MATH 309: Introduction to Analysis II

Equivalence of the Completeness of $\mathbb{R}$ and the Bolzano-Weierstrass Theorem

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Abstract

In any introductory course on Real Analysis, one of the fundamental properties of $\mathbb{R}-$*completeness*$-$is taken to be axiomatically true. This then serves as the starting point for further development of foundational results in real analysis, including the Bolzano-Weierstrass Theorem, Nested Interval Property, Monotone Convergence Theorem, and others. That said, it can be shown that these results, among others, are equivalent to the completeness property of real numbers. Thus, instead of assuming the completeness property, one can take any of these to be axiomatically true, and develop an identical theory of real numbers from the ground up. This small project aims to present an original proof of this claim for the Bolzano-Weierstrass Property.

Bolzano-Weierstrass Theorem $\Rightarrow$ Completeness of $\mathbb{R}$

We must first prove that the Bolzano-Weierstrass Theorem implies Archimedean property. This is an essential result that is usually proven using Completeness of $\mathbb{R}$.

Proof of Archimedean Property: Assume on the contrary that $\mathbb{N}$ is bounded. Then the sequence $x_n = n$ is a bounded sequence. By BWT, there exists a subsequence $x_{n_k} = n_k$ that is convergent to some $r \in \mathbb{R}$. Note that $n_{k+1} \geq n_k + 1$ for all $k \in \mathbb{N}$ (this property can be deduced from the order on $\mathbb{N}$ and without assuming the Archimidean property). Take $\epsilon = 0.5$. Then there exists a $k' \in \mathbb{N}$ such that $n_k \in (r-\epsilon, r+\epsilon)$ for all $k > k'$. This implies that $n_{k'+2} - n_{k'+1} < 2\epsilon = 1 \Rightarrow n_{k'+2} < n_{k'+1} + 1$. This is in contradiction to the property from before. Thus $\mathbb{N}$ has to be unbounded from above, and we can similarly argue for $\mathbb{Z}$ (which is additionally unbounded from below). ■

Moreover, this gives us that the sequence $\{2^{-n}\}$ converges to 0 (taking $2^n > n$ for sufficiently large n), and that the series $\sum_{i=1}^{\infty} 2^{-i}$ converges to 1 in the usual way. We will now proceed with the main proof.

Main argument: Let $A \subseteq \mathbb{R}$ be any non-empty, bounded above subset of $\mathbb{R}$. Consider the set $U = \{z \in \mathbb{Z} : z \text{ is an upper bound of A}\}$. As A is non-empty, and $\mathbb{Z}$ is unbounded from above in $\mathbb{R}$, we must have that U is non-empty. Furthermore, given the order on U induced by $\mathbb{Z}$, U is well-ordered. Thus there exists $b \in U$ that is the least element in U. Set $a = b - 1 \in \mathbb{Z}$. Thus we have that that $a \in \mathbb{Z}$ is not an upper bound of A, but $a + 1$ is. Now we proceed further.
It is assumed that A does not have a maximal element; otherwise, $\sup(A) = \max(A) \in \mathbb{R}$ trivially. Now

consider the construction of the following sequence:

$$x_n = \begin{cases} a & \text{if } n = 0 \\ x_{n-1} + 2^{-n} & \text{if } x_{n-1} \text{ is not an upper bound of } A \\ x_{n-1} - 2^{-n} & \text{if } x_{n-1} \text{ is an upper bound of } A \end{cases}$$

Firstly, we shall prove that $\{x_n\}$ is a Cauchy sequence. Consider any $\varepsilon > 0 \Rightarrow$ there exists $n_0 \in \mathbb{N}$ such that $2^{-n} < \varepsilon$ for all $n \geq n_0 \Rightarrow \sum_{l=n+1}^{\infty} 2^{-l} = 2^{-n} < \varepsilon$ for all $n > n_0$. We can see that $|x_n - x_m| \leq \sum_{l=n+1}^{m} 2^{-l} < \sum_{l=n+1}^{\infty} 2^{-l} < \varepsilon$, where $m > n > n_0 \Rightarrow \{x_n\}$ is a Cauchy sequence.
By the assumption of the Bolzano-Weierstrass Theorem, x_n is convergent, and $\{x_n\} \to r$ for some $r \in \mathbb{R}$.

