Directed Reading Project: *An Introduction to Manifolds* by Loring W. Tu

Abdullah Ahmed (2024-10-0035)

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This document is structured chapter-wise as follows: A summarizing flowchart (sometimes with additional remarks) pertaining to links between various concepts discussed in a single chapter, followed by original solutions to specifically instructional chapter problems and exercises. Also note that some distinct chapters that link well together have been combined into one. Happy reading![1]

Contents

1 Euclidean Spaces

1.1 Smooth Functions on a Euclidean Space

$$\boxed{\begin{array}{l} f \;\; is\; C^\infty \\ at\; p \in \mathbb{R}^n \end{array}} \quad \Longleftarrow \quad \boxed{\begin{array}{l} f\; is\; real - analytic \\ in\; some\; nbhd.\; U \\ containing\; p \in \mathbb{R}^n \end{array}}$$

The converse only holds partially as in *Lemma* 1.4 from the book that follows:

Lemma 1.4 (Taylor's theorem with remainder). *Let f be a C^∞ function on an open subset $U \subset \mathbb{R}^n$ that is star-shaped with respect to a point $p = (p^1, \ldots.., p^n)$ in U. Then there are functions $g_1(x), \ldots..., g_n(x) \in C^\infty(U)$ such that*

$$f(x) = f(p) + \sum_{i=1}^{n} (x^i - p^i) g_i(x), \quad g_i(p) = \frac{\partial f}{\partial x^i}(p).$$

Problem 1.6* Taylor's theorem with remainder to order 2
Prove that if $f : \mathbb{R}^2 \to \mathbb{R}$ is C^∞, then there exist C^∞ functions g_{11}, g_{12}, g_{22} on $\mathbb{R}^2$ such that:

$$\begin{aligned} f(x,y) = f(0,0) &+ \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y \\ &+ x^2 g_{11}(x,y) + xy g_{12}(x,y) + y^2 g_{22}(x,y). \end{aligned}$$

Solution: Let $f \in C^\infty(\mathbb{R}^2)$. Then by choosing $p = (0,0)$ in the lemma above we have:

$$f(x,y) = f(0,0) + x g_1(x,y) + y g_2(x,y),$$

where $g_1(x,y), g_2(x,y) \in C^\infty(\mathbb{R}^2)$ and $g_1(0,0) = \dfrac{\partial f}{\partial x}(0,0)$ and $g_2(0,0) = \dfrac{\partial f}{\partial y}(0,0)$. Thus we can apply the same remainder theorem on g_1 and g_2 to get:

$$f(x,y) = f(0,0) + x(g_1(0,0) + x h_1(x,y) + y h_2(x,y)) + y(g_2(0,0) + x l_1(x,y) + y l_2(x,y))$$
$$\Rightarrow f(x,y) = f(0,0) + x\frac{\partial f}{\partial x}(0,0) + y\frac{\partial f}{\partial y}(0,0) + x^2 g_{11}(x,y) + xy g_{12}(x,y) + y^2 g_{22}(x,y),$$

where $h_1(x,y) = g_{11}(x,y)$, $l_2(x,y) = g_{22}(x,y)$ and $h_2(x,y) + l_1(x,y) = g_{12}(x,y)$ are all C^∞ over $\mathbb{R}^n$.

Problem 1.7* A function with a removable singularity
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^∞ function with $f(0,0) = \partial f/\partial x(0,0) = \partial f/\partial y(0,0) = 0$. Define:

$$g(t,u) = \begin{cases} \dfrac{f(t,tu)}{t} & t \neq 0 \\ 0 & t = 0. \end{cases}$$

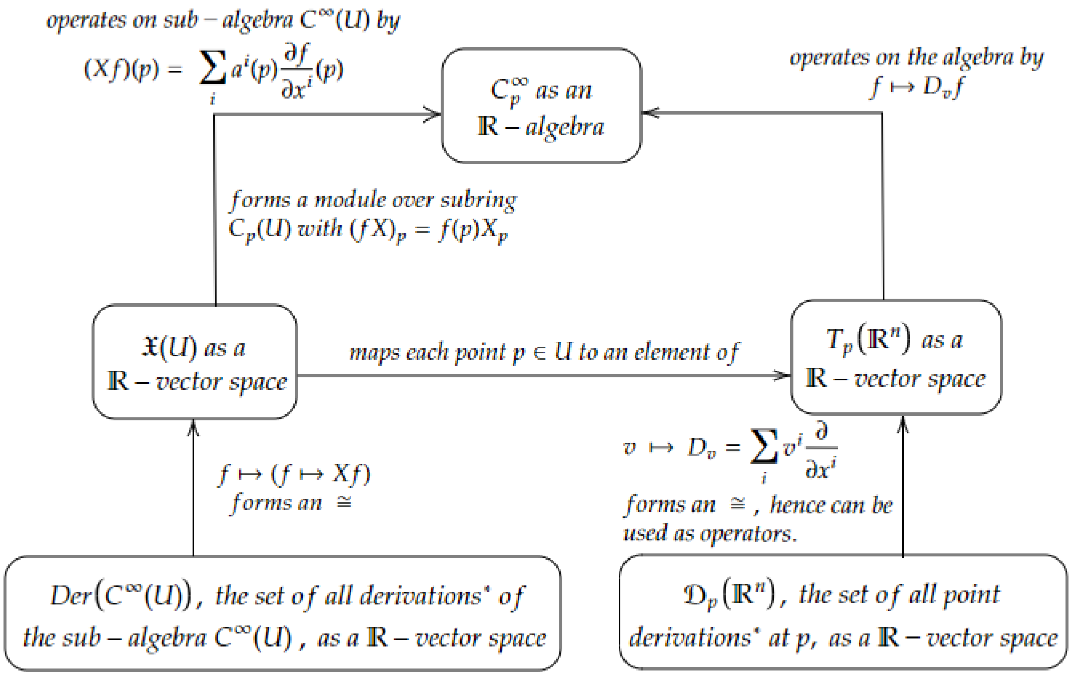
Prove that $g(t,u)$ is C^∞ for $(t,u) \in \mathbb{R}^2$.

Solution: Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be the C^∞ function given by $h(t,u) = (t,tu)$. We also have from the previous problem:

$$\begin{aligned} g(t,u) &= \frac{1}{t}\Big(t^2 g_{11}(t,tu) + t^2 u g_{12}(t,tu) + t^2u^2 g_{22}(t,tu)\Big) \\ &= t g_{11}(t,tu) + tu g_{12}(t,tu) + tu^2 g_{22}(t,tu) = k(t,u), \end{aligned}$$

for $t \neq 0$. As $g_{ij}(t,tu) = (g_{ij} \circ h)(t,u)$ is a composition of C^∞ functions, it is also $C^\infty \Rightarrow k(t,u)$ is C^∞. Also, as $k(0,u) = g(0,u) = 0$, we can see that $k(t,u) = g(t,u)$ for all $(t,u) \in \mathbb{R}^2$. Thus $g(t,u)$ is C^∞ as well.

1.2 Tangent Vectors in $\mathbb{R}^n$ as Derivations



*The general idea of a derivation is that of a linear operator that satisfies some form of the *Leibniz rule* that we are familiar with for derivatives. Also, strictly speaking, we consider the elements of $C^\infty(U)$ modulo the equivalence relation $(f, U) \sim (g, V) \Leftrightarrow f_{|U \cap V} = g_{|U \cap V}$.

Problem 2.4 Product of derivations

Let A be an algebra over a field K. If D_1 and D_2 are derivations of A, show that $D_1 \circ D_2$ is not necessarily a derivation (it is if D_1 or $D_2 = 0$), but $D_1 \circ D_2 - D_2 \circ D_1$ is always a derivation of A.

Solution: Assume D_1 and D_2 are derivations over an algebra A over a field K. Now consider:

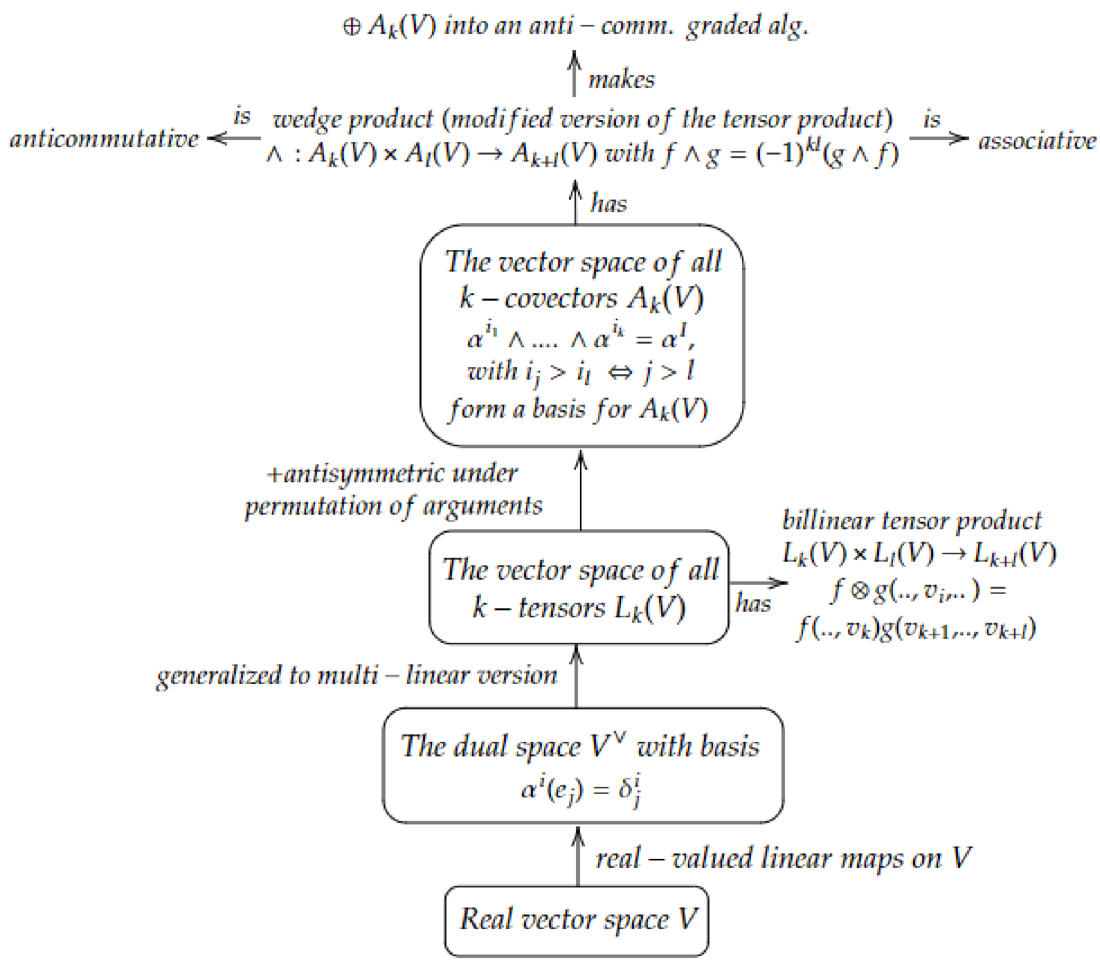
$$\begin{aligned}(D_1 \circ D_2)(ab) = D_1(D_2(ab)) &\overset{\text{leib.rule}}{=} D_1((D_2a)b + a(D_2b)) \overset{\text{lin.}}{=} D_1((D_2a)b) + D_1(a(D_2b)) \\ &= (D_1 \circ D_2)(a)b + (D_2a)(D_1b) + (D_1a)(D_2b) + a(D_1 \circ D_2)(b),\end{aligned}$$

where we can see clearly that the *Leibniz rule* is not satisfied. On the other hand, we can see that:

$$\begin{aligned}(D_1 \circ D_2 - D_2 \circ D_1)(ab) &= (D_1 \circ D_2)(a)b + (D_2a)(D_1b) + (D_1a)(D_2b) + a(D_1 \circ D_2)(b) \\ &\quad -((D_2 \circ D_1)(a)b + (D_1a)(D_2b) + (D_2a)(D_1b) + a(D_2 \circ D_1)(b)) \\ &= (D_1 \circ D_2 - D_2 \circ D_1)(a)b + a(D_1 \circ D_2 - D_2 \circ D_1)(b),\end{aligned}$$

thus $(D_1 \circ D_2 - D_2 \circ D_1)$ satisfies the *Leibniz rule* and is a derivation.

1.3 The Exterior Algebra of Multicovectors



Problem 3.4 A characterization of alternating k-tensors
Let f be a k-tensor on a vector space V. Prove that f is alternating if and only if f changes sign whenever two successive arguments are interchanged.

Solution: Assume we have an alternating k-tensor f on a vector space $V \Rightarrow \sigma f = \mathrm{sgn}(\sigma) f$ for all $\sigma \in S_k$. Let $\sigma = (i\ \ i+1)$Then we have:

$$f(..., v_{i+1}, v_i, ...) = \sigma f(..., v_i, v_{i+1}, ...) = \mathrm{sgn}(\sigma) f(..., v_i, v_{i+1}, ...) = -f(..., v_i, v_{i+1}, ...).$$

Now assume f changes sign whenever two successive arguments are interchanged. Consider an arbitrary permutation $\tau \in S_k$ and the corresponding ordered tuple $(\tau(1), ..., \tau(k))$. Starting from 1$= \tau(i_1)$ for some i_1, we swap its place with successive members to the left in the tuple above, until it reaches the first index. We then do the same for the next integer 2 up until $k-1$, such that the tuple is sorted by the end of this process. In this way, as each swap consists of a smaller integer being swapped with a larger one to the left of it, we can see that the total number of successive swaps made is precisely equal to the number of *inversions*, thus giving us:

$$\tau f = f(v_{\tau(1)}, ..., v_{\tau(k)}) = (-1)^{\text{no. of inv.}} f(v_1, ..., v_k) = \text{sgn}(\tau) f(v_1, ..., v_k)$$

Problem 3.10* Linear independence of covectors

Let α_1,...,α_k be 1-covectors on a vector space V. Show that $\alpha_1\wedge$,...,$\wedge\alpha_k$= 0 if and only if α_1,...,α_k are linearly independent in the dual space $V^\vee$.

Solution: Assume we are given that $\alpha^1 \wedge .... \wedge \alpha^k \neq 0$. Assume now on the contrary that $\{\alpha^i\}$ is a linearly dependent set in $V^\vee \Rightarrow \exists j$ such that $\alpha^j = \sum_{i\neq j}^k c_i^j \alpha^i$ for some $c_i^j \in \mathbb{R}$. Thus we have:

$$\alpha^1 \wedge ... \wedge \alpha^j \wedge ... \wedge \alpha^k = \alpha^1 \wedge ...(\sum_{i\neq j}^k c_i^j \alpha^i)... \wedge \alpha^k \overset{\text{lin.}}{=} \sum_{i\neq j}^k c_i^j (\alpha^1 \wedge ...\alpha^i ... \wedge \alpha^k\ ) = 0,$$

as for all terms in the sum above, at least one other superscript is equal to i, and $\alpha^i \wedge \alpha^i = 0$ by anti-commutativity of the wedge product. This is a contradiction, and thus $\{\alpha^i\}$ is a linearly independent set.
Now assume that the set $\{\alpha^i\}$ is linearly independent$\Rightarrow$ it can be extended to a basis $\alpha^1, ..., \alpha^n$ of $V^\vee$ where $n = \dim(V)$. Thus $\alpha^1 \wedge .... \wedge \alpha^k$ is a basis vector for $A_k(V)$ and hence cannot be zero.

