The Complex Numbers

The student has been known from previous study that the set of real numbers (R) is the largest set of numbers that Includes the natural numbers (N) , integers (Z) , rational numbers (Q) and irrational numbers, We'll study greater set of numbers that Includes the real numbers and non-real numbers (**imaginary**) and called this set (**Complex Numbers**) which symbolized (**C**)

Before the appearance of complex numbers were negative numbers under the square root ($\sqrt{-x}$) is not defined so we assume the existence of ($i=\sqrt{-1}$) that called the imaginary number and therefore we can now write any negative number under the square root as follows:

$$\sqrt{-2} = \sqrt{2}.\sqrt{-1} = \sqrt{2}\,i$$
$$\sqrt{-3} = \sqrt{3}.\sqrt{-1} = \sqrt{3}\,i$$
$$\sqrt{-4} = \sqrt{4}.\sqrt{-1} = 2i$$
$$\sqrt{-5} = \sqrt{5}.\sqrt{-1} = \sqrt{5}\,i$$
$$\sqrt{-9} = \sqrt{9}.\sqrt{-1} = 3i$$

$$\Rightarrow \quad \boxed{\sqrt{-x} = \sqrt{x}\,i}$$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

Definition : The Complex Number Is the number which consists of a **real part (a)** and **imaginary part (b)** :

$$C = a + bi \qquad : \qquad a,b \in R \quad , \quad i = \sqrt{-1}$$

i^n

$$i^0 = 1 \qquad i^1 = i \qquad i^2 = -1 \qquad i^3 = -i$$

$$i^{4n+r} = i^r \quad : \quad n \in Z$$

Examples :

$i^5 == i^{4+1} = i^4 . i^1 = i^1 = i$

$i^{17} = i^{4(4)} . i^1 = i^1 = i$

$i^{98} = i^{4(24)} . i^2 = i^2 = -1$

$i^{563} = i^{4(140)} . i^3 = i^3 = -i$

$i^{216} = i^{4(54)} . i^0 = i^0 = 1$

$i^{257} = i^{4(64)} . i^1 = i^1 = i$

$i^{138} = i^{4(34)} . i^2 = i^2 = -1$

$i^{409} = i^{4(102)} . i^1 = i^1 = i$

$i^{-3} = \frac{1}{i^3} = \frac{i^4}{i^3} = i$

$i^{-34} = \frac{1}{i^{34}} = \frac{1}{i^{4(8)} . i^2} = \frac{1}{i^2} = \frac{i^4}{i^2} = i^2 = -1$

$i^{-345} = \frac{1}{i^{345}} = \frac{1}{i^{4(86)} . i^1} = \frac{i^4}{i^1} = i^3 = -i$

$i^{4n+23} = i^{4(n+5)} . i^3 = i^3 = -i$

$i^{20n+62} = i^{4(5n+15)} . i^2 = i^2 = -1$

$i^{44n-17} = \frac{i^{44n}}{i^{17}} = \frac{i^{4(11n)}}{i^{4(4)} . i^1} = \frac{1}{i^1} = \frac{i^4}{i^1} = i^3 = -i$

H.W. :

1) $i^{55} =$

2) $i^{122} =$

3) $i^{87} =$

4) $i^{420} =$

5) $i^{-30} =$

6) $i^{-67} =$

1) $i^{-99} =$

2) $i^{-88} =$

3) $i^{4n+5} =$

4) $i^{12n+22} =$

5) $i^{44n} =$

6) $i^{20n-47} =$

$5 = 5 + 0i \quad \rightarrow$ Pure Real Number

$3i = 0 + 3i \quad \rightarrow$ Pure Imaginary Number

The Addition and the Subtraction of Complex Numbers :

Examples :

1) $(5+2i)+(3-4i)=8-2i$
2) $(7i+3)-(1-2i)=7i+3-1+2i=2+9i$

The Multiplication of Complex Numbers :

$$(a+bi)(x+yi) = ax+ayi+bxi+byi^2 = (ax-by)+(ay+bx)i$$

Examples :

1) $(5+2i)(2-i) = 10-5i+4i-2i^2 = 10-5i+4i+2 = 12-i$
2) $(1+3i)(2-6i) = 2-6i+6i-18i^2 = 2+18 = 20+0i$
3) $(5+i)^2 = 25+10i+i^2 = 25+10i-1 = 24+10i$

H.W. :

1) $(7+i)(1-2i)=$

2) $(4-3i)^2 =$

3) $(3-i)^2 =$

The Equality Of Complex Numbers

The Two Complex Numbers are Equal if and only if the (rp) of the first one equal to the (rp) of the other one and the (ip) of the first one equal to the (ip) of the other one

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Example : Find the real value of $x, y \in R$ which satisfied the equation :

a) $2x + 2i - 1 = 1 + yi + i$

$rp = rp \quad \Rightarrow \quad 2x - 1 = 1 \quad \Rightarrow \quad 2x = 2 \quad \Rightarrow \quad \boxed{x = 1}$

$ip = ip \quad \Rightarrow \quad 2 = y + 1 \quad \Rightarrow \quad 2 - 1 = y \quad \Rightarrow \quad \boxed{y = 1}$

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b) $2y - 2xi + 4 + i = -2 + 3i$

$rp = rp \quad \Rightarrow \quad 2y + 4 = -2 \quad \Rightarrow \quad 2y = -6 \quad \Rightarrow \quad \boxed{y = -3}$

$ip = ip \quad \Rightarrow \quad -2x + 1 = 3 \quad \Rightarrow \quad -2x = 2 \quad \Rightarrow \quad \boxed{x = -1}$

M.Q.) $y + 7i = (3x + i)(x + 2i)$

$y + 7i = 3x^2 + 6xi + xi + 2i^2$

$y + 7i = 3x^2 + 7xi - 2$

$y = 3x^2 - 2 \quad .....(1)$

$7 = 7x \quad \Rightarrow \boxed{x = 1}$

$\xrightarrow{in\ (1)} \quad y = 3(1)^2 - 2 \quad \Rightarrow \boxed{y = 1}$

Special Cases :

$$(1+i)^2 = 1+2i-1 = 2i \quad \Rightarrow \quad \boxed{(1+i)^2 = 2i}$$

$$(1-i)^2 = 1-2i-1 = -2i \quad \Rightarrow \quad \boxed{(1-i)^2 = -2i}$$

Example : Calculate $(1+i)^6$ as $(a+bi)$

Solution : $(1-i)^6 = \left((1-i)^2\right)^3 = (\cancel{1}-2i-\cancel{1})^3 = (-2i)^3 = (-)^3(2)^3(i)^3 = -8(-i) = 8i$

Example : Calculate $(1+i)^9$ as $(a+bi)$

Solution :

$$(1+i)^9 = (1+i)^8.(1+i) = \left((1+i)^2\right)^4.(1+i) = (\cancel{1}+2i-\cancel{1})^4.(1+i)$$
$$= (2i)^4.(1+i) = 16i^4.(1+i) = 16.(1+i) = 16+16i$$

H.W. :

1) $(1+i)^3 =$

2) $(1+i)^8 =$

3) $(1-i)^7 =$

4) $(1-i)^{15} =$

5) $(1+i)^5-(1-i)^5 = \left((1+i)^2\right)^2(1+i)-\left((1-i)^2\right)^2(1-i)$

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6) $(1+i)^6-(1-i)^6 =$

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7) $\dfrac{(1+i)^7}{8} =$

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8) $\dfrac{(1+i)^{15}}{128} =$

Example : *Analyze the numbers into two factors in* $\mathbb{C}$:

$$(x^2+9) \quad , \quad (1+4y^2) \quad , \quad 10 \quad , \quad 125$$

Solution :

1) $x^2+9 = x^2-9i^2 = (x-3i)(x+3i)$

2) $1+4y^2 = 1-4y^2i^2 = (1-2yi)(1+2yi)$

3) $10 = 1+9 = 1-9i^2 = (1-3i)(1+3i)$

4) $125 = 100+25 = 100-25i^2 = (10-5i)(10+5i)$

H.W. : *Analyze the numbers into two factors in* $\mathbb{C}$:

73

29

90

85

61

50

x^2+4

$16+25y^2$

The Conjugate of Complex Number : ($\overline{C}$)

To find the conjugate of Complex Number we reverse only the sign of (ip) :

$\forall a,b \in R \quad , \quad C = a+bi \quad \Rightarrow \quad \overline{C} = a-bi$

Examples :

1) $\overline{2+3i} = 2-3i$
2) $\overline{3i} = -3i$
3) $\overline{5} = 5$
4) $\overline{i-5} = -i-5$
5) $\overline{\left(\dfrac{7-2i}{1+4i}\right)} = \dfrac{7+2i}{1-4i}$
6) $\overline{\left(\dfrac{3-5i}{i}\right)} = \dfrac{3+5i}{-i}$

Conclusion :

$$\boxed{(a+bi)(a-bi) = a^2+b^2}$$

Examples :

1) $(2+3i)(2-3i) = 4+9 = 13$
2) $(5-i)(5+i) = 25+1 = 26$
3) $(x+2i)(x-2i) = x^2+4$
4) $(2a+\sqrt{7}\,i)(2a-\sqrt{7}\,i) = 4a^2+7$

H.W. :

1) $(4+5i)(4-5i) =$

2) $(2-i)(2+i) =$

3) $(x-4i)(x+4i) =$

4) $(\sqrt{a}+3bi)(\sqrt{a}-3bi) =$

conjugate for each other $\rightarrow \quad (\quad) = \overline{(\quad)}$

Example : If $\dfrac{x-yi}{1+5i}$, $\dfrac{3-2i}{i}$ are conjugate for each other , find $x, y \in R$

Solution :

$$\frac{x-yi}{1+5i} = \overline{\left(\frac{3-2i}{i}\right)} \Rightarrow \frac{x-yi}{1+5i} = \frac{3+2i}{-i}$$

$$-i(x-yi) = (3+2i)(1+5i)$$
$$-xi + yi^2 = 3 + 15i + 2i + 10i^2$$
$$-xi - y = -7 + 17i$$

$rp = rp \rightarrow -y = -7 \rightarrow y = 7$

$ip = ip \rightarrow -x = 17 \rightarrow x = -17$

Example : If $\dfrac{1+i}{3-2i}$, $\dfrac{3-i}{x+yi}$ are conjugate for each other , find $x, y \in R$

Solution :

$$\frac{3-i}{x+yi} = \overline{\left(\frac{1+i}{3-2i}\right)} \Rightarrow \frac{3-i}{x+yi} = \frac{1-i}{3+2i}$$

$$(x+yi)(1-i) = 9 + 6i - 3i + 2$$
$$x - xi + yi - yi^2 = 11 + 3i$$
$$x + y - xi + yi = 11 + 3i$$

$x + y = 11 \quad ...(1)$

$-x + y = 3 \quad ...(2) \quad +$

$2y = 14 \rightarrow \boxed{y = 7} \xrightarrow{\text{put in }(1)} x + 7 = 11 \rightarrow \boxed{x = 4}$

M.Q. : If $\frac{a+bi}{2-3i}$, $\frac{1-i}{i}$ are conjugate for each other , find $a, b \in R$

M.Q. : If $\frac{x+yi}{1+4i}$, $\frac{2i}{1-i}$ are conjugate for each other , find $x, y \in R$

M.Q. : *If* $\frac{-2}{x+yi}$, $\frac{1-5i}{3-2i}$ *are conjugate for each other , find* $x, y \in R$

Example : Find the multiplication inverse of $(c = 2-2i)$ and write as algebritic form

Solution : $\frac{1}{c} = \frac{1}{2-2i} \cdot \frac{2+2i}{2+2i} = \frac{2+2i}{4+4} = \frac{2}{8} + \frac{2}{8}i = \frac{1}{4} + \frac{1}{4}i$

Example : Find the multiplication inverse of $(c = 3-4i)$ and write as algebritic form

Solution : $\frac{1}{c} = \frac{1}{3-4i} \cdot \frac{3+4i}{3+4i} = \frac{3+4i}{9+16} = \frac{3}{25} + \frac{4}{25}i$

M.Q. : Find the multiplication inverse of $(c = 9-3i)$ and write as algebritic form

Note : The conjugate satisfies the following :

1) $\overline{c_1 \pm c_2} = \overline{c_1} \pm \overline{c_2}$

2) $\overline{c_1 . c_2} = \overline{c_1} . \overline{c_2}$

3) $\overline{\left(\dfrac{c_1}{c_2}\right)} = \dfrac{\overline{c_1}}{\overline{c_2}}$

4) $\overline{\overline{c}} = c$

5) $c = a + bi \rightarrow c.\overline{c} = a^2 + b^2$

6) $c \in R \rightarrow \overline{c} = c$

Example : *If* $c_1 = 5 - i$, $c_2 = 2 + 3i$ *prove that :*

1) $\overline{c_1 + c_2} = \overline{c_1} + \overline{c_2}$ 2) $\overline{c_1 . c_2} = \overline{c_1} . \overline{c_2}$ 3) $\overline{\left(\dfrac{c_1}{c_2}\right)} = \dfrac{\overline{c_1}}{\overline{c_2}}$

Solution :

1) $\overline{c_1 + c_2} = \overline{(5-i)+(2+3i)} = \overline{7+2i} = 7 - 2i$

$\overline{c_1} + \overline{c_2} = \overline{5-i} + \overline{2+3i} = 5 + i + 2 - 3i = 7 - 2i$

$\therefore \overline{c_1 + c_2} = \overline{c_1} + \overline{c_2}$

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2) $\overline{c_1 . c_2} = \overline{(5-i)(2+3i)} = \overline{10 + 15i - 2i + 3} = \overline{13 + 13i} = 13 - 13i$

$\overline{c_1} . \overline{c_2} = \overline{5-i} \; . \; \overline{2+3i} = (5+i)(2-3i) = 10 - 15i + 2i + 3 = 13 - 13i$

$\therefore \overline{c_1 . c_2} = \overline{c_1} . \overline{c_2}$

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3) $\overline{\left(\dfrac{c_1}{c_2}\right)} = \overline{\left(\dfrac{5-i}{2+3i} . \dfrac{2-3i}{2-3i}\right)} = \overline{\left(\dfrac{10 - 15i - 2i - 3}{4+9}\right)} = \overline{\left(\dfrac{7-17i}{13}\right)} = \dfrac{7+17i}{13}$

$\dfrac{\overline{c_1}}{\overline{c_2}} = \dfrac{\overline{5-i}}{\overline{2+3i}} = \dfrac{5+i}{2-3i} . \dfrac{2+3i}{2+3i} = \dfrac{10 + 15i + 2i - 3}{4+9} = \dfrac{7+17i}{13}$

$\therefore \overline{\left(\dfrac{c_1}{c_2}\right)} = \dfrac{\overline{c_1}}{\overline{c_2}}$

H.W. : *If* $c_1 = 11 + 10i$, $c_2 = 4 - i$ *prove that :*

1) $\overline{c_1 + c_2} = \overline{c_1} + \overline{c_2}$ 2) $\overline{c_1 . c_2} = \overline{c_1} . \overline{c_2}$ 3) $\overline{\left(\frac{c_1}{c_2}\right)} = \frac{\overline{c_1}}{\overline{c_2}}$

Example prove that : $\dfrac{13i+27}{13-27i}=i$

$-i^2=1$

Solution :

$$LHS=\frac{13i+27}{13-27i}=\frac{13i-27i^2}{13-27i}=\frac{i(13-27i)}{(13-27i)}=i=RHS$$

H.W. : prove that : $\dfrac{14i-58}{7+29i}=2i$

Example prove that : $\dfrac{(1+i)^{2025}}{(1-i)^{2023}}=2$

Solution :

$$LHS=\frac{(1+i)^{2025}}{(1-i)^{2023}}=\frac{(-i^2+i)^{2025}}{(1-i)^{2023}}=\frac{i^{2025}(-i+1)^{2025}}{(1-i)^{2023}}$$

$$=\frac{i^{4(506)}.i^1.(1-i)^{2025}}{(1-i)^{2023}}=i(1-i)^2=i(-2i)=-2i^2=2=RHS$$

H.W. : prove that : $\dfrac{(2-i)^{2025}}{(1+2i)^{2023}}=4+3i$

Example : Find the value of $x, y \in R$ which satisfied the equation : $(2+xi)(-x+i)=\dfrac{9y^2+49}{3y+7i}$

Solution :

$$(2+xi)(-x+i)=\boxed{\dfrac{9y^2+49}{3y+7i}}$$

Second Method

$$\frac{9y^2+49}{3y+7i}=\frac{9y^2-49i^2}{3y+7i}$$

$$=\frac{(3y-7i)\,\cancel{(3y+7i)}}{\cancel{(3y+7i)}}$$

$$=3y-7i$$

$$-2x+2i-x^2i-x=\frac{9y^2+49}{3y+7i}\cdot\frac{3y-7i}{3y-7i}$$

$$-3x-x^2i+2i=\frac{(9y^2+49)(3y-7i)}{(9y^2+49)}$$

Multiply the denominator first

$$-3x-x^2i+2i=3y-7i$$

$$-3x=3y \xrightarrow{\div(3)} y=-x \quad \text{......(1)}$$

$$-x^2+2=-7 \longrightarrow x^2=9$$

$$\text{either} \quad x=3 \xrightarrow{in(1)} y=-3$$

$$\text{or} \quad x=-3 \xrightarrow{in(1)} y=3$$

M.Q. Find the value of $x, y \in R$ which satisfied the equation : $(x+2i)(x-i)=\dfrac{121+9y^2}{11+3yi}$