Lemma: *There does not exist $m \in \mathbb{N}$ such that $\{x_n\}$ is strictly decreasing for all $n \geq m$.*

Proof: Assume on the contrary such an m exists. Then, as the sequence is initially increasing, there exists a smallest $\tilde{m} \in \mathbb{N}$ such that $\{x_n\}$ is strictly decreasing for all $n \geq \tilde{m}$ and $x_{\tilde{m}-1} < x_{\tilde{m}}$. That is, there is an index $\tilde{m}$ where the sequence starts to become strictly decreasing. Clearly, by construction, $x_{\tilde{m}-1}$ is not an upper bound, and $x_{\tilde{m}} = x_{\tilde{m}-1} + 2^{-\tilde{m}}$ is an upper bound. Thus there exists $\tilde{a} \in A$ such that $x_{\tilde{m}} > \tilde{a} > x_{\tilde{m}-1}$. As the sequence is strictly decreasing now, we have that x_n is an upper bound for all $n > \tilde{m} - 1$. Moreover, $\{x_n\}$ converges to $x_{\tilde{m}} - \sum_{i=\tilde{m}+1}^{\infty} 2^{-i} = x_{\tilde{m}} - 2^{-\tilde{m}} = x_{\tilde{m}-1}$. Take $\epsilon = \tilde{a} - x_{\tilde{m}-1} > 0$. Then there exists $\tilde{n} \in \mathbb{N}_{\geq \tilde{m}}$ such that $\tilde{a} > x_n > x_{\tilde{m}-1}$ for all $m > \tilde{n}$. This is a contradiction as x_n must be an upper bound for all $n > \tilde{n} > \tilde{m} - 1$. Hence, the sequence never becomes strictly decreasing. ■

Now we have two cases.

Case 1: *There exists $m \in \mathbb{N}$ such that $\{x_n\}$ is strictly increasing for all $n \geq m$.*
Then, by the well-ordering principle, there exists a least index $\tilde{m} \in \mathbb{N}$ such that x_n is strictly increasing for all $n \geq \tilde{m}$.

Sub-Case 1: $\tilde{m} = 1$.

Then we show that $\sup(A) = r = (a+1) \in \mathbb{R}$. Firstly, we have x_n is not an upper bound of A for all $n \in \mathbb{N}$, and thus $x_n = a + \sum_{l=1}^{n} 2^{-l} \Rightarrow x_n$ converges to $(a+1)$. We already have that $(a+1)$ is an upper bound of A by assumption. Moreover, for any $\varepsilon > 0$, there exists $\bar{n} \in \mathbb{N}$ such that $(a+1) - x_n < \varepsilon$ for all $n > \bar{n}$, where x_n are not upper bounds of $A \Rightarrow [(a+1) - \varepsilon]$ is not an upper bound of A for all $\varepsilon > 0 \Rightarrow \sup(A) = (a+1) \in \mathbb{R}$. ■

Sub-Case 2: $\tilde{m} > 1$.

Then we show that $\sup(A) = r = x_{\tilde{m}-1}$. Firstly, we have by construction that $x_{\tilde{m}-1} > x_{\tilde{m}}$, and that $x_{\tilde{m}-1}$ is an upper bound. Also as the sequence is strictly decreasing for all $n \geq \tilde{m}$, we have that x_n is not an upper bound for all $n > \tilde{m} - 1$. Moreover, $\{x_n\}$ converges to $x_{\tilde{m}} + \sum_{i=\tilde{m}+1}^{\infty} 2^{-i} = x_{\tilde{m}} + 2^{-\tilde{m}} = x_{\tilde{m}-1}$. Now take any $\epsilon > 0$. Then there exists $\tilde{n} \in \mathbb{N}_{\geq \tilde{m}}$ such that $x_{\tilde{m}-1} - \epsilon < x_n < x_{\tilde{m}-1}$ for all $n > \tilde{n}$. As x_n is not an upper bound for all $n > \tilde{n} > \tilde{m} - 1$, $x_{\tilde{m}-1} - \epsilon$ is not either. As $\epsilon > 0$ was arbitrary, we have that $\sup(A) = x_{\tilde{m}-1} \in \mathbb{R}$. ■

Case 2: *There does not exist an $m \in \mathbb{N}$ such that $\{x_n\}$ is strictly increasing for all $n \geq m$.*

This assumption allows us to build the following subsequences:

$$x_{n_k} = \text{subsequence of all upper bounds in } x_n$$

$$x_{n_l} = \text{subsequence of all non-upper bounds in } x_n$$

as we can always find elements of $\{x_n\}$ that are upper bounds or non-upper bounds due to $\{x_n\}$ never becoming a strictly increasing or a strictly decreasing sequence. Moreover, as subsequences of $\{x_n\}$, they also converge to $r \in \mathbb{R}$. We show that this once again implies that $\sup(A) = r \in \mathbb{R}$.