Problem 3.11* Exterior multiplication

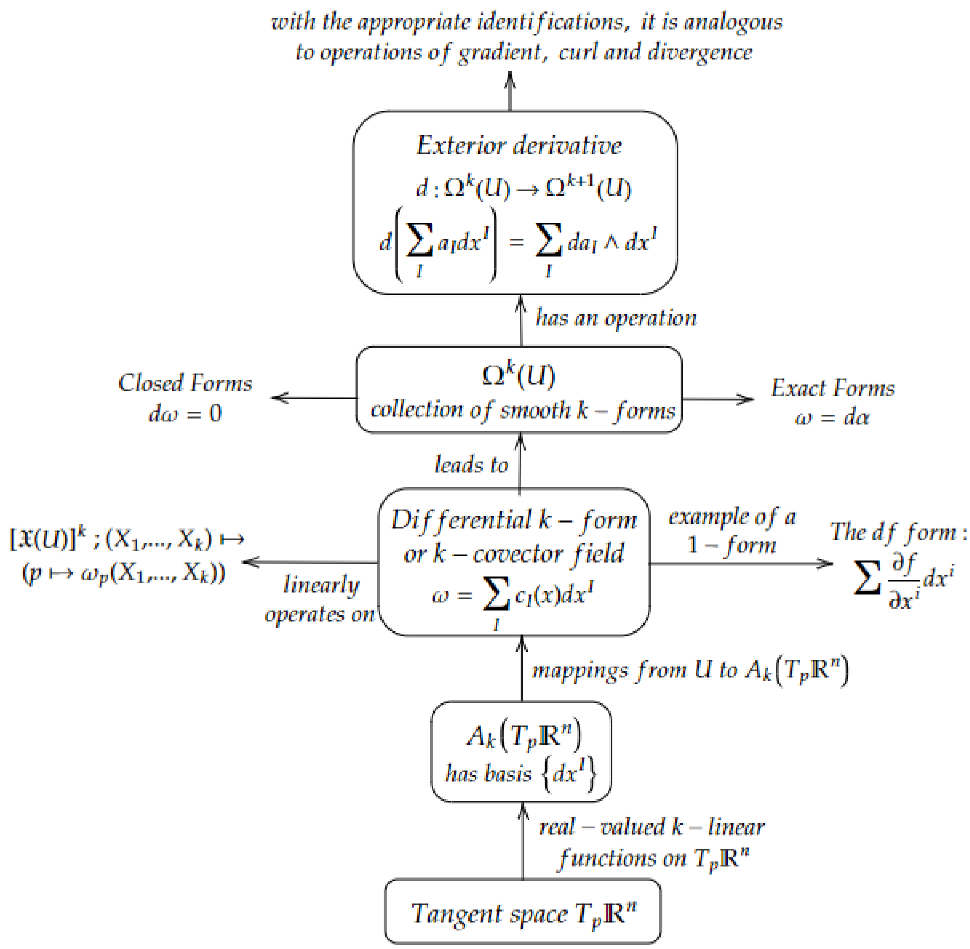
Let α be a nonzero 1-covector and γ a k-covector on a finite-dimensional vector space V. Show that $\alpha \wedge \gamma = 0$ if and only if $\gamma = \alpha \wedge \beta$ for some $(k-1)$-covector β on V.

Solution: Let α be a 1-covector and γ be a k-covector.
Assume $\gamma = \alpha \wedge \beta$ for some $(k-1)$-covector $\beta \Rightarrow \alpha \wedge \gamma = \alpha \wedge \alpha \wedge \beta = 0$.
Now assume $\alpha \wedge \gamma = 0$. Denote α by β_1, and extend it to a basis $\beta_1, ..., \beta_n$ of $V^\vee$ of dimension $n \Rightarrow \beta^I$ forms a basis for $A_k(V)$ where I varies over all strictly increasing multi-indices of length $k \Rightarrow \gamma = \sum_J c_J \beta^J$ for some $c_J \in \mathbb{R}$. The sum can be split up as follows:

$$\begin{aligned} \gamma &= \sum_{1\in J} c_J \beta^J + \sum_{1\notin J} c_J \beta^J \\ \Rightarrow \alpha \wedge \gamma &= \sum_{1\in J} c_J (\alpha \wedge \beta^J) + \sum_{1\notin J} c_J (\alpha \wedge \beta^J) = \sum_{1\notin J} c_J (\alpha \wedge \beta^J) = 0 \qquad (1) \end{aligned}$$

As $\alpha \wedge \beta^J$ are linearly independent in $A_{k+1}(V)$ for $1 \notin J$, (1) $\Rightarrow c_J = 0$ for all J such that $1 \notin J \Rightarrow \gamma = \sum_{1\in J} c_J \beta^J = \alpha \wedge \beta$ for some $(k-1)$-covector β.

1.4 Differential Forms on $\mathbb{R}^n$



Probelem 4.7 Commutator for derivations and anti-derivations
Let $A = \oplus_{k=-\infty}^{k=0} A_k$ be a graded algebra over a field K with $A_k = 0$ for $k < 0$. Let m be an integer. A superderivation of A of degree m is a K-linear map $D : A \to A$ such that for all k, $D(A_k) \subseteq A_{k+m}$ and for all $a \in A_k$ and $b \in A_l$,

$$D(ab) = (Da)b + (-1)^{km} a(Db)$$

If D_1 and D_2 are superderivations of A of respective degrees m_1 and m_2, define their commutator to be $[D_1, D_2] = D_1 \circ D_2 - (-1)^{m_1 m_2} D_2 \circ D_1$. Show that $[D_1, D_2]$ is a superderivation of degree $m_1 + m_2$.

Solution:

$$\begin{aligned} (D_1 \circ D_2)(ab) &= D_1((D_2 a)b + (-1)^{km_2} a D_2 b \\ &= (D_1 \circ D_2)(a)b + (-1)^{m_1(k+m_2)}(D_2 a)(D_1 b) \\ +(-1)^{km_2}&((D_1 a)(D_2 b) + (-1)^{km_1} a(D_1 \circ D_2)(b)) \end{aligned}$$

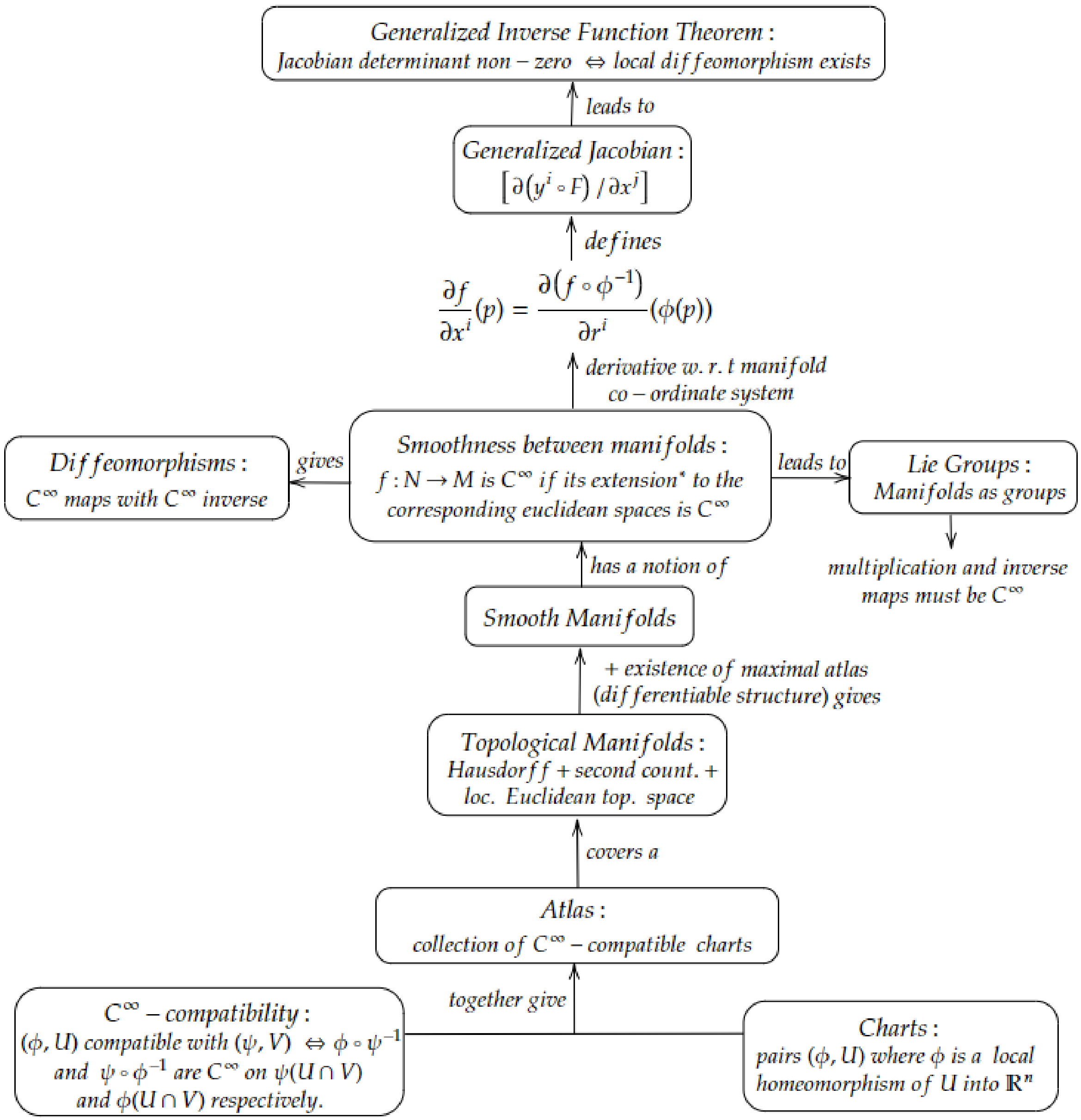
Similarly, we can calculate $D_2 \circ D_1(ab)$, by switching D_1 with D_2 and m_1 with m_2, and finally get after obvious manipulations:

$$\begin{aligned} [D_1, D_2](ab) &= (D_1 \circ D_2 - (-1)^{m_1 m_2} D_2 \circ D_1)(ab) \\ &= [D_1, D_2](a)b - (-1)^{k(m_1+m_2)} a[D_1, D_2]b, \end{aligned}$$

implying that $[D_1, D_2]$ is a superderivation of degree $m_1 + m_2$.

2 Manifolds

2.1 Smoothness on Manifolds



*Extension here alludes to a common idea of having the manifold locally inherit the smoothness from its associated euclidean space through its atlas. Given any point p and corresponding charts in the differentiable structure (ϕ, U) and (ψ, V) with $p \in U \subseteq N$ and $f(p) \in V \subseteq M$, we can *extend* f to the map $\psi \circ f \circ \phi^{-1} : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$, with $A = \phi(f^{-1}(V) \cap U)$.

As mentioned above, ideas of differentiability in the general setting of manifolds is *inherited* from their associated euclidean spaces-a relatively intuitive way to go about introducing differential structure on them. As is known, there exists a stronger notion of differentiability for complex-valued functions, i.e, *complex differentiability.* One could tweak the definition to impose that the charts be *holomorphic* instead, have the manifold inherit the notions of complex differentiability. One can assume that this would produce nicer results as compared to smooth manifolds, as is the case when results in complex analysis are compared to real analysis. As it turns out, this is true and this special class of manifolds are known as *complex manifolds*, and they have a very deep and active theory associated with them.

Problem 5.4* Existence of a coordinate neighbourhood
Let $\{(U_\alpha, \phi_\alpha)\}$ be the maximal atlas on a manifold M. For any open set U in M and a point $p \in U$, prove the existence of a coordinate open set U_α such that $p \in U_\alpha \subseteq U$.

Solution: Let $\mathfrak{A} = \{(U_\alpha, \phi_\alpha)\}$ be the maximal atlas of the smooth manifold M. Let U be an open set containing a point p. As $p \in M \Rightarrow \exists\, (U_\alpha, \phi_\alpha) \in \mathfrak{A}$ such that $p \in U_\alpha$. Then we have that $p \in V = U \cap U_\alpha \subseteq U$ is open. Clearly the restriction $\phi_\alpha|_V$ is local homeomorphism $\Rightarrow$ $(V, \phi_\alpha|_V)$ is a chart. As (U_α, ϕ_α) is C^∞-compatible with all other charts in $\mathfrak{A}$, so is $(V, \phi_\alpha|_V)$ as $V \subseteq U_\alpha \Rightarrow (V, \phi_\alpha|_V) \in \mathfrak{A}$ with $p \in V \subseteq U$.

Proposition 5.18 An atlas for a product manifold
If $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_i, \psi_i)\}$ are C^∞ atlases for the manifolds M and N of dimensions m and n, respectively, then the collection $\{(U_\alpha \times V_i, \phi_\alpha \times \psi_i : U\alpha \times V_i \to \mathbb{R}^m \times \mathbb{R}^n)\}$ of charts is a C^∞ atlas on $M \times N$. Therefore, $M \times N$ is a C^∞ manifold of dimension $m+n$.

Proof: Let M and N be m and n dimensional manifolds respectively, with atlases $\mathfrak{A}_M = \{(U_\alpha, \phi_\alpha)\}$ and $\mathfrak{A}_N = \{(V_i, \psi_i)\}$. We claim that $\mathfrak{A} = \{(U_\alpha \times V_i, \phi_\alpha \times \psi_i : U_\alpha \times V_i \to \mathbb{R}^m \times \mathbb{R}^n)\}$ is an atlas for $N \times M$, which is hence a $n+m$ dimensional manifold.
Clearly as $\mathfrak{A}$ covers $M \times N$, we must only check C^∞-compatibility. Let $(U_\alpha \times V_i, \phi_\alpha \times \psi_i)$ and $(U_\beta \times V_j, \phi_\beta \times \psi_j)$ be two arbitrary charts in $\mathfrak{A}$. We can see that:

$$(\phi_\alpha \times \psi_i) \circ (\phi_\beta \times \psi_j)^{-1} = (\phi_\alpha \circ \phi_\beta^{-1}) \times (\psi_i \circ \psi_j^{-1}),$$

is C^∞ on $(U_\alpha \times V_i) \cap (U_\beta \times V_j) = (U_\alpha \cap U_\beta) \times (V_i \cap V_j)$ as each component is C^∞ due to C^∞-compatibility among $\mathfrak{A}_M$ and $\mathfrak{A}_N$. A similar argument can be made for $(\phi_\beta \times \psi_j) \circ (\phi_\alpha \times \psi_i)^{-1}$. Hence $\mathfrak{A}$ is an atlas.

Problem 6.1 Differentiable structures on $\mathbb{R}$
Let $\mathbb{R}$ be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}, \phi = \mathbb{1} : \mathbb{R} \to \mathbb{R})$, and let $\mathbb{R}'$ be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}, \psi : \mathbb{R} \to \mathbb{R})$, where $\psi(x) = x^{1/3}$.
(a) Show that these two differentiable structures are distinct.

(b) Show that there is a diffeomorphism between $\mathbb{R}$ and $\mathbb{R}'$.

Solution: Assume we have two differentiable structures on $\mathbb{R}$, namely $\mathbb{R}_1$ and $\mathbb{R}_2$, generated by the maximal atlas of $(\mathbb{R}, \phi = id$ and $(\mathbb{R}, \psi(x) = x^{1/3}$ respectively. We can see that as $\phi^{-1} \circ \psi = \psi$ is not a C^∞ map from $\mathbb{R} \to \mathbb{R}$, the two differentiable structures are distinct. But we can check that $f : \mathbb{R}_1 \to \mathbb{R}_2$ defined by $f(x) = x^3$ is a diffeomorphism between the two:

$$\psi \circ f \circ \phi^{-1} = id \text{ is } C^\infty \text{ on } \phi(f^{-1}(\mathbb{R}) \cap \mathbb{R}) = \mathbb{R} \text{ and,}$$
$$\phi \circ f^{-1} \circ \psi^{-1} = id \text{ is } C^\infty \text{ on } \psi(f^{-1}(\mathbb{R}) \cap \mathbb{R}) = \mathbb{R}.$$

Exercise 6.18 Smoothness of a map to a Cartesian product
Let M_1, M_2, and N be manifolds of dimensions m_1, m_2, and n respectively. Prove that a map $(f_1, f_2) : N \to M_1 \times M_2$ is C^∞ if and only if $f_i : N \to M_i$, $i = 1, 2$, are both C^∞.

