Particle Filters

an overview

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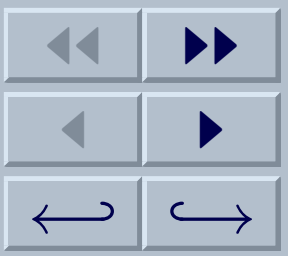

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a tutorial

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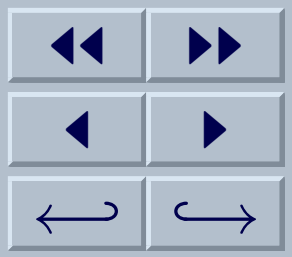


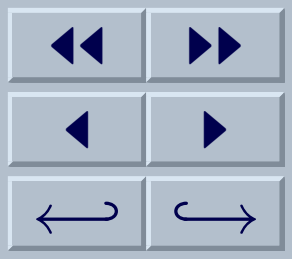
1 Introduction

An increasing number of researchers is using a family of techniques and algorithms called

- *condensation algorithms*
- *bootstrap filtering*
- *particle filters*
- *interacting particle approximations*
- *sequential Monte Carlo methods*
- *SIS, SIR, ASIR, RPF, . . .*

Time scale: last 10 years [e.g. Isard & Blake 1996; Kitagawa 1996; Gordon, Salmond & Smith 1993]

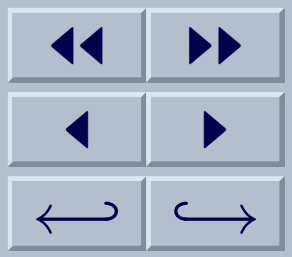
The question of this talk is: What is behind all that?



General Classification of Filter Strategies

Gaussian models:

- Kalman filter
- extended Kalman filter
- linear-update filter / linear regression filter / statistical linearization filter
  - unscented filter
  - central difference filter
  - divided difference filter
- assumed density filter / moment matching

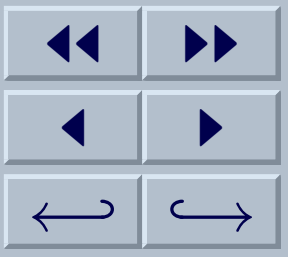


Mixture of Gaussian models:

- assumed density filter / pseudo-Bayes
- Gaussian-sum filter

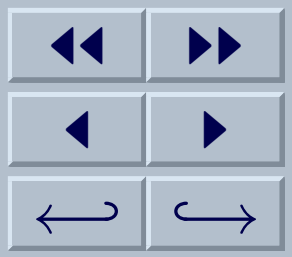
Nonparametric models:

- **particle filter class**
- histogram filter



Some Basic Remarks

- various applications: computer vision (i.e. tracking), control theory, econometrics (stock markets, monetary flow, interest rates), ...
- we deal with discrete time systems only
- no out-of-sequence measurements
- we are mainly interested in estimating the state at time k from measurements up to time $k' = k$ (opposite: smoothing $(k' > k)$ and prediction $(k' < k)$; furthermore k' need not be fixed...)
- no restrictions to linear processes or Gaussian noise!

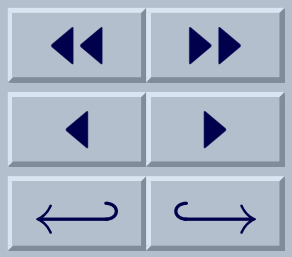


Overview of this Talk

- The Dynamic System Model
- Bayesian Filter Approach
- Optimal and Suboptimal Solutions
- The Particle Filter
- Experiments and Summary

– states of a system and state transition equation

– measurement equation



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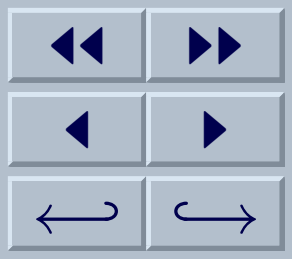
– estimation of the state

– probabilistic modelling

– Bayesian filter



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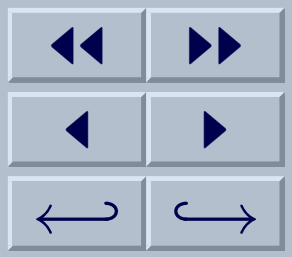
– filtered pdf can be written down easily, but it is not always tractable ($\rightarrow$ ugly integrals . . .)

– conditions under which optimal solutions exist: Kalman filter and grid-based filter

– what can be done in other cases: suboptimal approaches



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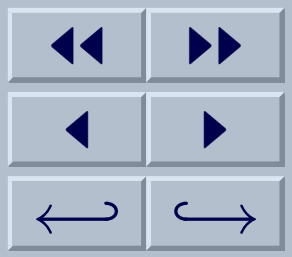
– standard particle filter

– various improved versions



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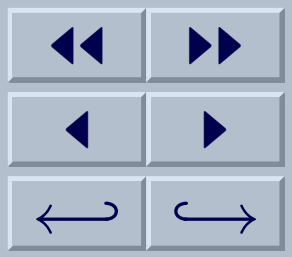
– some experimental data and conclusion



2 Dynamic System

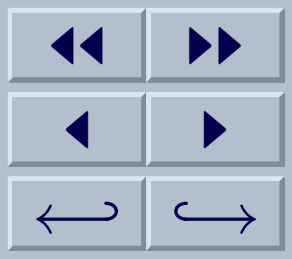
A dynamic system can be modelled with two equations:

State Transition or Evolution Equation

$$\mathbf{x}_k = f_k(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{v}_{k-1})$$

$f(\cdot,\cdot,\cdot)$: evolution function (possible non-linear)
$\mathbf{x}_k, \mathbf{x}_{k-1} \in \mathbb{R}^{n_x}$: current and previous state
$\mathbf{v}_{k-1} \in \mathbb{R}^{n_v}$: state noise (usually *not* Gaussian)
$\mathbf{u}_{k-1} \in \mathbb{R}^{n_u}$: known input