M.Q. Find the value of $x, y \in R$ which satisfied the equation :

a) $\frac{2+2i}{1-i} - (x-2iy) = (4-i)^2 + 1$

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b) $(x+iy)(1-\sqrt{-3}) = 2$

Don't multiply the brackets

c) $\frac{1-i}{1+i}x+(1+3i)^2y=(1-i)(1+3i)$

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d) $\frac{125}{11+2i}x+(1-i)^2y=11$

e) $\frac{6}{x+yi}+(2-i)^2=4-3i$

M.Q. *Prove that :* $\frac{(1-i).(1-i^2).(1-i^3)}{(1+i)^3+(1-i)^3}=-1$

LHS =

M.Q. *Write* $\frac{(1-i)^{13}}{64}$ *as ordinary expression of complex number*

EXERCISES (1-1)

Q1) Put each of the following in the ordinary expression for the complex numbers :

$$i^5 = i^4.i = i = 0+i$$

$$i^6 = i^4.i^2 = -1 = -1+0i$$

$$i^{124} = (i^4)^{31} = 1 = 1+0i$$

$$i^{999} = (i^4)^{249}.i^3 = -i = 0-i$$

$$i^{4n+1} = (i^4)^n.i = i = 0+i$$

$$(2+3i)^2+(12+2i) = (4+12i-9)+(12+2i) = (-5+12i)+(12+2i) = 7+14i$$

$$(10+3i)(6i) = 60i+18i^2 = 60i-18 = -18+60i$$

$$(1+i)^4-(1-i)^4 = [(1+i)^2]^2-[(1-i)^2]^2 = (1+2i-1)^2-(1-2i-1)^2$$
$$= (2i)^2-(-2i)^2 = (-4)-(-4) = 0 = 0+0i$$

$$\frac{12+i}{i} = \frac{12+i}{i}.\frac{-i}{-i} = \frac{-12i-i^2}{1} = 1-12i$$

$$\frac{3+4i}{3-4i} = \frac{3+4i}{3-4i}.\frac{3+4i}{3+4i} = \frac{9+24i-16}{9+16} = \frac{-7+24i}{25} = \frac{-7}{25}+\frac{24}{25}i$$

$$\frac{i}{2+3i} = \frac{i}{2+3i}.\frac{2-3i}{2-3i} = \frac{2i+3}{4+9} = \frac{3+2i}{13} = \frac{3}{13}+\frac{2}{13}i$$

$$\left(\frac{3+i}{1+i}\right)^3 = \left(\frac{3+i}{1+i}\times\frac{1-i}{1-i}\right)^3$$
$$= \left(\frac{3-3i+i+1}{1+1}\right)^3$$
$$= \left(\frac{4-2i}{2}\right)^3$$
$$= (2-i)^3$$
$$= (2-i)(2-i)^2$$
$$= (2-i)(4-4i-1)$$
$$= (2-i)(3-4i)$$
$$= 6-8i-3i-4$$
$$= 2-11i$$

$$\frac{2+3i}{1-i}\times\frac{1+4i}{4+i} = \frac{2+8i+3i-12}{4+i-4i+1}$$
$$= \frac{-10+11i}{5-3i}\cdot\frac{5+3i}{5+3i}$$
$$= \frac{-50-30i+55i-33}{25+9}$$
$$= \frac{-83+25i}{34}$$
$$= \frac{-83}{34}+\frac{25}{34}i$$

$$(1+i)^3+(1-i)^3 = (1+i)^2.(1+i)+(1-i)^2.(1-i)$$
$$= (1+2i-1).(1+i)+(1-2i-1).(1-i)$$
$$= (2i).(1+i)+(-2i).(1-i)$$
$$= \cancel{2i}+2i^2-\cancel{2i}+2i^2$$
$$= -2-2 \quad = -4 \quad = -4+0i$$

Q2) Find the real value of each x , y if you know that :

a) $y+5i=(2x+i)(x+2i)$

$$y+5i=2x^2+4xi+xi-2$$
$$y+5i=2x^2-2+5xi$$

$$rp=rp \quad \Rightarrow \quad y=2x^2-2 \quad ....(1)$$

$$ip=ip \quad \Rightarrow \quad 5=5x$$
$$\Rightarrow \quad x=1 \quad \xrightarrow{in\ (1)} \quad y=2(1)^2-2=0$$

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b) $8i=(x+2i)(y+2i)+1$

$$0+8i=xy+2xi+2yi-4+1$$

$$rp=rp \quad \Rightarrow \quad xy-3=0 \quad \Rightarrow \quad xy=3 \quad \Rightarrow \quad y=\frac{3}{x} \quad ......(1)$$

$$ip=ip \quad \Rightarrow \quad 2x+2y=8 \quad \Rightarrow \quad x+y=4 \quad ......(2)$$

$$\xrightarrow{(1)\ in\ (2)} \quad x+\frac{3}{x}=4 \quad]\times(x)$$
$$\Rightarrow \quad x^2+3=4x$$
$$\Rightarrow \quad x^2-4x+3=0$$
$$\Rightarrow \quad (x-1)(x-3)=0$$

$$either \quad x=1 \quad \xrightarrow{in\ (1)} \quad y=3$$
$$or \quad x=3 \quad \xrightarrow{in\ (1)} \quad y=1$$

c) $\left(\frac{1-i}{1+i}\right)+(x+yi)=(1+2i)^2$

$$\left(\frac{1-i}{1+i}\times\frac{1-i}{1-i}\right)+x+yi=1+4i-4$$

$$\left(\frac{1-i-i-1}{1+1}\right)+x+yi=-3+4i$$

$$\left(\frac{-2i}{2}\right)+x+yi=-3+4i$$

$$-i+x+yi=-3+4i$$

$$x+yi=-3+4i+i$$

$$x+yi=-3+5i \quad\Rightarrow\quad \boxed{x=-3} \quad\Rightarrow\quad \boxed{y=5}$$

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d) $\frac{2-i}{1+i}x+\frac{3-i}{2+i}y=\frac{1}{i}$

$$\frac{2x-xi}{1+i}+\frac{3y-yi}{2+i}=\frac{1}{i}$$

$$\frac{2x-xi}{1+i}\cdot\frac{1-i}{1-i}+\frac{3y-yi}{2+i}\cdot\frac{2-i}{2-i}=\frac{1}{i}\cdot\frac{-i}{-i}$$

$$\frac{2x-2xi-xi-x}{2}+\frac{6y-3yi-2yi-y}{5}=\frac{-i}{1}$$

$$\frac{x-3xi}{2}+\frac{5y-5yi}{5}=-i$$

$$\frac{x-3xi}{2}+y-yi=-i \qquad \}\times 2$$

$$x-3xi+2y-2yi=0-2i$$

$$rp=rp \quad\Rightarrow\quad x+2y=0 \quad\Rightarrow\quad 2y=-x \quad\Rightarrow\quad \boxed{y=\frac{-x}{2}}\ \ldots\ldots(1)$$

$$ip=ip \quad\Rightarrow\quad \boxed{-3x-2y=-2}\ \ldots\ldots(2)$$

$$\xrightarrow{(1)\ in\ (2)} -3x-2\left(\frac{-x}{2}\right)=-2 \rightarrow -3x+x=-2 \rightarrow -2x=-2 \rightarrow \boxed{x=1}$$

$$\xrightarrow{in\ (1)} \boxed{y=\frac{-1}{2}}$$

Q3) Prove that :

a) $\dfrac{1}{(2-i)^2}-\dfrac{1}{(2+i)^2}=\dfrac{8}{25}i$

$$LHS = \frac{1}{(2-i)^2}-\frac{1}{(2+i)^2}=\frac{1}{4-4i-1}-\frac{1}{4+4i-1}$$

$$=\frac{1}{3-4i}-\frac{1}{3+4i}=\frac{1}{3-4i}.\frac{3+4i}{3+4i}-\frac{1}{3+4i}.\frac{3-4i}{3-4i}$$

$$=\frac{3+4i}{9+16}-\frac{3-4i}{9+16}=\frac{3+4i-3+4i}{25}\quad=\frac{8}{25}i\quad=RHS$$

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b) $\dfrac{(1-i)^2}{1+i}+\dfrac{(1+i)^2}{1-i}=-2$

$$LHS=\frac{(1-i)^2}{1+i}+\frac{(1+i)^2}{1-i}=\frac{-2i}{1+i}+\frac{2i}{1-i}$$

$$=\frac{-2i}{1+i}.\frac{1-i}{1-i}+\frac{2i}{1-i}.\frac{1+i}{1+i}=\frac{-2i+2i^2}{1+1}+\frac{2i+2i^2}{1+1}$$

$$=\frac{-2i-2+2i-2}{2}\quad=\quad\frac{-4}{2}\quad=\quad-2\quad=\quad RHS$$

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c) $(1-i)(1-i^2)(1-i^3)=4$

$$\begin{aligned}LHS\quad &=(1-i)(1+1)(1+i)\\ &=(1+1)(1-i)(1+i)\\ &=(1+1)(1+1)=2\times2=4=\quad RHS\end{aligned}$$

Q4) Analyze the numbers into two factors in C :

$$29 = 25 + 4 = 25 - 4i^2 = (5 - 2i)(5 + 2i)$$

$$125 = 100 + 25 = 100 - 25i^2 = (10 - 5i)(10 + 5i)$$

$$125 = 121 + 4 = 121 - 4i^2 = (11 - 2i)(11 + 2i)$$

$$41 = 25 + 16 = 25 - 16i^2 = (5 - 4i)(5 + 4i)$$

$$85 = 81 + 4 = 81 - 4i^2 = (9 - 2i)(9 + 2i)$$

$$85 = 36 + 49 = 36 - 49i^2 = (6 - 7i)(6 + 7i)$$

Q5) If $\frac{3+i}{2-i}$, $\frac{6}{x+yi}$ are conjugate for each other, find $x, y \in R$

$$\frac{6}{x+yi} = \overline{\left(\frac{3+i}{2-i}\right)} \quad \Rightarrow \quad \frac{6}{x+yi} = \frac{3-i}{2+i}$$

$$\Rightarrow \quad (x+yi)(3-i) = 6(2+i)$$

$$\Rightarrow \quad (x+yi)(3-i) = 12 + 6i \qquad]\div(3-i)$$

$$\Rightarrow \quad (x+yi) = \frac{12+6i}{3-i} \times \frac{3+i}{3+i}$$

$$\Rightarrow \quad x+yi = \frac{36+12i+18i-6}{9+1}$$

$$\Rightarrow \quad x+yi = \frac{30+30i}{10}$$

$$\Rightarrow \quad x+yi = \frac{30}{10} + \frac{30}{10}i$$

$$\Rightarrow \quad x+yi = 3+3i \qquad \Rightarrow \qquad x = 3$$

$$y = 3$$

The Square Root of Complex Number

Example : Find the square roots of the complex number : $(8+6i)$

Solution :

$$Let \quad x+yi=\sqrt{8+6i}$$

$$(x+yi)^2=8+6i$$

$$x^2+2xyi-y^2=8+6i$$

$$x^2-y^2+2xyi=8+6i$$

$$x^2-y^2=8 \;\ldots\ldots\ldots\ldots\; (1)$$

$$2xy=6 \quad \Rightarrow \quad \boxed{y=\frac{3}{x}} \;\ldots\ldots\ldots\ldots\; (2)$$

$$x^2-\left(\frac{3}{x}\right)^2=8$$

$$x^2-\frac{9}{x^2}=8 \qquad]\times(x^2)$$

$$x^4-9=8x^2$$

$$x^4-8x^2-9=0$$

$$(x^2-9)(x^2+1)=0$$

$$either \qquad x^2=9 \quad \Rightarrow \quad \begin{cases} x=3 \xrightarrow{in(2)} y=1 \\ x=-3 \xrightarrow{in(2)} y=-1 \end{cases}$$

$$or \qquad x^2=-1 \quad \text{(neglected)}$$

$\therefore$ The square roots are : $(3+i)\,,\,(-3-i) \quad \xrightarrow{or} \quad \boxed{\pm(3+i)}$

Example : Find the square roots of the Complex Number (8i)

Solution :

$$Let \quad x + yi = \sqrt{8i}$$

$$(x+yi)^2 = 8i$$

$$x^2 - y^2 + 2xyi = 0 + 8i$$

$$x^2 - y^2 = 0 \quad \ldots\ldots\ldots\ldots (1)$$

$$2xy = 8 \quad \Rightarrow \quad \boxed{y = \frac{4}{x}} \quad \ldots\ldots\ldots\ldots (2)$$

$$x^2 - (\frac{4}{x})^2 = 0$$

$$x^2 - \frac{16}{x^2} = 0$$

$$x^4 - 16 = 0$$

$$(x^2-4)(x^2+4) = 0$$

$$either \quad x^2 = 4 \quad \Rightarrow \quad \begin{cases} x = -2 \xrightarrow{in(2)} y = -2 \\ x = \ \ 2 \xrightarrow{in(2)} y = \ \ 2 \end{cases}$$

$$or \quad x^2 = -4 \quad \text{(neglected)}$$

$\therefore$ The square roots are : $\pm(2+2i)$

H.W. Find the square roots of the Complex Number $(-18i)$

Example : Find the square roots of the Complex Number (i)

Solution :

$$Let \quad x+yi=\sqrt{i}$$
$$(x+yi)^2=i$$
$$x^2-y^2+2xyi=0+i$$
$$x^2-y^2=0 \quad .... \quad (1)$$
$$2xy=1 \quad \Rightarrow \quad \boxed{y=\frac{1}{2x}}$$

$$x^2-(\frac{1}{2x})^2=0$$
$$x^2-\frac{1}{4x^2}=0$$
$$4x^4-1=0$$
$$(2x^2-1)(2x^2+1)=0$$

$$either \quad 2x^2=1 \quad \Rightarrow \quad x^2=\frac{1}{2} \quad \Rightarrow \quad \begin{cases} x=\frac{1}{\sqrt{2}} \quad \rightarrow \quad y=\frac{1}{2.\frac{1}{\sqrt{2}}}=\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}} \\ x=\frac{-1}{\sqrt{2}} \quad \rightarrow \quad y=\frac{1}{2.\frac{-1}{\sqrt{2}}}=\frac{-\sqrt{2}}{2}=\frac{-1}{\sqrt{2}} \end{cases}$$

$$or \quad x^2=-1 \quad \text{(neglected)}$$

$\therefore$ *The square roots are :* $\pm(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)$

H.W. Find the square roots of the Complex Number $(-i)$

H.W. Find the square roots of the Complex Number $(5-12i)$

H.W. Find the square roots of the Complex Number $(12-16i)$

Exercises) Q3) Find the square roots of the complex numbers below :

a) $-6i$

Solution :

$$Let \quad x + yi = \sqrt{-6i}$$
$$(x+yi)^2 = -6i$$
$$x^2 - y^2 + 2xyi = 0 - 6i$$
$$x^2 - y^2 = 0 \quad \text{............} \quad (1)$$

$$2xy = -6 \quad \Rightarrow \quad \boxed{y = \frac{-3}{x}}$$

$$x^2 - \frac{9}{x^2} = 0$$
$$x^4 - 9 = 0$$
$$(x^2+3)(x^2-3) = 0$$

$$either \qquad x^2 = -3 \quad \text{(neglected)}$$

$$either \qquad x^2 = 3 \qquad \Rightarrow \begin{cases} x = -\sqrt{3} \quad \rightarrow \quad y = \dfrac{3}{\sqrt{3}} = \dfrac{\sqrt{3}.\sqrt{3}}{\sqrt{3}} = \sqrt{3} \\ x = \sqrt{3} \quad \rightarrow \quad y = -\dfrac{3}{\sqrt{3}} = -\dfrac{\sqrt{3}.\sqrt{3}}{\sqrt{3}} = -\sqrt{3} \end{cases}$$

