

# Directed Research Project Reading: *An Introduction to Manifolds* by Loring W. Tu

Abdullah Ahmed (2024-10-0035)

Lahore University of Management Sciences, Summer 2023

This document is structured chapter-wise as follows: A summarizing flowchart (sometimes with additional remarks) pertaining to links between various concepts discussed in a single chapter, followed by original solutions to specifically instructional chapter problems and exercises. Also note that some distinct chapters that link well together have been combined into one. Happy reading![1]

## Contents

<b>1</b>	<b>Euclidean Spaces</b>	<b>3</b>
1.1	Smooth Functions on a Euclidean Space . . . . .	3
1.2	Tangent Vectors in $\mathbb{R}^n$ as Derivations . . . . .	5
1.3	The Exterior Algebra of Multivectors . . . . .	6
1.4	Differential Forms on $\mathbb{R}^n$ . . . . .	8
<b>2</b>	<b>Manifolds</b>	<b>9</b>
2.1	Smoothness on Manifolds . . . . .	9
2.2	Quotients and Real Projective Plane . . . . .	12
<b>3</b>	<b>The Tangent Space</b>	<b>13</b>
3.1	The Tangent Space . . . . .	13
3.2	Submanifolds . . . . .	16
3.3	Categories and Functors . . . . .	18
3.4	The Rank of a Smooth Map . . . . .	20
3.5	The Tangent Bundle . . . . .	22
3.6	Bump Functions and Partitions of Unity . . . . .	24
3.7	Vector Fields . . . . .	26
<b>4</b>	<b>Lie Groups and Lie Algebras</b>	<b>28</b>
4.1	Lie Groups . . . . .	28
4.2	Lie Algebras . . . . .	30

<b>5</b>	<b>Differential Forms</b>	<b>32</b>
5.1	Differential 1-forms . . . . .	32
5.2	Differential $k$ -Forms . . . . .	34
5.3	The Exterior Derivative . . . . .	37
5.4	The Lie Derivative and Interior Multiplication . . . . .	41
<b>6</b>	<b>Integration</b>	<b>44</b>
6.1	Orientations . . . . .	44
6.2	Manifolds with Boundary . . . . .	46
6.3	Integration on Manifolds . . . . .	49
<b>7</b>	<b>De Rahm Theory</b>	<b>51</b>
7.1	De Rahm Cohomology . . . . .	51
7.2	The Long Exact Sequence in Cohomology . . . . .	52
7.3	The Mayer-Vietoris Sequence . . . . .	56
7.4	Homotopy Invariance . . . . .	58
7.5	Computation of the de Rahm Cohomology . . . . .	60
7.6	Proof of Homotopy Invariance . . . . .	63
<b>8</b>	<b>Acknowledgements and Final Note</b>	<b>64</b>

# 1 Euclidean Spaces

## 1.1 Smooth Functions on a Euclidean Space

$$\boxed{\begin{array}{l} f \text{ is } C^\infty \\ \text{at } p \in \mathbb{R}^n \end{array}} \iff \boxed{\begin{array}{l} f \text{ is real - analytic} \\ \text{in some nbhd. } U \\ \text{containing } p \in \mathbb{R}^n \end{array}}$$

The converse only holds partially as in *Lemma 1.4* from the book that follows:

**Lemma 1.4 (Taylor's theorem with remainder).** *Let  $f$  be a  $C^\infty$  function on an open subset  $U \subset \mathbb{R}^n$  that is star-shaped with respect to a point  $p = (p^1, \dots, p^n)$  in  $U$ . Then there are functions  $g_1(x), \dots, g_n(x) \in C^\infty(U)$  such that*

$$f(x) = f(p) + \sum_{i=1}^n (x^i - p^i) g_i(x), \quad g_i(p) = \frac{\partial f}{\partial x^i}(p).$$

**Problem 1.6\* Taylor's theorem with remainder to order 2**

Prove that if  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is  $C^\infty$ , then there exist  $C^\infty$  functions  $g_{11}, g_{12}, g_{22}$  on  $\mathbb{R}^2$  such that:

$$\begin{aligned} f(x, y) = f(0, 0) &+ \frac{\partial f}{\partial x}(0, 0)x + \frac{\partial f}{\partial y}(0, 0)y \\ &+ x^2 g_{11}(x, y) + xy g_{12}(x, y) + y^2 g_{22}(x, y). \end{aligned}$$

*Solution:* Let  $f \in C^\infty(\mathbb{R}^2)$ . Then by choosing  $p = (0, 0)$  in the lemma above we have:

$$f(x, y) = f(0, 0) + x g_1(x, y) + y g_2(x, y),$$

where  $g_1(x, y), g_2(x, y) \in C^\infty(\mathbb{R}^2)$  and  $g_1(0, 0) = \frac{\partial f}{\partial x}(0, 0)$  and  $g_2(0, 0) = \frac{\partial f}{\partial y}(0, 0)$ . Thus we can apply the same remainder theorem on  $g_1$  and  $g_2$  to get:

$$\begin{aligned} f(x, y) &= f(0, 0) + x(g_1(0, 0) + x h_1(x, y) + y h_2(x, y)) + y(g_2(0, 0) + x l_1(x, y) + y l_2(x, y)) \\ \Rightarrow f(x, y) &= f(0, 0) + x \frac{\partial f}{\partial x}(0, 0) + y \frac{\partial f}{\partial y}(0, 0) + x^2 g_{11}(x, y) + xy g_{12}(x, y) + y^2 g_{22}(x, y), \end{aligned}$$

where  $h_1(x, y) = g_{11}(x, y)$ ,  $l_2(x, y) = g_{22}(x, y)$  and  $h_2(x, y) + l_1(x, y) = g_{12}(x, y)$  are all  $C^\infty$  over  $\mathbb{R}^n$ .

**Problem 1.7\* A function with a removable singularity**

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a  $C^\infty$  function with  $f(0,0) = \partial f / \partial x(0,0) = \partial f / \partial y(0,0) = 0$ . Define:

$$g(t, u) = \begin{cases} \frac{f(t, tu)}{t} & t \neq 0 \\ 0 & t = 0. \end{cases}$$

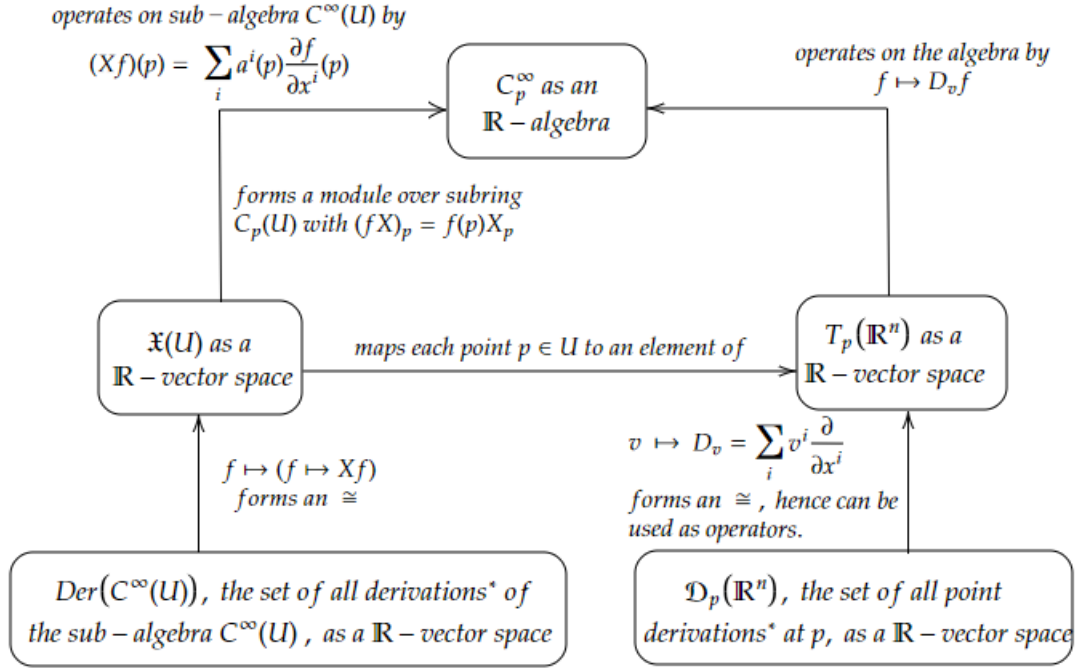
Prove that  $g(t, u)$  is  $C^\infty$  for  $(t, u) \in \mathbb{R}^2$ .

*Solution:* Let  $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the  $C^\infty$  function given by  $h(t, u) = (t, tu)$ . We also have from the previous problem:

$$\begin{aligned} g(t, u) &= \frac{1}{t} \left( t^2 g_{11}(t, tu) + t^2 u g_{12}(t, tu) + t^2 u^2 g_{22}(t, tu) \right) \\ &= t g_{11}(t, tu) + t u g_{12}(t, tu) + t u^2 g_{22}(t, tu) = k(t, u), \end{aligned}$$

for  $t \neq 0$ . As  $g_{ij}(t, tu) = (g_{ij} \circ h)(t, u)$  is a composition of  $C^\infty$  functions, it is also  $C^\infty \Rightarrow k(t, u)$  is  $C^\infty$ . Also, as  $k(0, u) = g(0, u) = 0$ , we can see that  $k(t, u) = g(t, u)$  for all  $(t, u) \in \mathbb{R}^2$ . Thus  $g(t, u)$  is  $C^\infty$  as well.

## 1.2 Tangent Vectors in $\mathbb{R}^n$ as Derivations



\*The general idea of a derivation is that of a linear operator that satisfies some form of the *Leibniz rule* that we are familiar with for derivatives. Also, strictly speaking, we consider the elements of  $C^\infty(U)$  modulo the equivalence relation  $(f, U) \sim (g, V) \Leftrightarrow f|_{U \cap V} = g|_{U \cap V}$ .

### Problem 2.4 Product of derivations

Let  $A$  be an algebra over a field  $K$ . If  $D_1$  and  $D_2$  are derivations of  $A$ , show that  $D_1 \circ D_2$  is not necessarily a derivation (it is if  $D_1$  or  $D_2 = 0$ ), but  $D_1 \circ D_2 - D_2 \circ D_1$  is always a derivation of  $A$ .

*Solution:* Assume  $D_1$  and  $D_2$  are derivations over an algebra  $A$  over a field  $K$ . Now consider:

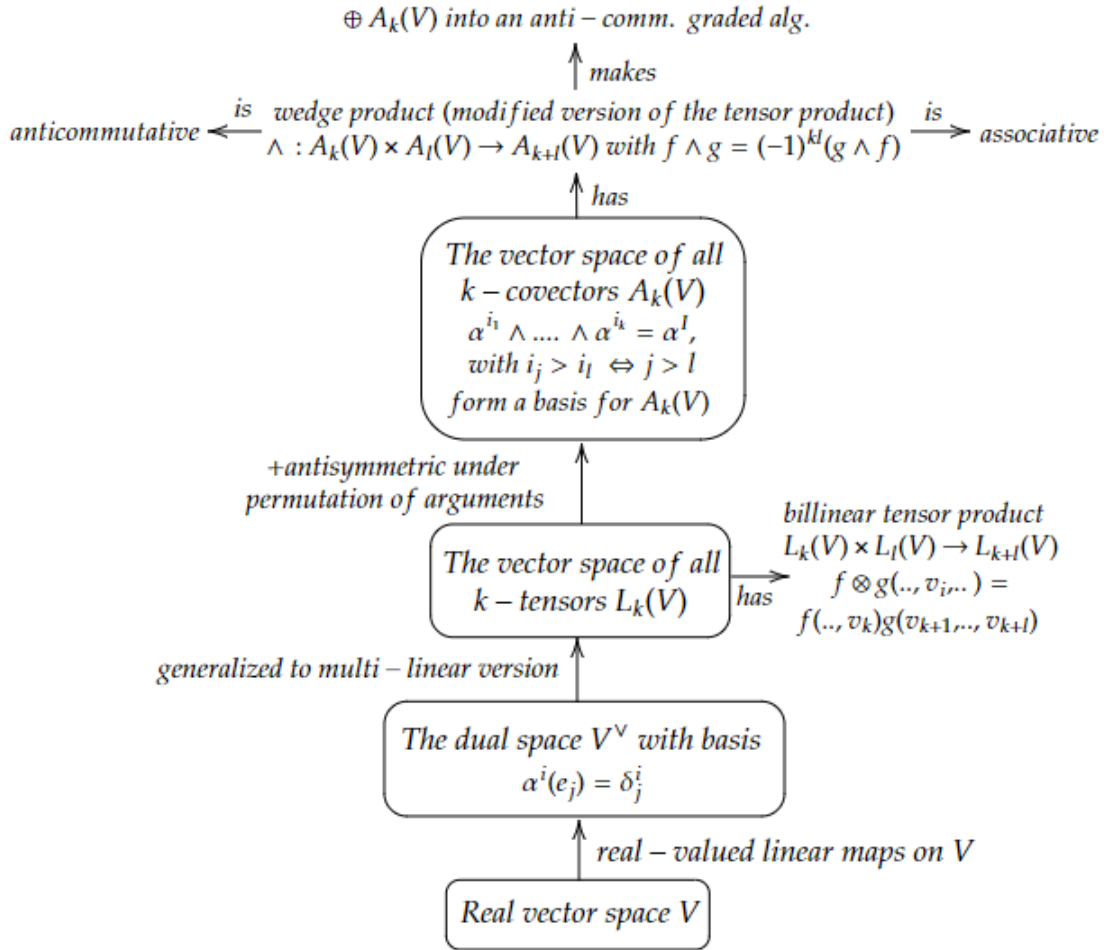
$$\begin{aligned} (D_1 \circ D_2)(ab) &= D_1(D_2(ab)) \stackrel{\text{leib. rule}}{=} D_1((D_2a)b + a(D_2b)) \stackrel{\text{lin.}}{=} D_1((D_2a)b) + D_1(a(D_2b)) \\ &= (D_1 \circ D_2)(a)b + (D_2a)(D_1b) + (D_1a)(D_2b) + a(D_1 \circ D_2)(b), \end{aligned}$$

where we can see clearly that the *Leibniz rule* is not satisfied. On the other hand, we can see that:

$$\begin{aligned} (D_1 \circ D_2 - D_2 \circ D_1)(ab) &= (D_1 \circ D_2)(a)b + (D_2a)(D_1b) + (D_1a)(D_2b) + a(D_1 \circ D_2)(b) \\ &\quad - ((D_2 \circ D_1)(a)b + (D_1a)(D_2b) + (D_2a)(D_1b) + a(D_2 \circ D_1)(b)) \\ &= (D_1 \circ D_2 - D_2 \circ D_1)(a)b + a(D_1 \circ D_2 - D_2 \circ D_1)(b), \end{aligned}$$

thus  $(D_1 \circ D_2 - D_2 \circ D_1)$  satisfies the *Leibniz rule* and is a derivation.

### 1.3 The Exterior Algebra of Multivectors



#### Problem 3.4 A characterization of alternating $k$ -tensors

*Solution:* Assume we have an alternating  $k$ -tensor  $f$  on a vector space  $V \Rightarrow \sigma f = \text{sgn}(\sigma)f$  for all  $\sigma \in S_k$ . Let  $\sigma = (i \ i+1)$ . Then we have:

$$f(\dots, v_{i+1}, v_i, \dots) = \sigma f(\dots, v_i, v_{i+1}, \dots) = \text{sgn}(\sigma)f(\dots, v_i, v_{i+1}, \dots) = -f(\dots, v_i, v_{i+1}, \dots).$$

Now assume  $f$  changes sign whenever two successive arguments are interchanged. Consider an arbitrary permutation  $\tau \in S_k$  and the corresponding ordered tuple  $(\tau(1), \dots, \tau(k))$ . Starting from  $1 = \tau(i_1)$  for some  $i_1$ , we swap its place with successive members to the left in the tuple above, until it reaches the first index. We then do the same for the next integer 2 up until  $k-1$ , such that the tuple is sorted by the end of this process. In this way, as each swap consists of a smaller integer being swapped with a larger one to the left of it, we can see that the total number of successive swaps made is precisely equal to the number of *inversions*, thus giving us:

$$\tau f = f(v_{\tau(1)}, \dots, v_{\tau(k)}) = (-1)^{\text{no. of inv.}} f(v_1, \dots, v_k) = \text{sgn}(\tau)f(v_1, \dots, v_k)$$

**Problem 3.10\* Linear independence of covectors**

*Solution:* Assume we are given that  $\alpha^1 \wedge \dots \wedge \alpha^k \neq 0$ . Assume now on the contrary that  $\{\alpha^i\}$  is a linearly dependent set in  $V^\vee \Rightarrow \exists j$  such that  $\alpha^j = \sum_{i \neq j}^k c_i^j \alpha^i$  for some  $c_i^j \in \mathbb{R}$ . Thus we have:

$$\alpha^1 \wedge \dots \wedge \alpha^j \wedge \dots \wedge \alpha^k = \alpha^1 \wedge \dots \left( \sum_{i \neq j}^k c_i^j \alpha^i \right) \wedge \dots \wedge \alpha^k \stackrel{\text{lin.}}{=} \sum_{i \neq j}^k c_i^j (\alpha^1 \wedge \dots \wedge \alpha^i \wedge \dots \wedge \alpha^k) = 0,$$

as for all terms in the sum above, at least one other superscript is equal to  $i$ , and  $\alpha^i \wedge \alpha^i = 0$  by anti-commutativity of the wedge product. This is a contradiction, and thus  $\{\alpha^i\}$  is a linearly independent set.

Now assume that the set  $\{\alpha^i\}$  is linearly independent  $\Rightarrow$  it can be extended to a basis  $\alpha^1, \dots, \alpha^n$  of  $V^\vee$  where  $n = \dim(V)$ . Thus  $\alpha^1 \wedge \dots \wedge \alpha^k$  is a basis vector for  $A_k(V)$  and hence cannot be zero.

**Problem 3.11\* Exterior multiplication**

*Solution:* Let  $\alpha$  be a 1-covector and  $\gamma$  be a  $k$ -covector.

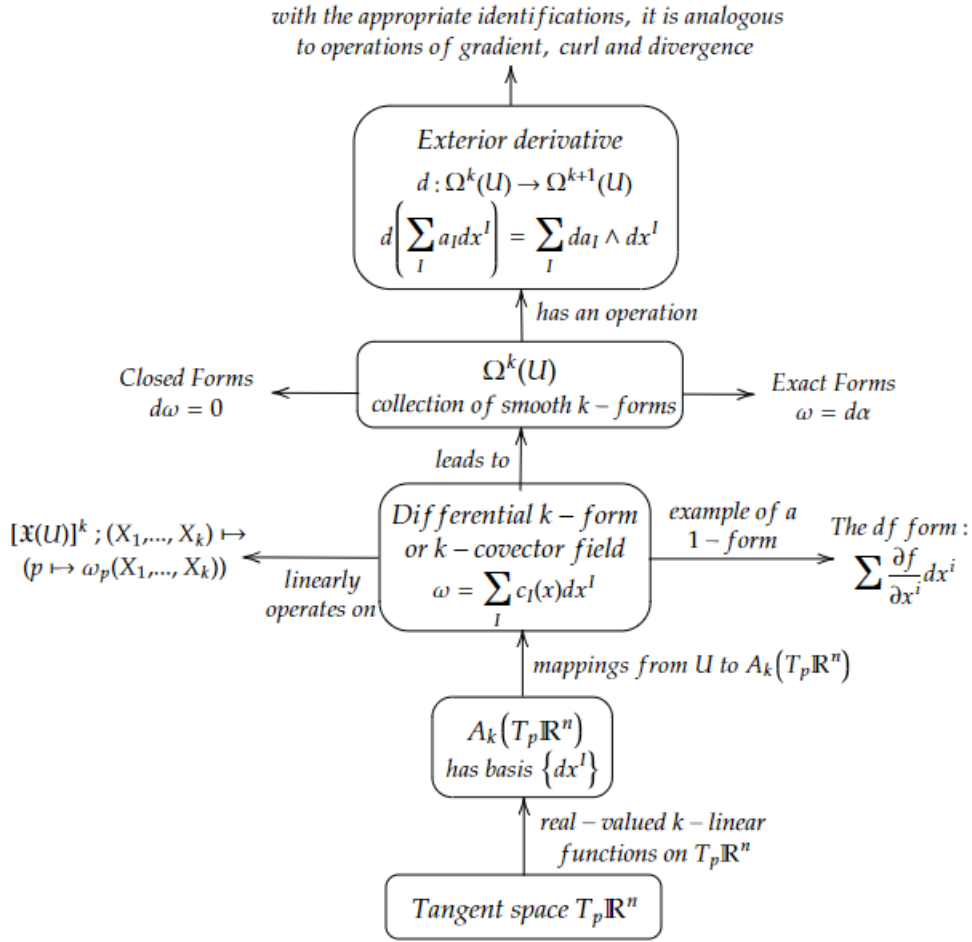
Assume  $\gamma = \alpha \wedge \beta$  for some  $(k-1)$ -covector  $\beta \Rightarrow \alpha \wedge \gamma = \alpha \wedge \alpha \wedge \beta = 0$ .

Now assume  $\alpha \wedge \gamma = 0$ . Denote  $\alpha$  by  $\beta_1$ , and extend it to a basis  $\beta_1, \dots, \beta_n$  of  $V^\vee$  of dimension  $n \Rightarrow \beta^I$  forms a basis for  $A_k(V)$  where  $I$  varies over all strictly increasing multi-indices of length  $k \Rightarrow \gamma = \sum_J c_J \beta^J$  for some  $c_J \in \mathbb{R}$ . The sum can be split up as follows:

$$\begin{aligned} \gamma &= \sum_{1 \in J} c_J \beta^J + \sum_{1 \notin J} c_J \beta^J \\ \Rightarrow \alpha \wedge \gamma &= \sum_{1 \in J} c_J (\alpha \wedge \beta^J) + \sum_{1 \notin J} c_J (\alpha \wedge \beta^J) = \sum_{1 \notin J} c_J (\alpha \wedge \beta^J) = 0 \quad (1) \end{aligned}$$

As  $\alpha \wedge \beta^J$  are linearly independent in  $A_{k+1}(V)$  for  $1 \notin J$ , (1)  $\Rightarrow c_J = 0$  for all  $J$  such that  $1 \notin J \Rightarrow \gamma = \sum_{1 \in J} c_J \beta^J = \alpha \wedge \beta$  for some  $(k-1)$ -covector  $\beta$ .

## 1.4 Differential Forms on $\mathbb{R}^n$



### Probleem 4.7 Commutator for derivations and anti-derivations

*Solution:*

$$\begin{aligned}
 (D_1 \circ D_2)(ab) &= D_1((D_2 a)b + (-1)^{km_2} a D_2 b) \\
 &= (D_1 \circ D_2)(a)b + (-1)^{m_1(k+m_2)} (D_2 a)(D_1 b) \\
 &\quad + (-1)^{km_2} ((D_1 a)(D_2 b) + (-1)^{km_1} a (D_1 \circ D_2)(b))
 \end{aligned}$$

Similarly, we can calculate  $D_2 \circ D_1(ab)$ , by switching  $D_1$  with  $D_2$  and  $m_1$  with  $m_2$ , and finally get after obvious manipulations:

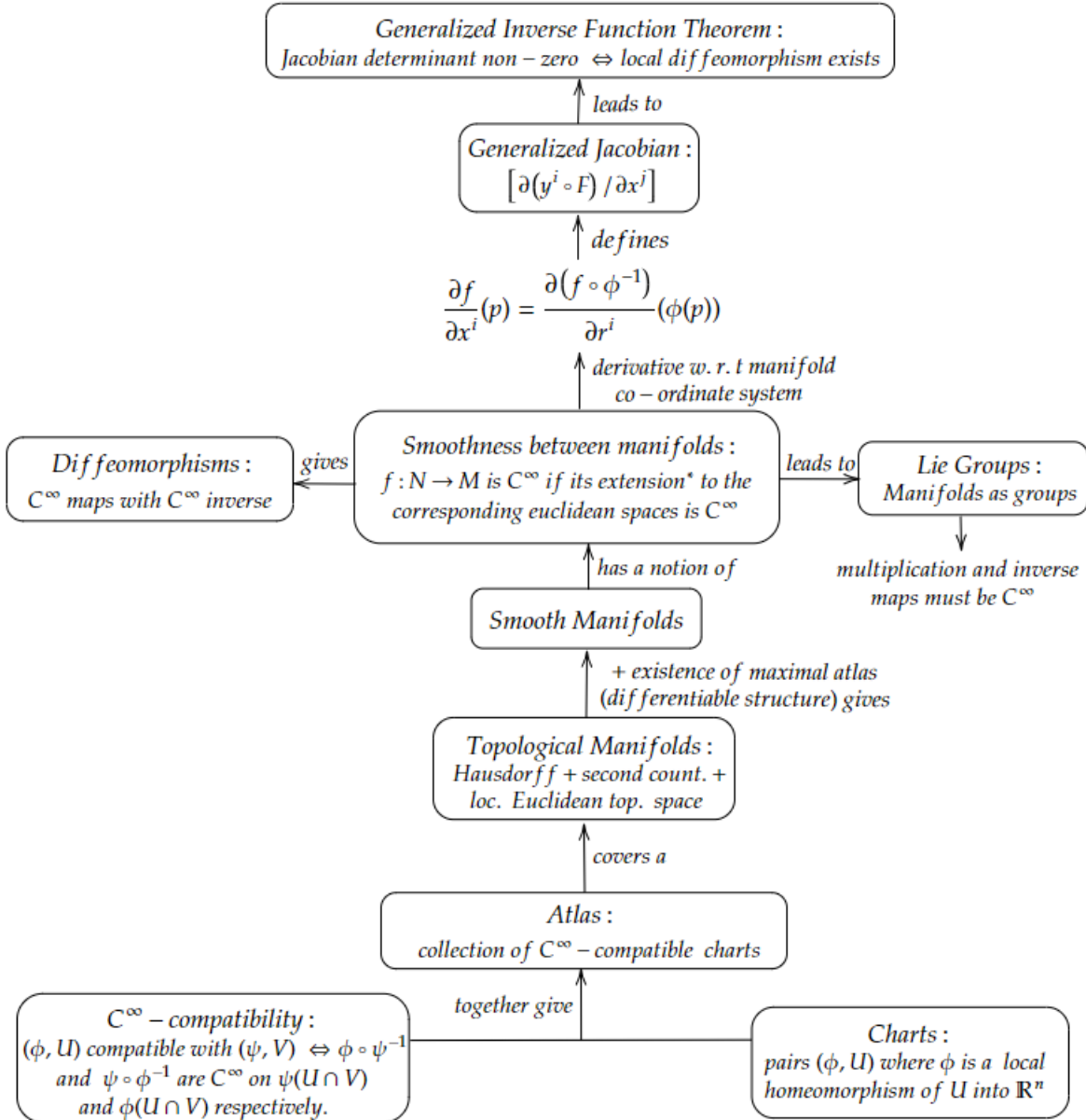
$$\begin{aligned}
 [D_1, D_2](ab) &= (D_1 \circ D_2 - (-1)^{m_1 m_2} D_2 \circ D_1)(ab) \\
 &= [D_1, D_2](a)b - (-1)^{k(m_1+m_2)} a [D_1, D_2]b,
 \end{aligned}$$

implying that  $[D_1, D_2]$  is a superderivation of degree  $m_1 + m_2$ .