Note: state only depends on previous state, i.e. first order Markov process

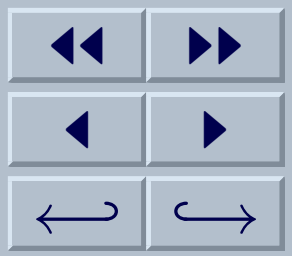


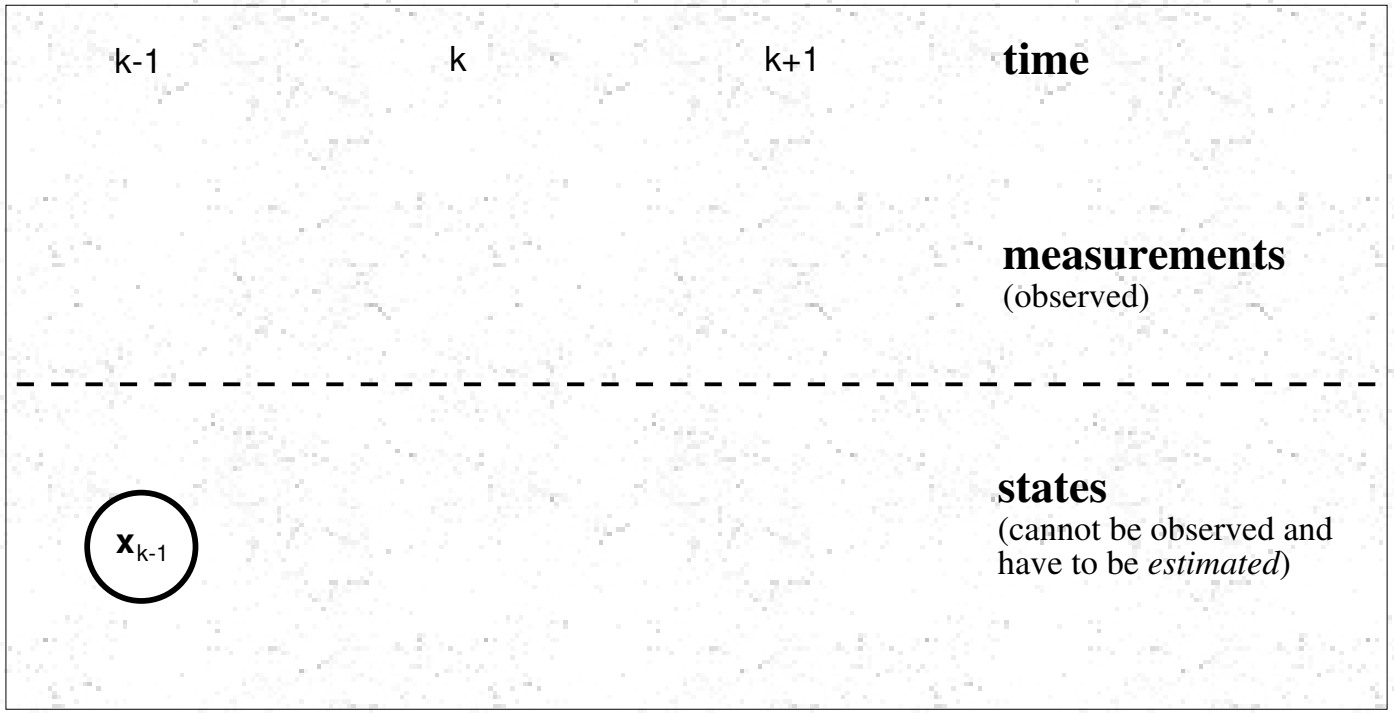
Measurement Equation

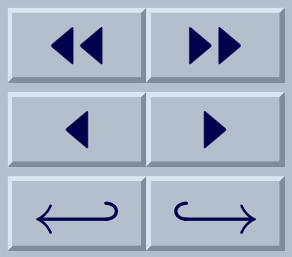
$$\mathbf{z}_k = h_k(\mathbf{x}_k, \mathbf{u}_k, \mathbf{n}_k)$$

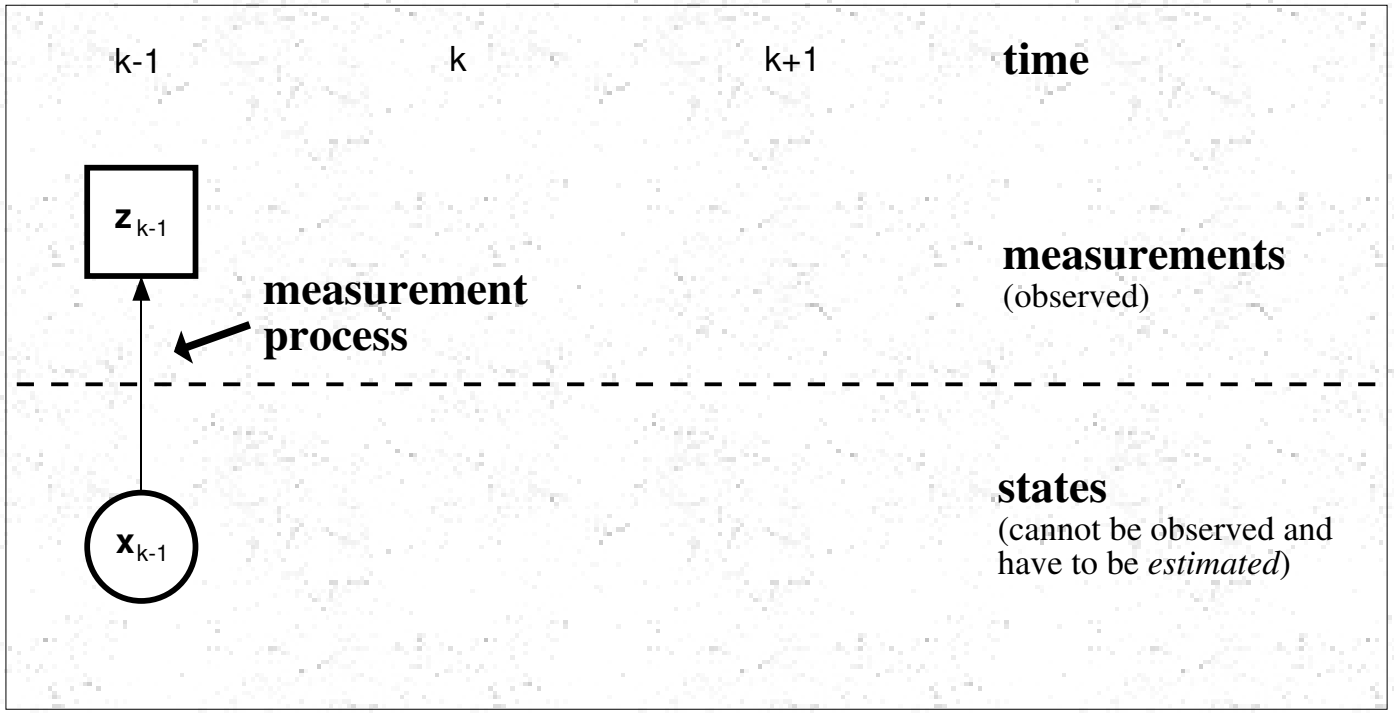
$h(\cdot,\cdot,\cdot)$: measurement function (possible non-linear)
$\mathbf{z}_k \in \mathbb{R}^{n_z}$: measurement
$\mathbf{x}_k \in \mathbb{R}^{n_x}$: state
$\mathbf{n}_k \in \mathbb{R}^{n_n}$: measurement noise (usually *not* Gaussian)
$\mathbf{u}_k \in \mathbb{R}^{n_u}$: known input

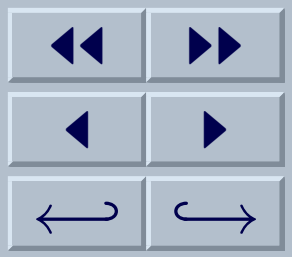
(dimensionality of state, measurement, input, state noise, and measurement noise can all be different!)