$\therefore$ The square roots are : $\pm(-\sqrt{3} + \sqrt{3}i)$

b) $7+24i$

Solution :

$$\text{Let} \quad x+yi=\sqrt{7+24i}$$
$$(x+yi)^2=7+24i$$
$$x^2-y^2+2xyi=7+24i$$
$$x^2-y^2=7 \quad \text{............} \quad (1)$$

$$2xy=24 \quad \Rightarrow \quad \boxed{y=\frac{12}{x}}$$

$$x^2-\frac{144}{x^2}=7$$
$$x^4-144=7x^2$$
$$x^4-7x^2-144=0$$
$$(x^2-16)(x^2+9)=0$$

$$\text{either} \qquad x^2=16 \quad \Rightarrow \quad \begin{cases} x=-4 & \rightarrow \quad y=-3 \\ x=\ \ 4 & \rightarrow \quad y=\ \ 3 \end{cases}$$

$$\text{or} \qquad x^2=-9 \quad \text{(neglected)}$$

$\therefore$ The square roots are : $\pm(-4-3i)$

c) $\dfrac{4}{1-\sqrt{3}i}$

Solution :

$$\frac{4}{1-\sqrt{3}i}\times\frac{1+\sqrt{3}i}{1+\sqrt{3}i} = \frac{4(1+\sqrt{3}i)}{1+3} = \frac{4(1+\sqrt{3}i)}{4} = 1+\sqrt{3}i$$

$$Let \quad x+yi=\sqrt{1+\sqrt{3}i}$$

$$(x+yi)^2 = 1+\sqrt{3}i$$

$$x^2-y^2+2xyi = 1+\sqrt{3}i$$

$$x^2-y^2 = 1 \quad \dots\dots\dots\dots (1)$$

$$2xy=\sqrt{3} \quad \Rightarrow \quad \boxed{y=\frac{\sqrt{3}}{2x}}$$

$$x^2-\frac{3}{4x^2}=1 \qquad] \times 4x^2$$

$$4x^4-3=4x^2$$

$$4x^4-4x^2-3=0$$

$$(2x^2-3)(2x^2+1)=0$$

$$either \quad x^2=\frac{3}{2} \Rightarrow \begin{cases} x=\dfrac{\sqrt{3}}{\sqrt{2}} \rightarrow & y=\dfrac{\sqrt{3}}{2(\frac{\sqrt{3}}{\sqrt{2}})}=\dfrac{\sqrt{2}.\sqrt{3}}{2.\sqrt{3}}=\dfrac{1}{\sqrt{2}} \\ x=-\dfrac{\sqrt{3}}{\sqrt{2}} \rightarrow & y=\dfrac{\sqrt{3}}{2(-\frac{\sqrt{3}}{\sqrt{2}})}=-\dfrac{\sqrt{2}.\sqrt{3}}{2.\sqrt{3}}=-\dfrac{1}{\sqrt{2}} \end{cases}$$

$$or \quad x^2=-\frac{1}{2} \quad \text{(neglected)}$$

$\therefore$ The square roots are : $\pm(\dfrac{\sqrt{3}}{\sqrt{2}}+\dfrac{1}{\sqrt{2}}i)$

The Quadratic Equation (Second Degree) in Complex Numbers

General form of quadratic equation

$$ax^2+bx+c=0 \quad \rightarrow \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The Quadratic Formula

Example : Solve the equation $x^2+4x+5=0$

Solution :

$a=1 \quad , \quad b=4 \quad , \quad c=5$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-4\pm\sqrt{16-4(5)}}{2(1)}$$

$$=\frac{-4\pm\sqrt{-4}}{2}=\frac{-4\pm 2i}{2}=\frac{-4}{2}\pm\frac{2i}{2}=-2\pm i$$

$$S.S.=\{-2+i \ , \ -2-i\}$$

H.W. Solve the equation $z^2-6z+13=0$

Example : *Solve the equation :* $z^2+4=0$

Solution :

$z^2+4=0$

$z^2-4i^2=0$

$(z-2i)(z+2i)=0$

either $z=2i$ *or* $z=-2i$

$\therefore S.S.=\{-2i, 2i\}$

Example : *Solve the equation :* $4z^2+49=0$

Solution :

$4z^2+49=0$

$4z^2-49i^2=0$

$(2z-7i)(2z+7i)=0$

either $2z=7i \rightarrow z=\frac{7}{2}i$

or $2z=-7i \rightarrow z=-\frac{7}{2}i$

$\therefore S.S.=\left\{\frac{7}{2}i, -\frac{7}{2}i\right\}$

M.Q. *Solve the equation :* $Z^4+13Z^2+36=0$

Solution :

$Z^4+13Z^2+36=0$

$(Z^2+4)(Z^2+9)=0$

either $Z^2=-4 \rightarrow Z=\pm 2i$

or $Z^2=-9 \rightarrow Z=\pm 3i$

$\therefore S.S.=\{2i, -2i, 3i, -3i\}$

H.W. *Solve the equations :*

1) $z^2+9=0$

2) $9z^2+64=0$

3) $Z^4+29Z^2+100=0$

Example : *Solve the equation :* $z^2 + 6zi + 7 = 0$

Solution :

$z^2 + 6zi - 7i^2 = 0$

$(z + 7i)(z - i) = 0$

either $z = -7i$

or $z = i$ $\qquad \therefore S.S. = \{-7i, i\}$

2nd method

$a = 1 \quad , \quad b = 6i \quad , \quad c = 7$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6i \pm \sqrt{-36 - 4(1)(7)}}{2} = \frac{-6i \pm \sqrt{-36 - 28}}{2}$$

$$= \frac{-6i \pm \sqrt{-64}}{2} = \frac{-6i \pm 8i}{2} = \frac{-6i}{2} \pm \frac{8i}{2} = -3i \pm 4i$$

either $z = -3i + 4i = i$

or $z = -3i - 4i = -7i$ $\qquad \therefore S.S. = \{-7i, i\}$

H.W. *Solve the equation :* $z^2 - 8zi + 9 = 0$

Example : *Solve the equation :* $z^2-5z+7-i=0$

Solution :

$a=1 \quad , \quad b=-5 \quad , \quad c=7-i$

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{5\pm\sqrt{25-4(7-i)}}{2}=\frac{5\pm\sqrt{25-28+4i}}{2}=\boxed{\frac{5\pm\sqrt{-3+4i}}{2}}$$

$Let \quad x+yi=\sqrt{-3+4i}$

$x^2+2xyi-y^2=-3+4i$

$x^2-y^2=-3 \quad ...... (1)$

$2xy=4 \quad \rightarrow \quad \boxed{y=\frac{2}{x}}$

$$\left[x^2-\frac{4}{x^2}=-3\right]\times(x^2)$$

$x^4-4=-3x^2$

$x^4+3x^2-4=0$

$(x^2+4)(x^2-1)=0$

$either \quad x^2=-4$ (neglected)

$$or \quad x^2=1 \quad \Rightarrow \begin{cases} x=1 \rightarrow y=2 \rightarrow (1+2i) \\ x=-1 \rightarrow y=-2 \rightarrow (-1-2i) \end{cases}$$

$$Z=\frac{5\pm(1+2i)}{2}=\begin{cases} \dfrac{5+1+2i}{2}=\dfrac{6+2i}{2}=\dfrac{6}{2}+\dfrac{2i}{2}=3+i \\[2ex] \dfrac{5-1-2i}{2}=\dfrac{4-2i}{2}=\dfrac{4}{2}-\dfrac{2i}{2}=2-i \end{cases}$$

$\therefore \quad S.S.=\{(3+i)\ ,\ (2-i)\}$

H.W. *Solve the equation :* $z^2 - 3z + 11 - 3i = 0$

M.Q. *Solve the equation :* $Z^2 + 2i(3-2i) = 3Z$

Exercises) Q1) *Solve the equations :*

a) $z^2 = -12 \quad \Rightarrow \quad z = \pm\sqrt{-12} \quad \Rightarrow \quad z = \pm 2\sqrt{3}\,i \quad \Rightarrow \quad z = 0 \pm 2\sqrt{3}\,i$

$$S.S. = \{\ (0 + 2\sqrt{3}\,i)\ ,\ (0 - 2\sqrt{3}\,i)\ \}$$

---

b) $Z^2 - 3z + 3 + i = 0 \qquad a = 1 \qquad b = -3 \qquad c = 3 + i$

$$Z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(3+i)}}{2(1)} = \frac{3 \pm \sqrt{9 - 12 - 4i}}{2}$$

$$= \frac{3 \pm \sqrt{-3 - 4i}}{2} \quad \ldots\ldots \ (1)$$

$x + yi = \sqrt{-3 - 4i}$

$\Rightarrow \quad x^2 - y^2 + 2xyi = -3 - 4i$

$\Rightarrow \quad x^2 - y^2 = -3 \ldots\ldots\ldots\ldots\ldots (2)$

$\Rightarrow \quad 2xy = -4 \quad \Rightarrow \quad \boxed{y = \frac{-2}{x}}$

$\Rightarrow \quad x^2 - \frac{4}{x^2} = -3 \qquad \} \times (x^2)$

$\Rightarrow \quad x^4 + 3x^2 - 4 = 0$

$\Rightarrow \quad (x^2 + 4)(x^2 - 1) = 0$

either $\quad x^2 = -4$

or $\quad x^2 = 1 \quad \Rightarrow \begin{cases} x = 1 \ \rightarrow \ y = -2 \ \rightarrow \ (1 - 2i) \\ x = -1 \ \rightarrow \ y = 2 \ \rightarrow \ (-1 + 2i) \end{cases}$

$$Z = \frac{3 \pm (1 - 2i)}{2} = \begin{cases} \dfrac{3 + (1 - 2i)}{2} = \dfrac{3 + 1 - 2i}{2} = \dfrac{4 - 2i}{2} = 2 - i \\[2ex] \dfrac{3 - (1 - 2i)}{2} = \dfrac{3 - 1 + 2i}{2} = \dfrac{2 + 2i}{2} = 1 + i \end{cases}$$

$$\therefore \quad S.S. = \{\ (1 + i)\ ,\ (2 - i)\ \}$$

c) $2z^2 - 5z + 13 = 0$

$a = 2 \qquad b = -5 \qquad c = 13$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5 \pm \sqrt{25 - 4(2)(13)}}{2(2)}$$

$$= \frac{5 \pm \sqrt{25 - 104}}{4} = \frac{5 \pm \sqrt{-79}}{4} = \frac{5}{4} \pm \frac{\sqrt{79}}{4} i$$

$$\therefore \quad S.S. = \left\{ \left(\frac{5}{4} + \frac{\sqrt{79}}{4} i\right), \left(\frac{5}{4} - \frac{\sqrt{79}}{4} i\right) \right\}$$

---

d) $z^2 + 2z + i(2 - i) = 0$

$z^2 + 2z + 2i + 1 = 0$

$a = 1 \;,\; b = 2 \;,\; c = 1 + 2i$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{(2)^2 - 4(1)(1+2i)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4 - 4 - 8i}}{2}$$

$$= \frac{-2 \pm \sqrt{-8i}}{2} \quad \ldots\ldots \quad (1)$$

$x + yi = \sqrt{-8i}$

$x^2 - y^2 + 2xyi = 0 - 8i$

$x^2 - y^2 = 0 \ldots\ldots\ldots\ldots\ldots (2)$

$2xy = -8 \quad \Rightarrow \quad y = \dfrac{-4}{x}$

$x^2 - \dfrac{16}{x^2} = 0 \qquad \} \times (x^2)$

$x^4 - 16 = 0$

$(x^2 + 4)(x^2 - 4) = 0$

either $x^2 = -4$ *neglected*

or $x^2 = 4 \Rightarrow \begin{cases} x = 2 \rightarrow y = -2 \rightarrow (2 - 2i) \\ x = -2 \rightarrow y = 2 \rightarrow (-2 + 2i) \end{cases}$

$$Z = \frac{-2 \pm (2 - 2i)}{2} = \begin{cases} \dfrac{-2 + 2 - 2i}{2} = \dfrac{-2i}{2} = -i \\[2ex] \dfrac{-2 - 2 + 2i}{2} = \dfrac{-4 + 2i}{2} = \dfrac{-4}{2} + \dfrac{2i}{2} = -2 + i \end{cases}$$

e) $4z^2+25=0$

$$4z^2-25i^2=0$$
$$(2z-5i)(2z+5i)=0$$

$either \quad 2z-5i=0 \quad \Rightarrow \quad z=\frac{5}{2}i$

$or \quad 2z+5i=0 \quad \Rightarrow \quad z=-\frac{5}{2}i$

$Solution\ set=\{\ \frac{5}{2}i\ ,\ -\frac{5}{2}i\ \}$

2^{nd} method :

$z^2=-\frac{25}{4} \quad \Rightarrow \quad z=\pm\frac{5}{2}i$

$Solution\ set=\{\ \frac{5}{2}i\ ,\ -\frac{5}{2}i\ \}$

---

f) $z^2-2zi+3=0$

$$z^2-2zi-3i^2=0$$
$$(z-3i)(z+i)=0$$

$either \quad z=3i$

$or \quad z=-i$

$Solution\ set=\{\ 3i\ ,-i\ \}$

The General Formula of The Quadratic Equation

$$x^2-(L+m)x+(L\times m)=0$$

$$x^2-sx+p=0$$

L = root$_1$ = one of the two solution which satisfied the quadratic equation

M = root$_2$ = the other solution

Example : Find the Quadratic Equation which has two roots : $4+i\ ,\ 3-2i$

Solution :

$s=L+m=4+i+3-2i=7-i$

$p=L\times m=(4+i)(3-2i)=12-8i+3i+2=14-5i$

$$x^2-sx+p=0$$
$$x^2-(7-i)x+(14-5i)=0$$

Example : Find the Quadratic Equation which has two roots : $5+2i\ ,\ 5-2i$

Solution :

$s=L+m=5+2i+5-2i=10$

$p=L\times m=(5+2i)(5-2i)=25+4=29$

$$x^2-sx+p=0$$
$$x^2-10x+29=0$$

Note : If the Quadratic Equation has real coefficients then the two roots will conjugate for each other :

$$real\ coefficient \quad \Leftrightarrow \quad L \xleftrightarrow{conjugate} M$$

Example : Find the Quadratic Equation which has real coefficients , one of its roots $(3-4i)$

Solution :

$$real\ coefficient \quad \Rightarrow \quad L = \overline{m}$$

$$L = 3-4i \quad \Rightarrow \quad m = 3+4i$$

$$s = 3+4i+3-4i = 6$$

$$p = (3+4i)(3-4i) = 9+16 = 25$$

$$x^2 - s\,x + p = 0$$

$$\rightarrow \quad x^2 - 6x + 25 = 0$$

H.W. Find the quadratic equation which has two roots :

a) $4-i \ , \ 3+i$

b) $6+i \quad , \quad 3-2i$

---

c) $4-3\sqrt{2}i \quad , \quad 4+3\sqrt{2}i$

---

d) $2+\sqrt{-1} \quad , \quad 5-3i^7$

H.W. Find the Quadratic Equation which has real coefficients, one of its roots is:

a) $11-2i$

b) $5+\sqrt{5}\,i$

c) $\frac{2}{3}+\frac{\sqrt{-7}}{3}$

Exercises) Q2) Find the quadratic equations which has two roots $(m \ , L)$:

a) $m = 1 + 2i \quad , \quad L = 1 - i$

Solution :

$m + L = (1+2i) + (1-i) = \boxed{2+i}$

$m \times L = (1+2i).(1-i) = 1 - i + 2i + 2 = \boxed{3+i}$

$x^2 - (m + L)x + (m \times L) = 0$

$\therefore \quad x^2 - (2+i)x + (3+i) = 0$

b) $m = \dfrac{3-i}{1+i} \quad , \quad L = (3-2i)^2$

Solution :

$m = \dfrac{3-i}{1+i} \qquad = \dfrac{3-i}{1+i} \times \dfrac{1-i}{1-i} = \dfrac{3-3i-i-1}{1+1} = \dfrac{2-4i}{2} = \quad 1-2i$

$L = (3-2i)^2 \qquad = 9 - 12i - 4 = \quad 5 - 12i$

$m + L = 1 - 2i + 5 - 12i = \boxed{6 - 14i}$

$m \times L = (1-2i).(5-12i) = 5 - 12i - 10i - 24 = \boxed{-19 - 22i}$

$x^2 - (m + L)x + (m \times L) = 0$

$\therefore \quad x^2 - (6 - 14i)x + (-19 - 22i) = 0$

Q4) Find the Quadratic Equation which has real coefficients, one of its roots is:

a) i

Solution : The other root is $-i$

$M+L=i+(-i)=\boxed{0}$ $\quad$ $M\times L=(i).(-i)=\boxed{1}$

$x^2-(M+L)x+(M\times L)=0$

$x^2-(0)x+1=0 \quad \rightarrow \quad x^2+1=0$

---

b) $5-i$

Solution : The other root is $5+i$

$M+L=(5-i)+(5+i)=\boxed{10}$

$M\times L=(5-i).(5+i)=25+1=\boxed{26}$

$x^2-(M+L)x+(M\times L)=0$

$x^2-10x+26=0$

---

c) $\dfrac{\sqrt{2}+3i}{4}$

Solution : The other root is $\dfrac{\sqrt{2}-3i}{4}$

$$M+L=\frac{\sqrt{2}+3i}{4}+\frac{\sqrt{2}-3i}{4}=\frac{\sqrt{2}+3i+\sqrt{2}-3i}{4}=\frac{2\sqrt{2}}{4}=\frac{\sqrt{2}}{2}=\boxed{\frac{1}{\sqrt{2}}}$$

$$M\times L=\frac{\sqrt{2}+3i}{4}\times\frac{\sqrt{2}-3i}{4}=\frac{2+9}{16}=\boxed{\frac{11}{16}}$$

$$x^2-(M+L)x+(M\times L)=0$$

$$x^2-\frac{1}{\sqrt{2}}x+\frac{11}{16}=0$$

FINDING VARIABLES
FROM ROOTS OF QUADRATIC EQUATION

Example: If $(2-2i)$ was one root for equation $Z^2 - aZ + (10+6i) = 0$, then what is the value of (a) and what is the other root ?

