Symplectic Geometry and Toric Varieties

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Preface

Symplectic geometry has its roots in the 19th-century works of Hamilton and Lagrange in classical mechanics. The phase space of a closed physical system, say, a particle moving inside a sphere, must satisfy Hamilton's equations. These are a set of partial differential equations which govern the evolution of such systems. On the other hand, a symplectic manifold is an even dimensional manifold equipped with a non-degenerate 2-form. It turns out that the mathematical structure Hamilton's equations impose on the phase space is the same natural symplectic structure that the cotangent bundle of a smooth manifold supports. It is at this point that the study of symplectic geometry takes its own form and departs from its applications restricted to classical mechanics. Contemporary symplectic geometry now has links with low-dimensional topology, complex geometry, dynamical systems, and, as we will see by the end of this report, even algebraic combinatorics.

Introduction

This expository work assumes foundational knowledge of Differential Topology, including smooth manifolds, tangent/cotangent bundles, differential forms, and De Rham Cohomology. As such, it starts directly with an introduction to basic notions in symplectic geometry, including symplectic forms, symplectomorphisms, and Lagrangian submanifolds. Then, using these tools, the first significant results regarding the local behavior of symplectic manifolds are discussed through the Moser and Darboux theorems. The report then moves on to discuss the odd-dimensional analogous contact structure, and produces similar results to classify its local behavior. Finally the first part of this report comes to an end with a discussion on the intersection of complex and symplectic geometry, with Kähler manifolds receiving particular attention.

The second part of this report shifts focus to Hamiltonian geometry. It starts off with a discussion of notions like Hamiltonian and symplectic flows and vector fields, and generalizes the ideas through the language of group actions. In this effort, Lie theory is introduced, and the role of moment maps is signified. The symmetries of these Hamiltonian actions are then used to introduce the powerful idea of symplectic reduction. Finally, the report ends with a discussion on the classification of a special class of Hamiltonian spaces called symplectic toric manifolds. The culmination of this endeavour is in Delzant's theorem, which establishes an unexpected connection of symplectic geometry to algebraic combinatorics.

Acknowledgements

For this report, I have followed Dr. Ana Cannas da Silva's "Lecture Notes on Symplectic Geometry" very closely, with the occasional reference to proofs or ideas in Dr. Vladimir Arnold's "Mathematical Methods of Classical Mechanics". I have also consulted Dr. Alexander Kirillov Jr.'s "Introduction to Lie Groups and Lie Algebras" as a secondary resource to understand some important ideas in Lie theory. I am thankful to each of the three mathematicians for making their understanding of the subject accessible to everyone.

I am thankful to my supervisor, Dr. Shaheen Nazir, for her invaluable insights that have continuously guided me through problems I was stuck on. I am also grateful to Dr. Haniya Azam for introducing me to the amazing resource that are Dr. da Silva's notes. I would also like to appreciate Dr. Imran Anwar for taking out the time to attend my long presentations on the topic. All in all, I am very humbled to have received the support and guidance of the Mathematics department at LUMS over the last 4 years. Thank you for continuously coming up with unique ways to push me to grow mathematically, whether it be through the introduction of special courses, the John Conway seminars, or the Math Circles. Despite the small size of the department, I believe that the people that inhabit it have more than made up for it by their zeal for mathematics. The passion is contagious, and I hope to carry it with me in the future.

Contents

Chapter 1

Symplectic Structure

1.1 Symplectic Vector Spaces

Let V be an n-dimensional vector space over $\mathbb{R}$ with basis $\{w_1, w_2, \ldots, w_n\}$.

Definition 1.1.1. *A bilinear form $\Omega : V \times V \to \mathbb{R}$ that satisfies $\Omega(v, u) = -\Omega(u, v)$ for all $u, v \in V$ is called skew-symmetric.*

We know that every bilinear form L can be represented in the form of a matrix $\mathfrak{L}$ with $L(u, v) = u^T \mathfrak{L} v$ and $[\mathfrak{L}]_{ij} = L(w_i, w_j)$ considering the basis above. It is easy to see that every skew-symmetric bilinear form must have a skew-symmetric matrix representation. Moreover, we also have the following important result:

Theorem 1.1.2. *Let Ω be a skew-symmetric bilinear form on a real vector space V. Then there exists a basis in which the matrix for Ω is of the form:*

$$\begin{pmatrix} 0 & Id & 0 & \ldots & 0 \\ -Id & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

More precisely, the basis is given by $\{e_1, e_2, \ldots, e_m, f_1, f_2, \ldots, f_m, u_1, u_2, \ldots, u_k\}$ where $\Omega(e_i, f_j) = \delta_{ij}$, $\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0$ for all i, j, and $\Omega(u_i, v) = 0$ for all $v \in V$.

Example 1.1.3. Let M be a smooth 6-dimensional manifold and let T_pM be the tangent space at $p \in M$. Let $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ be a 2-form on M. Then $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is a skew-symmetric bilinear form on the 6-dimensional real vector space T_pM. It is easy to see that in the basis $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}\}$ of T_pM, the matrix representation for this 2-form is:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

It can also be seen that the basis $\{\frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}), \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}), \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}), \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}), \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}\}$ also gives the same matrix. So the basis in the theorem above is not unique, as expected. △

Now consider the linear map $\tilde{\Omega} : V \to V^*$ induced by Ω defined as: $\tilde{\Omega}(v) = \Omega(v, .)$.

Definition 1.1.4. *Let (V, Ω) be a pair consisting of a real vector space V and an associated skew-symmetric bilinear form Ω. If ker$(\tilde{\Omega}) = \{0\}$, Ω is called* **non-degenerate** *and (V, Ω) is called a symplectic vector space.*

Note that this condition is equivalent to span$\{u_1, u_2, \ldots, u_k\} = \{0\}$, where u_i's are the basis elements given in Theorem 1.1.2. This implies that V must be even-dimensional. Clearly, (T_pM, ω_p) was not symplectic, but (T_pN, ω_p) with $N = \{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_4}\}$. Moreover, non-degenerate bilinear forms have matrix representations of the form:

$$\begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$$

There is also a natural notion of equivalence among symplectic vector spaces:

Definition 1.1.5. *Two symplectic vector spaces (V_1, Ω_1) and (V_2, Ω_2) are called symplectomorphic if there exists a vector space isomorphism $\phi : V_1 \to V_2$ that preserves the symplectic structure.*

By preservation of symplectic structure it is meant that the pullback $\phi^*\Omega_2 = \Omega_1$. Now we are finally in a position to define the symplectic manifold.

1.2 Symplectic Manifolds

Definition 1.2.1. *A differential 2-form on a manifold M is called a symplectic form if and only if it is closed and ω_p is a non-degenerate bilinear form on T_pM for all $p \in M$.*

Definition 1.2.2. *A pair (M, ω) is called a symplectic manifold if ω is a symplectic form on M.*

Since $\dim T_pM = \dim M$, symplectic manifold must be even-dimensional. It can be observed that a 2-form $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$ with coordinates $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ is a symplectic form on $\mathbb{R}^{2n}$. Moreover, we can also replace the condition of non-degeneracy with an equivalent and more informative one:

Lemma 1.2.3. *A differential 2-form $\omega \in \Omega^2(M)$ is non-degenerate if and only if $\dim M = 2n$ and ω^n is non-vanishing, that is, ω^n is a volume form.*

Proof. First note that by Theorem 1.1.2, for $\dim T_pM = 2n$, there exists a basis $\{e_1, ...e_m, f_1, ..., f_m, u_1, ..., u_{2n-2m}\}$ for T_pM such that $\omega_p(u_i, ;) = 0$, $\omega_p(e_i, f_j) = \delta_{ij}$ and $\omega_p(e_i, e_j) = \omega_p(f_i, f_j) = 0$ for all i, j. This gives us that $\omega_p = \sum_{i=1}^{n} e_i^* \wedge f_i^*$ where e_i^*, f_i^* are part of corresponding dual basis for T_p^*M.
Now assume ω_p is non-degenerate for all $p \in M$. Then we have that $\dim M = 2n$ and $m = n$ from before:

$$\begin{aligned}
\underbrace{\omega_p \wedge .... \wedge \omega_p}_{n-\text{times}} &= \sum_{i_1,..,i_n}^{n} e^{i_1} \wedge f^{i_1} \wedge ...... \wedge e^{i_n} \wedge f^{i_n}. \\
&= \sum_{i_1 \neq i_2 .... \neq i_n}^{n} e^{i_1} \wedge f^{i_1} \wedge ...... \wedge e^{i_n} \wedge f^{i_n} \\
&= \sum_{\sigma \in S_n} e^{\sigma(1)} \wedge f^{\sigma(1)} \wedge ...... \wedge e^{\sigma(n)} \wedge f^{\sigma(n)}.
\end{aligned}$$

Now we can see that any element of the sum above is invariant under the following switching of successive pairs of 2-forms: $e^{\sigma(1)} \wedge f^{\sigma(1)} \wedge .... \wedge e^{\sigma(k)} \wedge f^{\sigma(k)} \wedge e^{\sigma(k+1)} \wedge f^{\sigma(k+1)} .... \wedge e^{\sigma(n)} \wedge f^{\sigma(n)} = e^{\sigma(1)} \wedge f^{\sigma(1)} \wedge .... \wedge e^{\sigma(k+1)} \wedge f^{\sigma(k+1)} \wedge e^{\sigma(k)} \wedge f^{\sigma(k)} .... \wedge e^{\sigma(n)} \wedge f^{\sigma(n)}$. Thus each element in the sum above is equivalent, and hence:

$$\underbrace{\omega_p \wedge .... \wedge \omega_p}_{n-\text{times})} = n!(e^1 \wedge f^1 \wedge ..... \wedge e^n \wedge f^n) \neq 0,$$

at any point $p \in M$. Thus ω^n is a volume form on M.
Now assume on the other hand that ω_p is degenerate for some $p \in M$, where $\dim M = 2n$. Then we have that $2n - 2m \geq 2$, and thus $n \geq m + 1$. We have, by a similar argument to before, $\omega_p^m = m!(e^1 \wedge f^1 \wedge ..... \wedge e^m \wedge f^m) \Rightarrow \omega_p^{m+1} = 0$ by anti-commutativity. Thus $\omega_p^n = 0$ and ω^n is not a volume form. □

Our notion for equivalent symplectic vector spaces carries over to symplectic manifolds:

Definition 1.2.4. *Two symplectic manifolds* (M_1, ω_1) *and* (M_2, ω_2) *are symplectomorphic if there exists a diffeomorphism* $\phi : M_1 \to M_2$ *with* $\phi^*\omega_2 = \omega_1$. The map ϕ is called a *symplectomorphism.*

One can ask a bunch of questions now: When does a symplectic form exist on an even-dimensional manifold? And if it does, when is it unique up to a symplectomorphism? That is, when is (M, ω_1) symplectomorphic to (M, ω_2)? Finally, does there at least exist some local commonality among all symplectic structures? As we will see, some of these questions have been fully answered, while others remain open problems to this day. For now, let us explore the existence of symplectic structures in different instances.

Example 1.2.5. Consider the Mobius strip. Assume it has a symplectic form ω. Then we must have ω is a volume form, which then implies that the Mobius strip is orientable. This is a contradiction, and so the Mobius strip must not support a symplectic structure in the first place. △

This gives us a more general result.

Proposition 1.2.6. *Every* $2n$*-dimensional symplectic manifold* (M, ω) *is orientable with orientation form* ω^n.

Is this a sufficient condition? Symplectic geometry would be very boring if it was: consider the orientable surface S^{2n}. Assume there exists a symplectic form ω on S^{2n}. Then ω^n is a volume form on S^{2n}. We can see that if this form is exact, we can apply Stoke's Theorem to get:

$$\text{vol}(S^{2n}) = \int_{S^{2n}} \omega^n = \int_{\partial S^{2n}} \alpha = 0,$$

where ω^n is assumed to be exact and equal to $d\alpha$. This is contradictory as S^{2n} is compact and hence $\text{vol}(S^{2n}) > 0$. Thus ω^n is not exact, which implies that ω is also not exact (if $\omega = d\alpha$, then $\omega^n = (d\alpha)^n = d(\alpha \wedge (d\alpha)^{n-1})$ is exact). Thus the de Rham cohomology class $[\omega] \neq 0$. If $n > 1$, this is contradictory to the fact that $H^2(S^{2n}) = 0$. Thus for $n > 1$, such an ω must not exist in the first place, and S^{2n} must not support a symplectic structure. That said, it is not difficult to see that S^2 supports a symplectic structure with its canonical area form being a symplectic form ($\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$).

Although we have seen smooth manifolds where no symplectic structure exists, there always exists an associated manifold with an intrinsic symplectic structure: the cotangent bundle. Recall that the cotangent bundle T^*M of an n-dimensional manifold M has local coordinates defined by $(x, \eta) \in T^*M \mapsto (x_1, ..., x_n, \eta_1, ..., \eta_n) \in \mathbb{R}^{2n}$. Define intrinsically the **Liouville** 1-form α on T^*M by $\alpha_{(x,\eta)} = \pi^*\eta$, where π is the canonical projection of T^*M onto M. Clearly, this definition is coordinate-independent. Furthermore, define the closed 2-form $\omega = -d\alpha$. Easy calculation shows that in any coordinate chart $(U, x_1, ..., x_n, \eta_1, ...\eta_n)$, $\alpha = \sum_{i=1}^n \eta_i dx_i$ and $\omega = \sum_{i=1}^n dx_i \wedge d\eta_i$. Through an identical calculation to that of Lemma 1.2.3, we have that ω is symplectic and (T^*M, ω) is a symplectic manifold.

It is clear that T^*M is a symplectic manifold intrinsically linked to the differential structure of M. Thus, it is no surprise that diffeomorphic spaces have symplectomorphic cotangent bundles:

Theorem 1.2.7. *Let $f : M_1 \to M_2$ be a diffeomorphism. Then there exists a symplectomorphism $g : (T^*M_1, \omega_1) \to (T^*M_2, \omega_2)$.*

Proof. Define $g = \bar{f} : (T^*M_1, \omega_1) \to (T^*M_2, \omega_2)$ by $\bar{f}(x, \eta) = (f(x), (f^{-1})^*\eta)$. First we show that this is a diffeomorphism. It is clear that $\bar{f}$ is bijective due to the bijectivity of f and the pullback f^*. Now consider any $p \in U = (x_1, ..., x_n, \eta_1, ..., \eta_n)$ and $\bar{f}(p) \in V = (y_1, ..., y_n, \bar{\eta}_1, ..., \bar{\eta}_n)$ To argue the smoothness of $\bar{f}$ at p, we have that one of the component functions f is smooth by definition, and for $(f^{-1})^*$ we have:

$$\begin{aligned}(f^{-1})^*\eta = \sum_{i=1}^n (\eta_i \circ f^{-1}) d(x_i \circ f^{-1}) &= \sum_{i,j}^n (\eta_i \circ f^{-1}) \frac{\partial (x_i \circ f^{-1})}{\partial y_j} dy_j \\ &= \sum_{j=1}^n \left(\sum_{i=1}^n (\eta_i \circ f^{-1}) \frac{\partial (x_i \circ f^{-1})}{\partial y_j} \right) dy_j.\end{aligned}$$

So $\eta_i(p) \mapsto \eta_i(p) \frac{\partial (x_i \circ f^{-1})}{\partial y_j}(f(p))$, which is clearly a smooth mapping. The same can be argued for $\bar{f}^{-1} = (f^{-1}, f^*)$, and thus $\bar{f}$ is a diffeomorphism. Finally, it is also not too hard to see that as $\pi_2 \circ \bar{f} = f \circ \pi_1$, and given the intrinsic definition of the Liouville form α above, $\bar{f}^*\alpha_2 = \alpha_1$ and hence $\bar{f}^*\omega_2 = \omega_1$. [2] □

We can in fact show a partial converse:

Proposition 1.2.8. *Any symplectomorphism $g : (T^*M, \omega) \to (T^*M, \omega)$ that preserves α must be the lift of some diffeomorphism.*

Proof. Let $p = (x, \eta)$ and $g(p) = (y, \zeta)$. As $g^*\alpha = \alpha$, we have that at p, $g^*\alpha_{g(p)} = \alpha_p$ and hence $g^*(\pi^*\zeta_y) = \pi^*\eta_x$. Then $(\pi \circ g)^*\zeta_y = \pi^*\eta_x$ and $(\pi \circ g)^*(\lambda\zeta)_y = \pi^*(\lambda\eta)_x$ for any $\lambda \in \mathbb{R}$. This implies $g^*(\alpha_{(y,\lambda\zeta)}) = \alpha_{(x,\lambda\eta)}$. As g is bijective, we know that there exists $q = (z, \beta) \in T^*M$ such that $g(q) = (y, \lambda\zeta)$ and hence $g^*(\alpha_{(y,\lambda\zeta)}) = \alpha_{(z,\beta)}$. Given the explicit local form of α from before, it is obvious that $(z, \beta) = (x, \lambda\eta)$. Hence $g(x, \lambda\eta) =$

$(y, \lambda\zeta)$. Specifically taking $\lambda = 0$, we have that $g(x, \eta) = (y, \zeta) \Rightarrow g(x, 0) = (y, 0)$. This implies $\pi(g(x, \eta)) = y$ for all $\eta \in T_x^*M$. Thus we can simply define a diffeomorphism $h : M \to M_0 \subset T^*M$ with $h(x) = (x, 0)$, and have $f : M \to M$ be a diffeomorphism such that $f = h^{-1} \circ g|_{M_0} \circ h$. Clearly $\pi \circ g = f \circ \pi$. Taking p as before, we can see that the lift of f gives $\bar{f}(p) = (f(x), \gamma) = (y, \gamma)$ where $f^*\gamma_y = \eta_x \Rightarrow (f \circ \pi)^*\gamma_y = \pi^*(f^*\gamma_y) = \pi^*\eta_x = (\pi \circ g)^*\zeta_y$. Using the relation between f and g, we get $(\pi \circ g)^*\gamma_y = (\pi \circ g)^*\zeta_y$. As $\pi \circ g$ is a subjective submersion, we must have $\gamma_y = \zeta_y$ and hence $g = \bar{f}$. $\square$

Now although we have made some progress on our questions from before, we are still far from addressing the general question of symplectomorphic manifolds. To address this, we will first take a detour to understand some important sub-structures of symplectic manifolds.

