

# MATH 309: Introduction to Analysis II

## Equivalence of the Completeness of $\mathbb{R}$ and the Bolzano-Weierstrass Theorem

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### Bolzano-Weierstrass Theorem $\Rightarrow$ Completeness of $\mathbb{R}$

We must first prove that the Bolzano-Weierstrass Theorem implies Archimedean property. This is an essential result that is usually proven using Completeness of  $\mathbb{R}$ .

**Sub-Proof:** Assume on the contrary that  $\mathbb{N}$  is bounded. Then the sequence  $x_n = n$  is a bounded sequence. By BWT, there exists a subsequence  $x_{n_k} = n_k$  that is convergent to some  $r \in \mathbb{R}$ . Note that  $n_{k+1} \geq n_k + 1$  for all  $k \in \mathbb{N}$  (this property can be deduced from the order on  $\mathbb{N}$  and without assuming the Archimidean property). Take  $\epsilon = 0.5$ . Then there exists a  $k' \in \mathbb{N}$  such that  $n_k \in (r - \epsilon, r + \epsilon)$  for all  $k > k'$ . This implies that  $n_{k'+2} - n_{k'+1} < 2\epsilon = 1 \Rightarrow n_{k'+2} < n_{k'+1} + 1$ . This is in contradiction to the property from before. Thus  $\mathbb{N}$  has to be unbounded from above, and we can similarly argue for  $\mathbb{Z}$  (which is additionally unbounded from below).

Moreover, this gives us that the sequence  $\{2^{-n}\}$  converges to 0 (taking  $2^n > n$  for sufficiently large  $n$ ), and that the series  $\sum_{i=1}^{\infty} 2^{-i}$  converges to 1 in the usual way. We will now proceed with the main proof.

**Proof:** Let  $A \subseteq \mathbb{R}$  be any non-empty, bounded above subset of  $\mathbb{R}$ . Consider the set  $U = \{z \in \mathbb{Z} : z \text{ is an upper bound of } A\}$ . As  $A$  is non-empty, and  $\mathbb{Z}$  is unbounded from above in  $\mathbb{R}$ , we must have that  $U$  is non-empty. Furthermore, given the order on  $U$  induced by  $\mathbb{Z}$ ,  $U$  is well-ordered. Thus there exists  $b \in U$  that is the least element in  $U$ . Set  $a = b - 1 \in \mathbb{Z}$ . Thus we have that  $a \in \mathbb{Z}$  is not an upper bound of  $A$ , but  $a + 1$  is. Now we proceed further.

It is assumed that  $A$  does not have a maximal element; otherwise,  $\sup(A) = \max(A) \in \mathbb{R}$  trivially. Now consider the construction of the following sequence:

$$x_n = \begin{cases} a & \text{if } n = 0 \\ x_{n-1} + 2^{-n} & \text{if } x_{n-1} \text{ is not an upper bound of } A \\ x_{n-1} - 2^{-n} & \text{if } x_{n-1} \text{ is an upper bound of } A \end{cases}$$

Firstly, we shall prove that  $\{x_n\}$  is a Cauchy sequence. Consider any  $\epsilon > 0 \Rightarrow$  there exists  $n_0 \in \mathbb{N}$  such that  $2^{-n} < \epsilon$  for all  $n \geq n_0 \Rightarrow \sum_{l=n+1}^{\infty} 2^{-l} = 2^{-n} < \epsilon$  for all  $n > n_0$ . We can see that  $|x_n - x_m| \leq \sum_{l=n+1}^m 2^{-l} < \sum_{l=n+1}^{\infty} 2^{-l} < \epsilon$ , where  $m > n > n_0 \Rightarrow \{x_n\}$  is a Cauchy sequence.

By the assumption of the Bolzano-Weierstrass Theorem,  $x_n$  is convergent, and  $\{x_n\} \rightarrow r$  for some  $r \in \mathbb{R}$ .

