MATH 309: Introduction to Analysis II

Equivalence of the Completeness of $\mathbb{R}$ and the Bolzano-Weierstrass Theorem

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2024-10-0035

Fall 2022

Bolzano-Weierstrass Theorem $\Rightarrow$ Completeness of $\mathbb{R}$

We must first prove that the Bolzano-Weierstrass Theorem implies Archimedian property. This is an essential result that is usually proven using Completeness of $\mathbb{R}$.

Sub-Proof: Assume on the contrary that $\mathbb{N}$ is bounded. Then the sequence $x_n = n$ is a bounded sequence. By BWT, there exists a subsequence $x_{n_k} = n_k$ that is convergent to some $r \in \mathbb{R}$. Note that $n_{k+1} \geq n_k + 1$ for all $k \in \mathbb{N}$ (this property can be deduced from the order on $\mathbb{N}$ and without assuming the Archimidean property). Take $\epsilon = 0.5$. Then there exists a $k' \in \mathbb{N}$ such that $n_k \in (r - \epsilon, r + \epsilon)$ for all $k > k'$. This implies that $n_{k'+2} - n_{k'+1} < 2\epsilon = 1 \Rightarrow n_{k'+2} < n_{k'+1} + 1$. This is in contradiction to the property from before. Thus $\mathbb{N}$ has to be unbounded from above, and we can similarly argue for $\mathbb{Z}$ (which is additionally unbounded from below).

Moreover, this gives us that the sequence $\{2^{-n}\}$ converges to 0 (taking $2^n > n$ for sufficiently large n), and that the series $\sum_{i=1}^{\infty} 2^{-i}$ converges to 1 in the usual way. We will now proceed with the main proof.

Proof: Let $A \subseteq \mathbb{R}$ be any non-empty, bounded above subset of $\mathbb{R}$. Consider the set $U = \{z \in \mathbb{Z} : z \text{ is an upper bound of A}\}$. As A is non-empty, and $\mathbb{Z}$ is unbounded from above in $\mathbb{R}$, we must have that U is non-empty. Furthermore, given the order on U induced by $\mathbb{Z}$, U is well-ordered. Thus there exists $b \in U$ that is the least element in U. Set $a = b - 1 \in \mathbb{Z}$. Thus we have that that $a \in \mathbb{Z}$ is not an upper bound of A, but $a + 1$ is. Now we proceed further.
It is assumed that A does not have a maximal element; otherwise, $\sup(A) = \max(A) \in \mathbb{R}$ trivially. Now consider the construction of the following sequence:

$$x_n = \begin{cases} a & \text{if } n = 0 \\ x_{n-1} + 2^{-n} & \text{if } x_{n-1} \text{ is not an upper bound of } A \\ x_{n-1} - 2^{-n} & \text{if } x_{n-1} \text{ is an upper bound of } A \end{cases}$$

Firstly, we shall prove that $\{x_n\}$ is a Cauchy sequence. Consider any $\varepsilon > 0 \Rightarrow$ there exists $n_0 \in \mathbb{N}$ such that $2^{-n} < \varepsilon$ for all $n \geq n_0 \Rightarrow \sum_{l=n+1}^{\infty} 2^{-l} = 2^{-n} < \varepsilon$ for all $n > n_0$. We can see that $|x_n - x_m| \leq \sum_{l=n+1}^{m} 2^{-l} < \sum_{l=n+1}^{\infty} 2^{-l} < \varepsilon$, where $m > n > n_0 \Rightarrow \{x_n\}$ is a Cauchy sequence.
By the assumption of the Bolzano-Weierstrass Theorem, x_n is convergent, and $\{x_n\} \to r$ for some $r \in \mathbb{R}$.

Lemma: There does not exist $m \in \mathbb{N}$ such that $\{x_n\}$ is strictly decreasing for all $n \geq m$.