Solution: Let $f = (f_1, f_2) : N \to M_1 \times M_2$, and $\mathfrak{A}_{M_1} = \{(U_\alpha, \phi_\alpha)\}$, $\mathfrak{A}_{M_2} = \{(V_\alpha, \psi_\alpha)\}$, and $\mathfrak{A}_N = \{(W_\alpha, \chi_\alpha)\}$ be atlases for M_1, M_2 and N respectively. Correspondingly, by Proposition 5.18 above, $\mathfrak{A} = \{(U_\alpha \times V_\alpha, \phi_\alpha \times \psi_\alpha)$ is an atlas for $M_1 \times M_2$.
Assume f is $C^\infty \Rightarrow f \circ \pi_i = f_i$ is C^∞ as a composition of C^∞ maps.
Now assume f_1 and f_2 are both C^∞. Then we can see that for any $p \in W_\alpha$ and $f(p) \in U_\alpha \times V_\alpha$:

$$(\phi_\alpha \times \psi_\alpha) \circ f \circ \chi^{-1} = (\phi_\alpha \circ f_1 \circ \chi^{-1}) \times (\psi_\alpha \circ f_2 \circ \chi^{-1}),$$

is C^∞ at p as its component maps are C^∞.

2.2 Quotients and Real Projective Plane

Not much was to be discussed here apart from what is already known from an introductory course in Topology and Algebraic Geometry.

Problem 7.5* Orbit space of a continuous group action
Suppose a right action of a topological group G on a topological space S is continuous; this simply means that the map $S \times G \to S$ describing the action is continuous. Define two points x, y of S to be equivalent if they are in the same orbit; i.e., there is an element $g \in G$ such that $y = xg$. Let S/G be the quotient space; it is called the orbit space of the action. Prove that the projection map $\pi : S \to S/G$ is an open map.

Solution: Assume a group G acts on S continuously, i.e, the map $\bullet : S \times G \to S$ is continuous.
Consider any open set $U \subseteq S$. We must show that $\pi(U)$ is open in the quotient topology on $S/G \Leftrightarrow \pi^{-1}(\pi(U))$ is open in S. For any $[p] \in \pi(U)$, $\pi^{-1}([p]) = \bullet^{-1}(\{p\}) \Rightarrow \pi^{-1}(\pi(U)) = \bullet^{-1}(U)$ which is open.

Problem 7.6 Quotient of $\mathbb{R}$ by $2\pi\mathbb{Z}$
Let the additive group $2\pi\mathbb{Z}$ act on $\mathbb{R}$ on the right by $x \cdot 2\pi n = x + 2\pi n$, where n is an integer. Show that the orbit space $\mathbb{R}/2\pi\mathbb{Z}$ is a smooth manifold.

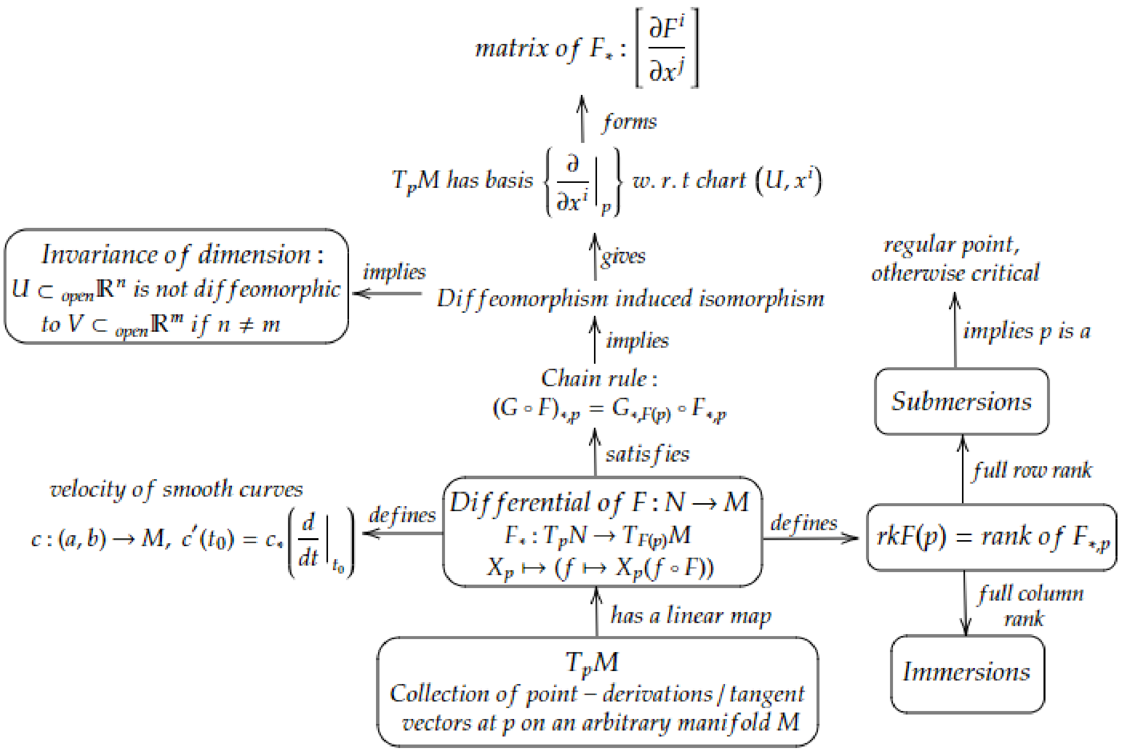
Solution: We know that $\mathbb{R}/2\pi\mathbb{Z}$ is homeomorphic to S^1, which is a smooth manifold with some atlas $\mathfrak{A} = \{(U_\alpha, \phi_\alpha)\}$. Let f be a homeomorphism from $\mathbb{R}/2\pi\mathbb{Z}$ to S^1. Then we claim $\mathfrak{A}' = \{(f^{-1}(U_\alpha), \phi_\alpha \circ f)\}$ is an atlas for $\mathbb{R}/2\pi\mathbb{Z}$. It is is easy to see that $\mathfrak{A}'$ covers $\mathbb{R}/2\pi\mathbb{Z}$ and that $\phi_\alpha \circ f$ are local homeomorphisms. We can also see that for any two charts $(f^{-1}(U_\alpha), \phi_\alpha \circ f), (f^{-1}(U_\beta), \phi_\beta \circ f) \in \mathfrak{A}'$:

$$(\phi_\alpha \circ f) \circ (\phi_\beta \circ f)^{-1} = (\phi_\alpha \circ f) \circ (f^{-1} \circ \phi_\beta^{-1}) = \phi_\alpha \circ \phi_\beta^{-1},$$

is C^∞ on $(\phi_\beta \circ f)(f^{-1}(U_\alpha) \cap f^{-1}(U_\beta)) = \phi(U_\alpha \cap U_\beta)$, by C^∞-compatibility in $\mathfrak{A}$. The same holds for $(\phi_\beta \circ f) \circ (\phi_\alpha \circ f)^{-1}$. Thus the two arbitrary charts are C^∞-compatible, and $\mathbb{R}/2\pi\mathbb{Z}$ is a smooth manifold. (Note that the argument is general enough to hold for any two homeomorphic spaces.)

3 The Tangent Space

3.1 The Tangent Space



Problem 8.1* Differential of a map:
Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the map:

$$(u, v, w) = F(x, y) = (x, y, xy).$$

Let $p = (x, y) \in \mathbb{R}^2$. Compute $F_*(\partial/\partial x|p)$ as a linear combination of $\partial/\partial u$, $\partial/\partial v$, and $\partial/\partial w$ at $F(p)$.

Solution: As $\left.\frac{\partial}{\partial u}\right|_{F(p)}$, $\left.\frac{\partial}{\partial v}\right|_{F(p)}$, and $\left.\frac{\partial}{\partial w}\right|_{F(p)}$ form a basis for $T_{F(p)}\mathbb{R}^3$, we have:

$$F_*(\partial/\partial x|_p) = a_u(\partial/\partial u|_{F(p)}) + a_v(\partial/\partial u|_{F(p)}) + a_w(\partial/\partial u|_{F(p)}).$$

Acting both sides on u, v, and w we get:

$$\begin{aligned} \partial/\partial x|_p(u\circ F) &= 1 = a_u \\ \partial/\partial x|_p(v\circ F) = \partial/\partial x|_p(y) &= 0 = a_v \\ \partial/\partial x|_p(w\circ F) = \partial/\partial x|_p(xy) &= y = a_w \end{aligned}$$

Thus $F_*(\partial/\partial x|_p) = (\partial/\partial u|_{F(p)}) + y(\partial/\partial u|_{F(p)})$.

Proposition 8.15 (Velocity of a curve in local coordinates)
Let $c :]a,b[\to M$ be a smooth curve, and let $(U, x_1, ..., x_n)$ be a coordinate chart about $c(t)$. Write $c_i = x_i \circ c$ for the ith component of c in the chart. Then $c'(t)$ is given by

$$c'(t) = \sum_{i=1}^{n} \dot{c}^i(t) \frac{\partial}{\partial x^i}\bigg|_{c(t)}.$$

Proof: Let $c : (a,b) \to M$ be a smooth curve, with $(U, x^1, ..., x^m)$ as a coordinate chart around $c(t)$. Then we have:

$$c'(t) = c_*\left(\frac{d}{dt}\bigg|_t\right) = \sum_{i=1}^{m} a^i \frac{\partial}{\partial x^i}\bigg|_{c(t)}.$$

Acting both sides on x^j:

$$\begin{aligned} \frac{d}{dt}\bigg|_t (x^j \circ c) = \dot{c}(t) &= \sum_{i=1}^{m} a^i \delta_i^j = a^j \\ \Rightarrow c'(t) &= \sum_{i=1}^{m} \dot{c}^i(t) \frac{\partial}{\partial x^i}\bigg|_{c(t)} \end{aligned}$$

Problem 8.7* Tangent space to a product
If M and N are manifolds, let $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ be the two projections. Prove that for $(p,q) \in M \times N$, $(\pi_{1_*}, \pi_{2_*}) : T_{(p,q)}(M \times N) \to T_pM \times T_qN$ is an isomorphism.

Solution: Consider the following coordinate charts: $(U, x^1, ..., x^m)$ in M and $(V, y^1, ..., y^n)$ in N, where $p \in U$ and $q \in V$. The corresponding coordinate chart in $M \times N$ is $(U \times V, \tilde{x}^1, ..., \tilde{x}^m, \tilde{y}^1, ..., \tilde{y}^m)$, where $\tilde{x}^i = x^i \circ \pi_1$ and $\widetilde{y^i} = y^i \circ \pi_2 \Rightarrow$ the basis for $T_p(M \times N)$ is $\left\{ \frac{\partial}{\partial \tilde{x}^1}\bigg|_{(p,q)}, ..., \frac{\partial}{\partial \tilde{x}^m}\bigg|_{(p,q)}, \frac{\partial}{\partial \tilde{y}^1}\bigg|_{(p,q)}, ..., \frac{\partial}{\partial \tilde{y}^n}\bigg|_{(p,q)} \right\}$. On the other hand, the basis for $T_pM \times T_pN$ is $\left\{ \left(\frac{\partial}{\partial x^1}\bigg|_p, \mathbf{0}\bigg|_q \right), ..., \left(\frac{\partial}{\partial x^m}\bigg|_p, \mathbf{0}\bigg|_q \right), \left(\mathbf{0}\bigg|_p, \frac{\partial}{\partial y^1}\bigg|_q \right), ..., \left(\mathbf{0}\bigg|_p, \frac{\partial}{\partial y^n}\bigg|_q \right) \right\}$. We show that (π_{1*}, π_{2*}) maps $\left.\dfrac{\partial}{\partial \tilde{x}^i}\right|_{(p,q)} \mapsto \left(\left.\dfrac{\partial}{\partial x^i}\right|_p, \left.\mathbf{0}\right|_q\right)$ and $\left.\dfrac{\partial}{\partial \tilde{y}^i}\right|_{(p,q)} \mapsto \left(\left.\mathbf{0}\right|_p, \left.\dfrac{\partial}{\partial y^i}\right|_q\right)$, hence mapping a basis to a basis, and thus being an isomorphism:

$$(\pi_{1*}, \pi_{2*})\left(\left.\frac{\partial}{\partial \tilde{x}^i}\right|_{(p,q)}\right) = \sum_{i=1}^m a^i \left(\left.\frac{\partial}{\partial x^i}\right|_p, \left.\mathbf{0}\right|_q\right) + \sum_{i=1}^n b^i \left(\left.\mathbf{0}\right|_p, \left.\frac{\partial}{\partial y^i}\right|_q\right)$$
$$\Rightarrow \left(\pi_{1*}\left(\left.\frac{\partial}{\partial \tilde{x}^i}\right|_{(p,q)}\right), \pi_{2*}\left(\left.\frac{\partial}{\partial \tilde{x}^i}\right|_{(p,q)}\right)\right) = \sum_{i=1}^m a^i \left(\left.\frac{\partial}{\partial x^i}\right|_p, \left.\mathbf{0}\right|_q\right) + \sum_{i=1}^n b^i \left(\left.\mathbf{0}\right|_p, \left.\frac{\partial}{\partial y^i}\right|_q\right),$$

where we can compare entry wise and act on the coordinate functions:

$$\pi_{1*}\left(\left.\frac{\partial}{\partial \tilde{x}^i}\right|_{(p,q)}\right)(x^j) = \sum_{i=1}^m a^i \left.\frac{\partial}{\partial x^i}\right|_p (x^j)$$
$$\Rightarrow \left.\frac{\partial}{\partial \tilde{x}^i}\right|_{(p,q)} (x^j \circ \pi_1) = \sum_{i=1}^m a^i \delta_i^j = a^j$$
$$\Rightarrow \left.\frac{\partial}{\partial r^i}\right|_{\phi(p,q)} (x^j \circ \pi_1 \circ (\phi^{-1} = (\tilde{x}^1, ..., \tilde{x}^m, \tilde{y}^1, ..., \tilde{y}^n)^{-1}) = \left.\frac{\partial}{\partial r^i}\right|_{\phi(p,q)} (r^j) = \delta_i^j = a^j.$$

Similarly, one can find that $b^i = 0 \Rightarrow (\pi_{1*}, \pi_{2*})\left(\left.\dfrac{\partial}{\partial \tilde{x}^i}\right|_{(p,q)}\right) = \left(\left.\dfrac{\partial}{\partial x^i}\right|_p, \left.\mathbf{0}\right|_q\right)$. The same arguments can be applied to basis elements of the form $\left.\dfrac{\partial}{\partial \tilde{y}^i}\right|_{(p,q)}$ to show $(\pi_{1*}, \pi_{2*})\left(\left.\dfrac{\partial}{\partial \tilde{y}^i}\right|_{(p,q)}\right) = \left(\left.\mathbf{0}\right|_p, \left.\dfrac{\partial}{\partial y^i}\right|_q,\right)$.

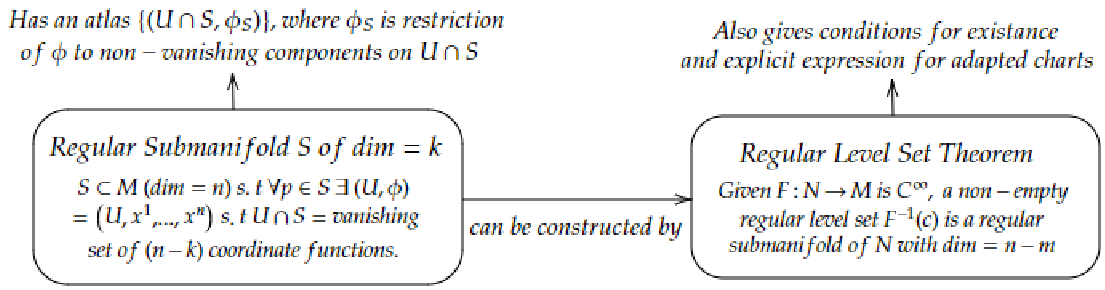
Problem 8.10 Local Maxima
Prove that a local maximum of a C^∞ function $f : M \to \mathbb{R}$ is a critical point of f.