Assume r is not an upper bound. Then there exists $a_0 \in A$ such that $a_0 > r$. Take $\varepsilon = a_0 - r$. By convergence of x_{n_k}, there exists $k_0 \in \mathbb{N}$ such that $x_{n_k} - r < a_0 - r$ for all $k > k_0 \Rightarrow x_{n_k} < a_0$. This is a contradiction as x_{n_k} is an upper bound of A. Hence, r is an upper bound of A.

Now consider any $\varepsilon > 0$. We have that there exists $l_0 \in \mathbb{N}$ such that $r - x_{n_l} < \varepsilon$ for all $l > l_0 \Rightarrow x_{n_l} > r - \varepsilon$. Thus as x_{n_l} is not an upper bound of A, $r - \varepsilon$ is not either for all $\varepsilon > 0$. Combining this with Proposition 5 that r is an upper bound of A, we finally have that $r = \sup(A) \in \mathbb{R}$. ■

Thus, $\mathbb{R}$ is complete.

Completeness of $\mathbb{R} \Rightarrow$ Bolzano-Weierstrass Theorem

Main Argument: Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}$ such that $|x_n| < M$ for all $n \in \mathbb{N}$. This implies $\sup\{x_n\} = r \in \mathbb{R}$ exists. We now have several cases:

Case 1: *$r \neq x_n$ for all $n \in \mathbb{N}$.*

Consider the construction of the following subsequence: Take $x_{n_1} = x_k$ for any $k \in \mathbb{N}$. Set $\delta_1 = \dfrac{\min\{|x_i - r|\}}{2}$ where $1 \leq i \leq n_1$. We know that $\delta_1 > 0$ as $r > x_n$ for all $n \in \mathbb{N}$ by assumption. Furthermore, as $r = \sup\{x_n\}$, there exists $\tilde{n} \in \mathbb{N}$ such that $|x_{\tilde{n}} - r| < \delta_1$. Clearly, by construction, $\bar{n} > n_1$. Set $x_{n_2} = x_{\tilde{n}}$. Set $\delta_2 = \dfrac{\min\{|x_i - r|\}}{2} < \delta_1/2$ where $1 \leq i \leq n_2$, and repeat to build a subsequence x_{n_k}. We can see that $|x_{n_k} - r| < \delta_{k-1}$ for all $k \in \mathbb{N}$. As $\delta_k < \dfrac{\delta_{k-1}}{2}$ for all $k \in \mathbb{N} \Rightarrow \delta_k \to 0 \Rightarrow x_{n_k} \to r$. ■

Case 2: *$r = x_m$ for some $m \in \mathbb{N}$.*

Sub-Case 1: *There exists $\tilde{n} \in \mathbb{N}$ such that $\sup\{x_n : n > \tilde{n}\} \notin \{x_n : n > \tilde{n}\}$.*

Repeat Case 1 for the sequence x_n where $n > \tilde{n}$ to find a subsequence x_{n_k} that converges to $\sup\{x_n : n > \tilde{n}\} \in \mathbb{R}$. We can do this as $\{x_n : n > \tilde{n}\}$ is also bounded. ■

Sub-Case 2: *For all $\tilde{n} \in \mathbb{N}$, $\sup\{x_n : n > \tilde{n}\} \in \{x_n : n > \tilde{n}\}$.*

Consider the following subsequence: Take $x_{n_1} = x_1$. Set $x_{n_2} = \sup\{x_n : n > n_1(= 1)\}$ as $\sup\{x_n : n > n_1(= 1)\} \in \{x_n : n > n_1\}$ by assumption. Similarly, set $x_{n_3} = \sup\{x_n : n > n_2\}$, and continue the process to build a subsequence $x_{n_k} = \sup\{x_n : n > n_{k-1}\}$,. This is a monotonically non-increasing subsequence in $\{x_n\}$ which is bounded below. Thus x_{n_k} is convergent by the Monotone Convergence Theorem (we can assume this theorem by completeness of $\mathbb{R}$ as usual). ■

Thus, there always exists a subsequence x_{n_k} of x_n that converges.