1.3 Lagrangian Submanifolds

Recall that a submanifold X of M is defined to be a manifold with an immersive homeomorphism $i : X \to M$ where $i(X)$ is closed in M.

Definition 1.3.1. *Let (M, ω) be a 2n-dimensional symplectic manifold. Then a submanifold Y of dimension n with $i^*\omega = \omega|_{T_pY} \equiv 0$ is called a Lagrangian submanifold.*

Example 1.3.2. A simple example of a Lagrangian submanifold is the *zero section* (see [2]) of T^*M given by $M_0 = \{(x, \eta) \in T^*M |\, \eta = 0 \text{ in } T_x^*M\}$. This is clearly an n-dimensional submanifold of T^*M defined by $\eta_i = 0$ for all i. Moreover, $\alpha|_{T_pM_0} \equiv 0 \Rightarrow \omega|_{T_pM_0} \equiv 0$, and so M_0 is a Lagrangian submanifold of T^*M. One can in fact prove a more general result stating that for a 1-form μ on M, the section $M_\mu = \{(x, \mu_x) \in T^*M\}$ is a Lagrangian submanifold of T^*M if and only if μ is closed, see [2]. This gives a whole family of Lagrangian submanifolds that can be *generated* by functions $f \in C^\infty(M)$ as M_{df}. $\triangle$

We now unravel the connection between our question about symplectomorphisms and Lagrangian submanifolds. Consider a diffeomorphism $\phi : (M_1, \omega_1) \to (M_2, \omega_2)$ where $\dim M_1 = \dim M_2 = 2n$. It induces a map $\psi : M_1 \to M_1 \times M_2$ given by $\psi(p) = (p, \phi(p))$. The image of ψ is the graph of the function ϕ, and is a closed embedding of M_1 into $M_1 \times M_2$. This gives us that M_1 is a $2n$-dimensional submanifold of $M_1 \times M_2$. We now impose on $M_1 \times M_2$ a symplectic structure given by the *twisted* closed 2-form $\bar{\omega} = \pi_1^*\omega_1 - \pi_2^*\omega_2$, where π_i is projection onto M_i. One can easily check that this is indeed a symplectic form. Considering $\psi^*\bar{\omega} = (\pi_1 \circ \psi)^*\omega_1 - (\pi_2 \circ \psi)^*\omega_2 = \omega_1 - \phi^*\omega_2$, we can see that ϕ is a symplectomorphism if and only if $\psi^*\bar{\omega} = 0$ and $\psi(M_1)$ is a Lagrangian submanifold of $(M_1 \times M_2, \bar{\omega})$. This gives us a new approach to the problem of finding symplectomorphisms! see for more details,[2].

Let us try to apply this. Consider $M_1 = (T^*X_1, \omega_1)$ and $M_2 = (T^*X_2, \omega_2)$ for some manifolds X_1 and X_2. We shall now work backwards; that is, look for Lagrangian submanifolds of $(M_1 \times M_2 \cong T^*(X_1 \times X_2), \bar{\omega})$. We know that it is easy to find Lagrangian submanifolds of $(M_1 \times M_2 \cong T^*(X_1 \times X_2), \omega = \pi_1^*\omega_1 + \pi_2^*\omega_2)$, where ω is just the canonical 2-form for the cotangent bundle $T^*(X_1 \times X_2)$. Assume we have such a Lagrangian submanifold Y. Then defining $\sigma : M_1 \times M_2 \to M_1 \times M_2$ as $\sigma(x, y, \eta_1, \eta_2) = (x, y, \eta_1, -\eta_2)$, we can see that $\sigma(Y)$ is a Lagrangian submanifold of $(M_1 \times M_2, \bar{\omega})$. So if we start off with $Y = (M_1 \times M_2)_{df}$ for some $f \in C^\infty(X_1 \times X_2)$, we get $\sigma(Y) = (x, y, d_x f, -d_y f)$. And so, if this is the graph of a diffeomorphism $\phi : M_1 \to M_2$, then ϕ must be a symplectomorphism. Thus, $\phi(x, \eta) = (y, \zeta)$ must satisfy the following *Hamilton* equations:

$$\begin{aligned} \eta_i &= \frac{\partial f}{\partial x_i}(x, y) \\ \zeta_i &= -\frac{\partial f}{\partial y_i}(x, y). \end{aligned}$$

By the implicit function theorem, for a solution $y = \phi_1(x, \eta)$ to exist for the first differential equation, we must have $\left[\dfrac{\partial^2 f}{\partial y_j \partial x_i}\right]_{i,j}$ is invertible locally.

Example 1.3.3. Consider a Riemannian manifold (X, g) that is *geodesically convex* and *geodesically complete*. This implies that for every $(x, v) \in TX$ there exists a unique minimizing geodesic of constant velocity v starting at x given as $\exp(x, v) : \mathbb{R} \to X$. If we define $f : X \times X \to \mathbb{R}$ as $f(x, y) = -\frac{1}{2}d(x, y)^2$, we can show that the symplectomorphism generated by f can be identified (through g) with the *geodesic flow* on X (the endomorphism $(x, v) \mapsto \exp(x, v)(1)$ of TX). Firstly, recall that the identification between TX and T^*X is given as $(x, v) \leftrightarrow g_x(v, ;)$. Thus the Hamilton equations modify to:

$$\begin{aligned} g_x(v, ;) &= d_x f \\ g_y(w, ;) &= -d_y f, \end{aligned}$$

where we aim to find (y, w) as function of (x, v). If we act both sides of the first equation on $\frac{d\exp(x,v)}{dt}(0) = v$, we get:

$$\begin{aligned} g_x(v, v) = d_x f(v) &= \frac{d}{dt}(f(\exp(x, v)(t), y))(0) \\ &= -\frac{1}{2}\frac{d}{dt}(d(\exp(x, v)(t), y)^2)(0). \end{aligned}$$

We can see that if we take $y = \exp(x,v)(1)$ we get by definition of the Riemannian distance that $d(\exp(x,v)(t), \exp(x,v)(1)) = \int_t^1 \sqrt{g_x\left(\frac{d\exp(x,v)}{dt}, \frac{d\exp(x,v)}{dt}\right)} = \int_t^1 \sqrt{g_x(v,v)} = \sqrt{g_x(v,v)}(1-t)$. Plugging this into the equation above confirms that this is indeed the unique solution to the first Hamilton equation. Moreover, if we act $\frac{d\exp(x,v)}{dt}(1) = v$ on both sides of the second equation, we get:

$$\begin{aligned} g_x(w,v) = -d_y f(v) &= \frac{d}{dt}(f(\exp(x,v)(t), \exp(x,v)(0)))(1) \\ &= \frac{1}{2}\frac{d}{dt}(d(\exp(x,v)(t), \exp(x,v)(0))^2)(1). \end{aligned}$$

Through a similar calculation as before, we get $d(\exp(x,v)(t), \exp(x,v)(0)) = -t\sqrt{g_x(v,v)}$, and plugging this into the equation above gives $g_x(w,v) = g_x(v,v)$. Moreover, if we consider any orthogonal $\bar{v} \in T_yX$ to v, we get $g_x(w,\bar{v}) = \frac{1}{2}\frac{d}{dt}(d(\exp(y,\bar{v})(t), \exp(x,v)(0))^2)(0)$. As $\exp(x,v)$ is a minimizing geodesic, we have that $d(\exp(y,\bar{v})(t), \exp(x,v)(0))$ is minimum when the two geodesics intersect at $t = 0$. Thus the derivative expression above is 0 and $g(w,\bar{v}) = 0$ for all orthogonal $\bar{v} \in T_yX$ to v. This implies $g(w, ;) = g(v, ;)$. By non-degeneracy of g, the identification we made before is bijective, and hence $w = v = \exp(x,v)(1)$. And so, the symplectomorphism f can be identified with the endomorphism $(x,v) \mapsto \exp(x,v)(1)$ of TX. $\triangle$

Now that we have some familiarity with the construction and finding of symplectomorphisms, we can start to explore questions about the existence of various kinds of equivalences between symplectic structures.

1.4 Symplectic Equivalences

We first recall few definitions and ideas.

Definition 1.4.1. *An isotopy is a map $\rho : M \times \mathbb{R} \to M$ such that $\rho_t = \rho(;,t) : M \to M$ is a diffeomorphism for all t, and $\rho_0 = id_M$.*

Every isoptopy gives an associated time-dependent vector field v_t given by $v_t = \frac{d\rho_t}{dt} \circ \rho_t^{-1}$. For a compact manifold, the converse is also true: for any time-dependent vector field v_t, there exists an isotopy satisfying the differential equation $\frac{d\rho_t}{dt} = v_t \circ \rho_t$, with initial condition $\rho_0 = id_M$. By Picard–Lindelof theorem, this also exists locally for non-compact manifolds. This allows us to define the *Lie derivative* along v_t as a map $\mathfrak{L}_{v_t} : \Omega^k(M) \to \Omega^k(M)$ given by $\mathfrak{L}_{v_t}\omega = \frac{d}{dt}(\rho_t^*\omega)|_{t=0}$, where ρ is the local isotopy associated with v_t. The Lie derivative satisfies the *Cartan magic formula:* $\mathfrak{L}_{v_t}\omega = i_{v_t}d\omega + di_{v_t}\omega$, and as $\rho_{h+t} = \rho_t \circ \rho_h$ (by uniqueness of solution to the differential equation from before), it also satisfies $\frac{d}{dt}\rho_t^*\omega|_{t=\tau} = \rho_\tau^* \mathfrak{L}_{v_t}\omega$. By the chain rule, this gives $\frac{d}{dt}\rho_t^*\omega_t = \rho_t^*(\mathfrak{L}_{v_t}\omega_t + \frac{d\omega_t}{dt})$.

Also recall that the *normal space* to a k-dimensional submanifold $S \subset M$ at $p \in S$ is defined as $n-k$ dimensional quotient vector space $N_pS = T_pM/T_pS$. This gives us the *normal bundle* NS, an n-dimensional manifold. It is not hard to see that S embeds into NS as the zero-section as well. In fact, we also have the following well-known result that states that these embeddings are locally the same:

Theorem 1.4.2. (Tubular neighborhood theorem) *Let $S \subset M$ be a submanifold of M. There exists a neighborhood U_0 of S in NS, and a neighborhood U of S in M, and a diffeomorphism $\phi : U_0 \to U$ with $\phi|_S = id_S$. Moreover, U_0 in convex in each N_pS.*

This gives us that S is a deformation retract of U and hence the inclusion induces an isomorphism of de Rham cohomologies. Now we return to the question from before, but slightly altered: *Does there exist a symplectomorphism between (M, ω_0) and (M, ω_1) that is homotopic to the identity?* The reason we have added this extra condition is that it restricts our problem to the specific case where $[\omega_0] = [\omega_1]$. Naturally, this problem is easier.

Define $S_c = \{\text{symplectic forms } \omega \text{ on } M \text{ with } [\omega] = c\}$ endowed with the C^∞ topology. Assume $\omega_0, \omega_1 \in S_c$, and that there exists a smooth straight line path connecting both of them in S_c, i.e, $\omega_t = t\omega_0 + (1-t)\omega_1$ is symplectic for all $t \in [0,1]$. Moreover, as $\omega_0 - \omega_1 = d\mu$ for some 1-form μ on M, we have $\omega_t = \omega_1 + td\mu \in S_c$ for all t. We claim that this is a sufficient condition for the existence of a symplectomorphism that is homotopic to the identity.

Proof. Before we prove that the existence of a straight line path is sufficient, we note in hindsight that if it is true, then (M, ω_t) is symplectomorphic to (M, ω_0) for all t by reparameterizing the path ω_t. All of these implications can be satisfied if there exists an isotopy ρ with $\rho_t^*\omega_t = \omega_0$. Looking for the existence of such an isotopy is exactly what is known as *Moser's trick*. If we additionally assume that M is compact, we can instead look for the associated vector field v_t, and integrate to get ρ_t. We have the following condition on v_t as in [2]:

$$\frac{d}{dt}(\rho_t^*\omega_t) = 0 = \rho_t^*\left(\mathfrak{L}_{v_t}\omega_t + \frac{d\omega_t}{dt}\right) \Leftrightarrow di_{v_t}\omega_t + d\mu = 0.$$

We can find a solution for this easily by solving for v_t such that $i_{v_t}\omega_t + \mu = 0$ (as ω_t is nondegenerate). Thus, we obtain the isotopy we wanted. $\square$

One can generalize this argument to any $\omega_0, \omega_1 \in S_c$ that lie in the same path-connected component:

Theorem 1.4.3. (Moser Theorem) *Let M be compact manifold with symplectic forms ω_0 and ω_1 connected by a smooth family of symplectic forms ω_t, with $0 \leq t \leq 1$, such that $[\omega_t] = c$ is constant. Then there exists an isotopy ρ of M with $\rho_t^* \omega_t = \omega_0$*

Let's apply this.

Proposition 1.4.4. *Let M a compact and oriented 2-manifold. Then each non-zero 2-cohomology class has a unique symplectic representative.*

Proof. Consider an orientable and compact 2-dimensional manifold M with areas forms ω_1 and ω_2. Clearly, these are symplectic forms. Assume further that $[\omega_1] = [\omega_2]$. First, let us show that ω_1 and ω_2 determine the same orientation. Assume they do not. Then $\omega_1 = -f\omega_2$ for some smooth $f > 0$. Moreover, we have $\int_M \omega_1 - \omega_2 = \int_M (1+f)\omega_1 > 0$. This is a contradiction as $\omega_1 - \omega_2 = d\mu$ is exact, and hence by Stoke's theorem $\int_M \omega_1 - \omega_2 = 0$. Thus ω_1 and ω_2 must determine the same orientation, and $\omega_1 = f\omega_2$ for some smooth $f > 0$. Now consider the smooth straight line path $\omega_t = t\omega_1 + (1-t)\omega_2 = (1 - t + ft)\omega_2$. As $t \in [0, 1]$, ω_t can only vanish at some point p if $f(p) = (t-1)/t \leq 0$. This is a contradiction. Hence ω_t is symplectic for all t. By Moser Theorem, the symplectic structure induced by ω_1 and ω_2 is the same. □

Moser Theorem also allows us to find a symplectic equivalence if two different symplectic forms agree on a compact submanifold. Recall the tubular neighborhood theorem−if X to be a compact submanifold of M, then there is a neighborhood U of X in M such that X is a deformation retract of U. If ω_1 and ω_2 are two symplectic forms that agree on X, i.e, $i^*(\omega_1 - \omega_2)|_X = 0$, then $(\omega_1 - \omega_2)|_U = d\mu$ is exact by homotopy invariance of the de Rham cohomology. This allows one to define a smooth family $\omega_t = \omega_2 + td\mu$ that vanishes on X. After some shrinking of neighborhoods, and applying Moser's Trick again, we have another theorem by Moser:

Theorem 1.4.5. (Relative Moser Theorem)*Let X be a compact submanifold of a symplectic manifold M with two symplectic forms ω_1 and ω_2 agreeing on X. Then there exists neighborhoods U and V of X in M, and a diffeomorphism $\phi : U \to V$ with $\phi^*\omega_2 = \omega_1$ and $\phi|_X = id_X$.*

If we take $X = p$, and use the canonical symplectic basis for the T_pM, we finally get the grand theorem classifying symplectic structures locally:

Theorem 1.4.6. (Darboux Theorem) *Let (M, ω) be a symplectic manifold, and let p be any point in M. Then we can find a coordinate system $(U, x_1, ..., x_n, y_1, ...y_n)$ centered at p such that on U:*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

If X is a compact Lagrangian submanifold of both (M, ω_1) and (M, ω_2), i.e, $i^*\omega_1 = i^*\omega_2 = 0$, then relative Moser applies identically to give what is known as the *Weinstein Lagrangian Neighborhood Theorem*. This theorem is imperative in helping us classify Lagrangian structures locally. But first, we prove a small result.

Lemma 1.4.7. *Let (V, Ω) be symplectic vector space with a lagrangian subspace U. Then there exists a canonical isomorphism between V/U and U^*.*

Proof. Define $\tilde{\Omega} : V/U \to U^*$ given by $\tilde{\Omega}([v]) = \Omega(v, ;)$. This is well-defined as for any $v \oplus u$, $\Omega(v \oplus u, ;) = \Omega(v, ;) + \Omega(u, ;) = \Omega(v, ;)$ as $\Omega|_U = 0$. Moreover, it is a non-degenerate pairing, and so the map above is an isomorphism. $\square$

Note that this allows us to canonically identify NX and T^*X. Now recall that every manifold X is a Lagrangian submanifold of (T^*X, ω_0) as the zero section. Assume X is also a Lagrangian submanifold of (M, ω). By using the tubular neighborhood theorem with the identification $NX \cong T^*X$, followed by Weinstein's Lagrangian neighborhood theorem, we obtain neighborhoods U of X in (T^*X, ω_0), V of X in M, and a composite symplectomorphism $\phi : (U, \omega_0) \to (V, \omega)$ that is identity on X[2]. This implies that every Lagrangian embedding is symplectically locally equivalent to the embedding of a manifold as the zero-section of its cotangent space! This classification is called the *Weinstein tubular neighborhood theorem.*This result has important applications.