Solution: Assume $p \in M$ is a local maxima of the C^∞ function $f : M \to \mathbb{R}$, where p is contained in the chart $(U, x^1, ..., x^n)$. Let $X_p = \dfrac{\partial}{\partial x^i}$, and choose a curve $c(t)$ in M such that $c(0) = p$ and $c'(0) = X_p$. Then we have:

$$\left.\frac{\partial f}{\partial x^i}\right|_p = X_p f = c'(0) f = \left.\frac{d}{dt}\right|_0 (f \circ c) = 0,$$

where the last step follows from the fact that $f \circ c : (-\epsilon, \epsilon) \to \mathbb{R}$ is a differentiable real-valued function that has a local maxima at 0. As i was arbitrary, we have that p is a critical point.

3.2 Submanifolds



9.4* Regular Submanifolds
Suppose that a subset S of $\mathbb{R}^2$ has the property that locally on S one of the coordinates is a C^∞ function of the other coordinate. Show that S is a regular submanifold of $\mathbb{R}^2$.

Solution: Consider any $p \in S \Rightarrow \exists$ a neighbourhood U of p s.t one of the coordinates, either x or y is a C^∞ function of the other. Assume WLOG that $x = p(y)$, which gives us that $\exists$ a chart $V \subset U, x = p(y), y)$. Consider the function $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ s.t $F(x, y) = (x - p(y), y)$. We can see that the $det\left[\dfrac{\partial F^i}{\partial x^j}(q)\right] = 1 \neq 0$ for all $q \in U \Rightarrow F$ is a diffeomorphism and $(U, x - p(y), y)$ is a chart around p s.t $U \cap S = Z(x - p(y))$. Thus, as p was arbitrary, S is a regular submanifold of codimension 1.

Problem 9.6 Euler's formula
A polynomial $F(x_0, ..., x_n) \in \mathbb{R}[x_0, ..., x_n]$ is homogeneous of degree k if it is a linear combination of monomials $x_0^{i_0} ... x_n^{i_n}$ of degree $\sum_j i_j = k$. Let $F(x_0, ..., x_n)$ be a homogeneous polynomial of degree k. Clearly, for any $t \in \mathbb{R}$,

$$F(tx_0, ..., tx_n) = t^k F(x_0, ..., x_n).$$

Show that

$$\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i} = kF.$$

Solution: Assume $F \in S[x_0, x_1, ..., x_n]$ is a homogenous polynomial of degree k. Then $F = \sum_\alpha c_\alpha x^\alpha$, where α are multi-indices depicting different monomials of degree k (for example for $\alpha = (1, 0, 3) = x_0^1 x_1^0 x_2^3$ is a monomial of degree 4 in $S[x_0, x_1, x_2]$). Then we have:

$$\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i} = \sum_{i=0}^{n} x_i \frac{\partial \sum_\alpha c_\alpha x^\alpha}{\partial x_i} = \sum_\alpha c_\alpha \sum_{i=0}^{n} x_i \frac{\partial x^\alpha}{\partial x_i} = \sum_\alpha c_\alpha \sum_{i=0}^{n} \alpha_i x^\alpha = \sum_\alpha k c_\alpha x^\alpha$$
$$= kF.$$

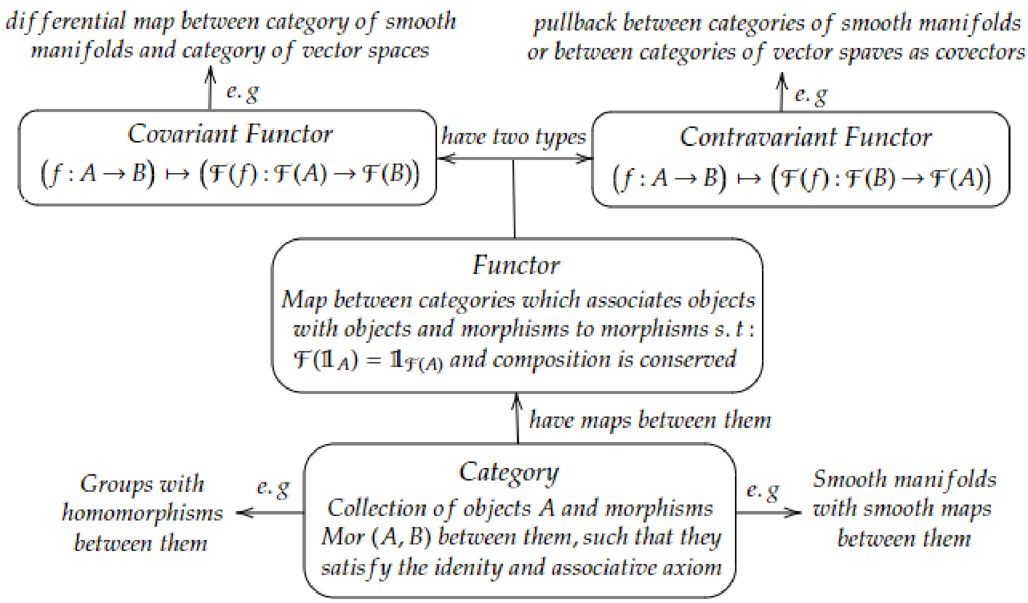
Problem 9.7 Smooth projective hypersurfaces
Show that the hypersurface $Z(F)$ defined by $F(x_0, x_1, x_2) = 0$ is smooth if $\partial F/\partial x_0, \partial F/\partial x_1$, and $\partial F/\partial x_2$ are not simultaneously zero on $Z(F)$.

Solution: Assume we have a projective hypersurface defined by $F^{-1}(0) = Z(F(x_0, x_1, x_2)) \subset \mathbb{RP}^2$. We need to show that given $\left(\dfrac{\partial F}{\partial x_i}\right)$ are not simultaneously zero on $F^{-1}(0)$, $F^{-1}(0)$ is a regular level set and hence the projective hypersurface is smooth. For this, we need to calculate the Jacobian determinant of the function $F : \mathbb{RP}^n \to \mathbb{R}$ for all $p \in F^{-1}(0)$. Given the standard atlas on $\mathbb{RP}^n$, assume WLOG that $p \in F^{-1}(0) \cap U_0$ and that the respective chart is $(U_0, \phi) = \left(U_0, x = \dfrac{x_1}{x_0}, y = \dfrac{x_2}{x_0}\right)$ (Note that $\phi^{-1}(x, y) = [1 : x : y]$). Then we have:

$$\frac{\partial}{\partial x}\bigg|_p F = \frac{\partial}{\partial r^1}\bigg|_{\phi(p)} (F \circ \phi^{-1}) = \frac{\partial F}{\partial x_1}(p).$$

Similary one can find that $\dfrac{\partial}{\partial y}\bigg|_p F = \dfrac{\partial F}{\partial x_2}(p)$. Now assume on the contrary that the Jacobian determinant is zero, i.e, $\dfrac{\partial F}{\partial x_1}(p) = \dfrac{\partial F}{\partial x_2}(p) = 0$. Then by Problem 9.6, we can see that as $x_0 \neq 0$ and $F(p) = 0 \Rightarrow \dfrac{\partial F}{\partial x_0}(p) = 0$, a contradiction. Hence the Jacobian determinant must be non-zero $\Rightarrow p$ is a regular point. One can similarly argue this for $p \in F^{-1}(0) \cap U_1$ or $F^{-1}(0) \cap U_2$, to show that $F^{-1}(0)$ is a regular level set. Hence the projective hypersurface is smooth.

3.3 Categories and Functors



Problem 10.1 Differential of the inverse map
If $F : N \to M$ is a diffeomorphism of manifolds and $p \in N$, prove that $(F^{-1})_*, F(p) = (F_*, p)^{-1}$.

Solution: Assume we have a diffeomorphism $F : N \to M$ and that $p \in N$. Then we have the differential maps $F_{*,p} : T_pN \to T_{F(p)}M$ and $(F^{-1})_{*,F(p)} : T_{F(p)}M \to T_pN$ such that:

$$((F^{-1})_{*,F(p)} \circ F_{*,p})(X_p)f = (F_{*,p}(X_p))(f \circ F^{-1}) = X_p(f \circ F^{-1} \circ F) = X_p(f),$$

for arbitrary $X_p \in T_pN$ and $f \in C^\infty(N)$. Thus $((F^{-1})_{*,F(p)} \circ F_{*,p})(X_p) = X_p \Rightarrow (F^{-1})_{*,F(p)} \circ F_{*,p} = \mathbb{1}_{T_pN}$. Similarly, it can be shown that $F_{*p} \circ (F^{-1})_{*,F(p)} = \mathbb{1}_{T_{F(p)}N}$. Thus $(F^{-1})_{*,F(p)} = (F_{*,p})^{-1}$ is an isomorphism of vector spaces T_pN and $T_{F(p)}M$.

Proposition 10.3 (Isomorphism under a functor)
Let $\mathfrak{F} : \mathfrak{C} \to \mathfrak{D}$ be a functor from a category $\mathfrak{C}$ to a category $\mathfrak{D}$. If $f : A \to B$ is an isomorphism in $\mathfrak{C}$, then $\mathfrak{F}(f) : \mathfrak{F}(A) \to \mathfrak{F}(B)$ is an isomorphism in $\mathfrak{D}$.

Proof: Let $\mathfrak{F} : \mathfrak{C} \to \mathfrak{D}$ be a covariant functor from a category $\mathfrak{C}$ to category $\mathfrak{D}$. Assume $f \in Mor(A, B)$ is an isomorphism in $\mathfrak{C} \Rightarrow \exists\, g \in Mor(B, A)$ s.t $g \circ f = \mathbb{1}_A$ and $f \circ g = \mathbb{1}_B$. Now $\mathfrak{F}(f) : \mathfrak{F}(A) \to \mathfrak{F}(B)$ and $\mathfrak{F}(g) : \mathfrak{F}(B) \to \mathfrak{F}(A)$ are morphisms in $\mathfrak{D}$. Then we have:

$$\mathfrak{F}(f) \circ \mathfrak{F}(g) = \mathfrak{F}(f \circ g) = \mathfrak{F}(\mathbb{1}_B) = \mathbb{1}_{\mathfrak{F}(B)},$$

and similarly $\mathfrak{F}(g) \circ \mathfrak{F}(f) = \mathbb{1}_{\mathfrak{F}(A)}$. Thus $\mathfrak{F}(f)$ is an isomorphism in $\mathfrak{D}$. One can almost identically prove this for the case where $\mathfrak{F}$ is a contravariant functor.

Problem 10.7 Pullback in the top dimension
Show that if $L : V \to V$ is a linear operator on a vector space V of dimension n, then the pullback $L_* : A_n(V) \to A_n(V)$ is multiplication by the determinant of L.

Solution: Given a linear transformation $L : V \to V$ where V is a vector space of dimension n, the pullback $L^* : A_n(V) \to A_n(V)$ is defined as $L^*f(v_1, ..., v_n) = f(L(v_1), ..., L(v_n))$. It is easy to see that given f is alternating, $f(v_1, ..., v_n) = 0$ if the set $\{v_1, ..., v_n\}$ is linearly dependent: assume $v_k = \sum_{i \neq k} c^i v_i$ for some k. Then we have:

$$f(v_1, ..., v_k, ..., v_n) = f(v_1, ..., \sum_{i \neq k} c^i v_i, ..., v_n) = \sum_{i \neq k} c^i f(v_1, ..., v_i, ... v_n) = 0$$

We also know that if $\{v_i\}$ is linearly dependent, then so is $\{L(v_i)\}$. Thus $f(v_1, ..., v_n) = 0 = L^*f(v_1, ..., v_n) = Det(L)L^*f(v_1, ..., v_n)$ trivially holds. Now we assume that the set $\{v_i\}$ is strictly linearly independent $\Rightarrow$ it is a basis for V. Assume the matrix for L in this basis $\Rightarrow$ $L(v_i)$ are the columns of this matrix. We know that the determinant function is also a $n-$covector, and that:

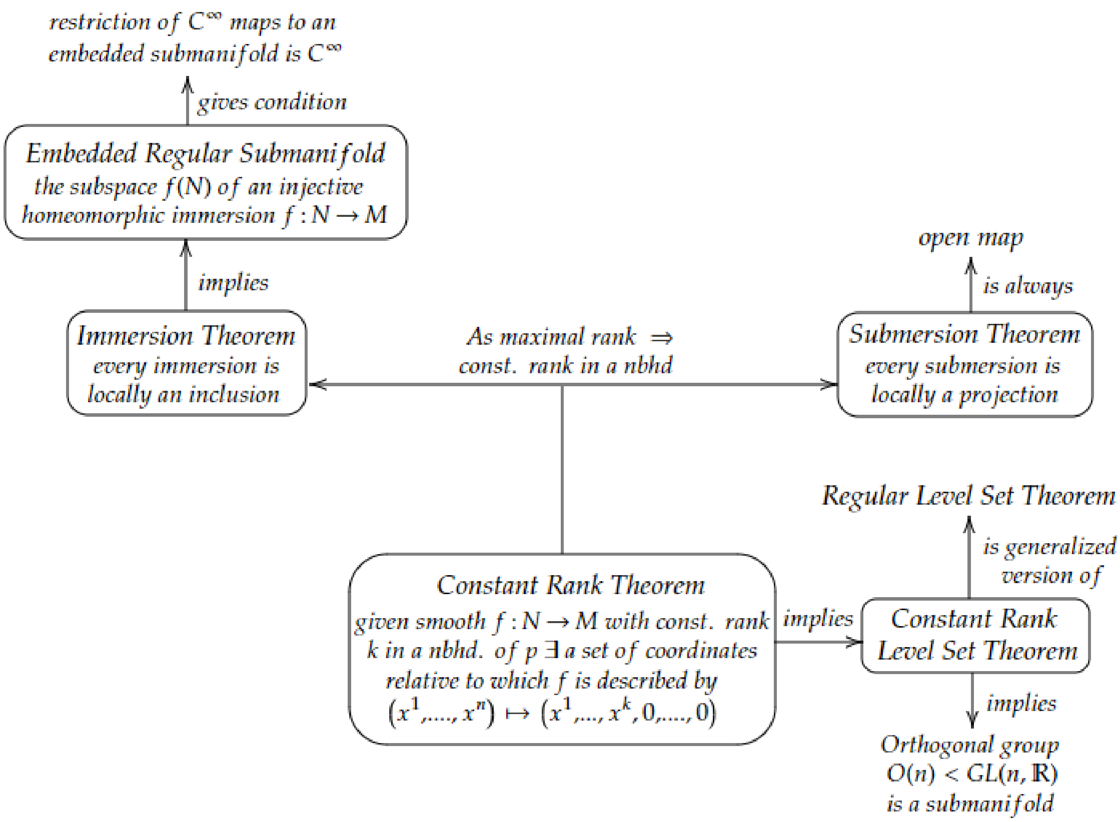
$$d = Det(L) = Det(L(v_1), ..., L(v_n)) = d \times Det(v_1, ..., v_n),$$

where $Det(v_1, ..., v_n) = Det(\mathbb{I}) = 1$. Note that the factoring out of d in the last step above follows from the well known *row operations*. These operations are based only on the alternating property of the determinant function, and hence can just as easily be generalized to any arbitrary $k-$covector, hence giving:

$$L^*f(v_1, ..., v_n) = f(L(v_1), ..., L(v_n)) = d \times f(v_1, ..., v_n),$$

where $d = Det(L)$.