Sub-Proof: Assume on the contrary such an m exists. Then, as the sequence is initially increasing, there exists a smallest $\tilde{m} \in \mathbb{N}$ such that $\{x_n\}$ is strictly decreasing for all $n \geq \tilde{m}$ and $x_{\tilde{m}-1} < x_{\tilde{m}}$. That is, there is an index $\tilde{m}$ where the sequence starts to become strictly decreasing. Clearly, by construction, $x_{\tilde{m}-1}$ is not an upper bound, and $x_{\tilde{m}} = x_{\tilde{m}-1} + 2^{-\tilde{m}}$ is an upper bound. Thus there exists $\tilde{a} \in A$ such that $x_{\tilde{m}} > \tilde{a} > x_{\tilde{m}-1}$. As the sequence is strictly decreasing now, we have that x_n is an upper bound for all $n > \tilde{m} - 1$. Moreover, $\{x_n\}$ converges to $x_{\tilde{m}} - \sum_{i=\tilde{m}+1}^{\infty} 2^{-i} = x_{\tilde{m}} - 2^{-\tilde{m}} = x_{\tilde{m}-1}$. Take $\epsilon = \tilde{a} - x_{\tilde{m}-1} > 0$. Then there exists $\tilde{n} \in \mathbb{N}_{\geq \tilde{m}}$ such that $\tilde{a} > x_n > x_{\tilde{m}-1}$ for all $m > \tilde{n}$. This is a contradiction as x_n must be an upper bound for all $n > \tilde{n} > \tilde{m} - 1$. Hence, the sequence never becomes strictly decreasing.

Now we have two cases.

Case 1: There exists $m \in \mathbb{N}$ such that $\{x_n\}$ is strictly increasing for all $n \geq m$.
Then, by the well-ordering principle, there exists a least index $\tilde{m} \in \mathbb{N}$ such that x_n is strictly increasing for all $n \geq \tilde{m}$.

Sub-Case 1: $\tilde{m} = 1$.

Then we show that $\sup(A) = r = (a+1) \in \mathbb{R}$. Firstly, we have x_n is not an upper bound of A for all $n \in \mathbb{N}$, and thus $x_n = a + \sum_{l=1}^{n} 2^{-l} \Rightarrow x_n$ converges to $(a+1)$. We already have that $(a+1)$ is an upper bound of A by assumption. Moreover, for any $\varepsilon > 0$, there exists $\bar{n} \in \mathbb{N}$ such that $(a+1) - x_n < \varepsilon$ for all $n > \bar{n}$, where x_n are not upper bounds of $A \Rightarrow [(a+1)-\varepsilon]$ is not an upper bound of A for all $\varepsilon > 0 \Rightarrow \sup(A) = (a+1) \in \mathbb{R}$.

Sub-Case 2: $\tilde{m} > 1$.

Then we show that $\sup(A) = r = x_{\tilde{m}-1}$. Firstly, we have by construction that $x_{\tilde{m}-1} > x_{\tilde{m}}$, and that $x_{\tilde{m}-1}$ is an upper bound. Also as the sequence is strictly decreasing for all $n \geq \tilde{m}$, we have that x_n is not an upper bound for all $n > \tilde{m} - 1$. Moreover, $\{x_n\}$ converges to $x_{\tilde{m}} + \sum_{i=\tilde{m}+1}^{\infty} 2^{-i} = x_{\tilde{m}} + 2^{-\tilde{m}} = x_{\tilde{m}-1}$. Now take any $\epsilon > 0$. Then there exists $\tilde{n} \in \mathbb{N}_{\geq \tilde{m}}$ such that $x_{\tilde{m}-1} - \epsilon < x_n < x_{\tilde{m}-1}$ for all $n > \tilde{n}$. As x_n is not an upper bound for all $n > \tilde{n} > \tilde{m}-1$, $x_{\tilde{m}-1} - \epsilon$ is not either. As $\epsilon > 0$ was arbitrary, we have that $\sup(A) = x_{\tilde{m}-1} \in \mathbb{R}$.

Case 2: There does not exist an $m \in \mathbb{N}$ such that $\{x_n\}$ is strictly increasing for all $n \geq m$.

This assumption allows us to build the following subsequences:

$$x_{n_k} = \text{subsequence of all upper bounds in } x_n$$

$$x_{n_l} = \text{subsequence of all non-upper bounds in } x_n$$

as we can always find elements of $\{x_n\}$ that are upper bounds or non-upper bounds due to $\{x_n\}$ never becoming a strictly increasing or a strictly decreasing sequence. Moreover, as subsequences of $\{x_n\}$, they also converge to $r \in \mathbb{R}$. We show that this once again implies that $\sup(A) = r \in \mathbb{R}$.