Example 1.4.8. Consider $\text{Sympl}(M, \omega) = \{f : M \xrightarrow{\cong} M | f^*\omega = \omega\}$ in the C^1 topology. Take $f \in \text{Sympl}(M, \omega)$ that is in a sufficiently small neighborhood of $id \in \text{Sympl}(M, \omega)$. We have that the graphs $\Gamma_f, \Gamma_{id} \cong M$ are lagrangian submanifolds of $(M \times M, \bar{\omega})$. By Weinstein's tubular neighborhood theorem, as f is sufficiently C^1 close to id, its graph is symplectically embedded into T^*M as the image of a smooth section $M_\mu = \{(p, \mu_p) \in T^*M\}$[2], where μ is closed and C^1-close to the zero form. On the other hand, if we take μ to be a closed 1-form that is C^1-close to the zero form, its section M_u will lie in a sufficiently small neighborhood of the zero section in U_2. Thus ϕ^{-1} is defined on M_μ. As μ is C^1-close to the zero form, $\phi^{-1}(M_\mu)$ also smoothly intersects $\{p\} \times M$ once for all p at $g(p)$. Thus, g is a diffeomorphism of M and $M_\mu \cong \Gamma_g$. As μ is closed, g is a symplectomorphism. This gives us that sufficiently small C^1 neighborhood of id in

Sympl(M, ω) is homeomorphic to a similar neighborhood of the zero form in the space of closed 1-forms on M. △

Example 1.4.9. Consider a compact symplectic manifold M with vanishing de Rham cohomology of degree 1. Take any $f \in \text{Sympl}(M, \omega)$ that is C^1-close to identity. Then, by the observation above, it is generated by $h \in C^\infty(M)$ through the recipe described in section 1.3. Now, as h is defined on a compact manifold, and hence achieves a maximum and minimum, it must have at least two critical points where $dh_p = 0$. Via the zero-section embedding discussed before, we have that Γ_f intersects Γ_{id} at two points[2]. Thus f must have two fixed points. △

Example 1.4.10. Similar to above, we can now also say something about Lagrangian intersections. Assume we have two compact Lagrangian submanifolds X and Y with vanishing de Rham cohomology of degree 1. Assume further that they are C^1-close, i.e, there exists a diffeomorphism $f : X \to Y$ that is C^1-close to the inclusion $i : X \to M$. Then, by Weinstein's tubular neighborhood theorem, there exist neighborhoods U_1 of X in M, U_2 of the zero-section of T^*X, and symplectomorphism ϕ between the two. We can assume $Y \subset U_1$. Now, lifting the diffeomorphism f, we can embed Y into U_2 as the section of a closed 1-form on X. We can see that μ must vanish for at least two points as X is compact and $H^1_{deRham}(X) = 0$. Via ϕ^{-1}, this corresponds to X and Y intersecting in at least two points. △

We have come a long way now in understanding the local behavior of symplectic structures. That said, this is just the tip of the iceberg as far as contemporary research in this area is concerned. Moreover, many questions remain open problems to this day, and have even led to developments of new theories. One such problem is along the themes of the examples above about fixed points of symplectomorphisms called *Arnold Conjecture*, and how it led to the development of *Morse theory* in the form of *Floer homology.* As we cannot hope to cover all of the above in such depth, we shall, for now, take a detour to explore another important idea in symplectic geometry, and that is *contact structure.*

Chapter 2

Contact Structure

2.1 Contact Manifolds

Definition 2.1.1. *A contact element on a manifold M is a pair (p, H_p) where $p \in M$ and $H_p \subset T_pM$ is a tangent hyperplane.*

If we define a field of contact elements $H : p \mapsto H_p$, it is not hard to see that H is locally defined by the kernel of a 1-form α.

Definition 2.1.2. *A contact structure on M is a smooth field of tangent hyperplanes $H \subset TM$ such that, for any locally defining 1-form α, $d\alpha|_H$ is symplectic. The pair (M, H) is called a contact manifold and α is called a local contact form.*

It is easy to see that for a contact manifold, $\dim H_p = \dim(\ker \alpha_p) = 2n$ for all p, and $T_pM = \ker\alpha_p \oplus \ker d\alpha_p$ (where α is a locally defining 1-form) is $2n+1$-dimensional. As will be seen moving ahead, contact structures are often considered to be the 'odd' counterpart of symplectic structures. As a first glimpse into this, we show that the symplectic condition above can be replaced by a more similar one we saw in symplectic manifolds:

Proposition 2.1.3. *Let H be a field of hyperplanes on a manifold. Then H is a contact structure if and only if $\alpha \wedge (d\alpha)^n$ is non-vanishing for any locally defining 1-form α.*

Proof. Assume α is locally defining 1-form for $p \in M$. We can decompose T_pM and choose a basis $\{e_1, ...e_{2n+1}\}$ such that $H_p = \text{span}(e_2, ..., e_{2n+1})$. We can see that:

$$\begin{aligned}
\alpha \wedge (d\alpha)^n(e_1, ..., e_{2n+1}) \neq 0 &\Leftrightarrow \alpha_p(e_1)(d\alpha_p)^n(e_2, ..., e_{2n+1}) \neq 0 \\
&\Leftrightarrow (d\alpha_p)^n(e_2, ..., e_{2n+1}) \neq 0 \\
&\Leftrightarrow (d\alpha_p)^n|_{H_p} \neq 0 \\
&\Leftrightarrow (d\alpha)^n|_H \text{ is symplectic.}
\end{aligned}$$

Note that if α is a global form, then $\alpha \wedge (d\alpha)^n$ is a volume form on M, and hence M is orientable. And it is not hard to see that if TM/H is an orientable quotient line bundle with orientation form $\bar{\alpha}$, then α (the corresponding form on M) is a global contact form. $\square$

Drawing another analogy to symplectic structures, a *contactomorphism* is defined as a diffeomorphism ϕ such that $\phi_* H = H$. We can also define a canonical vector field R for contact structures satisfying $i_R d\alpha = 0$ and $i_R \alpha = 1$ for any locally defining 1-form α. It is called the *Reeb vector field*, and given the decomposition of the tangent space from before, we can see that it is unique up to multiplication by a non-vanishing smooth function. Moreover, the flow of the Reeb vector field is a family of contactomorphisms preserving the contact form. Let us look at some examples now to solidify our understanding.

Example 2.1.4. Consider $\mathbb{R}^{2n+1}$ with coordinates $(x_1, .., x_n, y_1, ..., y_n, z)$. Let $\alpha = \sum_{i=1}^n (x_i dy_i) + dz$. Then as $d\alpha = \sum_{i=1}^n (dx_i \wedge dy_i)$, we can see that $\alpha \wedge (d\alpha)^n = dx_1 \wedge ... \wedge dx_n \wedge ... dy_1 \wedge dy_n \wedge dz$ is a volume form. Hence $\mathbb{R}^{2n+1}$ admits a contact structure for $n \geq 0$. The corresponding field of hyperplanes is given by $\ker\alpha_p$ as $H_{\tilde{p}} = \{\sum_{i=1}^n (a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial y_i}) - \sum_{i=1}^n (b_i \tilde{x}_i) \frac{\partial}{\partial z} | a_i, b_i \in \mathbb{R}\}$. The corresponding Reeb vector field is given by $\frac{\partial}{\partial z}$, and its flow is given by translation in the z-direction: $\rho_t(\tilde{p}) = \tilde{p} + t(0, 0, ..., 1)$. It is clear that this preserves the contact form. In fact, any translational diffeomorphism by a vector of the form $(0, ..., 0, \bar{y}_1, \bar{y}_2, ..., \bar{z})$ is contactomorphism. $\triangle$

Just like the cotangent bundle T^*M is a canonical symplectic structure associated with any manifold M, we can also define a canonical contact structure associated with M. Define the following equivalence relation on ($T^*M\setminus$ zero section) given by $(p, \zeta) \sim (p, \zeta') \Leftrightarrow \zeta = \lambda\zeta'$ for some $\lambda \in \mathbb{R}\setminus\{0\}$. We shall show that $\mathbb{P}^*M = (T^*M\setminus \text{ zero section})/\sim$ has a contact structure.

Let $\mathfrak{C} = \{(p, \chi_p) | \chi_p \text{ is a hyperplane in } T_pM\}$, the collection of all contact elements in M. It can be naturally identified with $\mathbb{P}^*M$ under the well-defined map $\phi(p, [\zeta]) = (p, \ker\zeta)$. This also allows us to import the smooth structure of $\mathbb{P}^*M$ (given by $M \times \mathbb{RP}^{n-1}$) to $\mathfrak{C}$ to make it a $2n-1$-dimensional manifold. Now define a natural field of hyperplanes on $\mathfrak{C}$ given by $\mathfrak{H}_{(p,\chi_p)} = (\pi_*)^{-1}\chi_p$ where π is the projection of $\mathfrak{C}$ onto M. Under the differential of the map (now a diffeomorphism) ϕ from before, we get a field of hyperplanes $\mathbb{H}$ on $\mathbb{P}^*M$. In fact, we can show that $\mathbb{H}_{(p,[\zeta])} = \ker(\tilde{\pi}^*\zeta)$, where $\tilde{\pi}$ is the projection of $\mathbb{P}^*M$ onto M. This is not hard to see: $\ker\tilde{\pi}^*\zeta = (\tilde{\pi}_*)^{-1}\ker\zeta = ((\pi \circ \phi)_*)^{-1}\ker\zeta = (\phi_*)^{-1}((\pi_*)^{-1}(\ker\zeta)) = \mathbb{H}_{(p,[\zeta])}$ by definition. Not that the defining 1-form above is very similar to the canonical 1-from α on the cotangent bundle. This we expand on now.

Let us first clarify the coordinate structure on $\mathbb{P}^*M$. Let $(p, \chi_p) \in \mathfrak{C}$. Choose the coordinates $(x_1, ..., x_n)$ in a neighborhood U of $p \in M$ to be such that $\chi_p = \text{span}(\frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n})$. Now import these coordinates to $\mathbb{P}^*M$ via ϕ^{-1}. Given its definition, at $\phi^{-1}(p, \chi_p)$, at least one $\zeta_i \neq 0$. Given the fact that $\frac{\partial}{\partial x_1} \notin \ker\chi_p$, we can see that $\zeta_1 \neq 0$. Then $\zeta_i \neq 0$ in some neighborhood U of p as well. Thus, similar to what we do in projective space, we can scale our coordinates such that $\zeta_1 = 1$ in U and so $(x_1, ..., x_n, \zeta_2, ..., \zeta_n)$ play the role of our coordinates in U. If we define $\alpha = dx_1 + \sum_{i=2}^n \zeta_i dx_i$ in U, we can see that as $\alpha_{(p,[\zeta])} = \pi^*\zeta_p$ in some smaller neighborhood contained in U, α is a locally defining 1-form for $\mathbb{H}$. Moreover $d\alpha = \sum_{i=2}^n d\zeta_i dx_i$, and so $(d\alpha)^n = (n-1)! d\zeta_2 \wedge dx_2 \wedge \ldots d\zeta_n \wedge dx_n$. Given that $\mathbb{H}_(p, [\zeta]) = \text{span}(\frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial \zeta_1}, ..., \frac{\partial}{\partial \zeta_n})$, we can see that $d\alpha$ is nondegenerate on it. Hence α is a contact form, and $(\mathbb{P}^*M, \mathbb{H})$ is contact a manifold. By the identification above $(\mathfrak{C}, \mathfrak{H})$ is also a contact manifold, and it is canonical.

Example 2.1.5. Let $M = \mathbb{R}^3$. Then $T^*M \cong \mathbb{R}^6$ and $\mathfrak{C} \cong \mathbb{P}^*M \cong \mathbb{R}^3 \times \mathbb{RP}^2$ is the canonical contact manifold. $\triangle$

Example 2.1.6. Let $M = S^1 \times S^1 \cong \mathbb{T}^2$. Then $T^*M \cong \mathbb{T}^2 \times \mathbb{R}^2$, and $\mathbb{P}^*M \cong \mathbb{T}^2 \times \mathbb{RP}^1$ is the canonical contact manifold. $\triangle$

Now one could ask if there exists a local classification for contact structures similar to symplectic manifolds. To show this, we first define a technique that helps connect contact and symplectic structures directly: *symplectization*.

Theorem 2.1.7. *Let (M,H) be a $(2n-1)$-dimenisonal contact manifold with contact form α. Define $\tilde{M} = M \times \mathbb{R}$, with projection $\pi : \tilde{M} \to M$ and τ the additional coordinate on $\mathbb{R}$. Then $\omega = d(e^\tau \pi^*\alpha)$ is symplectic on $\tilde{M}$, and $(\tilde{M}, \omega)$ is called the symplectization of (M, H).*

Proof. We have:

$$\begin{aligned} \omega &= d(e^\tau \pi^*\alpha) = e^\tau (d\tau \wedge \pi^*\alpha + \pi^*(d\alpha)) \\ \Rightarrow \omega^n &= e^{n\tau}(nd\tau \wedge \pi^*\alpha \wedge \pi^*(d\alpha)^{n-1} + \pi^*(d\alpha)^n) \\ &= ne^{n\tau}(d\tau \wedge \pi^*(\alpha \wedge (d\alpha)^{n-1})). \end{aligned}$$

As $\alpha \wedge (d\alpha)^{n-1}$ is a volume form on M, it is not hard to see that ω^n above is also a volume form on $\tilde{M}$. Hence $(\tilde{M}, \omega)$ is a symplectic manifold. $\square$

Now we prove the contact version of Darboux's Theorem.

Theorem 2.1.8. *Let (M,H) be a contact manifold and $p \in M$. Then there exists a neighborhood U of p with coordinates $(x_1, ..., x_n, y_1, ..., y_n, z)$ on which $\alpha = \sum_{i=1}^n x_i dy_i + dz$.*

Proof. Symplectize (M, H) and identify $M \cong M \times \{0\} \subset M \times \mathbb{R} = \tilde{M}$. As $d(e^\tau \pi^* \alpha)$ is symplectic on $\tilde{M}$, there exists a neighborhood $\tilde{U}$ of $(p, 0) \in \tilde{M}$ such that $d(e^\tau \pi^* \alpha) = \sum_{i=1}^n dx_i \wedge dy_i + d\tau \wedge dz = d(\sum_{i=1}^n x_i dy_i + \tau dz)$. Thus, by local exactness of closed forms, there exists a smaller neighborhood in $\tilde{U}$ such that $e^\tau \pi^* \alpha = \sum_{i=1}^n x_i dy_i + \tau dz + d\beta$ for some $v \in C^\infty(\tilde{M})$. It is clear from here that as there is no $d\tau$ term on the left side, β does not depend on τ. Moreover, restricting these forms to M where $\pi = id$ and $\tau = 0$, we get $\alpha = \sum_{i=1}^n x_i dy_i + d\beta$. We only need to now show that $(x_1, ..., x_n, y_1, ..., y_n, \beta)$ forms a coordinate system on $\pi(\tilde{U})$ containing p. It is sufficient to check that $\frac{\partial \beta}{\partial z} \neq 0$ for the Jacobian to be invertible. If we have otherwise, then $\frac{\partial}{\partial z} \in \ker\alpha$. But then we have that $d\alpha$ is degenerate on $\ker\alpha$, which contradicts it being a contact form. Hence $\frac{\partial \beta}{\partial z} \neq 0$, and we are done.[1] $\square$

We have now started to explore how symplectic manifolds have natural relationships to other structures (for instance, contact structure). Moving ahead, we shall continue in this direction, and now discuss the link between symplectic and complex geometry. This is usually the starting point for how modern-day symplectic geometry is studied.

Chapter 3

Almost Complex Structure

3.1 Complex Structure

Definition 3.1.1. *A complex structure on a vector space V is a linear map $J : V \to V$ with $J^2 = -Id$.*

J above plays a role that is similar to multiplication by the complex number i, and hence gives the structure of a complex vector space to V. To establish a connection between symplectic and complex structures, we introduce the following notion of compatibility between the two: *A complex structure J on a symplectic vector space (V, Ω) is considered compatible if $G_J(u, v) := \Omega(u, Jv)$ is a positive inner product on V.* This essentially boils down to checking two conditions:

Lemma 3.1.2. J is Ω-compatible if and only if $\Omega(Ju, Jv) = \Omega(u, v)$ and $\Omega(u, Ju) > 0$ for all $u \neq 0$.

Proof. First note that J is bijective. As such, we can take (Ju, v) to an arbitrary element of $V \times V$. Then we have:

$$G_J(Ju, v) = G_J(v, Ju) \Leftrightarrow \Omega(Ju, Jv) = \Omega(v, -u) = \Omega(u, v)$$

and $G_J(u, u) > 0$ for all $u \neq 0 \Leftrightarrow \Omega(u, Ju) > 0$ for all $u \neq 0$. □

Now the question arises as to whether such a compatible complex structure J exists for every symplectic vector space (V, Ω). It is not hard to see that it does: Take $e_1, ..., e_n, f_1, ..., f_n$ to be the canonical symplectic basis, and define $J(e_i) = f_i$ and $J(f_i) = -e_i$. This is Ω-compatible. In fact, given the first property in Lemma 3.1.2, the converse is true as well: *If J is Ω-compatible, then there exists a symplectic basis of the form $e_1, ..., e_n, Je_1, ..., Je_n$.* Given any complex structure J, this allows us to construct an Ω

such that J is Ω-compatible.

There is also a canonical (basis-independent) method of constructing a Ω-compatible complex structure J. Let G be any positive inner product on (V, Ω). Given that $\Omega(u, ;)$ and $G(u, ;)$ are isomorphism from V to V^*, there exists an operator A on V such that $\Omega(u, v) = G(Au, v)$. One can then perform polar decomposition on $A = \sqrt{AA^*}J$, where J turns out to be our desired complex structure.[2] The canonical nature of this construction implies that for smooth family of symplectic vector spaces (V_t, Ω_t), we may construct a corresponding smooth family of Ω_t-compatible complex structures J_t.

We now explore the connections between complex and symplectic structures on vector spaces. Let J be a compatible complex structure on a symplectic vector space (V, Ω). Let $e_1, ..., e_n, Je_1, ..., Je_n$ be the symplectic basis mentioned above. Considering V as vector space over $\mathbb{C}$ through J, we can see that this new vector space $V^{\mathbb{C}}$ has basis $e_1, ..., e_n$. Now let L be the matrix representation of an element of $GL(n, \mathbb{C})$. Let us separate the real and 'imaginary' parts of $L = X + J \circ Y (= X + iY)$ where $X, Y \in GL(n, \mathbb{R})$. This allows us to interpret L as an element of $GL(2n, \mathbb{R})$ acting on V as the original $\mathbb{R}$-vector space as follows: We have that $L(e_i) = X(e_i) + J(Y(e_i))$ and $L(J(e_i)) = J(L(e_i)) = J(X(e_i)) - Y(e_i)$, and so can be represented by the real $2n \times 2n$ matrix: $\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$. Note that a matrix of this form is orthogonal if and only if correspondingly $X + iY$ is unitary. This interpretation of $GL(n, \mathbb{C})$ allows us to study interactions between complex and symplectic structures.