3.4 The Rank of a Smooth Map



Problem 11.3* Critical points of a smooth map on a compact manifold
Show that a smooth map f from a compact manifold N to $\mathbb{R}^m$ has a critical point.

Solution: Consider a map $f : N \to \mathbb{R}^m$ where N is a compact manifold. Assume f has no critical points, that is, it is a submersion$\Rightarrow$ f is an open map. Then $f(N)$ is an open subset of $\mathbb{R}^m$. Also, as N is compact and f in continuous, $f(N)$ is compact in $\mathbb{R}^m \Rightarrow f(N)$ is closed in $\mathbb{R}^m$. Also, as $\mathbb{R}^m$ is not compact itself, $f(N) \neq \mathbb{R}^m$. Thus, $\mathbb{R}^m = f(N) \cup f(N)^c$ forms a separation of $\mathbb{R}^m \Rightarrow$ this is a contradiction as $\mathbb{R}^m$ is connected. Thus f must have a critical point.

Problem 11.4 Differential of an inclusion map
On the upper hemisphere of the unit sphere S^2, we have the coordinate map $\phi = (u, v)$, where $u(a, b, c) = a$ and $v(a, b, c) = b$. So the derivations $\partial/\partial u|_p, \partial/\partial v|_p$ are tangent vectors of $S^2 atanypointp = (a, b, c)$ on the upper hemisphere. Let $i : S^2$ß$\mathbb{R}^3$ be the inclusion and x, y, z the standard coordinates on $\mathbb{R}^3$. The differential $i_* : T_pS^2 \to T_p\mathbb{R}^3$

maps $\partial/\partial u|_p, \partial/\partial v|_p$ into $T_p\mathbb{R}^3$. Thus,

$$i_*\left(\frac{\partial}{\partial u}\bigg|_p\right) = \alpha^1\left(\frac{\partial}{\partial x}\bigg|_p\right) + \beta^1\left(\frac{\partial}{\partial y}\bigg|_p\right) + \gamma^1\left(\frac{\partial}{\partial z}\bigg|_p\right)$$
$$i_*\left(\frac{\partial}{\partial u}\bigg|_p\right) = \alpha^2\left(\frac{\partial}{\partial x}\bigg|_p\right) + \beta^2\left(\frac{\partial}{\partial y}\bigg|_p\right) + \gamma^2\left(\frac{\partial}{\partial z}\bigg|_p\right)$$

for some constants $\alpha^i, \beta^i, \gamma^i$. Find $(\alpha^i, \beta^i, \gamma^i)$ for $i = 1, 2$.

Solution: Assume we have the coordinate map $\phi(a,b,c) = (u,v)$, with $u(a,b,c) = a$ and $v(a,b,c) = b$ on the upper hemisphere of $S^2 \subset \mathbb{R}^3$. Then we have for $i_* : T_pS^2 \to T_p\mathbb{R}^3$ at $p = (a,b,c)$:

$$\begin{aligned}
i_*\left(\frac{\partial}{\partial u}\bigg|_p\right) &= \alpha^1\left(\frac{\partial}{\partial x}\bigg|_p\right) + \beta^1\left(\frac{\partial}{\partial y}\bigg|_p\right) + \gamma^1\left(\frac{\partial}{\partial z}\bigg|_p\right)\\
\Rightarrow i_*\left(\frac{\partial}{\partial u}\bigg|_p\right)(x) &= \alpha^1\left(\frac{\partial}{\partial x}\bigg|_p\right)(x) + \beta^1\left(\frac{\partial}{\partial y}\bigg|_p\right)(x) + \gamma^1\left(\frac{\partial}{\partial z}\bigg|_p\right)(x)\\
\Rightarrow \left(\frac{\partial}{\partial u}\bigg|_p\right)(x\circ i) &= \left(\frac{\partial}{\partial r^1}\bigg|_{(a,b)}\right)(x\circ i\circ\phi^{-1}) = \left(\frac{\partial}{\partial r^1}\bigg|_{(a,b)}\right)(r^1) = 1 = \alpha^1.
\end{aligned}$$

Similarly:

$$\begin{aligned}
i_*\left(\frac{\partial}{\partial u}\bigg|_p\right)(y) &= \alpha^1\left(\frac{\partial}{\partial x}\bigg|_p\right)(y) + \beta^1\left(\frac{\partial}{\partial y}\bigg|_p\right)(y) + \gamma^1\left(\frac{\partial}{\partial z}\bigg|_p\right)(y)\\
\Rightarrow \left(\frac{\partial}{\partial u}\bigg|_p\right)(y\circ i) &= \left(\frac{\partial}{\partial r^1}\bigg|_{(a,b)}\right)(y\circ i\circ\phi^{-1}) = \left(\frac{\partial}{\partial r^1}\bigg|_{(a,b)}\right)(r^2) = 0 = \beta^1,
\end{aligned}$$

and:

$$\begin{aligned}
i_*\left(\frac{\partial}{\partial u}\bigg|_p\right)(z) &= \alpha^1\left(\frac{\partial}{\partial x}\bigg|_p\right)(z) + \beta^1\left(\frac{\partial}{\partial y}\bigg|_p\right)(z) + \gamma^1\left(\frac{\partial}{\partial z}\bigg|_p\right)(z)\\
\Rightarrow \left(\frac{\partial}{\partial u}\bigg|_p\right)(z\circ i) &= \left(\frac{\partial}{\partial r^1}\bigg|_{(a,b)}\right)(z\circ i\circ\phi^{-1}) = \left(\frac{\partial}{\partial r^1}\bigg|_{(a,b)}\right)(\sqrt{1-(r^1)^2-(r^2)^2}) = -\frac{a}{c} = \gamma^1.
\end{aligned}$$

Similarly, one can see that:

$$i_*\left(\frac{\partial}{\partial v}\bigg|_p\right) = \alpha^2\left(\frac{\partial}{\partial x}\bigg|_p\right) + \beta^2\left(\frac{\partial}{\partial y}\bigg|_p\right) + \gamma^2\left(\frac{\partial}{\partial z}\bigg|_p\right),$$

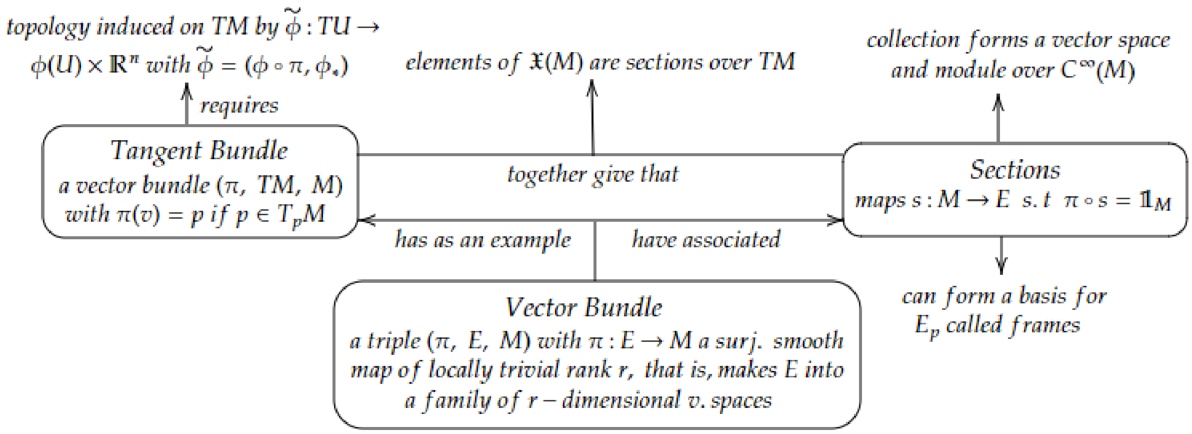
with $\alpha^2 = 0$, $\beta^2 = 1$, $\gamma^2 = -\dfrac{b}{c}$.

Problem 11.5 One-to-one immersion of a compact manifold

Prove that if N is a compact manifold, then a one-to-one immersion $f : N \to M$ is an embedding.

Solution: Assume $f : N \to M$ is a one-to-one immersion (hence continuous) with N as a compact manifold. Then the induced map $\tilde{f} : N \to f(N)$ is continuous and bijective. Now take any closed set $C \subset N \Rightarrow C$ is compact $\Rightarrow \tilde{f}(C)$ is compact in $f(N)$. As $f(N)$ is Hausdorff under the subspace topology, $\tilde{f}(C)$ is closed in $f(N)$. Thus $\tilde{f}^{-1}$ is continuous, and $\tilde{f}$ is a homeomorphism of N with its image $f(N)$. Thus f is an embedding.

3.5 The Tangent Bundle



Problem 12.2 Transition functions for the total space of the tangent bundle
Let $(U, \phi) = (U, x_1, ..., x_n)$ and $(V, \psi) = (V, y_1, ..., y_n)$ be overlapping coordinate charts on a manifold M. They induce coordinate charts $(TU, \tilde{\phi})$ and $(TV, \tilde{\psi})$ on the total space TM of the tangent bundle, with transition function $\tilde{\psi} \circ \tilde{\phi}^{-1}$:

$$(x_1, ..., x_n, a_1, ..., a_n) \mapsto (y_1, ..., y_n, b_1, ..., b_n).$$

Compute the Jacobian matrix of the transition function$\tilde{\psi} \circ \tilde{\phi}^{-1}$ at $\phi(p)$ and its determinant.

Solution: Consider two overlapping coordinate charts on a manifold M given by $(U, x^1, ..., x^n) = (U, \phi)$ and $(V, x^1, ..., x^n) = (U, \psi)$. They induce coordinate charts $(TU, \tilde{\phi})$ and $(TV, \tilde{\psi})$ on TM. This gives us the transition function $F = \tilde{\psi} \circ \tilde{\phi}^{-1}$ with $(x^1(p), ..., x^n(p), a^1, ..., a^n) \mapsto (y^1(p), ..., y^n(p), b^1, ..., b^n)$ where $\sum_i a^i \frac{\partial}{\partial x^i} = \sum_i b^i \frac{\partial}{\partial y^i}$. Not that if we apply y^j to both sides of this relation, we get:

$$\sum_i a^i \frac{\partial y^j}{\partial x^i} = b^j \tag{1}$$

Now we can get to calculating the Jacobian matrix of this transition functions. Firstly, note that by definition, $\frac{\partial F^i}{\partial r^j} = 0$ for $1 \leq i \leq n$ and $n+1 \leq j \leq 2n$, or for $n+1 \leq i \leq 2n$ and $1 \leq j \leq n$, i.e, b^i only depend on a^i and y^i only depend on x^i. We can see that for $n + 1 \leq i, j \leq 2n$, (1) gives us that:

$$\frac{\partial F^i}{\partial r^j} = \frac{\partial b^i}{\partial a^j} = \frac{\partial y^i}{\partial x^j}.$$

Furthermore, for $1 \leq i, j \leq n$:

$$\frac{\partial F^i}{\partial r^j} = \frac{\partial (y^i \circ \phi^{-1})}{\partial r^j} = \frac{\partial y^i}{\partial x^j}.$$

Thus the Jacobian matrix is the following block matrix: $\begin{pmatrix} \left[\dfrac{\partial y^i}{\partial x^j}\right] & 0 \\ 0 & \left[\dfrac{\partial y^i}{\partial x^j}\right] \end{pmatrix}.$

It is not hard to see that the Jacobian determinant is then $(\det[\partial y^i/\partial x^j])^2$.

Proposition 12.9(ii) Smoothness of scalar multiplication
Let s and t be C^∞ sections of a C^∞ vector bundle $\pi : E \to M$ and let f be a C^∞ real-valued function on M. Then the product $fs : M \to E$ defined by:

$$(fs)(p) = f(p)s(p) \in E_p, p \in M,$$

a C^∞ section of E

Solution: Assume s is a C^∞ section of a C^∞ vector bundle $\pi : E \to M$, and f is a C^∞ real-valued function on M. We want to show that $(fs) : M \to E$ defined as $(fs)(p) = f(p)s(p)$ is a C^∞ section of E. Firstly, it is clear that it is indeed a section of E, as E_p is closed under scalar multiplication for all p. Now consider any point $p \in M$, and a corresponding trivializing open set $\pi(p) \in U$ with trivialization $\phi \Rightarrow \exists (V, \psi)$ with $p \in V \subseteq \pi^{-1}(U)$. As s is $C^\infty \Rightarrow ((\psi \times \mathbb{1}) \circ \phi \circ s \circ \psi^{-1})$ is C^∞. We can see that:

$$((\psi \times \mathbb{1}) \circ \phi \circ s \circ \psi^{-1})(\psi(p)) = (p, c^1, ..., c^n),$$

for some C^∞ functions c^i. Now if we instead consider $((\psi \times \mathbb{1}) \circ \phi \circ (fs) \circ \psi^{-1})$, as ϕ is $\mathbb{R}$-linear:

$$((\psi \times \mathbb{1}) \circ \phi \circ s \circ \psi^{-1})(\psi(p)) = (p, f(p)c^1, ..., f(p)c^n),$$

which is also C^∞ as the product (fc^i) is C^∞ for all i. Thus (fs) is a C^∞ section on E.

Problem 12.4 Coefficients relative to a smooth frame
Let $\pi : E \to M$ be a C^∞ vector bundle and $s_1, ..., s_r$ a C^∞ frame for E over an open set U in M. Then every $e \in \pi^{-1}(U)$ can be written uniquely as a linear combination

$$e = \sum_{j=1}^{r} c^j(e)s_j(p), , , , , , p \in \pi(e)$$

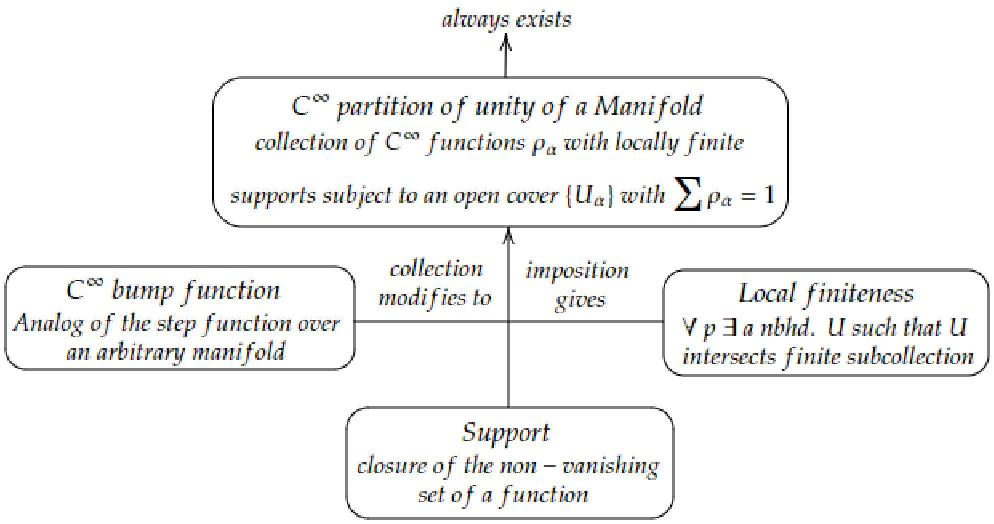
Prove that $c_j : \pi^{-1}U \to \mathbb{R}$ is C^∞ for $j = 1, ..., r$.