Solution : Assume $L = 2-2i$ and $M =$ Other root

$$Z^2 - (\quad a \quad)\, Z + (10+6i) = 0$$
$$Z^2 - (L+M)\, Z + (L\times M) = 0$$

$$L\times M = 10+6i$$
$$(2-2i)\times M = 10+6i \qquad]\div(2-2i)$$

$$M = \frac{10+6i}{2-2i}\times\frac{2+2i}{2+2i} = \frac{20+20i+12i-12}{4+4} = \frac{8+32i}{8} = \boxed{1+4i}$$ The other root

$$a = L+M \quad\Rightarrow\quad 2-2i + 1+4i = a \quad\Rightarrow\quad a = \boxed{3+2i}$$

Note : We can put the root $(2-2i)$ instead of (Z) in the equation

M.Q. If $(3-4i)$ was one root for equation $x^2 - nx + 10 - 5i = 0$, then what is the other root and what is the value of (n) ?

Solution : Assume $L = 3-4i$ and $M =$ Other root

$$x^2 - (\ n\ )\ x + (10-5i) = 0$$
$$x^2 - (L+M)x + (L\times M) = 0$$

$$L\times M = 10-5i$$
$$(3-4i)\times M = 10-5i$$

$$M = \frac{10-5i}{3-4i}\cdot\frac{3+4i}{3+4i} = \frac{30+40i-15i+20}{9+16} = \frac{50+25i}{25} = \boxed{2+i}$$ The other root

$$n = L+M \quad\Rightarrow\quad 3-4i+2+i = n \quad\Rightarrow\quad n = \boxed{5-3i}$$

Exercises) Q5) If $(3+i)$ is one of the two roots of the equation $x^2 - ax + (5+5i) = 0$

Find the value of (a) and find the other root

Solution : Assume $L = 3+i$ and $M = \text{Other root}$

$$x^2 - (\quad a \quad)x + (5+5i) = 0$$

$$x^2 - (L+M)x + (L \times M) = 0$$

$$L \times M = 5+5i$$

$$(3+i) \times M = 5+5i$$

Note

We can put the root $(3+i)$ instead of (x) in the equation

$$M = \frac{5+5i}{3+i} \cdot \frac{3-i}{3-i} = \frac{15-5i+15i+5}{9+1} = \frac{20+10i}{10} = \boxed{2+i}$$ The other root

$$a = L + m \quad \Rightarrow \quad (3+i)+(2+i) = a \quad \Rightarrow \quad a = \boxed{5+2i}$$

H.W. If $(4-3i)$ was one root for equation $x^2 - a\,x + 9 + 12i = 0$ then what is the value of (a) and find the other root

M.Q. If $(3+2i)$ was one root for equation $x^2-(4-2i)x+(a)=0$, then what is the value of (a) and what is the other root ?

Solution : Assume $L=3+2i$ and $M=$ other root

$$x^2-(4-2i)x+(\quad a \quad)=0$$
$$x^2-(L+M)x+(L\times M)=0$$

$$L+M=4-2i$$

$$M=(4-2i)-L=4-2i-3-2i=\boxed{1-4i}$$ The other root

$$a=L\times M=(3+2i).(1-4i)=3-12i+2i+8=\boxed{11-10i}$$

Example: If $(1+2i)$ is one of the two roots of the equation $az^2-2bz+25=0$, Find the value of $a, b\in R$

Solution : $L=1+2i \quad \Rightarrow \quad m=1-2i$

$$az^2-2bz+25=0 \quad]\div a$$
$$z^2-\boxed{\frac{2b}{a}}z+\boxed{\frac{25}{a}}=0$$

Coefficient of z^2 most equal to (1)

$$L\times m=\frac{25}{a}$$
$$(1+2i)(1-2i)=\frac{25}{a}$$
$$1+4=\frac{25}{a}$$
$$5a=25 \rightarrow \boxed{a=5}$$

$$L+m=\frac{2b}{a}$$
$$1+2i+1-2i=\frac{2b}{a}$$
$$2=\frac{2b}{5}$$
$$10=2b \rightarrow \boxed{b=5}$$

M.Q. If one of the roots of quadratic equation $x^2-3ix-6x+c=0$ is twice than the other root, then find c

Solution : Assume $L=$ first root $\rightarrow$ the other root ($M=2L$)

$$x^2-3ix-6x+c=0$$
$$x^2-(3i+6)\,x+(\ c\ )=0$$
$$x^2-(L+M)\,x+(L\times M)=0$$

$$L+M=L+2L=3L$$

$3L=3i+6 \rightarrow L=2+i$ The first root

$\rightarrow M=2L=2(2+i)=4+2i$ The other root

$c=L.M=(2+i).(4+2i)=8+4i+4i-2 \rightarrow \boxed{c=6+8i}$

H.W. If one of the roots of quadratic equation $x^2-4x-4xi+c=0$ is three times of the other root, then find c

Example: If $(2-4i)$ is one of the two roots of the equation $2x^2-x-bx+c-6=0$,

Find the value of $b,\ c\in R$

Solution : $L=2-4i \quad \Rightarrow \quad m=2+4i$

$2x^2-x(1+b)+(c-6)=0 \quad]\div 2$

Coefficient of x^2 most equal to (1)

$x^2-\left(\frac{1+b}{2}\right)x+\left(\frac{c-6}{2}\right)=0$

$L\times m=\frac{c-6}{2}$

$(2+4i)(2-4i)=\frac{c-6}{2}$

$4+16=\frac{c-6}{2}$

$40=c-6 \rightarrow \boxed{c=46}$

$L+m=\frac{1+b}{2}$

$2+4i+2-4i=\frac{1+b}{2}$

$4=\frac{1+b}{2}$

$8=1+b \rightarrow \boxed{b=7}$

M.Q. If $(1+2i)$ is one of the two roots of the equation $2x^2-2x-bx+a-7=0$,

Find the value of $a,\ b\in R$

THE CUBE ROOTS OF ONE

$$z^3 = 1 \quad \Rightarrow \quad z^3 - 1 = 0 \quad \Rightarrow \quad (z-1)(z^2+z+1) = 0$$

$$1 \qquad -\frac{1}{2}+\frac{\sqrt{3}}{2}i \qquad -\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

$$\omega \qquad \omega^2$$

$$z^3 = 1 \quad \Rightarrow \quad z = \begin{cases} 1 \\ \omega \\ \omega^2 \end{cases}$$

$$\boxed{\overline{a+b\omega} = a+b\omega^2}$$

$$\overline{3+2\omega} = 3+2\omega^2$$

Properties :

1) The two roots are conjugate for each other ($-\frac{1}{2}+\frac{\sqrt{3}}{2}i \;,\; -\frac{1}{2}-\frac{\sqrt{3}}{2}i$)

2) Multiplying the imaginary roots is equal to (1) :

$$\omega \,.\, \omega^2 = \omega^3 = 1 \quad \Rightarrow \quad \boxed{\omega = \frac{1}{\omega^2}} \quad \boxed{\omega^2 = \frac{1}{\omega}}$$

3) Sum of three roots equal to (0)

$$1+\omega+\omega^2 = 0$$

$$\Rightarrow \quad 1+\omega = -\omega^2$$

$$\Rightarrow \quad 1+\omega^2 = -\omega$$

$$\Rightarrow \quad 1 = -\omega-\omega^2$$

$$\vdots \qquad \vdots \qquad \vdots$$

SUM OF TWO ROOTS = -(THIRD ROOT)

Note

$$\omega-\omega^2 = \omega^2-\omega = \pm\sqrt{3}\,i$$

Examples:

$\omega^3 = 1$

$\omega^4 = \omega^3 . \omega^1 = 1 . \omega = \omega$

$\omega^8 = (\omega^3)^2 . \omega^2 = \omega^2$

$\omega^{36} = (\omega^3)^{12} . \omega^0 = 1$

$\omega^{100} = (\omega^3)^{33} . \omega^1 = \omega$

$$\omega^{-20} = \frac{1}{\omega^{20}} = \frac{1}{(\omega^3)^6 . \omega^2} = \frac{1}{\omega^2} = \omega$$

$$\omega^{-133} = \frac{1}{\omega^{133}} = \frac{1}{(\omega^3)^{44} . \omega^1} = \frac{1}{\omega} = \omega^2$$

$$\omega^{-237} = \frac{1}{\omega^{237}} = \frac{1}{(\omega^3)^{76} . \omega^0} = \frac{1}{1} = 1$$

$$\omega^{6n+52} = (\omega^3)^{2n} . \omega^{52} = (1)^{2n} . (\omega^3)^{17} . \omega^1 = \omega$$

$\boxed{\omega^{3n+r} = \omega^r}$

Similar coaffeciants :

1) $7 + 2\omega^2 + 2\omega = 7 + 2(\omega^2 + \omega) = 7 + 2(-1) = 7 - 2 = 5$

2) $5 + 3\omega^2 + 3\omega = 5 + 3(\omega^2 + \omega) = 5 + 3(-1) = 5 - 3 = 2$

3) $4 - 9\omega^2 - 9\omega = 4 - 9(\omega^2 + \omega) = 4 - 9(-1) = 4 + 9 = 13$

$\boxed{\omega^2 + \omega = -1}$

Different coaffeciants :

1) $4 + 3\omega^2 + 2\omega = 4 + 3(-1 - \omega) + 2\omega = 4 - 3 - 3\omega + 2\omega = 1 - \omega$

2) $10 + 8\omega + 7\omega^2 = 10 + 8\omega + 7(-1 - \omega) = 10 + 8\omega - 7 - 7\omega = 3 + \omega$

$\boxed{\omega^2 = -1 - \omega}$

3) $7\omega^{6n} + \frac{3}{\omega^2} + \frac{2}{\omega} = 7(\omega^3)^{2n} + 3\omega + 2\omega^2$

$= 7(1)^{2n} + 3\omega + 2(-1 - \omega)$

$= 7 + 3\omega - 2 - 2\omega$

$= 5 + \omega$

Second Method

$= 7(\omega^3)^{2n} + 3\omega + 2\omega^2$
$= 7 + 3\omega + 2\omega^2$
$= 5 + 2 + \omega + 2\omega + 2\omega^2$
$= 5 + \omega + 2(1 + \omega + \omega^2)$
$= 5 + \omega + 2(0)$
$= 5 + \omega$

Example: **Simplify :** $(1-2\omega^2-\frac{2}{\omega^2})(7+\frac{3}{\omega}+3\omega)$

Solution :

$$(1-2\omega^2-\frac{2}{\omega^2})(7+\frac{3}{\omega}+3\omega) = (1-2\omega^2-2\omega)\,(7+3\omega^2+3\omega)$$

$$= [1-2(\omega^2+\omega)]\,[7+3(\omega^2+\omega)]$$

$$= (1+2)(7-3) \quad =3\times 4 \quad =12$$

M.Q. **Prove that :** $-3(1+\frac{2}{\omega^2}+\omega^2)(1+\omega-\frac{5}{\omega})=18$

Solution :

$$LHS=-3(1+\frac{2}{\omega^2}+\omega^2)(1+\omega-\frac{5}{\omega})=-3(1+\omega^2+2\omega)(1+\omega-5\omega^2)$$

$$=-3(-\omega+2\omega)(-\omega^2-5\omega^2)$$

$$=-3(\omega)(-6\omega^2)=18\omega^3=18=RHS$$

H.W. **Simplify :** $(7+\frac{5}{\omega}+5\omega)(3-2\omega^2-\frac{2}{\omega^2})$

Example: **Prove that :** $(6+5\omega^4+7\omega^2)^2=-3$

Solution :

$$LHS=(6+5\omega^4+7\omega^2)^2=[6+5\omega^3.\omega+7(-1-\omega)]^2$$
$$=[6+5\omega-7-7\omega]^2$$
$$=(-1-2\omega)^2$$
$$=1+4\omega+4\omega^2$$
$$=1+4(\omega+\omega^2)$$
$$=1+4(-1)$$
$$=1-4 \qquad = -3 \qquad = RHS$$

H.W. **Prove that :** $(4\omega^{3n}+\frac{6}{\omega}+5\omega^4)^2=-3\omega^2$

M.Q. **Prove that :** $(5-\frac{5}{\omega^2+1}+\frac{3}{\omega^2})^6=64$

Solution :

$$LHS=(5-\frac{5}{-\omega}+\frac{3}{\omega^2})^6=(5+5\omega^2+3\omega)^6=\left(5(1+\omega^2)+3\omega\right)^6$$

$$=(-5\omega+3\omega)^6=(-2\omega)^6=64\omega^6=64=RHS$$

General Exercises) Q2) Simplify : $\left(3\omega^{9n}+\frac{5}{\omega^5}+\frac{4}{\omega^4}\right)^6$ **, such that** $n\in Z$

Solution :

$$\left[3(\omega^3)^{3n}+\frac{5}{\omega^2}+\frac{4}{\omega}\right]^6=[3(1)^{3n}+5\omega+4\omega^2]^6$$

$$=[3+5\omega+4(-1-\omega)]^6$$

$$=(3+5\omega-4-4\omega)^6$$

$$=(\omega-1)^6 \quad =\left[(\omega-1)^2\right]^3 \quad =\left[\omega^2-2\omega+1\right]^3$$

$$=(-\omega-2\omega)^3$$

$$=(-3\omega)^3 \quad =-27$$

M.Q. Simplify : $(3\omega^{12n}+\frac{5}{\omega^8}+\frac{4}{\omega^{10}})^6$ **, such that** $n\in Z$

Solution :

$$\left[3\omega^{12n}+\frac{5}{\omega^8}+\frac{4}{\omega^{10}}\right]^6=\left[3(\omega^3)^{4n}+\frac{5}{\omega^2}+\frac{4}{\omega}\right]^6=\left[3(1)^{4n}+5\omega+4\omega^2\right]^6$$

$$=\left[3+5\omega+4(-1-\omega)\right]^6=\left[3+5\omega-4-4\omega\right]^6$$

$$=\left[(\omega-1)^2\right]^3=\left[\omega^2-2\omega+1\right]^3$$

$$=\left[-\omega-1-2\omega+1\right]^3=\left[-3\omega\right]^3=-27\,\omega^3=-27$$

Example: Prove that : $\left(\frac{3\omega-4}{3-4\omega^2}\right)^3=1$

Solution :

$\boxed{\omega^3=1}$

$$LHS=\left(\frac{3\omega-4}{3-4\omega^2}\right)^3=\left(\frac{3\omega-4\omega^3}{3-4\omega^2}\right)^3=\left(\frac{\omega\cancel{(3-4\omega^2)}}{\cancel{(3-4\omega^2)}}\right)^3=\omega^3=1=RHS$$

M.Q. **Prove that :** $\left(\dfrac{5\omega^2 i-1}{5+i\omega}\right)^6=-1$

Solution :

$i^2=-1$

$$LHS=\left(\frac{5\omega^2 i-1}{5+i\omega}\right)^6=\left(\frac{5\omega^2 i+i^2}{5+i\omega}\right)^6=\left(\frac{5\omega^2 i+\omega^3 i^2}{5+i\omega}\right)^6=\left(\frac{\omega^2 i(5+\omega i)}{(5+i\omega)}\right)^6$$

$$=\omega^{12}.\,i^6=\omega^{3(4)}.i^4.i^2=-1=RHS$$

H.W. **Prove that :** $\left(\dfrac{7+5\omega^2}{7\omega+5}-\dfrac{3-2\omega}{3\omega^2-2}\right)^4=9$

M.Q. **Find the quadratic equation which has two roots:** $(2-2\omega-2\omega^2)^2\ ,\ (2\omega+2\omega^2-1)^2$

Solution :

$$L=(2\omega+2\omega^2-1)^2=\left[2(\omega+\omega^2)-1\right]^2=\left[2(-1)-1\right]^2=\left[-2-1\right]^2=9$$

$$M=(2-2\omega-2\omega^2)^2=\left[2-2(\omega+\omega^2)\right]^2=\left[2-2(-1)\right]^2=\left[2+2\right]^2=16$$

$$L+M=9+16=25$$
$$L\times M=9\times16=144$$

$\therefore$ The quadratic equation is : $x^2-25x+144=0$

M.Q. **Find the quadratic equation which has two roots:** $\frac{1}{\omega}\ ,\ \frac{1+3\omega}{\omega^2+3}$

Solution :

$$L=\frac{1}{\omega}=\frac{\omega^3}{\omega}=\omega^2$$

$$M=\frac{1+3\omega}{\omega^2+3}=\frac{\omega^3+3\omega}{\omega^2+3}=\frac{\omega(\omega^2+3)}{(\omega^2+3)}=\omega$$

$$L+M=\omega^2+\omega=-1$$

$$L\times M=\omega^2\times\omega=\omega^3=1$$

$\therefore$ The quadratic equation is : $z^2-(-1)z+(1)=0 \longrightarrow z^2+z+1=0$