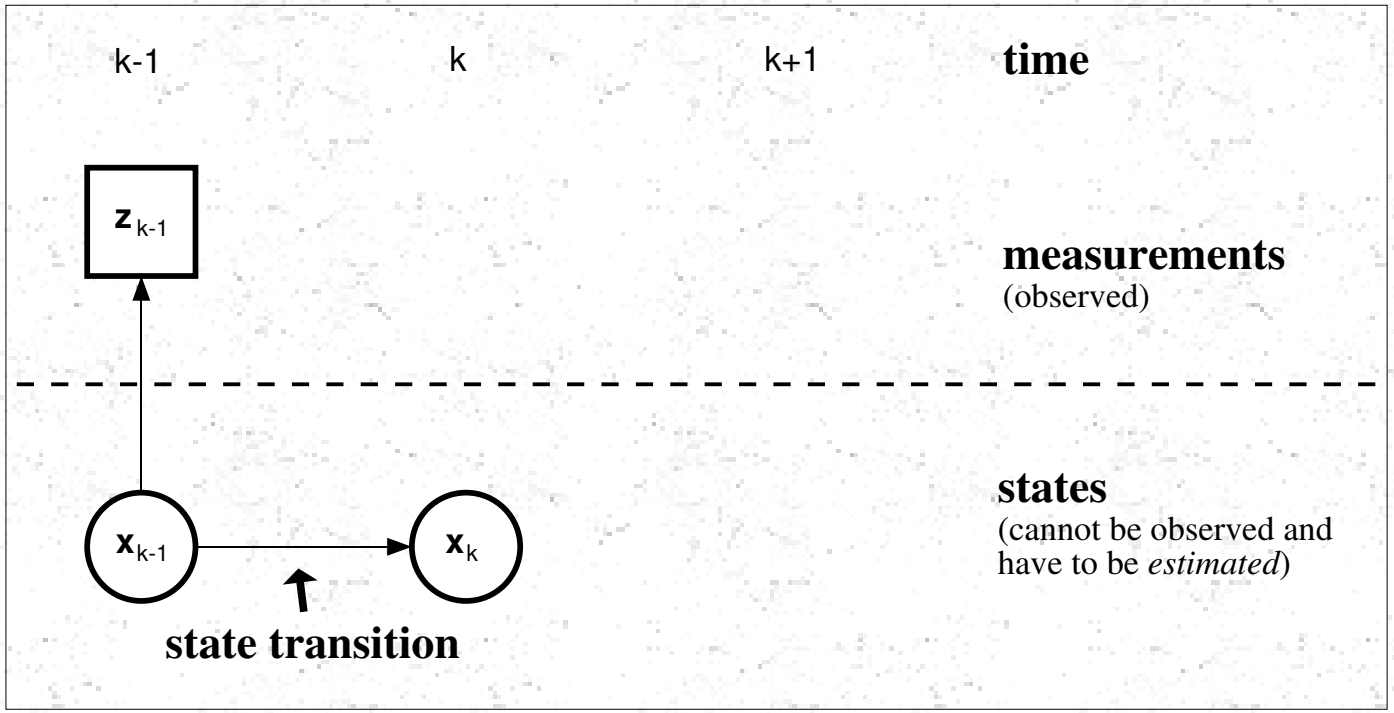


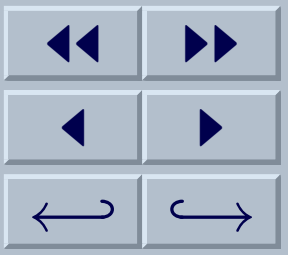


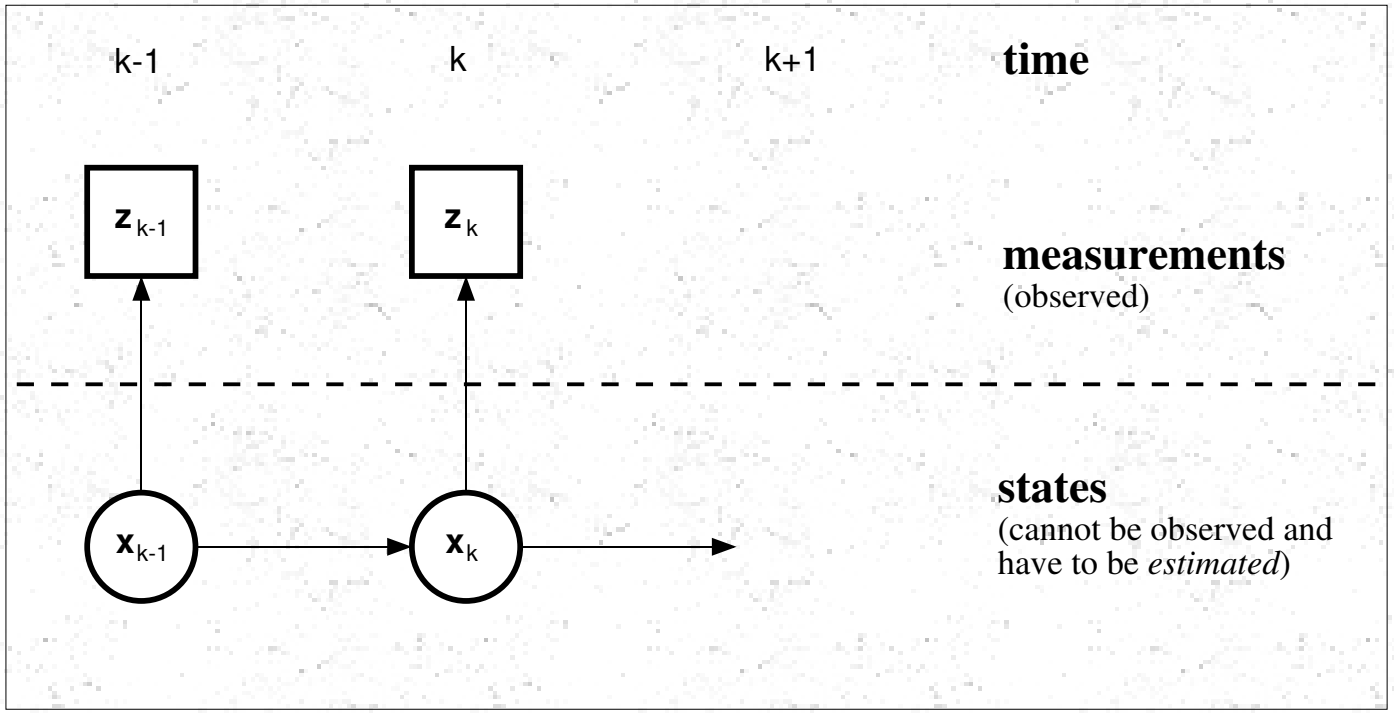


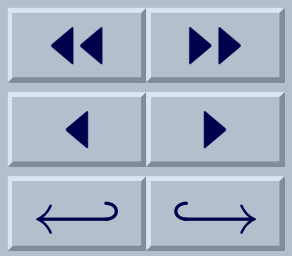


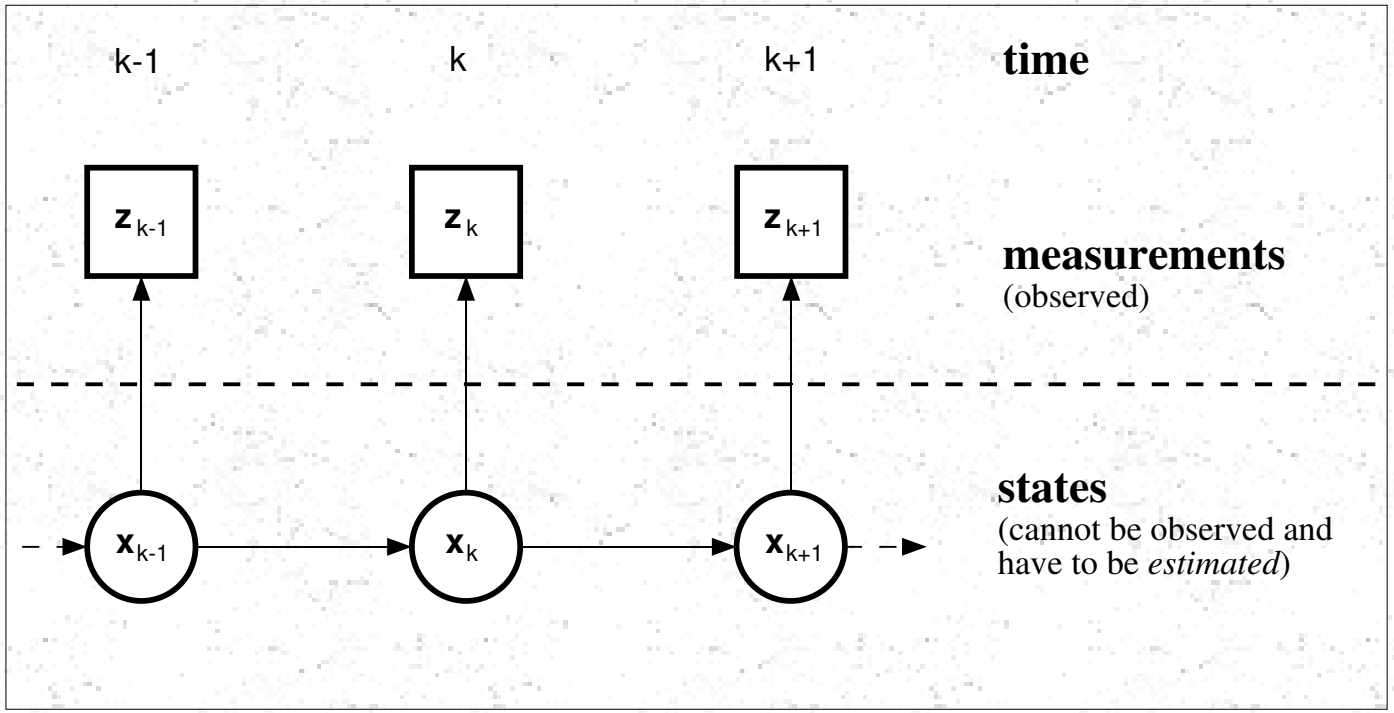


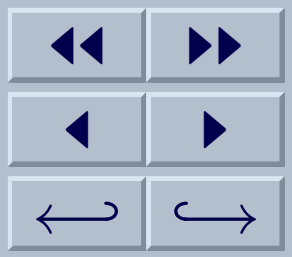


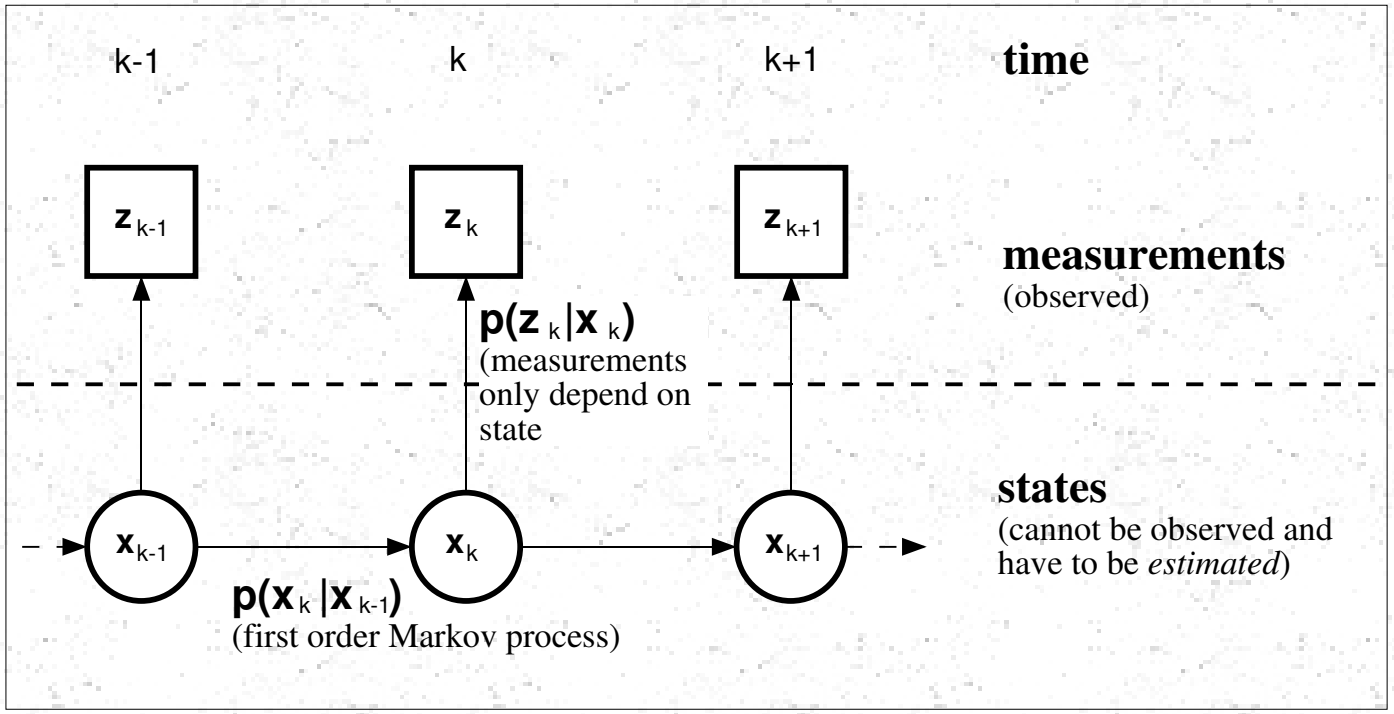








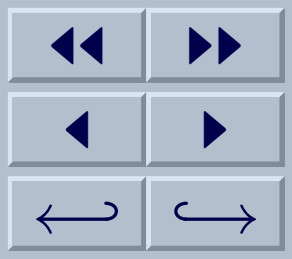




Assumptions:

The observations are conditionally independent given the state: $p(\mathbf{z}_k|\mathbf{x}_k)$.

Hidden Markov Model (HMM):
$p(\mathbf{x}_0)$ given and $p(\mathbf{x}_k|\mathbf{x}_{k-1})$ defines state transition probability for $k \geq 1$.



3 Bayesian Filters

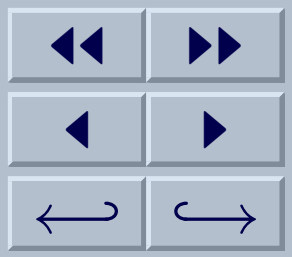
Estimating the Posterior

Bayesian approach: We attempt to construct the posterior pdf of the state given all measurements.

$\Rightarrow$ can be termed a complete solution to the estimation problem because all available information is used; from the pdf, an optimal estimate can theoretically be found for any criterion.

in detail: We seek estimates of $\mathbf{x}_k$ based on all available measurements up to time k (abbreviated as $\mathbf{z}_{1:k}$) by constructing the posterior $p(\mathbf{x}_k|\mathbf{z}_{1:k})$.

Assumption: initial state pdf (prior) $p(\mathbf{x}_0)$ is given



The Use of Knowing the Posterior

Let $f_k : \mathbb{R}^{(k+1)\times n_x} \rightarrow \mathbb{R}$ be any arbitrary (integrable) function that can depend