## 2 Manifolds

### 2.1 Smoothness on Manifolds



\*Extension here alludes to a common idea of having the manifold locally inherit the smoothness from its associated euclidean space through its atlas. Given any point  $p$  and corresponding charts in the differentiable structure  $(\phi, U)$  and  $(\psi, V)$  with  $p \in U \subseteq N$  and  $f(p) \in V \subseteq M$ , we can *extend*  $f$  to the map  $\psi \circ f \circ \phi^{-1} : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ , with  $A = \phi(f^{-1}(V) \cap U)$ .

As mentioned above, ideas of differentiability in the general setting of manifolds is *inherited* from their associated euclidean spaces—a relatively intuitive way to go about introducing differential structure on them. As is known, there exists a stronger notion of differentiability for complex-valued functions, i.e., *complex differentiability*. One could tweak the definition to impose that the charts be *holomorphic* instead, have the manifold inherit the notions of complex differentiability. One can assume that this would produce nicer results as compared to smooth manifolds, as is the case when results in complex analysis are compared to real analysis. As it turns out, this is true and this special class of manifolds are known as *complex manifolds*, and they have a very deep and active theory associated with them.

**Problem 5.4\* Existence of a coordinate neighbourhood**

*Solution:* Let  $\mathfrak{A} = \{(U_\alpha, \phi_\alpha)\}$  be the maximal atlas of the smooth manifold  $M$ . Let  $U$  be an open set containing a point  $p$ . As  $p \in M \Rightarrow \exists (U_\alpha, \phi_\alpha) \in \mathfrak{A}$  such that  $p \in U_\alpha$ . Then we have that  $p \in V = U \cap U_\alpha \subseteq U$  is open. Clearly the restriction  $\phi_\alpha|_V$  is local homeomorphism  $\Rightarrow (V, \phi_\alpha|_V)$  is a chart. As  $(U_\alpha, \phi_\alpha)$  is  $C^\infty$ -compatible with all other charts in  $\mathfrak{A}$ , so is  $(V, \phi_\alpha|_V)$  as  $V \subseteq U_\alpha \Rightarrow (V, \phi_\alpha|_V) \in \mathfrak{A}$  with  $p \in V \subseteq U$ .

**Proposition 5.18 An atlas for a product manifold**

*Proof:* Let  $M$  and  $N$  be  $m$  and  $n$  dimensional manifolds respectively, with atlases  $\mathfrak{A}_M = \{(U_\alpha, \phi_\alpha)\}$  and  $\mathfrak{A}_N = \{(V_i, \psi_i)\}$ . We claim that  $\mathfrak{A} = \{(U_\alpha \times V_i, \phi_\alpha \times \psi_i : U_\alpha \times V_i \rightarrow \mathbb{R}^m \times \mathbb{R}^n)\}$  is an atlas for  $N \times M$ , which is hence a  $n + m$  dimensional manifold.

Clearly as  $\mathfrak{A}$  covers  $M \times N$ , we must only check  $C^\infty$ -compatibility. Let  $(U_\alpha \times V_i, \phi_\alpha \times \psi_i)$  and  $(U_\beta \times V_j, \phi_\beta \times \psi_j)$  be two arbitrary charts in  $\mathfrak{A}$ . We can see that:

$$(\phi_\alpha \times \psi_i) \circ (\phi_\beta \times \psi_j)^{-1} = (\phi_\alpha \circ \phi_\beta^{-1}) \times (\psi_i \circ \psi_j^{-1}),$$

is  $C^\infty$  on  $(U_\alpha \times V_i) \cap (U_\beta \times V_j) = (U_\alpha \cap U_\beta) \times (V_i \cap V_j)$  as each component is  $C^\infty$  due to  $C^\infty$ -compatibility among  $\mathfrak{A}_M$  and  $\mathfrak{A}_N$ . A similar argument can be made for  $(\phi_\beta \times \psi_j) \circ (\phi_\alpha \times \psi_i)^{-1}$ . Hence  $\mathfrak{A}$  is an atlas.

**Problem 6.1 Differentiable structures on  $\mathbb{R}$**

*Solution:* Assume we have two differentiable structures on  $\mathbb{R}$ — $\mathbb{R}_1$  and  $\mathbb{R}_2$ —generated by the maximal atlas of  $(\mathbb{R}, \phi = id)$  and  $(\mathbb{R}, \psi(x) = x^{1/3})$  respectively. We can see that as  $\phi^{-1} \circ \psi = \psi$  is not a  $C^\infty$  map from  $\mathbb{R} \rightarrow \mathbb{R}$ , the two differentiable structures are distinct. But we can check that  $f : \mathbb{R}_1 \rightarrow \mathbb{R}_2$  defined by  $f(x) = x^3$  is a diffeomorphism between the two:

$$\begin{aligned} \psi \circ f \circ \phi^{-1} &= id \text{ is } C^\infty \text{ on } \phi(f^{-1}(\mathbb{R}) \cap \mathbb{R}) = \mathbb{R} \text{ and,} \\ \phi \circ f^{-1} \circ \psi^{-1} &= id \text{ is } C^\infty \text{ on } \psi(f^{-1}(\mathbb{R}) \cap \mathbb{R}) = \mathbb{R}. \end{aligned}$$

**Exercise 6.18 Smoothness of a map to a Cartesian product**

*Solution:* Let  $f = (f_1, f_2) : N \rightarrow M_1 \times M_2$ , and  $\mathfrak{A}_{M_1} = \{(U_\alpha, \phi_\alpha)\}$ ,  $\mathfrak{A}_{M_2} = \{(V_\alpha, \psi_\alpha)\}$ ,

and  $\mathfrak{A}_N = \{(W_\alpha, \chi_\alpha)\}$  be atlases for  $M_1$ ,  $M_2$  and  $N$  respectively. Correspondingly, by Proposition 5.18 above,  $\mathfrak{A} = \{(U_\alpha \times V_\alpha, \phi_\alpha \times \psi_\alpha)\}$  is an atlas for  $M_1 \times M_2$ .

Assume  $f$  is  $C^\infty \Rightarrow f \circ \pi_i = f_i$  is  $C^\infty$  as a composition of  $C^\infty$  maps.

Now assume  $f_1$  and  $f_2$  are both  $C^\infty$ . Then we can see that for any  $p \in W_\alpha$  and  $f(p) \in U_\alpha \times V_\alpha$ :

$$(\phi_\alpha \times \psi_\alpha) \circ f \circ \chi^{-1} = (\phi_\alpha \circ f_1 \circ \chi^{-1}) \times (\psi_\alpha \circ f_2 \circ \chi^{-1}),$$

is  $C^\infty$  at  $p$  as its component maps are  $C^\infty$ .

## 2.2 Quotients and Real Projective Plane

Not much was to be discussed here apart from what is already known from an introductory course in Topology and Algebraic Geometry.

### Problem 7.5\* Orbit space of a continuous group action

*Solution:* Assume a group  $G$  acts on  $S$  continuously, i.e, the map  $\bullet : S \times G \rightarrow S$  is continuous.

Consider any open set  $U \subseteq S$ . We must show that  $\pi(U)$  is open in the quotient topology on  $S/G \Leftrightarrow \pi^{-1}(\pi(U))$  is open in  $S$ . For any  $[p] \in \pi(U)$ ,  $\pi^{-1}([p]) = \bullet^{-1}(\{p\}) \Rightarrow \pi^{-1}(\pi(U)) = \bullet^{-1}(U)$  which is open.

### Problem 7.6 Quotient of $\mathbb{R}$ by $2\pi\mathbb{Z}$

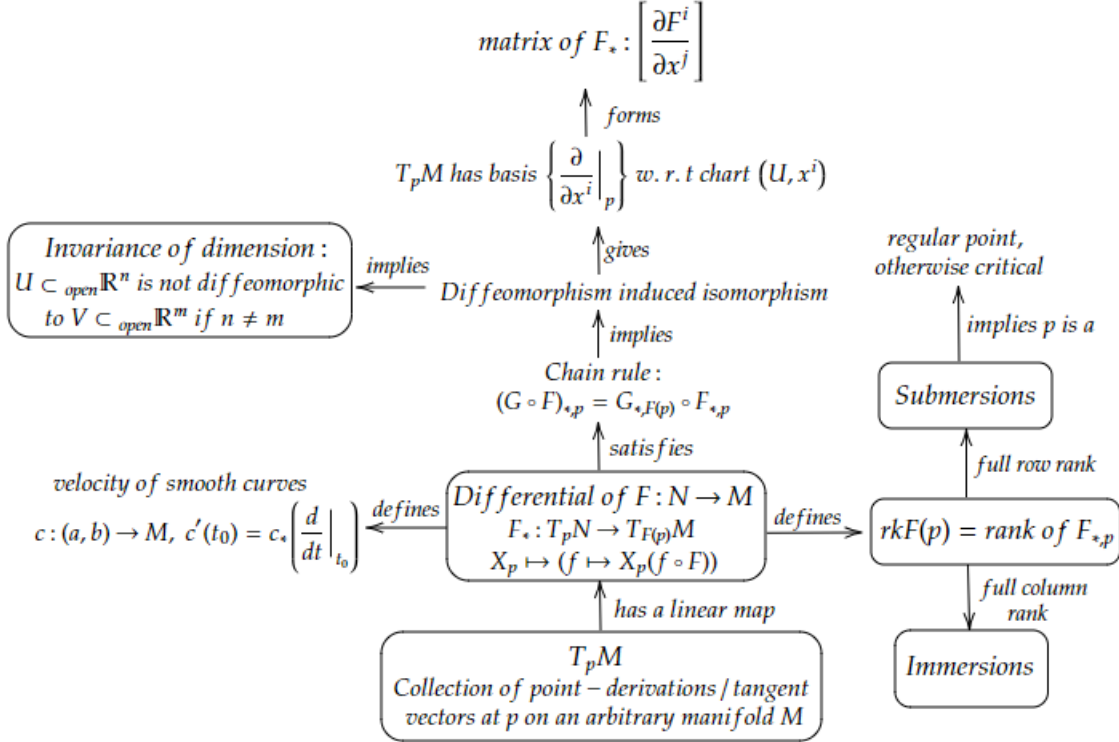
*Solution:* We know that  $\mathbb{R}/2\pi\mathbb{Z}$  is homeomorphic to  $S^1$ , which is a smooth manifold with some atlas  $\mathfrak{A} = \{(U_\alpha, \phi_\alpha)\}$ . Let  $f$  be a homeomorphism from  $\mathbb{R}/2\pi\mathbb{Z}$  to  $S^1$ . Then we claim  $\mathfrak{A}' = \{(f^{-1}(U_\alpha), \phi_\alpha \circ f)\}$  is an atlas for  $\mathbb{R}/2\pi\mathbb{Z}$ . It is easy to see that  $\mathfrak{A}'$  covers  $\mathbb{R}/2\pi\mathbb{Z}$  and that  $\phi_\alpha \circ f$  are local homeomorphisms. We can also see that for any two charts  $(f^{-1}(U_\alpha), \phi_\alpha \circ f), (f^{-1}(U_\beta), \phi_\beta \circ f) \in \mathfrak{A}'$ :

$$(\phi_\alpha \circ f) \circ (\phi_\beta \circ f)^{-1} = (\phi_\alpha \circ f) \circ (f^{-1} \circ \phi_\beta^{-1}) = \phi_\alpha \circ \phi_\beta^{-1},$$

is  $C^\infty$  on  $(\phi_\beta \circ f)(f^{-1}(U_\alpha) \cap f^{-1}(U_\beta)) = \phi(U_\alpha \cap U_\beta)$ , by  $C^\infty$ -compatibility in  $\mathfrak{A}$ . The same holds for  $(\phi_\beta \circ f) \circ (\phi_\alpha \circ f)^{-1}$ . Thus the two arbitrary charts are  $C^\infty$ -compatible, and  $\mathbb{R}/2\pi\mathbb{Z}$  is a smooth manifold. (Note that the argument is general enough to hold for any two homeomorphic spaces.)

### 3 The Tangent Space

#### 3.1 The Tangent Space



#### Problem 8.1\* Differential of a map:

*Solution:* As  $\frac{\partial}{\partial u} \Big|_{F(p)}$ ,  $\frac{\partial}{\partial v} \Big|_{F(p)}$ , and  $\frac{\partial}{\partial w} \Big|_{F(p)}$  form a basis for  $T_{F(p)} \mathbb{R}^3$ , we have:

$$F_*(\partial/\partial x|_p) = a_u(\partial/\partial u|_{F(p)}) + a_v(\partial/\partial v|_{F(p)}) + a_w(\partial/\partial w|_{F(p)}).$$

Acting both sides on  $u$ ,  $v$ , and  $w$  we get:

$$\partial/\partial x|_p(u \circ F) = 1 = a_u$$

$$\partial/\partial x|_p(v \circ F) = \partial/\partial x|_p(y) = 0 = a_v$$

$$\partial/\partial x|_p(w \circ F) = \partial/\partial x|_p(xy) = y = a_w$$

Thus  $F_*(\partial/\partial x|_p) = (\partial/\partial u|_{F(p)}) + y(\partial/\partial w|_{F(p)})$ .

#### Proposition 8.15 (Velocity of a curve in local coordinates)

*Proof:* Let  $c: (a, b) \rightarrow M$  be a smooth curve, with  $(U, x^1, \dots, x^m)$  as a coordinate chart

around  $c(t)$ . Then we have:

$$c'(t) = c_* \left( \frac{d}{dt} \Big|_t \right) = \sum_{i=1}^m a^i \frac{\partial}{\partial x^i} \Big|_{c(t)}.$$

Acting both sides on  $x^j$ :

$$\begin{aligned} \frac{d}{dt} \Big|_t (x^j \circ c) &= \dot{c}(t) = \sum_{i=1}^m a^i \delta_i^j = a^j \\ \Rightarrow c'(t) &= \sum_{i=1}^m \dot{c}^i(t) \frac{\partial}{\partial x^i} \Big|_{c(t)} \end{aligned}$$

**Problem 8.7\* Tangent space to a product**

*Solution:* Consider the following coordinate charts:  $(U, x^1, \dots, x^m)$  in  $M$  and  $(V, y^1, \dots, y^n)$  in  $N$ , where  $p \in U$  and  $q \in V$ . The corresponding coordinate chart in  $M \times N$  is  $(U \times V, \tilde{x}^1, \dots, \tilde{x}^m, \tilde{y}^1, \dots, \tilde{y}^n)$ , where  $\tilde{x}^i = x^i \circ \pi_1$  and  $\tilde{y}^i = y^i \circ \pi_2 \Rightarrow$  the basis for  $T_p(M \times N)$  is  $\left\{ \frac{\partial}{\partial \tilde{x}^1} \Big|_{(p,q)}, \dots, \frac{\partial}{\partial \tilde{x}^m} \Big|_{(p,q)}, \frac{\partial}{\partial \tilde{y}^1} \Big|_{(p,q)}, \dots, \frac{\partial}{\partial \tilde{y}^n} \Big|_{(p,q)} \right\}$ . On the other hand, the basis for  $T_p M \times T_p N$  is  $\left\{ \left( \frac{\partial}{\partial x^1} \Big|_p, \mathbf{0} \Big|_q \right), \dots, \left( \frac{\partial}{\partial x^m} \Big|_p, \mathbf{0} \Big|_q \right), \left( \mathbf{0} \Big|_p, \frac{\partial}{\partial y^1} \Big|_q \right), \dots, \left( \mathbf{0} \Big|_p, \frac{\partial}{\partial y^n} \Big|_q \right) \right\}$ . We show that  $(\pi_{1*}, \pi_{2*})$  maps  $\frac{\partial}{\partial \tilde{x}^i} \Big|_{(p,q)} \mapsto \left( \frac{\partial}{\partial x^i} \Big|_p, \mathbf{0} \Big|_q \right)$  and  $\frac{\partial}{\partial \tilde{y}^i} \Big|_{(p,q)} \mapsto \left( \mathbf{0} \Big|_p, \frac{\partial}{\partial y^i} \Big|_q \right)$ , hence mapping a basis to a basis, and thus being an isomorphism:

$$\begin{aligned} (\pi_{1*}, \pi_{2*}) \left( \frac{\partial}{\partial \tilde{x}^i} \Big|_{(p,q)} \right) &= \sum_{i=1}^m a^i \left( \frac{\partial}{\partial x^i} \Big|_p, \mathbf{0} \Big|_q \right) + \sum_{i=1}^n b^i \left( \mathbf{0} \Big|_p, \frac{\partial}{\partial y^i} \Big|_q \right) \\ \Rightarrow \left( \pi_{1*} \left( \frac{\partial}{\partial \tilde{x}^i} \Big|_{(p,q)} \right), \pi_{2*} \left( \frac{\partial}{\partial \tilde{x}^i} \Big|_{(p,q)} \right) \right) &= \sum_{i=1}^m a^i \left( \frac{\partial}{\partial x^i} \Big|_p, \mathbf{0} \Big|_q \right) + \sum_{i=1}^n b^i \left( \mathbf{0} \Big|_p, \frac{\partial}{\partial y^i} \Big|_q \right), \end{aligned}$$

where we can compare entry wise and act on the coordinate functions:

$$\begin{aligned} \pi_{1*} \left( \frac{\partial}{\partial \tilde{x}^i} \Big|_{(p,q)} \right) (x^j) &= \sum_{i=1}^m a^i \frac{\partial}{\partial x^i} \Big|_p (x^j) \\ \Rightarrow \frac{\partial}{\partial \tilde{x}^i} \Big|_{(p,q)} (x^j \circ \pi_1) &= \sum_{i=1}^m a^i \delta_i^j = a^j \\ \Rightarrow \frac{\partial}{\partial r^i} \Big|_{\phi(p,q)} (x^j \circ \pi_1 \circ (\phi^{-1} = (\tilde{x}^1, \dots, \tilde{x}^m, \tilde{y}^1, \dots, \tilde{y}^n)^{-1})) &= \frac{\partial}{\partial r^i} \Big|_{\phi(p,q)} (r^j) = \delta_i^j = a^j. \end{aligned}$$

Similarly, one can find that  $b^i = 0 \Rightarrow (\pi_{1*}, \pi_{2*}) \left( \frac{\partial}{\partial \tilde{x}^i} \Big|_{(p,q)} \right) = \left( \frac{\partial}{\partial x^i} \Big|_p, \mathbf{0} \Big|_q \right)$ . The same

arguments can be applied to basis elements of the form  $\frac{\partial}{\partial \tilde{y}^i} \Big|_{(p,q)}$  to show  $(\pi_{1*}, \pi_{2*}) \left( \frac{\partial}{\partial \tilde{y}^i} \Big|_{(p,q)} \right) = \left( \mathbf{0} \Big|_p, \frac{\partial}{\partial y^i} \Big|_q \right)$ .

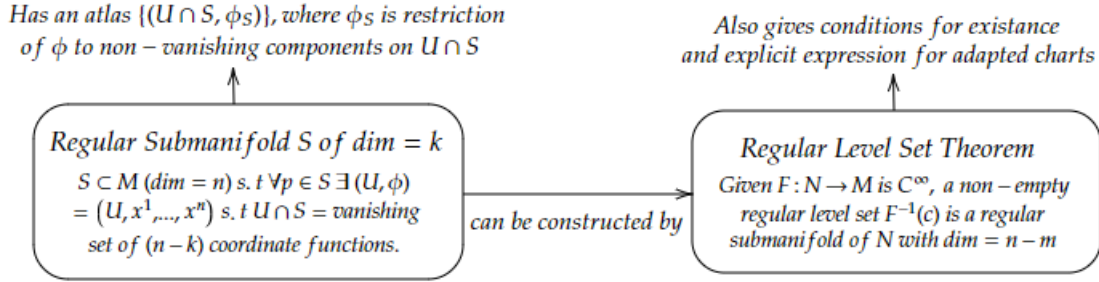
**Problem 8.10 Local Maxima**

*Solution:* Assume  $p \in M$  is a local maxima of the  $C^\infty$  function  $f : M \rightarrow \mathbb{R}$ , where  $p$  is contained in the chart  $(U, x^1, \dots, x^n)$ . Let  $X_p = \frac{\partial}{\partial x^i}$ , and choose a curve  $c(t)$  in  $M$  such that  $c(0) = p$  and  $c'(0) = X_p$ . Then we have:

$$\frac{\partial f}{\partial x^i} \Big|_p = X_p f = c'(0) f = \frac{d}{dt} \Big|_0 (f \circ c) = 0,$$

where the last step follows from the fact that  $f \circ c : (-\epsilon, \epsilon) \rightarrow \mathbb{R}$  is a differentiable real-valued function that has a local maxima at 0. As  $i$  was arbitrary, we have that  $p$  is a critical point.

## 3.2 Submanifolds



### 9.4\* Regular Submanifolds

*Solution:* Consider any  $p \in S \Rightarrow \exists$  a neighbourhood  $U$  of  $p$  s.t. one of the coordinates, either  $x$  or  $y$  is a  $C^\infty$  function of the other. Assume WLOG that  $x = p(y)$ , which gives us that  $\exists$  a chart  $V \subset U, x = p(y), y$ . Consider the function  $F: U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$  s.t.  $F(x, y) = (x - p(y), y)$ . We can see that the  $\det \left[ \frac{\partial F^i}{\partial x^j}(q) \right] = 1 \neq 0$  for all  $q \in U \Rightarrow F$  is a diffeomorphism and  $(U, x - p(y), y)$  is a chart around  $p$  s.t.  $U \cap S = Z(x - p(y))$ . Thus, as  $p$  was arbitrary,  $S$  is a regular submanifold of codimension 1.

### Problem 9.6 Euler's formula

*Solution:* Assume  $F \in S[x_0, x_1, \dots, x_n]$  is a homogenous polynomial of degree  $k$ . Then  $F = \sum_{\alpha} c_{\alpha} x^{\alpha}$ , where  $\alpha$  are multi-indices depicting different monomials of degree  $k$  (for example for  $\alpha = (1, 0, 3) = x_0^1 x_1^0 x_2^3$  is a monomial of degree 4 in  $S[x_0, x_1, x_2]$ ). Then we have:

$$\sum_{i=0}^n x_i \frac{\partial F}{\partial x_i} = \sum_{i=0}^n x_i \frac{\partial \sum_{\alpha} c_{\alpha} x^{\alpha}}{\partial x_i} = \sum_{\alpha} c_{\alpha} \sum_{i=0}^n x_i \frac{\partial x^{\alpha}}{\partial x_i} = \sum_{\alpha} c_{\alpha} \sum_{i=0}^n \alpha_i x^{\alpha} = \sum_{\alpha} k c_{\alpha} x^{\alpha} = kF.$$

### Problem 9.7 Smooth projective hypersurfaces

*Solution:* Assume we have a projective hypersurface defined by  $F^{-1}(0) = Z(F(x_0, x_1, x_2)) \subset \mathbb{RP}^2$ . We need to show that given  $\left( \frac{\partial F}{\partial x_i} \right)$  are not simultaneously zero on  $F^{-1}(0)$ ,  $F^{-1}(0)$  is a regular level set and hence the projective hypersurface is smooth. For this, we need to calculate the Jacobian determinant of the function  $F: \mathbb{RP}^n \rightarrow \mathbb{R}$  for all  $p \in F^{-1}(0)$ . Given the standard atlas on  $\mathbb{RP}^n$ , assume WLOG that  $p \in F^{-1}(0) \cap U_0$  and that the respective chart is  $(U_0, \phi) = \left( U_0, x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0} \right)$  (Note that  $\phi^{-1}(x, y) = [1 : x : y]$ ). Then we have:

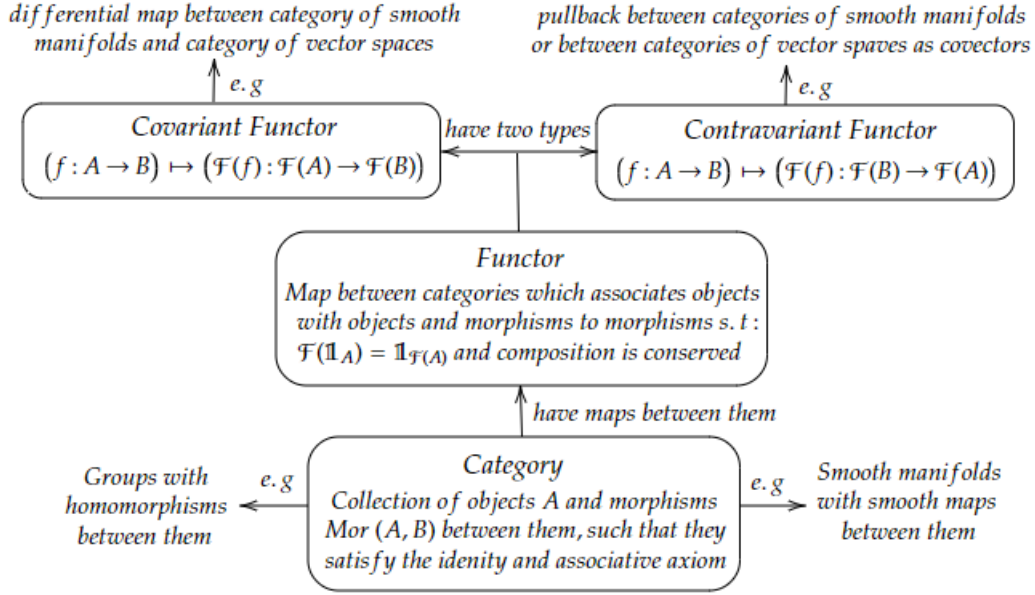
$$\frac{\partial}{\partial x} \Big|_p F = \frac{\partial}{\partial r^1} \Big|_{\phi(p)} (F \circ \phi^{-1}) = \frac{\partial F}{\partial x_1}(p).$$

Similary one can find that  $\frac{\partial}{\partial y} \Big|_p F = \frac{\partial F}{\partial x_2}(p)$ . Now assume on the contrary that the



Jacobian determinant is zero, i.e.,  $\frac{\partial F}{\partial x_1}(p) = \frac{\partial F}{\partial x_2}(p) = 0$ . Then by Problem 9.6, we can see that as  $x_0 \neq 0$  and  $F(p) = 0 \Rightarrow \frac{\partial F}{\partial x_0}(p) = 0$ , a contradiction. Hence the Jacobian determinant must be non-zero  $\Rightarrow p$  is a regular point. One can similarly argue this for  $p \in F^{-1}(0) \cap U_1$  or  $F^{-1}(0) \cap U_2$ , to show that  $F^{-1}(0)$  is a regular level set. Hence the projective hypersurface is smooth.

