x-0) - 1(y-0) - (z-0) = 0 \qquad or \qquad x - y - z = 0$$

The Harmonic Function

Definition: The function $f(x,y)$ is sais to be harmonic function if it satisfies Laplace's equation i.e.,

$$f_{xx} + f_{yy} = 0$$

everywhere on D. This is usually written as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{where} \quad \nabla^2 f = 0$$

Example(32): The following functions are harmonic

- The function $f(x,y) = e^x \sin y$
- The function $f(x,y) = \cos x \sin(-y)$
- The function $f(x,y) = \ln(x^2+y^2)$ define on $R^2 - (0,0)$
- The function $f(x,y) = (x^2+y^2)^{1-n/2}$ define on $R^2 - (0,0)$ for $n > 2$

Theorem: If $u = u(x,y), v = v(x,y)$ two functions on the variables x, y , satisfying the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

And assume that $u_{yx}, u_{xy}, v_{yx}, v_{xy}$ are continuous functions . Then u, v are harmonic unction.

Proof: $\because \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$

$\therefore$ differentiating with respect to x

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad \rightarrow (1)$$

$$\because \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ differentiating with respect to y

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \qquad \rightarrow (2)$$

By adding (1) and(2)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x}$$

From continuity condition

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore$ u is harmonic

Similarly , for proving that v is harmonic

$$\because \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$\therefore$ differentiating with respect to x

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$\therefore \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} \qquad \rightarrow (3)$$

$$\because \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$\therefore$ differentiating with respect to y

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$\therefore \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} \qquad \rightarrow (4)$$

By adding (3) and(4)

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 u}{\partial x \partial y}$$

From continuity condition

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\therefore v$ is harmonic

Theorem:

If $f = f(u,v)$ is a harmonic function in the variables u, v and $u = u(x,y), v = v(x,y)$ satisfying the Cauchy–Riemann equations . Then $f = f(x,y)$ is harmonic in the variables x, y

Proof:

Since $f = f(u,v)$ is a harmonic function in the variables u, v

$$\therefore \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = 0 \qquad \rightarrow (1)$$

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

$\therefore$ differentiating with respect to x

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\right)$$

$$=\frac{\partial f}{\partial u}\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\right)+\frac{\partial f}{\partial v}\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v}\right)$$

$$=\frac{\partial f}{\partial u}\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\left[\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u}\right)\frac{\partial v}{\partial x}\right]$$

$$+\frac{\partial f}{\partial v}\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial x}\left[\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 f}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2+2\frac{\partial^2 f}{\partial u\partial v}\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}+\frac{\partial^2 f}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2+\frac{\partial f}{\partial u}\frac{\partial^2 u}{\partial x^2}+\frac{\partial f}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

From Cauchy–Riemann equations.

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 f}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2-2\frac{\partial^2 f}{\partial u\partial v}\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}+\frac{\partial^2 f}{\partial v^2}\left(\frac{\partial u}{\partial y}\right)^2+\frac{\partial f}{\partial u}\frac{\partial^2 u}{\partial x^2}-\frac{\partial f}{\partial v}\frac{\partial^2 u}{\partial x\partial y}\rightarrow(2)$$

Similarly

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial^2 f}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2+2\frac{\partial^2 f}{\partial u\partial v}\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}+\frac{\partial^2 f}{\partial v^2}\left(\frac{\partial u}{\partial x}\right)^2-\frac{\partial f}{\partial u}\frac{\partial^2 u}{\partial x^2}+\frac{\partial f}{\partial v}\frac{\partial^2 u}{\partial y\partial x}\rightarrow(3)$$

By adding (2) and (3)

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]\left[\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}\right]$$

By using (1) $\quad \therefore \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$

Homogeneous function

Definition: A homogeneous function of two variables x and y is a real-valued function $f(x,y)$ that satisfies the condition $f(\lambda x,\lambda y)=\lambda^n f(x,y)$ for some constant n and all real numbers λ. The constant n is called the **degree of homogeneity**.