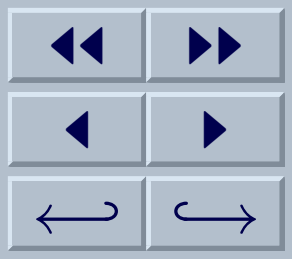
- on all components of the state $\mathbf{x}$
- on the whole trajectory in state-space

Examples: This function can be an estimator for the current state or for future observations.

Then we can compute its expectation using

$$\mathrm{E}\left[f_k(\mathbf{x}_{0:k})\right] = \int f(\mathbf{x}_{0:k}) p(\mathbf{x}_{0:k}|\mathbf{z}_{1:k}) d\mathbf{x}_{0:k}$$

MMSE estimate of state: $\hat{\mathbf{x}} = \mathrm{E}\left[\mathbf{x}_k\right]$. Other estimates that can be computed: median, modes, confidence intervals, kurtosis, ...



Recursive Filters

recursive filters (i.e. sequential update of previous estimate) $\leftrightarrow$ batch processing (computation with all data in one step)

not only faster: allow on-line processing of data (lower storage costs, rapid adaption to changing signals characteristics)

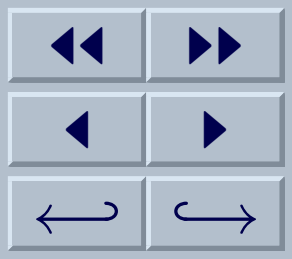
essentially consist of two steps:

prediction step: $p(\mathbf{x}_{k-1}|\mathbf{z}_{1:k-1}) \rightarrow p(\mathbf{x}_k|\mathbf{z}_{1:k-1})$

(usually deforms / translates / spreads state pdf due to noise)

update step: $p(\mathbf{x}_k|\mathbf{z}_{1:k-1}), \mathbf{z}_k \rightarrow p(\mathbf{x}_k|\mathbf{z}_{1:k})$

(combines likelihood of current measurement with predicted state; usually concentrates state pdf)



General Prediction-Update Framework

Assume that pdf $p(\mathbf{x}_{k-1}|\mathbf{z}_{1:k-1})$ is available at time $k-1$.