Proposition 3.1.3. *The intersection of any two of Sp(2n), O(2n), GL(n,$\mathbb{C}$), is U(n) (identified as an element of GL(2n,$\mathbb{R}$) as above).*

Proof. It is clear from the last remark above that $GL(n, \mathbb{C}) \cap O(2n) = U(n)$. Let us compute the intersection of $Sp(2n)$ and $O(2n)$. Recall that $Sp(2n)$ is the set of symplectomorphisms of the standard symplectic euclidean space, and $O(2n)$ is the set of matrices L such that $L^{-1} = L^T$. In the symplectic basis, a matrix $A \in Sp(2n)$ must be such that $A^T \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix} A = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$. If A is additionally an orthogonal matrix, then we have $A^{-1} \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix} A = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$, which implies that the columns of A form an orthonormal basis $v_1, ..., v_n, w_1, ..., w_n$ such that $\begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix} v_i = w_i$ and

$\begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix} w_i = -v_i$, that is A is of the form $\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$. Moreover as $A^T A = Id$, we have $XX^T + YY^T = Id$ and $XY^T - YX^T = 0$. This gives us that $(X+iY)(X+iY)^\dagger = Id$. Hence A is unitary. The reverse inclusion follows similarly.

For a matrix $A \in Sp(2n) \cap GL(n, \mathbb{C})$, it is of the form $\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$ and $A^T \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix} A = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix} = A^{-1} \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix} A$. As $\begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$ is invertible, we get $AA^T = Id$. The reverse inclusion holds similarly. $\square$

3.2 Almost Complex Manifolds

Definition 3.2.1. *An **almost complex structure** on a manifold M is smooth field J of complex structures on the tangent spaces such that $J_x : T_x M \to T_x M$ and $J_x^2 = -Id$ for all $x \in M$. The pair (M, J) is called an **almost complex manifold.***

As was the case in symplectic vector spaces, we analogously have that an almost complex structure J on a symplectic manifold (M, ω) is considered *compatible* if g with $g_x : T_x M \times T_x M \to \mathbb{R}$ with $g(u, v) = \omega_x(u, J_x v)$ is a riemannian metric on M. After we have chosen any riemannian metric (which always exists), the existence of a compatible almost complex structure for any symplectic manifold follows from the canonical recipe discussed before. Its smoothness is also obvious due to coordinate independence.

By the recipe above, changing the choice of the riemannian metric may change the resulting compatible almost complex structure. How different can they be? The following proposition answers the question.

Theorem 3.2.2. *The set of all almost compatible complex structures on a symplectic manifold (M, ω) is path-connected.*

Proof. Consider any two almost compatible complex structures J_1 and J_2, with respective riemannian metrics g_1, g_2. Then $g_t = tg_1 + (1-t)g_2$ for $t \in [0, 1]$ is a family of riemannian metrics. Applying the canonical recipe of polar decomposition to (ω, g_t) gives a smooth family of almost compatible complex structures joining J_1 and J_2.[2] $\square$

At this point we have three structures defined on a symplectic manifold: a riemannian metric g, a symplectic form ω, and an almost complex structure J. It is easy to see that we can create any one using the other two: $g(u, v) = \omega(u, Jv)$, $\omega(u, v) = g(Ju, v)$, and $J(u) = \tilde{g}^{-}(\tilde{\omega}(u))$ where $\tilde{g}(u) = g(u, ;)$ and $\tilde{\omega}(u) = \omega(u, ;)$. Any triple (g, ω, J) that are related as above are considered a **compatible triple**. This remarkable three-way compatibility also allows us to show another result mirroring the above:

Theorem 3.2.3. *The set of all symplectic structures compatible with an almost complex manifold (M, J) is path-connected.*

Proof. Taking any two compatible symplectic forms ω_0 and ω_1, and corresponding riemannian metrics g_0 and g_1, define $\omega_t(u, v) = tg_0(Ju, v) + (1 - t)g_1(Ju, v)$. $\square$

We can further expand on the study of almost complex manifolds by considering the natural complexification of its tangent space $TM \otimes \mathbb{C}$, and extending J linearly as $J(v \otimes c) = (Jv) \otimes c$. This allows J to take on eigenvalues $+i, -i$, and partition the tangent space into the respective eigenspaces $T_{1,0} \oplus T_{0,1}$. It is not hard to see that $T_{1,0} = \{v \otimes 1 - Jv \otimes i | v \in TM\}$ and $T_{0,1} = \{v \otimes 1 + Jv \otimes i | v \in TM\}$. The elements of $T_{1,0}$ and $T_{0,1}$ are respectively called *J-holomorphic tangent vectors* and *J-antiholomorphic tangent vectors*. Similarly, the cotangent bundle can be complexified and split as $T^*M \otimes \mathbb{C} \cong T^{1,0} \oplus T^{0,1} = (T_{1,0})^* \oplus (T_{0,1})^*$. It is not hard to see again that $T^{1,0} = \{\zeta \otimes 1 - (\zeta \circ J) \otimes i | \zeta \in T^*M\}$ and $T^{0,1} = \{\zeta \otimes 1 + (\zeta \circ J) \otimes i | \zeta \in T^*M\}$.

Now we can define the set of *complex-valued forms* $\Omega^k(M; \mathbb{C})$. Define $\Lambda^{l,m} = \Lambda^l(T^{1,0}) \wedge \Lambda^m(T^{0,1})$, and then define $\Lambda^k(T^*M \otimes \mathbb{C}) = \oplus_{l+m=k}\Lambda^{l,m}$. Then define the set of forms of type (l, m) as $\Omega^{l,m}(M; \mathbb{C})$ =sections of $\Lambda^{l,m}$, and $\Omega^k(M; \mathbb{C}) = \oplus_{l+m=k}\Omega^{l,m}(M; \mathbb{C})$ = set of sections of $\Lambda^k(T^*M \otimes \mathbb{C})$. The exterior derivative $d : \Omega^k(M; \mathbb{C}) \to \Omega^{k+1}(M; \mathbb{C})$ as a map can have different projections onto different form types. Most notably used are $\partial = \pi^{l+1,m} \circ d : \Omega^{l,m}(M; \mathbb{C}) \to \Omega^{l+1,m}(M; \mathbb{C})$ and $\bar{\partial} = \pi^{l,m+1} \circ d : \Omega^{l,m}(M; \mathbb{C}) \to \Omega^{l,m+1}(M; \mathbb{C})$. Note that $d = \partial + \bar{\partial}$ on smooth real-valued functions $\Omega^{0,0}(M; \mathbb{C})$. And in the case $d = \partial + \bar{\partial}$, we have $0 = d^2 = \partial\bar{\partial} + \bar{\partial}\partial + \partial^2 + \bar{\partial}^2$. Considering the direct sum decomposition above, we get $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$. Similar to the de Rham complexes formed by the exterior derivative d, the operator $\bar{\partial}$ forms a differential complex that leads to the formation of **Dolbeault cohomology**, where $H^{l,m}_{Dolbeault}(M) := \dfrac{\text{ker } \bar{\partial} : \Omega^{l,m} \to \Omega^{l,m+1}}{\text{im } \bar{\partial} : \Omega^{l,m-1} \to \Omega^{l,m}}$.

All this said, we still have not explicitly unraveled any connections to **complex manifolds**, i.e, manifolds that are locally biholomorphic to $\mathbb{C}^n$. In fact, we shall now see that every complex manifold is almost complex, allowing us to utilize all that we have discussed above.

3.3 Complex Manifolds

Definition 3.3.1. *A **complex manifold** of dimension n is topological manifold M equipped with a complete complex atlas $\mathfrak{A} = \{(U_\alpha \subset M, V_\alpha \subset \mathbb{C}^n, \phi_\alpha : U\alpha \to V_\alpha)\}$ where $\cup_\alpha U_\alpha = M$ and ϕ_α are biholomorphisms. Moreover, the transition maps are also biholomorphic.*

Example 3.3.2. Consider n-dimensional projective space $\mathbb{CP}^n$. Let U_i be the open set defined by $z_i \neq 0$, and define the biholomorphism $\phi_i : U_i \to \mathbb{C}^n$ by $\phi_i([z_0, ..., z_n]) = (\frac{z_0}{z_i}, ..., \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, ..., \frac{z_n}{z_i})$. It is easy to check that the collection $\mathfrak{A} = \{U_i, \phi_i\}$ forms a smooth atlas.

Theorem 3.3.3. *Every complex manifold has a canonical almost complex structure.*

Proof. Let M be an n-dimensional complex manifold with complex coordinates $(z_1, ..., z_n)$. Writing each $z_j = x_j + iy_j$, we get $(x_1, ..., x_n, y_1, ..., y_n)$ as real coordinates of M. Thus the tangent space at any point $p \in M$ has a real-basis $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_n}\}$. We can locally define a complex structure $J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}$ and $J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i}$. The fact that this is well-defined on all of M boils down to utilizing the *Cauchy-Riemann Equations* on the biholomorphic transition maps between intersecting coordinate charts.[2] $\square$

How does the differential structure we defined on almost complex manifolds before look like for complex manifolds? First notice that the complexification of the tangent space, in this case, is just given by $T_pM \otimes \mathbb{C} = \mathbb{C}\text{-span}\{\frac{\partial}{\partial x_j}|_p, \frac{\partial}{\partial y_j}|_p\}$. Solving for the eigenspaces of J, we get $T_{1,0} = \mathbb{C}\text{-span}\{\frac{\partial}{\partial x_j}|_p - i\frac{\partial}{\partial y_j}|_p\}$ and $T_{0,1} = \mathbb{C}\text{-span}\{\frac{\partial}{\partial x_j}|_p + i\frac{\partial}{\partial y_j}|_p\}$. Thus the corresponding complexified tangent bundle $T^{1,0}$ is given by $\mathbb{C}$-span $\{dx_j + idy_j\}$ and $T^{0,1} = \mathbb{C}$-span $\{dx_j - idy_j\}$. Letting $\frac{\partial}{\partial z_j}|_p = \frac{\partial}{\partial x_j}|_p - i\frac{\partial}{\partial y_j}|_p$, $\frac{\partial}{\partial \bar{z}_j}|_p = \frac{\partial}{\partial x_j}|_p + i\frac{\partial}{\partial y_j}|_p$, $dz_j = dx_j + idy_j$ and $d\bar{z}_j = dx_j - idy_j$, we get $T_{1,0} = \mathbb{C}\text{-span}\{\frac{\partial}{\partial z_j}\}$, $T_{0,1} = \mathbb{C}\text{-span}\{\frac{\partial}{\partial \bar{z}_j}\}$, $T^{1,0} = \mathbb{C}\text{-span}\{dz_j\}$ and $T^{0,1} = \mathbb{C}\text{-span}\{d\bar{z}_j\}$. The latter gives us a suitable basis to deduce that the general (l, m)-form looks like $\sum_{|J|=l,|L|=m} b_{J,K} dz_J \wedge d\bar{z}_K$ where J and K are multi-indices. In fact, the existence of such a basis implies that $d = \partial + \bar{\partial}$ on all forms on complex manifolds, where $\partial\beta = \sum_j \frac{\partial\beta}{\partial z_j} dz_j$ and $\bar{\partial}\beta = \sum_j \frac{\partial\beta}{\partial \bar{z}_j} dz_j$.[2]

Is the converse true? That is, if $d = \partial + \bar{\partial}$ on all complex-valued forms on an almost complex manifold, is it necessarily complex? If it is, such a J is called *integrable.* It was proven in 1957 in the *Newlander-Nirenberg theorem* that this converse is true. In fact, they introduced a much more tangible condition as well: the *Nijenhuis tensor* $\mathfrak{N}(v, w) = [Jv, Jw] - J[v, Jw] - J[Jv, w] - [v, w]$ is identically zero if and only if J is integrable. We prove one side of this assertion.

Proposition 3.3.4. *Assume M is a complex manifold and J is its canonical complex structure. Then $\mathfrak{N}$ vanishes everywhere.*

Proof. Consider the coordinates $(z_1, ..., z_n)$ with $z_j = x_j + iy_j$. Then $dz_j|_p = dx_j|_p + idy_j|_p$, and the real and imaginary parts form a basis for the $2n$-dimensional real vector space T_p^*M. Moreover, note that dz_j is J-holomorphic ($dz_j \circ J = idz_j$) as $dz_j \in T^{1,0}$. If we evaluate $dz_j(\mathfrak{N}(v, w))$ we get:

$$\begin{aligned} dz_j(\mathfrak{N}(v, w)) &= [Jv, Jw]z_j - J[v, Jw]z_j - J[Jv, w]z_j - [v, w]z_j \\ &= Jv(Jwz_j) - Jw(Jvz_j) - i(v(Jwz_j) - Jw(vz_j)) \\ &\quad - i(Jv(wz_j) - w(Jvz_j)) - (v(wz_j) - w(vz_j)) \\ &= iJv(wz_j) - iJw(vz_j) + v(wz_j) + iJw(vz_j) - iJv(wz_j) - w(vz_j) - v(wz_j) + w(vz_j) \\ &= 0, \end{aligned}$$

where we have used the complex linearity of the map $f \mapsto vf$ to pull out the factors of i. As this holds for all j and points $p \in M$, and dz_j is a basis for T_p^*M, we must have $\mathfrak{N} \equiv 0$. □

Example 3.3.5. Assume the map $v \mapsto [v, w]$ is complex linear ($[Jv, w] = J[v, w]$). Then $\mathfrak{N}(v, w) = [Jv, Jw] - [Jv, Jw] - [-v, w] - [v, w] \equiv 0$, and J is integrable. △

Example 3.3.6. Assume M is a surface with an almost complex structure J. Then $\{v_p, J_pv_p\}$ forms a basis for any $v_p \in T_pM/\{0\}$. Consider then $\mathfrak{N}(v, v), \mathfrak{N}(Jv, Jv), \mathfrak{N}(v, Jv)$. All of them equal 0. Hence $\mathfrak{N} \equiv 0$, and J is integrable. Thus as any orientable surface is symplectic, and hence admits an almost complex structure, we can conclude that every orientable surface is a complex manifold. △

Note that in the proof of Proposition 3.3.3, the key point was the existence of *J-holomorphic functions* z_j whose differentials satisfied $dz_j \circ J = idz_j$. These naturally occur for complex manifolds, but are rare for almost complex manifolds. This is where we come to a crossroads. In studying symplectic manifolds with even non-integrable almost complex structures, the modern-day technique that was introduced by *Mikhael Gromov* utilizes *pseudo-holomorphic curves* that satisfy $df \circ i = J \circ df$. On the other, there is the study of symplectic manifolds that permit a complex manifold structure. Such a special class of manifolds, called *Kähler* manifolds, have a very rich contemporary study that we delve into next.

3.4 Kähler Manifolds

Definition 3.4.1. *A **Kähler manifold** is a symplectic manifold* (M, ω) *with an integrable compatible almost complex structure.*

The symplectic form ω is also then called the *Kähler form*. Also, now that we have a complex manifold structure on M, we have the same decomposition of the differential structure on M given as $\Omega^l(M;\mathbb{C}) = \oplus_{l+m=k}\Omega^{l,m}$, $d = \partial + \bar{\partial}$, and basis $\{dz_j, d\bar{z}_j\}$ for $\Omega^1(M;\mathbb{C})$. As ω is a 2-form, it must be of the form $\omega = \sum_{j,k} a_{jk} dz_j \wedge dz_k + b_{jk} dz_j \wedge d\bar{z}_k + c_{jk} d\bar{z}_j \wedge d\bar{z}_k$, for smooth complex-valued coefficients a_{jk}, b_{jk}, c_{jk}. As the canonical almost complex structure J is compatible, we must have $J^*\omega = \omega$. Using the fact that $J^* dz_j = i dz_j$ and $J^* d\bar{z}_k = -i d\bar{z}_k$, it is not hard to see that $a_{jk} = c_{jk} = 0$ for all j, k, and thus $\omega \in \Omega^{1,1}(M;\mathbb{C})$. Imposing the other conditions, that is, ω being closed, nondegenerate, and of course, real-valued, we get the following equivalent characterization for Kähler forms[2]:

Proposition 3.4.2. *Kähler forms* ω *are* ∂*- and* $\bar{\partial}$*-closed* $(1,1)$*-forms that, on a coordinate chart* $(U, z_1, ..., z_n)$ *are given by* $\omega = \frac{i}{2}\sum_{j,k} b_{jk} dz_j \wedge d\bar{z}_k$*, with* $b_{jk}(p)$ *being a positive-definite hermitian matrix for all* $p \in U$*.*

Example 3.4.3. The canonical Kähler structure is $(\mathbb{C}^n, \omega_0)$ with $\omega_0 = \frac{i}{2}\sum_j dz_j \wedge d\bar{z}_j$.△

Example 3.4.4. By Example 3.3.6, every Riemann surface, that is a one-dimensional complex manifold and two-dimensional orientable real manifold, is Kähler with the area form as the Kähler form. △

This definition also implies that given two Kähler forms ω_1 and ω_2, any convex combination $\omega_t = t\omega_1 + (1-t)\omega_2$ is symplectic as well (as convex combinations of positive-definite matrices is positive-definite). This gives us the following:

Theorem 3.4.5. *Let* M *be a compact complex manifold with Kähler forms* ω_1 *and* ω_2 *such that* $[\omega_1]_{\text{de Rham}} = [\omega_2]_{\text{de Rham}}$*. Then* (M, ω_1) *is symplectomorphic to* (M, ω_2)*, and thus both induce the same Kähler structure on* M*.*

Proof. Apply Moser Theorem coupled with the observation above. □

Similar to symplectic forms, one can ask if there is a canonical local characterization for Kähler forms. We delve into this next.