Example(33): The function

$$f(x,y)= xy^2 \sin^{-1}\left(\frac{y}{x}\right)+\frac{y^4}{x}\log\left(\frac{x}{y}\right)$$

Is a homogeneous function of degree three , since

$$f(\lambda x,\lambda y)=(\lambda x)(\lambda^2 y^2)\sin^{-1}\left(\frac{\lambda y}{\lambda x}\right)+\frac{\lambda^4 y^4}{\lambda x}\log\left(\frac{\lambda x}{\lambda y}\right)$$

$$=\lambda^3 xy^2 \sin^{-1}\left(\frac{y}{x}\right)+\lambda^3\frac{y^4}{x}\log\left(\frac{x}{y}\right)$$

$$=\lambda^3\left[xy^2 \sin^{-1}\left(\frac{y}{x}\right)+\frac{y^4}{x}\log\left(\frac{x}{y}\right)\right]$$

$$f(\lambda x,\lambda y)=\lambda^3 f(x,y)$$

Euler Theorem for homogeneous function

Theorem: Let $f(x,y)$ be a homogeneous function of degree n in the variables x,y . Then

(1) $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$

(2) $x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=n(n-1)f$

Proof:

Let $f(x,y)$ be a homogeneous function of degree n in the variables x,y

$$\therefore f(\lambda x,\lambda y)=\lambda^n f(x,y) \qquad \rightarrow (1)$$

Put $u=\lambda x, v=\lambda y$, where $\frac{\partial u}{\partial \lambda}=x,\frac{\partial v}{\partial \lambda}=\lambda$

$\therefore f(u,v)=\lambda^n f(x,y)$

Differentiate both sides w.r.to λ

$$\frac{\partial}{\partial \lambda} f(u, v) = n\lambda^{n-1} f(x, y)$$

$$\therefore \frac{\partial f}{\partial u}\frac{\partial u}{\partial \lambda} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial \lambda} = n\lambda^{n-1} f(x, y)$$

$$\therefore x\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v} = n\lambda^{n-1} f(x, y)$$

At $\lambda = 1$ we have $u = x, v = y$

$$\therefore x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y) \qquad \rightarrow (2)$$

From (1) differentiate both sides w.r. to x

$$x\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x} + y\frac{\partial^2 f}{\partial y \partial x} = n\frac{\partial f}{\partial x}$$

$$\therefore x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial y \partial x} = (n-1)\frac{\partial f}{\partial x}$$

Multiply by x

$$\therefore x^2\frac{\partial^2 f}{\partial x^2} + xy\frac{\partial^2 f}{\partial y \partial x} = (n-1)x\frac{\partial f}{\partial x} \qquad \rightarrow (3)$$

From (2) differentiate both sides w.r. to y

$$\therefore x\frac{\partial^2 f}{\partial x \partial y} + y\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial y} = (n)\frac{\partial f}{\partial y}$$

$$\therefore x\frac{\partial^2 f}{\partial x \partial y} + y\frac{\partial^2 f}{\partial y^2} = (n-1)\frac{\partial f}{\partial y}$$

Multiply by y

$$\therefore xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = (n-1)y\frac{\partial f}{\partial y} \qquad \rightarrow (4)$$

By adding (3) and (4)

$$\therefore x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = (n-1)\left[x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} \right]$$

From the first part of the theorem , we have

$$\therefore x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = n(n-1) f$$

Example(34): If

$$f(x, y) = x^2 y^3 \sin^{-1}\left(\frac{y}{x} \right) + \frac{y^6}{x} \tan^{-1}\left(\frac{x}{y} \right) + x^5 \log\left(\frac{x}{y} \right)$$

Prove that :