Solution: Let $\pi : E \to M$ be a C^∞ vector bundle and $s^1, ..., s^r$ be a C^∞ frame for E over an open set U and $t_1, ..., t_r$ be the frame of trivialization. Then for any $e \in \pi^{-1}(U)$, $e = \sum_{j=1}^r b^j(e)t_j(p) = \sum_{j=1}^r c^j(e)s_j(p)$. We show that $t_j : \pi^{-1}(U) \to \mathbb{R}$ is C^∞, and hence so are s_j. Consider the following action of the trivialization ϕ of E over U:

$$\phi(e) = \phi(\sum_{j=1}^r b^j(e)t_j(p)) \overset{\text{lin.}}{=} \sum_{j=1}^r b^j(e)\phi(t_j(p)) = \sum_{j=1}^r b^j(e)(\pi(e), e_j) = (\pi(e), b^1(e), ..., b^r(e)),$$

where b^j are components of ϕ. As ϕ is C^∞, so are the b^j. Now we can take $s_i(p) = \sum_j d_i^j t_j(p) \Rightarrow \sum_j b^j(e)t_j(p) = \sum_{i,j} c^i(e)d_i^j t_j(p) \Rightarrow b^j(e) = \sum_i c^i(e)d_i^j$. This equation is equivalent to a change of basis, and so the matrix $[d_i^j]$ is invertible. As the matrix $[d_i^j]$ consists of C^∞ functions, its inverse multiplied by the column vector $[b^j]$ will give that c^i are also C^∞.

3.6 Bump Functions and Partitions of Unity



The idea of a C^∞ partition is that it gives an explicit connection between the local and global properties of a smooth manifold. One can understand this somewhat intuitively considering that ρ_α are 'local' functions (vanish outside U_α) with the global property of their sum being consistenly equal to 1.

Lemma 13.5* Support of a finite sum

Let $f : M \to \mathbb{R}$ be a C^∞ function on a manifold M. If N is another manifold and $\pi : M \times N \to M$ is the projection onto the first factor, prove that:

$$\mathrm{supp}(\pi^* f) = (\mathrm{supp} f) \times N.$$

Proof: Define $F = \sum_i \rho_i$. Consider any $p \in F^{-1}(\mathbb{R}^\times)$. This implies that $\rho_j(p) \neq 0$ for some $j \Rightarrow p \in \rho_j^{-1}(\mathbb{R}^\times)$. Thus $F^{-1}(\mathbb{R}^\times) \subseteq \bigcup_i \rho_i^{-1}(\mathbb{R}^\times) \Rightarrow \mathbf{cl}(F^{-1}(\mathbb{R}^\times)) \subseteq \mathbf{cl}(\bigcup_i \rho_i^{-1}(\mathbb{R}^\times)) = \bigcup_i \mathbf{cl}(\rho_i^{-1}(\mathbb{R}^\times)) \Rightarrow \mathrm{supp}(\sum_i \rho_i) \subseteq \bigcup_i \mathrm{supp}(\rho_i)$.

Problem 13.2* Locally finite family and compact set

Let $\{A_\alpha\}$ be a locally finite family of subsets of a topological space S. Show that every compact set K in S has a neighborhood W that intersects only finitely many of the A_α.

Solution: For every $p \in K$, take a neighbourhood U_p that intersects finitely many of $\{A_\alpha\}$. Then $\{U_p\}_{p \in K}$ forms an open cover of K. As K is compact$\Rightarrow$ there exists a finite subcover $\{U_i\}_{i=1}^n$. Then $\bigcup_i U_i$ is neighbourhood of K that intersects finitely many $\{A_\alpha\}$.

Problem 13.3 Smooth Urysohn Lemma

(a)Let A and B be two disjoint closed sets in a manifold M. Find a C^∞ function f on M such that f is identically 1 on A and identically 0 on B.

(b)Let A be a closed subset and U an open subset of a manifold M. Show that there is a C^∞ function f on M such that f is identically 1 on A and supp $f \subseteq U$.

Solution(a): Given two disjoint closed sets A and B of a manifold M, consider the open cover $\{M - A, M - B\}$. By Theorem 13.7, there exists a C^∞ partition of unity $\{\rho_{M-A}, \rho_{M-B}\}$ subordinate to the open cover. Thus as $\text{supp}(\rho_{M-B}) \subseteq (M - B) \Rightarrow \rho_{M-B}(B) = 0$. Similarly $\rho_{M-A}(A) = 0$. Moreover, as $\rho_{M-A} + \rho_{M-B} = 1 \Rightarrow \rho M - B(A) = \{1\}$. Thus ρ_{M-B} is a C^∞ function that is identically 1 on A and identically 0 on B.

Solution(b): Let A be a closed subset of M and U be an open set containing A. Then $\{U, M - A\}$ is an open cover of $M \Rightarrow$ by Theorem 13.7, there exists a C^∞ partition $\{\rho_U, \rho_{M-A}\}$ subordinate to the open cover. As $\rho_{M-A}(A) = \{0\}$ and $\rho_U + \rho_{M-A} = 1 \Rightarrow$, $\rho_U(A) = \{1\}$. Thus ρ_U is a C^∞ function that is identically 1 on A with $\text{supp}(\rho_U) \subseteq U$ by definition.

Problem 13.4 Support of the pullback of a function

Let $F : N \to M$ be a C^∞ map of manifolds and $h : M \to \mathbb{R}$ a C^∞ real-valued function. Prove that supp $F^*h \subseteq F^{-1}(\text{supp } h)$.

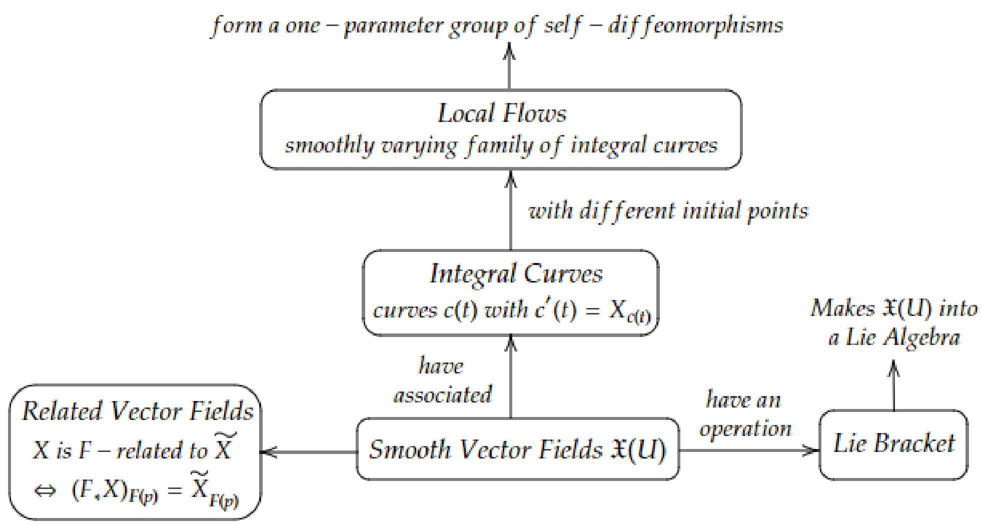
Solution: Let $F : N \to M$ and $h : M \to \mathbb{R}$ be C^∞. We can see that $(F^*h)^{-1}(\mathbb{R}^\times) = F^{-1}(h^{-1}(\mathbb{R}^\times)) \subseteq F^{-1}(\mathbf{cl}(h^{-1}(\mathbb{R}^\times))) = F^{-1}(\text{supp}(h)) \Rightarrow \mathbf{cl}(F^{-1}(h^{-1}(\mathbb{R}^\times))) \subseteq \mathbf{cl}(F^{-1}(\text{supp}(h))) = F^{-1}(\text{supp}(h))$ as $F^{-1}(\text{supp}(h))$ is the continuous preimage of a closed set, and hence closed.

Problem 13.6 Pullback of a partition of unity

Suppose $\{\rho_\alpha\}$ is a partition of unity on a manifold M subordinate to an open cover $\{U_\alpha\}$ of M and $F : N \to M$ is a C^∞ map. Prove that the collection of supports $\{\text{supp } F^*\rho_\alpha\}$ is locally finite and the collection of functions $\{F^*\rho_\alpha\}$ is a partition of unity on N subordinate to the open cover $\{F^{-1}(U_\alpha)\}$ of N.

Solution: Take $\{\rho_\alpha\}$ to be a partition of unity on a manifold M subordinate to an open cover $\{U_\alpha\}$ and $F : N \to M$ be C^∞. Consider the collection $\{\text{supp}(F^*\rho_\alpha)\}$. We know from Problem 13.4 that $\text{supp}(F^*\rho_\alpha) \subseteq F^{-1}(\text{supp}(\rho_\alpha))$. Thus if we are able to show that the collection $\{F^{-1}(\text{supp}(\rho_\alpha))\}$ is locally finite, so is $\{\text{supp}(F^*\rho_\alpha)\}$. Consider any $p \in N \Rightarrow F(p) \in M$. As $\{\text{supp}(\rho_\alpha)\}$ is locally finite, there exists a neighbourhood U of $F(p)$ such that U intersects only finitely many supports $\Rightarrow F^{-1}(U)$ is neighbourhood of p that intersects finitely many $F^{-1}(\text{supp}(\rho_\alpha)) \Rightarrow \{F^{-1}(\text{supp}(\rho_\alpha))\}$ is locally finite and so is $\{\text{supp}(F^*\rho_\alpha)\}$. It is trivial to see that $F^*\rho_\alpha$ are also non-negative functions, and as $\sum_\alpha \rho_\alpha = 1$, this implies that $\sum_\alpha (F^*\rho_\alpha) = \sum_\alpha (\rho_\alpha \circ F) = 1$. Finally, we can see that $\text{supp}(F^*\rho_\alpha) \subseteq F^{-1}(\text{supp}(\rho_\alpha)) \subseteq F^{-1}(U_\alpha)$ from the initial assumption about $\{\rho_\alpha\}$. Thus, $\{F^*\rho_\alpha\}$ forms a partition of unity on N subordinate to the open cover $\{F^{-1}(U_\alpha)\}$.

3.7 Vector Fields



Problem 14.2 Vector field on an odd sphere
Let $x_1, y_1, ..., x_n, y_n$ be the standard coordinates on $\mathbb{R}^{2n}$. The unit sphere S^{2n-1} in $\mathbb{R}^{2n}$ is defined by the equation $\sum_i^n (x^i)^2 + (y^i)^2 = 1$. Show that:

$$X = \sum_i^n -y^i \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial y^i}$$

is a nowhere-vanishing smooth vector field on S^{2n-1}.

Solution: The gradient vector on S^{2n-1} is defined as $v = (2x^1, ..., 2x^n, 2y^1, ..., 2y^n)$ at a point $p = (x^1, ..., x^n, y^1, ..., y^n)$. Representing the vector field X at point p as the vector $X_p = (-y^1, ..., -y^n, x^1, ..., x^n)$, we can see that $\sum_i v^i X_p^i = \sum_i (-y^i x^i + y^i x^i) = 0$. Thus X_p is tangent to S^{2n-1}, and hence, as p was arbitrary, X defines vector field on S^{2n-1}. We can also see that it is non-vanishing, as for it to vanish $y^i = x^i = 0$ for all $i \Rightarrow \sum_i (x^i)^2 + (y^i)^2 = 0 \neq 1$, which is a contradiction as the point must lie on S^{2n-1}. To check the smoothness of the vector field, consider the following atlas on S^{2n-1}: $\{(U_{i1} = \{p \in S^{2n-1} : x^i > 0\}, \pi_{x^i}), (U_{i2} = \{p \in S^{2n-1} : x^i < 0\}, \pi_{x^i}), (V_{i1} = \{p \in S^{2n-1} : y^i > 0\}, \pi_{y^i}), (V_{i2} = \{p \in S^{2n-1} : y^i < 0\}, \pi_{y^i})\}_{i=1}^n$. Consider now any point $p \in S^{2n-1}$. WLOG assume $p \in U_{11}$ with chart $\pi_{x_1} = (x^2, ..., x^n, y^1, ... y^n)$. The vector field in terms of these coordinates is then $X = x^1 \frac{\partial}{\partial y^1} + \sum_{i=2}^n (-y^i \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial y^i})$. Here we must note that as $x^1 \circ \pi_{x^1}^{-1}(x^2, ..., x^n, y^1, ..., y^n) = \sqrt{(x^2)^2 + ... + (x^n)^2 + (y^1)^2 + ... + (y^n)^2)}$ is C^∞, x^1 is C^∞ at p. The other coefficients are trivially smooth, giving that the X is a smooth vector field on S^{2n-1}.

Problem 14.4 Integral curves in the plane

Find the integral curves of the vector field on $\mathbb{R}^2$ *given by:*

$$X_{(x,y)} = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

Solution: Consider the vector field $X_{(x,y)} = x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y}$ on $\mathbb{R}^2$. Then its integral curve $c(t) = (x(t), y(t))$ is given by the following differential equations: $\dot{x}(t) = x$ and $\dot{y}(t) = -y$. These are standard ODEs with solutions $x(t) = e^t + c$ and $y(t) = e^{-t} + d$ for some constants c, d. Taking our initial point to be $p = (x_0, y_0)$, this gives $c = x_0 - 1$ and $d = y_0 - 1$.

Problem 14.13 Vector field under a diffeomorphism
Let $F : N \to M$ *be a* C^∞ *diffeomorphism of manifolds. Prove that if* g *is a* C^∞ *function and* X *a* C^∞ *vector field on* N*, then:*

$$F_*(gX) = (g \circ F^{-1})F_*X.$$

Solution: Let $F : N \to M$ be a diffeomorphism, and g be a C^∞ function and X be a C^∞ vector field on N. Consider now two vector fields: $F_*(gX)$ and $(g \circ F^{-1})F_*X$. We prove these are equal by acting them on an arbitrary C^∞ function f on M at $F(p)$:

$$(F_*(gX))_{F(p)}f = (gX)_p(f \circ F) = g(p)X_p(f \circ F).$$
$$((g \circ F^{-1})F_*X)_{F(p)}f = (g \circ F^{-1})(F(p))(F_*X)_{F(p)}f = g(p)X_p(f \circ F).$$

Problem 14.14 Lie bracket under a diffeomorphism
Let $F : N \to M$ *be a* C^∞ *diffeomorphism of manifolds. Prove that if* X *and* Y *are* C^∞ *vector fields on* N*, then:*

$$F_*[X, Y] = [F_*X, F_*Y].$$

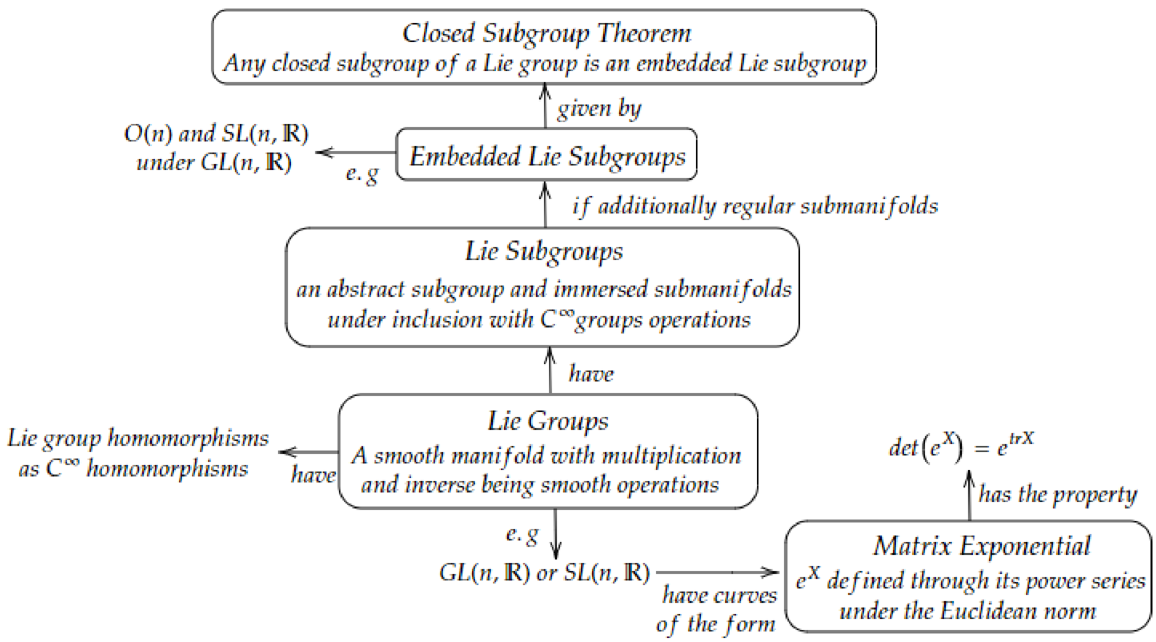
Solution: Let $F : N \to M$ be a C^∞ diffeomorphism, with X and Y as C^∞ vector fields on N. Consider the followng related vectors fields :$F_*[X, Y]$ and $[F_*X, F_*Y]$. Let p be an arbitrary point in N contained in a chart $(U, x^1, ...x^n)$, and $X = \sum_i a_i(p)\dfrac{\partial}{\partial x^i}$, $Y = \sum_i b_i(p)\dfrac{\partial}{\partial x^i}$. We prove the following are equal by acting them on an arbitrary C^∞

function f on M at $F(p)$:

$$\begin{aligned}
(F_*[X,Y])_{F(p)}f &= [X,Y]_p(f\circ F) = (X_pY - Y_pX)(f\circ F)\\
[F_*X, F_*Y]_{F(p)}f &= (F_*X)_{F(p)}(F_*Y)f - (F_*Y)_{F(p)}(F_*X)f\\
&= X_p(((F_*Y)f)\circ F) - Y_p(((F_*X)f)\circ F)\\
&= X_p((F_*Y)_F f) - Y_p((F_*X)_F f) = X_p(Y(f\circ F)) - Y_p(X(f\circ F))\\
&= (X_pY - Y_pX)(f\circ F).
\end{aligned}$$

4 Lie Groups and Lie Algebras

4.1 Lie Groups



Problem 15.3 Identity component of a Lie Group
The identity component G_0 of a Lie group G is the connected component of the identity element e in G. Let μ and i be the multiplication map and the inverse map of G.
(a) For any $x \in G_0$, show that $\mu(\{x\} \times G_0 \subseteq G$.
(b) Show that $i(G_0) \subseteq G_0$.
(c) Show that G_0 is an open subset of G.
(d) Prove that G_0 is itself a Lie group.