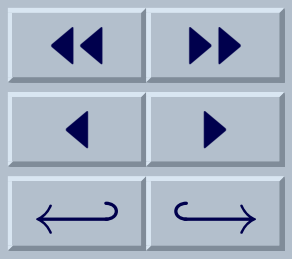
Prediction step: (using Chapman-Kolmogoroff equation)

$$p(\mathbf{x}_k|\mathbf{z}_{1:k-1}) = \int p(\mathbf{x}_k|\mathbf{x}_{k-1})p(\mathbf{x}_{k-1}|\mathbf{z}_{1:k-1})d\mathbf{x}_{k-1} \quad (1)$$

This is the prior of the state $\mathbf{x}_k$ at time k *without knowledge of the measurement* $\mathbf{z}_k$, i.e. the probability *given only previous measurements*.

Update step: (compute posterior pdf from predicted prior pdf and new measurement)

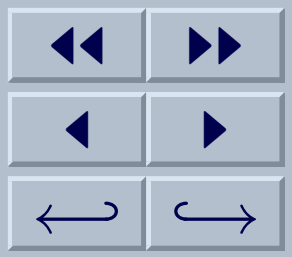
$$p(\mathbf{x}_k|\mathbf{z}_{1:k}) = \frac{p(\mathbf{z}_k|\mathbf{x}_k)p(\mathbf{x}_k|\mathbf{z}_{1:k-1})}{p(\mathbf{z}_k|\mathbf{z}_{1:k-1})} \quad (2)$$



Let us prove formula (2) (just in order to train calculations with joint and conditional probabilities. . .)

$$p(\mathbf{x}_k|\mathbf{z}_{1:k}) = \frac{p(\mathbf{z}_{1:k}|\mathbf{x}_k)p(\mathbf{x}_k)}{p(\mathbf{z}_{1:k})}$$

(Bayes rule)



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& p(\mathbf{x}_k|\mathbf{z}_{1:k}) \\
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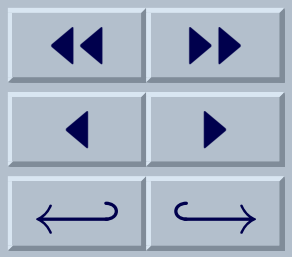
&= \frac{p(\mathbf{z}_k, \mathbf{z}_{1:k-1}|\mathbf{x}_k)p(\mathbf{x}_k)}{p(\mathbf{z}_k, \mathbf{z}_{1:k-1})}
\end{aligned}
$$

(separate $p(\mathbf{z}_{1:k})$ into $p(\mathbf{z}_k, \mathbf{z}_{1:k-1})$)



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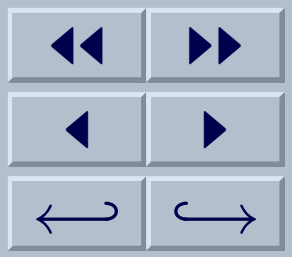
&= \frac{p(\mathbf{z}_k|\mathbf{z}_{1:k-1}, \mathbf{x}_k)p(\mathbf{z}_{1:k-1}|\mathbf{x}_k)p(\mathbf{x}_k)}{p(\mathbf{z}_k|\mathbf{z}_{1:k-1})p(\mathbf{z}_{1:k-1})}
\end{aligned}
$$

(factorize joint probability: $p(a,b|c) = p(a|b,c) \cdot p(b|c)$ and $p(a,b) = p(a|b) \cdot p(b)$)



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&= \frac{p(\mathbf{z}_k|\mathbf{z}_{1:k-1}, \mathbf{x}_k)p(\mathbf{x}_k|\mathbf{z}_{1:k-1})p(\mathbf{z}_{1:k-1})p(\mathbf{x}_k)}{p(\mathbf{z}_k|\mathbf{z}_{1:k-1})p(\mathbf{z}_{1:k-1})p(\mathbf{x}_k)}
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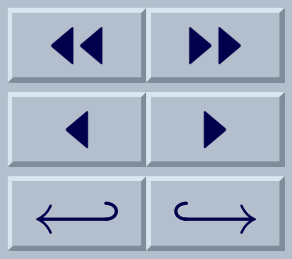
(Bayes rule)



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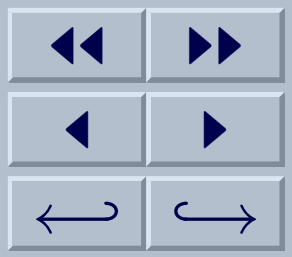
&= \frac{p(\mathbf{z}_k|\mathbf{z}_{1:k-1}, \mathbf{x}_k)p(\mathbf{x}_k|\mathbf{z}_{1:k-1})p(\mathbf{z}_{1:k-1})p(\mathbf{x}_k)}{p(\mathbf{z}_k|\mathbf{z}_{1:k-1})p(\mathbf{z}_{1:k-1})p(\mathbf{x}_k)} \\
&= \frac{p(\mathbf{z}_k|\mathbf{x}_k)p(\mathbf{x}_k|\mathbf{z}_{1:k-1})}{p(\mathbf{z}_k|\mathbf{z}_{1:k-1})}
\end{aligned}
$$

(independence of observations; cancelling out terms)



The Structure of the Update Equation

$$p(\mathbf{x}_k|\mathbf{z}_{1:k}) = \frac{p(\mathbf{z}_k|\mathbf{x}_k) \cdot p(\mathbf{x}_k|\mathbf{z}_{1:k-1})}{p(\mathbf{z}_k|\mathbf{z}_{1:k-1})}$$

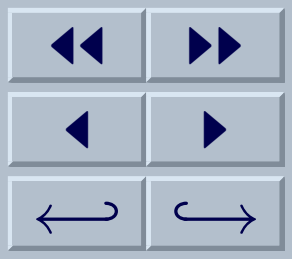
$$\text{posterior} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}$$

prior: given by prediction equation

likelihood: given by observation model

evidence: the normalizing constant in the denominator

$$p(\mathbf{z}_k|\mathbf{z}_{1:k-1}) = \int p(\mathbf{z}_k|\mathbf{x}_k) p(\mathbf{x}_k|\mathbf{z}_{1:k-1}) d\mathbf{x}_k$$

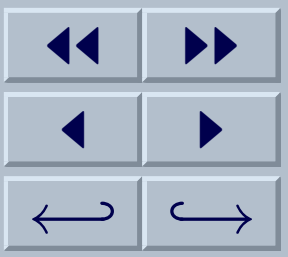


This theoretically allows an optimal Bayesian solution (in the sense of computing the posterior pdf).

Problem: only a conceptual solution; integrals are not tractable.

But: in some restricted cases, an optimal solution is possible. Two optimal solutions (under restrictive assumptions):

- (standard) Kalman filter
- grid-based filter



4 Kalman Filter

Introduction

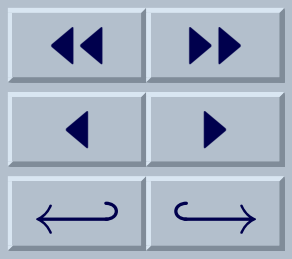
Assumptions:

- posterior at time $k-1$, i.e. $p(\mathbf{x}_{k-1}|\mathbf{z}_{k-1})$, is Gaussian
- dynamic system characterized by

$$\mathbf{x}_k = \mathbf{F}_k\mathbf{x}_{k-1} + \mathbf{G}_k\mathbf{v}_{k-1}$$
$$\mathbf{z}_k = \mathbf{H}_k\mathbf{x}_k + \mathbf{J}_k\mathbf{n}_k$$

- both noise vectors Gaussian (covariance matrices are $\mathbf{Q}_{k-1}$ and $\mathbf{R}_k$)

Then new posterior $p(\mathbf{x}_k|\mathbf{z}_k)$ is Gaussian, too, and can be computed using simple linear equations.



optimal solution, but *highly restrictive* assumptions must hold

Prediction Equation

At time $k-1$: $p(\mathbf{x}_{k-1}|\mathbf{z}_{1:k-1}) = \mathcal{N}(\mathbf{m}_{k-1|k-1}, \mathbf{P}_{k-1|k-1})$