(I) $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 5 f$

(II) $x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 20 f$

Solution: By testing the homogeneous of the function

$$f(\lambda x, \lambda y) = (\lambda^2 x^2)(\lambda^3 y^3) \sin^{-1}\left(\frac{\lambda y}{\lambda x} \right) + \frac{\lambda^6 y^6}{\lambda x} \tan^{-1}\left(\frac{\lambda x}{\lambda y} \right) + \lambda^5 x^5 \log\left(\frac{\lambda x}{\lambda y} \right)$$

$$\therefore f(\lambda x, \lambda y) = \lambda^5 x^2 y^3 \sin^{-1}\left(\frac{y}{x} \right) + \lambda^5 \frac{y^6}{x} \tan^{-1}\left(\frac{x}{y} \right) + \lambda^5 x^5 \log\left(\frac{x}{y} \right)$$

$$\therefore f(\lambda x, \lambda y) = \lambda^5 f(x, y)$$

$\therefore$ The function is homogeneous of degree 5

Then by using Euler's theorem , we have

(I) $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 5 f$

Also

(II) $x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 5(5-1) f = 20 f$

Example(35): If $u = \log v$ where v is homogeneous function of degree n in the variables x, y .

Prove that : $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=n$

Solution: Since v is homogeneous function of degree n in the variables x, y.

From Euler's theorem

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=nv \qquad \to (1)$$

$$\therefore u=\log v$$

$$\therefore \frac{\partial u}{\partial x}=\frac{\partial u}{\partial v}\frac{\partial v}{\partial x}\Rightarrow\frac{\partial u}{\partial x}=\frac{1}{v}\frac{\partial v}{\partial x}$$

Multiply by x

$$\therefore x\frac{\partial u}{\partial x}=\frac{x}{v}\frac{\partial v}{\partial x} \qquad \to (2)$$

Also

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial v}\frac{\partial v}{\partial y}\Rightarrow\frac{\partial u}{\partial y}=\frac{1}{v}\frac{\partial v}{\partial y}$$

Multiply by y

$$\therefore y\frac{\partial u}{\partial y}=\frac{y}{v}\frac{\partial v}{\partial y} \qquad \to (3)$$

By adding (2) and (3)

$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{v}\left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}\right)$$

From (1) , we have

$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{v}(nv)=n$$

Exercises on Homogeneous functions

(1) If $w=\left(\dfrac{x-y}{x+y}\right)^n$ prove that

(a) $x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=0$

(b) $x^2\dfrac{\partial^2 w}{\partial x^2}+2xy\dfrac{\partial^2 w}{\partial x\partial y}+y^2\dfrac{\partial^2 w}{\partial y^2}=0$

(2)) If $f(x,y)=(x^2-y^2)g\left(\dfrac{x}{y}\right)$ prove that

(a) $xf_x+yf_y=2f$

(b) $x^2f_{xx}+2xyf_{xy}+y^2f_{yy}=2f$

(3) Prove that the function $z=x\cos\left(\dfrac{y}{x}\right)+y\tan\left(\dfrac{y}{x}\right)$

Satisfying the relation

$$x^2z_{xx}+2xyz_{xy}+y^2z_{yy}=0$$

(4) If $z=x^4y^2\sin^{-1}\left(\dfrac{y}{x}\right)$ prove that:

(I) $xz_x+yz_y=6z$

(II) $x^2z_{xx}+2xyz_{xy}+y^2z_{yy}=30z$

(5) If $f(x,y)=\sqrt[3]{x^3+y^3}$ Prove that:

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=f$$

(6) If $z=x\Phi\left(\dfrac{y}{x}\right)$ Prove that:

$$x^2z_{xx}+2xyz_{xy}+y^2z_{yy}=0$$

(7) Let $z=f(x,y)$ where x,y are two homogeneous functions of degree P on the variables u,v .

Prove that :

$$u^2 f_{uu} + 2ufz_{uv} + v^2 f_{vv} + uf_u + vf_v = P^2\left(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy} + xf_x + yf_y\right)$$

(8) If $f = f(x_1, x_2, x_3)$ is homogeneous function of degree n .