M.Q. **Find the quadratic equation which has two roots:** $\frac{\omega}{3-\omega^2}\ ,\ \frac{\omega^2}{3-\omega}$

Solution :

$$L+M=\frac{\omega}{3-\omega^2}+\frac{\omega^2}{3-\omega}=\frac{3\omega-\omega^2+3\omega^2-\omega^4}{(3-\omega^2)(3-\omega)}=\frac{3\omega-\omega^2+3\omega^2-\omega}{9-3\omega-3\omega^2+\omega^3}$$

$$=\frac{2\omega+2\omega^2}{9-3(\omega+\omega^2)+1}=\frac{2(\omega+\omega^2)}{10-3(-1)}=\frac{2(-1)}{10+3}=\frac{-2}{13}$$

$$L\times M=\frac{\omega}{3-\omega^2}.\frac{\omega^2}{3-\omega}=\frac{\omega^3}{(3-\omega^2)(3-\omega)}=\frac{1}{9-3\omega-3\omega^2+\omega^3}$$

$$=\frac{1}{9-3(\omega+\omega^2)+1}=\frac{1}{10-3(-1)}=\frac{1}{10+3}=\frac{1}{13}$$

$\therefore$ The quadratic equation is : $x^2-(\frac{-2}{13})x+(\frac{1}{13})=0 \rightarrow x^2+\frac{2}{13}x+\frac{1}{13}=0$

M.Q. Find the quadratic equation which has two roots: $(i-\frac{5}{\omega})$, $(i-\frac{5}{\omega^2})$

M.Q. Find the Quadratic Equation which has real coefficients , one of its roots $\frac{7+i\omega+i\omega^2}{2+i\omega^4+i\omega^5}$

M.Q. ***Find the square roots of the Complex Number:*** $Z = \dfrac{7 + \omega i + \omega^2 i}{1 - \omega i - \omega^2 i}$

M.Q. **Find the real value of** $x, y \in R$ **which satisfied the equation:**

$$x + yi = \left(\sqrt{\omega + \omega^{17}} + \sqrt{\omega + \omega^{38}}\right)^2 - \frac{3+i}{1+i}$$

Solution :

$$x + yi = \left(\sqrt{\omega + (\omega^3)^5.\omega^2} + \sqrt{\omega + (\omega^3)^{12}.\omega^2}\right)^2 - \frac{3+i}{1+i}.\frac{1-i}{1-i}$$

$$x + yi = \left(\sqrt{\omega + \omega^2} + \sqrt{\omega + \omega^2}\right)^2 - \frac{3-3i+i+1}{1+1}$$

$$x + yi = \left(\sqrt{-1} + \sqrt{-1}\right)^2 - \frac{4-2i}{2}$$

$$x + yi = (i+i)^2 - (\frac{4}{2} - \frac{2i}{2})$$

$$x + yi = (2i)^2 - (2-i)$$

$$x + yi = -4 - 2 + i$$

$$x + yi = -6 + i$$

$\boxed{x = -6}$ $\boxed{y = 1}$

H.W. **Find** $x, y \in R$ **which satisfied the equation:** $x + \omega yi = \dfrac{125 - i}{26 - 5\omega i + \omega}$

Solution :

$\boxed{-i = +i^3}$

$$x + \omega yi = \frac{125 + i^3}{26 - 5\omega i + \omega}$$

$$x + \omega yi = \frac{125 + \omega^3 i^3}{26 - 5\omega i + \omega}$$

Example: **Prove that :** $\dfrac{(-\omega^3-\omega).(3-i^2).(1+\omega^8)}{(1+i)^3+(1-i)^3}=1$

Solution :

$$LHS = \frac{(-\omega^3-\omega).(3-i^2).(1+\omega^8)}{(1+i)^3+(1-i)^3} = \frac{(-1-\omega).(3+1).(1+\omega^2)}{(1+i)^2(1+i)+(1-i)^2(1-i)}$$

$$= \frac{(\omega^2).(4).(-\omega)}{(2i)(1+i)+(-2i)(1-i)}$$

$$= \frac{-4\omega^3}{2i+2i^2-2i+2i^2}$$

$$= \frac{-4}{-2-2} = \frac{-4}{-4} = 1 = RHS$$

M.Q. **Prove that :** $\left(\dfrac{1}{\omega}-\dfrac{1}{\omega^2}\right)^2\left(2+\dfrac{2}{\omega}\right)\left(\dfrac{-1}{1+\omega^2}\right)=6$

M.Q. **Prove that :** $\left[\frac{1}{1+i}-\frac{1}{1-i}\right]^{100}=\left[\frac{2+3\omega}{2\omega^2+3}+\frac{4\omega^2+1}{4+\omega}\right]^{200}$

M.Q. If $3+2\omega$ was one root for equation $x^2-ax+7=0$, then what is the other root and what is the value of $a\in R$

H.W. **Prove that :** $\frac{19}{3-2\omega}=5+2\omega$

H.W. **Prove that :** $\frac{3\omega-1}{\omega^2-2i}=1+2\omega i$

EXERCISES (1-3)

Q1) Simplify the following :

a) $\omega^{64}=(\omega^3)^{21}.\omega=\omega$

b) $\omega^{-325}=\dfrac{1}{\omega^{325}}=\dfrac{1}{(\omega^3)^{108}.\omega}=\dfrac{1}{\omega}=\omega^2$

c) $\dfrac{1}{(1+\omega^{-32})^{12}}=\dfrac{1}{(1+\frac{1}{\omega^{32}})^{12}}=\dfrac{1}{(1+\frac{1}{\omega^{2}})^{12}}=\dfrac{1}{(1+\omega)^{12}}=\dfrac{1}{(-\omega^2)^{12}}=\dfrac{1}{\omega^{24}}=\dfrac{1}{1}=1$

d) $(1+\omega^2)^{-4}=(-\omega)^{-4}=\dfrac{1}{(-\omega)^4}=\dfrac{1}{\omega^4}=\dfrac{1}{\omega}=\omega^2$

e) $\omega^{9n+5}=\omega^{9n}.\omega^5=(\omega^9)^n.\omega^2=(1)^n.\omega^2=\omega^2$

Q2) Find the quadratic equation which has two roots :

a) $1+\omega^2 \quad , \quad 1+\omega$

Solution :

$L+M=(1+\omega^2)+(1+\omega)=(-\omega)+(-\omega^2)=1$

$L\times M=(1+\omega^2).(1+\omega)=(-\omega).(-\omega^2)=\omega^3=1$

$x^2-(M+L)x+(M\times L)=0$

$\therefore$ The quadratic equation is : $x^2-x+1=0$

b) $\dfrac{\omega}{2-\omega^2}$, $\dfrac{\omega^2}{2-\omega}$

Solution :

$$L+M=\frac{\omega}{2-\omega^2}+\frac{\omega^2}{2-\omega}=\frac{\omega(2-\omega)+\omega^2(2-\omega^2)}{(2-\omega^2)(2-\omega)}$$

$$=\frac{2\omega-\omega^2+2\omega^2-\omega}{4-2\omega-2\omega^2+1}=\frac{\omega+\omega^2}{5-2(\omega+\omega^2)}=\frac{-1}{5+2}=\boxed{\frac{-1}{7}}$$

$$L\times M=\frac{\omega}{2-\omega^2}\times\frac{\omega^2}{2-\omega}=\frac{\omega^3}{(2-\omega^2).(2-\omega)}=\boxed{\frac{1}{7}}$$

$$x^2-(M+L)x+(M\times L)=0$$

$\therefore$ **The quadratic equation is :** $x^2-(\frac{-1}{7})x+(\frac{1}{7})=0 \qquad \}\times 7$

$$7x^2+x+1=0$$

---

c) $\dfrac{3i}{\omega^2}$, $\dfrac{-3\omega^2}{i}$

Solution :

$$\frac{3i}{\omega^2}=\boxed{3i\omega}$$

$$\frac{-3\omega^2}{i}.\frac{-i}{-i}=\boxed{3i\omega^2}$$

$$L+M=3i\omega+3i\omega^2=3i(\omega+\omega^2)=3i(-1)=\boxed{-3i}$$

$$L\times M=(3i\omega)\times(3i\omega^2)=9i^2\omega^3=9(-1)(1)=\boxed{-9}$$

$$x^2-(M+L)x+(M\times L)=0$$

$\therefore$ **The quadratic equation is :** $x^2+3ix-9=0$

Q3) **If** $Z^2+Z+1=0$ **then find the value of :** $\dfrac{1+3Z^{10}+3Z^{11}}{1-3Z^7-3Z^8}$

Solution :

$$Z^2+Z+1=0$$

$$a=1 \quad , \quad b=1 \quad , \quad c=1$$

$$Z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{3}i}{2}=\begin{cases}\dfrac{-1}{2}+\dfrac{\sqrt{3}}{2}i\\[2ex]\dfrac{-1}{2}-\dfrac{\sqrt{3}}{2}i\end{cases}$$

$$\Rightarrow \quad either \quad Z=\omega \quad or \quad Z=\omega^2$$

$when \quad Z=\omega :$

$$\frac{1+3Z^{10}+3Z^{11}}{1-3Z^7-3Z^8}=\frac{1+3\omega^{10}+3\omega^{11}}{1-3\omega^7-3\omega^8}=\frac{1+3\omega+3\omega^2}{1-3\omega-3\omega^2}=\frac{1+3(\omega+\omega^2)}{1-3(\omega+\omega^2)}=\frac{1-3}{1+3}=\frac{-2}{4}=\frac{-1}{2}$$

$when \quad Z=\omega^2 :$

$$\frac{1+3Z^{10}+3Z^{11}}{1-3Z^7-3Z^8}=\frac{1+3\omega^{20}+3\omega^{22}}{1-3\omega^{14}-3\omega^{16}}=\frac{1+3\omega^2+3\omega}{1-3\omega^2-3\omega}=\frac{1+3(\omega^2+\omega)}{1-3(\omega^2+\omega)}=\frac{1-3}{1+3}=\frac{-2}{4}=\frac{-1}{2}$$

H.W. If $Z^2+Z+1=0$ then find the value of : $\dfrac{7-3Z^{10}-3Z^{11}}{5+3Z^{7}+3Z^{8}}$

Q3) Prove that :

a) $\left(\frac{1}{2+\omega}-\frac{1}{2+\omega^2}\right)^2=-\frac{1}{3}$

Solution :

$$LHS=\left(\frac{1}{2+\omega}-\frac{1}{2+\omega^2}\right)^2=\left(\frac{2+\omega^2-2-\omega}{(2+\omega)(2+\omega^2)}\right)^2=\frac{(\omega^2-\omega)^2}{(4+2\omega^2+2\omega+\omega^3)^2}$$

$$=\frac{\omega^4-2\omega^3+\omega^2}{(4+2(\omega^2+\omega)+1)^2}=\frac{\omega-2+\omega^2}{(5-2)^2}=\frac{-2-1}{(3)^2}=\frac{-3}{9}=-\frac{1}{3}\quad RHS$$

---

b) $\frac{\omega^{14}+\omega^7-1}{\omega^{10}+\omega^5-2}=\frac{2}{3}$

Solution :

$$LHS=\frac{\omega^{14}+\omega^7-1}{\omega^{10}+\omega^5-2}=\frac{\overbrace{\omega^2+\omega}^{-1}-1}{\underbrace{\omega+\omega^2}_{-1}-2}=\frac{(-1)-1}{(-1)-2}=\frac{-2}{-3}=\frac{2}{3}\quad RHS$$

---

c) $\left(1-\frac{2}{\omega^2}+\omega^2\right)\left(1+\omega-\frac{5}{\omega}\right)=18$

Solution :

$$LHS=\left(1-\frac{2}{\omega^2}+\omega^2\right)\left(1+\omega-\frac{5}{\omega}\right)=\left(1-2\omega+\omega^2\right)\left(1+\omega-5\omega^2\right)$$

$$=\left(-\omega-2\omega\right)\left(-\omega^2-5\omega^2\right)=\left(-3\omega\right)\left(-6\omega^2\right)=18\,\omega^3=18\quad RHS$$

---

d) $(1+\omega^2)^3+(1+\omega)^3=-2$

Solution :

$$LHS=(1+\omega^2)^3+(1+\omega)^3=(-\omega)^3+(-\omega^2)^3=(-\omega^3)+(-\omega^6)$$

$$=(-1)+(-1)=-2\quad RHS$$

Geometric Representation of Complex Numbers

((Argand shape))

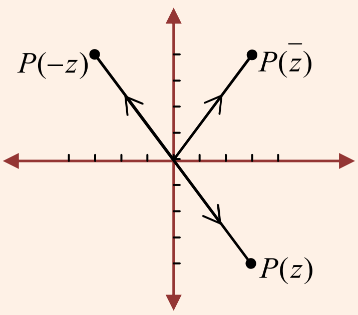
Each of complex numbers is represent a point in Complex plane

$$\underset{\text{algebraic form}}{z = a + bi} \rightarrow \underset{\text{Cartesian form}}{P(z) = (a,b)}$$

z	P (z)	P ($\bar{z}$)	P (- z)
$1-2i$	$(1,-2)$	$(1,2)$	$(-1,2)$
$3i-5$	$(-5,3)$	$(-5,-3)$	$(5,-3)$
$-4i$	$(0,-4)$	$(0,4)$	$(0,4)$
6	$(6,0)$	$(6,0)$	$(-6,0)$

Example : If $Z = 3-4i$ Represent Z , $-Z$, $\bar{Z}$ on Argand shape

Solution :



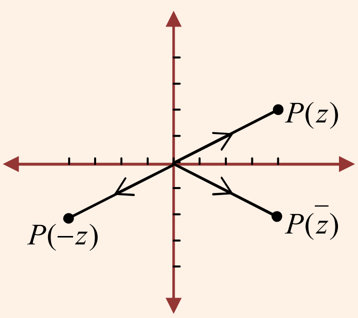
$P(Z) = (3,-4)$

$P(-Z) = (-3,4)$

$P(\bar{Z}) = (3 , 4)$

Q3) If $Z = 4+2i$ Represent Z , $-Z$, $\bar{Z}$ on Argand shape

Solution :



$P(Z) = (4,2)$

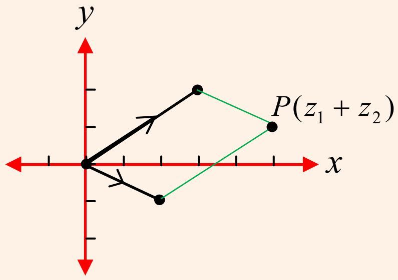
$P(-Z) = (-4,-2)$

$P(\bar{Z}) = (4, -2)$

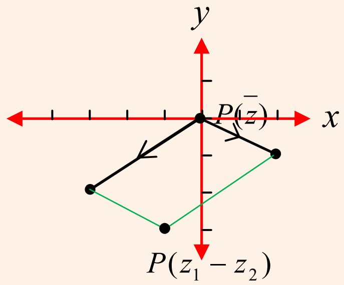
Example : If $z_1 = 2 - i$, $z_2 = 3 + 2i$ **Represent** $z_1 - z_2$, $z_1 + z_2$ **on Argand shape**

Solution :

$P(z_1) = (2,-1)$, $P(z_2) = (3,2)$



$P(z_1 + z_2) = (2,-1) + (3,2) = (5,1)$



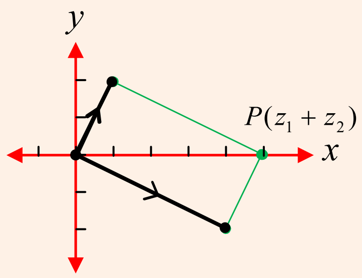
$P(z_1 - z_2) = P(z_1) + P(-z_2)$

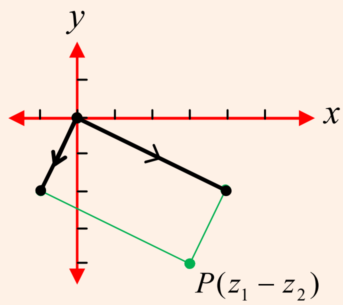
$= (2,-1) + (-3,-2) = (-1,-3)$

Q4) If $z_1 = 4 - 2i$, $z_2 = 1 + 2i$ **Represent on Argand shape :**
$z_1 + z_2$, $z_1 - z_2$, $-3z_2$, $2z_1$

Solution :