### 3.3 Categories and Functors



#### Problem 10.1 Differential of the inverse map

*Solution:* Assume we have a diffeomorphism  $F : N \rightarrow M$  and that  $p \in N$ . Then we have the differential maps  $F_{*,p} : T_p N \rightarrow T_{F(p)} M$  and  $(F^{-1})_{*,F(p)} : T_{F(p)} M \rightarrow T_p N$  such that:

$$((F^{-1})_{*,F(p)} \circ F_{*,p})(X_p)f = (F_{*,p}(X_p))(f \circ F^{-1}) = X_p(f \circ F^{-1} \circ F) = X_p(f),$$

for arbitrary  $X_p \in T_p N$  and  $f \in C^\infty(N)$ . Thus  $((F^{-1})_{*,F(p)} \circ F_{*,p})(X_p) = X_p \Rightarrow (F^{-1})_{*,F(p)} \circ F_{*,p} = \mathbb{1}_{T_p N}$ . Similarly, it can be shown that  $F_{*,p} \circ (F^{-1})_{*,F(p)} = \mathbb{1}_{T_{F(p)} M}$ . Thus  $(F^{-1})_{*,F(p)} = (F_{*,p})^{-1}$  is an isomorphism of vector spaces  $T_p N$  and  $T_{F(p)} M$ .

#### Proposition 10.3 (Isomorphism under a functor)

*Proof:* Let  $\mathfrak{F} : \mathfrak{C} \rightarrow \mathfrak{D}$  be a covariant functor from a category  $\mathfrak{C}$  to category  $\mathfrak{D}$ . Assume  $f \in \text{Mor}(A, B)$  is an isomorphism in  $\mathfrak{C} \Rightarrow \exists g \in \text{Mor}(B, A)$  s.t.  $g \circ f = \mathbb{1}_A$  and  $f \circ g = \mathbb{1}_B$ . Now  $\mathfrak{F}(f) : \mathfrak{F}(A) \rightarrow \mathfrak{F}(B)$  and  $\mathfrak{F}(g) : \mathfrak{F}(B) \rightarrow \mathfrak{F}(A)$  are morphisms in  $\mathfrak{D}$ . Then we have:

$$\mathfrak{F}(f) \circ \mathfrak{F}(g) = \mathfrak{F}(f \circ g) = \mathfrak{F}(\mathbb{1}_B) = \mathbb{1}_{\mathfrak{F}(B)},$$

and similarly  $\mathfrak{F}(g) \circ \mathfrak{F}(f) = \mathbb{1}_{\mathfrak{F}(A)}$ . Thus  $\mathfrak{F}(f)$  is an isomorphism in  $\mathfrak{D}$ . One can almost identically prove this for the case where  $\mathfrak{F}$  is a contravariant functor.

#### Problem 10.7 Pullback in the top dimension

*Solution:* Given a linear transformation  $L : V \rightarrow V$  where  $V$  is a vector space of dimension  $n$ , the pullback  $L^* : A_n(V) \rightarrow A_n(V)$  is defined as  $L^*f(v_1, \dots, v_n) =$

$f(L(v_1), \dots, L(v_n))$ . It is easy to see that given  $f$  is alternating,  $f(v_1, \dots, v_n) = 0$  if the set  $\{v_1, \dots, v_n\}$  is linearly dependent: assume  $v_k = \sum_{i \neq k} c^i v_i$  for some  $k$ . Then we have:

$$f(v_1, \dots, v_k, \dots, v_n) = f(v_1, \dots, \sum_{i \neq k} c^i v_i, \dots, v_n) = \sum_{i \neq k} c^i f(v_1, \dots, v_i, \dots, v_n) = 0$$

We also know that if  $\{v_i\}$  is linearly dependent, then so is  $\{L(v_i)\}$ . Thus  $f(v_1, \dots, v_n) = 0 = L^*f(v_1, \dots, v_n) = \text{Det}(L)L^*f(v_1, \dots, v_n)$  trivially holds. Now we assume that the set  $\{v_i\}$  is strictly linearly independent  $\Rightarrow$  it is a basis for  $V$ . Assume the matrix for  $L$  in this basis  $\Rightarrow L(v_i)$  are the columns of this matrix. We know that the determinant function is also a  $n$ -covector, and that:

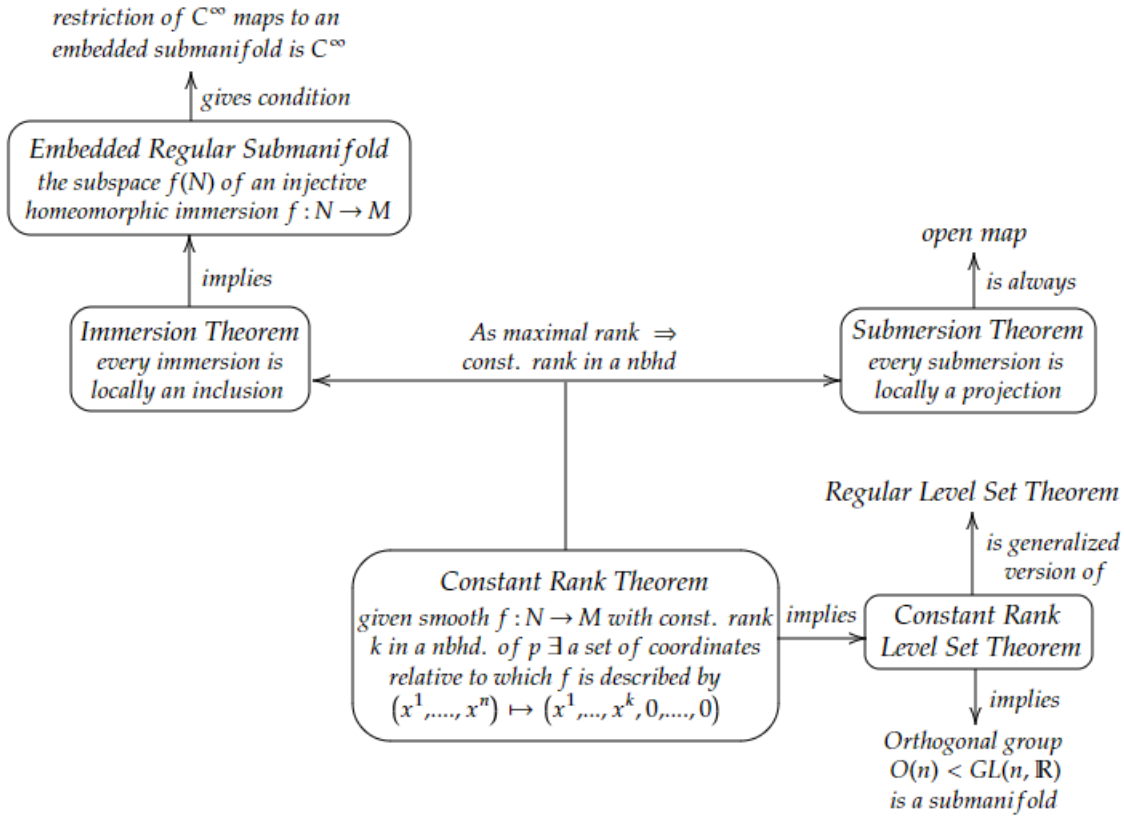
$$d = \text{Det}(L) = \text{Det}(L(v_1), \dots, L(v_n)) = d \times \text{Det}(v_1, \dots, v_n),$$

where  $\text{Det}(v_1, \dots, v_n) = \text{Det}(\mathbb{I}) = 1$ . Note that the factoring out of  $d$  in the last step above follows from the well known *row operations*. These operations are based only on the alternating property of the determinant function, and hence can just as easily be generalized to any arbitrary  $k$ -covector, hence giving:

$$L^*f(v_1, \dots, v_n) = f(L(v_1), \dots, L(v_n)) = d \times f(v_1, \dots, v_n),$$

where  $d = \text{Det}(L)$ .

### 3.4 The Rank of a Smooth Map



#### Problem 11.3\* Critical points of a smooth map on a compact manifold

*Solution:* Consider a map  $f: N \rightarrow \mathbb{R}^m$  where  $N$  is a compact manifold. Assume  $f$  has no critical points, that is, it is a submersion  $\Rightarrow f$  is an open map. Then  $f(N)$  is an open subset of  $\mathbb{R}^m$ . Also, as  $N$  is compact and  $f$  is continuous,  $f(N)$  is compact in  $\mathbb{R}^m \Rightarrow f(N)$  is closed in  $\mathbb{R}^m$ . Also, as  $\mathbb{R}^m$  is not compact itself,  $f(N) \neq \mathbb{R}^m$ . Thus,  $\mathbb{R}^m = f(N) \cup f(N)^c$  forms a separation of  $\mathbb{R}^m \Rightarrow$  this is a contradiction as  $\mathbb{R}^m$  is connected. Thus  $f$  must have a critical point.

#### Problem 11.4 Differential of an inclusion map

*Solution:* Assume we have the coordinate map  $\phi(a, b, c) = (u, v)$ , with  $u(a, b, c) = a$  and  $v(a, b, c) = b$  on the upper hemisphere of  $S^2 \subset \mathbb{R}^3$ . Then we have for  $i_*: T_p S^2 \rightarrow T_p \mathbb{R}^3$  at  $p = (a, b, c)$ :

$$\begin{aligned}
 i_* \left( \frac{\partial}{\partial u} \Big|_p \right) &= \alpha^1 \left( \frac{\partial}{\partial x} \Big|_p \right) + \beta^1 \left( \frac{\partial}{\partial y} \Big|_p \right) + \gamma^1 \left( \frac{\partial}{\partial z} \Big|_p \right) \\
 \Rightarrow i_* \left( \frac{\partial}{\partial u} \Big|_p \right) (x) &= \alpha^1 \left( \frac{\partial}{\partial x} \Big|_p \right) (x) + \beta^1 \left( \frac{\partial}{\partial y} \Big|_p \right) (x) + \gamma^1 \left( \frac{\partial}{\partial z} \Big|_p \right) (x) \\
 \Rightarrow \left( \frac{\partial}{\partial u} \Big|_p \right) (x \circ i) &= \left( \frac{\partial}{\partial r^1} \Big|_{(a,b)} \right) (x \circ i \circ \phi^{-1}) = \left( \frac{\partial}{\partial r^1} \Big|_{(a,b)} \right) (r^1) = 1 = \alpha^1.
 \end{aligned}$$

Similarly:

$$\begin{aligned} i_* \left( \frac{\partial}{\partial u} \Big|_p \right) (y) &= \alpha^1 \left( \frac{\partial}{\partial x} \Big|_p \right) (y) + \beta^1 \left( \frac{\partial}{\partial y} \Big|_p \right) (y) + \gamma^1 \left( \frac{\partial}{\partial z} \Big|_p \right) (y) \\ \Rightarrow \left( \frac{\partial}{\partial u} \Big|_p \right) (y \circ i) &= \left( \frac{\partial}{\partial r^1} \Big|_{(a,b)} \right) (y \circ i \circ \phi^{-1}) = \left( \frac{\partial}{\partial r^1} \Big|_{(a,b)} \right) (r^2) = 0 = \beta^1, \end{aligned}$$

and:

$$\begin{aligned} i_* \left( \frac{\partial}{\partial u} \Big|_p \right) (z) &= \alpha^1 \left( \frac{\partial}{\partial x} \Big|_p \right) (z) + \beta^1 \left( \frac{\partial}{\partial y} \Big|_p \right) (z) + \gamma^1 \left( \frac{\partial}{\partial z} \Big|_p \right) (z) \\ \Rightarrow \left( \frac{\partial}{\partial u} \Big|_p \right) (z \circ i) &= \left( \frac{\partial}{\partial r^1} \Big|_{(a,b)} \right) (z \circ i \circ \phi^{-1}) = \left( \frac{\partial}{\partial r^1} \Big|_{(a,b)} \right) (\sqrt{1 - (r^1)^2 - (r^2)^2}) = -\frac{a}{c} = \gamma^1. \end{aligned}$$

Similarly, one can see that:

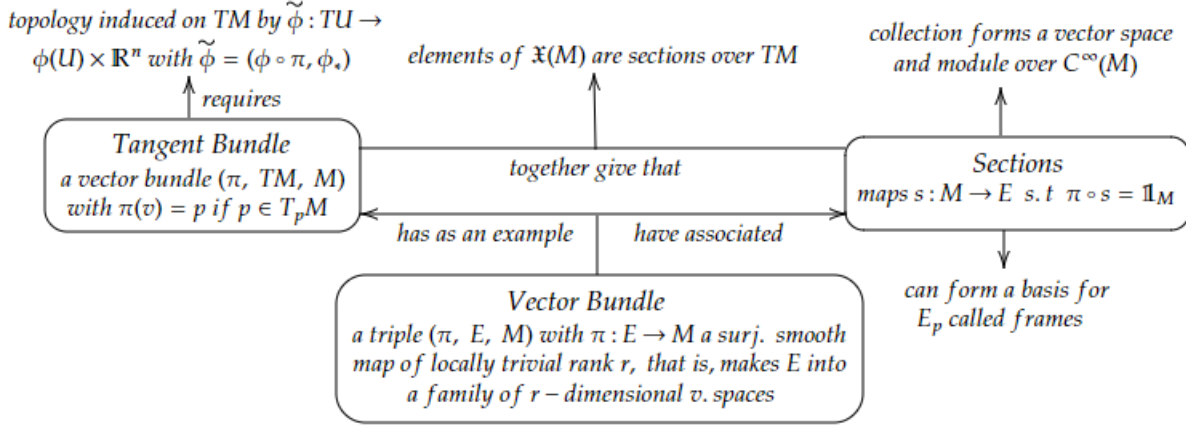
$$i_* \left( \frac{\partial}{\partial v} \Big|_p \right) = \alpha^2 \left( \frac{\partial}{\partial x} \Big|_p \right) + \beta^2 \left( \frac{\partial}{\partial y} \Big|_p \right) + \gamma^2 \left( \frac{\partial}{\partial z} \Big|_p \right),$$

with  $\alpha^2 = 0$ ,  $\beta^2 = 1$ ,  $\gamma^2 = -\frac{b}{c}$ .

**Problem 11.5 One-to-one immersion of a compact manifold**

*Solution:* Assume  $f : N \rightarrow M$  is a one-to-one immersion (hence continuous) with  $N$  as a compact manifold. Then the induced map  $\tilde{f} : N \rightarrow f(N)$  is continuous and bijective. Now take any closed set  $C \subset N \Rightarrow C$  is compact  $\Rightarrow \tilde{f}(C)$  is compact in  $f(N)$ . As  $f(N)$  is Hausdorff under the subspace topology,  $\tilde{f}(C)$  is closed in  $f(N)$ . Thus  $\tilde{f}^{-1}$  is continuous, and  $\tilde{f}$  is a homeomorphism of  $N$  with its image  $f(N)$ . Thus  $f$  is an embedding.

### 3.5 The Tangent Bundle



#### Problem 12.2 Transition functions for the total space of the tangent bundle

*Solution:* Consider two overlapping coordinate charts on a manifold  $M$  given by  $(U, x^1, \dots, x^n) = (U, \phi)$  and  $(V, x^1, \dots, x^n) = (U, \psi)$ . They induce coordinate charts  $(TU, \tilde{\phi})$  and  $(TV, \tilde{\psi})$  on  $TM$ . This gives us the transition function  $F = \tilde{\psi} \circ \tilde{\phi}^{-1}$  with  $(x^1(p), \dots, x^n(p), a^1, \dots, a^n) \mapsto (y^1(p), \dots, y^n(p), b^1, \dots, b^n)$  where  $\sum_i a^i \frac{\partial}{\partial x^i} = \sum_i b^i \frac{\partial}{\partial y^i}$ .

Not that if we apply  $y^j$  to both sides of this relation, we get:

$$\sum_i a^i \frac{\partial y^j}{\partial x^i} = b^j \quad (1)$$

Now we can get to calculating the Jacobian matrix of this transition functions. Firstly, note that by definition,  $\frac{\partial F^i}{\partial r^j} = 0$  for  $1 \leq i \leq n$  and  $n+1 \leq j \leq 2n$ , or for  $n+1 \leq i \leq 2n$  and  $1 \leq j \leq n$ , i.e,  $b^i$  only depend on  $a^i$  and  $y^i$  only depend on  $x^i$ . We can see that for  $n+1 \leq i, j \leq 2n$ , (1) gives us that:

$$\frac{\partial F^i}{\partial r^j} = \frac{\partial b^i}{\partial a^j} = \frac{\partial y^i}{\partial x^j}.$$

Furthermore, for  $1 \leq i, j \leq n$ :

$$\frac{\partial F^i}{\partial r^j} = \frac{\partial (y^i \circ \phi^{-1})}{\partial r^j} = \frac{\partial y^i}{\partial x^j}.$$

Thus the Jacobian matrix is the following block matrix:

$$\begin{pmatrix} \left[ \frac{\partial y^i}{\partial x^j} \right] & 0 \\ 0 & \left[ \frac{\partial y^i}{\partial x^j} \right] \end{pmatrix}$$

It is not hard to see that the Jacobian determinant is then  $(\det[\partial y^i / \partial x^j])^2$ .

**Proposition 12.9(ii) Smoothness of scalar multiplication**

*Solution:* Assume  $s$  is a  $C^\infty$  section of a  $C^\infty$  vector bundle  $\pi : E \rightarrow M$ , and  $f$  is a  $C^\infty$  real-valued function on  $M$ . We want to show that  $(fs) : M \rightarrow E$  defined as  $(fs)(p) = f(p)s(p)$  is a  $C^\infty$  section of  $E$ . Firstly, it is clear that it is indeed a section of  $E$ , as  $E_p$  is closed under scalar multiplication for all  $p$ . Now consider any point  $p \in M$ , and a corresponding trivializing open set  $\pi(p) \in U$  with trivialization  $\phi \Rightarrow \exists(V, \psi)$  with  $p \in V \subseteq \pi^{-1}(U)$ . As  $s$  is  $C^\infty \Rightarrow ((\psi \times \mathbb{1}) \circ \phi \circ s \circ \psi^{-1})$  is  $C^\infty$ . We can see that:

$$((\psi \times \mathbb{1}) \circ \phi \circ s \circ \psi^{-1})(\psi(p)) = (p, c^1, \dots, c^n),$$

for some  $C^\infty$  functions  $c^i$ . Now if we instead consider  $((\psi \times \mathbb{1}) \circ \phi \circ (fs) \circ \psi^{-1})$ , as  $\phi$  is  $\mathbb{R}$ -linear:

$$((\psi \times \mathbb{1}) \circ \phi \circ s \circ \psi^{-1})(\psi(p)) = (p, f(p)c^1, \dots, f(p)c^n),$$

which is also  $C^\infty$  as the product  $(fc^i)$  is  $C^\infty$  for all  $i$ . Thus  $(fs)$  is a  $C^\infty$  section on  $E$ .

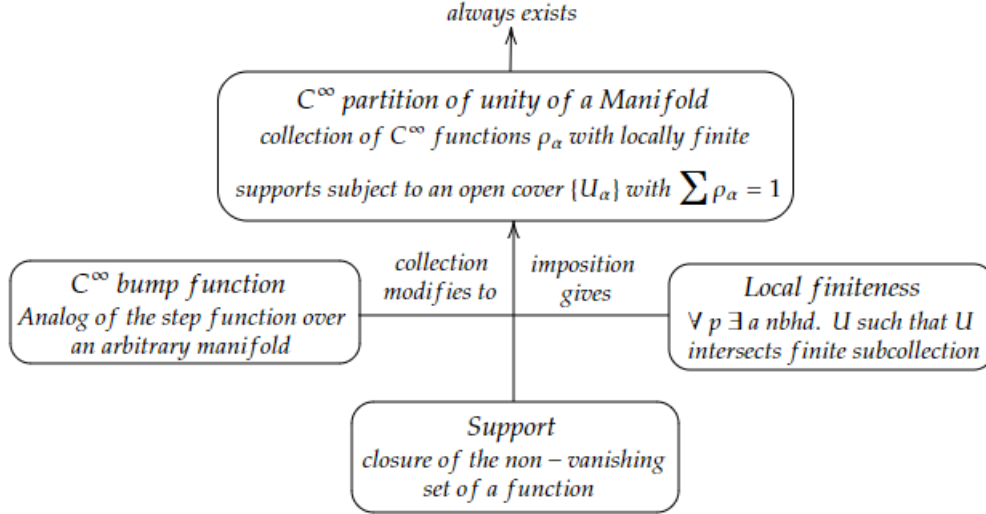
**Problem 12.4 Coefficients relative to a smooth frame**

*Solution:* Let  $\pi : E \rightarrow M$  be a  $C^\infty$  vector bundle and  $s^1, \dots, s^r$  be a  $C^\infty$  frame for  $E$  over an open set  $U$  and  $t_1, \dots, t_r$  be the frame of trivialization. Then for any  $e \in \pi^{-1}(U)$ ,  $e = \sum_{j=1}^r b^j(e)t_j(p) = \sum_{j=1}^r c^j(e)s_j(p)$ . We show that  $t_j : \pi^{-1}(U) \rightarrow \mathbb{R}$  is  $C^\infty$ , and hence so are  $s_j$ . Consider the following action of the trivialization  $\phi$  of  $E$  over  $U$ :

$$\phi(e) = \phi\left(\sum_{j=1}^r b^j(e)t_j(p)\right) \stackrel{\text{lin.}}{=} \sum_{j=1}^r b^j(e)\phi(t_j(p)) = \sum_{j=1}^r b^j(e)(\pi(e), e_j) = (\pi(e), b^1(e), \dots, b^r(e)),$$

where  $b^j$  are components of  $\phi$ . As  $\phi$  is  $C^\infty$ , so are the  $b^j$ . Now we can take  $s_i(p) = \sum_j d_i^j t_j(p) \Rightarrow \sum_j b^j(e)t_j(p) = \sum_{i,j} c^i(e)d_i^j t_j(p) \Rightarrow b^j(e) = \sum_i c^i(e)d_i^j$ . This equation is equivalent to a change of basis, and so the matrix  $[d_i^j]$  is invertible. As the matrix  $[d_i^j]$  consists of  $C^\infty$  functions, its inverse multiplied by the column vector  $[b^j]$  will give that  $c^i$  are also  $C^\infty$ .

### 3.6 Bump Functions and Partitions of Unity



The idea of a  $C^\infty$  partition is that it gives an explicit connection between the local and global properties of a smooth manifold. One can understand this somewhat intuitively considering that  $\rho_\alpha$  are 'local' functions (vanish outside  $U_\alpha$ ) with the global property of their sum being consistently equal to 1.

#### Lemma 13.5\* Support of a finite sum

*Proof:* Define  $F = \sum_i \rho_i$ . Consider any  $p \in F^{-1}(\mathbb{R}^\times)$ . This implies that  $\rho_j(p) \neq 0$  for some  $j \Rightarrow p \in \rho_j^{-1}(\mathbb{R}^\times)$ . Thus  $F^{-1}(\mathbb{R}^\times) \subseteq \bigcup_i \rho_i^{-1}(\mathbb{R}^\times) \Rightarrow \text{cl}(F^{-1}(\mathbb{R}^\times)) \subseteq \text{cl}(\bigcup_i \rho_i^{-1}(\mathbb{R}^\times)) = \bigcup_i \text{cl}(\rho_i^{-1}(\mathbb{R}^\times)) \Rightarrow \text{supp}(\sum_i \rho_i) \subseteq \bigcup_i \text{supp}(\rho_i)$ .

#### Problem 13.2\* Locally finite family and compact set

*Solution:* For every  $p \in K$ , take a neighbourhood  $U_p$  that intersects finitely many of  $\{A_\alpha\}$ . Then  $\{U_p\}_{p \in K}$  forms an open cover of  $K$ . As  $K$  is compact  $\Rightarrow$  there exists a finite subcover  $\{U_i\}_{i=1}^n$ . Then  $\bigcup_i U_i$  is neighbourhood of  $K$  that intersects finitely many  $\{A_\alpha\}$ .

#### Problem 13.3 Smooth Urysohn Lemma

*Solution(a):* Given two disjoint closed sets  $A$  and  $B$  of a manifold  $M$ , consider the open cover  $\{M - A, M - B\}$ . By Theorem 13.7, there exists a  $C^\infty$  partition of unity  $\{\rho_{M-A}, \rho_{M-B}\}$  subordinate to the open cover. Thus as  $\text{supp}(\rho_{M-B}) \subseteq (M - B) \Rightarrow \rho_{M-B}(B) = 0$ . Similarly  $\rho_{M-A}(A) = 0$ . Moreover, as  $\rho_{M-A} + \rho_{M-B} = 1 \Rightarrow \rho_{M-B}(A) = \{1\}$ . Thus  $\rho_{M-B}$  is a  $C^\infty$  function that is identically 1 on  $A$  and identically 0 on  $B$ .

*Solution(b):* Let  $A$  be a closed subset of  $M$  and  $U$  be an open set containing  $A$ . Then  $\{U, M - A\}$  is an open cover of  $M \Rightarrow$  by Theorem 13.7, there exists a  $C^\infty$  partition  $\{\rho_U, \rho_{M-A}\}$  subordinate to the open cover. As  $\rho_{M-A}(A) = \{0\}$  and  $\rho_U + \rho_{M-A} = 1 \Rightarrow \rho_U(A) = \{1\}$ . Thus  $\rho_U$  is a  $C^\infty$  function that is identically 1 on  $A$  with  $\text{supp}(\rho_U) \subseteq U$  by definition.



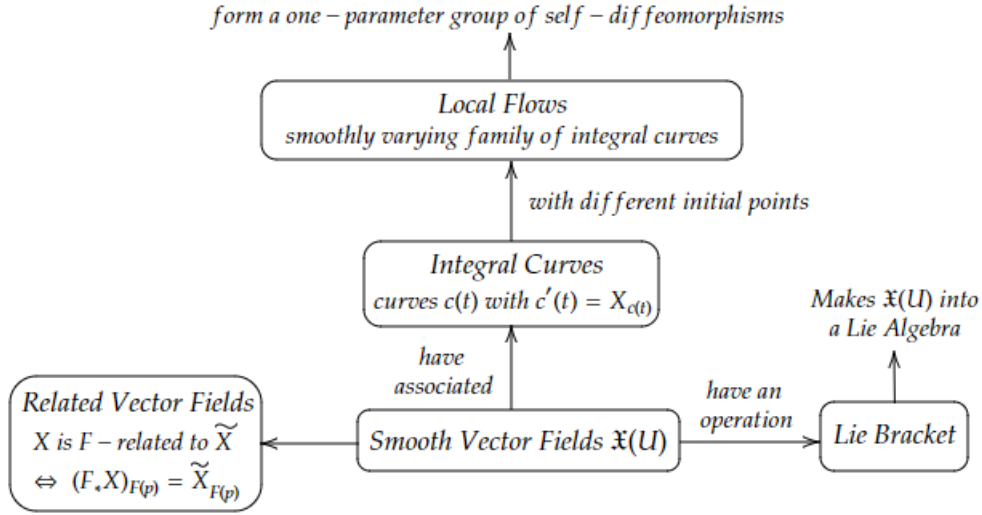
**Problem 13.4 Support of the pullback of a function**

*Solution:* Let  $F : N \rightarrow M$  and  $h : M \rightarrow \mathbb{R}$  be  $C^\infty$ . We can see that  $(F^*h)^{-1}(\mathbb{R}^\times) = F^{-1}(h^{-1}(\mathbb{R}^\times)) \subseteq F^{-1}(\text{cl}(h^{-1}(\mathbb{R}^\times))) = F^{-1}(\text{supp}(h)) \Rightarrow \text{cl}(F^{-1}(h^{-1}(\mathbb{R}^\times))) \subseteq \text{cl}(F^{-1}(\text{supp}(h))) = F^{-1}(\text{supp}(h))$  as  $F^{-1}(\text{supp}(h))$  is the continuous preimage of a closed set, and hence closed.

