Symplectic geometry and Toric Varieties

Part I: A walk through Symplectic geometry

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Table of Contents

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Quick recap of definitions

- A C^∞ *manifold* M is a second countable, Hausdorff, locally Euclidean space with an atlas $\mathfrak{A} = \{U_\alpha \subset M, V_\alpha \subset \mathbb{R}^n, \phi_\alpha : U_\alpha \to V_\alpha\}$ with $\cup_\alpha U_\alpha = M$, ϕ_α are homeomorphisms, and $\phi_\alpha^{-1} \circ \phi_\beta$ are smooth.

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- The *tangent space* T_pM at a point $p \in M$ is defined to be the vector space of all derivations at p. It can be identified with the basis $\{\frac{\partial}{\partial x_i}\}_{i=1}^n$ where $\{x_i\}_{i=1}^n$ are the coordinates at p. The dual vector space T_p^*M is called the *cotangent space* at p.

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- The *tangent bundle* $TM = \sqcup_p T_pM$ over an n-dimensional manifold M, is a $2n$-dimensional manifold with coordinates $\{x_i, c_i\}_{i=1}^n$ where c_i are the coefficients of $v \in T_{p=(x_1,\dots,x_n)}M$. The *cotangent bundle* $T^*M = \sqcup_p T_p^*M$ is constructed similarly.

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- The *kth de Rham cohomology* of M is $H^k(M) := \frac{\{\text{closed k-forms on } M\}}{\{\text{exact k- forms on } M\}}$.

Table of Contents

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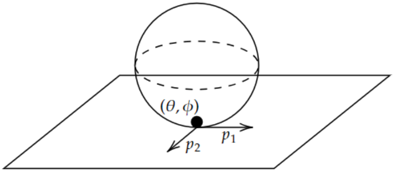

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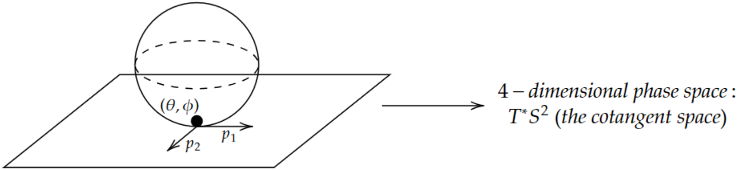

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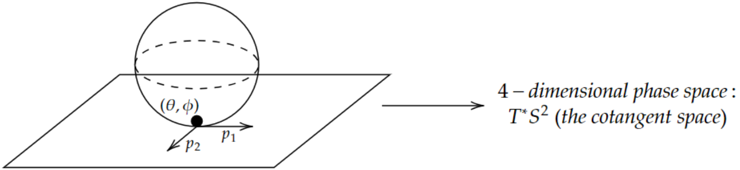

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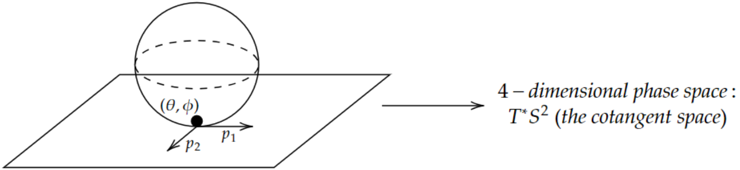

- Hamilton's equations and the laws of physics impose restrictions on the structure of the phase space.
- This resulting structure is exactly equivalent to the symplectic structure.

Symplectic Vector Space

Theorem

Let Ω be a skew-symmetric bilinear form on a real vector space V. Then there exists a basis $\{e_1, ..., e_n, f_1, ..., f_n, u_1, ..., u_k\}$ of V such that:
(i) $\Omega(e_i, e_j) = \Omega(f_i, f_j) = \Omega(u_i, ;) = 0$
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- Example: Any even-dimensional vector space V can be given a symplectic structure.

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A manifold M is symplectic iff it is equipped with a differential 2-form ω (called the symplectic form) such that:
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- Single most important symplectic manifold.

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- Natural question: How does one check for the existence of a symplectomorphism between arbitrary symplectic manifolds?

Lagrangian submanifolds

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- Converse also holds, thus giving a criterion for a diffeomorphism to be a symplectomorphism.
- Lagrangian submanifolds provide a geometric method of studying symplectomorphisms, and is especially effective in fixed point theory.

Important results

Theorem (Moser)

For a compact manifold M, let $S_c = \{\text{symplectic forms } \omega \text{ on } M \mid [\omega] = c\}$ for some $c \in H^2_{de\ Rham}(M)$. Then all symplectic forms on a path-connected component of S_c are symplectomorphic.

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Let (M, ω) be a symplectic manifold and let $p \in M$. Then there exists a coordinate chart $(U, x_1, ..., x_n, y_1, ..., y_n)$ centered at p such that $\omega|_U = \sum_i dx_i \wedge dy_i$.

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- No local invariants exist for symplectic manifolds.

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For a compact manifold M, let $S_c = \{\text{symplectic forms } \omega \text{ on } M \mid [\omega] = c\}$ for some $c \in H^2_{de\ Rham}(M)$. Then all symplectic forms on a path-connected component of S_c are symplectomorphic.

Theorem (Darboux Theorem, Weinstein)

Let (M, ω) be a symplectic manifold and let $p \in M$. Then there exists a coordinate chart $(U, x_1, ..., x_n, y_1, ..., y_n)$ centered at p such that $\omega|_U = \sum_i dx_i \wedge dy_i$.

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- Any compact orientable 3-manifold admits a contact structure (Martinet, 1971).

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Table of Contents

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- $\bar{\partial}^2 = \partial^2 = 0$, and hence allows for the formation of the *Dolbeault Cohomology*.

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- This definition allows for a natural definition for a basis of 1-forms: $T^{1,0} = \mathbb{C}\text{-span}\{dz_j\}$ and $T^{0,1} = \mathbb{C}\text{-span}\{d\bar{z}_j\}$, and thus $d = \partial + \bar{\partial}$.

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Let (M, J) be an almost complex manifold, and the Nijenhuis tensor $\mathfrak{N}(v, w) = [Jv, Jw] - J[v, Jw] - J[Jv, w] - [v, w]$. Then J is integrable if and only if $\mathfrak{N} \equiv 0 \Leftrightarrow d = \partial + \bar{\partial}$.

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- In particular, one can see that any orientable surface is symplectic, hence almost complex, and by the theorem above, also a complex manifold.

Kähler Manifolds

Definition

A Kähler manifold is a symplectic manifold (M, ω) with an integrable compatible almost complex structure. ω is then called a Kähler form.

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- In particular a Kähler form is a $(1,1)$-form that is ∂- and $\bar{\partial}$-closed, and on a coordinate chart $(U, z_1, ..., z_n)$ looks like $\omega = \frac{i}{2} \sum_{j,k} b_{j,k} dz_j \wedge d\bar{z}_k$ with $b_{jk}(p)$ being a positive definite hermitian matrix for all $p \in M$.

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- In particular, $[\omega_1] = [\omega_2] \Rightarrow (M, \omega_1) \cong (M, \omega_2)$.

Local Kähler structure

Definition

Let M be a complex manifold. A function $f \in C^\infty(M;\mathbb{R})$ is strictly plurisubharmonic if on each local chart $(U, z_1, ..., z_n)$, the matrix $\left[\frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}(p)\right]$ is positive-definite for all $f \in U$.

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- Example: The above principle is used to construct the *Fubini-Study* Kähler form on $\mathbb{CP}^n$, with Kähler potential $f(z) = \log(|z|^2 + 1)$.

Thank you![2][1]

V.I Arnold.
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