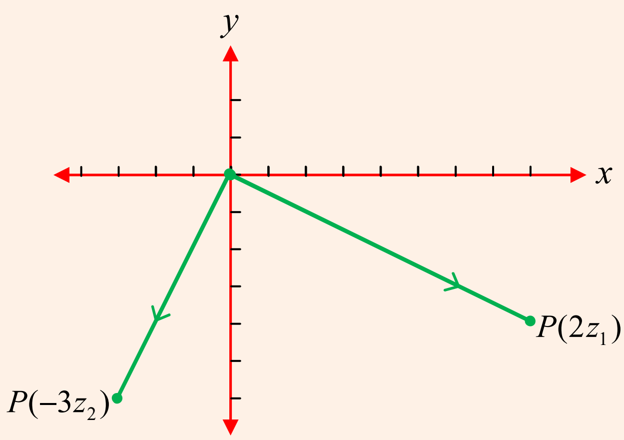
$P(z_1) = (4,-2)$, $P(z_2) = (1,2)$





$P(z_1 + z_2) = (4,-2) + (1,2)$
$= (5,0)$

$P(z_1 - z_2) = P(z_1) + P(-z_2)$
$= (4,-2) + (-1,-2)$
$= (3,-4)$



$P(-3z_2) = -3(1,2) = (-3,-6)$

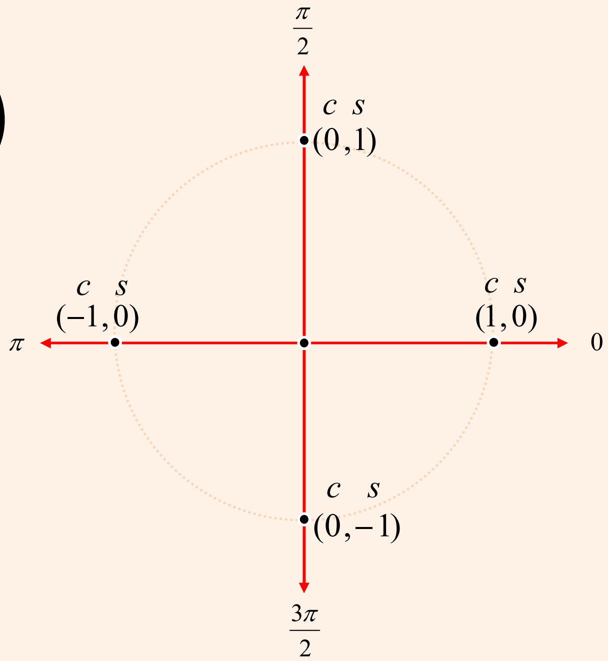
$P(2z_1) = 2(4,-2) = (8,-4)$

SPECIAL ANGLES & TRIGONOMETRY

$$\sin^2 x + \cos^2 x = 1$$

$$\sin(2x) = 2\sin x \cos x$$

$$\cos(2x) = \cos^2 x - \sin^2 x$$



	$0^o = 0$	$90^o = \frac{\pi}{2}$	$180^o = \pi$	$270^o = \frac{3\pi}{2}$
sin	0	1	0	-1
cos	1	0	-1	0

$$\tan x = \frac{\sin x}{\cos x}$$

	$30^o = \frac{\pi}{6}$	$60^o = \frac{\pi}{3}$	$45^o = \frac{\pi}{4}$
sin	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$

$$\cot x = \frac{\cos x}{\sin x}$$

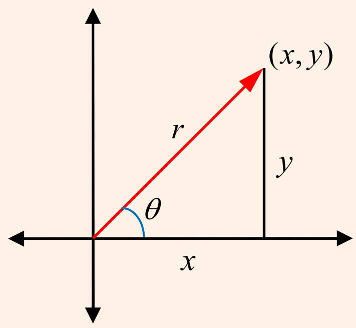
THE POLAR FORM of Complex Numbers

In the Cartesian plane we can write the complex number (x + yi) as (x , y). If we sketch a vector (r) from the origin point to (x , y) and symbolized the angle between the vector and x-axis (θ) , so we can write the polar form of (x , y) as :

$$r^2 = x^2 + y^2 \quad \Rightarrow \quad r = \sqrt{x^2 + y^2}$$

$$\operatorname{mod}(z) = r = \| z \| \qquad (Modulus)$$

$$\arg(z) = \theta \qquad (Argument)$$



$$\cos\theta = \frac{x}{r} = \frac{x}{\| z \|} \qquad \sin\theta = \frac{y}{r} = \frac{y}{\| z \|}$$

Note : After finding the (r) and the (θ) we can write a complex number as a polar form :

$$Z = r(\cos\theta + i\sin\theta)$$

Argument

$$\arg(z) = \theta$$

Modulus

$$\operatorname{mod}(z) = \| z \| = r$$

Example: Write the polar form of : $5\sqrt{3}+5i$

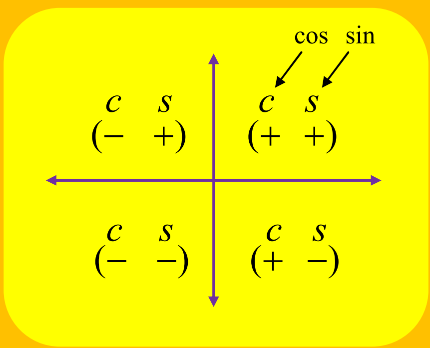
Solution :

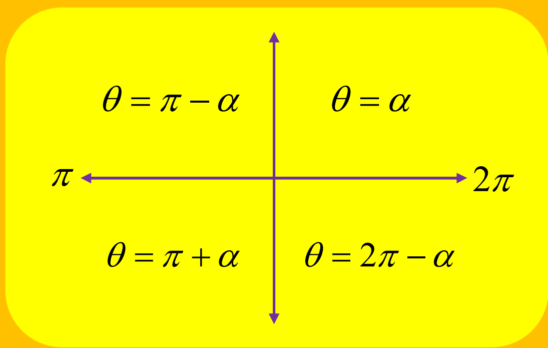
$Z=(5\sqrt{3}\,,5)$ $\qquad$ $r=\|Z\|=\sqrt{x^2+y^2}=\sqrt{75+25}=\sqrt{100}=10$

$$\left.\begin{array}{l}\cos\theta=\dfrac{x}{r}=\dfrac{\sqrt{3}}{2}\\[2ex]\sin\theta=\dfrac{y}{r}=-\dfrac{1}{2}\end{array}\right\rangle \Rightarrow \quad \alpha=30^o=\frac{\pi}{6}$$

$$Z \;=\; r(\cos\theta+i\sin\theta) \;=\; 10(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$$

Note : *We can determine the quadrant which is located (θ) through the sign of ($\sin\theta$) and ($\cos\theta$) and if we assume that (α) is the angle of attribution, we conclude :*





Example: Write the polar form of : $2\sqrt{3}-2i$

Solution :

$Z=(2\sqrt{3}\,,-2)$ $\qquad$ $r=\sqrt{x^2+y^2}=\sqrt{12+4}=\sqrt{16}=4$

$$\left.\begin{array}{l}\cos\theta=\dfrac{x}{r}=\dfrac{\sqrt{3}}{2}\\[2ex]\sin\theta=\dfrac{y}{r}=-\dfrac{1}{2}\end{array}\right\rangle \Rightarrow \alpha=\frac{\pi}{6}\ (4^{\text{th}}\text{ quadrant}) \qquad \theta=2\pi-\alpha=2\pi-\frac{\pi}{6}=\frac{11\pi}{6}$$

$$Z \;=\; r(\cos\theta+i\sin\theta) \;=\; 4(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6})$$

Example: Write the polar form of : $-9-9i$

Solution :

$Z=(-9,-9)$ $\qquad r=\sqrt{x^2+y^2}=\sqrt{81+81}=\sqrt{81\times 2}=9\sqrt{2}$

$$\left.\begin{aligned}\cos\theta &= \frac{-9}{9\sqrt{2}}=\frac{-1}{\sqrt{2}}\\ \sin\theta &= \frac{-9}{9\sqrt{2}}=\frac{-1}{\sqrt{2}}\end{aligned}\right\rangle \Rightarrow \quad \theta=\pi+\frac{\pi}{4}=\frac{5\pi}{4}$$

$$Z = r(\cos\theta+i\sin\theta) = 9\sqrt{2}\,(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4})$$

H.W. Write the polar form of : $4-4\sqrt{3}i$

H.W. Write the polar form of : $-1-\sqrt{-3}$

H.W. Write the polar form of : $i-\sqrt{3}$

Note
$i-\sqrt{3} \rightarrow (-\sqrt{3}, 1)$

H.W. Write the polar form of : $\frac{1}{2}-\frac{\sqrt{3}}{2}i$

M.Q. Write the polar form of : $\dfrac{1-3i^2}{1-\omega i-\omega^2 i}$

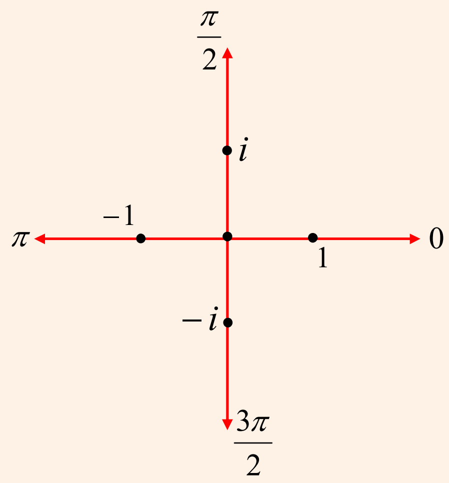
$\dfrac{1+3}{1-i(\omega+\omega^2)} \to \dfrac{4}{1+i}$

M.Q. Find the modulus and argument of : $Z=\dfrac{4+2i\omega+2i\omega^2}{3-i\omega^2-i\omega}$

Example: Write the polar form of :

a) 1 b) i c) -1 d) $-i$

Solution :



$$1 = (\cos 0 + i\sin 0)$$

$$-1 = (\cos\pi + i\sin\pi)$$

$$i = (\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$$

$$-i = (\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2})$$

Note : Depending on the previous information we can directly write the pure real numbers and the pure imaginary numbers as the polar form :

$$\begin{aligned} 5 &= 5\times 1 &&= 5(\cos 0 + i\sin 0) \\ 9i &= 9\times i &&= 9(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) \\ -3 &= 3\times(-1) &&= 3(\cos\pi + i\sin\pi) \\ -3i &= 3\times(-i) &&= 3(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}) \end{aligned}$$

Note : Sometimes we need use the modulus like this :

$$\|x+yi\| = \sqrt{x^2+y^2}$$

$$\|9+2i\| = \sqrt{(9)^2+(2)^2} = \sqrt{81+4} = \sqrt{85}$$

$$\|7-i\| = \sqrt{(7)^2+(-1)^2} = \sqrt{49+1} = \sqrt{50} = 5\sqrt{2}$$

$$\|3-4i\| = \sqrt{(3)^2+(-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

DeMoivre's Theorem :

Not Fractional Exponents

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \qquad \forall\ n \in Z \quad , \quad \theta \in R$$

Example: Calculate by using DeMoivre's Theorem :

$$(\cos\frac{3\pi}{16} + i\sin\frac{3\pi}{16})^4 = \cos(4)\frac{3\pi}{16} + i\sin(4)\frac{3\pi}{16} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

$$(\cos\frac{5\pi}{21} + i\sin\frac{5\pi}{21})^7 = \cos(7)\frac{5\pi}{21} + i\sin(7)\frac{5\pi}{21} = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$(\cos\frac{7\pi}{18} + i\sin\frac{7\pi}{18})^3 = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6} = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$

$$(\cos\frac{3\pi}{10} + i\sin\frac{3\pi}{10})^5 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = 0 + i(-1) = -i$$

$$(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8})^{-2} = \cos(-2)\frac{3\pi}{8} + i\sin(-2)\frac{3\pi}{8}$$

$$= \cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$$

Note

$\sin(-\theta) = -\sin(\theta)$

$\cos(-\theta) = \cos(\theta)$

Generalization :

$$(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$$

Note : In the case that the numerator is greater than twice the denominator $(\frac{17\pi}{6},\frac{20\pi}{3},\frac{49\pi}{4},\frac{11\pi}{4},\frac{9\pi}{4})$ it means that there are extra cycles $(2\pi,4\pi,6\pi,....)$, so we subtract these extra cycles as examples below :

$$(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})^{19} = \cos\frac{19\pi}{4}+i\sin\frac{19\pi}{4}$$

$$= \cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}$$

$$\boxed{\frac{19\pi}{4}-4\pi=\frac{19\pi-16\pi}{4}=\frac{3\pi}{4}}$$

$$= -\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$$

$$(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})^{15} = \cos\frac{15\pi}{4}+i\sin\frac{15\pi}{4}$$

$$= \cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}$$

$$\boxed{\frac{15\pi}{4}-2\pi=\frac{15\pi-8\pi}{4}=\frac{7\pi}{4}}$$

$$= \frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$$

$$(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6})^{-2} = \cos\frac{11\pi}{3}-i\sin\frac{11\pi}{3}$$

$$= \cos\frac{5\pi}{3}-i\sin\frac{5\pi}{3}$$

$$\boxed{\frac{11\pi}{3}-2\pi=\frac{11\pi-6\pi}{3}=\frac{5\pi}{3}}$$

$$= \frac{1}{2}+\frac{\sqrt{3}}{2}i$$

Note : There is another way (dividing the numerator by twice of the denominator and make the remainder of the division in the numerator)

H.W. Calculate by using DeMoivre's Theorem (or generalization) :

1) $(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6})^2$

2) $(\cos\frac{3\pi}{20}+i\sin\frac{3\pi}{20})^5$

3) $(\cos\frac{\pi}{9}+i\sin\frac{\pi}{9})^6$

4) $(\cos\frac{3\pi}{14}+i\sin\frac{3\pi}{14})^{-7}$

5) $(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})^{23}$

6) $(\cos\frac{3\pi}{4}+i\sin\frac{7\pi}{4})^{-6}$

Example: *Calculate by using DeMoivre's Theorem :* $(1+i)^{11}$

Solution :

$$Z=(1+i)=(1,1)$$

$$r=\sqrt{x^2+y^2}=\sqrt{(1)^2+(1)^2}=\sqrt{2}$$

$$\left.\begin{array}{l}\cos\theta=\dfrac{1}{\sqrt{2}}\\[2ex] \sin\theta=\dfrac{1}{\sqrt{2}}\end{array}\right\rangle \Rightarrow \quad \theta=\frac{\pi}{4}$$

in 1^{st} quadrant

$$Z = r(\cos\theta+i\sin\theta) = \sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$$

$$Z^{11} = \left(\sqrt{2}\right)^{11}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})^{11}$$

$$\frac{11\pi}{4}-2\pi=\frac{11\pi-8\pi}{4}=\frac{3\pi}{4}$$

$$(1+i)^{11} = \left(\sqrt{2}\right)^{10}\left(\sqrt{2}\right)(\cos\frac{11\pi}{4}+i\sin\frac{11\pi}{4})$$

$$=32\sqrt{2}(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4})$$

$$\frac{3\pi}{4}=135^\circ \quad \text{in } 2^{nd} \text{ quadrant}$$

$$=32\sqrt{2}(-\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$$

$$=32\sqrt{2}(\frac{-1}{\sqrt{2}}+i\frac{1}{\sqrt{2}})$$

$$=32(-1+i)$$

$$=-32+32i$$

M.Q. *Calculate by using DeMoivre's Theorem :* $(-1-\sqrt{-1})^{-3}$

Solution :

$$Z=-1-i=(-1,-1)$$

$$r=\sqrt{x^2+y^2}=\sqrt{(-1)^2+(-1)^2}=\sqrt{2}$$

$$\left.\begin{array}{l}\cos\theta=\dfrac{-1}{\sqrt{2}}\\[2ex] \sin\theta=\dfrac{-1}{\sqrt{2}}\end{array}\right\rangle \Rightarrow \quad \theta=\pi+\frac{\pi}{4}=\frac{5\pi}{4}$$

in 3^{rd} quadrant

$$Z = r(\cos\theta+i\sin\theta) = \sqrt{2}\,(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4})$$

$$Z^{-3} = \left(\sqrt{2}\right)^{-3}(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4})^{-3}$$

$$\frac{15\pi}{4}-2\pi=\frac{15\pi-8\pi}{4}=\frac{7\pi}{4}$$

$$(-1-i)^{-3} = \frac{1}{\left(\sqrt{2}\right)^3}\,(\cos\frac{15\pi}{4}-i\sin\frac{15\pi}{4})$$

$$=\frac{1}{2\sqrt{2}}\,(\cos\frac{7\pi}{4}-i\sin\frac{7\pi}{4})$$

$$=\frac{1}{2\sqrt{2}}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$$

$\frac{7\pi}{4}=315^\circ$ in 4^{th} quadrant

$$=\frac{1}{2\sqrt{2}}(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)$$

$$=\frac{1}{4}+\frac{1}{4}i$$

M.Q. *Calculate by using DeMoivre's Theorem :* $(1+i)^{-5}$

M.Q. *Calculate by using DeMoivre's Theorem :* $(-\sqrt{3}+i)^5$

M.Q. *Calculate by using DeMoivre's Theorem :* $\dfrac{1}{(1-\sqrt{3}\,i)^4}$

M.Q. Find the Quadratic Equation which has real coefficients , one of its roots is the complex number which its modulus (2) and argument ($\frac{5\pi}{3}$)

Corollary of DeMoivre's Theorem :