**Problem 13.6 Pullback of a partition of unity**

*Solution:* Take  $\{\rho_\alpha\}$  to be a partition of unity on a manifold  $M$  subordinate to an open cover  $\{U_\alpha\}$  and  $F : N \rightarrow M$  be  $C^\infty$ . Consider the collection  $\{\text{supp}(F^*\rho_\alpha)\}$ . We know from Problem 13.4 that  $\text{supp}(F^*\rho_\alpha) \subseteq F^{-1}(\text{supp}(\rho_\alpha))$ . Thus if we are able to show that the collection  $\{F^{-1}(\text{supp}(\rho_\alpha))\}$  is locally finite, so is  $\{\text{supp}(F^*\rho_\alpha)\}$ . Consider any  $p \in N \Rightarrow F(p) \in M$ . As  $\{\text{supp}(\rho_\alpha)\}$  is locally finite, there exists a neighbourhood  $U$  of  $F(p)$  such that  $U$  intersects only finitely many supports  $\Rightarrow F^{-1}(U)$  is neighbourhood of  $p$  that intersects finitely many  $F^{-1}(\text{supp}(\rho_\alpha)) \Rightarrow \{F^{-1}(\text{supp}(\rho_\alpha))\}$  is locally finite and so is  $\{\text{supp}(F^*\rho_\alpha)\}$ . It is trivial to see that  $F^*\rho_\alpha$  are also non-negative functions, and as  $\sum_\alpha \rho_\alpha = 1$ , this implies that  $\sum_\alpha (F^*\rho_\alpha) = \sum_\alpha (\rho_\alpha \circ F) = 1$ . Finally, we can see that  $\text{supp}(F^*\rho_\alpha) \subseteq F^{-1}(\text{supp}(\rho_\alpha)) \subseteq F^{-1}(U_\alpha)$  from the initial assumption about  $\{\rho_\alpha\}$ . Thus,  $\{F^*\rho_\alpha\}$  forms a partition of unity on  $N$  subordinate to the open cover  $\{F^{-1}(U_\alpha)\}$ .

### 3.7 Vector Fields



#### Problem 14.2 Vector field on an odd sphere

*Solution:* The gradient vector on  $S^{2n-1}$  is defined as  $v = (2x^1, \dots, 2x^n, 2y^1, \dots, 2y^n)$  at a point  $p = (x^1, \dots, x^n, y^1, \dots, y^n)$ . Representing the vector field  $X$  at point  $p$  as the vector  $X_p = (-y^1, \dots, -y^n, x^1, \dots, x^n)$ , we can see that  $\sum_i v^i X_p^i = \sum_i (-y^i x^i + y^i x^i) = 0$ . Thus  $X_p$  is tangent to  $S^{2n-1}$ , and hence, as  $p$  was arbitrary,  $X$  defines vector field on  $S^{2n-1}$ . We can also see that it is non-vanishing, as for it to vanish  $y^i = x^i = 0$  for all  $i \Rightarrow \sum_i (x^i)^2 + (y^i)^2 = 0 \neq 1$ , which is a contradiction as the point must lie on  $S^{2n-1}$ . To check the smoothness of the vector field, consider the following atlas on  $S^{2n-1}$ :  $\{(U_{i1} = \{p \in S^{2n-1} : x^i > 0\}, \pi_{x^i}), (U_{i2} = \{p \in S^{2n-1} : x^i < 0\}, \pi_{x^i}), (V_{i1} = \{p \in S^{2n-1} : y^i > 0\}, \pi_{y^i}), (V_{i2} = \{p \in S^{2n-1} : y^i < 0\}, \pi_{y^i})\}_{i=1}^n$ . Consider now any point  $p \in S^{2n-1}$ . WLOG assume  $p \in U_{11}$  with chart  $\pi_{x^1} = (x^2, \dots, x^n, y^1, \dots, y^n)$ . The vector field in terms of these coordinates is then  $X = x^1 \frac{\partial}{\partial y^1} + \sum_{i=2}^n (-y^i \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial y^i})$ . Here we must note that as  $x^1 \circ \pi_{x^1}^{-1}(x^2, \dots, x^n, y^1, \dots, y^n) = \sqrt{(x^2)^2 + \dots + (x^n)^2 + (y^1)^2 + \dots + (y^n)^2}$  is  $C^\infty$ ,  $x^1$  is  $C^\infty$  at  $p$ . The other coefficients are trivially smooth, giving that the  $X$  is a smooth vector field on  $S^{2n-1}$ .

#### Problem 14.4 Integral curves in the plane

*Solution:* Consider the vector field  $X_{(x,y)} = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$  on  $\mathbb{R}^2$ . Then its integral curve  $c(t) = (x(t), y(t))$  is given by the following differential equations:  $\dot{x}(t) = x$  and  $\dot{y}(t) = -y$ . These are standard ODEs with solutions  $x(t) = e^t + c$  and  $y(t) = e^{-t} + d$  for some constants  $c, d$ . Taking our initial point to be  $p = (x_0, y_0)$ , this gives  $c = x_0 - 1$  and  $d = y_0 - 1$ .

#### Problem 14.13 Vector field under a diffeomorphism

*Solution:* Let  $F : N \rightarrow M$  be a diffeomorphism, and  $g$  be a  $C^\infty$  function and  $X$  be a  $C^\infty$  vector field on  $N$ . Consider now two vector fields:  $F_*(gX)$  and  $(g \circ F^{-1})F_*X$ . We

prove these are equal by acting them on an arbitrary  $C^\infty$  function  $f$  on  $M$  at  $F(p)$ :

$$\begin{aligned}(F_*(gX))_{F(p)}f &= (gX)_p(f \circ F) = g(p)X_p(f \circ F). \\ ((g \circ F^{-1})F_*X)_{F(p)}f &= (g \circ F^{-1})(F(p))(F_*X)_{F(p)}f = g(p)X_p(f \circ F).\end{aligned}$$

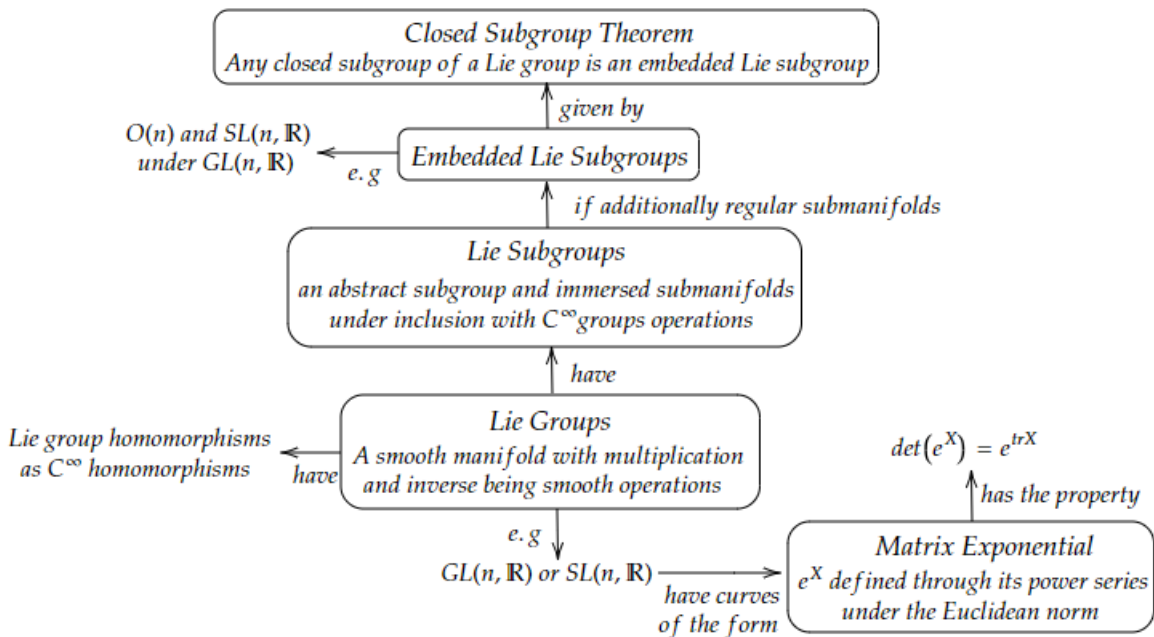
**Problem 14.14 Lie bracket under a diffeomorphism**

*Solution:* Let  $F : N \rightarrow M$  be a  $C^\infty$  diffeomorphism, with  $X$  and  $Y$  as  $C^\infty$  vector fields on  $N$ . Consider the following related vectors fields  $F_*[X, Y]$  and  $[F_*X, F_*Y]$ . Let  $p$  be an arbitrary point in  $N$  contained in a chart  $(U, x^1, \dots, x^n)$ , and  $X = \sum_i a_i(p) \frac{\partial}{\partial x^i}$ ,  $Y = \sum_i b_i(p) \frac{\partial}{\partial x^i}$ . We prove the following are equal by acting them on an arbitrary  $C^\infty$  function  $f$  on  $M$  at  $F(p)$ :

$$\begin{aligned}(F_*[X, Y])_{F(p)}f &= [X, Y]_p(f \circ F) = (X_pY - Y_pX)(f \circ F) \\ [F_*X, F_*Y]_{F(p)}f &= (F_*X)_{F(p)}(F_*Y)f - (F_*Y)_{F(p)}(F_*X)f \\ &= X_p(((F_*Y)f) \circ F) - Y_p(((F_*X)f) \circ F) \\ &= X_p((F_*Y)_F f) - Y_p((F_*X)_F f) = X_p(Y(f \circ F)) - Y_p(X(f \circ F)) \\ &= (X_pY - Y_pX)(f \circ F).\end{aligned}$$

## 4 Lie Groups and Lie Algebras

### 4.1 Lie Groups



#### Problem 15.3 Identity component of a Lie Group

*Solution:*

(i) Clearly, for the identity connected component  $G_0$  of the Lie Group  $G$ ,  $\{x\} \times G_0$  is connected. Thus, as  $\mu$  is smooth,  $\mu(\{x\} \times G_0)$  is connected and contains  $x \in G_0$ . Thus  $\mu(\{x\} \times G_0) \subseteq G_0$ .

(ii) Through a similar argument to above  $i(G_0)$  is connected and contains  $e \in G_0$ . Thus  $i(G_0) \subseteq G_0$ .

(iii) Consider any point  $g \in G_0$  contained in a chart  $(U, \phi) \Rightarrow \phi(g) \in \phi(U) \subseteq \mathbb{R}^n$ . As  $\mathbb{R}^n$  is locally connected, there exists a connected neighbourhood  $V \subseteq \phi(U)$  of  $\phi(g) \Rightarrow \phi^{-1}(V)$  is a connected neighbourhood of  $g \Rightarrow \phi^{-1}(V) \subseteq G_0$ . Thus  $G_0$  is open.

(iv) As  $G_0$  is an open subset of a smooth manifold, it is also a smooth manifold. The restrictions of the multiplication map and the inverse map are also  $C^\infty$  onto  $G_0$ . Furthermore, by (i) and (ii),  $G_0$  is closed under multiplication and inverses. Thus  $G_0$  is a Lie Group.

#### Problem 15.4\* Open subgroup of a connected Lie Group

*Solution:* Consider an open subgroup  $H$  of a connected Lie group  $G$ . Assume  $H \neq G$ . Consider the smooth map  $l_g$  of left-multiplication by  $g \in G$ . Then as  $H = l_g^{-1}(gH)$ ,  $gH$  is open for all  $g$ . It is well-known that  $\{gH\}_{g \in G}$  forms a partition of  $G$ , and hence a separation of  $G$ . This is a contradiction as  $G$  is connected  $\Rightarrow H = G$ .

#### Problem 15.7\* Differential of the determinant map at A

*Solution:* Consider the following curve in  $GL(n, \mathbb{R})$  with  $c(t) = Ae^{tX} \Rightarrow c(0) = A$  and  $c'(0) = AX$ . Thus:

$$\det_{*,A}(AX) = \frac{d}{dt} \Big|_{t=0} \det(c(t)) = \frac{d}{dt} \Big|_{t=0} \det(A) \det(e^{tX}) = \frac{d}{dt} \Big|_{t=0} \det(A) e^{t \operatorname{tr} X} = \det(A) \operatorname{tr} X.$$

### Problem 15.10 Orthogonal Group

*Solution:* Consider the map  $f : GL(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$  given by  $f(A) = A^T A$ . Considering the general linear group as a subset of euclidean space,  $f$  is a polynomial map  $\Rightarrow$  it is smooth. Thus  $f^{-1}(I) = O(n)$  is a closed set. One can also note that  $[A^T A]_{ij} = \sum_k a_{ki} a_{kj}$ . Thus  $[A^T A]_{ii} = \sum_k (a_{ki})^2 \Rightarrow \operatorname{tr}(A^T A) = \sum_i \sum_k (a_{ki})^2 = \|A\|^2 \Rightarrow \operatorname{tr}(I) = n = \|A\|^2$ . Thus  $O(n)$  is a bounded subset of euclidean space  $\Rightarrow$  it is compact.

### Problem 15.12 Unitary group

*Solution:* Define  $f : GL(n, \mathbb{C}) \rightarrow H$  given by  $f(A) = \bar{A}^T A$  where  $H$  is the set of Hermitian matrices. We can see that  $H$ , as a vector space over  $\mathbb{R}$ , is of dimension  $(2n(n-1)/2) + n = n^2$  as the diagonal must be real. In exact analogy to Example 15.6,  $f^{-1}(I)$  is a regular level set, which implies that  $U(n)$  is a regular submanifold of  $GL(n, \mathbb{C})$  of  $\dim(U(n)) = 2n^2 - n^2 + n - n = n^2$ .

### Problem 15.13 Special unitary group $SU(2)$

*Solution:* Consider the following arbitray matrix  $M \in SU(2)$  given by  $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \Rightarrow$

$$M^{-1} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \text{ as } \det(M) = 1. \text{ As we have } \bar{M}^T = M^{-1} \Rightarrow \bar{a} = d \text{ and } c = -\bar{b}.$$

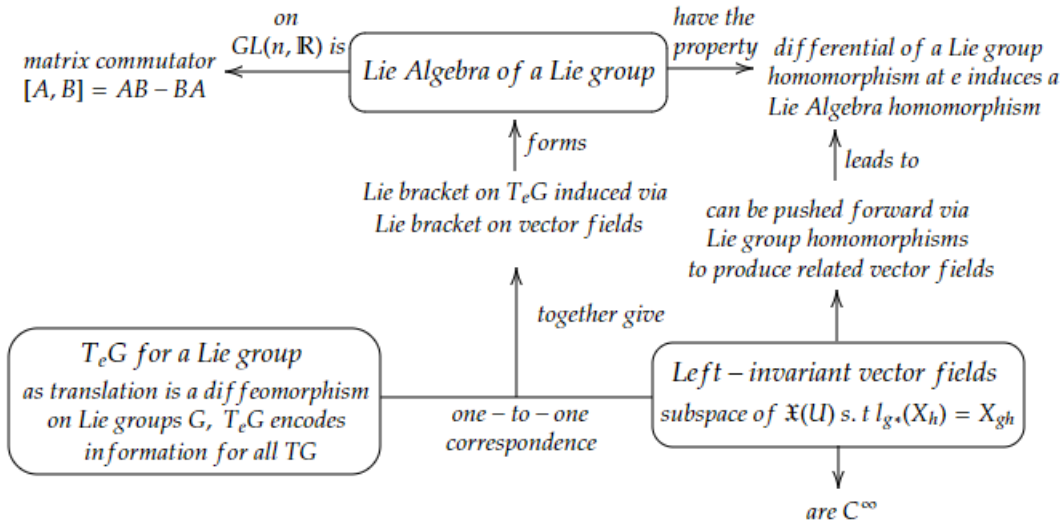
Thus  $M = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$  with  $a\bar{a} + b\bar{b} = 1$ . Now define a map  $f : GL(n, \mathbb{C}) \rightarrow S^3$  given by

$$f(M) = (\operatorname{Re}(a), \operatorname{Im}(a), \operatorname{Re}(b), \operatorname{Im}(b)) = \frac{1}{2}(a + \bar{a}, a - \bar{a}, b + \bar{b}, b - \bar{b}) = \frac{1}{2}(a + d, a - d, b - c, b + c). \text{ Clearly, this is a well-defined map as } \operatorname{Re}(a)^2 + \operatorname{Im}(a)^2 + \operatorname{Re}(b)^2 + \operatorname{Im}(b)^2 = |a|^2 + |b|^2 = a\bar{a} + b\bar{b} = 1. \text{ This map is also } C^\infty \text{ as it consists of polynomials, and has a}$$

$C^\infty$  inverse:  $g : S^3 \rightarrow GL(n, \mathbb{C})$  given by  $g(a, b, c, d) = \begin{pmatrix} (a+b)/2 & (d-c)/2 \\ (c+d)/2 & (a-b)/2 \end{pmatrix}$ . Thus

$GL(n, \mathbb{C})$  is diffeomorphic to  $S^3$ .

## 4.2 Lie Algebras



### Problem 16.2 Lie Algebra of a unitary group

*Solution:* Recall that a unitary matrix  $U$  satisfies the property  $\overline{U}^T U = I$ . Consider any tangent vector  $X \in T_I U(n)$ , and define a curve  $c(t)$  in  $U(n)$  with  $c(0) = I$  and  $c'(0) = X$ . Then we have for all  $t$ :

$$\overline{c(t)}^T c(t) = I \Rightarrow \frac{d}{dt}(\overline{c(t)}^T c(t)) = 0 \Rightarrow (\overline{c(t)}^T)' c(t) + \overline{c(t)}^T c'(t) = 0.$$

Plugging in  $t = 0$  and noting that  $(\overline{c(t)}^T)' = (\overline{c'(t)})^T$ , we get  $\overline{X}^T + X = 0 \Rightarrow X$  is a skew-Hermitian matrix. Also, as argued before,  $\dim(U(n)) = n^2$  is also the dimension of the vector space of skew-Hermitian matrices,  $H$ . Thus  $U(n) = H$ .

### Problem 16.6 Left-invariant vector fields on a circle

*Solution:* We will take  $S^1 = \{e^{it} = 0 \leq t < 2\pi\}$  as a group under standard multiplication, or addition modulo  $2\pi$  on  $t$ . Consider the following atlas for  $S^1 = \{(U_{t \neq 0}, t), (U_{t \neq \pi}, t)\}$ .

Now consider the action of  $l_{g*} \left( \frac{\partial}{\partial t} \Big|_0 \right)$  on the function  $t$  (taking  $l_g$  as addition on  $t$  by  $g$ ):

$$\begin{aligned} l_{g*} \left( \frac{\partial}{\partial t} \Big|_0 \right) &= c_g \frac{\partial}{\partial t} \Big|_g \\ \Rightarrow l_{g*} \left( \frac{\partial}{\partial t} \Big|_0 \right) t &= c_g \frac{\partial}{\partial t} \Big|_g t \\ \Rightarrow \frac{\partial}{\partial t} \Big|_0 (t \circ l_g) &= c_g \\ \Rightarrow c_g &= 1. \end{aligned}$$

As this holds for any  $g$ , any left-invariant vector field must be of the form  $c \frac{\partial}{\partial t}$  where  $c$  is a constant.

### Problem 16.8 Parallelizable manifolds

*Solution:*

( $\Rightarrow$ ) Assume a manifold  $M$  is trivial  $\Rightarrow TM \cong M \times \mathbb{R}^n$ . Set  $s_i : M \rightarrow TM \cong M \times \mathbb{R}^n$  given by  $s_i(p) = (p, e_i)$ , where  $e_i$  is the standard basis for  $\mathbb{R}^n$ . Clearly,  $\{s_i\}$  forms a smooth frame on  $M$ .

( $\Leftarrow$ ) Assume there exists a smooth frame  $\{s_i\}_{i=1}^n$  on  $M$ . Then for any  $p \in M$  and  $v \in T_p M \Rightarrow v = \sum_i c_{p_i}(v) s_i(p)$  for some real-valued function  $c_{p_i}(v)$ . Define the following map:  $\tilde{f} : TM \rightarrow M \times \mathbb{R}^n$  given by  $\tilde{f}(p, v) = (p, c_{p_1}(v), \dots, c_{p_n}(v))$ . Clearly it is bijective, and we can also show that it is  $C^\infty$  as for any chart  $\psi$  on  $TM$ ,  $(\psi \times \mathbb{1}) \circ \tilde{f} \circ \phi^{-1} \circ (\psi \times \mathbb{1})^{-1} = \mathbb{1}$ , where  $\phi$  is the trivialization of  $TM$  over the appropriate open set. Thus  $(\mathbb{1}_M, \tilde{f})$  is a bundle isomorphism between  $TM$  and  $M \times \mathbb{R}^n$ .

### Problem 16.9 Parallelizability of a Lie Group

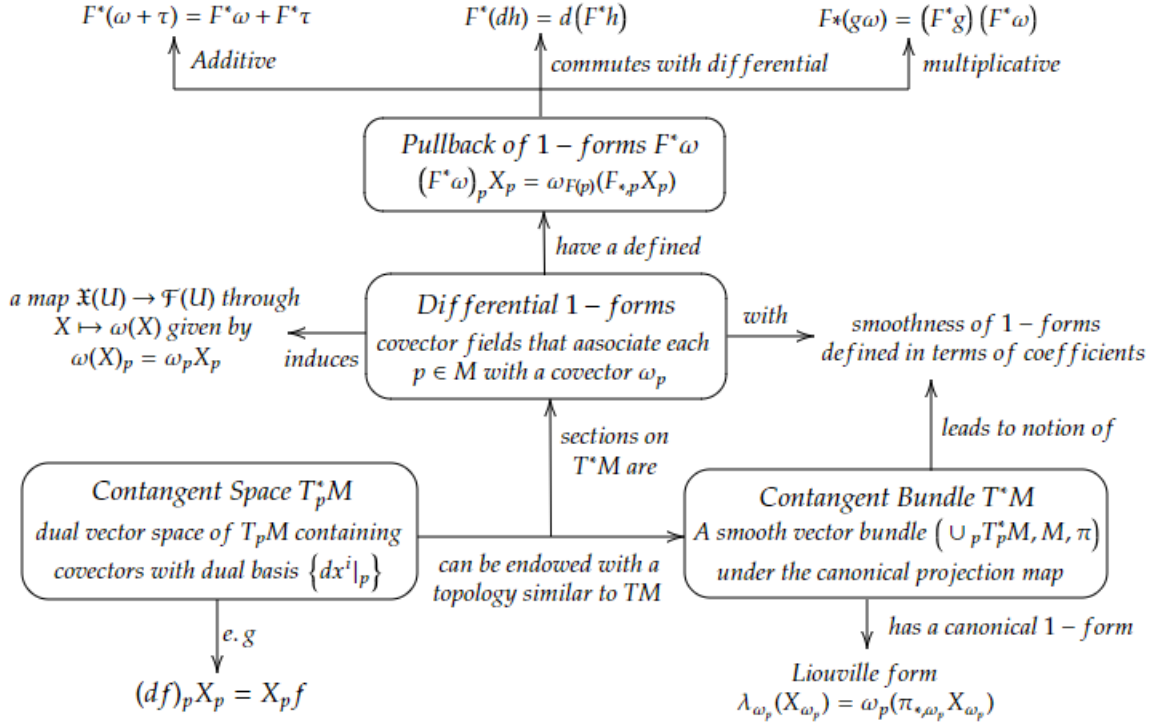
*Solution:* Let  $(U, x^1, \dots, \{e_i\})$  be a basis for the  $T_e G$  where  $G$  is a Lie group. Define  $s_i : G \rightarrow TG$  as  $s_i(g) = (g, l_{g*}(e_i))$ . Clearly,  $s_i$  is smooth for all  $i$ . As  $l_g$  is a diffeomorphism, we know that  $l_{g*}$  is an isomorphism between  $T_e G$  and  $T_g G \Rightarrow$  it maps basis to basis. Thus  $\{l_{g*}(e_i)\}$  forms a basis for  $T_g G$  for all  $g$ . Thus  $\{s_i\}$  forms a smooth frame for  $G$  and thus  $G$  is a parallelizable.

### Problem 16.11 The adjoint representation

*Solution:* Firstly, we can see that  $c_a \circ c_b(g) = c_a(bgb^{-1}) = a(bgb^{-1})a^{-1} = (ab)g(ab)^{-1} = c_{ab}(g)$ . Thus we now have  $Ad(ab) = c_{ab*} = (c_a \circ c_b)_* = (c_{a*} \circ c_{b*})$ . Thus  $Ad : G \rightarrow GL(\mathfrak{g})$  is a homomorphism. Furthermore, identifying  $GL(\mathfrak{g})$  as a subset of euclidean space, we can see that  $Ad_{ij}(a) = \frac{\partial c_a^i}{\partial x^j}(e)$ . We can rewrite  $c_a(x)$  as  $\mu(a, \mu(x, a^{-1}))$  which is  $C^\infty$ , so is  $Ad_{ij}(a)$  as stated above. Thus  $Ad : G \rightarrow GL(\mathfrak{g})$  is  $C^\infty$ .

## 5 Differential Forms

### 5.1 Differential 1-forms



#### Problem 17.3 Pullback of a one-form on $S^1$

*Solution:* Let  $\omega = -ydx + xdy$  as a one-form on  $S^1$ , and  $S^1$  with the standard coordinate chart  $\phi(e^{it}) = t$ . Then we have for any  $g$  and  $p = (p_x, p_y) \in S^1$ :

$$(l_g^*\omega)_p \left( \frac{d}{dt} \Big|_p \right) = \omega_{l_g(p)} \left( l_{g*} \frac{d}{dt} \Big|_p \right) = \omega_{l_g(p)} \left( \left( \frac{d(t \circ l_g)}{dt} \Big|_p \right) \frac{d}{dt} \Big|_{l_g(p)} \right) = \omega_{l_g(p)} \left( \frac{d}{dt} \Big|_{l_g(p)} \right).$$

Now computing:

$$\omega_p \left( \frac{d}{dt} \Big|_p \right) = -p_y \frac{dx}{dt} \Big|_p + p_x \frac{dy}{dt} \Big|_p = p_y \sin(t)|_p + p_x \cos(t)|_p = p_y^2 + p_x^2 = 1.$$

As the answer is independent of  $p$ , we have that  $\omega_{l_g(p)} \left( \frac{d}{dt} \Big|_{l_g(p)} \right) = 1$  as well. As  $p$  was arbitrary, we have thus  $l_g^*\omega = \omega$ .

#### Problem 17.4 Liouville form on the cotangent bundle

*Solution:* The Liouville form is given by  $\lambda_{\omega_p} = \pi^*\omega_p$ . Given the coordinate basis  $(\bar{x}^1, \dots, \bar{x}^n, c_1, \dots, c_n)$ , we have that the basis for  $T^*(T^*M)$  is  $\{d\bar{x}^1, \dots, d\bar{x}^n, dc^1, \dots, dc^n\}$ .