Definition 3.4.6. *Let* M *be an* n*-dimensional complex manifold. Then* $f \in C^\infty(M;\mathbb{R})$ *is **strictly plurisubharmonic** (s.p.s.h for short) if on each coordinate chart* $(U, z_1, ..., z_n)$*, the matrix* $\left[\frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}(p)\right]$ *is positive-definite for all* $p \in M$*.*

Proposition 3.4.7. *Let M be a complex manifold, and $f \in C^\infty(M;\mathbb{R})$ be s.p.s.h. Then the (1,1)-form $\omega = \frac{i}{2}\partial\bar{\partial} f$ is a Kähler form.*

Proof. We first show that it is closed.

$$d\omega = \frac{i}{2}(\partial + \bar{\partial})\partial\bar{\partial} f = \frac{i}{2}(\partial^2\bar{\partial} f + \bar{\partial}\partial\bar{\partial} f) = -\frac{i}{2}\partial\bar{\partial}^2 f = 0.$$

Moreover, in any coordinate chart $(U, z_1, ..., z_n)$, $\omega = \frac{i}{2}\sum_{j,k} b_{jk} dz_j \wedge d\bar{z}_k$ where $b_{jk} = \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}(p)$. By assumption of f being s.p.s.h, and Proposition 3.4.2, ω is clearly Kähler. □

Such a function f is often called a **Kähler potential**.

Example 3.4.8. Taking $f : \mathbb{C}^n \to \mathbb{R}$ with $f(z_1, ..., z_n) = \sum_i |z_i|^2$ to be the Kähler potential, the corresponding Kähler form is the standard symplectic form on $\mathbb{R}^{2n}$. △

In fact, it is a well-known result that every Kähler form is locally defined by a Kähler potential.

Theorem 3.4.9. *Let ω be a closed real-valued (1,1)-form on a complex manifold M and let $p \in M$. Then there exists a neighborhood U of p and $f \in C^\infty(U;\mathbb{R})$ such that, on U, $\omega = \frac{i}{2}\partial\bar{\partial} f$.*

We discuss now an important instance of a Kähler manifold, $\mathbb{CP}^n$, that connects complex geometry to algebraic geometry. For this, first consider $f : \mathbb{C}^n \to \mathbb{R}$ given by $f(z) = \log(|z|^2 + 1)$, where $|z|^2 = \sum_i z_i \bar{z}_i$ is the magnitude of vector $z = (z_1, ..., z_n)$.

Lemma 3.4.10. *f as defined above is s.p.s.h.*

Proof. We have $f(z) = \log(\sum_i z_i\bar{z}_i + 1)$. Computing $\dfrac{\partial^2 f}{\partial z_j \partial \bar{z}_k} = \dfrac{\delta_{jk}(|z|^2+1) - z_k\bar{z}_j}{(|z|^2+1)^2}$. To check that this is positive definite, we check that $v^\dagger \left[\dfrac{\partial^2 f}{\partial z_j \partial \bar{z}_k}\right] v > 0$ for all non-zero $v \in \mathbb{C}^n$. This reduces to showing that $\sum_{j,k} \bar{v}_j \dfrac{\partial^2 f}{\partial z_j \partial \bar{z}_k} v_k > 0$.

$$\begin{aligned}\sum_{j,k} \bar{v}_j \frac{\delta_{jk}(|z|^2+1) - z_k\bar{z}_j}{(|z|^2+1)^2} v_k &= \frac{1}{(|z|^2+1)^2}(|v|^2|z|^2 + |v|^2 - \sum_{j,k} \bar{v}_j z_k \bar{z}_j v_k) \\ &= \frac{1}{(|z|^2+1)^2}(|v|^2 + (|v|^2|z|^2 - |(v.z)|^2)).\end{aligned}$$

By the Cauchy-Schwarz inequality, the last term is non-negative, and hence the whole expression is positive. □

This implies that the form $\omega_{FS} = \frac{i}{2}\partial\bar{\partial}f$ is a Kähler form. This form is known as the *Fubini-Study form* on $\mathbb{C}^n$. We show that this form, under pullback by the coordinate charts on $\mathbb{CP}^n$, forms global Kähler form on $\mathbb{CP}^n$. Recall now the complex manifold structure of $\mathbb{CP}^n$ from Example 3.3.2.

Proposition 3.4.11. *Let ω_{FS} be the Fubini-Study form on $\mathbb{C}^n$. Then the form $\tilde{\omega}_{FS}$ on $\mathbb{CP}^n$ defined by $\phi_i^*\omega_{FS}$ in each U_i is well-defined and Kähler.*

Proof. First, we argue it is well-defined. For this we need to show that on the intersection of any two coordinated charts U_i and U_j, we have that $\phi_i^*\omega_{FS} = \phi_j^*\omega_{FS}$. Consider the transition map ϕ_{01} between the two coordinate charts U_0 and U_1, given by $\phi_1 \circ \phi_0^{-1} = \phi_{01} : \phi_0(U_0 \cap U_1) \to \phi_1(U_0 \cap U_1)$ given by $\phi_{01}(z_1, ..., z_n) = \frac{1}{z_1}(1, z_2, ..., z_n)$. Note that $\phi_0(U_0 \cap U_1) = \phi_1(U_0 \cap U_1) = \{(z_1, ..., z_n) \in \mathbb{C}^n | z_1 \neq 0\}$. We calculate $\phi_{01}^*\omega_{FS} = \omega FS$. For this, first consider $\phi_{01}^* f$, where f is as defined above. We can see that:

$$\begin{aligned}\phi_{01}^* f(z) = \log(|f(z)|^2 + 1) = \log\left(\frac{1}{|z_1|^2}(1 + \sum_{i\geq 2}|z_i|^2) + 1\right) &= \log\left(\frac{1}{|z_1|^2}(1 + \sum_{i\geq 2}|z_i|^2 + z_1^2)\right) \\ &= \log(|z|^2 + 1) - \log(z_1) - \log(\bar{z_1}).\end{aligned}$$

Thus $\partial\bar{\partial}\phi_{01}^* f = \partial\bar{\partial}f$. Similar to the exterior derivative, it is not hard to see that the pullback commutes with ∂ and $\bar{\partial}$. Thus, $\phi_{01}^*\partial\bar{\partial}f = \partial\bar{\partial}f$ and $\phi_{01}^*\omega_{FS} = \omega_{FS}$. This then implies that $\phi_0^*\omega_{FS} = \phi_1^*\omega_{FS}$ on $U_0 \cap U_1$. This argument can be generalized to any intersecting U_i and U_j, to give that $\tilde{\omega}_{FS}$ is well-defined.
Also, $\tilde{\omega}_{FS}$ is a (1,1)-form that is, by commutation of the pullback with ∂ and $\bar{\partial}$, also closed. Moreover, because of how the Jacobian of each ϕ_i is the identity, we have that $\phi_i^* f$ is also s.p.s.h, which implies $\phi_i^*\omega_{FS}$ are Kähler forms. These glue together at the intersecting coordinate charts and give $\mathbb{CP}^n$ a total Kähler structure $\tilde{\omega}_{FS}$, called the Fubini-Study structure of complex projective space. □

Example 3.4.12. We can derive the local expression for the Fubini-Study form $\tilde{\omega}_{FS}$ for $\mathbb{CP}^1$ neatly. Let us do it for U_0, fixing $z_0 = 1$ and letting z_1 denote the local coordinate. Firstly, the Fubini-Study form on $\mathbb{C}^1$ is given by $\omega_{FS} = \frac{i}{2}\frac{1}{(|z|^2+1)}dz \wedge d\bar{z}$ using the working from Lemma 3.4.8. The pullback $\phi_0^*\omega_{FS}$ only replaces the coordinate z with z_1. Separating into the real and imaginary parts of the complex coordinate, we get:

$$\tilde{\omega}_{FS} = \frac{i}{2}\frac{(dx + idy) \wedge (dx - idy)}{(x^2 + y^2 + 1)^2} = \frac{i}{2}\frac{-2idx \wedge dy}{(x^2 + y^2 + 1)^2} = \frac{dx \wedge dy}{(x^2 + y^2 + 1)^2}.$$

One can even compute the area of $\mathbb{CP}^1$ w.r.t $\tilde{\omega}_{FS}$. Note that $U_0 \cup \{\infty\} = \mathbb{CP}^1$, hence it suffices to calculate the area of U_0:

$$\int_{\mathbb{R}^2} \frac{dx \wedge dy}{(x^2+y^2+1)^2} = \int_0^{2\pi} \int_0^{\infty} \frac{r dr \wedge d\theta}{(r^2+1)^2} = \pi.$$

$\triangle$

This ends our exploration of the structural connections between symplectic and complex geometry. Many questions remained unanswered here, including, for example, whether every manifold with a symplectic and complex structure has a Kähler structure. Does every symplectic manifold admit some compatible complex manifold structure? Does every complex manifold admit some compatible symplectic structure? And then there are some **Hodge theoretic** questions regarding the Kähler structure of complex projective space. Some of these questions have answers, while others do not at this point in time. Either way, it is an important area of research for those working in symplectic or complex geometry, mathematical physics, and algebraic geometry.

Transition: We will now significantly change gears. These first 3 chapters have given us an overview of the different types of differential and geometric structures that relate to symplectic manifolds. Now we shift to studying symplectic structures through functions and group actions defined on the manifolds themselves. More specifically, we will explore the *dynamics* on a symplectic manifold through its vector fields.

Chapter 4

Prelude to Hamiltonian Geometry

4.1 Hamiltonian vector fields

Definition 4.1.1. *A vector field X on a symplectic manifold (M, ω) is called* ***symplectic*** *if $i_X\omega$ is closed. If $i_X\omega$ is additionally exact, X is called* ***Hamiltonian****.*

Note that every symplectic vector field preserves the symplectic structure along its flows: $\mathfrak{L}_X\omega = di_X\omega + i_X d\omega = 0$. In particular, $\rho_t^*\omega = \omega$ for all t, where ρ_t is the flow of X. Moreover, for a Hamiltonian vector field X_H, the corresponding function $H \in C^\infty(M)$ such that $i_X\omega = dH$ is called a *Hamiltonian function* of X_H. This function is also preserved by the flow of X_H.

Example 4.1.2. $X_h = \dfrac{\partial}{\partial\theta}$ is a Hamiltonian vector field on $(S^2, \omega = d\theta \wedge dh)$, with h being the height function on S^2. The flow of X_h are rotations about the vertical axis, which preserves h. △

Example 4.1.3. Consider a particle moving in $\mathbb{R}^n$. To study its trajectory in the phase space $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$ with coordinates (p, q) being the respective position and momentum, we must have that the trajectory preserve an *energy function* $H : \mathbb{R}^{2n} \to \mathbb{R}$. That is, the trajectory $c(t) = (p(t), q(t))$ must be an integral curve of the Hamiltonian vector field induced by H. With the standard symplectic form $\omega = \sum_i dp_i \wedge dq_i$ on the phase space, this boils down exactly to solving Hamilton's equations:

$$\begin{aligned} \frac{dq_i(t)}{dt} &= \frac{\partial H}{\partial p_i} \\ \frac{dp_i(t)}{dt} &= -\frac{\partial H}{\partial q_i} \end{aligned}$$

△

It is also obvious by local exactness of differential forms, that every symplectic vector field is locally Hamiltonian. That said, is it possible to do better and produce a global Hamiltonian vector field from the symplectic vector field? Yes.

Proposition 4.1.4. *If X and Y are symplectic vector fields on a symplectic manifold (M, ω), then $[X, Y]$ is a Hamiltonian vector field with Hamiltonian function $\omega(Y, X)$.*

Here, $W = [X, Y]$ is the *Lie bracket* of the two vector fields X and Y, i.e, the unique vector field W satisfying $\mathfrak{L}_W f = \mathfrak{L}_X(\mathfrak{L}_Y f) - \mathfrak{L}_Y(\mathfrak{L}_X f)$ for all $f \in C^\infty(M)$. The proof of the above requires a small result about Lie brackets:

Lemma 4.1.5. *For any form α, $i_{[X,Y]}\alpha = \mathfrak{L}_X i_Y \alpha - i_Y \mathfrak{L}_X \alpha$.*

Proof. The proof of this proceeds in a similar way to the Cartan magic formula: as both sides are antiderivations in the exterior algebra of differential forms, we only need to show equality for α a 0-form and an exact 1-form. For $\alpha \in C^\infty(M)$, both sides are zero. For $\alpha = df$ for where $f \in C^\infty(M)$, we have on the LHS:

$$\begin{aligned} i_{[X,Y]}\alpha = i_{[X,Y]}df = df([X,Y]) = \mathfrak{L}_{[X,Y]}(f) &= \mathfrak{L}_X(\mathfrak{L}_Y f) - \mathfrak{L}_Y(\mathfrak{L}_X f) \\ &= \mathfrak{L}_X(i_Y df) - \mathfrak{L}_Y(i_X df) \\ &= \mathfrak{L}_X(i_Y \alpha) - i_Y d i_X df = \mathfrak{L}_X(i_Y \alpha) + i_Y(\mathfrak{L}_X \alpha). \end{aligned}$$

□

Now it becomes clear that $i_{[X,Y]}\omega = \mathfrak{L}_X i_Y \omega - i_Y \mathfrak{L}_X \omega = d i_X i_Y \omega + i_X d i_Y \omega - i_Y d i_X \omega - i_Y i_X d\omega = d\omega(Y, X)$, assuming X and Y are symplectic. This proves the proposition above. It is not hard to see that the Lie bracket operation on the vector space of vector fields is a bilinear, anti-symmetric form that satisfies the property: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for any $X, Y, Z \in \chi(M)$. The last property is known as the *Jacobi identity*, and the Lie bracket makes $\chi(M)$ into what is called a *Lie algebra*. In fact, $\chi^{\text{symp}}(M)$ is a *Lie subalgebra* (closed under the Lie bracket) of $\chi(M)$, and $\chi^{\text{ham}}(M)$ is Lie subalgebra of $\chi^{\text{symp}}(M)$. We shall see later that this Lie algebraic structure is imperative to study Hamiltonian geometry.

4.2 Lie Group Actions

Definition 4.2.1. *A Lie group is smooth manifold G that is also a group under some smooth multiplication $\cdot : G \times G \to G$, with a smooth inverse map $g \mapsto g^{-1}$.*

Example 4.2.2. $\mathbb{R}$ is a Lie group under standard addition. △

Example 4.2.3. A lot of matrix groups are Lie groups under matrix multiplication: $GL(V)$, $O(n)$, $U(n)$, $SO(n)$, $SU(n)$. $\triangle$

For any Lie group G and $g \in G$, we define the smooth left-multiplication map $L_g : a \mapsto ga$. A vector field X on G is *left-invariant* if $(L_g)_* X = X$ for all $g \in G$, i.e, $X_{ga} = (L_g)_{*,a} X_a$ for all $g, a \in G$. In fact, it can be checked that the set of left-invariant vector fields $\mathfrak{g}$ on G is a vector space that is closed under the standard Lie bracket defined before, and thus, forms a Lie algebra. In fact, it can be naturally identified with T_eG (where e is the identity element of G) through $v \in T_eG \leftrightarrow X \in \chi(M)$ such that $X_g = (L_g)_* v$ for all $g \in G$. Thus, the Lie algebra $\mathfrak{g} \cong T_eG$ can be associated with any Lie group G.

The Lie algebra $\mathfrak{g}$ above is one of the most well-studied objects in Lie theory. As seen above, it has multiple ways of being identified and interpreted, and we introduce two more standard ones: the *adjoint* and *coadjoint representation* of $\mathfrak{g}$ over itself. Consider the conjugation map $C_g : G \to G$ given as $a \mapsto gag^{-1}$. The differential of this map at e can be labelled as $Ad_g : T_eG \to T_eG$, and is an automorphism. Under the identification, Ad_g is an automorphism of $\mathfrak{g}$. Moreover, we have the following result which allows us to retrieve the *adjoint representation* of G through the homomorphism $Ad : G \to GL(\mathfrak{g})$ with $g \mapsto Ad_g$ (follows from $C_g \circ C_h = C_{gh}$). We can similarly obtain a *coadjoint representation* by defining $Ad^*_g \in GL(\mathfrak{g}^*)$ through $Ad^*_g(\zeta)(X) = \zeta(Ad_{g^{-1}}(X))$ for all $X \in \mathfrak{g}$, and $\zeta \in \mathfrak{g}^*$.

The group structure of Lie groups allows us to define their action on a manifold M through the corresponding diffeomorphism group Diff(M), where $g \in G \mapsto \psi_g \in$Diff$(M)$ is a homomorphism, and $g \,.\, p = \psi_g(p)$. Let us look at a detailed example of this:

Example 4.2.4. Let $\mathfrak{H}$ be the space of $n \times n$ complex hermitian matrices, with a $U(n)$-action by conjugation defined on it: $A \,.\, \zeta = A\zeta A^{-1}$, where $\zeta \in \mathfrak{H}$ and $A \in U(n)$. Clearly, by the spectral theorem, $A, B \in \mathfrak{H}$ lie in the same orbit iff they have the same eigenvalues. Thus, the orbits of this action are the manifolds H_λ consisting of all $n \times n$ complex hermitian matrices with spectrum $\lambda \in \mathbb{R}^n$.
We can also talk about the stabilizer of an element of H_λ. If λ consists of k distinct entries with the ith distinct eigenvalue being repeated n_i times, assume that $\lambda = (\lambda_1, \lambda_1, ..., \lambda_k, \lambda_k)$ is ordered such that the repititions are grouped. Consider now the diagonal matrix $[\Lambda]_{ii} = \lambda_i$, with $\Lambda \in \mathfrak{H}_\lambda$. For any unitary basis transformation such that $U^\dagger \Lambda U = \Lambda$, it is clear that the rows of U must consist of the eigenvectors of Λ (in this case, these are just your standard basis vectors). Moreover, the ith row eigenvector must correspond to the eigenvalue Λ_{ii}. This implies that the matrix U is block diagonal,

consisting of k blocks corresponding to each eigenvalue. Moreover, the ith block is of the form of an $n_i \times n_i$ permutation matrix, with entries being complex numbers z with $|z| = 1$ (unitary condition).

$$\begin{pmatrix} P_{n_1} & & & \\ & P_{n_2} & & \\ & & \ddots & \\ & & & P_{n_k} \end{pmatrix}$$

Thus $\mathrm{Stab}_{U(n)}(\Lambda)$ consists of all unitary matrices of the form above. Now for any other arbitrary matrix $H \in \mathfrak{H}_\lambda$, we have that $H = A\Lambda A^{-1}$ for some $A \in U(n)$. It is easy to see that this implies that $\mathrm{Stab}_{U(n)}(H) = A\mathrm{Stab}_{U(n)}(\Lambda)A^{-1}$.