Fractional Exponents

$$Z^{\frac{1}{n}} = r^{\frac{1}{n}}(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n})$$

$$k = 0\,,1\,,2\,,\ldots\,,(n-1)$$

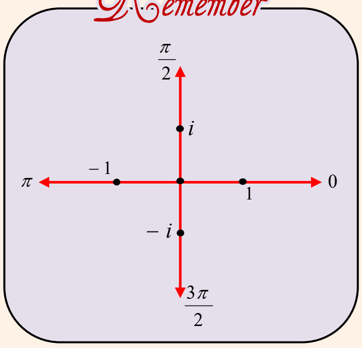
Example: *Find cube roots of* (-8) *by using DeMoivre's Theorem :*

Solution :

$$Z = \sqrt[3]{-8} = 8^{\frac{1}{3}}(-1)^{\frac{1}{3}} = 2(\cos\pi + i\sin\pi)^{\frac{1}{3}}$$

$$\theta = \pi \quad , \quad n = 3 \quad , \quad k = 0,1,2$$

Remember



$$z = 2(\cos\frac{\pi+2k\pi}{3} + i\sin\frac{\pi+2k\pi}{3})$$

when $k = 0:$

$$Z_1 = 2\,(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = 2\,(\frac{1}{2} + \frac{\sqrt{3}}{2}i) = \boxed{1+\sqrt{3}\,i}$$

when $k = 1:$

$$Z_2 = 2\,(\cos\frac{3\pi}{3} + i\sin\frac{3\pi}{3}) = 2\,(\cos\pi + i\sin\pi) = 2(-1) = \boxed{-2}$$

when $k = 2:$

$$Z_3 = 2\,(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3})$$

$$= 2\,(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}) = 2\,(\frac{1}{2} - \frac{\sqrt{3}}{2}i) = \boxed{1-\sqrt{3}\,i}$$

$$S.S. = \left\{-2\ ,\ 1+\sqrt{3}\,i\ ,\ 1-\sqrt{3}\,i\right\}$$

Example: *Calculate by using DeMoivre's Theorem :* $x^3 + 1 = 0$

Solution :

$x^3 = -1$

$x = (-1)^{\frac{1}{3}}$

$x = (\cos\pi + i\sin\pi)^{\frac{1}{3}} \quad \Rightarrow \quad n = 3 \quad , \quad k = 0,1,2$

$x = (1)^{\frac{1}{3}}(\cos\frac{\pi + 2k\pi}{3} + i\sin\frac{\pi + 2k\pi}{3})$

when $k = 0:$

$x = (\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = \boxed{\frac{1}{2} + \frac{\sqrt{3}}{2}i}$

when $k = 1:$

$x = (\cos\frac{\pi + 2\pi}{3} + i\sin\frac{\pi + 2\pi}{3})$

$= \cos\pi + i\sin\pi = \boxed{-1}$

when $k = 2:$

$x = (\cos\frac{\pi + 4\pi}{3} + i\sin\frac{\pi + 4\pi}{3})$

$= \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$

$= \cos\frac{\pi}{3} - i\sin\frac{\pi}{3} = \boxed{\frac{1}{2} - \frac{\sqrt{3}}{2}i}$

$$S.S. = \left\{(\frac{1}{2} + \frac{\sqrt{3}}{2}i) \ , \ -1 \ , \ (\frac{1}{2} - \frac{\sqrt{3}}{2}i)\right\}$$

Example: *Find cube roots of (64i) by using DeMoivre's Theorem*

Solution :

$$\text{Let}\quad z=\sqrt[3]{64i}=(64\times i)^{\frac{1}{3}}=\left[64(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})\right]^{\frac{1}{3}}=(64)^{\frac{1}{3}}(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})^{\frac{1}{3}}$$

$$\theta=\frac{\pi}{2}\qquad n=3\qquad k=0,1,2$$

$$z=(64)^{\frac{1}{3}}(\cos\frac{\frac{\pi}{2}+2k\pi}{3}+i\sin\frac{\frac{\pi}{2}+2k\pi}{3})$$

$$=4(\cos\frac{\pi+4k\pi}{6}+i\sin\frac{\pi+4k\pi}{6})$$

Note

$$\frac{\frac{\pi}{2}\times 2+2k\pi\times 2}{3\times 2}=\frac{\pi+4k\pi}{6}$$

$$\text{when } k=0:\quad z_1=4\,(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})=4\,(\frac{\sqrt{3}}{2}+\frac{1}{2}i)=2\sqrt{3}+2i$$

$$\text{when } k=1:\quad z_2=4\,(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6})=4(-\frac{\sqrt{3}}{2}+\frac{1}{2}i)=-2\sqrt{3}+2i$$

$$\text{when } k=2:\quad z_3=4\,(\cos\frac{9\pi}{6}+i\sin\frac{9\pi}{6})=4\,(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2})=4(-i)=-4i$$

The solution set $=\left\{-4i\ ,\ -2\sqrt{3}+2i\ ,\ 2\sqrt{3}+2i\right\}$

H.W. : *Find square roots of* $(\frac{1}{2}+\frac{\sqrt{3}}{2}i)$ *by using DeMoivre's Theorem*

M.Q. By using corollary DeMoivre's Theorem find square roots of $(9i)$

(By using DeMoivre's Theorem solve the equation : $z^2 - 9i = 0$)

Example: *Write the polar form of* $(\sqrt{3}+i)^2$ *then find the five roots*

Solution :

Find the polar form of $(\sqrt{3}+i)^{\frac{2}{5}}$

Let $Z=\sqrt{3}+i \rightarrow (\sqrt{3},1) \Rightarrow r=\sqrt{3+1}=\sqrt{4}=2$

Note

$$(\quad)^{\frac{a}{b}}=\left[(\quad)^a\right]^{\frac{1}{b}}$$

$$\left.\begin{aligned}\cos\theta&=\frac{\sqrt{3}}{2}\\ \sin\theta&=\frac{1}{2}\end{aligned}\right\} \Rightarrow \theta=\frac{\pi}{6} \qquad \therefore\ Z=2\,(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$$

$$(\sqrt{3}+i)^2=2^2\,(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})^2=4\,(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$$

$$\left(Z^2\right)^{\frac{1}{5}} = 4^{\frac{1}{5}}(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})^{\frac{1}{5}}$$

$\theta=\frac{\pi}{3}$, $n=5$, $k=0,1,2,3,4$

$$(Z)^{\frac{2}{5}}=\sqrt[5]{4}\,(\cos\frac{\frac{\pi}{3}+2k\pi}{5}+i\sin\frac{\frac{\pi}{3}+2k\pi}{5})$$

$$=\sqrt[5]{4}\,(\cos\frac{\pi+6k\pi}{15}+i\sin\frac{\pi+6k\pi}{15})$$

Note

$$\frac{\frac{\pi}{3}\times 3+2k\pi\times 3}{5\times 3}=\frac{\pi+6k\pi}{15}$$

when $k=0 \rightarrow Z_1=\sqrt[5]{4}(\cos\frac{\pi}{15}+i\sin\frac{\pi}{15})$

when $k=1 \rightarrow Z_2=\sqrt[5]{4}(\cos\frac{7\pi}{15}+i\sin\frac{7\pi}{15})$

when $k=2 \rightarrow Z_3=\sqrt[5]{4}(\cos\frac{13\pi}{15}+i\sin\frac{13\pi}{15})$

when $k=3 \rightarrow Z_4=\sqrt[5]{4}(\cos\frac{19\pi}{15}+i\sin\frac{19\pi}{15})$

when $k=4 \rightarrow Z_5=\sqrt[5]{4}(\cos\frac{25\pi}{15}+i\sin\frac{25\pi}{15}) = \sqrt[5]{4}(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3})$

M.Q. *Calculate by using DeMoivre's Theorem :* $(\sqrt{3}+i)^{\frac{-3}{2}}$

Note

$(\quad)^{\frac{-3}{2}} = \left[(\quad)^{-3}\right]^{\frac{1}{2}}$

EXERCISES (1-5)

Q1) Calculate :

$$a)\ \left[\cos\frac{5}{24}\pi + i\sin\frac{5}{24}\pi\right]^4 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$$

$$\frac{5\pi}{6} = 150^\circ \quad (\text{in } 2^{nd} \text{ quadrant})$$

$$= -\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$$

$$= -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$b)\ \left[\cos\frac{7}{12}\pi + i\sin\frac{7}{12}\pi\right]^{-3} = \cos\frac{7\pi}{4} - i\sin\frac{7\pi}{4}$$

$$\frac{7\pi}{4} = 315^\circ \quad (\text{in } 4^{th} \text{ quadrant})$$

$$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

---

2) Calculate by using DeMoivre's Theorem (or generalization) **:**

a) $(1-i)^7$

$$Z = 1 - i = (1,-1)$$

$$r = \sqrt{x^2+y^2} = \sqrt{1+1} = \sqrt{2}$$

$$\left.\begin{aligned}\cos\theta &= \frac{1}{\sqrt{2}} \\ \sin\theta &= -\frac{1}{\sqrt{2}}\end{aligned}\right\rangle \Rightarrow \quad \alpha = 45^\circ = \frac{\pi}{4} \quad \Rightarrow \quad \theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

$$\frac{49\pi}{4} - 12\pi = \frac{49\pi - 48\pi}{4} = \frac{\pi}{4}$$

$$Z = r(\cos\theta + i\sin\theta) = \sqrt{2}(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4})$$

$$Z^7 = (\sqrt{2})^7(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4})^7$$

$$= 2^3\sqrt{2}(\cos\frac{49\pi}{4} + i\sin\frac{49\pi}{4})$$

$$= 8\sqrt{2}\,(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$$

$$= 8\sqrt{2}\,(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i) \quad = 8 + 8i$$

$$\begin{aligned}(1-i)^7 &= \left((1-i)^2\right)^3(1-i) \\ &= (-2i)^3(1-i) \\ &= (8i)(1-i) \\ &= 8i + 8\end{aligned}$$

b) $(\sqrt{3}+i)^{-9}$

$Z=(\sqrt{3},1)$

$r=\sqrt{x^2+y^2}=\sqrt{3+1}=\sqrt{4}=2$

$$\left.\begin{array}{l}\cos\theta=\dfrac{\sqrt{3}}{2}\\ \sin\theta=\dfrac{1}{2}\end{array}\right\rangle \Rightarrow \quad \theta=\frac{\pi}{6}$$

$$Z = r(\cos\theta+i\sin\theta) = 2(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$$

$$Z^{-9}=2^{-9}(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})^{-9}$$

$$=\frac{1}{2^9}(\cos\frac{3\pi}{2}-i\sin\frac{3\pi}{2}) \quad =\frac{1}{512}\left[0-i(-1)\right] \quad =\frac{1}{512}i$$

---

3) Calculate

a) $\dfrac{(\cos 2\theta+i\sin 2\theta)^5}{(\cos 3\theta+i\sin 3\theta)}$ b) $(\cos\theta+i\sin\theta)^8(\cos\theta-i\sin\theta)^4$

Solution :

a) $$\frac{(\cos 2\theta+i\sin 2\theta)^5}{(\cos 3\theta+i\sin 3\theta)} = \frac{(\cos\theta+i\sin\theta)^{10}}{(\cos\theta+i\sin\theta)^3} = (\cos\theta+i\sin\theta)^7 = \cos 7\theta+i\sin 7\theta$$

---

b) $$(\cos\theta+i\sin\theta)^8(\cos\theta-i\sin\theta)^4 = (\cos\theta+i\sin\theta)^8(\cos\theta+i\sin\theta)^{-4}$$
$$=(\cos\theta+i\sin\theta)^4$$
$$=\cos 4\theta+i\sin 4\theta$$

M.Q. *Prove that* : $\dfrac{(\cos 2\theta + i\sin 2\theta)^5}{(\cos 4\theta + i\sin 4\theta)^2} - (\cos\theta + i\sin\theta)^2 = 0$

M.Q. *Prove that* : $\left[\dfrac{(\cos 3\theta + i\sin 3\theta)^4}{(\cos 5\theta + i\sin 5\theta)^2}\right](\cos\theta - i\sin\theta)^2 = 1$

M.Q. *Calculate* : $\dfrac{(\cos 2\theta - i\sin 2\theta)^{-5}}{(\cos 5\theta + i\sin 5\theta)^2} + 1$

M.Q. *Simplified* : $(\cos\theta - i\sin\theta)^4 . \dfrac{(\cos 5\theta + i\sin 5\theta)^2}{(\cos 3\theta + i\sin 3\theta)^2}$

4) *Find square roots of* $(-1+\sqrt{3}i)$ *by using DeMoivre's Theorem*

$$Z=-1+\sqrt{3}i=(-1,\sqrt{3}) \qquad , \qquad r=\sqrt{1+3}=2$$

$$\left.\begin{aligned}\cos\theta&=\frac{-1}{2}\\ \sin\theta&=\frac{\sqrt{3}}{2}\end{aligned}\right\} \Rightarrow \alpha=60^{o}=\frac{\pi}{3} \Rightarrow \theta=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$$

$$\Rightarrow Z=r(\cos\theta+i\sin\theta)=2(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})$$

$$Z^{\frac{1}{2}} = 2^{\frac{1}{2}}(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})^{\frac{1}{2}}$$

$$\theta=\frac{2\pi}{3} \qquad n=2 \qquad , \qquad k=0,1$$

$$Z^{\frac{1}{2}} = \sqrt{2}(\cos\frac{\frac{2\pi}{3}+2k\pi}{2}+i\sin\frac{\frac{2\pi}{3}+2k\pi}{2})$$

$$= \sqrt{2}(\cos\frac{2\pi+6k\pi}{6}+i\sin\frac{2\pi+6k\pi}{6})$$

Note

$$\frac{\frac{2\pi}{3}\times 3+2k\pi\times 3}{2\times 3}=\frac{2\pi+6k\pi}{6}$$

when $k=0$:

$$Z_1 = \sqrt{2}(\cos\frac{\cancel{2}\pi}{\cancel{6}}+i\sin\frac{\cancel{2}\pi}{\cancel{6}}) = \sqrt{2}(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$$

$$= \sqrt{2}(\frac{1}{2}+\frac{\sqrt{3}}{2}i) = \boxed{\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{\sqrt{2}}i}$$

when $k=1$:

$$Z_2 = \sqrt{2}(\cos\frac{\cancel{8}\pi}{\cancel{6}}+i\sin\frac{\cancel{8}\pi}{\cancel{6}}) = \sqrt{2}(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3})$$

$$= \sqrt{2}(-\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}) = \sqrt{2}(-\frac{1}{2}-\frac{\sqrt{3}}{2}i) = \boxed{-\frac{1}{\sqrt{2}}-\frac{\sqrt{3}}{\sqrt{2}}i}$$

$$S.S.=\left\{ \frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{\sqrt{2}}i \ , \ -\frac{1}{\sqrt{2}}-\frac{\sqrt{3}}{\sqrt{2}}i \right\}$$

5) Find three roots of $(27i)$ *by using DeMoivre's Theorem*

Q1) Solve the equation : $x^3 - 27i = 0$

Q2) Solve the equation : $\frac{x^3}{3} - 9i = 0$

$$Z = \sqrt[3]{27i} = (27i)^{\frac{1}{3}} = 27^{\frac{1}{3}}(i)^{\frac{1}{3}} = 3(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})^{\frac{1}{3}}$$

$\theta = \frac{\pi}{2}$ $\quad n = 3 \quad , \quad k = 0,1,2$

Note

$$\frac{\frac{\pi}{2} \times 2 + 2k\pi \times 2}{3 \times 2} = \frac{\pi + 4k\pi}{6}$$

$$Z = 3(\cos\frac{\frac{\pi}{2} + 2k\pi}{3} + i\sin\frac{\frac{\pi}{2} + 2k\pi}{3}) = 3(\cos\frac{\pi + 4k\pi}{6} + i\sin\frac{\pi + 4k\pi}{6})$$

when $k = 0$:

$$Z_1 = 3(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = 3(\frac{\sqrt{3}}{2} + \frac{1}{2}i) = \boxed{\frac{3\sqrt{3}}{2} + \frac{3}{2}i}$$

when $k = 1$:

$$Z_2 = 3(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}) = 3(-\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = \boxed{\frac{-3\sqrt{3}}{2} + \frac{3}{2}i}$$

when $k = 2$:

$$Z_3 = 3(\cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}) = 3(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}) = 3(-i) = \boxed{-3i}$$

$$S.S. = \left\{ \frac{3\sqrt{3}}{2} + \frac{3}{2}i \ , \ \frac{-3\sqrt{3}}{2} + \frac{3}{2}i \ , \ -3i \right\}$$

6) *Find four roots of* (-16) *by using DeMoivre's Theorem*

Q1) Solve the equation : $x^4 + 16 = 0$

Q2) Solve the equation : $\frac{x^4}{2} + 8 = 0$

$$Z = \sqrt[4]{-16} = (-16)^{\frac{1}{4}} = 16^{\frac{1}{4}}(-1)^{\frac{1}{4}} = 2(\cos\pi + i\sin\pi)^{\frac{1}{4}}$$