Thus  $\lambda_{\omega_p} = \sum_i a_i d\bar{x}^i|_{\omega(p)} + \sum_j b_j d\bar{c}^j|_{\omega(p)}$ . Assuming  $\omega_p = \sum_i f_i(p) dx^i|_p$ , first we shall act on  $\frac{d}{d\bar{c}^k}\Big|_{\omega_p}$  :

$$\begin{aligned} (\pi^*\omega_p) \frac{d}{d\bar{c}^k}\Big|_{\omega(p)} &= \left( \sum_i a_i d\bar{x}^i|_{\omega(p)} + \sum_j b_j d\bar{c}^j|_{\omega(p)} \right) \frac{d}{d\bar{c}^k}\Big|_{\omega(p)} \\ \Rightarrow \omega_p \left( \pi_* \frac{d}{d\bar{c}^k}\Big|_{\omega(p)} \right) &= \omega_p \left( \sum_j \frac{d(x^j \circ \pi)}{d\bar{c}^k} \frac{d}{dx^j}\Big|_p \right) = 0 = b_k. \end{aligned}$$

Now acting on  $\frac{d}{d\bar{x}^k}\Big|_{\omega_p}$ , we similarly get:

$$\begin{aligned} (\pi^*\omega_p) \frac{d}{d\bar{x}^k}\Big|_{\omega(p)} &= \left( \sum_i a_i d\bar{x}^i|_{\omega(p)} + \sum_j b_j d\bar{c}^j|_{\omega(p)} \right) \frac{d}{d\bar{x}^k}\Big|_{\omega(p)} \\ \Rightarrow \omega_p \left( \pi_* \frac{d}{d\bar{x}^k}\Big|_{\omega(p)} \right) &= \omega_p \left( \sum_j \frac{d(x^j \circ \pi)}{d\bar{x}^k} \frac{d}{dx^j}\Big|_p \right) = \omega_p \left( \frac{d}{dx^k} \right) = f_k(p) = a_k. \end{aligned}$$

Thus  $\lambda_{\omega_p} = \sum_i f_i(p) d\bar{x}^i|_{\omega_p}$ . Furthermore, we can see that the coefficients of  $\lambda$  are given by  $f_i \circ \pi$  which is a  $C^\infty$  function on  $T^*M$ . Hence  $\lambda$  is  $C^\infty$ .

### Proposition 17.11 Pullback of a sum and a product

*Solution:* For an arbitrary vector  $X_p$ :

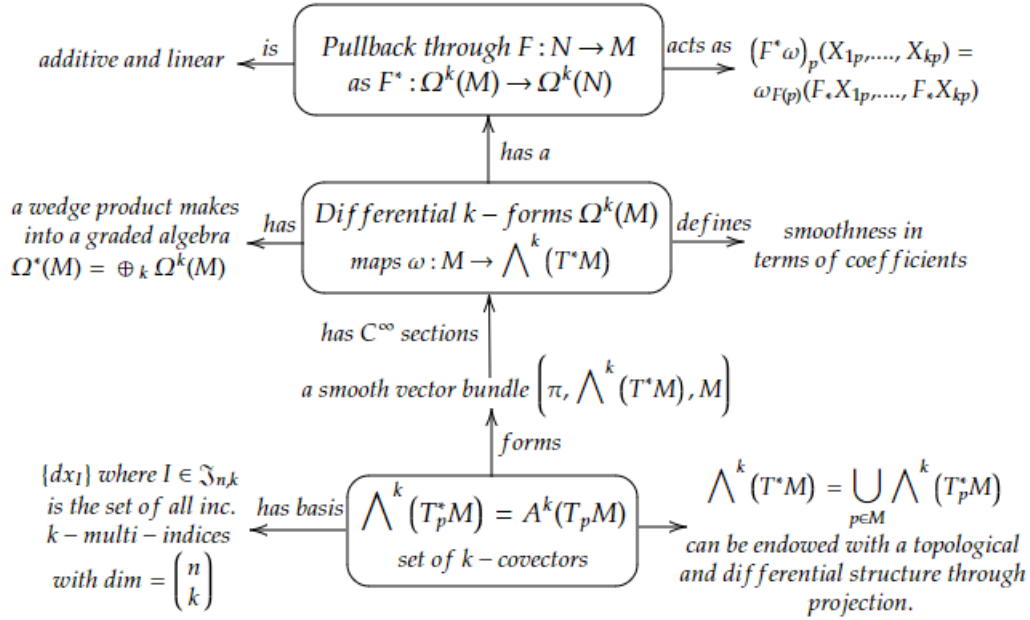
$$\begin{aligned} F^*((\omega + \tau)_{F(p)})X_p &= (\omega + \tau)_{F(p)}(F_{*,p}X_p) \\ &= \omega_{F(p)}(F_{*,p}X_p) + \tau_{F(p)}(F_{*,p}X_p) \\ &= F^*(\omega_{F(p)})(X_p) + F^*(\tau_{F(p)})(X_p), \end{aligned}$$

thus giving that  $F^*(\omega + \tau) = F^*\omega + F^*\tau$  Similarly:

$$F^*(g\omega)_{F(p)}X_p = g_{F(p)}\omega_{F(p)}(F_{*,p}X_p) = (F^*g)_p F^*(\omega_{F(p)})X_p,$$

giving that  $F^*(g\omega) = (F^*g)(F^*\omega)$ .

## 5.2 Differential $k$ -Forms



### Proposition 18.7 Characterization of a smooth $k$ -form

*Solution:* Take an arbitrary  $k$ -form  $\omega = \sum a_I dx^I$ . Assume it is smooth, that is, the coefficient functions  $a_I$  are smooth. Now consider an arbitrary set of  $k$  smooth vector fields  $\{X_1, \dots, X_k\}$  and the resulting function  $\omega(X_1, \dots, X_n)$ . Evaluating it at a point  $p$  with local coordinates  $(U, x_1, \dots, x_n)$ :

$$\omega(X_1, \dots, X_n)(p) = \omega_p(X_{1p}, \dots, X_{np}) = \left( \sum_I a_I(p) dx^I|_p \right) \left( \sum_{i_1 \leq n} c^{i_1}(p) \partial_{i_1}, \dots, \sum_{i_n \leq n} c^{i_n}(p) \partial_{i_n} \right),$$

with  $\partial_i = \frac{\partial}{\partial x^i} \Big|_p$  and some smooth coefficient functions  $c^{ij}$ . Further using multilinearity:

$$\begin{aligned} \omega(X_1, \dots, X_n)(p) &= \sum_{i_1, \dots, i_n} c^{i_1}(p) \dots c^{i_n}(p) \left( \sum_I a_I dx^I(\partial_{i_1}, \dots, \partial_{i_n}) \right) \\ &= \sum_J \sum_I c^J(p) a_I(p) dx^I(\partial_J) = \sum_J \sum_I c^J(p) a_I(p) \delta_J^I, \end{aligned}$$

where we have denoted  $c^J(p) = c^{i_1}(p) \dots c^{i_n}(p)$  and  $\partial_J = (\partial_{i_1}, \dots, \partial_{i_n})$  with  $J$  as the multi-index set  $\{i_1, \dots, i_n\}$ . Clearly  $c^J$  is  $C^\infty$ , hence so is  $\omega(X_1, \dots, X_n)$ .

Now assume for that for all  $k$  smooth vector fields  $\omega(X_1, \dots, X_n)$  is smooth. We aim to show this implies that  $\omega$  is smooth. Consider any point  $p$  in a chart  $(U, x_1, \dots, x_n)$  centered around  $p$ , and a corresponding set of vector fields defined on  $U$  given by  $X_i = \partial_i$ . Each of these can be extended to  $C^\infty$  vector fields  $\tilde{X}_i$  on  $M$  such that they

agree with  $\partial_i$  on a nbhd.  $V$  of  $p$ . Now evaluating  $\omega(\tilde{X}_{i_1}, \dots, \tilde{X}_{i_k})(p)$  in  $V$ :

$$\omega(\tilde{X}_{i_1}, \dots, \tilde{X}_{i_k})(p) = \omega_p(\partial_J) = \sum_I a_I(p) dx^I(\partial_J) = a_I(p).$$

As  $\omega(\tilde{X}_{i_1}, \dots, \tilde{X}_{i_k})$  is  $C^\infty \Rightarrow a_I$  is  $C^\infty$  about  $p$  as well. Thus  $\omega$  is a  $C^\infty$   $k$ -form.

**Problem 18.5 Support of a linear combination**

*Solution:* Consider a set of linearly independent  $k$ -forms  $\{\omega^i\}_i \Rightarrow \sum_i a_i(p) \omega_p^i = 0 \Leftrightarrow a_i(p) = 0$  for all  $i$ . Thus for any set of  $C^\infty$  functions  $a_i$ :

$$Z\left(\sum_i a_i \omega^i\right) = \cap_i Z(a_i) \Rightarrow Z\left(\sum_i a_i \omega^i\right)^c = \left(\cap_i Z(a_i)\right)^c = \bigcup_i Z(a_i)^c.$$

Thus  $\text{supp} \sum_i a_i \omega^i = \bigcup_i \text{supp } a_i$  as the closure distributes over a finite union.

**Problem 18.6\* Locally finite collection of supports**

*Solution:* Let  $\tau$  be a topological space. In it, consider a locally finite collection of closed sets  $\{C_\alpha\}$ , and a compact set  $K$ . We show that  $K$  intersects only finitely many  $C_\alpha$ . For each  $p \in K$ , choose a nbhd  $U_p$  of  $p$  such that  $U_p$  intersects only finitely many  $C_\alpha$ . Thus  $\{U_p\}_{p \in K}$  forms an open cover of  $K \Rightarrow$  there exist finitely many  $U_i$  such that  $\{U_i\}$  covers  $K \Rightarrow K$  intersects finitely many  $C_\alpha$ . Now replace  $\tau$  with a smooth manifold  $M$ , the  $C_\alpha$  with  $\text{supp } \rho_\alpha$  and  $K$  with  $\text{supp } \omega$ , where  $\omega$  is a smooth differential  $k$ -form with compact support. This gives us that  $\rho_\alpha \omega$  is non-zero for only finitely many  $\alpha$ .

**Problem 18.8\* Pullback by a surjective submersion**

*Solution:* Consider a surjective submersion between two manifolds  $\pi : \bar{M} \rightarrow M$ . It's pullback is a map  $\pi^* : \Omega^*(M) \rightarrow \Omega^*(\bar{M})$ . It is an algebra homomorphism by Proposition 18.11. We further prove that it's kernel is trivial, that is, it is also injective. Assume  $\pi^*(\omega) = 0$  where  $\omega$  is a smooth  $k$ -form on  $M$ . Consider any arbitrary point  $p \in M$  and any arbitrary set of  $k$  vectors  $\{X_1, \dots, X_k\}$  such that  $X_i \in T_p M$ . By assumption, there exists  $q \in \bar{M}$  and  $Y_i \in T_q \bar{M}$  such that  $\pi(q) = p$  and  $\pi_{*,q} Y_i = X_i$ . Thus we have:

$$(\pi^* \omega)_q(Y_1, \dots, Y_k) = 0 \Rightarrow \omega_p(\pi_{*,q} Y_1, \dots, \pi_{*,q} Y_k) = \omega_p(X_1, \dots, X_k) = 0.$$

As  $X_i$  were chosen arbitrarily  $\Rightarrow \omega = 0$ .

**Problem 18.9 Bi-invariant top forms on a compact, connected Lie group**

*Solution(a):* Given a left-invariant  $n$ -form on  $G$ , we aim to show that  $r_a^* \omega$  is also

left-invariant for any  $a \in G$ . Take any point  $p \in M$  and set of  $n$  vectors  $X_i \in T_p M$ :

$$\begin{aligned}
(l_g^*(r_a^*\omega))_p(X_1, \dots, X_n) &= (r_a^*\omega)_{gp}(l_{g*}X_1, \dots, l_{g*}X_n) \\
&= \omega_{gpa}(r_{a*}(l_{g*}X_1), \dots, r_{a*}(l_{g*}X_n)) \\
&= \omega_{gpa}((r_a \circ l_g)_*X_1, \dots, (r_a \circ l_g)_*X_n) \\
&= \omega_{gpa}((l_g \circ r_a)_*X_1, \dots, (l_g \circ r_a)_*X_n) \\
&= \omega_{gpa}(l_{g*}(r_{a*}X_1), \dots, l_{g*}(r_{a*}X_n)) \\
&= (r_a^*(l_g^*\omega))_p(X_1, \dots, X_n) \\
&= (r_a^*\omega)_p(X_1, \dots, X_n).
\end{aligned}$$

*Solution(b):* As  $\dim \Omega^n(G)^G = 1$ , we can see that  $r_a^*\omega = f(a)\omega$ . Now consider  $r_{ab}^*\omega = f(ab)\omega$ . We can see that as  $r_{ab} = r_b \circ r_a \Rightarrow r_{ab}^* = r_b^* \circ r_a^*$ . Thus  $r_{ab}^*\omega = f(b)f(a)\omega \Rightarrow f(ab) = f(a)f(b)$ . Thus  $f : G \rightarrow \mathbb{R}^\times$  is a group homomorphism.

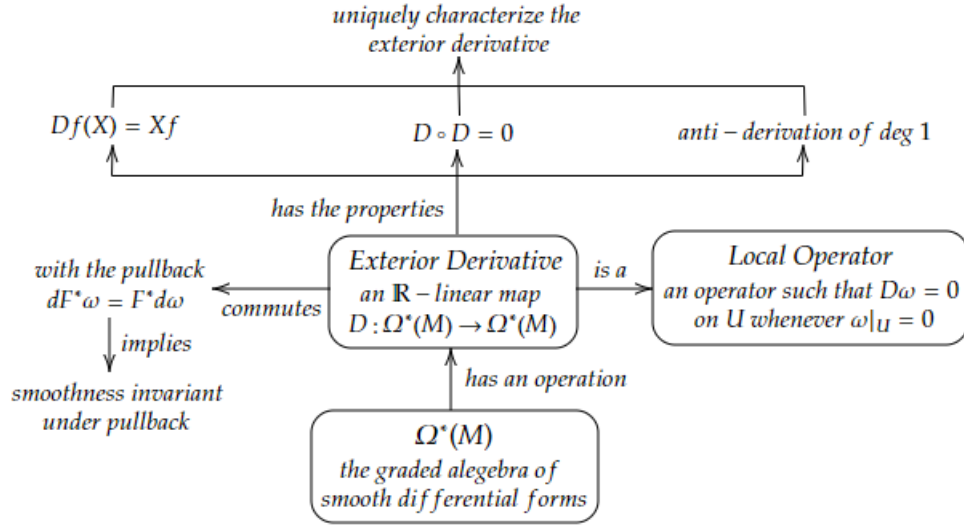
*Solution(c):* First note that  $f(a)\omega_e = (r_a^*\omega)_e = r_a^*(\omega_e) = r_a^*l_{a^{-1}}^*(\omega_e) = (r_a \circ l_{a^{-1}})^*(\omega_e) = c_{a^{-1}}^*(\omega_e)$ , where  $c_{a^{-1}}$  is the conjugation map. Let  $e \in G$  belong to a chart  $(U, x^1, \dots, x^n)$ , giving  $T_e G$  the basis  $\{\partial_i\}$ . As  $c_{a^{-1}*} : T_e G \rightarrow T_e G$  is linear map, let it be represented by the matrix  $[a_{ij}]$  using the basis  $\{\partial_i\}$ . Furthermore let  $\omega_e = r dx^1|_e \wedge \dots \wedge dx^n|_e$ . Thus we have  $f(a)\omega_e(b_1\partial_1, \dots, b_n\partial_n) = f(a)r\Pi_i(b_i\partial_i)_i = f(a)r\Pi_i b_i$ . On the other hand we have:

$$c_{a^{-1}}^*(\omega_e)(b_1\partial_1, \dots, b_n\partial_n) = \omega_e(c_{a^{-1}*}(b_1\partial_1), \dots, c_{a^{-1}*}(b_n\partial_n)) = r\Pi_i(c_{a^{-1}*}(b_i\partial_i))_i = r\Pi_i(a_{ii}b_i).$$

Comparing both sides of the starting equation, we get  $f(a) = \Pi_i a_{ii}$ . Thus we now define a map  $h : GL(\mathfrak{g}) \rightarrow \mathbb{R}$  given by  $h([a_{ij}]) = \Pi_i a_{ii}$ . Clearly  $h$  is  $C^\infty$ . Now consider the  $C^\infty$  composition  $h \circ \text{Ad} \circ i : G \rightarrow \mathbb{R}$ . After smoothly restricting the codomain to the regular submanifold  $\mathbb{R}^\times$ , one can see that  $f = h \circ \text{Ad} \circ i$ , and hence is also  $C^\infty$ .

*Solution(d):*  $f(G)$  is the image of a connected compact set, and hence should be connected and compact in  $\mathbb{R}^\times$ . In this case, it is equivalent to have  $f(G)$  be a closed and bounded interval, that is, of the form  $[a, b] - \{0\}$ . Moreover, it should be multiplicatively closed and closed under inverses as  $f$  is a group homomorphism. Firstly, we can see that for any  $b > 1$  or  $b < 0$ , we have  $b^2 > b \Rightarrow b^2 \notin [a, b]$ , thus  $0 \leq b \leq 1$ . As  $1 \in f(G) \Rightarrow b = 1$ . Now assume  $a \neq b \Rightarrow$  there exists some  $0 < r < 1$  in  $f(G)$ . But  $\frac{1}{r} > 1 \Rightarrow \frac{1}{r} \in f(G)$ . This is a contradiction, hence  $a = b$  and  $f(G) = 1$ . Thus  $r_a^*\omega = \omega$  for all  $a \in G$ .

### 5.3 The Exterior Derivative



#### Problem 19.4 Pullback of a restriction

*Solution:* We have:

$$(F|_{F^{-1}(U)})^*(\omega|_U) = (F|_{F^{-1}(U)})^*(i_U^*\omega) = (i \circ F|_{F^{-1}(U)})^*(\omega).$$

On the other hand:

$$(F^*\omega)|_{F^{-1}(U)} = i_{F^{-1}(U)}^*(F^*\omega) = (F \circ i_{F^{-1}(U)})^*\omega.$$

As  $i \circ F|_{F^{-1}(U)} \equiv F \circ i_{F^{-1}(U)}$ , we have  $(F|_{F^{-1}(U)})^*(\omega|_U) = (F^*\omega)|_{F^{-1}(U)}$ .

#### Problem 19.6 Local Operators

*Solution:* ( $\Rightarrow$ ) Assume an operator  $L : \Omega^*(M) \rightarrow \Omega^*(M)$  is a local operator. Consider any  $\omega \in \Omega^*(M)$ . Let  $p \in \text{supp } \omega^c \Rightarrow \omega_p = 0$ . As  $\text{supp } \omega^c$  is an open nbhd. of  $p \Rightarrow L(\omega) \equiv 0$  in  $\text{supp } \omega^c$  containing  $p$ . Thus  $p \notin \text{supp } L(\omega) \Rightarrow \text{supp } L(\omega) \subseteq \text{supp } \omega$ . ( $\Leftarrow$ ) Assume the operator  $L : \Omega^*(M) \rightarrow \Omega^*(M)$  is support-decreasing. Take any  $\omega \in \Omega^*(M)$  that is identically zero in an open set  $U \Rightarrow U \subseteq \text{supp } \omega^c \subseteq (\text{supp } L(\omega))^c$ . Thus  $L(\omega) \equiv 0$  on  $U$  as well. Thus  $L$  is a local operator.

#### Problem 19.7 Derivations of $C^\infty$ function are local operators

*Solution:* Consider any  $\omega \in \Omega^*(M)$  that vanishes on some open set  $U$ . Take any point  $p \in U$  and a corresponding  $C^\infty$  bump function  $f$  at  $p$  supported in  $U$ . Clearly  $f\omega \equiv 0$ . Thus we get:

$$0 = D(f\omega) = Df \wedge \omega + f \wedge (D\omega).$$

Evaluating both sides at  $p$ , we get  $(D\omega)_p = 0$ . Thus  $D\omega \equiv 0$  on  $U$ .

**Problem 19.8 Nondegenerate 2-forms**

*Solution(a):* With  $\omega = \sum_{j=1}^n dx^j \wedge dy^j$ , consider the following:

$$\begin{aligned} \omega \wedge \dots \wedge \omega (\text{n times}) &= \sum_{i_1, \dots, i_n}^n dx^{i_1} \wedge dy^{i_1} \wedge \dots \wedge dx^{i_n} \wedge dy^{i_n}. \\ &= \sum_{i_1 \neq i_2 \dots \neq i_n}^n dx^{i_1} \wedge dy^{i_1} \wedge \dots \wedge dx^{i_n} \wedge dy^{i_n} \\ &= \sum_{\sigma \in S_n} dx^{\sigma(1)} \wedge dy^{\sigma(1)} \wedge \dots \wedge dx^{\sigma(n)} \wedge dy^{\sigma(n)}. \end{aligned}$$

Now we can see that any element of the sum above is invariant under the following switching of successive pairs of 2-forms:  $dx^{\sigma(1)} \wedge dy^{\sigma(1)} \wedge \dots \wedge dx^{\sigma(k)} \wedge dy^{\sigma(k)} \wedge dx^{\sigma(k+1)} \wedge dy^{\sigma(k+1)} \dots \wedge dx^{\sigma(n)} \wedge dy^{\sigma(n)} = dx^{\sigma(1)} \wedge dy^{\sigma(1)} \wedge \dots \wedge dx^{\sigma(k+1)} \wedge dy^{\sigma(k+1)} \wedge dx^{\sigma(k)} \wedge dy^{\sigma(k)} \dots \wedge dx^{\sigma(n)} \wedge dy^{\sigma(n)}$ . Thus each element in the sum above is equivalent, and hence:

$$\omega \wedge \dots \wedge \omega (\text{n times}) = n!(dx^1 \wedge dy^1 \wedge \dots \wedge dx^n \wedge dy^n) \neq 0.$$

Thus  $\omega$  is non-degenerate.

*Solution(b):* Recall the Liouville form  $\lambda_{\omega_p} = \sum_i (f_i \circ \pi)(\omega_p) d\tilde{x}^i$  where  $\omega_p = \sum_i f_i(p) dx^i$  locally in a chart  $(U, x^1, \dots, x^n)$  around  $p \Rightarrow$  the corresponding coordinates around  $\omega_p$  are  $\tilde{x}^i = x^i \circ \pi, \dots, \tilde{x}^n, f_1, \dots, f^n$ . We can see that:

$$d\lambda = \sum_i d(f_i \circ \pi) \wedge d\tilde{x}^i = \sum_i \sum_j \left( \frac{\partial f_i}{\partial \tilde{x}^j} d\tilde{x}^j + \frac{\partial f_i}{\partial c_j} df^j \right) \wedge d\tilde{x}^i = \sum_i dc^i \wedge d\tilde{x}^i.$$

This is exactly analogous to the differential form in (a), and hence, is non-degenerate.

**Problem 19.11 A  $C^\infty$  nowhere-vanishing form on a smooth hypersurface**

*Solution(a):* We have:

$$\begin{aligned} f(x, y) &= 0 \\ \Rightarrow df &= 0 \Rightarrow f_x dx + f_y dy = 0 \end{aligned}$$

Using this we can construct the following one-form:

$$\omega = \begin{cases} \frac{dy}{f_x} & f_x \neq 0 \\ -\frac{dx}{f_y} & f_y \neq 0. \end{cases}$$

This is well-defined as  $f_x$  and  $f_y$  never vanish simultaneously due to  $f$  being smooth, and for  $f_x$  and  $f_y$  both non-zero,  $\frac{dy}{f_x} = -\frac{dx}{f_y}$ . Clearly, by definition,  $\omega$  is a nowhere

vanishing 1-form.

*Solution(b)*: Similarly, for a smooth hypersurface  $g(x, y, z) = 0$  we have  $g_x dx + g_y dy + g_z dz = 0$ . Now wedge both sides with  $dz$  to get  $g_x dz \wedge dx - g_y dy \wedge dz = 0$ . Similarly, by wedging by  $dx$  and  $dy$ , we get  $g_y dx \wedge dy - g_z dz \wedge dx = 0$  and  $-g_x dx \wedge dy + g_z dy \wedge dz = 0$ . Now we can define the following 2-form:

$$\omega = \begin{cases} \frac{dx \wedge dy}{f_z} & f_z \neq 0 \\ \frac{dy \wedge dz}{f_x} & f_x \neq 0 \\ \frac{dz \wedge dx}{f_y} & f_y \neq 0. \end{cases}$$

This is also well-defined as  $g_x, g_y$ , and  $g_z$  are never simultaneously zero as  $g$  is smooth, and for any pair of  $f_x, f_y$  or  $f_z$  that are simultaneously non-zero, we have that  $\frac{dx \wedge dy}{f_z} = \frac{dy \wedge dz}{f_x} = \frac{dz \wedge dx}{f_y}$ . Clearly  $\omega$  is nowhere vanishing.