Solution:
(i) Clearly, for the identity connected component G_0 of the Lie Group G, $\{x\} \times G_0$ is connected. Thus, as μ is smooth, $\mu(\{x\} \times G_0)$ is connected and contains $x \in G_0$. Thus $\mu(\{x\} \times G_0) \subseteq G_0$.
(ii) Through a similar argument to above $i(G_0)$ is connected and contains $e \in G_0$. Thus $i(G_0) \subseteq G_0$.
(iii) Consider any point $g \in G_0$ contained in a chart $(U, \phi) \Rightarrow \phi(g) \in \phi(U) \subseteq \mathbb{R}^n$. As $\mathbb{R}^n$ is locally connected, there exists a connected neighbourhood $V \subseteq \phi(U)$ of $\phi(p) \Rightarrow \phi^{-1}(V)$ is a connected neighbourhood of $p \Rightarrow \phi^{-1}(V) \subseteq G_0$. Thus G_0 is open.
(iv) As G_0 is an open subset of a smooth manifold, it is also a smooth manifold. The restrictions of the multiplication map and the inverse map are also C^∞ onto G_0. Furthermore, by (i) and (ii), G_0 is closed under multiplication and inverses. Thus G_0 is a Lie Group.

Problem 15.4* Open subgroup of a connected Lie Group
Prove that an open subgroup H of a connected Lie group G is equal to G.

Solution: Consider an open subgroup H of a connected Lie group G. Assume $H \neq G$. Consider the smooth map l_g of left-mutliplication by $g \in G$. Then as $H = l_g^{-1}(gH)$, gH is open for all g. It is well-known that $\{gH\}_{g \in G}$ forms a partition of G, and hence a separation of G. This is a contradiction as G is connected $\Rightarrow H = G$.

Problem 15.7* Differential of the determinant map at A
Show that the differential of the determinant map det: $GL(n, \mathbb{R}) \to \mathbb{R}$ at $A \in GL(n, \mathbb{R})$ is given by:

$$\det_{*,A}(AX) = (\det A)\mathrm{tr}X \text{ for } X \in \mathbb{R}^{n \times n}.$$

Solution: Consider the following curve in $GL(n, \mathbb{R})$ with $c(t) = Ae^{tX} \Rightarrow c(0) = A$ and $c'(0) = AX$. Thus:

$$det_{*,A}(AX) = \frac{d}{dt}\bigg|_{t=0} det(c(t)) = \frac{d}{dt}\bigg|_{t=0} det(A)det(e^{tX}) = \frac{d}{dt}\bigg|_{t=0} det(A)e^{t\,trX} = det(A)trX.$$

Problem 15.10 Orthogonal Group
Show that the orthogonal group $O(n)$ is compact.

Solution: Consider the map $f : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$ given by $f(A) = A^T A$. Considering the general linear group as a subset of euclidean space, f is a polynomial map $\Rightarrow$ it is smooth. Thus $f^{-1}(I) = O(n)$ is a closed set. One can also note that $[A^T A]_i j = \sum_k a_{ki} a_{kj}$. Thus $[A^T A]_i i = \sum_k (a_{ki})^2 \Rightarrow tr(A^T A) = \sum_i \sum_k (a_{ki})^2 = ||A||^2 \Rightarrow tr(I) = n = ||A||^2$. Thus $O(n)$ is a bounded subset of euclidean space $\Rightarrow$ it is compact.

Problem 15.12 Unitary group
Show that $U(n)$ is a regular submanifold of $GL(n, \mathbb{C})$ and that $\dim U(n) = n^2$

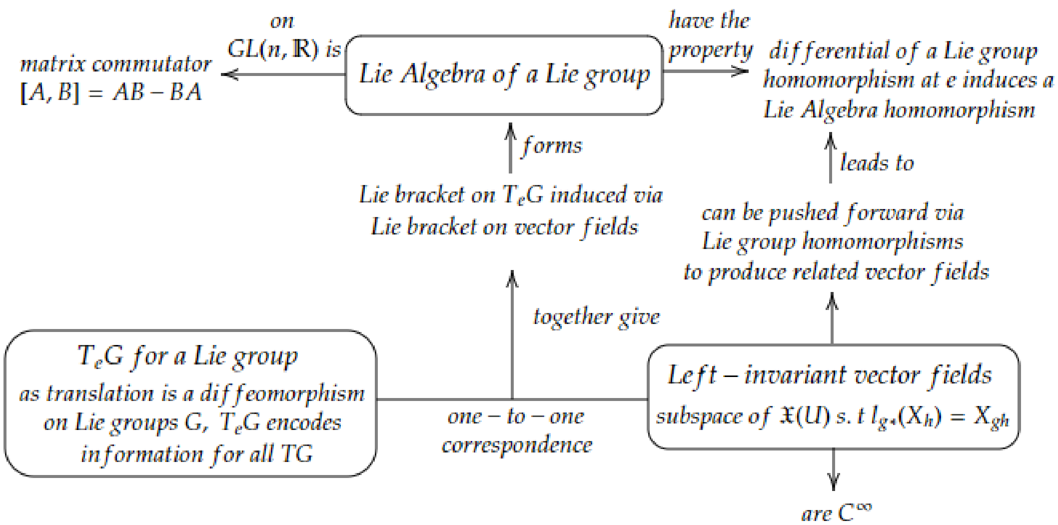
Solution: Define $f : GL(n, \mathbb{C}) \to H$ given by $f(A) = \bar{A}^T A$ where H is the set of Hermitian matrices. We can see that H, as a vector space over $\mathbb{R}$, is of dimension $(2n(n-1)/2) + n = n^2$ as the diagonal must be real. In exact analogy to Example 15.6, $f^{-1}(I)$ is a regular level set, which implies that $U(n)$ is a regular submanifold of $GL(n, \mathbb{C})$ of $\dim(U(n)) = 2n^2 - n^2 + n - n = n^2$.

Problem 15.13 Special unitary group SU(2)
Show that $SU(2)$ is diffeomorphic to the three-dimensional sphere.

Solution: Consider the following arbitray matrix $M \in SU(2)$ given by $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \Rightarrow$

$M^{-1} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ as $det(M) = 1$. As we have $\bar{M}^T = M^{-1} \Rightarrow \bar{a} = d$ and $c = -\bar{b}$. Thus $M = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$ with $a\bar{a} + b\bar{b} = 1$. Now define a map $f : GL(n, \mathbb{C}) \to S^3$ given by $f(M) = (Re(a), Im(a), Re(b), Im(b)) = \dfrac{1}{2}(a+\bar{a}, a-\bar{a}, b+\bar{b}, b-\bar{b}) = \dfrac{1}{2}(a+d, a-d, b-c, b+c)$. Clearly, this is a well-defined map as $Re(a)^2 + Im(a)^2 + Re(b)^2 + Im(b)^2 = |a|^2 + |b|^2 = a\bar{a} + b\bar{b} = 1$. This map is also C^∞ as it consists of polynomials, and has a C^∞ inverse: $g : S^3 \to GL(n, \mathbb{C})$ given by $g(a, b, c, d) = \begin{pmatrix} (a+b)/2 & (d-c)/2 \\ (c+d)/2 & (a-b)/2 \end{pmatrix}$. Thus $GL(n, \mathbb{C})$ is diffeomorphic to S^3.

4.2 Lie Algebras



Problem 16.2 Lie Algebra of a unitary group
Show that the tangent space at the identity I of the unitary group $U(n)$ is the vector space of $n \times n$ skew-Hermitian matrices.

Solution: Recall that a unitary matrix U satisfies the property $\overline{U}^T U = I$. Consider any tangent vector $X \in T_I U(n)$, and define a curve $c(t)$ in $U(n)$ with $c(0) = I$ and $c'(0) = X$. Then we have for all t:

$$\overline{c(t)}^T c(t) = I \Rightarrow \frac{d}{dt}(\overline{c(t)}^T c(t)) = 0 \Rightarrow (\overline{c(t)}^T)' c(t) + \overline{c(t)}^T c(t)' = 0.$$

Plugging in $t = 0$ and noting that $(\overline{c(t)}^T)' = (\overline{c(t)'})^T$, we get $\overline{X}^T + X = 0 \Rightarrow X$ is a skew-Hermitian matrix. Also, as argued before, $dim(U(n)) = n^2$ is also the dimension of the vector space of skew-Hermitian matrices, H. Thus $U(n) = H$.

Problem 16.6 Left-invariant vector fields on a circle
Find the left-invariant vector fields on S^1.

Solution: We will take $S^1 = \{e^{it} = 0 \leq t < 2\pi\}$ as a group under standard multiplication, or addition modulo 2π on t. Consider the following atlas for $S^1 = \{(U_{t \neq 0}, t), (U_{t \neq \pi}, t)\}$. Now consider the action of $l_{g*}\left(\frac{\partial}{\partial t}\bigg|_0\right)$ on the function t (taking l_g as addition on t by

g):

$$\begin{aligned} l_{g*}\left(\frac{\partial}{\partial t}\bigg|_0\right) &= c_g \frac{\partial}{\partial t}\bigg|_g \\ \Rightarrow l_{g*}\left(\frac{\partial}{\partial t}\bigg|_0\right) t &= c_g \frac{\partial}{\partial t}\bigg|_g t \\ \Rightarrow \frac{\partial}{\partial t}\bigg|_0 (t \circ l_g) &= c_g \\ \Rightarrow c_g &= 1. \end{aligned}$$

As this holds for any g, any left-invariant vector field must be of the form $c\dfrac{\partial}{\partial t}$ where c is a constant.

Problem 16.8 Parallelizable manifolds
If M is a manifold of dimension n, show that parallelizability is equivalent to the existence of a smooth frame $X_1, ..., X_n$ on M.

Solution:
($\Rightarrow$) Assume a manifold M is trivial $\Rightarrow TM \cong M \times \mathbb{R}^n$. Set $s_i : M \to TM \cong M \times \mathbb{R}^n$ given by $s_i(p) = (p, e_i)$, where e_i is the standard basis for $\mathbb{R}^n$. Clearly, $\{s_i\}$ forms a smooth frame on M.
($\Leftarrow$) Assume there exists a smooth frame $\{s_i\}_{i=1}^n$ on M. Then for any $p \in M$ and $v \in T_pM \Rightarrow v = \sum_i c_{p_i}(v)s_i(p)$ for some real-valued function $c_{p_i}(v)$. Define the following map: $\tilde{f} : TM \to M \times \mathbb{R}^n$ given by $f(p, v) = (p, c_{p_1}(v), ..., c_{p_n}(v))$. Clearly it is bijective, and we can also show that it is C^∞ as for any chart ψ on TM, $(\psi \times \mathbb{1}) \circ \tilde{f} \circ \phi^{-1} \circ (\psi \times \mathbb{1})^{-1} = \mathbb{1}$, where ϕ is the trivialization of TM over the appropriate open set. Thus $(\mathbb{1}_M, \tilde{f})$ is be an a bundle isomorphism between TM and $M \times \mathbb{R}^n$.

Problem 16.9 Parallelizability of a Lie Group
Show that every Lie group is parallelizable.

Solution: Let $\{e_i\}$ be a basis for the T_eG where G is a Lie group. Define $s_i : G \to TG$ as $s_i(g) = (g, l_{g*}(e_i))$. Clearly, s_i is smooth for all i. As l_g is a diffeomorphism, we know that l_{g*} is an isomorphism between T_eG and $T_gG \Rightarrow$ it maps basis to basis. Thus $\{l_{g*}(e_i)\}$ forms a basis for T_gG for all g. Thus $\{s_i\}$ forms a smooth frame for G and thus G is a parallelizable.

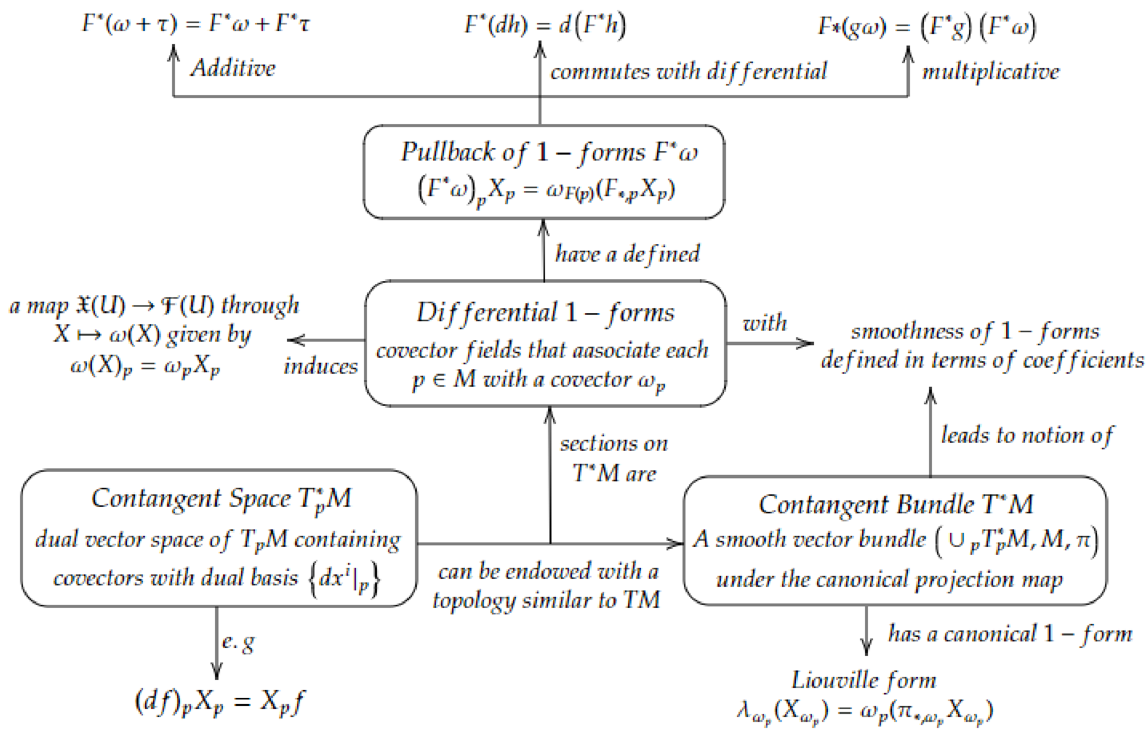
Problem 16.11 The adjoint representation
Let G be a Lie group of dimension n with Lie algebra $\mathfrak{g}$ and c_a be the conjugation map by $a \in G$. Show that the map $Ad : G \to GL(\mathfrak{g})$ defined by $Ad(a) = c_{a*}$ is a C^∞ group homomorphism.