Inserting into (1) yields

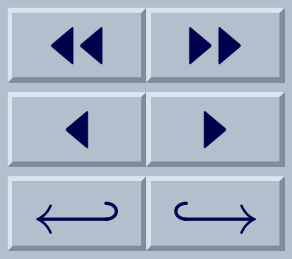
$$p(\mathbf{x}_k|\mathbf{z}_{1:k-1}) = \mathcal{N}(\mathbf{m}_{k|k-1}, \mathbf{P}_{k|k-1})$$

with

$$\mathbf{m}_{k|k-1} = \mathbf{F}_k \mathbf{m}_{k-1|k-1}$$

and

$$\mathbf{P}_{k|k-1} = \mathbf{G}_k \mathbf{Q}_{k-1} \mathbf{G}_k^T + \mathbf{F}_k \mathbf{P}_{k-1|k-1} \mathbf{F}_k^T$$



Update Equation

Inserting into (2) yields

$$p(\mathbf{x}_k|\mathbf{z}_{1:k}) = \mathcal{N}(\mathbf{m}_{k|k}, \mathbf{P}_{k|k})$$

with

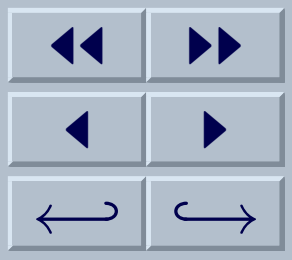
$$\mathbf{m}_{k|k} = \mathbf{m}_{k|k-1} + \mathbf{K}_k(\mathbf{z}_k - \underbrace{\mathbf{H}_k\mathbf{m}_{k|k-1}}_{\text{estimated } \hat{\mathbf{z}}_k})$$

and

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{K}_k\mathbf{H}_k\mathbf{P}_{k|k-1}$$

Kalman Gain:

$$\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}_k^T(\underbrace{\mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^T + \mathbf{J}_k\mathbf{R}_k\mathbf{J}_k^T}_{\text{Cov}[\hat{\mathbf{z}}_k]})^{-1}$$



5 Grid-Based Filter

Introduction

Assumptions:

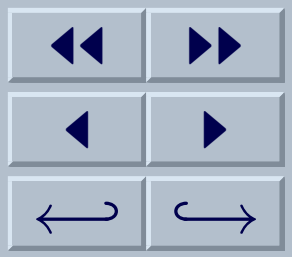
- state space is discrete
- number of different states (N_s) is limited
  (Note: implicitly includes discreteness)

Suppose at time $k-1$ we have states $\mathbf{x}^i$ with $i = 1, \ldots, N_s$.
Conditional probability of these states:

$$Pr(\mathbf{x}_{k-1} = \mathbf{x}^i | \mathbf{z}_{1:k-1}) = w^i_{k-1|k-1}$$

Then the (old) posterior at time $k-1$ is given by:

$$p(\mathbf{x}_{k-1} | \mathbf{z}_{1:k-1}) = \sum_{i=1}^{N_s} w^i_{k-1|k-1} \, \delta(\mathbf{x}_{k-1} - \mathbf{x}^i)$$



Results (Summary)

Both the (new) prior and the (new) posterior have the same structure: a sum of weighted Dirac peaks:

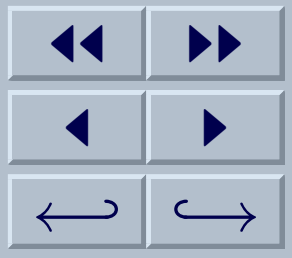
$$p(\mathbf{x}_k|\mathbf{z}_{1:k-1}) = \sum_{i=1}^{N_s} w_{k|k-1}^i \, \delta(\mathbf{x}_{k-1} - \mathbf{x}^i)$$

$$p(\mathbf{x}_k|\mathbf{z}_{1:k}) = \sum_{i=1}^{N_s} w_{k|k}^i \, \delta(\mathbf{x}_{k-1} - \mathbf{x}^i)$$

Note: extension to different sets of states for each time step

$$\{\mathbf{x}^i\} : i = 1, \ldots, N_s \quad \longrightarrow \quad \{\mathbf{x}_k^i\} : i = 1, \ldots, N_{s,k}$$

with time-varying index k is easily possible; the 'allowed' states need not be constant.



Prediction Equation (in Detail)

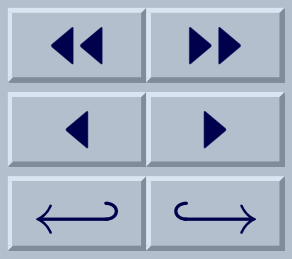
Inserting into (1) yields

$$p(\mathbf{x}_k|\mathbf{z}_{1:k-1}) = \sum_{i=1}^{N_s}\sum_{j=1}^{N_s} w_{k-1|k-1}^{j} p(\mathbf{x}^i|\mathbf{x}^j)\delta(\mathbf{x}_{k-1}-\mathbf{x}^i)$$

$$= \sum_{i=1}^{N_s} w_{k|k-1}^{i}\delta(\mathbf{x}_{k-1}-\mathbf{x}^i)$$

where $w_{k|k-1}^{i} = \sum_{j=1}^{N_s} w_{k-1|k-1}^{j} p(\mathbf{x}^i|\mathbf{x}^j)$

(new) prior weights = old posterior weights, reweighted using state transition probabilities



Update Equation (in Detail)

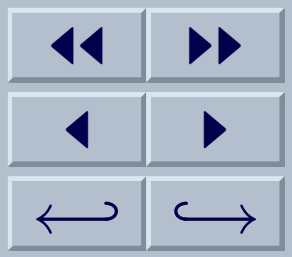
Inserting into (2) yields

$$p(\mathbf{x}_k|\mathbf{z}_{1:k}) = \sum_i^{N_s} w_{k|k}^i \delta(\mathbf{x}_{k-1} - \mathbf{x}^i)$$

where $w_{k|k}^i = \frac{w_{k|k-1}^i p(\mathbf{z}_k|\mathbf{x}^i)}{\sum_j^{N_s} w_{k|k-1}^j p(\mathbf{z}_k|\mathbf{x}^j)}$.

Note: denominator only needed for normalization

posterior weights = prior weights, reweighted using likelihoods



6 Particle Filter

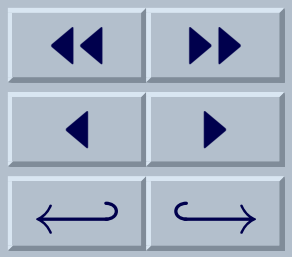
Suboptimal Approximations

If we want to preserve Kalman filter principle...

- Extended Kalman Filter (EKF)
- Unscented Kalman Filter (UKF)

...we get better results,

BUT: *we cannot get rid off Gaussian approximations*

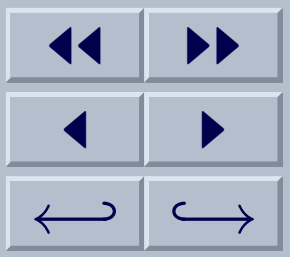


EKF / UKF:

All these approaches fail if we have

- bimodal / multimodal pdfs
- heavily skewed pdfs

We need a more general scheme to tackle these problems.



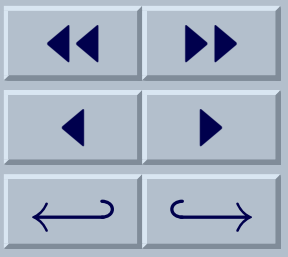
Particle Filter – General Concept

Many different names (do you remember the introduction?) but the general concept is rather simple:

PARTICLE FILTER:
If we cannot solve the integrals required for a Bayesian recursive filter analytically . . . we represent the posterior probabilities by a set of randomly chosen weighted samples.