*Solution(c)*: Assume we have a smooth hypersurface given by  $f(x^1, \dots, x^{n+1}) = 0$  in  $\mathbb{R}^{n+1}$ . Then by taking the exterior derivative on both sides, we get  $\sum_i f_{x^i} dx^i = 0$ . Now define the following  $n$ -form:

$$\omega = \begin{cases} \frac{dx^2 \wedge \dots \wedge dx^n}{f_{x^1}} & f_{x^1} \neq 0 \\ \cdot \\ \cdot \\ \cdot \\ (-1)^{i-1} \frac{dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n}{f_{x^i}} & f_{x^i} \neq 0 \\ \cdot \\ \cdot \\ \cdot \\ (-1)^{n-1} \frac{dx^1 \wedge \dots \wedge dx^{n-1}}{f_{x^n}} & f_{x^n} \neq 0, \end{cases}$$

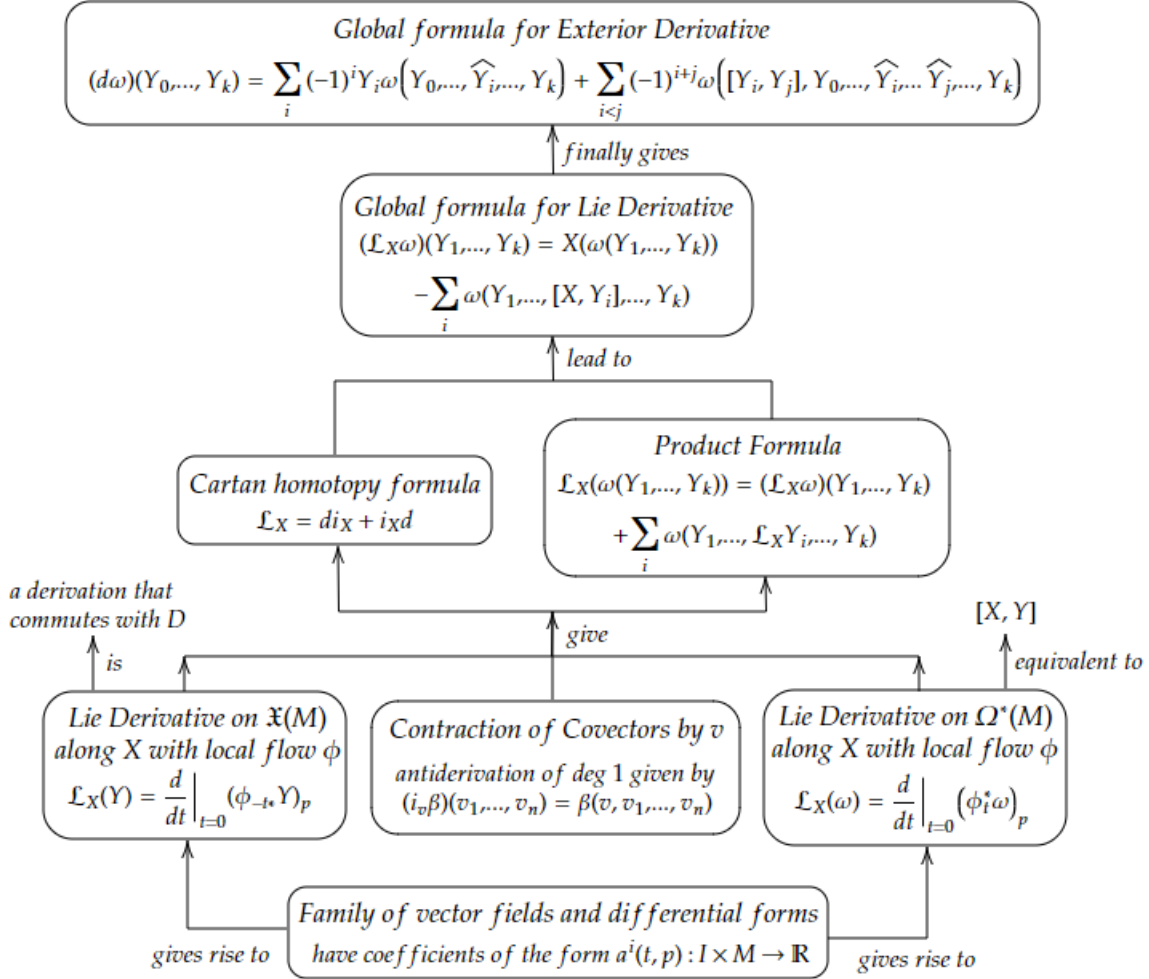
To argue that this is a well-defined form, first note that not all of  $f_{x^i}$  can be zero simultaneously, due to  $f$  being smooth. Now we claim that for any pairs  $f_{x^i}$  and  $f_{x^j}$  being non-zero simultaneously,  $(-1)^{i-1} \frac{dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n}{f_{x^i}} = (-1)^{j-1} \frac{dx^1 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^n}{f_{x^j}}$ , thus making  $\omega$  well-defined. Other than that, it is easy to see that  $\omega$  is a nowhere vanishing differential form. Now consider taking the wedge product with  $dx^1 \wedge \dots \wedge$

$\widehat{dx}^i \wedge \dots \wedge \widehat{dx}^j \wedge \dots \wedge dx^n$  on both sides of  $\sum_i f_{x^i} dx^i = 0$ , for arbitrary  $i < j$ . We get:

$$\begin{aligned}
& f_{x^i} \dots \wedge \widehat{dx}^i \wedge \dots \wedge \widehat{dx}^j \wedge \dots \wedge dx^n \wedge dx^i + f_{x^j} \dots \wedge \widehat{dx}^i \wedge \dots \wedge \widehat{dx}^j \wedge \dots \wedge dx^n \wedge dx^j = 0 \\
\Rightarrow & (-1)^{n-i-1} f_{x^i} dx^1 \wedge \dots \wedge \widehat{dx}^j \wedge \dots \wedge dx^n + (-1)^{n-j} f_{x^j} dx^1 \wedge \dots \wedge \widehat{dx}^i \wedge \dots \wedge dx^n = 0 \\
\Rightarrow & (-1)^{n-i-1} f_{x^i} dx^1 \wedge \dots \wedge \widehat{dx}^j \wedge \dots \wedge dx^n = (-1)^{n-j-1} f_{x^j} dx^1 \wedge \dots \wedge \widehat{dx}^i \wedge \dots \wedge dx^n \\
\Rightarrow & (-1)^j (1/f_{x^j}) dx^1 \wedge \dots \wedge \widehat{dx}^j \wedge \dots \wedge dx^n = (-1)^i (1/f_{x^i}) dx^1 \wedge \dots \wedge \widehat{dx}^i \wedge \dots \wedge dx^n \\
\Rightarrow & (-1)^{i-1} \frac{dx^1 \wedge \dots \wedge \widehat{dx}^i \wedge \dots \wedge dx^n}{f_{x^i}} = (-1)^{j-1} \frac{dx^1 \wedge \dots \wedge \widehat{dx}^j \wedge \dots \wedge dx^n}{f_{x^j}}.
\end{aligned}$$



## 5.4 The Lie Derivative and Interior Multiplication



### Problem 20.3\* Derivative of a smooth family of vector fields

*Solution:* Consider an arbitrary smooth family of vector fields  $X_t \in \mathfrak{X}(M)$ . Consider a point  $p \in M$  and two charts  $(U, x^1, \dots, x^n)$  and  $(V, y^1, \dots, y^n)$  containing  $p$ . Then  $X_t(p) = \sum_i a^i(t, p) \frac{\partial}{\partial x^i} = \sum_i b^i(t, p) \frac{\partial}{\partial y^i}$ , where  $b_i(t, p) = \sum_j a^j(t, p) \frac{\partial y^i}{\partial x^j}(p)$ . Now the derivative of  $X_t$  w.r.t the parameter  $t$  is defined by the derivative of the coefficient

functions. Thus we have:

$$\begin{aligned}
\frac{d}{dt}X_t &= \sum_i \frac{d}{dt}b^i(t, p) \frac{\partial}{\partial y^i} = \sum_i \frac{d}{dt} \left( \sum_j a^j(t, p) \frac{\partial y^i}{\partial x^j} \right) \frac{\partial}{\partial y^i} \\
&= \frac{d}{dt} \left( \sum_i \left( \sum_j a^j(t, p) \frac{\partial y^i}{\partial x^j} \right) \frac{\partial}{\partial y^i} \right) \\
&= \frac{d}{dt} \left( \sum_i a^i(t, p) \frac{\partial}{\partial x^i} \right) \\
&= \sum_i \frac{d}{dt} a^i(t, p) \frac{\partial}{\partial x^i}
\end{aligned}$$

Thus the derivative is independent of the choice of local coordinates.

**Problem 20.6\* Global formula for the exterior derivative**

*Solution:* It is clear the the given formula:

$$(d\omega)(Y_0, \dots, Y_k) = \sum_{i=0}^k (-1)^i Y_i \omega(Y_0, \dots, \widehat{Y}_i, \dots, Y_k) + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \omega([Y_i, Y_j], Y_0, \dots, \widehat{Y}_i, \dots, \widehat{Y}_j, \dots, Y_k),$$

holds for  $k=1$  by Proposition 20.13. Now assume it holds for all  $r < k$ , where  $r \in \mathbb{N}$ . For  $r = k$  we have:

$$(d\omega)(Y_0, \dots, Y_k) = (i_{Y_0} d\omega)(Y_1, \dots, Y_k) = (\mathfrak{L}_{Y_0} \omega)(Y_1, \dots, Y_k) - (di_{Y_0} \omega)(Y_1, \dots, Y_k),$$

by the Cartan homotopy formula. Now using the global defintion of the Lie derivative along with the assumption that the above formula holds for  $r = k - 1$ :

$$\begin{aligned}
(d\omega)(Y_0, \dots, Y_k) &= Y_0(\omega)(Y_1, \dots, Y_k) - \sum_{i=1}^r \omega(Y_1, \dots, [Y_0, Y_i], \dots, Y_k) \\
&\quad - \sum_{i=1}^r (-1)^{i-1} Y_i (i_{Y_0} \omega)(Y_1, \dots, \widehat{Y}_i, \dots, \widehat{Y}_j, \dots, Y_k) \\
&\quad - \sum_{1 \leq i < j \leq k} (-1)^{(i-1)+(j-1)} (i_{Y_0} \omega)([Y_i, Y_j], Y_1, \dots, \widehat{Y}_i, \dots, \widehat{Y}_j, \dots, Y_k) \\
\Rightarrow (d\omega)(Y_0, \dots, Y_k) &= Y_0(\omega)(Y_1, \dots, Y_k) - \sum_{i=1}^r \omega(Y_1, \dots, [Y_0, Y_i], \dots, Y_k) \\
&\quad + \sum_{i=1}^r (-1)^i Y_i \omega(Y_0, Y_1, \dots, \widehat{Y}_i, \dots, \widehat{Y}_j, \dots, Y_k) \\
&\quad - \sum_{1 \leq i < j \leq k} (-1)^{i+j} \omega(Y_0, [Y_i, Y_j], \dots, \widehat{Y}_i, \dots, \widehat{Y}_j, \dots, Y_k).
\end{aligned}$$

Now note that we can combine the first term into the second sum. Furthermore, we reindex the first sum by  $j$ :

$$\begin{aligned}
\Rightarrow (d\omega)(Y_0, \dots, Y_k) &= \sum_{i=0}^r (-1)^i Y_i \omega(Y_0, Y_1, \dots, \widehat{Y}_i, \dots, \widehat{Y}_j, \dots, Y_k) + \sum_{j=1}^r (-1)^j \omega([Y_0, Y_j], Y_1, \dots, Y_k) \\
&\quad + \sum_{1 \leq i < j \leq k} (-1)^{i+j} \omega([Y_i, Y_j], Y_0, \dots, \widehat{Y}_i, \dots, \widehat{Y}_j, \dots, Y_k) \\
&= \sum_{i=0}^k (-1)^i Y_i \omega(Y_0, \dots, \widehat{Y}_i, \dots, Y_k) + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \omega([Y_i, Y_j], Y_0, \dots, \widehat{Y}_i, \dots, \widehat{Y}_j, \dots, Y_k)
\end{aligned}$$

Thus the global formula holds for all  $r \in \mathbb{N}$ .

### Problem 20.7 $\mathfrak{L}$ -Linearity and the Lie Derivative

*Solution:* Using the Cartan homotopy formula we have:

$$\begin{aligned}
\mathfrak{L}_{fX}\omega &= di_{fX}\omega + i_{fX}d\omega = d(fi_X\omega) + fi_X(d\omega) \\
&= df \wedge i_X\omega + fd(i_X\omega) + fi_X(d\omega) = df \wedge i_X\omega + f\mathfrak{L}_X\omega.
\end{aligned}$$

### Problem 20.8 Bracket of the Lie Derivative and Interior Multiplication

*Solution:* Using the global formula for Lie derivatives, we get for arbitrary  $\omega \in \Omega^k(M)$  and  $Y, Y_1, \dots, Y_{k-1} \in \mathfrak{X}(M)$ :

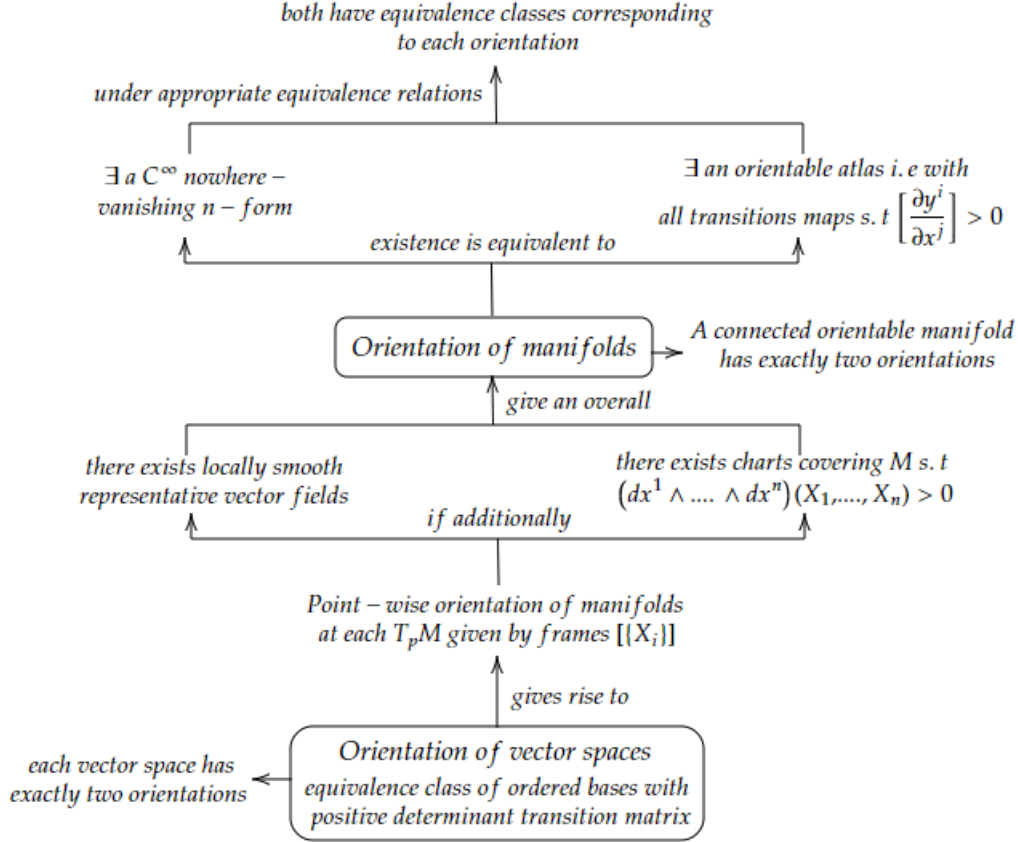
$$\begin{aligned}
\mathfrak{L}_X i_Y(\omega)(Y_1, \dots, Y_{k-1}) - i_Y \mathfrak{L}_X(\omega)(Y_1, \dots, Y_{k-1}) &= X(i_Y\omega(Y_1, \dots, Y_{k-1})) \\
&\quad - \sum_{i=1}^{k-1} i_Y \omega(Y_1, \dots, [X, Y_i], \dots, Y_{k-1}) \\
&\quad - \mathfrak{L}_X(\omega)(Y, Y_1, \dots, Y_{k-1}).
\end{aligned}$$

Further expanding:

$$\begin{aligned}
[\mathfrak{L}_X, i_Y](\omega)(Y_1, \dots, Y_{k-1}) &= X(\omega(Y, Y_1, \dots, Y_{k-1})) - \sum_{i=1}^{k-1} \omega(Y, Y_1, \dots, [X, Y_i], \dots, Y_{k-1}) \\
&\quad - X(\omega(Y, Y_1, \dots, Y_{k-1})) + \sum_{i=1}^{k-1} \omega(Y, Y_1, \dots, [X, Y_i], \dots, Y_{k-1}) \\
&\quad + \omega([X, Y], Y_1, \dots, Y_{k-1}) \\
&= i_{[X, Y]}\omega(Y_1, \dots, Y_{k-1}).
\end{aligned}$$

## 6 Integration

### 6.1 Orientations



#### Problem 21.1\* Locally constant map on a connected space

*Solution:* Consider any arbitrary  $y \in Y$ . Let  $s \in f^{-1}(y)$ . Then there exists a nbhd.  $U$  of  $s$  such that  $f(U) = f(s) = y \Rightarrow U \subseteq f^{-1}(y)$ . Thus  $f^{-1}(y)$  is open  $\Rightarrow S = \cup_{y \in Y} f^{-1}(y)$  is a decomposition of  $S$  into disjoint open sets. As  $S$  is connected, we must have that only one of these sets is non-empty  $\Rightarrow S = f^{-1}(y)$  for some  $y \in Y$ . Thus  $f$  is constant.

#### Problem 21.2 Continuity of pointwise orientations

*Solution:* ( $\Rightarrow$ ) Assume the a pointwise orientaton  $[(X_1, \dots, X_n)]$  on a manifold  $M$  is continuous. Consider any arbitrary  $p \in M$ . By Lemma 21.4, there a exists a chart  $(U, \phi) = (U, x^1, \dots, x^n)$  around  $p$  such that  $(dx^1 \wedge \dots \wedge dx^n)(X_1, \dots, X_n) > 0$ . Letting  $X_i = \sum_j a_j^i(q) \frac{\partial}{\partial x^j} \Big|_q$  in  $U$ , the previous statement implies  $\det[a_j^i(q)] > 0$  for all  $q \in U$ . Now considering the differential  $\phi_{*,q} : T_q M \rightarrow \mathbb{R}^n$ , we have trivially that the matrix of  $\phi_{*,q}$  is the identity matrix  $\rightarrow \phi_*(X_{i,q}) = \sum_j a_j^i(q) \frac{\partial}{\partial x^j}$ . Thus the transition matrix between  $(\phi_* X_{1,q}, \dots, \phi_* X_{n,q})$  and  $\left( \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right)$  is  $[a_j^i]$ , that has positive determinant.

Thus  $(\phi_*X_{1,q}, \dots, \phi_*X_{n,q}) \sim \left( \frac{\partial}{\partial r^1}, \dots, \frac{\partial}{\partial r^n} \right)$ . The exact reverse argument gives  $(\Leftarrow)$ .

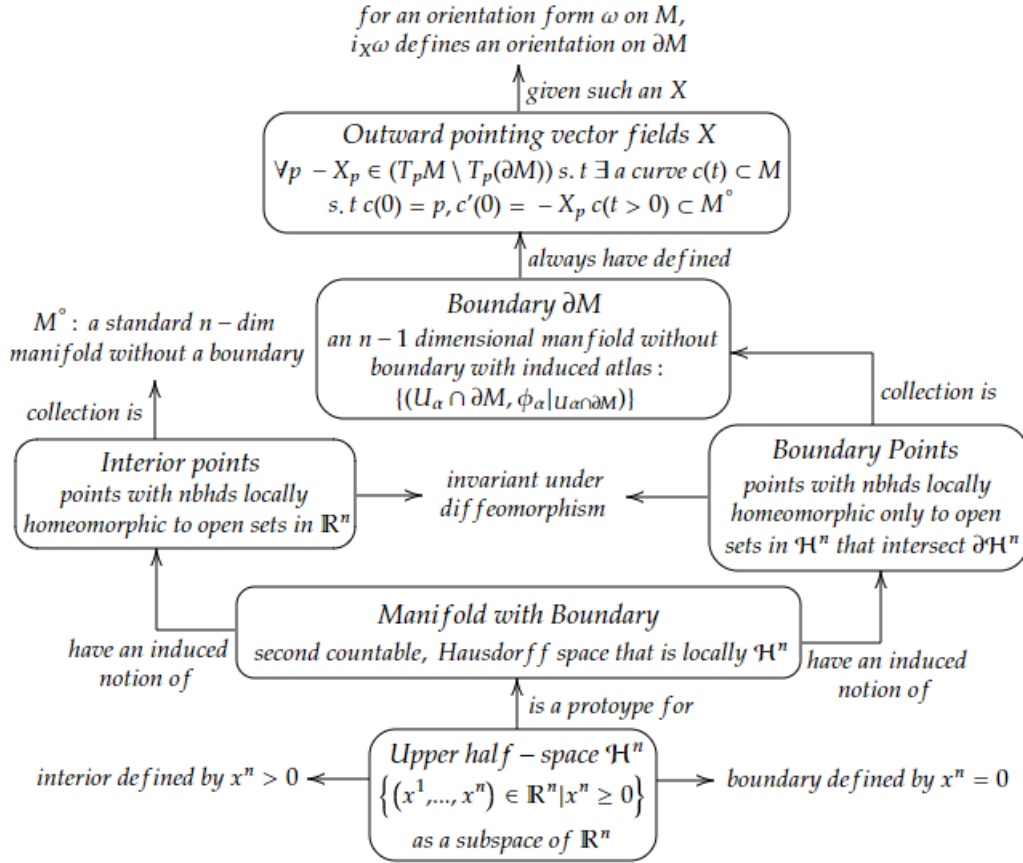
**Problem 21.8 Orientability of a parallelizable manifold**

*Solution:* Referring to Problem 16.8, every parallelizable manifold has a smooth frame  $X_1, \dots, X_n$ , thus implying that  $[(X_1, \dots, X_n)]$  is an orientation. Thus every parallelizable manifold is orientable. Note that this also implies that every Lie group, as it is parallelizable, is also orientable.

**Problem 21.8 Orientability of the total space of the tangent bundle**

*Solution:* Consider any two overlapping charts on the total space  $(TU, \tilde{\phi}) = (TU, x^1 \circ \pi, \dots, x^n \circ \pi, c^1, \dots, c^n)$  and  $(TV, \tilde{\psi}) = (TV, y^1 \circ \pi, \dots, y^n \circ \pi, b^1, \dots, b^n)$  which correspond to the charts on the base space  $(U, \phi) = (U, x^1, \dots, x^n)$  and  $(V, \psi) = (V, y^1, \dots, y^n)$  and corresponding coefficient functions  $c^i$  and  $b^i$  respectively. Referring to Problem 12.2, we can see that the jacobian determinant of the transition map between the two charts on  $TM$  is  $\left[ \frac{\partial y^i}{\partial x^j} \right]^2$ , which is everywhere positive (cannot be 0 as the transition map is a diffeomorphism). Thus  $TM$  is always orientable.

## 6.2 Manifolds with Boundary



### Problem 22.3\* Inward-pointing vectors at the boundary

*Solution:* ( $\Rightarrow$ ) Assume  $X_p \in T_p M$  is an inward pointing vector and  $(U, x^1, \dots, x^n)$  is a chart centered around  $p$ . Then we have that there exists an  $\epsilon > 0$  and curve  $c(t)$  in  $M$  such that  $c(0) = p$ ,  $c'(0) = X_p$ , and  $c((0, \epsilon)) \subseteq M^\circ$ . As  $X_p = \sum_i a^i \frac{\partial}{\partial x^i} \Big|_p$ , we can see

that  $a_n = X_p(x_n) = c'(0)(x_n) = \frac{d}{dt} \Big|_0 (x_n \circ c)$ . We know that  $(x_n \circ c) : \mathbb{R} \rightarrow \mathbb{R}$  is such that  $(x_n \circ c)(0) = x_n(p) = 0$ , as  $p \in \partial M$ . Moreover, as  $c((0, \epsilon)) \subseteq M^\circ \Rightarrow (x_n \circ c)((0, \epsilon)) \subseteq \mathcal{H}^\circ$ . Thus  $(x_n \circ c)(t) > 0$  for  $t \in (0, \epsilon) \Rightarrow a_n = \frac{d}{dt} \Big|_0 (x_n \circ c) > 0$ .

( $\Leftarrow$ ) Assume  $X_p \in T_p M$  is such that for every chart  $(U, x^1, \dots, x^n)$  centered around  $p \in \partial M$ , the coefficient of  $(\partial/\partial x^n)_p = a_n$  is positive. Correspondingly, there exists a chart  $(U \cap \partial M, x^1|_{U \cap \partial M}, \dots, x^{n-1}|_{U \cap \partial M})$  around  $p$  in  $\partial M$ . Thus for any  $Y \in T_p(\partial M)$ ,  $Y(x_n|_{U \cap \partial M}) = 0$  as  $x_n \equiv 0$  on  $\partial M$ . But on the other hand  $X_p(x_n) = a_n > 0$ . Thus  $X_p \notin T_p(\partial M)$ . Moreover, we can construct a curve  $c : (-\alpha, \alpha) \rightarrow M$  for  $\alpha > 0$  such that  $c(0) = p$  and  $c'(0) = X_p$ . We can see that  $0 < a_n = X_p(x_n) = c'(0)(x_n) = \frac{d}{dt} \Big|_0 (x_n \circ c)$ . Thus there exists an  $0 < \epsilon < \alpha$  such that  $(x_n \circ c)(\delta) - (x_n \circ c)(0) = (x_n \circ c)(\delta) > 0$  for  $0 < \delta < \epsilon$ . Thus  $(x_n \circ c)((0, \epsilon)) \subseteq \mathcal{H}^\circ \Rightarrow c((0, \epsilon)) \subseteq M^\circ$ . Thus restricting  $c$  to  $[0, \epsilon)$ , we

can see that  $X_p$  is an inward pointing vector.

### Problem 22.5 Boundary orientation

*Solution(a):* It is trivial to see that  $i_X\tau = i_Xf\omega = fi_X\omega$ .

*Solution(b):* Assume  $i_X\omega$  corresponds to the orientation of  $\partial M$  given by the frame  $[(X_1, \dots, X_{n-1})]$ , that is,  $i_X\omega(X_1, \dots, X_{n-1}) = \omega(X, X_1, \dots, X_{n-1}) > 0$ . Consider any point  $p$  with the centralized chart  $(U, x^1, \dots, x^n)$ . Take  $\omega = f dx^1 \wedge \dots \wedge dx^n$  for a nowhere-vanishing smooth function  $f$  in  $U$ . We then have  $\omega(X, X_1, \dots, X_n)(p) = f(p) dx^1 \wedge \dots \wedge dx^n(X, X_1, \dots, X_n)(p) = (-1)^{n-1} f dx^1 \wedge \dots \wedge dx^n(X_1, \dots, X_{n-1}, X)(p)$ . As  $X$  is an outward pointing vector field, we know the coefficient of  $(\partial/\partial x^n)_p$  in  $X_p$  is negative, say  $a(p)$ . Furthermore, as  $X_{i_p} \in T_p\partial M$ ,  $X_{i_p}(x_n) = 0$  for all  $i$ . Thus we can finally see that  $i_X\omega(X_1, \dots, X_{n-1})(p) = a(p)f(p)(-1)^{n-1} dx^1 \wedge \dots \wedge dx^{n-1}(X_1, \dots, X_{n-1})$ . Similarly, we can see that for another outward pointing vector field  $Y$ ,  $i_Y\omega(X_1, \dots, X_{n-1}) = b(p)f(p)(-1)^{n-1} dx^1 \wedge \dots \wedge dx^{n-1}(X_1, \dots, X_{n-1})$  for another negative function  $b(p)$  that is the coefficient of  $(\partial/\partial x^n)_p$  in  $Y_p$ . Thus we can see that  $i_Y\omega(X_1, \dots, X_{n-1})$  has the same parity as  $i_X\omega(X_1, \dots, X_n)$ , and thus  $i_Y\omega \sim i_X\omega$ .