$\theta = \pi \qquad n = 4 \quad , \qquad k = 0,1,2,3$

$$Z = 2(\cos\frac{\pi + 2k\pi}{4} + i\sin\frac{\pi + 2k\pi}{4})$$

when $k = 0 :\quad Z_1 = 2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = 2(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i) = \boxed{\sqrt{2} + \sqrt{2}i}$

when $k = 1 :\quad Z_2 = 2(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}) = 2(-\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = \boxed{-\sqrt{2} + \sqrt{2}i}$

when $k = 2 :\quad Z_3 = 2(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}) = 2(-\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}) = \boxed{-\sqrt{2} - \sqrt{2}i}$

when $k = 3 :\quad Z_4 = 2(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}) = 2(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}) = \boxed{\sqrt{2} - \sqrt{2}i}$

$$S.S. = \left\{(\sqrt{2} - \sqrt{2}i) \;,\; (-\sqrt{2} - \sqrt{2}i) \;,\; (-\sqrt{2} + \sqrt{2}i) \;,\; (\sqrt{2} + \sqrt{2}i)\right\}$$

7) Find six roots of $(-64i)$ *by using DeMoivre's Theorem*

$$Z = \sqrt[6]{-64i} = (-64i)^{\frac{1}{6}} = 64^{\frac{1}{6}}(-i)^{\frac{1}{6}} = 2(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2})^{\frac{1}{6}}$$

$$\theta = \frac{3\pi}{2} \qquad n = 6 \quad , \qquad k = 0,1,2,3,4,5$$

$$Z = 2(\cos\frac{\frac{3\pi}{2} + 2k\pi}{6} + i\sin\frac{\frac{3\pi}{2} + 2k\pi}{6})$$

$$= 2(\cos\frac{3\pi + 4k\pi}{12} + i\sin\frac{3\pi + 4k\pi}{12})$$

Note

$$\frac{\frac{3\pi}{2} \times 2 + 2k\pi \times 2}{6 \times 2} = \frac{3\pi + 4k\pi}{12}$$

when $k = 0$: $Z_1 = 2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = \sqrt{2} + \sqrt{2}i$

when $k = 1$: $Z_2 = 2(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12})$

when $k = 2$: $Z_3 = 2(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12})$

when $k = 3$: $Z_4 = 2(\cos\frac{15\pi}{12} + i\sin\frac{15\pi}{12}) = 2(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4})$

$$= 2(-\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}) = -\sqrt{2} - \sqrt{2}i$$

when $k = 4$: $Z_5 = 2(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12})$

when $k = 5$: $Z_6 = 2(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12})$

General Exercises

Q1) Find the real values of $x, y \in R$ if : $\dfrac{y}{1+i} = \dfrac{x^2+4}{x+2i}$

Solution :

$$\frac{y}{1+i} = \frac{x^2+4}{x+2i} \cdot \frac{x-2i}{x-2i} \quad \Rightarrow \quad \frac{y}{1+i} = \frac{(x^2+4)(x-2i)}{(x^2+4)}$$

$$\Rightarrow \quad \frac{y}{1+i} = x-2i$$

$$\Rightarrow \quad y = x + xi - 2i + 2$$

$$\Rightarrow \quad y + 0i = x + 2 + xi - 2i$$

$rp = rp \quad \Rightarrow \quad y = x + 2 \quad ......(1)$

$ip = ip \quad \Rightarrow \quad 0 = x - 2 \qquad \Rightarrow \quad x = 2 \xrightarrow{(1)} y = 4$

M.Q. Find the real values of $x, y \in R$ if : $\dfrac{y}{1+i} = \dfrac{x^2+9}{x+3i}$

2) If $z = \dfrac{1-\sqrt{3}\,i}{1+\sqrt{-3}}$, find $z^{\frac{1}{2}}$ by using DeMoivre's Theorem

Solution :

$$Z = \frac{1-\sqrt{3}i}{1+\sqrt{3}i}\cdot\frac{1-\sqrt{3}i}{1-\sqrt{3}i} = \frac{1-\sqrt{3}i-\sqrt{3}i-3}{4} = \frac{-2-2\sqrt{3}i}{4} = -\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

$$\therefore\ Z = (-\frac{1}{2},-\frac{\sqrt{3}}{2}) \quad\Rightarrow\quad r = \sqrt{\frac{1}{4}+\frac{3}{4}} = \sqrt{\frac{4}{4}} = 1$$

$$\left.\begin{aligned}\cos\theta &= -\frac{1}{2}\\ \sin\theta &= -\frac{\sqrt{3}}{2}\end{aligned}\right\} \Rightarrow\quad \theta = \pi+\frac{\pi}{3} = \frac{4\pi}{3} \qquad \therefore\ Z = (\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3})$$

$$Z^{\frac{1}{2}} = (\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3})^{\frac{1}{2}}$$

$$\boxed{\theta = \frac{4\pi}{3}\quad , \quad n = 2 \quad , \quad k = 0,1}$$

$$Z^{\frac{1}{2}} = (\cos\frac{\frac{4\pi}{3}+2k\pi}{2}+i\sin\frac{\frac{4\pi}{3}+2k\pi}{2})$$

$$= (\cos\frac{4\pi+6k\pi}{6}+i\sin\frac{4\pi+6k\pi}{6})$$

Note

$$\frac{\frac{4\pi}{3}\times 3+2k\pi\times 3}{2\times 3} = \frac{4\pi+6k\pi}{6}$$

$$When\ \ k=0:\quad Z_1 = (\cos\frac{4\pi}{6}+i\sin\frac{4\pi}{6})$$

$$= (\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}) = (-\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}) = \boxed{-\frac{1}{2}+\frac{\sqrt{3}}{2}i}$$

$$When\ \ k=1:\quad Z_2 = (\cos\frac{10\pi}{6}+i\sin\frac{10\pi}{6})$$

$$= (\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}) = (\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}) = \boxed{\frac{1}{2}-\frac{\sqrt{3}}{2}i}$$

$$S.S. = \left\{ -\frac{1}{2}+\frac{\sqrt{3}}{2}i\ ,\ \frac{1}{2}-\frac{\sqrt{3}}{2}i \right\}$$

M.Q. If $z=\cos\theta+i\sin\theta$ prove that : $\dfrac{z^n}{1+z^{2n}}=\dfrac{1}{2\cos n\theta}$

Solution :

$$LHS=\frac{z^n}{1+z^{2n}}=\frac{1}{\frac{1+z^{2n}}{z^n}}=\frac{1}{\frac{1}{z^n}+\frac{z^{2n}}{z^n}}=\frac{1}{z^{-n}+z^n}$$

$$=\frac{1}{(\cos\theta+i\sin\theta)^{-n}+(\cos\theta+i\sin\theta)^n}$$

$$=\frac{1}{\cos n\theta\cancel{-i\sin n\theta}+\cos n\theta\cancel{+i\sin n\theta}}$$

$$=\frac{1}{\cos n\theta+\cos n\theta}=\frac{1}{2\cos n\theta}=RHS$$

M.Q. If $z=\cos 2x+i\sin 2x$ prove that : $\dfrac{2}{1+z}=1-i\tan x$

Solution :

$$LHS=\frac{2}{1+z}=\frac{2}{1+\cos 2x+i\sin 2x}=\frac{2}{1+\cos^2 x-\sin^2 x+i2\sin x\cos x}$$

$$=\frac{2}{(1-\sin^2 x)+\cos^2 x+i2\sin x\cos x}=\frac{2}{\cos^2 x+\cos^2 x+i2\sin x\cos x}$$

$$=\frac{2}{2\cos^2 x+i2\sin x\cos x}=\frac{\cancel{2}}{\cancel{2}\cos x(\cos x+i\sin x)}$$

$$=\frac{1}{\cos x(\cos x+i\sin x)}=\frac{(\cos x+i\sin x)^{-1}}{\cos x}$$

$$=\frac{\cos x-i\sin x}{\cos x}=\frac{\cos x}{\cos x}-\frac{i\sin x}{\cos x}=1-i\tan x=RHS$$

M.Q. If $Z=\cos\theta+i\sin\theta$ prove that : $(1+\overline{Z})Z=1+Z$

Solution :

$$LHS=(1+\overline{Z})Z$$
$$=Z+\overline{Z}.Z$$
$$=\cos\theta+i\sin\theta+(\overline{\cos\theta+i\sin\theta})(\cos\theta+i\sin\theta)$$
$$=\cos\theta+i\sin\theta+(\cos\theta-i\sin\theta)(\cos\theta+i\sin\theta)$$
$$=\cos\theta+i\sin\theta+\cos^2\theta+\cancel{i\cos\theta\sin\theta}-\cancel{i\cos\theta\sin\theta}-i^2\sin^2\theta$$
$$=\cos\theta+i\sin\theta+(\cos^2\theta+\sin^2\theta)$$
$$=\cos\theta+i\sin\theta+(1)$$
$$=1+\cos\theta+i\sin\theta=1+Z=RHS$$

M.Q. If $Z=-2+bi$ is complex number its argument $\frac{4\pi}{3}$, Find the value of b

Solution :

$$-2+bi=r\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$$
$$-2+bi=r\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$$
$$-2+bi=\frac{-r}{2}-\frac{r\sqrt{3}}{2}i$$
$$-2=\frac{-r}{2}\quad\rightarrow\quad r=4$$
$$b=\frac{-r\sqrt{3}}{2}\quad\rightarrow\quad b=\frac{-4\sqrt{3}}{2}\quad\rightarrow\quad b=-2\sqrt{3}$$

Second Method

$x=-2\quad,\quad y=b$

$$\tan\frac{4\pi}{3}=\frac{\sin\frac{4\pi}{3}}{\cos\frac{4\pi}{3}}=\frac{\frac{-\sqrt{3}}{2}}{\frac{-1}{2}}=\sqrt{3}$$

$$\tan\frac{4\pi}{3}=\frac{y}{x}\quad\rightarrow\quad\sqrt{3}=\frac{b}{-2}\quad\rightarrow\quad b=-2\sqrt{3}$$

M.Q. Simplified: $\dfrac{(1-\sqrt{3}\,i)^{15}}{(1-i)^{20}}$

Solution :

$z_1 = 1-\sqrt{3}\,i \qquad r = \sqrt{1+3} = \sqrt{4} = 2$

$$\left.\begin{array}{l} \cos\theta = \dfrac{x}{r} = \dfrac{1}{2} \\ \sin\theta = \dfrac{y}{r} = \dfrac{-\sqrt{3}}{2} \end{array}\right\rangle \Rightarrow \quad \theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

$$z_1 = 2\,(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3})$$

$z_2 = 1-i \qquad r = \sqrt{1+1} = \sqrt{2}$

$$\left.\begin{array}{l} \cos\theta = \dfrac{x}{r} = \dfrac{1}{\sqrt{2}} \\ \sin\theta = \dfrac{y}{r} = \dfrac{-1}{\sqrt{2}} \end{array}\right\rangle \Rightarrow \quad \theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

$$z_2 = \sqrt{2}\,(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4})$$

$$\frac{(1-\sqrt{3}\,i)^{15}}{(1-i)^{20}} = \frac{(2)^{15}\,(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3})^{15}}{(\sqrt{2})^{20}\,(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4})^{20}}$$

$$= \frac{2^{15}\,(\cos 25\pi + i\sin 25\pi)}{2^{10}\,(\cos 35\pi + i\sin 35\pi)} = 2^5 . \frac{\cancel{(\cos\pi + i\sin\pi)}}{\cancel{(\cos\pi + i\sin\pi)}} = 32$$

M.Q. If $z_1 = 1+\sqrt{3}\,i$, $z_2 = 1-i$ **find the value of :** $\left(\frac{z_1}{z_2}\right)^{20}$

Solution :

$z_1 = 1+\sqrt{3}\,i \qquad r = 2 \qquad \theta = \frac{\pi}{3} \qquad z_1 = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$

$z_2 = 1-i \qquad r = \sqrt{2} \qquad \theta = \frac{7\pi}{4} \qquad z_2 = \sqrt{2}(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4})$

$$(\frac{z_1}{z_2})^{20} = (\frac{1+\sqrt{3}i}{1-i})^{20} = \left(\frac{2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})}{\sqrt{2}(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4})}\right)^{20}$$

$$= \frac{2^{20}(\cos\frac{20\pi}{3} + i\sin\frac{20\pi}{3})}{2^{10}(\cos 35\pi + i\sin 35\pi)}$$

$$\boxed{\frac{20\pi}{3} - 6\pi = \frac{20\pi - 18\pi}{3} = \frac{2\pi}{3}}$$

$$= 2^{10}.\frac{(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})}{\cos\pi + i\sin\pi}$$

$$= 2^{10}.\frac{(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)}{-1}$$

$$= 2^{10}(\frac{1}{2} - \frac{\sqrt{3}}{2}i)$$

$$= \frac{2^{10}}{2}(1-\sqrt{3}i) \qquad = 2^{9}(1-\sqrt{3}i) \qquad = 512(1-\sqrt{3}i)$$

M.Q. Prove that : $\left(\dfrac{1+\cos x+i\sin x}{1+\cos x-i\sin x}\right)^6=\cos 6x+i\sin 6x$

Solution :

$$LHS=\left(\frac{1+\cos x+i\sin x}{1+\cos x-i\sin x}\right)^6=\left(\frac{\cos^2 x+\sin^2 x+\cos x+i\sin x}{1+\cos x-i\sin x}\right)^6$$

$$=\left(\frac{(\cos^2 x-i^2\sin^2 x)+\cos x+i\sin x}{1+\cos x-i\sin x}\right)^6$$

$$=\left(\frac{(\cos x+i\sin x)(\cos x-i\sin x)+(\cos x+i\sin x)}{1+\cos x-i\sin x}\right)^6$$

$$=\left(\frac{(\cos x+i\sin x)\cancel{(\cos x-i\sin x+1)}}{\cancel{(\cos x-i\sin x+1)}}\right)^6$$

$$=(\cos x+i\sin x)^6=\cos 6x+i\sin 6x=RHS$$

M.Q. Find $x,y\in R$ which satisfied the equation : $\dfrac{y^2+4yi+5}{y-i}=(2x+i)(x+2i)$

Solution :

$$\frac{y^2+4yi-5i^2}{y-i}=(2x+i)(x+2i)$$

$$\frac{(y+5i)(y-i)}{(y-i)}=2x^2+4xi+xi+2i^2$$

$$y+5i=2x^2+5xi-2$$

$$\boxed{y=2x^2-2}\quad\dots(1)$$

$$5x=5\Rightarrow \boxed{x=1}$$

$$\xrightarrow{in\,(1)} y=2(1)^2-2$$

$$\longrightarrow \boxed{y=0}$$

Q1) Find the algebritic form of complex number : $\dfrac{i}{(\sqrt{2}+i)^2}+\dfrac{i}{(\sqrt{2}-i)^2}$

Q2) If $a+ib=\dfrac{7-4i}{2+i}$, $a,b\in R$ find the value of : $\sqrt{2a-ib}$

Q3) Prove that : $\left(\dfrac{3\omega^2+2\omega+7}{2\omega^2+7\omega+3}+\dfrac{5-8\omega}{8-5\omega^2}\right)^4=9$

Q4) Prove that : $(5+\omega)(\dfrac{1}{5+4\omega+3\omega^2}-\dfrac{1}{3\omega+2\omega^2})=4$

Q5) Prove that : $\left(\dfrac{\frac{1}{\omega}-4\omega-\frac{1}{\omega^2}-4\omega^2}{9\omega^2-6}\right)^{-3}=27$

Q6) Prove that : $\dfrac{(2+11\omega+2\omega^2)(2+11\omega^2+2\omega^4)}{(1-\frac{5}{\omega^2}+\omega^2)(1+\omega-\frac{5}{\omega})}=\dfrac{9}{4}$

Q7) Prove that : $\dfrac{10+\omega^2}{4+\omega+2\omega^2}=4+\omega$

Q8) Prove that : $\dfrac{6\omega+2\omega i+3i}{2\omega+i}-\dfrac{2}{\omega^2-2i}=3$

Q9) Solve the equation in C : $4^x-2^{x+\omega}-2^{x+\omega^2}+\dfrac{1}{2}=0$

Q10) Find the algebritic form of complex number which modulus (3) and argument $\dfrac{\pi}{3}$

Q11) Find the Quadratic Equation whose has real coefficients , one of its roots is the complex number which its modulus (2) and argument $(\dfrac{-\pi}{6})$

Q12) Write the ordinary form of complex number whose real part (4) unit and argument $(\dfrac{7\pi}{3})$

Q13) If the argument of complex number $(a+\sqrt{3}\,i)$ is $\dfrac{-23\pi}{3}$, find the value of a

Q14) Find the complex number whose modulus is (5) and its imaginary part is bigger than its real part by (1)

Q15) Calculate by using DeMoivre's Theorem : $(\sin\dfrac{\pi}{12}-i\cos\dfrac{\pi}{12})^4$

Q16) Simplified: $(\cos\dfrac{13\pi}{8}+i\sin\dfrac{13\pi}{8})(\cos\dfrac{9\pi}{8}-i\sin\dfrac{9\pi}{8})$