**Lemma:** There does not exist  $m \in \mathbb{N}$  such that  $\{x_n\}$  is strictly decreasing for all  $n \geq m$ .

**Sub-Proof:** Assume on the contrary such an  $m$  exists. Then, as the sequence is initially increasing, there exists a smallest  $\tilde{m} \in \mathbb{N}$  such that  $\{x_n\}$  is strictly decreasing for all  $n \geq \tilde{m}$  and  $x_{\tilde{m}-1} < x_{\tilde{m}}$ . That is, there is an index  $\tilde{m}$  where the sequence starts to become strictly decreasing. Clearly, by construction,  $x_{\tilde{m}-1}$  is not an upper bound, and  $x_{\tilde{m}} = x_{\tilde{m}-1} + 2^{-\tilde{m}}$  is an upper bound. Thus there exists  $\tilde{a} \in A$  such that  $x_{\tilde{m}} > \tilde{a} > x_{\tilde{m}-1}$ . As the sequence is strictly decreasing now, we have that  $x_n$  is an upper bound for all  $n > \tilde{m} - 1$ . Moreover,  $\{x_n\}$  converges to  $x_{\tilde{m}} - \sum_{i=\tilde{m}+1}^{\infty} 2^{-i} = x_{\tilde{m}} - 2^{-\tilde{m}} = x_{\tilde{m}-1}$ . Take  $\epsilon = \tilde{a} - x_{\tilde{m}-1} > 0$ . Then there exists  $\tilde{n} \in \mathbb{N}_{\geq \tilde{m}}$  such that  $\tilde{a} > x_n > x_{\tilde{m}-1}$  for all  $n > \tilde{n}$ . This is a contradiction as  $x_n$  must be an upper bound for all  $n > \tilde{n} > \tilde{m} - 1$ . Hence, the sequence never becomes strictly decreasing.

Now we have two cases.

**Case 1:** There exists  $m \in \mathbb{N}$  such that  $\{x_n\}$  is strictly increasing for all  $n \geq m$ .

Then, by the well-ordering principle, there exists a least index  $\tilde{m} \in \mathbb{N}$  such that  $x_n$  is strictly increasing for all  $n \geq \tilde{m}$ .

**Sub-Case 1:**  $\tilde{m} = 1$ .

Then we show that  $\sup(A) = r = (a+1) \in \mathbb{R}$ . Firstly, we have  $x_n$  is not an upper bound of  $A$  for all  $n \in \mathbb{N}$ , and thus  $x_n = a + \sum_{l=1}^n 2^{-l} \Rightarrow x_n$  converges to  $(a+1)$ . We already have that  $(a+1)$  is an upper bound of  $A$  by assumption. Moreover, for any  $\epsilon > 0$ , there exists  $\bar{n} \in \mathbb{N}$  such that  $(a+1) - x_n < \epsilon$  for all  $n > \bar{n}$ , where  $x_n$  are not upper bounds of  $A \Rightarrow [(a+1) - \epsilon]$  is not an upper bound of  $A$  for all  $\epsilon > 0 \Rightarrow \sup(A) = (a+1) \in \mathbb{R}$ .

**Sub-Case 2:**  $\tilde{m} > 1$ .

Then we show that  $\sup(A) = r = x_{\tilde{m}-1}$ . Firstly, we have by construction that  $x_{\tilde{m}-1} > x_{\tilde{m}}$ , and that  $x_{\tilde{m}-1}$  is an upper bound. Also as the sequence is strictly decreasing for all  $n \geq \tilde{m}$ , we have that  $x_n$  is not an upper bound for all  $n > \tilde{m} - 1$ . Moreover,  $\{x_n\}$  converges to  $x_{\tilde{m}} + \sum_{i=\tilde{m}+1}^{\infty} 2^{-i} = x_{\tilde{m}} + 2^{-\tilde{m}} = x_{\tilde{m}-1}$ . Now take any  $\epsilon > 0$ . Then there exists  $\tilde{n} \in \mathbb{N}_{\geq \tilde{m}}$  such that  $x_{\tilde{m}-1} - \epsilon < x_n < x_{\tilde{m}-1}$  for all  $n > \tilde{n}$ . As  $x_n$  is not an upper bound for all  $n > \tilde{n} > \tilde{m} - 1$ ,  $x_{\tilde{m}-1} - \epsilon$  is not either. As  $\epsilon > 0$  was arbitrary, we have that  $\sup(A) = x_{\tilde{m}-1} \in \mathbb{R}$ .

**Case 2:** There does not exist an  $m \in \mathbb{N}$  such that  $\{x_n\}$  is strictly increasing for all  $n \geq m$ .

This assumption allows us to build the following subsequences:

$$x_{n_k} = \text{subsequence of all upper bounds in } x_n$$

$$x_{n_l} = \text{subsequence of all non-upper bounds in } x_n$$

as we can always find elements of  $\{x_n\}$  that are upper bounds or non-upper bounds due to  $\{x_n\}$  never becoming a strictly increasing or a strictly decreasing sequence. Moreover, as subsequences of  $\{x_n\}$ , they also converge to  $r \in \mathbb{R}$ . We show that this once again implies that  $\sup(A) = r \in \mathbb{R}$ .