This action by conjugation of the Lie group $U(n)$ also allows us to utilize the Lie algebra $T_I U(n) = \mathfrak{u}(n) \cong i\mathfrak{H}$ of skew-symmetric matrices to induce corresponding vector fields on $\mathfrak{H}$. Consider any $X \in \mathfrak{u}(n)$, and $\zeta \in \mathfrak{H}$. Define the map $\phi_\zeta : U(n) \to \mathfrak{H}$ by $U \mapsto U\zeta U^\dagger$. Let $c : \mathbb{R} \to U(n)$ such that $c(0) = I$ and $c'(0) = X$. Then $\phi_\zeta \circ c(t) = c(t)\zeta c(t)^\dagger$, and $(\phi_\zeta \circ c)'(t) = c(t)^\dagger \zeta c'(t) - c(t)^\dagger c'(t) c(t)^\dagger \zeta c(t)$. Thus $(\phi_\zeta \circ c)'(0) = \phi_{\zeta_*}(X) = X\zeta - \zeta X = [X, \zeta]$. Thus the corresponding vector field on $\mathfrak{H}$ is given by $\tilde{X}_\zeta = [X, \zeta]$.

Moreover, $\mathfrak{H}_\lambda$ carries a particular manifold structure. Note that for any $H \in \mathfrak{H}_\lambda$, $H = U^\dagger \Lambda U$ for some $U \in U(n)$. Thus, H is primarily defined by its corresponding unitary matrix U of eigenvectors. More generally, for possibly repeating eigenvalues as before, it is defined by the set of corresponding orthogonal eigenspaces. This allows for a natural identification with the manifold structure of $\Pi_{i=1}^{k} \mathrm{Gr}_{n_i}(\mathbb{C}^{n-s_i})$, where $s_i = \sum_{j=1}^{i} n_j$. $\triangle$

4.3 Moment Maps

We can recast the idea of Hamiltonian and symplectic vector fields in the language of Lie group actions. In particular, for any complete vector field X on M, the corresponding flow $\rho : t \in \mathbb{R} \mapsto \rho_t$ gives an $\mathbb{R}$-action on M. Moreover, assuming the appropriate smoothness of any $\mathbb{R}$-action on a manifold M, one can integrate to get a complete vector field that induces it. If the vector field is either symplectic or Hamiltonian (or equivalently, ρ_t is a symplectomorphism or Hamiltonian diffeomorphism for all t), the corresponding action is called symplectic or Hamiltonian respectively.

Can the above definition of symplectic and Hamiltonian actions be generalized to arbitrary Lie group actions? Note that the definition of a symplectic action can generalized, but the Hamiltonian action is dependent on the existence of an integrable vector field associated with the one-parameter family of diffeomorphisms. A well-defined analog of this is required for the case when the Lie group action is not an $\mathbb{R}$-action. For this, we introduce the idea of *moment maps.*

Definition 4.3.1. *A symplectic action* $\psi : G \to \mathrm{Sympl}(M)$ *is Hamiltonian if there exists a map* $\mu : M \to \mathfrak{g}^*$ *satisfying:*
(i) For each $X \in \mathfrak{g}$*, set* $\mu^X : p \in M \mapsto \mu_p(X) \in \mathbb{R}$ *and* $\tilde{X} \in \chi(M)$ *as being generated by* $\{\exp\,(tX) | t \in \mathbb{R}\}$*. Then* μ^X *is Hamiltonian in the usual sense for* $\tilde{X}$*.*
(ii) $\mu \circ \psi_g = \mathrm{Ad}_g^* \circ \mu$ *for all* $g \in G$*. (equivariance condition)*
Then (M, ω, G, μ) *is called a* ***Hamiltonian G-space*** *and* μ *is a* ***moment map****.*

Let us try and make sense of the definition above and see how it applies for $\mathbb{R}$-actions. For $G = \mathbb{R}$, $\mathfrak{g} \cong \mathfrak{g}^* \cong \mathbb{R}$. Then for the standard basis vector $1 \in \mathfrak{g}$, it generates the constant left-invariant vector field $X = 1$ on G. The integral curve through 0 of X is given by $\exp(tX) = t$. Then $\tilde{X}$ is the standard vector field generated by the $\mathbb{R}$-action that we considered before. Moreover, $\mu^1(p) = \mu_p(1) = \mu_p \Rightarrow \mu^1 = \mu$. Thus, according to (i), we must have that $d\mu = i_{\tilde{X}}\omega$, for some $\mu \in C^\infty(M)$. Similarly, as $\mathrm{Ad}_g^* = id$ for $g \in \mathbb{R}$, condition (ii) becomes $\mu \circ \psi_g = \mu \Leftrightarrow \mathfrak{L}_{\tilde{X}}\mu = 0$. This is the standard invariance property of Hamiltonian actions. Thus, we recover our criteria for Hamiltonian actions for $G = \mathbb{R}$. Note that the primary idea of this definition was to find a suitable $\mathbb{R}$-analog for the action of any general Lie group G. This is done by choosing an integral curve in G, dependent on an initial choice of $X \in \mathfrak{g}$, and considering the one-parameter family of diffeomorphisms it represents.

Example 4.3.2. Let $\omega = \sum_i r_i dr_i \wedge d\theta_i$ be the standard symplectic form on $\mathbb{C}^n$ expressed in polar coordinates. Consider the $\mathbb{T}^n$-action given by $\psi_\alpha(z) = (e^{ik_1\alpha_1}z_1, ..., e^{ik_n\alpha_n}z_n)$, where $\alpha \in \mathbb{T}^n$. In polar coordinates, this amounts to $(r_i, \theta_i) \mapsto (r_i, \theta_i + k_i\alpha_i)$. Clearly, $\psi_\alpha^*\omega = \sum_i r_i dr_i \wedge d(\theta_i + k_i\alpha_i) = \omega$, and thus the action is symplectic. We also claim that $\mu : \mathbb{C}^n \to \mathbb{R}^n$ with $\mu(z) = -\frac{1}{2}(k_1r_1^2, ..., k_nr_n^2) + c$ is a moment map for the action for any $c \in \mathbb{R}^n$. For $X = \sum_i a_i \frac{\partial}{\partial \theta_i} \in (\mathfrak{t}^n)^* \cong \mathbb{R}^n$, $\mu^X = -\sum_i \frac{1}{2} k_i r_i^2 a_i$, and $\exp(tX) = t(a_1, ..., a_n)$, thus giving $\tilde{X} = \sum_i k_i a_i \frac{\partial}{\partial \theta_i}$. We can see that $i_{\tilde{X}}\omega = d\mu^X$. Moreover, μ is clearly invariant under the action of $\mathbb{T}^n$, thus satisfying the two conditions above. The action is Hamiltonian. $\triangle$

Example 4.3.3. Consider the natural action of $U(n)$ on $\mathbb{C}^n$ with the standard symplectic form $\omega = \sum_j dx^j \wedge dy_j$. This action is symplectic by Proposition 3.1.3. We claim that $\mu(z) = \frac{i}{2}zz^* \in \mathfrak{u}(n)$ is a *comoment map.* This comoment map gives us a natural moment map through the identification $\mathfrak{u}(n) \cong \mathfrak{u}^*(n)$ using the inner product $(A, B) = \mathrm{tr}(A^*B)$. Let us check this. First note that the Lie algebra $\mathfrak{u}(n)$ is the space of $n \times n$ skew-hermitian matrices $X = V + iW$ where $V = -V^T$ and $W = W^T$ are real matrices.

This induces a natural vector field $\tilde{X}_z = Xz$ on $\mathbb{C}^n$. Letting $z_k = x_k + iy_k$, we get that $\tilde{X}_z = (Vx - Wy) + i(Vy + Wx)$. Consequently:

$$i_{\tilde{X}}\omega = \sum_i (Vx - Wy)_i dy_i - (Vy + Wx)_i dx_i = \sum_{i,j} (V_{ij}x_j - W_{ij}y_j)dy_i - (V_{ij}y_j + W_{ij}x_j)dx_i$$

On the other hand:

$$\begin{aligned}\mu^X(z) = \frac{i}{2}\mathrm{tr}(zz^*X) = \frac{i}{2}z^*Xz &= \frac{i}{2}(x^T - iy^T)(V + iW)(x + iy) \\ &= \frac{i}{2}(x^TVx + ix^TVy + ix^TWx - x^TWy - iy^TVx + y^TVy + y^TWx + iy^TWy)\end{aligned}$$

Using the fact that each term is scalar, and hence symmetric, we can set some terms to zero and group others:

$$\begin{aligned}\mu^X(z) &= y^TVx - \frac{1}{2}x^TWx - \frac{1}{2}y^TWy \\ &= \sum_{i,j} y_iV_{ij}x_j - \frac{1}{2}x_iW_{ij}x_j - \frac{1}{2}y_iW_{ij}y_j,\end{aligned}$$

where the first term can be rewritten as $-y_jV_{ij}x_i$ by reordering of indices and antisymmetry of V. After using this along with the symmetry of W, it is not hard to see that $d\mu^X = i_{\tilde{X}}\omega$. Moreover, equivariance of μ is follows as $\mu \circ \psi_U = \frac{i}{2}Uzz^*U^* = \frac{i}{2}Uzz^*U^{-1} = \mathrm{Ad}_U \circ \mu$. Thus, the action is Hamiltonian. △

We also have the following useful lemma regarding moment maps:

Lemma 4.3.4. *Let G have a Hamiltonian action on (M_1, ω_1) and (M_2, ω_2) with moment maps μ_1 and μ_2. Extend the action to $(M_1 \times M_2, \omega = \mathrm{pr}_1^*\omega_1 + \mathrm{pr}_2^*\omega_2)$ by $\psi_g(p_1, p_2) = ((\psi_1)_g(p_1), (\psi_2)_g(p_2))$. Then $\mu = \mu_1 \circ \mathrm{pr}_1 + \mu_2 \circ \mathrm{pr}_2$ is a moment map for this action.*

Proof. For any $X \in \mathfrak{g}$, as $T_p(M_1 \times M_2) \cong T_p(M_1) \times T_p(M_2)$, and $\mathrm{pr}_i \circ \psi_g = (\psi_i)_g \circ \mathrm{pr}_i$ we have that $(\mathrm{pr}_i)_*(\tilde{X}) = \tilde{X}_i$, where $\tilde{X}_i$ is the vector field generated by the individual actions. Thus we have:

$$\begin{aligned}i_{\tilde{X}}\omega = i_{\tilde{X}}(pr_1^*\omega_1 + pr_2^*\omega_2) &= \omega_1(pr_{1_*}\tilde{X}, ;) + \omega_2(pr_{2_*}\tilde{X}, ;) \\ &= \omega_1(\tilde{X}_1, ;) + \omega_2(\tilde{X}_2, ;) \\ &= i_{\tilde{X}_1}\omega_i + i_{\tilde{X}_2}\omega_2 = d\mu_1 + d\mu_2 = d\mu.\end{aligned}$$

Moreover, as Ad_g^* is linear, we also have $\mu \circ \psi_g = \mu_1 \circ \mathrm{pr}_1 \circ \psi_g + \mu_2 \circ \mathrm{pr}_2 \circ \psi_g = \mu_1 \circ (\psi_1)_g \circ \mathrm{pr}_1 + \mu_2 \circ (\psi_2)_g \circ \mathrm{pr}_2 = (\mathrm{Ad}_g^* \circ \mu_1) \circ \mathrm{pr}_1 + (\mathrm{Ad}_g^* \circ \mu_2) \circ \mathrm{pr}_2 = \mathrm{Ad}_g^* \circ \mu$. □

Example 4.3.5. Consider the natural action of $U(k)$ on $(\mathbb{C}^{k\times n}, \omega_0)$. Note that this action can be identified with n copies of the action of $U(k)$ on $\mathbb{C}^k$ as in Example 4.3.3. Thus, by the lemma above, the comoment map μ for this action can be given as $\mu(A) = \frac{i}{2}\sum_{j=1}^{n} z_j z_j^* = \frac{i}{2}AA^*$, where $z_i \in \mathbb{C}^k$ are the columns of $A \in \mathbb{C}^{k\times n}$. △

All of this said, it is still unclear how exactly to find moment maps, and whether they are unique. This we will have a brief look into now.

4.4 Uniqueness and Existence of Moment Maps

It is clear that if $H^1_{\text{de Rham}}(M) = 0$, then $\chi^{ham}(M) = \chi^{symp}(M)$. Motivated by this, we define the *Chevalley cohomology*, which helps us measure the the more general *Hamiltonianness* of a symplectic action.

Definition 4.4.1. *Let C^k be the set of alternating k-linear maps on $\mathfrak{g}$ and define $\delta : C^k \to C^{k+1}$ given by $\delta c(X_0, ..., X_k) = \sum_{i<j}(-1)^{i+j}c([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ...X_k)$. Then the* ***Chevalley cohomology groups*** *are defined by $H^k(\mathfrak{g}; \mathbb{R}) = \dfrac{ker\ \delta : C^k \to C^{k+1}}{im\ \delta : C^{k-1} \to C^k}$.*

Let us confirm if the above cohomology groups are well-defined by checking if $\delta^2 = 0$:

$$\begin{aligned}
\delta^2 c(X_0, ..., X_{k+1}) &= \sum_{i<j}(-1)^{i+j}\delta c([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ...X_{k+1}) \\
&= \sum_{i<j<m<l}(-1)^{i+j+m+l-2}c([X_k, X_l], [X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j ..., \hat{X}_m, ..., \hat{X}_l ... X_{k+1}) \\
&+ \sum_{i<m<j<l}(-1)^{i+j+m+l-1}c([X_k, X_l], [X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_m ..., \hat{X}_j, ..., \hat{X}_l ... X_{k+1}) \\
&+ \sum_{i<m<l<j}(-1)^{i+j+m+l}c([X_k, X_l], [X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_m ..., \hat{X}_l, ..., \hat{X}_j ... X_{k+1}) \\
&+ \sum_{m<i<j<l}(-1)^{i+j+m+l}c([X_k, X_l], [X_i, X_j], X_0, ..., \hat{X}_m, ..., \hat{X}_i ..., \hat{X}_j, ..., \hat{X}_l ... X_{k+1}) \\
&+ \sum_{m<i<l<j}(-1)^{i+j+m+l+1}c([X_k, X_l], [X_i, X_j], X_0, ..., \hat{X}_m, ..., \hat{X}_i ..., \hat{X}_l, ..., \hat{X}_j ... X_{k+1}) \\
&+ \sum_{m<l<i<j}(-1)^{i+j+m+l+2}c([X_k, X_l], [X_i, X_j], X_0, ..., \hat{X}_m, ..., \hat{X}_l ..., \hat{X}_i, ..., \hat{X}_j ... X_{k+1}) \\
&+ \sum_{i<j<l}(-1)^{i+j+l-1}c([[X_i, X_j], X_l], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., \hat{X}_l ... X_{k+1}) \\
&+ \sum_{i<l<j}(-1)^{i+j+l}c([[X_i, X_j], X_l], X_0, ..., \hat{X}_i, ..., \hat{X}_l, ..., \hat{X}_j ... X_{k+1}) \\
&+ \sum_{l<i<j}(-1)^{i+j+l+1}c([[X_i, X_j], X_l], X_0, ..., \hat{X}_l, ..., \hat{X}_i, ..., \hat{X}_j ... X_{k+1}).
\end{aligned}$$

It is not hard to see that the 1st and 6th, 2nd and 4th, 3rd and 5th terms cancel by the alternating property of c after switching indices $i \leftrightarrow m$ and $j \leftrightarrow l$. The last three terms cancel in the first argument by the Jacobi identity after cyclically permuting the indices $i \to j \to l$. Thus $\delta^2 = 0$.
It can be shown that, in the case of compact and connected Lie groups, the Chevalley cohomology coincides with the De Rham Cohomology.[2] Moreover, we have the following main result:

Proposition 4.4.2. *If $H^1(\mathfrak{g};\mathbb{R}) = H^2(\mathfrak{g};\mathbb{R}) = 0$, then any symplectic G-action is Hamiltonian.[2]*

Moroever, it is not hard to see that $c \in \mathfrak{g}^*$ has $\delta c = 0$ if and only if c vanishes on the *commutator ideal* of $\mathfrak{g} = \{[X,Y]$ for all $X, Y \in \mathfrak{g}\}$. This condition of $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ defines a special class of Lie groups that are widely studied today, called *semisimple* Lie groups[3]. Moreover, it can be shown that moment maps are unique up to constants $c \in [\mathfrak{g},\mathfrak{g}]^0$, and thus completely unique when $H^1(\mathfrak{g};\mathbb{R}) = 0$[2].

A proper treatment of the ideas above requires a bit of abstract Lie theory that we will not be diverted by for now. Nevertheless, the main results have been stated, and we can thus finally begin to see the utility of generalizing the notion of a Hamiltonian action. This will be seen most clearly as we discuss ***symplectic reduction***, where the symmetries of the action allow for the consideration of a reduced space where information about the action is more succinct and simple.

4.5 Symplectic Reduction

Theorem 4.5.1. *Let (M, ω, G, μ) be Hamiltonian G-space for a compact Lie group G. Assume G acts freely on $\mu^{-1}(0)$. Then:*
(i) the orbit space $M_{red} = \mu^{-1}(0)/G$ is a manifold.
(ii) there is a symplectic form ω_{red} on M_{red} with $i^*\omega = \pi^*\omega_{red}$, where $i : \mu^{-1}(0) \to M$ is inclusion, and $\pi : \mu^{-1}(0) \to M_{red}$ is canonical projection.

We shall now sketch a proof of (ii) from the theorem above.