And $u = \dfrac{x_1}{x_3}, v = \dfrac{x_2}{x_3}, w = x_3$

Prove that : $w\dfrac{\partial f}{\partial w} = n\, f$

Taylor series in several variables

The Taylor series may also be generalized to functions of more than one variable . For example, for a function $z = f(x, y)$ that depends on two variables, x and y, the **Taylor series** to second order about the point (a, b) is

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy} + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

Example(36): Find the Taylor series generated by

$f(x, y) = (x + y^2 + 1)^{1/2}$ at $(a,b) = (1,0)$.

Solution: We first need to computes all the necessary partial derivatives:

$$f_x = \frac{1}{2}(x + y^2 + 1)^{-\frac{1}{2}}$$

$$f_y = y(x + y^2 + 1)^{-\frac{1}{2}}$$

$$f_{xx} = -\frac{1}{4}(x + y^2 + 1)^{-\frac{3}{2}}$$

$$f_{yy} = (x+y^2+1)^{-\frac{1}{2}} - y^2(x+y^2+1)^{-\frac{3}{2}}$$

$$f_{xy} = f_{yx} = -\frac{y}{2}(x+y^2+1)^{-\frac{3}{2}}$$

Evaluating these derivatives at the origin gives the Taylor coefficients

$$f(1,0) = \sqrt{2}$$

$$f_x(1,0) = \frac{\sqrt{2}}{4}$$

$$f_y(1,0) = 0$$

$$f_{xx}(1,0) = -\frac{1}{8\sqrt{2}}$$

$$f_{yy}(1,0) = \frac{1}{\sqrt{2}}$$

$$f_{xy}(1,0) = f_{yx}(1,0) = 0$$

Substituting these values in to the general formula

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] +$$
$$\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy} + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

produces

$$f(1,0) = \sqrt{2} + \left[(x-1)\,\frac{\sqrt{2}}{4} + (y-0)\,0\right] + \frac{1}{2!}\left[(x-1)^2\,\frac{-1}{8\sqrt{2}} + 2(x-1)(y-0)\,0 + (y-0)^2\,\frac{1}{\sqrt{2}}\right] + \cdots$$

$$= \sqrt{2} + \frac{\sqrt{2}}{4}(x-1) + \frac{1}{2\sqrt{2}}y^2 + \frac{-1}{16\sqrt{2}}(x-1)^2 + \cdots.$$

Example(37): Find the Taylor series generated by $f(x,y) = e^x \log(y+1)$ at $(a,b) = (0,0)$.

Solution: We first need to computes all the necessary partial derivatives:

$$f_x = e^x \log(y+1)$$

$$f_y = \frac{e^x}{1+y}$$

$$f_{xx} = e^x \log(y+1)$$

$$f_{yy} = -\frac{e^x}{(1+y)^2}$$

$$f_{xy} = f_{yx} = \frac{e^x}{(1+y)^2}$$

Evaluating these derivatives at the origin gives the Taylor coefficients

$$f(0,0) = 0$$

$$f_x(0,0) = 0$$

$$f_y(0,0) = 1$$

$$f_{xx}(0,0) = 0$$

$$f_{yy}(0,0) = -1$$

$$f_{xy}(0,0) = f_{yx}(0,0) = 1$$

Substituting these values in to the general formula

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] +$$
$$\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy} + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

produces

$$f(0,0) = 0 + \left[(x-0)0 + (y-0)1\right] + \frac{1}{2!}\left[(x-0)^2\, 0 + 2(x-0)(y-0)\, 1 + (y-0)^2\, (-1)\right] + \cdots$$

$$= y + x\,y - \frac{y^2}{2} + \cdots.$$

Since $\log(y+1)$ is analytic in $|\,y\,| < 1$, we have

$$e^x \log(y+1) = y + x\,y - \frac{y^2}{2} + \cdots$$