Assume  $r$  is not an upper bound. Then there exists  $a_0 \in A$  such that  $a_0 > r$ . Take  $\epsilon = a_0 - r$ . By convergence of  $x_{n_k}$ , there exists  $k_0 \in \mathbb{N}$  such that  $x_{n_k} - r < a_0 - r$  for all  $k > k_0 \Rightarrow x_{n_k} < a_0$ . This is a contradiction as  $x_{n_k}$  is an upper bound of  $A$ . Hence,  $r$  is an upper bound of  $A$ .

Now consider any  $\varepsilon > 0$ . We have that there exists  $l_0 \in \mathbb{N}$  such that  $r - x_{n_l} < \varepsilon$  for all  $l > l_0 \Rightarrow x_{n_l} > r - \varepsilon$ . Thus as  $x_{n_l}$  is not an upper bound of  $A$ ,  $r - \varepsilon$  is not either for all  $\varepsilon > 0$ . Combining this with Proposition 5 that  $r$  is an upper bound of  $A$ , we finally have that  $r = \sup(A) \in \mathbb{R}$ . Thus,  $\mathbb{R}$  is complete.

## Completeness of $\mathbb{R} \Rightarrow$ Bolzano-Weierstrass Theorem

**Proof:** Let  $\{x_n\}$  be a bounded sequence in  $\mathbb{R}$  such that  $|x_n| < M$  for all  $n \in \mathbb{N}$ . This implies  $\sup\{x_n\} = r \in \mathbb{R}$  exists.

**Case 1:**  $r \neq x_n$  for all  $n \in \mathbb{N}$ .

Consider the construction of the following subsequence: Take  $x_{n_1} = x_k$  for any  $k \in \mathbb{N}$ . Set  $\delta_1 = \frac{\min\{|x_i - r|\}}{2}$  where  $1 \leq i \leq n_1$ . We know that  $\delta_1 > 0$  as  $r > x_n$  for all  $n \in \mathbb{N}$  by assumption. Furthermore, as  $r = \sup\{x_n\}$ , there exists  $\tilde{n} \in \mathbb{N}$  such that  $|x_{\tilde{n}} - r| < \delta_1$ . Clearly, by construction,  $\tilde{n} > n_1$ . Set  $x_{n_2} = x_{\tilde{n}}$ . Set  $\delta_2 = \frac{\min\{|x_i - r|\}}{2} < \delta_1/2$  where  $1 \leq i \leq n_2$ , and repeat to build a subsequence  $x_{n_k}$ . We can see that  $|x_{n_k} - r| < \delta_{k-1}$  for all  $k \in \mathbb{N}$ . As  $\delta_k < \frac{\delta_{k-1}}{2}$  for all  $k \in \mathbb{N} \Rightarrow \delta_k \rightarrow 0 \Rightarrow x_{n_k} \rightarrow r$ .

**Case 2:**  $r = x_m$  for some  $m \in \mathbb{N}$ .

**Sub-Case 1:** There exists  $\tilde{n} \in \mathbb{N}$  such that  $\sup\{x_n : n > \tilde{n}\} \notin \{x_n : n > \tilde{n}\}$ .

Repeat Case 1 for the sequence  $x_n$  where  $n > \tilde{n}$  to find a subsequence  $x_{n_k}$  that converges to  $\sup\{x_n : n > \tilde{n}\} \in \mathbb{R}$ . We can do this as  $\{x_n : n > \tilde{n}\}$  is also bounded.

**Sub-Case 2:** For all  $\tilde{n} \in \mathbb{N}$ ,  $\sup\{x_n : n > \tilde{n}\} \in \{x_n : n > \tilde{n}\}$ .

Consider the following subsequence: Take  $x_{n_1} = x_1$ . Set  $x_{n_2} = \sup\{x_n : n > n_1 (= 1)\}$  as  $\sup\{x_n : n > n_1 (= 1)\} \in \{x_n : n > n_1\}$  by assumption. Similarly, set  $x_{n_3} = \sup\{x_n : n > n_2\}$ , and continue the process to build a subsequence  $x_{n_k} = \sup\{x_n : n > n_{k-1}\}$ . This is a monotonically non-increasing subsequence in  $\{x_n\}$  which is bounded below. Thus  $x_{n_k}$  is convergent by the Monotone Convergence Theorem.

Thus, there always exists a subsequence  $x_{n_k}$  of  $x_n$  that converges. ■