Lemma 4.5.2. *Let $\mathfrak{g}_p$ be the Lie algebra of the stabilizer of $p \in M$. Let $\mu_{*,p} : T_pM \to \mathfrak{g}^*$ be the differential at p of the corresponding moment map. Then* $\ker \mu_{*,p} = (T_p\mathfrak{O}_p)^{\omega_p}$ *and* $\operatorname{im} \mu_{*,p} = \mathfrak{g}_p^0$*, where $\mathfrak{O}_p$ is the G-orbit through p, and $\mathfrak{g}_p^0 \subset \mathfrak{g}^*$ is the annihilator of $\mathfrak{g}_p$.*

Proof. Recall that $d\mu = i_{\tilde{X}}\omega$ for all $X \in \mathfrak{g}$. This can be rewritten as $\omega_p(\tilde{X}_p, v) = \mu_{*,p}(v)(X)$ for all $X \in \mathfrak{g}$. Consider any $v \in (T_p\mathfrak{O}_p)^{\omega_p}$. As $\tilde{X}$ is generated by the action of G, we have that $\tilde{X}_p \in T_p\mathfrak{O}_p$ for all $p \in M$. Thus $\mu_{*,p}(v)(X) = \omega_p(\tilde{X}_p, v) = 0$ for all $X \in \mathfrak{g}$. Thus $\mu_{*,p}(v) = 0$. The other direction holds similarly to get ker $\mu_{*,p} = (T_p\mathfrak{O}_p)^{\omega}_p$. Now consider any $v \in T_pM$, and $X \in \mathfrak{g}_p$. Because of the triviality of the G-action, we have $\tilde{X} = 0$ and naturally $\mu_{*,p}(v)(X) = 0$ for all $X \in \mathfrak{g}_p$. Thus im $\mu_{*,p} \subset \mathfrak{g}_p^0$. Considering dimensions we have codim ker $\mu_{*,p}$ =dim $\mathfrak{O}_p \cong G/G_p$, which has dimension equal to the codimension of the stabilizer G_p. Similarly, $\mathfrak{g}_p^0$ has dimension equal to the codimension of the stabilizer G_p. Thus dim im $\mu_{*,p}$ =dim $\mathfrak{g}_p^0$, and we are done. □

Note that if the action of G on $\mu^{-1}(0)$ is free, then $\mu_{*,p}$ is surjective for all $p \in \mu^{-1}(0)$, and 0 is a regular values of μ. Thus $\mu^{-1}(0)$ is a closed submanifold of M with codimension equal to dimension of G. Moreover, $\mu_{*,p} = 0$ for all $p \in \mu^{-1}(0)$. This gives that ker $\mu_{*,p} = T_p\mu^{-1}(0)$. Moreover, by equivariance of μ, we have that $\mu(g \centerdot p) = Ad_g^*(\mu(p)) = 0$. This implies that $\mathfrak{O}_p \subset \mu^{-1}(0)$, and hence $T_p\mathfrak{O}_p$ is istropic. It is not hard to check that this then allows us to canonically define a symplectic structure ω_{red} on the quotient $T_p\mu^{-1}(0)/T_p\mathfrak{O}_p \cong T_{[p]}M_{red}$ equivalent to the one mentioned in (ii).[2] The symplectic manifold (M_{red}, ω_{red}) is called the ***symplectic quotient*** or ***reduced space*** of (M, ω).

Example 4.5.3. Consider the standard action of $U(k)$ on $\mathbb{C}^{k\times n}$ with moment map $\mu(A) = \frac{i}{2}AA^* - \frac{i}{2}\mathrm{Id}$. Then $\mu^{-1}(0) = \{A \in \mathbb{C}^{k\times n} | AA^* = \mathrm{Id}\}$. Note that $\mu^{-1}(0)$ is preserved under the action. Consider then the quotient $\mu^{-1}(0)/U(k)$. First note that any element in $\mu^{-1}(0)$ defined a k-dimensional subspace of $\mathbb{C}^n$ through its row space. Moreover, as any $U \in U(k)$ is invertible, UA has the same row space as $A \in \mathbb{C}^{k\times n}$. Thus $\mu^{-1}(0)/U(k) = \mathrm{Gr}(k, n)$. △

Example 4.5.4. Consider the usual rotational action of S^1-action on $(\mathbb{C}^{n+1}, \omega_0)$ defined before in Example 4.3.2. A corresponding moment map is $\mu(z) = \frac{-1}{2}|z|^2 + \frac{1}{2}$, and $\mu^{-1}(0) \cong S^{2n+1}$. Furthermore, consider the quotient S^{2n+1}/S^1. Any two elements in S^{2n-1} define the same 1-dimensional subspace iff they differ by a complex multiplicative factor of modulus 1, i.e, by e^{it} for some $t \in \mathbb{R}$. This gives us the natural diffeomorphism $M_{red} = \mu^{-1}(0)/S^1 \cong \mathbb{CP}^n$. Moreover, we can check that $\omega_{red} = \omega_{FS}$, the Fubini-Study form. Consider the natural projection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ given by $(z_0, z_1, ..., z_n) \mapsto [z_0 : z_1 : ... : z_n]$. Assume we are looking at this projection locally in a neighbourhood where $z_0 \neq 0$. Then we know that the Fubini-Study form ω_{FS} is locally defined by $\frac{i}{2}\partial\bar{\partial}\phi_0^*(\log(|z|^2 + 1))$ where $\phi_0([z_0 : z_1 : ... : z_n]) = \frac{1}{z_0}(z_1, ..., z_n)$. We can see now that:

$$\begin{aligned}\pi^*\omega_{FS} = \frac{i}{2}\partial\bar{\partial}\pi^*\phi_0^* \log(|z|^2 + 1) &= \frac{i}{2}\partial\bar{\partial} \log(|\phi_0(\pi(z))|^2 + 1) \\ &= \frac{i}{2}\partial\bar{\partial}(\log(|z|^2) - \log(z_0) - \log(\bar{z_0})) = \frac{i}{2}\partial\bar{\partial} \log(|z|^2).\end{aligned}$$

Set $\tilde{\pi} : S^{2n+1} \to \mathbb{CP}^n$. Then $\tilde{\pi} = i \circ \pi$, and $\tilde{\pi}^*\omega_{FS} = \pi^*\omega_{FS}|_{S^{2n+1}}$. Using the calculation in Lemma 3.4.10, we can see that this restricted 2-form expressed in local coordinates is equal to the restriction $\omega_0|_{S^{2n+1}}$. Hence, the Fubini-Study structure satisfies the conditions of a symplectic quotient as above. $\triangle$

One could ask if it is possible to reduce more generally across other level sets. Notice that the main property that allows this symplectic reduction is that $T_p\mathfrak{O}_p \subset T_p\mu^{-1}(0)$, and hence is isotropic. If this condition holds for any other $\zeta \in \mathfrak{g}^*$ as well (after checking whether G preserves $\mu^{-1}(\zeta)$), then reduction can be done using $\mu^{-1}(\zeta)$ instead. For example, any $\mathbb{T}^n$-action has a trivial adjoint representation, and hence preserves any level set $\mu^{-1}(\zeta)$ by equivariance. Moreover, we can also talk about cases where the action is not free on a particular level set $\mu^{-1}(\zeta)$. In this case, ζ being regular is enough to imply a finite stabilizer, and hence an ***orbifold*** structure on $\mu^{-1}(\zeta)/G$. That is, the reduced space will locally carry the structure of a Euclidean space quotient by a finite group action. .

With this final piece of machinery in our toolkit, we can now begin to work on the initial aim and final chapter of this project: unraveling the Delzant polytope and toric variety correspondence.

Chapter 5

Delzant's Theorem

5.1 Symplectic toric manifolds

We divert our attention now to a special class of Hamiltonian actions, and that is toric (or $\mathbb{T}^m$) actions. These are the simplest actions we can consider whose study is not trivial. In fact, symplectic toric manifolds will turn out to be a special sub-class of such Hamiltonian spaces, and their study will be of particular interest to us.

First, we begin with the discussion of a major result about Hamiltonian toric actions.

Theorem 5.1.1. (Atiyah-Guillemin-Sternberg Convexity Theorem) *Let (M, ω) be a compact, connected symplectic manifold with a Hamiltonian $\mathbb{T}^m$-action and corresponding moment map $\mu : M \to \mathbb{R}^m$. Then $\mu(M)$ is convex polyhedron, with the vertices being the images of the fixed points of the action.*

Proof. The proof requires some *Morse theory*, and details can be found in [2]. □

Example 5.1.2. For the rotational action of S^1 on S^2, the fixed points are the north and south poles. With the moment map being the height function $\mu = h$, we can see that $\mu(S^2)$ is a closed interval with endpoints being the image of the north and south poles. Also note that the stabilizer of the preimage of the open line segment is trivial (0-dimensional), while the stabilizer of the preimage of the endpoints is S^1 (1-dimensional). △

Example 5.1.3. Consider the $\mathbb{T}^3$-action on $\mathbb{CP}^3$ given by $(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot [z_0, z_1, z_2, z_3] = [z_0, z_1 e^{i\theta_1}, z_2 e^{i\theta_2}, z_3 e^{i\theta_3}]$. This has moment map $(\mu[z])_i = -\dfrac{|z_i|^2}{2|z|^2}$. Note that as we can factor out the rotational exponential from any of the arguments, the fixed points of this action are given by $[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]$. The images of these points under the moment map are $(0, 0, 0), (-0.5, 0, 0), (0, -0.5, 0), (0, 0, -0.5)$. We can see that

for $z_0 \neq 0$, we can take $z_0 = 1$, and we have $\mu_1([z]) + \mu_2([z]) + \mu_3([z]) = -\frac{1}{2}\frac{s}{s+1}$, for some $s \in \mathbb{R}_{\geq 0}$. Thus the image $\mu(\mathbb{CP}^3|_{z_0 \neq 0})$ is the union of planes (restricted to the negative orthant) given by $\mu_1 + \mu_2 + \mu_3 = a$ for $-\frac{1}{2} < a \leq 0$. For $z_0 = 0$, and not all of z_1, z_2, z_3 zero, the image is the plane $\mu_1 + \mu_2 + \mu_3 = -\frac{1}{2}$ restricted to the negative orthant. Thus the total image is indeed the tetrahedron with vertices mentioned above.
We also calculate the stabilizers of each face. The vertices, being 0-dimensional faces, have a 3-dimensional stabilizer $\mathbb{T}^3$. Let us consider now the 1-dimensional edges. They are characterized by $\{x = y = 0\}, \{x = z = 0\}, \{z = y = 0\}, \{2x + 2y = -1\}, \{2x + 2z = -1\}, \{2y + 2z = -1\}$. The preimages are respectively $\{z_1 = z_2 = 0\}, \{z_1 = z_3 = 0\}, \{z_2 = z_3 = 0\}, \{z_0 = z_3 = 0\}, \{z_0 = z_2 = 0\}, \{z_0 = z_1 = 0\}$. These have a 2-dimensional stabilizer $\mathbb{T}^2$. The 2-dimensional faces are given by $\{x = 0\}, \{y = 0\}, \{z = 0\}, \{2x + 2y + 2z = -1\}$, and their preimages $\{z_1 = 0\}, \{z_2 = 0\}, \{z_3 = 0\}, \{z_0 = 0\}$ have a 1-dimensional stabilizer S^1. The 3-dimensional face is given by the interior: $\{-1 < 2x + 2y + 2z < 0\} \cap \{x, y, z \neq 0\}$, with preimage $\{z_0, z_1, z_2, z_3 \neq 0\}$ that has 0-dimensional stabilizer $\{e\}$. $\triangle$

In the two examples above, we make the curious observation that dim face $F+$ dim $\mathrm{Stab}(\mu^{-1}(F)) =$ dim $\mu(M)$. This observation we shall address later. For now, we discuss some implications of the convexity theorem.

Definition 5.1.4. *An **effective** action of a group G on a manifold M is an action such that $\cap_{p \in M} G_p = \{e\}$, where G_p is the stabilizer of p.*

Lemma 5.1.5. *If a smooth toric action is effective, then it is locally free at some point.*

Proof. Assume the stabilizer for any point $p \in M$ is not finite. Then G_p has to be dense in $\mathbb{T}^m$ (otherwise, if there exists a sizeable ϵ-neighbourhood disjoint from the stabilizer, G_p, being a group, will have to be finite.) As the action is smooth when restricted onto p, we must have $G_p = \mathbb{T}^m$. Thus, if the action is not locally free at any point, then $\cap_{p \in M} G_p = \mathbb{T}^m$. This is a contradiction, and thus the action is locally free for at least one $p \in M$. $\square$

If effectiveness holds, the lemma implies that the moment map is a submersion at p, and hence $\mu(p)$ lies in some open neighborhood contained in $\mu(M)$. That is, $\mu(M)$ is a non-degenerate convex polytope. This gives us the following implication:

Corollary 5.1.6. *An effective Hamiltonian action of $\mathbb{T}^m$ on a compact, connected symplectic manifold has at least $m + 1$ fixed points.*

Note that the effectiveness condition on a $\mathbb{T}^m$-action is very natural because of the following reason: If an action is not effective, then the set $\{d\mu_i\}$ will be linearly dependent.

Choosing a_i such that $\sum_i a_i d\mu_i \equiv 0$, we can set $X = \sum_i a_i \frac{\partial}{\partial x_i} \in \mathfrak{g}$. We will naturally have $d\mu^X = 0$, and hence $\tilde{X} = 0$. Thus we can ignore the action of $\mathbb{T}^m$ along this direction, and instead consider the significant action of the $(m-1)$ sub-torus left over.

Theorem 5.1.7. *Let $(M, \omega, \mathbb{T}^m, \mu)$ be Hamiltonian $\mathbb{T}^m$-space, where the action is effective. Then dim $M \geq 2m$.*

Proof. As the adjoint action is trivial, we have that μ is $\mathbb{T}^m$-invariant. Thus $\mu_*(T_p\mathfrak{O}_p) = 0$, and $T_p\mathfrak{O}_p \subset (T_p\mathfrak{O}_p)^\omega$ is isotropic. Thus dim $M \geq$dim O_p. For p such that the action is locally free at the point, $\mathfrak{O}_p$ dimension m. Thus dim $M \geq 2m$. □

This brings us to the special class of Hamiltonian $\mathbb{T}^m$-spaces.

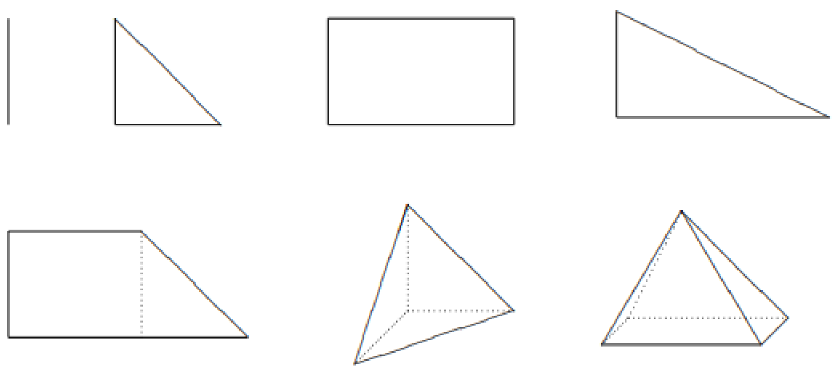
Definition 5.1.8. *A **symplectic toric manifold** is a compact and connected symplectic manifold (M, ω), equipped with an effective Hamiltonian $\mathbb{T}^m$-action, where dim $M = 2m$.*

Example 5.1.9. The examples discussed above in this section are all symplectic toric manifolds. △

5.2 Delzant Construction

Definition 5.2.1. *A **Delzant polytope** $\triangle \subset \mathbb{R}^n$ is a convex polytope that is:*
*(i) **simple**, i.e, has n edges meeting each vertex;*
*(ii) **rational**, i.e, each edge has an integral direction vector $u_i \in \mathbb{Z}^n$;*
*(iii) **smooth**, i.e, the direction vectors u_i of each of the n edges meeting at a vertex p can be chosen such that they form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$.*

Example 5.2.2. Consider the figures below.



It is clear that the last polytope is not Delzant. Moreover, the top-right polytope is also not Delzant in general. Isosceles triangles are, but thinner triangles might not be. For example, assume it is a $30-60-90$ triangle. Then the edge given by the hypotenuse is not rational, neither is it smooth. In general, even if the hypotenuse were rational, but extended more so in one direction, it would fail the smoothness condition. $\triangle$

Being a special class of polyhedra, Delzant polytopes can be expressed as the intersection of closed halfspaces $\triangle = \{x \in \mathbb{R}^n | \langle x, v_i \rangle \leq \lambda_i,\ i = 1, 2, ..., d\}$ for $\lambda_i \in \mathbb{R}$ and $v_i \in \mathbb{Z}^n$ being outward-pointing normal vectors to the d *facets* (i.e, $(n-1)$ dimensional faces) of $\triangle$. It is useful to assume that v_i are *primitive*, i.e, have coprime entries.

We now state the penultimate theorem of Delzant that we aim to understand.

Theorem 5.2.3. (Delzant, 1988) *Symplectic toric manifolds are classified by Delzant polytopes, that is, there exists a one-to-one correspondence:*

$$\{\textit{symplectic toric manifolds}\} \leftrightarrow \{\textit{Delzant Polytopes}\}$$
$$(M^{2n}, \omega, \mathbb{T}^n, \mu) \leftrightarrow \mu(M).$$

We shall mainly focus on the *Delzant construction* that proves subjectivity. For this, we introduce some important objects.