### Problem 22.9 Boundary orientation on a sphere

*Solution:* Consider the standard orientation in  $\mathbb{R}^{n+1}$  being restricted to the closed unit ball. This is given by the orientation form  $\omega = dx^1 \wedge \dots \wedge dx^{n+1}$ . Given the outward pointing vector field  $X = \sum x^i \partial/\partial x^i$ , the boundary orientation on  $S^n$  is then given by  $i_X\omega$ . First noting that  $i_X dx_i = X(x_i) = x^i$ :

$$\begin{aligned} i_X\omega &= i_X(dx^1 \wedge \dots \wedge dx^{n+1}) = x^i dx^2 \wedge \dots \wedge dx^{n+1} - dx^1 \wedge (i_X(dx^2 \wedge \dots \wedge dx^{n+1})) \\ &= x^i dx^2 \wedge \dots \wedge dx^{n+1} - x^2 dx^1 \wedge dx^3 \wedge \dots \wedge dx^{n+1} + dx^1 \wedge dx^2 \wedge (i_X(dx^3 \wedge \dots \wedge dx^{n+1})), \end{aligned}$$

where it is easy to see that successive applications of the antiderivation property gives us:

$$i_X\omega = \sum_{i=1}^{n+1} (-1)^{i-1} x^i dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^{n+1}.$$

### Problem 22.10 Orientation on the upper hemisphere of a sphere

*Solution(a):* We already established an orientation form  $\omega$  on the sphere  $S^n$  in the previous problem. The corresponding orientation on its open subset  $U$  will be given by the restriction of  $\omega$  onto  $U$ . This will be done by setting  $x^{n+1} = \sqrt{1 - \sum_{i=1}^n (x^i)^2}$

in  $\omega$ . Note that  $dx^{n+1} = \sum_i \frac{-x^i}{x^{n+1}} dx^i$ . Substituting this into  $\omega$ :

$$\begin{aligned}\omega|_U &= \sum_{i=1}^n (-1)^{i-1} x^i dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n \wedge \left( \sum_j \frac{-x^j}{x^{n+1}} dx^j \right) + (-1)^n x^{n+1} dx^1 \wedge \dots \wedge dx^n \\ &= \sum_{i=1}^n (-1)^n \frac{(x^i)^2}{x^{n+1}} dx^1 \wedge \dots \wedge dx^n + (-1)^n x^{n+1} dx^1 \wedge \dots \wedge dx^n \\ &= (-1)^n \frac{1}{x^{n+1}} dx^1 \wedge \dots \wedge dx^n.\end{aligned}$$

*Solution(b):*

$$\begin{aligned}\pi \text{ is orientation-preserving} &\Leftrightarrow [\pi^*(dx^1 \wedge \dots \wedge dx^n)] \sim [\omega] \\ &\Leftrightarrow [d(\pi^* x^1) \wedge \dots \wedge d(\pi^* x^n)] \sim [\omega] \\ &\Leftrightarrow [dx^1 \wedge \dots \wedge dx^n] \sim \left[ (-1)^n \frac{1}{x^{n+1}} dx^1 \wedge \dots \wedge dx^n \right] \\ &\Leftrightarrow (-1)^n \frac{1}{x^{n+1}} > 0 \text{ on } U \Leftrightarrow n \text{ is even.}\end{aligned}$$

**Problem 22.11** Antipodal map on a sphere and the orientability of  $\mathbb{R}P^n$

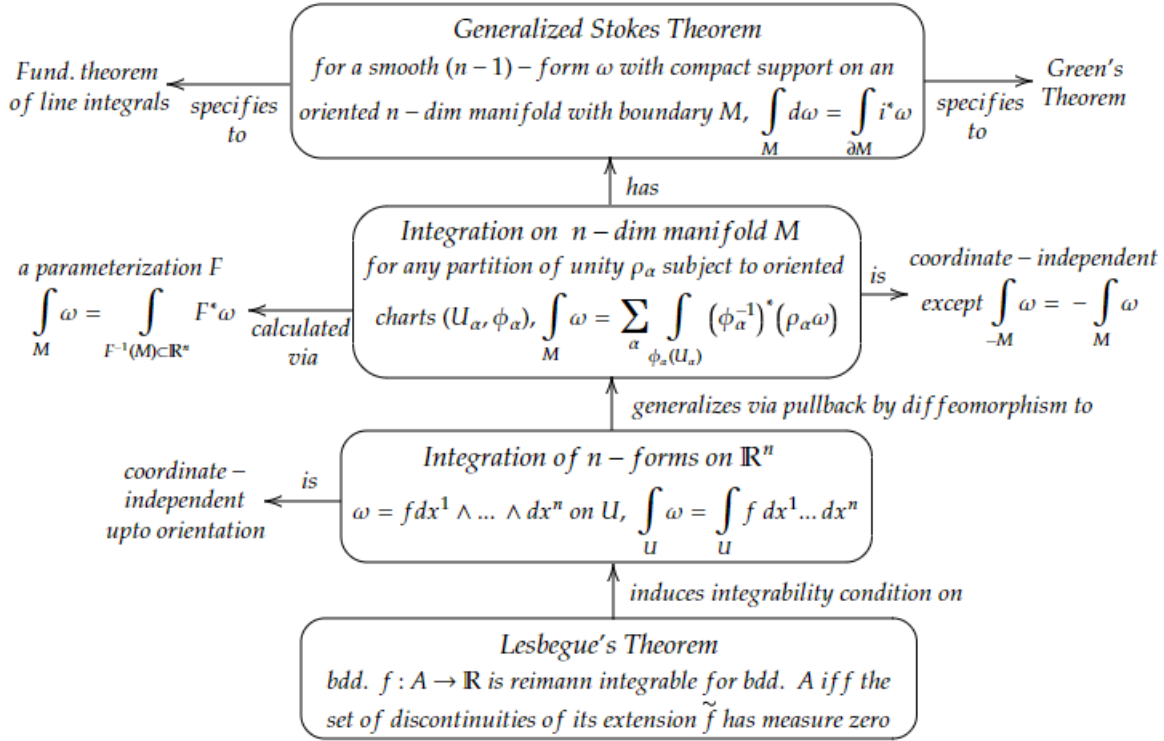
*Solution:*

$$\begin{aligned}a \text{ is orientation-preserving} &\Leftrightarrow [a^* \omega] \sim [\omega] \\ &\Leftrightarrow \left[ a^* \left( \sum_{i=1}^{n+1} (-1)^{i-1} x^i dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^{n+1} \right) \right] \sim [\omega] \\ &\Leftrightarrow \left[ \left( \sum_{i=1}^{n+1} (-1)^{i-1} (a^* x^i) d(a^* x^1) \wedge \dots \wedge \widehat{d(a^* x^i)} \wedge \dots \wedge d(a^* x^{n+1}) \right) \right] \sim [\omega] \\ &\Leftrightarrow \left[ (-1)^{n+1} \left( \sum_{i=1}^{n+1} (-1)^{i-1} x^i dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^{n+1} \right) \right] \sim [\omega] \\ &\Leftrightarrow [(-1)^{n+1} \omega] \sim [\omega] \Leftrightarrow n \text{ is odd.}\end{aligned}$$

We can now see that as  $\mathbb{R}P^n = S^n / \sim_a$ , the orientation  $\omega$  induces a consistent orientation on  $\mathbb{R}P^n$  for odd  $n$ .



### 6.3 Integration on Manifolds



#### Problem 23.3\* Integral under a diffeomorphism

*Solution:* Let  $\{U_\alpha, \phi_\alpha\}$  be an oriented atlas on  $M$ , with a partition of unity  $\{\rho_\alpha\}$  subject to  $U_\alpha$ . Correspondingly  $\{(F^{-1}(U_\alpha), F^* \phi_\alpha)\}$  forms an oriented atlas on  $N$ . Depending on whether  $F$  is orientation-preserving or orientation-reversing, the orientation given by this induced atlas is the same or opposite to the orientation of  $N$  respectively. Similarly, we also have an induced partition of unity on  $N$  given by  $\{F^* \rho_\alpha\}$  subject to  $F^{-1}(U_\alpha)$ . Thus we have:

$$\begin{aligned} \int_N F^* \omega &= \pm \sum_\alpha \int_{F^{-1}(U_\alpha)} (F^* \rho_\alpha) F^* \omega = \pm \sum_\alpha \int_{\phi(U_\alpha)} ((F^* \phi_\alpha)^{-1})^* ((F^* \rho_\alpha) F^* \omega) \\ &= \pm \sum_\alpha \int_{\phi(U_\alpha)} (\phi_\alpha^{-1})^* \circ (F^{-1})^* ((F^* \rho_\alpha) F^* \omega) \\ &= \pm \sum_\alpha \int_{\phi(U_\alpha)} (\phi_\alpha^{-1})^* (\rho_\alpha \omega) = \pm \int_M \omega. \end{aligned}$$

#### Problem 23.4\* Stoke's Theorem

*Solution:* First we prove the case for  $\mathbb{R}^n$ , considering the standard coordinates  $x^1, \dots, x^n$  and standard orientation. Take an  $n-1$  form  $\omega = \sum_i f_i dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n$  with compact support in  $\mathbb{R}^n$ . Then we can choose  $a > 0$  large enough such that  $[-a, a]^n$  contains the supports of  $f_i$  in its interior. Now computing  $d\omega = \sum_i (-1)^{i-1} (f_i)_{x^i} dx^1 \wedge \dots \wedge dx^n$

....  $\wedge dx^n$ . Now we have:

$$\begin{aligned}\int_{\mathbb{R}^n} d\omega &= \sum_i (-1)^{i-1} \int_{\mathbb{R}^n} (f_i)_{x^i} dx^1 \dots dx^n = \sum_i (-1)^{i-1} \int_{\mathbb{R}^{n-1}} \int_{-a}^a (f_i)_{x^i} dx^i dx^1 \dots \widehat{dx^i} \dots dx^n \\ &= \sum_i (-1)^{i-1} \int_{\mathbb{R}^{n-1}} (f_i)_{x^i=-a}^a dx^1 \dots \widehat{dx^i} \dots dx^n = 0.\end{aligned}$$

On the other hand, as the boundary of  $\mathbb{R}^n$  is empty,  $\int_{\partial \mathbb{R}^n} \omega = 0$  as well. Thus Stoke's Theorem holds for  $\mathbb{R}^n$ .

For  $\mathcal{H}^n$ , the steps are similar except that we choose an  $a > 0$  large enough such that  $[-a, a]^{n-1} \times [0, a]$  contains the support of  $f_i$  in its interior. Now we have:

$$\begin{aligned}\int_{\mathbb{R}^n} d\omega &= \sum_i (-1)^{i-1} \int_{\mathbb{R}^n} (f_i)_{x^i} dx^1 \dots dx^n \\ &= \sum_{i=1}^{n-1} (-1)^{i-1} \int_{\mathbb{R}^{n-1}} \int_{-a}^a (f_i)_{x^i} dx^i dx^1 \dots \widehat{dx^i} \dots dx^n + (-1)^{n-1} \int_{\mathbb{R}^{n-1}} \int_0^a (f_n)_{x^n} dx^n dx^1 \dots dx^{n-1} \\ &= (-1)^n \int_{\mathbb{R}^{n-1}} f_n(x^1, \dots, x^{n-1}, 0) dx^1 \dots dx^{n-1}.\end{aligned}$$

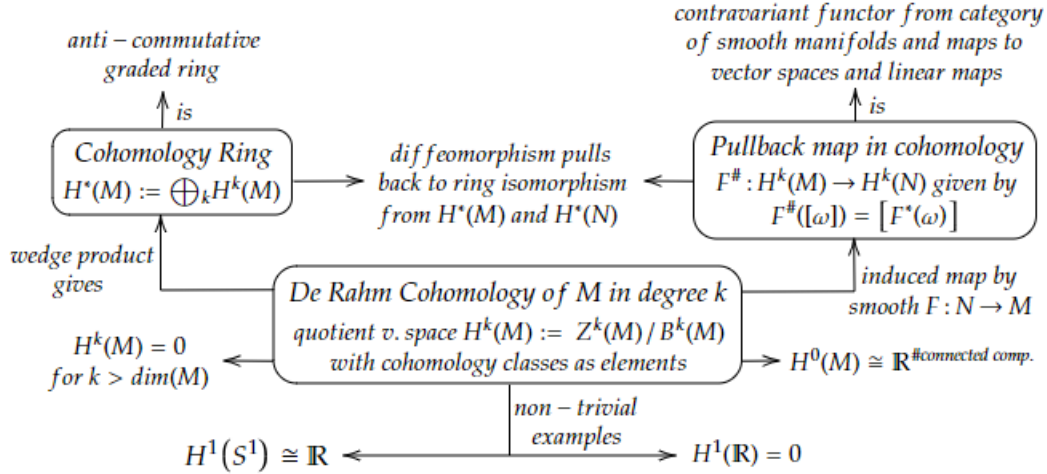
On the other hand, we can see that  $\partial \mathcal{H}^n$  is defined by  $x^n = 0$ , and hence has  $dx^n = 0$ . This gives that  $\omega|_{\partial \mathcal{H}^n} = f_n(x^1, \dots, x^{n-1}, 0) dx^1 \wedge \dots \wedge dx^{n-1}$ . Furthermore, the orientation on  $\partial \mathcal{H}^n$  is given by  $i_{-\partial/\partial x^n}(dx^1 \wedge \dots \wedge dx^n) = (-1)^n dx^1 \wedge \dots \wedge dx^{n-1}$ . Thus:

$$\int_{\partial \mathcal{H}^n} \omega = (-1)^n \int_{\mathbb{R}^{n-1}} f_n(x^1, \dots, x^{n-1}, 0) dx^1 \dots dx^{n-1}$$

as well. Thus Stoke's Theorem holds for  $\mathcal{H}^n$ .

## 7 De Rahm Theory

### 7.1 De Rahm Cohomology



#### Problem 24.1 Nowhere-vanishing 1-forms

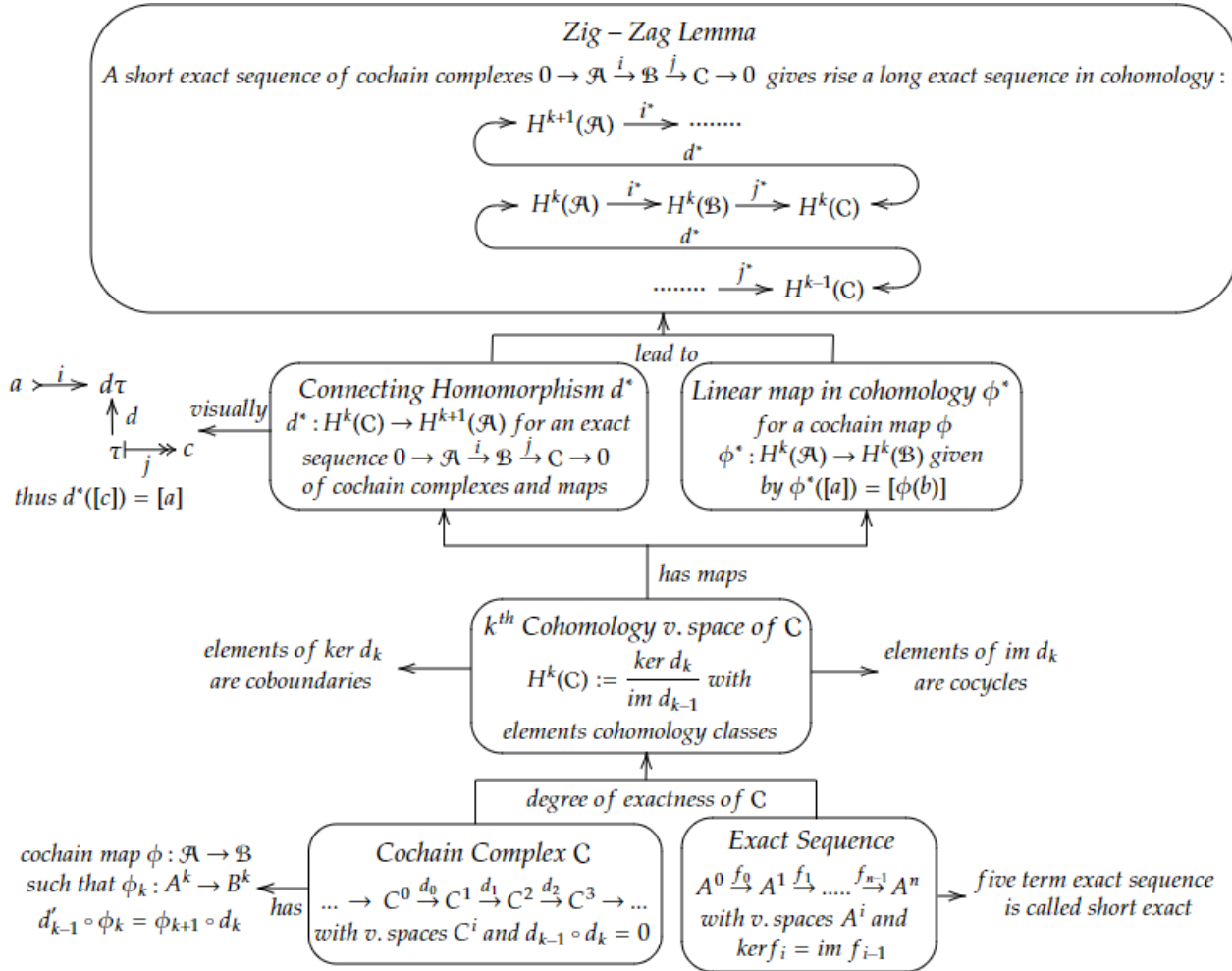
*Solution:* An nowhere-vanishing exact 1-form  $\alpha$  is of the form  $\alpha = df$  for some smooth function  $f$  on  $M$ . As  $M$  is compact  $\Rightarrow f(M)$  attains a maximum in  $\mathbb{R}$ . Let this be denoted by  $f(p)$  where  $p \in M$  is contained in some chart  $(U, x^1, \dots, x^n)$ . As  $p$  is also a local maxima, by Problem 8.10,  $p$  must be a critical point of  $f \Rightarrow \frac{\partial f}{\partial x^i} = 0$  for all  $i$ . Thus  $df_p = \alpha_p = 0$ . This is a contradiction as  $\alpha$  was nowhere-vanishing.

#### Problem 24.2 Cohomology in degree zero

*Solution:* As  $M$  is second countable, it admits a countable basis  $\{B_\alpha\}$ . Furthermore, as  $f$  is locally connected (due to being locally euclidean), each connected component of  $M$  is open. Let  $\mathcal{C}$  denote the set of connected components. To each  $C \in \mathcal{C}$ , we can associate a basis set  $B_\alpha$  contained in  $C$ . This is an injective map from  $\mathcal{C}$  to  $\{B_\alpha\}$  as each connected component is disjoint. Thus  $\mathcal{C}$  is at most countable.

Now as there are no non-zero exact forms of degree 0, we have  $H^0(M) \cong Z^0(M)$ , that is, the vector space of all closed 0-forms on  $M$ . A 0-form on  $M$  is a  $C^\infty$  function  $f$  on  $M$  such that  $df = 0$ . This implies that  $f$  is locally constant on  $M$ , and constant on each connected component. Thus  $f$  is defined by its value on each connected map, and this induces an isomorphism between  $Z^0(M)$  and  $\mathbb{R}^\infty$ . Thus  $H^0(M) \cong \mathbb{R}^\infty$ .

## 7.2 The Long Exact Sequence in Cohomology



### Exercise 25.5 Connecting homomorphism

*Solution:* The connecting homomorphism is given by the following diagram:

$$\begin{array}{ccc} a & \xrightarrow{i} & db \\ & \uparrow d & \\ & b & \xrightarrow{j} c \end{array}$$

where  $b \in B^k$  is any element such that  $j(b) = c$ , and  $a \in A^{k+1}$  is uniquely defined by  $i(a) = db$ . This gives  $d^*([c]) = [a]$ .

Now assume  $\exists \bar{b}$  such that  $j(\bar{b}) = c$ . Then we also have the following diagram:

$$\begin{array}{ccc}
\bar{a} & \xrightarrow{i} & d\bar{b} \\
& & \uparrow d \\
& & \bar{b} \xrightarrow{j} c
\end{array}$$

We aim to show that  $[a] = [\bar{a}]$ . We have that  $j(b - \bar{b}) = 0$ , thus  $(b - \bar{b}) \in \ker j$ . By exactness, we have that  $(b - \bar{b}) \in \text{im } i$ . Thus there exists an  $e \in A^k$  such that  $i(e) = b - \bar{b}$ . Thus  $i(de) = d(i(e)) = d(b - \bar{b})$ . This gives us the following commutative diagram:

$$\begin{array}{ccccc}
de & \xrightarrow{i} & d(b - \bar{b}) & & \\
\uparrow d & & \uparrow d & & \\
e & \xrightarrow{i} & (b - \bar{b}) & \xrightarrow{j} & 0
\end{array}$$

But we also have that  $i(a - \bar{a}) = d(b - \bar{b}) \Rightarrow$  as  $i$  is injective,  $de = a - \bar{a} \Rightarrow [a] = [\bar{a}]$ .

Now we check whether for a different representative of  $[c]$  of the form  $c + d\tau$  gives the same output  $[a]$ . We have the following diagram representing  $d^*([c + d\tau]) = [a']$ :

$$\begin{array}{ccc}
a' & \xrightarrow{i} & db' \\
& & \uparrow d \\
& & b' \xrightarrow{j} c + d\tau
\end{array}$$

Thus  $j(b - b') = d\tau$  for some  $\tau \in C^{k-1}$ . As  $j$  is surjective,  $\exists g \in B^{k-1}$  such that  $j(g) = \tau$ . This gives us the following diagram for  $d^*([d\tau]) = [a - a']$ :

$$\begin{array}{ccccc}
a - a' & \xrightarrow{i} & d(b - b') & & \\
& & \uparrow d & & \\
& & (b - b') & \xrightarrow{j} & d\tau \\
& & & & \uparrow d \\
& & & & g \xrightarrow{j} \tau
\end{array}$$

Thus we will have  $j(dg) = d(j(g)) = d\tau$ . Now, we can instead consider the computation of  $d^*([d\tau])$  using  $dg$  instead of  $(b - b')$ . This gives us  $d^*([d\tau]) = [l]$  for some  $l \in A^{k+1}$  such that:

$$\begin{array}{ccc}
l & \xrightarrow{i} & 0 \\
& \uparrow d & \\
dg & \xrightarrow{j} & d\tau \\
\uparrow d & & \uparrow d \\
g & \xrightarrow{j} & \tau
\end{array}$$

As  $i$  is injective  $\Rightarrow l = 0$ . As proven before, the choice of  $dg$  should not change the result as compared to before, thus  $[a - a'] = [0] \Rightarrow [a] = [a']$ . Thus the connecting homomorphism is well-defined.

### Problem 25.3 Long exact cohomology sequence

*Solution:* First we check exactness at  $H^k(\mathcal{B})$ .

Consider any  $[i(a)] \in \text{im } i^*$ . Then  $j^*([i(a)]) = [j(i(a))] = [0]$  as  $\text{im } i = \ker j$ . Thus  $\text{im } i^* \subseteq \ker j^*$ . Now take any  $[b] \in \ker j^* \Rightarrow j(b) = d\tau$  for some  $\tau \in C^{k-1}$ . As  $j$  is surjective  $\Rightarrow \exists b' \in B^{k-1}$  such that  $j(b') = \tau$ . Furthermore,  $j(db') = d(j(b')) = d\tau$ . This gives us the following diagram:

$$\begin{array}{ccc}
db' & \xrightarrow{j} & d\tau \\
\uparrow d & & \uparrow d \\
b' & \xrightarrow{j} & \tau
\end{array}$$

This gives us that  $j(b - db') = 0 \Rightarrow (b - db') \in \ker j = \text{im } i$ . Thus  $\exists a \in A^k$  such that  $i(a) = b - db'$ . Furthermore  $i(da) = d(i(a)) = db - ddb' = db = 0$  as  $b$  is a cocycle. Thus we have  $i^*([a]) = [b]$ . Thus  $[b] \in \text{im } i^* \Rightarrow \text{im } i^* = \ker j$ .

Now we check exactness at  $H^k(\mathcal{A})$ . Consider any  $d^*([c]) = [a] \in \text{im } d^*$  for any  $[c] \in H^{k-1}(C)$ . This is given by:

$$\begin{array}{ccc}
a & \xrightarrow{i} & d\tau \\
& \uparrow d & \\
\tau & \xrightarrow{j} & c
\end{array}$$

Clearly  $i^*([a]) = [i(a)] = [d\tau] = [0]$ . Thus  $[a] \in \ker i^*$  and  $\text{im } d^* \subseteq \ker i^*$ . Furthermore, take an arbitrary  $[a] \in \ker i^* \Rightarrow i(a) = d\tau$  for some  $\tau \in B^{k-1}$ . This trivially gives us:

$$\begin{array}{ccc}
 a & \xrightarrow{i} & d\tau \\
 & \uparrow d & \\
 \tau & \xrightarrow{j} & j(\tau)
 \end{array}$$

As  $d(j(\tau)) = j(d\tau) = j(i(a)) = 0$ ,  $j(\tau)$  is a cocycle. Hence  $d^*([j(\tau)]) = [a]$ , and hence  $\ker i^* = \operatorname{im} d^*$ .

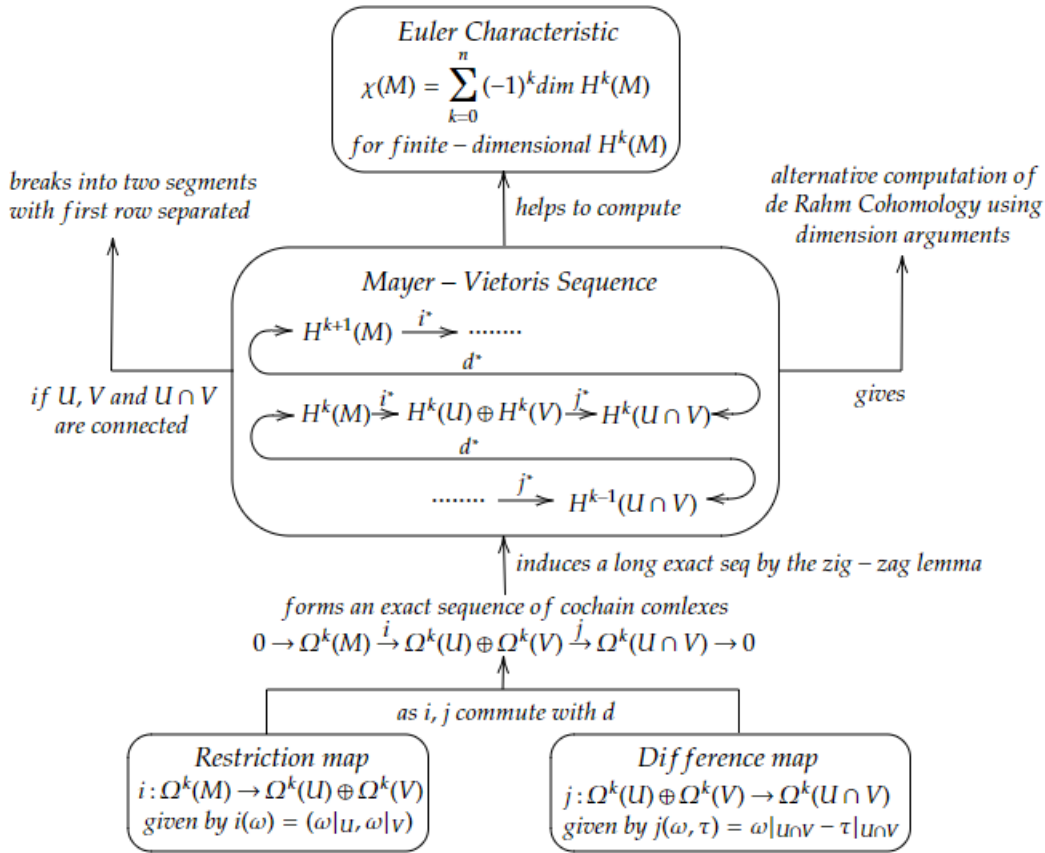
**Problem 25.4\* The snake lemma**

*Solution:* The commutative diagram can be extended into an almost identical extension given by:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & A^1 & \longrightarrow & B^1 & \longrightarrow & C^1 \longrightarrow 0 \\
 & & \uparrow \alpha & & \uparrow \beta & & \uparrow \gamma \\
 0 & \longrightarrow & A^0 & \longrightarrow & B^0 & \longrightarrow & C^0 \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

We can see that this is clearly a short exact cochain complex given by  $0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$ . Furthermore, we have that  $H^0(\mathcal{A}) := \frac{\ker \alpha}{\operatorname{im} 0} = \ker \alpha$ , and  $H^1(\mathcal{A}) := \frac{\ker 0}{\operatorname{im} \alpha} = \frac{A}{\operatorname{im} \alpha} = \operatorname{coker} \alpha$ . Similarly  $H^0(\mathcal{B}) = \ker \beta$ ,  $H^1(\mathcal{B}) = \operatorname{coker} \beta$ ,  $H^0(\mathcal{C}) = \ker \gamma$ ,  $H^1(\mathcal{C}) = \operatorname{coker} \gamma$ . Now applying the zig-zag lemma, we get the induced long exact sequence in the snake lemma.