Note: "randomly chosen" ≡ "Monte Carlo"
(we are playing roulette / throwing the dice)

Increasing number of samples ⇒ (almost sure) convergence to true pdf



Sequential Importance Sampling (SIS)

SIS is the basic framework for most particle filter algorithms. Let

$\{\mathbf{x}_{0:k}^i\}$: set of support points (samples, particles)

$i = 1, \ldots, N_s$

(whole trajectory for each particle!)

w_k^i : associated weights, normalized to $\sum_i w_k^i = 1$

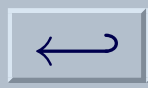
Then:

$$p(\mathbf{x}_k|\mathbf{z}_{1:k}) \approx \sum_{i=1}^{N_s} w_k^i \delta(\mathbf{x}_{0:k} - \mathbf{x}_{0:k}^i)$$

(discrete weighted approximation to the true posterior)








SIS (continued)

Usually we cannot draw samples $\mathbf{x}_k^i$ from $p(\cdot)$ directly. Assume we sample directly from a (different) *importance function* $q(\cdot)$. Our approximation is still correct (up to normalization) if

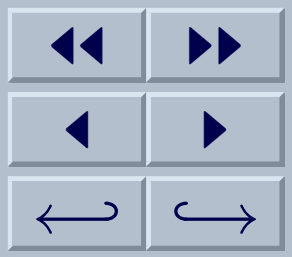
$$w_k^i \propto \frac{p(\mathbf{x}_{0:k}^i|\mathbf{z}_{1:k})}{q(\mathbf{x}_{0:k}^i|\mathbf{z}_{1:k})}$$

The trick: we can choose $q(\cdot)$ freely!

If the importance function is chosen to factorize such that

$$q(\mathbf{x}_{0:k}|\mathbf{z}_{1:k}) = q(\mathbf{x}_k|\mathbf{x}_{0:k-1},\mathbf{z}_{1:k})\, q(\mathbf{x}_{0:k-1}|\mathbf{z}_{1:k-1})$$

then one can augment old particles $\mathbf{x}_{0:k-1}^i$ by $\mathbf{x}_k \sim q(\mathbf{x}_k|\mathbf{x}_{0:k-1},\mathbf{z}_{1:k})$ to get new particles $\mathbf{x}_{0:k}^i$.



SIS (continued)

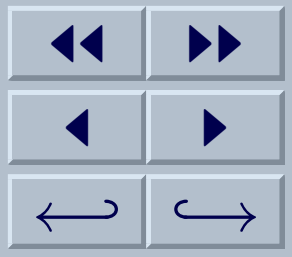
Weight update (after some lengthy computations. . .):

$$w_k^i = w_{k-1}^i \frac{p(\mathbf{z}_k|\mathbf{x}_k^i)\, p(\mathbf{x}_k^i|\mathbf{x}_{k-1}^i)}{q(\mathbf{x}_k^i|\mathbf{x}_{0:k-1}^i, \mathbf{z}_{1:k})} \tag{3}$$

Furthermore, if $q(\mathbf{x}_k|\mathbf{x}_{0:k-1}, \mathbf{z}_{1:k}) = q(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{z}_{1:k})$ (only dependent on *last* state and observations):

$$p(\mathbf{x}|\mathbf{z}_{1:k}) \approx \sum_{i=1}^{N_s} w_k^i\, \delta(\mathbf{x}_k - \mathbf{x}_k^i)$$

(and we need not preserve trajectories $\mathbf{x}_{0:k-1}^i$ and history of observations $\mathbf{z}_{1:k-1}$)



SIS Algorithm – Pseudo Code

$[\{\mathbf{x}_k^i, w_k^i\}_{i=1}^{N_s}]$ = **SIS**$(\{\mathbf{x}_{k-1}^i, w_{k-1}^i\}_{i=1}^{N_s}, \mathbf{z}_k)$

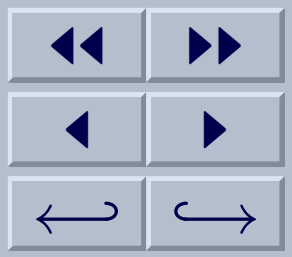
FOR $i = 1 : N_s$

draw $\mathbf{x}_k^i \sim q(\mathbf{x}_k | \mathbf{x}_{k-1}^i, \mathbf{z}_k)$

update weights according to (3)

END FOR

normalize weights to $\sum_{i=1}^{N_s} w_k^i = 1$



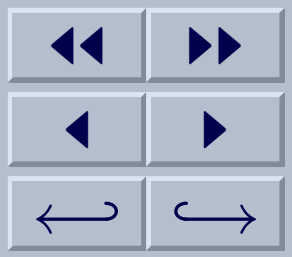
PROBLEM: Degeneracy Problem

Problem with SIS approach: after a few iterations, most particles have negligible weight (the weight is concentrated on a few particles only)

Counter measures:

- brute force: many, many samples N_s
- good choice of importance density
- resampling

Note: amount of degeneracy can be estimated based on variance of weights [Liu 1996].



Optimal Impotance Density:

It can be shown that the optimal importance density is given by

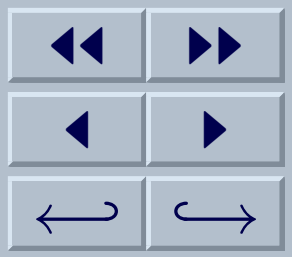
$$q(\mathbf{x}_k|\mathbf{x}_{k-1},\mathbf{z}_k)_{opt} = p(\mathbf{x}_k|\mathbf{x}_{k-1},\mathbf{z}_k)$$

Then

$$w_k^i = w_{k-1}^i \int p(\mathbf{z}_k|\mathbf{x}_k')p(\mathbf{x}_k'|\mathbf{x}_{k-1}^i)d\mathbf{x}_k'$$

Two major drawbacks: usually neither sampling from q_opt nor solving the integral in w_k^i is possible... (but in some special cases, it works)

Other alternative which is often convenient:
$q(\cdot) = p(\mathbf{x}_k|\mathbf{x}_{k-1})$ (prior). Easy to implement, but does not take measurements into account.



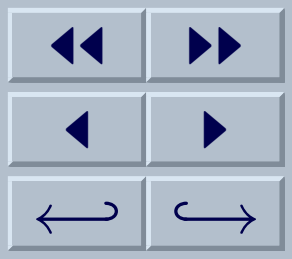
Resampling Approaches

Basic idea of resampling:

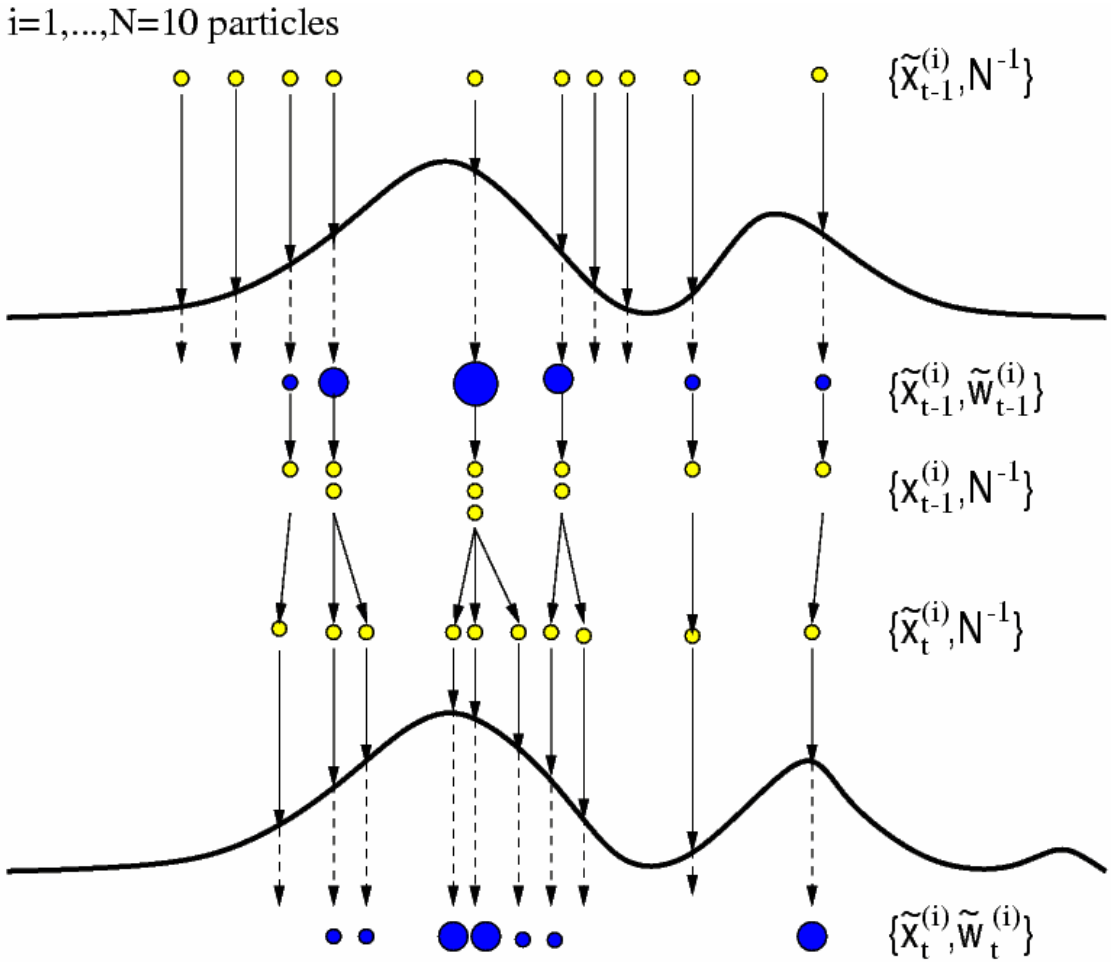
> Whenever degeneracy rises above threshold: replace old set of samples (+ weights) with new set of samples (+ weights), such that sample density better reflects posterior pdf.

This eliminates particles with low weight and chooses more particles in more probable regions.

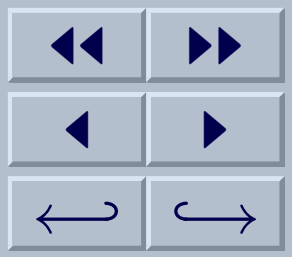
Complexity: possible in $O(N_s)$ operations



The resampling principle:



(graphics taken from Van der Merwe et al.)



General Particle Filter – Pseudo Code

$[\{\mathbf{x}_k^i, w_k^i\}_{i=1}^{N_s}]$ = **PF(**$\{\mathbf{x}_{k-1}^i, w_{k-1}^i\}_{i=1}^{N_s}$, $\mathbf{z}_k$**)**

FOR $i = 1 : N_s$

draw $\mathbf{x}_k^i \sim q(\mathbf{x}_k | \mathbf{x}_{k-1}^i, \mathbf{z}_k)$

update weights according to (3)

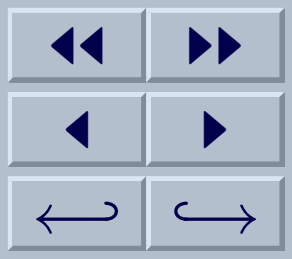
END FOR

normalize weights to $\sum_{i=1}^{N_s} w_k^i = 1$

IF degeneracy too high

resample $\{\mathbf{x}_k^i, w_k^i\}_{i=1}^{N_s}$

END IF



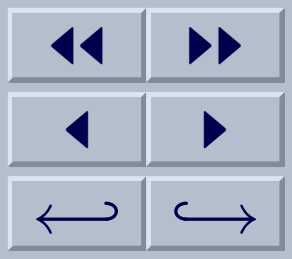
PROBLEM: Loss of Diversity

No degeneracy problem but new problem arises:

> Particles with high weight are selected more and more often, others die out slowly
> $\Rightarrow$ *loss of diversity* or *sample impoverishment*

For small process noise, all particles can collapse into a single point within a few iterations.

Other problem: resampling limits the ability to parallelize algorithm.



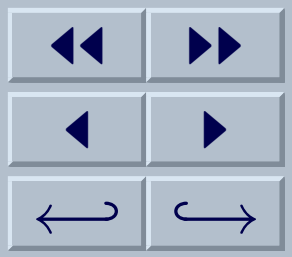
Other Particle Filter Variants

Methods to counteract loss of diversity and degeneracy problem:

- resample-move algorithm
- regularization
- Rao-Blackwellisation
- multiple Monte-Carlo

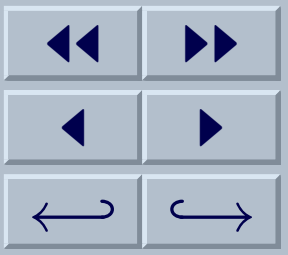
Other particle filter variants found in the literature:

- sampling importance resampling (SIR)
- auxiliary sampling importance resampling (ASIR)
- regularized particle filter (RPF)
- ...



7 Experiments

see videos...

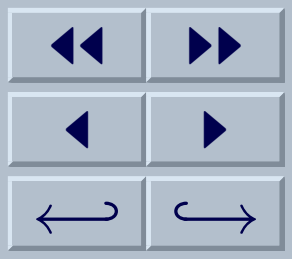


8 Summary

First of all: what I did *not* talk about...

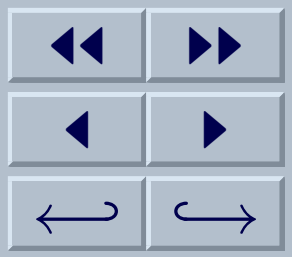
- speed of convergence
- number of samples needed
- complexity issues / tricks for speed-up of algorithms
- advanced particle filter variants in detail

$\Rightarrow$ refer to the literature if you want to know more



Advantages of particle filters (PFs):

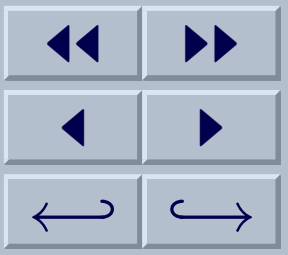
- can deal with non-linearities
- can deal with non-Gaussian noise
- can be implemented in $O(N_s)$
- mostly parallelizable
- easy to implement
- in contrast to HMM filters (state-space discretized to N fixed states) : PFs focus adaptively on probable regions of state-space



Thesis:

If you want to solve a filtering problem, then particle filters are the best filters you can use, much better than e.g. Kalman filters.

Right or **wrong**?



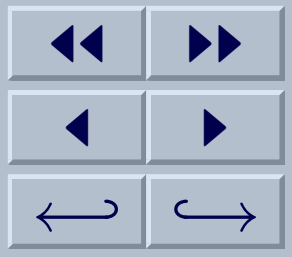
WRONG!

Particle filters include a random element; they only convergence to the true posterior pdf (almost surely) if $N_s \rightarrow \infty$.

Therefore: *If the assumptions for Kalman filters or grid-based filters are valid, no PF can outperform them!*

Additionally: depending on the dynamic model, Gaussian sum filters, unscented Kalman filters or extended Kalman filters may produce satisfactory results at lower computational cost.

(But you should at least try a PF; it is usually better than other suboptimal methods!)



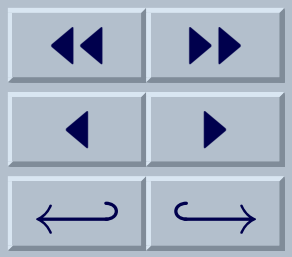
PF approaches proved their usefulness in a variety of applications.

But:

- choice of importance function $q(\cdot)$ is crucial in PF design
- large sample number N_s increases computational effort
- potential problems: *degeneracy* and *loss of diversity*

If these points are taken into account, then particle filters are an extremely powerful tool for filtering / estimation.

("black box usage" vs "know what you're doing!")



Thank you!

This presentation was made with LaTeX.
(try to write Bucureşti in Powerpoint. . .)