Let $\triangle = \{x \in \mathbb{R}^n | \langle x, v_i \rangle \leq \lambda_i,\ i = 1, 2, ..., d\}$ be an n-dimensional Delzant polytope with d facets. It is a given that $d \geq n$. Now, note that any such polytope can be formed by the intersection of the negative orthant $\mathbb{R}^d_-$ with an affine plane A (up to translation and rotation). Note that $\mathbb{R}^d_- = \{x \in \mathbb{R}^d | \langle x, e_i \rangle \leq 0\ i = 1, 2, ..., d\}$, where e_i is a standard basis vector in $\mathbb{R}^d$. We use this idea of a natural higher dimensional embedding to motivate our construction ahead. Define $\pi : \mathbb{R}^d \to \mathbb{R}^n$ be given by $\pi : e_i \mapsto v_i$. Then its dual $\pi^* : \mathbb{R}^n \to \mathbb{R}^d$ acts like inclusion, and hence maps our original polytope $\triangle$ to a polytope in $\mathbb{R}^d$. For it to precisely map onto a polytope contained in the negative orthant, we translate by $\lambda = (\lambda_1, ..., \lambda_d)$. We can then check that $\pi^*(\triangle) - \lambda = \mathbb{R}^d_- \cap A$, where A is the n-dimensional affine plane that is the image of $\pi^* - \lambda$:

$$(\pi^*(x) - \lambda) \in \mathbb{R}^d_- \Leftrightarrow \langle \pi^*(x), e_i \rangle - \langle \lambda, e_i \rangle \leq 0 \Leftrightarrow \langle x, \pi(e_i) = v_i \rangle \leq \lambda_i \Leftrightarrow x \in \triangle.$$

Now, we now that the negative orthant is the d-dimensional polytope that is the image of the moment map $\tilde{\phi} : \mathbb{C}^d \to \mathbb{R}^d$ given by $(z_1, ..., z_d) \mapsto -\pi(|z_1|^2, ..., |z_d|^2)$ for the standard coordinate-wise rotational action of $\mathbb{T}^d$ on $\mathbb{C}^d$. Coupling this with the observation above, we will aim to find a symplectic quotient of $(\mathbb{C}^d, \omega_0, \mathbb{T}^d, \phi)$ on which we can naturally restrict the original action given by $\phi = \tilde{\phi} + \lambda$. This will then become our required Hamiltonian G-space for $\triangle$. Let us proceed with this motivation in mind.

Lemma 5.2.4. *π maps $\mathbb{Z}^d$ onto $\mathbb{Z}^n$.*

Proof. For any vertex p, the direction vectors $\{u_i\}$ of the edges form a basis for $\mathbb{Z}^n$. It is obvious then that the corresponding normal vectors v_i span $\mathbb{Z}^n$ as well. $\square$

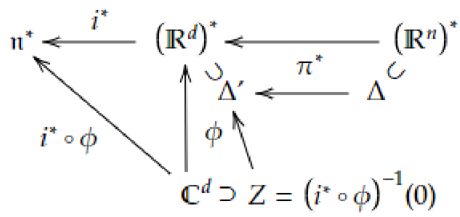
We now extend π to a subjective map between tori $\pi : \mathbb{T}^d \to \mathbb{T}^n$. Set $N = \ker \pi$ be a Lie subgroup of $\mathbb{T}^d$, with Lie algebra $\mathfrak{n}$. This way we produce the following exact sequences:

$$0 \to N \xrightarrow{i} \mathbb{T}^d \xrightarrow{\pi} \mathbb{T}^n \to 0$$
$$0 \to \mathfrak{n} \xrightarrow{i} \mathbb{R}^d \xrightarrow{\pi} \mathbb{R}^n \to 0$$
$$0 \to (\mathbb{R}^n)^* \xrightarrow{\pi^*} (\mathbb{R}^d)^* \xrightarrow{i^*} \mathfrak{n}^* \to 0$$

Note that as N is a closed Lie subgroup of $\mathbb{T}^d$ and hence an $(d-n)$-dimensional sub-torus, it is not hard to see that N acts on $\mathbb{C}^d$ in a Hamiltonian way with moment map $i^* \circ \phi$. We aim to perform symplectic reduction on this Hamiltonian N-space now. For this, we first prove that $Z = (i^* \circ \phi)^{-1}(0)$ is compact and freely acted upon by N.

Lemma 5.2.5. *Z is compact.*

Proof. We claim $\triangle' = \pi^*(\triangle) = \phi(Z)$:



To see that the above diagram holds true, note that $\phi(Z) = \phi((i^* \circ \phi)^{-1}(0)) = \phi(\phi^{-1}((i^*)^{-1}(0))) = \phi(\mathbb{C}^d) \cap \pi^*((\mathbb{R}^n)^*)$. And so, we only need to check that $\triangle'$ lies in the image of ϕ. This is easy to see as $\langle \pi^*(x), e_i \rangle = \langle x, \pi(e_i) \rangle = \langle x, v_i \rangle$. By π^* acting like inclusion, $\triangle'$ is a compact polytope. Additionally, with ϕ being proper, $\phi^{-1}(\triangle') = Z$ is also compact. $\square$

Lemma 5.2.6. *For any $\phi(z)$ belonging to an $(n-r)$-dimensional face of the polytope $\triangle'$, the stabilizer of z under the total action of $\mathbb{T}^d$ is an r-dimensional sub-torus.*

Proof. First, it must be noted that as the $\mathbb{T}^d$-action preserves ϕ, it preserves $\phi^{-1}(\triangle') = Z$. Now for any $\phi(z)$ belonging to an $(n-r)$-dimensional face of $\triangle'$, it must satisfy r different constraints given by $\langle \phi(z), e_{i_j} \rangle = \lambda_{i_j}$ for $1 \leq i_1, ..., i_r \leq d$. By definition of ϕ, this gives $z_{i_j} = 0$, and hence, it is easy to see that the stabilizer of z in $\mathbb{T}^d$ are elements with non-zero components only in the i_j indices. This is an r-dimensional sub-torus. $\square$

Note that we have already seen a similar example of the above in Example 5.1.3.
To have that N acts freely on $\triangle'$, we must have that the stabilizer is disjoint from N. This is not hard to see now. For any $z \in Z$, we must have that $\phi(z) \in F \subset \triangle'$, where F is a face. Hence, $\langle \phi(z), e_{i_j} \rangle = \lambda_{i_j}$ for some indices i_j. Note that as any face contains a vertex, the e_{i_j} must be normal vectors to facets connecting at some common vertex. Thus, πe_{i_j} are all linearly independent by the property of Delzant polytopes. As Stab(z) $\subset$span$\{e_{i_j}\}$, we have that $\pi(\text{Stab}(z) \setminus \{0\})$ is non-zero. Hence N is disjoint from all stabilizers, and the action of N is free on Z. Thus, performing symplectic reduction, we obtain a compact, connected $2n$-dimensional symplectic manifold $M_\triangle = Z/N$ with symplectic form $\omega_\triangle$ satisfying $p^*\omega_\triangle = j^*\omega_0$, where $p : Z \to M_\triangle$ is the canonical projection, and $j : Z \to \mathbb{C}^d$ is inclusion.

Proposition 5.2.7. *The symplectic quotient $(M_\triangle, \omega_\triangle)$ is a Hamiltonian $\mathbb{T}^n$-space with moment map μ such that $\mu(M_\triangle) = \triangle$.*

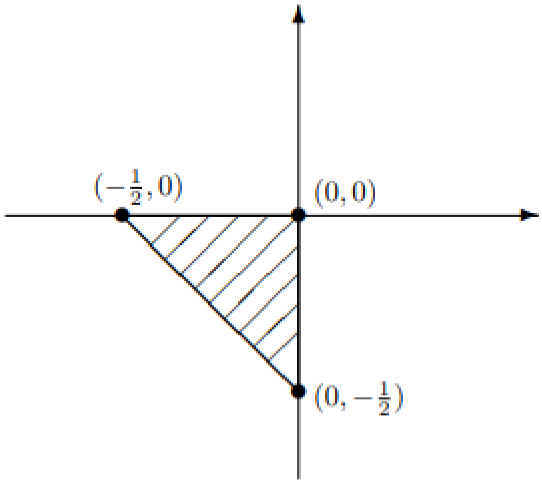
Proof. We have that $\mathbb{T}^d = \mathbb{T}^n \times N$ and hence $(\mathbb{R}^d)^* = (\mathbb{R}^n)^* \oplus \mathfrak{n}^*$. Define $\text{pr}_1 : (\mathbb{R}^d)^* \to \mathfrak{n}^*$ and $\text{pr}_2 : (\mathbb{R}^d)^* \to (\mathbb{R}^n)^*$ as projections. Then it is easy to see that the map $\text{pr}_2 \circ \phi \circ j : Z \to (\mathbb{R}^n)^*$ is constant on N-orbits by virtue of ϕ being preserved under $\mathbb{T}^d$-action. This implies that there exists a natural well-defined moment map μ for the sub-action of $\mathbb{T}^n$ (also effective) on $M_\triangle$ that satisfies $\mu \circ p = \text{pr}_2 \circ \phi \circ j$.

$$\begin{tikzcd}
(\mathbb{R}^n)^* \arrow[dr, shift left, "\pi^*"'] & \mathfrak{n}^* \\
M_\Delta \arrow[u, "\mu"] & (\mathbb{R}^d)^* \supset \phi(Z) = \pi^*(\Delta) \arrow[ul, shift left, "pr_2"'] \arrow[u, "pr_1"'] \\
Z \arrow[u, "p"] \arrow[r, "j"'] & \mathbb{C}^d \arrow[u, "\phi"']
\end{tikzcd}$$

It is clear that the image of μ is $\triangle$. $\square$

And that is it! We've been able to show a method of producing symplectic toric manifolds from Delzant polytopes. Note that we did not need to verify whether μ and $\omega_\triangle$ work together well to satisfy the properties of a Hamiltonian action. This is because of how these two objects have the same definition as their original compatible counterparts ϕ and ω except that they are applied to equivalence classes. That is, $d\phi^X = d\mu^{[X]}$ and $\omega(X, ;) = \omega_\triangle([X], ;)$ for $X \in (\mathbb{R}^n)^*$. Applying this whole process to an example is very straightforward.

Example 5.2.8. Consider the 2-dimensional polytope with vertices $(0, 0)$, $(-0.5, 0)$, and $(0, -0.5)$.



Here, $d = 3$, and the primitive outward-pointing normal vectors are given by $v_1 = (1, 0), v_2 = (1, 0)$ and $v_3 = (-1, -1)$. Thus $\triangle = \{\langle x, v_1\rangle \leq 0 | x \in \mathbb{R}^2\} \cap \{\langle x, v_2\rangle \leq 0 | x \in \mathbb{R}^2\} \cap \{\langle x, v_3\rangle \leq 0.5 | x \in \mathbb{R}^2\}$. Moreover, our map π is given by $\pi(1, 0, 0) = (1, 0)$, $\pi(0, 1, 0) = (0, 1)$ and $\pi(0, 0, 1) = (-1, -1)$. We can see that ker π =span$\{(1, 1, 1)\}$, and thus $i^*(x_1, x_2, x_3) = x_1 + x_2 + x_3$. Thus $N = \{(e^{2\pi it}, e^{2\pi it}, e^{2\pi it})\}$ acts on $\mathbb{C}^3$ with moment map $i^* \circ \phi(z) = -\pi(|z|^2) + 0.5$. Then $(i^* \circ \phi)^{-1}(0) = \{z \in \mathbb{C}^3 | |z|^2 = \frac{1}{2\pi}\} \cong S^5$. As $N \cong S^1$, we can see that $M_\triangle = \mathbb{CP}^2$! The corresponding Hamiltonian action of $\mathbb{T}^2$ on $\mathbb{CP}^2$ is naturally well-defined on the classes: $(e^{2\pi it_1}, e^{2\pi it_2}) \cdot [z_1, z_2, z_3] = [e^{2\pi it_1} z_1, e^{2\pi it_2} z_2, z_3]$, and moment map $\mu[z] = -\frac{1}{2}\left(\frac{|z_1|^2}{|z|^2}, \frac{|z_2|^2}{|z|^2}\right)$, with the property that $\mu \circ p = \mathrm{pr}_2 \circ \phi \circ j$.

An important application of the theorem is the classification of n-dimensional symplectic toric manifolds with a certain number of fixed points. We work out a full example of this now.

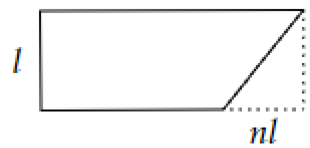
Example 5.2.9. Consider any $(M, \omega, \mathbb{T}^n, \mu)$ that is a 4-dimensional symplectic toric manifold with 4 fixed points under the action. This implies $\mu(M)$ is a 2-dimensional Delzant polytope with 4 vertices. Classifying these Delzant polytopes will allow us to classify the corresponding toric manifolds. We proceed to do this now. First, we discuss an important lemma:

Lemma 5.2.10. *There is a one-to-one correspondence between $SL(2, \mathbb{Z})$ and the set of all possible pairs of edge directions around a vertex in a 2-dimensional Delzant polytope.*

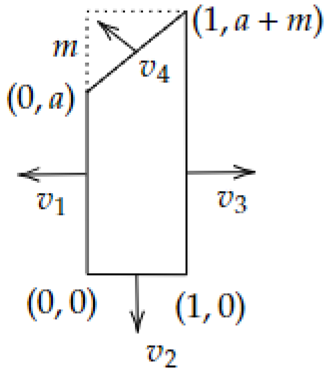
Proof. Consider a pair of vectors $v_1 = (a, b)$ and $v_2 = (c, d)$ satisfying the edge conditions about a vertex in a Delzant polytope $\triangle$. Thus we have that gcd(a, b)= gcd(c, d)=1 by

rationality, and by smoothness we have e_1 and e_2 belong to span$\{v_1, v_2\}$. That is, there exists $t_1, t_2, t_3, t_4 \in \mathbb{Z}$ such that $t_1 a + t_2 c = 1$, $t_1 b + t_2 d = 0$, $t_3 a + t_4 c = 0$, $t_3 b + t_4 d = 1$. Thus, if construct a matrix $M = (v_1^T, v_2^T)$, the matrix is invertible, and so the t_1, t_2 are unique. Moreover, this implies that gcd(a, c) =gcd$(b, d) = 1$ as well, and hence the only solutions are $t_1 = \pm d, t_2 = \mp b, t_3 = \mp c, t_4 = \pm a$. Correspondingly we have Det$(M) = ad - bc = \pm 1$. By switching columns if needed, we have that $M \in SL(2, \mathbb{Z})$. More precisely, whichever vector is less than π radians counterclockwise away from the other, will be placed in the second column for $+1$ determinant.
On the other hand, for a matrix $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL(2, \mathbb{Z})$, we have that $ad - bc = 1$. This automatically gives us that gcd(a, b) =gcd(c, d) =gcd(a, c) =gcd$(b, d) = 1$. Moreover, $M \begin{pmatrix} d \\ -b \end{pmatrix} = e_1$ and $M \begin{pmatrix} -c \\ a \end{pmatrix} = e_2$. Thus the pair $v_1 = (a, b)$ and $v_2 = (c, d)$ is valid as a pair of edge directions about a vertex in a 2-dimensional Delzant polytope. $\square$

Given the above, as $SL(2, \mathbb{Z})$ is a group, it is only natural to consider an equivalence among Delzant polytopes with respect to its action. Now we can begin our classification of 2-dimensional Delzant polytopes with 4 vertices.
Consider a 2-dimensional Delzant polytope $\triangle$, with one vertex at $(0, 0)$ (since Delzant polytopes are translational invariant). Going around counterclockwise, let us call the edge vectors v_1, v_2, v_3 and v_4. We will represent any vertex by its corresponding matrix of edge vectors: $A = (v_1^T, v_2^T), B = (v_2^T, v_3^T), C = (v_3^T, v_4^T), D = (v_4^T, v_1^T)$. We can act upon $\triangle$ by $A^{-1} \in SL(2, \mathbb{Z})$ to obtain new edge matrices with $\tilde{A} = Id, \tilde{B} = (e_2^T, \tilde{v}_3^T), \tilde{C} = (\tilde{v}_3^T, \tilde{v}_4^T), \tilde{D} = (\tilde{v}_4^T, e_1^T)$, where we have right angle at $\tilde{A}$. Applying the condition that Det$(\tilde{B})$ =Det$(\tilde{C})$ =Det$(\tilde{D}) = 1$, we get $v_{31} = -1, v_{42} = -1$ and $v_{32} v_{41} = 0$. If $v_{32} = 0$, then the vertex $\tilde{B}$ is at a right angle, and if $v_{41} = 0$, then vertex $\tilde{D}$ is at a right angle. This implies that any 2-dimensional Delzant polytope with 4 vertices must have two adjacent right angles, i.e, are right trapezoids with the condition that the difference in length of the parallel sides has to be an integer multiple of the base. Note that this automatically includes rectangles and squares.



Now using the Delzant construction on these polytopes, we obtain the *only* symplectic toric manifolds with 4 fixed points. Let us sketch the calculation for such a polytope with vertices $(0, 0), (1, 0), (0, a), (1, a + n)$ for $n \geq 0$.



Here we have $d = 4$, $n = 2$. The primitive outward-pointing vectors are $v_1 = (-1, 0), v_2 = (0, -1), v_3 = (1, 0), v_4 = (-m, 1)$, and the polytope $\triangle = \{\langle x, v_1\rangle \leq 0\} \cap \{\langle x, v_2\rangle \leq 0\} \cap \{\langle x, v_3\rangle \leq 1\} \cap \{\langle x, v_4\rangle \leq a\}$ where $x \in (\mathbb{R}^2)^*$. Now for $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ with $\pi(e_i) = v_i$, $\ker\pi = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ m \\ 1 \end{pmatrix} \right\}$. Thus we have $i^* : (\mathbb{R}^4)^* \to \mathfrak{n}^*$ is given by $i^*(x_1, x_2, x_3, x_4) = (x_1 + x_3, x_2 + mx_3 + x_4)$ and $(i^* \circ \phi)(z) = -\pi(|z_1|^2 + |z_3|^2, |z_2|^2 + m|z_3|^2 + |z_4|^2) + (1, m + a)$. This gives us that $Z = (i^* \circ \phi)^{-1}(0) = \{z \in \mathbb{C}^4 | |z_1|^2 + |z_3|^2 = 1, |z_2|^2 + m|z_3|^2 + |z_4|^2 = a + m\}$. Then our reduced symplectic toric manifold $M_\triangle = Z/\mathbb{T}^2$. For $m = 0$, that is when $\triangle$ is a rectangle or square, it is easy to see that $Z \cong S^3 \times S^3$, and $M_\triangle = \mathbb{CP}^1 \times \mathbb{CP}^1$. For $m > 0$, the corresponding, the toric complex surfaces $\mathfrak{H}_m$ that are produced are called *Hirzebruch surfaces*. For instance, $\mathfrak{H}_1 = \mathbb{CP}^2 \# -\mathbb{CP}^2$. $\triangle$

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