Assume r is not an upper bound. Then there exists $a_0 \in A$ such that $a_0 > r$. Take $\varepsilon = a_0 - r$. By convergence of x_{n_k}, there exists $k_0 \in \mathbb{N}$ such that $x_{n_k} - r < a_0 - r$ for all $k > k_0 \Rightarrow x_{n_k} < a_0$. This is a contradiction as x_{n_k} is an upper bound of A. Hence, r is an upper bound of A.

Now consider any $\varepsilon > 0$. We have that there exists $l_0 \in \mathbb{N}$ such that $r - x_{n_l} < \varepsilon$ for all $l > l_0 \Rightarrow x_{n_l} > r - \varepsilon$. Thus as x_{n_l} is not an upper bound of A, $r - \varepsilon$ is not either for all $\varepsilon > 0$. Combining this with Proposition 5 that r is an upper bound of A, we finally have that $r = \sup(A) \in \mathbb{R}$.
Thus, $\mathbb{R}$ is complete.

Completeness of $\mathbb{R} \Rightarrow$ Bolzano-Weierstrass Theorem

Proof: Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}$ such that $|x_n| < M$ for all $n \in \mathbb{N}$. This implies $\sup\{x_n\} = r \in \mathbb{R}$ exists.

Case 1: $r \neq x_n$ for all $n \in \mathbb{N}$.
Consider the construction of the following subsequence: Take $x_{n_1} = x_k$ for any $k \in \mathbb{N}$. Set $\delta_1 = \dfrac{\min\{|x_i - r|\}}{2}$ where $1 \leq i \leq n_1$. We know that $\delta_1 > 0$ as $r > x_n$ for all $n \in \mathbb{N}$ by assumption. Furthermore, as $r = \sup\{x_n\}$, there exists $\tilde{n} \in \mathbb{N}$ such that $|x_{\tilde{n}} - r| < \delta_1$. Clearly, by construction, $\bar{n} > n_1$. Set $x_{n_2} = x_{\tilde{n}}$. Set $\delta_2 = \dfrac{\min\{|x_i - r|\}}{2} < \delta_1/2$ where $1 \leq i \leq n_2$, and repeat to build a subsequence x_{n_k}. We can see that $|x_{n_k} - r| < \delta_{k-1}$ for all $k \in \mathbb{N}$. As $\delta_k < \dfrac{\delta_{k-1}}{2}$ for all $k \in \mathbb{N} \Rightarrow \delta_k \to 0 \Rightarrow x_{n_k} \to r$.

Case 2: $r = x_m$ for some $m \in \mathbb{N}$.

Sub-Case 1: There exists $\tilde{n} \in \mathbb{N}$ such that $\sup\{x_n : n > \tilde{n}\} \notin \{x_n : n > \tilde{n}\}$.

Repeat Case 1 for the sequence x_n where $n > \tilde{n}$ to find a subsequence x_{n_k} that converges to $\sup\{x_n : n > \tilde{n}\} \in \mathbb{R}$. We can do this as $\{x_n : n > \tilde{n}\}$ is also bounded.

Sub-Case 2: For all $\tilde{n} \in \mathbb{N}$, $\sup\{x_n : n > \tilde{n}\} \in \{x_n : n > \tilde{n}\}$.

Consider the following subsequence: Take $x_{n_1} = x_1$. Set $x_{n_2} = \sup\{x_n : n > n_1(= 1)\}$ as $\sup\{x_n : n > n_1(= 1)\} \in \{x_n : n > n_1\}$ by assumption. Similarly, set $x_{n_3} = \sup\{x_n : n > n_2\}$, and continue the process to build a subsequence $x_{n_k} = \sup\{x_n : n > n_{k-1}\}$,. This is a monotonically non-increasing subsequence in $\{x_n\}$ which is bounded below. Thus x_{n_k} is convergent by the Monotone Convergence Theorem.

Thus, there always exists a subsequence x_{n_k} of x_n that converges. ■