### 7.3 The Mayer-Vietoris Sequence



#### Exercise 26.3 Smooth extension of a function

*Solution:* Consider any  $p \in V$ . If  $p \in U \cap V$ , then there exists a chart around  $p$  contained in  $U \cap V$  over which both  $\rho_U$  and  $f$  are smooth. Thus  $\rho_U f$  is also smooth at  $p$ . If  $p \in V - (U \cap V)$  then there exists an open set  $W$  containing  $p$  such that  $W \cap \text{supp} \rho_U = \emptyset$ . Thus there exists a chart  $(Y, \phi)$  around  $p$  contained in  $W$  such that  $f_V|_Y = 0$  is smooth. thus  $f_V$  is smooth on  $V$ .

#### Problem 26.1 Short exact Mayer-Vietoris sequence

*Solution:* First we check exactness at  $\Omega^k(M)$ . For this we need to show that  $i$  is injective. Assume  $i(\omega) = (i_U^* \omega, i_V^* \omega) = (\omega|_U, \omega|_V) = (0, 0)$ . As  $U \cup V = M$ , this clearly means  $\omega = 0$  on  $M$ . Thus  $i$  is injective.

Now we check exactness at  $\Omega^k(U) \oplus \Omega^k(V)$ . Consider any  $i(\omega) = (\omega|_U, \omega|_V) \in \text{im } i$ . Then  $j(i(\omega)) = (\omega|_U)|_{U \cap V} - (\omega|_V)|_{U \cap V} = \omega|_{U \cap V} - \omega|_{U \cap V} = 0$ . Thus  $\text{im } i \subseteq \ker j$ . Now take any  $(\omega, \tau) \in \ker j \Rightarrow \omega|_{U \cap V} = \tau|_{U \cap V}$ . Consider a  $k$ -form  $\sigma$  on  $M$  well-defined by  $\sigma|_U = \omega$  and  $\sigma|_V = \tau$ . For any  $p \in M$ , we have that  $p \in U$  or  $p \in V$ . In either case, there exists a chart  $(W, \phi)$  around  $p$  contained in  $U$  or  $V$ . Then  $\sigma|_W = \omega|_W$  or  $\sigma|_W = \tau|_W$ . In both cases,  $\sigma$  is  $C^\infty$  at  $p$ . Thus  $\sigma \in \Omega^k(M)$ . Clearly  $i(\sigma) = (\omega, \tau)$ , thus  $(\omega, \tau) \in \text{im } i$ . Thus  $\text{im } i = \ker j$ .



**Problem 26.2 Alternating sum of dimensions**

*Solution:* We have at each step  $\dim A^k = \dim \ker d_k + \dim \operatorname{im} d_k = \dim \operatorname{im} d_{k-1} + \dim \operatorname{im} d_k$  by exactness. This gives:

$$\sum_{k=0}^m (-1)^k \dim A^k = \sum_{k=0}^m (-1)^k (\dim \operatorname{im} d_{k-1} + \dim \operatorname{im} d_k) = \dim \operatorname{im} d_{-1} + (-1)^m \dim \operatorname{im} d_m,$$

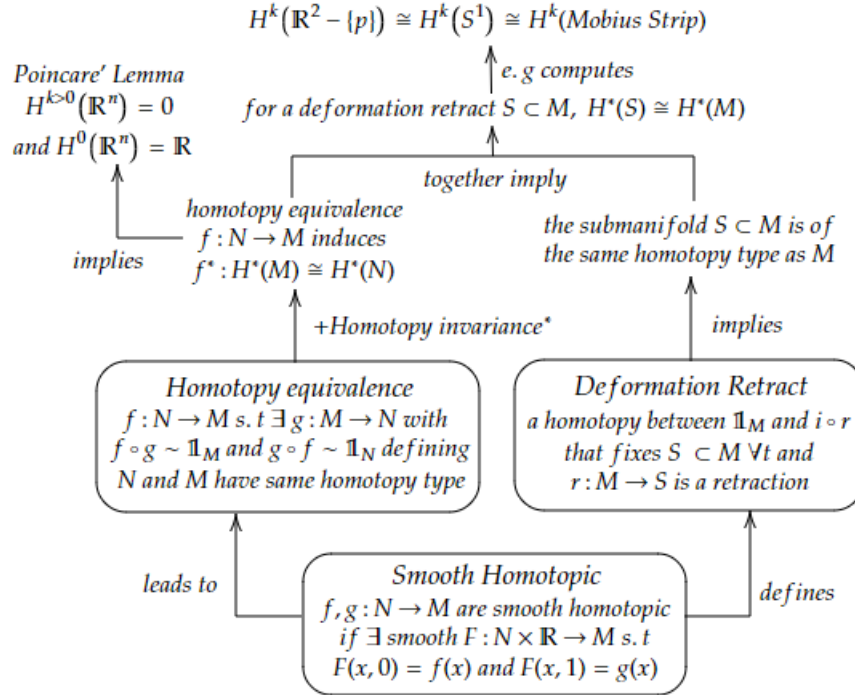
where the last step follows from the sum being a telescoping series. As  $d_{-1}$  and  $d_m$  are the zero maps, the sum equals 0.

**Exercise 26.5 Euler characteristics in terms of an open cover**

*Solution:* Given that the Mayer-Vietoris sequence is exact, we can apply Problem 26.2:

$$\begin{aligned} \sum_{k=0}^n (-1)^{3k} \dim H^k(M) + (-1)^{3k+1} (\dim H^k(U) + \dim H^k(V)) + (-1)^{3k+2} \dim H^k(U \cap V) &= 0 \\ \Rightarrow \sum_{k=0}^n (-1)^k (\dim H^k(M) - (\dim H^k(U) + \dim H^k(V)) + \dim H^k(U \cap V)) &= 0 \\ \Rightarrow \chi(M) - (\chi(U) + \chi(V)) + \chi(U \cap V) &= 0 \end{aligned}$$

## 7.4 Homotopy Invariance



\*Will be discussed in the last chapter.

### Problem 27.1 Homotopy equivalence

*Solution:* As  $M$  and  $N$  are homotopy equivalent, and  $N$  and  $P$  are homotopy equivalent, there exists homotopy equivalences  $f: M \rightarrow N$  and  $g: N \rightarrow P$  with  $h: N \rightarrow M$  and  $k: P \rightarrow N$  as their homotopic inverses respectively. Thus  $(h \circ f) \sim \mathbb{1}_M$ ,  $(f \circ h) \sim \mathbb{1}_N$ ,  $(k \circ g) \sim \mathbb{1}_N$  and  $(g \circ k) \sim \mathbb{1}_P$ . We want to show that  $g \circ f: M \rightarrow P$  is a homotopy equivalence with  $h \circ k: P \rightarrow M$  as a homotopic inverse.

To show that  $(h \circ k) \circ (g \circ f) \sim \mathbb{1}_M$ , we show first that  $(h \circ k) \circ (g \circ f) \sim (h \circ f)$ . Then by transitivity of the homotopic relation, our result follows. Define the following homotopy  $F: M \times \mathbb{R} \Rightarrow M$  given by  $F = h \circ G \circ (f \times \mathbb{1})$ , where  $G$  is a homotopy between  $(k \circ g)$  and  $\mathbb{1}_N$ . Clearly  $F$  is smooth, and we can check that that under  $F$ :

$$\begin{aligned} (x, 0) &\xrightarrow{f \times \mathbb{1}} (f(x), 0) \xrightarrow{G} (k \circ g)(f(x)) \xrightarrow{h} (h \circ k \circ g \circ f)(x) \\ (x, 1) &\xrightarrow{f \times \mathbb{1}} (f(x), 1) \xrightarrow{G} f(x) \xrightarrow{h} (h \circ f)(x) \end{aligned}$$

Thus  $F$  is a homotopy between  $(h \circ k) \circ (g \circ f)$  and  $h \circ f$ .

Similarly, we can show that  $(g \circ f) \circ (h \circ k) \sim (g \circ k) \sim \mathbb{1}_P$ . For this we construct the smooth homotopy  $H: P \times \mathbb{R} \rightarrow P$  given by  $H = g \circ K \circ (k \times \mathbb{1})$ , where  $K$  is a homotopy between  $(f \circ h)$  and  $\mathbb{1}_N$ . Now under  $H$ :

$$\begin{aligned}
(x, 0) &\xrightarrow{k \times \mathbb{1}} (k(x), 0) \xrightarrow{K} (f \circ h)(f(x)) \xrightarrow{g} (g \circ f \circ h \circ k)(x) \\
(x, 1) &\xrightarrow{k \times \mathbb{1}} (k(x), 1) \xrightarrow{G} k(x) \xrightarrow{h} (g \circ k)(x)
\end{aligned}$$

Thus  $H$  is a homotopy between  $(g \circ f) \circ (h \circ k)$  and  $(g \circ k)$ . This proves that  $M$  and  $P$  are homotopy equivalent.

### Problem 27.2 Contractibility and path-connectedness

*Solution:* Assume  $M$  is contractible. Choose any two points  $p, q \in M$ . We know that  $M$  and  $\{p\}$  have the same homotopy type. Thus the constant map  $f : M \rightarrow \{p\}$  is a homotopy equivalence with inclusion  $i$  as a homotopic inverse. Thus there exists a smooth homotopy between  $i \circ f$  and  $\mathbb{1}_M$  given by  $F(x, t) : M \times \mathbb{R} \rightarrow M$  such that  $F(x, 0) = p$  and  $F(x, 1) = x$  for all  $x \in M$ . Restricting  $F|_{x=q} : \mathbb{R} \rightarrow M$  is such that  $F|_{x=q}(0) = p$  and  $F|_{x=q}(1) = q$ . Restricting this further to the closed interval  $[0, 1]$  we obtain a continuous path from  $p$  to  $q$ . As these points were arbitrary,  $M$  is path-connected.

### Problem 27.3 Deformation retract from a cylinder to a circle

*Solution:* Consider the smooth map  $F : (S^1 \times \mathbb{R}) \times \mathbb{R} \rightarrow S^1 \times \mathbb{R}$  given by  $F((p, z), t) = (p, (1 - t)z)$ . We can see that  $F((p, z), 0) = (p, z)$ , and that  $F((p, 0), t) = (p, 0) \in S^1 \times \{0\}$  for all  $t$ . Furthermore  $F((p, z), 1) = (p, 0)$  is a retraction from  $S^1 \times \mathbb{R}$  to  $S^1 \times \{0\}$ . Thus the circle  $S^1 \times \{0\}$  is a deformation retract of the cylinder  $S^1 \times \mathbb{R}$ .

## 7.5 Computation of the de Rahm Cohomology

This chapter mainly involved computation of the Cohomology vector space of the torus as  $H^1(\Sigma) = \mathbb{R}^2$  and  $H^2(\Sigma) = \mathbb{R}$ . The following imperative result is also mentioned:

**Lemma 28.3.** *Suppose  $p$  is in a compact oriented surface  $M$  without boundary, and  $i : C \rightarrow M - \{p\}$  is the inclusion of a small circle around the puncture. Then the restriction map:*

$$i^* : H^1(M - \{p\}) \rightarrow H^1(C)$$

*is the zero map.*

This allows the computation of the de Rahm Cohomology of  $\Sigma_2$  to give  $H^1(\Sigma_2) = \mathbb{R}^4$  and  $H^2(\Sigma_2) = \mathbb{R}$ .

### Problem 28.1 Real Projective Plane

*Solution:* Consider  $\mathbb{R}P^2$  as  $D^2$  with anti-podal points identified with each other. Consider the open cover  $\{U, V\}$  where  $U$  consists of  $\mathbb{R}P^2$  with the center point  $p \in D^2$  removed, and  $V$  is an open disk containing  $p$ . We can see that  $U$  deformation retracts to  $S^1$  with anti-podal points identified, which is just  $\mathbb{R}P^1 \cong S^1$ . Thus  $U \sim S^1$ . Similarly,  $V$  is contractible, but  $U \cap V \sim S^1$ . Taking into account that  $\mathbb{R}P^2$  is connected, this gives us the following Mayer-Vietoris sequence:

$$0 \rightarrow \mathbb{R} \xrightarrow{\alpha_1} \mathbb{R}^2 \xrightarrow{\alpha_2} \mathbb{R} \xrightarrow{\alpha_3} H^1(\mathbb{R}P^2) \xrightarrow{\alpha_4} \mathbb{R} \xrightarrow{i^*} \mathbb{R} \xrightarrow{\alpha_5} H^2(\mathbb{R}P^2) \rightarrow 0 \rightarrow \dots$$

Firstly, it should be noted that  $i^*$  is restriction from  $H^1(U)$  to  $H^1(U \cap V)$ . We can show that  $i^*$  is an isomorphism by showing that  $i : U \cap V \rightarrow U$  is a homotopy equivalence. Define the map  $j : U \rightarrow U \cap V$  by  $j(r, \theta) = (b, \theta)$  for some  $b$  less than the radius of  $V$ . Note that this map is well-defined as for points on the boundary of  $D^2$  at  $r = 1$ ,  $j(1, \theta) = (b, 2\theta) = (b, 2\theta + 2\pi) = j(1, \theta + \pi)$ . It is easy to see that  $i \circ j \sim 1_U$  and  $j \circ i \sim 1_{U \cap V}$ .

Now that  $i^*$  is an isomorphism, we must have that  $\alpha_5 = 0 \Rightarrow H^2(\mathbb{R}P^2) = 0$ . Moreover as  $i^*$  is injective,  $\alpha_4 = 0$ . Thus we can decompose the above sequence into:

$$0 \rightarrow \mathbb{R} \xrightarrow{\alpha_1} \mathbb{R}^2 \xrightarrow{\alpha_2} \mathbb{R} \xrightarrow{\alpha_3} H^1(\mathbb{R}P^2) \rightarrow 0$$

Using Problem 26.2, we get that  $1 - 2 + 1 - \dim H^1(\mathbb{R}P^2) = 0 \Rightarrow H^1(\mathbb{R}P^2) = 0$ .

### Problem 28.2 The $n$ -sphere

*Solution:* Let us first compute the cohomology of  $S^2$ . Take a point  $p \in S^2$ , and consider the open cover  $\{U = S^2/\{p\}, V\}$ , where  $V$  is an open disk around  $p$  and hence is contractible. Furthermore,  $U \cap V \sim S^1$ , and  $U = S^2/\{p\} \cong \mathbb{R}^2$ . Now as  $U, V$  and  $U \cap V$  are all connected, we may begin the Mayer-Vietoris sequence from  $H^1(S^2)$  as

follows:

$$0 \rightarrow H^1(S^2) \xrightarrow{\beta} H^1(U) \oplus H^1(V) \xrightarrow{\alpha} H^1(S^1) \xrightarrow{d^*} H^2(S^2) \xrightarrow{\gamma} H^2(U) \oplus H^2(V) \rightarrow \dots$$

As  $H^k(U) = H^k(V) = 0$  for  $k > 0$ , and  $\beta$  is injective  $\Rightarrow H^1(S^2) = 0$ . Furthermore, as  $\alpha = 0 \Rightarrow d^*$  is injective. Also, as  $\gamma = 0 \Rightarrow d^*$  is also surjective and thus an isomorphism. Thus  $H^2(S^2) \cong \mathbb{R}$ , and  $H^k(S^2) = 0$  for all  $k > 2$ .

This leads us to make the following hypothesis:  $H^k(S^n) = \mathbb{R}$  for  $k = 0, n$  and is 0 otherwise. We will prove this inductively. Clearly it holds for  $k = 0, 1, 2$ . Assume it holds for all  $n < k$ . For  $S^k$ , consider a point  $p \in S^k$  and an open cover  $\{S^n/\{p\} \cong \mathbb{R}^n, V\}$ , where  $V$  is an open disk  $\text{int}(D^n)$  containing  $p$ . Thus we have that  $V$  is contractible and  $U \cap V \sim S^{n-1}$ . As  $U, V$  and  $U \cap V$  are all connected, we may start the Mayer-Vietoris sequence from  $H^1(S^k)$ . Now as  $H^l(S^{n-1}) = 0$  for  $l < n - 1$ , and  $H^l(U) = H^l(V) = 0$  by the inductive hypothesis, each row of the exact sequence can be separated into a exact sequence:

$$0 \rightarrow H^l(S^k) \rightarrow 0,$$

where  $l < n$ . This trivially implies that  $H^l(S^k) = 0$  for  $l < k$ . For  $l = k$ , we have the exact sequence:

$$0 \rightarrow H^{k-1}(S^{k-1}) \xrightarrow{d^*} H^k(S^k) \rightarrow 0 \rightarrow \dots$$

Clearly, by exactness,  $d^*$  is an isomorphism and thus  $H^k(S^k) \cong H^{k-1}(S^{k-1}) \cong \mathbb{R}$ .

### Problem 28.3 Cohomology of a multiply punctured plane

*Solution:* Consider the doubly punctured plane  $\mathbb{R}^2 - \{p, q\}$ . Cover  $\mathbb{R}^2 - \{p\}$  with  $\{U = \mathbb{R}^2 - \{p, q\}, V\}$ , where  $V$  is an open disc around  $q$ . Furthermore,  $U \cap V \sim S^1 \sim U \cap V$ , and  $V$  is contractible. Clearly  $U, V$  and  $U \cap V$  are all connected, and hence we can start the Mayer-Vietoris sequence from  $H^1(\mathbb{R}^2 - \{p\}) \cong H^1(S^1) \cong \mathbb{R}$  to get:

$$0 \rightarrow \mathbb{R} \xrightarrow{\beta} H^1(U) \xrightarrow{\alpha} \mathbb{R} \xrightarrow{d^*} 0 \rightarrow H^2(U) \rightarrow 0 \rightarrow \dots$$

where every row for  $k > 1$  becomes  $0 \rightarrow H^k(U) \rightarrow 0$ . Thus we have that  $H^k(U) = 0$  for  $k > 1$ . For  $k = 1$ , we can see that as  $\beta$  is injective,  $\dim \ker \alpha = 1$ . Furthermore as  $\alpha$  is surjective,  $\dim \text{im } \alpha = 1$ . Thus by the rank-nullity theorem,  $\dim H^1(U) = 2$ . Thus  $H^1(\mathbb{R}^2 - \{p, q\}) \cong \mathbb{R}^2$ .

This leads us to form the following hypothesis:  $H^1(\mathbb{R}^2 - \{p_1, \dots, p_n\}) \cong \mathbb{R}^n$ , and  $H^k(\mathbb{R}^2 - \{p_1, \dots, p_n\}) = 0$  for  $k > 1$  (it is obvious that  $H^0(\mathbb{R}^2 - \{p_1, \dots, p_n\}) \cong \mathbb{R}$  by connectedness). We prove this inductively. This clearly holds for  $n = 1, 2$  as before. Assume it holds for all  $n < k$ . Now for  $n = k$ , let  $\mathbb{R}^2 - \{p_1, \dots, p_{k-1}\}$  be covered by  $\{\mathbb{R}^2 - \{p_1, \dots, p_k\}, V\}$  where  $V$  is an open disc around  $p_k$ . Again,  $V$  is contractible and  $U \cap V \sim S^1$ . Furthermore, as  $U, V, U \cap V$  are all connected, we can start the

Mayer-Vietoris sequence as:

$$0 \rightarrow H^1(\mathbb{R}^2 - \{p_1, \dots, p_{k-1}\}) \cong \mathbb{R}^{k-1} \xrightarrow{\beta} H^1(\mathbb{R}^2 - \{p_1, \dots, p_k\}) \xrightarrow{\alpha} H^1(S^1) \cong \mathbb{R} \rightarrow 0 \rightarrow \dots$$

where it is clear that  $H^k(\mathbb{R}^2 - \{p_1, \dots, p_k\}) = 0$  for  $k > 1$ . Furthermore, as  $\beta$  is injective,  $\dim \ker \alpha = k - 1$ , while  $\dim \operatorname{im} \alpha = 1$  by surjectivity. Thus  $H^1(\mathbb{R}^2 - \{p_1, \dots, p_k\}) \cong \mathbb{R}^k$ .

**Problem 28.4 Cohomology of a surface of genus  $g$**

*Solution:* We hypothesize that  $H^k(\sum_n) = 0$  for  $k > 2$ ,  $H^0(\sum_n) = H^2(\sum_n) = \mathbb{R}$  and  $H^1(\sum_n) = \mathbb{R}^{2n}$ . We prove this inductively. This clearly holds for  $n = 1, 2$ . Assume it holds for all  $n < k$ . Now consider covering  $\{U = \sum_{k-1} - \{p\}, V\}$  of  $\sum_{k-1}$ , where  $p \in \sum_{k-1}$  and  $V$  is an open disk around it. Note that  $V$  is contractible, and that  $U \cap V \sim S^1$ . Furthermore, as  $U$ ,  $V$  and  $U \cap V$  are all connected, we can start the Mayer-Vietoris sequences as follows:

$$0 \rightarrow \mathbb{R}^{2k-2} \xrightarrow{\alpha} H^1(U) \xrightarrow{i^*} \mathbb{R} \xrightarrow{d^*} \mathbb{R} \xrightarrow{\gamma} H^2(U) \rightarrow 0 \rightarrow 0 \rightarrow H^3(U) \rightarrow 0 \rightarrow \dots$$

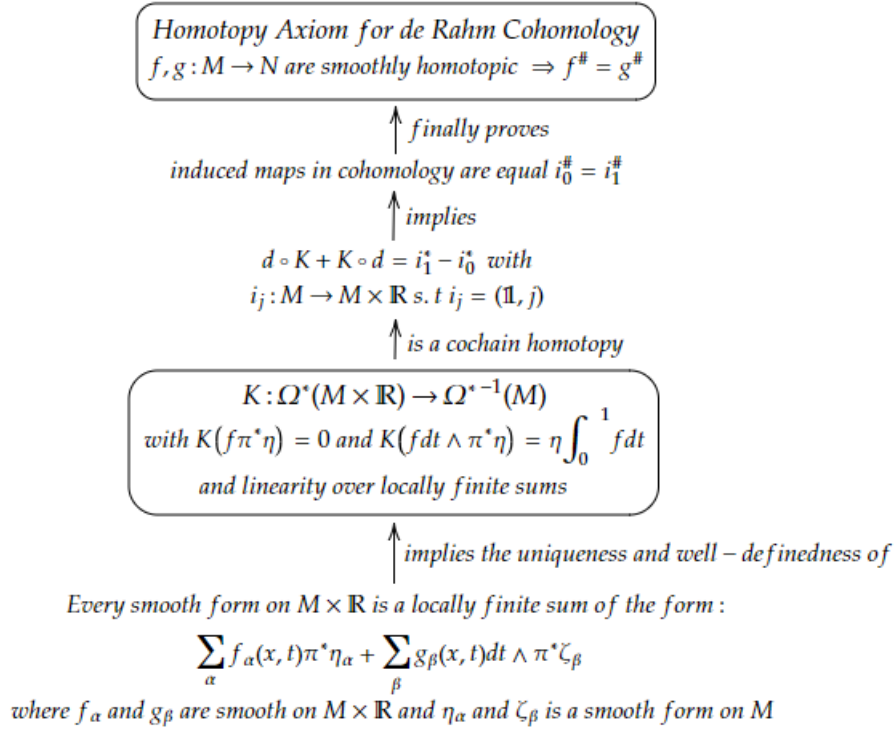
As  $i^*$  is restriction onto  $S^1$ , we know that  $i^*$  is the zero map  $\Rightarrow \alpha$  is surjective as well as injective. Thus  $H^1(U) \cong \mathbb{R}^{2k-2}$ . Furthermore,  $d^*$  is injective, and hence a surjection as well  $\Rightarrow \gamma$  is the zero map. This implies  $H^2(U) = 0$ . We can also see that trivially,  $H^l(U) = 0$  for all  $l > 2$  as well.

Now we proceed to calculate the cohomology of  $\sum_k$ . Cover it with  $\{U, V\}$  where  $U$  consists of one genus of the torus, and  $V$  covers the rest  $k - 1$  genres. Note that this way  $U \cap V \sim S^1$  and  $U \sim \sum_1 - \{p\}$ , and  $V \sim \sum_{k-1} - \{q\}$ . Furthermore, as  $U$ ,  $V$ , and  $U \cap V$  are all connected, we can start the Mayer-Vietoris sequence as:

$$0 \rightarrow H^1(\sum_k) \xrightarrow{\alpha} \mathbb{R}^{2k-2} \oplus \mathbb{R}^2 \xrightarrow{i^*} \mathbb{R} \xrightarrow{\beta} H^2(\sum_k) \rightarrow 0 \rightarrow 0 \rightarrow H^3(\sum_k) \rightarrow 0 \rightarrow \dots$$

As  $i^*$  is the zero map,  $\alpha$  is an isomorphism and hence  $H^1(\sum_k) \cong \mathbb{R}^{2k}$ . Furthermore  $\beta$  is an isomorphism too, giving  $H^2(\sum_k) \cong \mathbb{R}$ . We can trivially see that  $H^l(\sum_k) = 0$  for  $l > 2$ .

## 7.6 Proof of Homotopy Invariance



### Problem 29.2 Linearity of pullback over locally finite sums

*Solution:* Consider any point  $p \in N$ . Then  $h(p)$  contains a nbhd.  $V$  such that  $\sum \omega_{\alpha}$  is a finite sum over  $V$ . As  $h$  is  $C^{\infty} \Rightarrow h^{-1}(V)$  is an open nbhd. of  $p$ . Now we have:

$$\begin{aligned}
\left( h^* \left( \sum \omega_{\alpha} \right) \right) \Big|_{h^{-1}(V)} &= h^* \left( \left( \sum \omega_{\alpha} \right) \Big|_V \right) \\
&= \sum (h^* \omega_{\alpha} |_V) \\
&= \sum ((h^* \omega_{\alpha})|_{h^{-1}(V)}) \\
&= \left( \sum (h^* \omega_{\alpha}) \right) \Big|_{h^{-1}(V)}.
\end{aligned}$$

Thus as every point  $p \in M$  has a nbhd. where  $h^*(\sum \omega_{\alpha}) = \sum h^* \omega_{\alpha}$ , the equality holds over all of  $N$ .

## 8 Acknowledgements and Final Note

There were also some extra chapters in the appendix of the book, for instance on the Symplectic Group, which I did study but were not included in this report.

I would like to thank my supervisor, Dr. Shaheen Nazir, for continuously guiding me throughout the summer in this project.

This was my first extensive self-studying venture, and so it naturally consisted of many ups and downs. I would like to thank a certain person dear to me—Manahil—and her bookmark for helping me get through those periods of frustration and doubt.

## References

- [1] Loring W. Tu. *An introduction to manifolds*. Springer New York, 2011.