Solution: Firstly, we can see that $c_a \circ c_b(g) = c_a(bgb^{-1}) = a(bgb^{-1})a^{-1} = (ab)g(ab)^{-1} =$

$c_{ab}(g)$. Thus we now have $Ad(ab) = c_{ab_*} = (c_a \circ c_b)_* = (c_{a_*} \circ c_{b_*})$. Thus $Ad : G \to GL(\mathfrak{g})$ is a homomorphism. Furthermore, identifying $GL(\mathfrak{g})$ as a subset of euclidean space, we can see that $Ad_{ij}(a) = \dfrac{\partial c_a^i}{\partial x^j}(e)$. We can rewrite $c_a(x)$ as $\mu(a, \mu(x, a^{-1}))$ which is C^∞, so is $Ad_{ij}(a)$ as stated above. Thus $Ad : G \to GL(\mathfrak{g})$ is C^∞.

5 Differential Forms

5.1 Differential 1-forms



Problem 17.3 Pullback of a one-form on S^1
Let $\omega = -ydx + xdy$ be a 1-form on S^1, and let l_g be complex multiplication by g on S^1. Prove that $l_g^*\omega = \omega$ for all $g \in S^1$.

Solution: Let $\omega = -ydx + xdy$ as a one-form on S^1, and S^1 with the standard coordinate chart $\phi(e^{it}) = t$. Then we have for any g and $p = (p_x, p_y) \in S^1$:

$$(l_g^*\omega)_p\left(\frac{d}{dt}\Big|_p\right) = \omega_{l_g(p)}\left(l_{g*}\frac{d}{dt}\Big|_p\right) = \omega_{l_g(p)}\left(\left(\frac{d(t\circ l_g)}{dt}\Big|_p\right)\frac{d}{dt}\Big|_{l_g(p)}\right) = \omega_{l_g(p)}\left(\frac{d}{dt}\Big|_{l_g(p)}\right).$$

Now computing:

$$\omega_p\left(\frac{d}{dt}\Big|_p\right) = -p_y\frac{dx}{dt}\Big|_p + p_x\frac{dy}{dt}\Big|_p = p_y sin(t)|_p + p_x cost(t)_p = p_y^2 + p_x^2 = 1.$$

As the answer is independent of p, we have that $\omega_{l_g(p)}\left(\frac{d}{dt}\Big|_{l_g(p)}\right) = 1$ as well. As p was arbitrary, we have thus $l_g^*\omega = \omega$.

Problem 17.4 Liouville form on the cotangent bundle
Let $(U, \phi) = (U, x_1, ..., x_n)$ be a chart on a manifold M, and let:

$$(\pi^{-1}U, \bar{\phi}) = (\pi^{-1}U, \bar{x}_1, ..., \bar{x}_n, c_1, ..., c_n)$$

be the induced chart on the cotangent bundle T^*M. Find a formula for the Liouville form λ on $\pi^{-1}U$ in terms of the coordinates $\bar{x}_1, ..., \bar{x}_n, c_1, ..., c_n$. Also show that it is C^∞.

Solution: The Liouville form is given by $\lambda_{\omega_p} = \pi^*\omega_p$. Given the coordinate basis $(\bar{x}^1, ..., \bar{x}^n, c_1, ...c_n)$, we have that the basis for $T^*(T^*M)$ is $\{d\bar{x}^1, ...d\bar{x}^n, dc^1, ..., dc^n\}$. Thus $\lambda_{\omega_p} = \sum_i a_i d\bar{x}^i|_{\omega(p)} + \sum_j b_j dc^j|_{\omega(p)}$. Assuming $\omega_p = \sum_i f_i(p) dx^i|_p$, first we shall act on $\left.\frac{d}{dc^k}\right|_{\omega_p}$:

$$\begin{aligned}
(\pi^*\omega_p)\left.\frac{d}{dc^k}\right|_{\omega(p)} &= \left(\sum_i a_i d\bar{x}^i|_{\omega(p)} + \sum_j b_j dc^j|_{\omega(p)}\right)\left.\frac{d}{dc^k}\right|_{\omega(p)} \\
\Rightarrow \omega_p\left(\pi_*\left.\frac{d}{dc^k}\right|_{\omega(p)}\right) &= \omega_p\left(\sum_j \frac{d(x^j \circ \pi)}{dc^k}\left.\frac{d}{dx^j}\right|_p\right) = 0 = b_k.
\end{aligned}$$

Now acting on $\left.\frac{d}{d\bar{x}^k}\right|_{\omega_p}$, we similarly get:

$$(\pi^*\omega_p)\left.\frac{d}{d\bar{x}^k}\right|_{\omega(p)} = \left(\sum_i a_i d\bar{x}^i|_{\omega(p)} + \sum_j b_j d\bar{x}^j|_{\omega(p)}\right)\left.\frac{d}{d\bar{x}^k}\right|_{\omega(p)}$$
$$\Rightarrow \omega_p\left(\pi_*\left.\frac{d}{d\bar{x}^k}\right|_{\omega(p)}\right) = \omega_p\left(\sum_j \frac{d(x^j \circ \pi)}{d\bar{x}^k}\left.\frac{d}{dx^j}\right|_p\right) = \omega_p\left(\frac{d}{dx^k}\right) = f_k(p) = a_k.$$

Thus $\lambda_{\omega_p} = \sum_i f_i(p) d\bar{x}^i|_{\omega_p}$. Furthermore, we can see that the coefficients of λ are given by $f_i \circ \pi$ which is a C^∞ function on T^*M. Hence λ is C^∞.

Proposition 17.11 Pullback of a sum and a product
Show that the pullback by a C^∞ map $F : N \to M$ is additive over $\Omega^1(M)$ and multiplicative over $C^\infty(M)$.

Solution: For an arbitrary vector X_p:

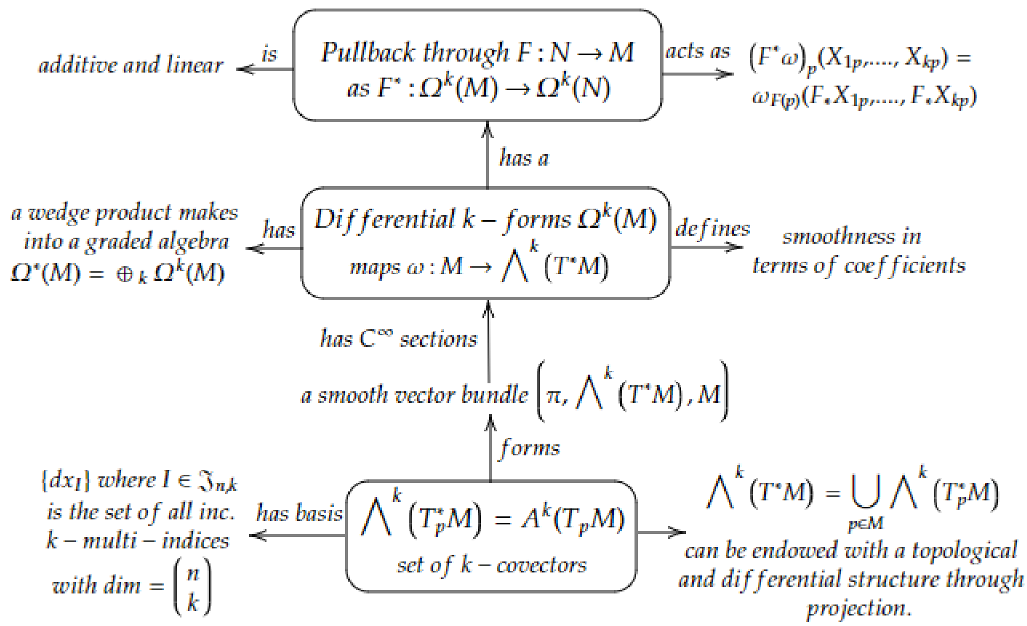
$$\begin{aligned}
F^*((\omega + \tau)_{F(p)})X_p &= (\omega + \tau)_{F(p)}(F_{*,p}X_p) \\
&= \omega_{F(p)}(F_{*,p}X_p)\tau_{F(p)}(F_{*,p}X_p) \\
&= F^*(\omega_{F(p)})(X_p) + F * (\tau_{F(p)})(X_p),
\end{aligned}$$

thus giving that $F^*(\omega + \tau) = F^*\omega + F^*\tau$ Similarly:

$$F^*(g\omega)_{F(p)}X_p = g_{F(p)}\omega_{F(p)}(F_{*,p}X_p) = (F * g)_p F^*(\omega_{F(p)})X_p,$$

giving that $F^*(g\omega) = (F^*g)(F^*\omega)$.

5.2 Differential k-Forms



Proposition 18.7 Characterization of a smooth k-form
A k-form ω is C^∞ on M if and only if for any k smooth vector fields $X_1, ..., X_k$ on M, the function $\omega(X_1, ..., X_k)$ is C^∞ on M.

Solution: Take an arbitrary $k-$form $\omega = \sum a_I dx^I$. Assume it is smooth, that is, the coefficient functions a_I are smooth. Now consider an arbitrary set of k smooth vector fields $\{X_1, ..., X_k\}$ and the resulting function $\omega(X_1, ..., X_n)$. Evaluating it at a point p with local coordinates $(U, x_1, ..., x_n)$:

$$\omega(X_1, ..., X_n)(p) = \omega_p(X_{1_p}, ...X_{n_p}) = \left(\sum_I a_I(p)dx^I|_p\right)\left(\sum_{i_1 \leq n} c^{i_1}(p)\partial_{i_1}, ..., \sum_{i_n \leq n} c^{i_n}(p)\partial_{i_n}\right),$$

with $\partial_i = \left.\dfrac{\partial}{\partial x^i}\right|_p$ and some smooth coefficient functions c^{i_j}. Further using multi-linearity:

$$\begin{aligned}\omega(X_1, ..., X_n)(p) &= \sum_{i_1,...,i_n} c^{i_1}(p)....c^{i_n}(p)\left(\sum_I a_I dx^I(\partial_{i_1}, ..., \partial_{i_n})\right)\\ &= \sum_J \sum_I c^J(p)a_I(p)dx^I(\partial_J) = \sum_J \sum_I c^J(p)a_I(p)\delta^I_J,\end{aligned}$$

where we have denoted $c^J(p) = c^{i_1}(p)....c^{i_n}(p)$ and $\partial_J = (\partial_{i_1}, ..., \partial_{i_n})$ with J as the multi-index set $\{i_1, .., i_n\}$. Clearly c^J is C^∞, hence so is $\omega(X_1, ...X_n)$.
Now assume for that for all k smooth vector fields $\omega(X_1, ..., X_n)$ is smooth. We aim

to show this implies that ω is smooth. Consider any point p in a chart $(U, x_1, ...x_n)$ centered around p, and a corresponding set of vector fields defined on U given by $X_i = \partial_i$. Each of these can be extended to C^∞ vector fields $\tilde{X}_i$ on M such that they agree with ∂_i on a nbhd. V of p. Now evaluating $\omega(\tilde{X}_{i_1}, ..., \tilde{X}_{i_k}))(p)$ in V:

$$\omega(\tilde{X}_{i_1}, ..., \tilde{X}_{i_k}))(p) = \omega_p(\partial_J) = \sum_I a_I(p) dx^I(\partial_J) = a_I(p).$$

As $\omega(\tilde{X}_{i_1}, ..., \tilde{X}_{i_k})$ is $C^\infty \Rightarrow a_I$ is C^∞ about p as well. Thus ω is a C^∞ $k-$form.

Problem 18.5 Support of a linear combination

Prove that if the k-forms $\omega_1, ..., \omega_r \in \Omega^k(M)$ are linearly independent at every point of a manifold M and $a_1, ..., a_r$ are C^∞ functions on M, then:

$$\text{supp} \sum_{i=1}^r a_i \omega_i = \bigcup_{i=1}^r \text{supp } a_i$$

Solution: Consider a set of linearly independent $k-$forms $\{\omega^i\}_i \Rightarrow \sum_i a_i(p) \omega^i_p = 0 \Leftrightarrow a_i(p) = 0$ for all i. Thus for any set of C^∞ functions a_i:

$$Z\left(\sum_i a_i \omega^i\right) = \cap_i Z(a_i) \Rightarrow Z\left(\sum_i a_i \omega^i\right)^c = \left(\bigcap_i Z(a_i)\right)^c = \bigcup_i Z(a_i)^c.$$

Thus supp$\sum_i a_i \omega^i = \bigcup_i$ supp a_i as the closure distributes over a finite union.

Problem 18.6* Locally finite collection of supports

Let $\{\rho_\alpha\}_{\alpha \in A}$ be a collection of functions on M and ω a C^∞ k-form with compact support on M. If the collection $\{\text{supp}\rho_\alpha\}_{\alpha \in A}$ of supports is locally finite, prove that $\rho_\alpha \omega \equiv 0$ for all but finitely many α.

Solution: Let τ be a topological space. In it, consider a locally finite collection of closed sets $\{C_\alpha\}$, and a compact set K. We show that K intersects only finitely many C_α. For each $p \in K$, choose a nbhd U_p of p such that U_p intersects only finitely many C_α. Thus $\{U_p\}_{p \in K}$ forms an open cover of $K \Rightarrow$ there exist finitely many U_i such that $\{U_i\}$ covers $K \Rightarrow K$ intersects finitely many C_α. Now replace τ with a smooth manifold M, the C_α with supp ρ_α and K with supp ω, where ω is a smooth differential $k-$form with compact support. This gives us that $\rho_\alpha \omega$ is non-zero for only finitely many α.

Problem 18.8* Pullback by a surjective submersion

Assuming that the pullback of a C^∞ form is C^∞, prove that if $\pi : \tilde{M} \to M$ is a surjective submersion, then the pullback map $\pi^* : \Omega^*(M) \to \Omega^*(\tilde{M})$ is an injective algebra homomorphism.

Solution: Consider a surjective submersion between two manifolds $\pi : \bar{M} \to M$. It's pullback is a map $\pi^* : \Omega^*(M) \to \Omega^*(\bar{M})$. It is an algebra homomorphism by Proposition 18.11. We further prove that it's kernel is trivial, that is, it is also injective. Assume $\pi^*(\omega) = 0$ where ω is a smooth $k-$form on M. Consider any arbitrary point $p \in M$ and any arbitrary set of k vectors $\{X_1, ..., X_k\}$ such that $X_i \in T_pM$. By assumption, there exists $q \in \bar{M}$ and $Y_i \in T_q\bar{M}$ such that $\pi(q) = p$ and $\pi_{*,q}Y_i = X_i$. Thus we have:

$$(\pi^*\omega)_q(Y_1, ..., Y_k) = 0 \Rightarrow \omega_p(\pi_{*,q}Y_1, ..., \pi_{*,q}Y_k) = \omega_p(X_1, ..., X_k) = 0.$$

As X_i were chosen arbitrarily $\Rightarrow \omega = 0$.

Problem 18.9 Bi-invariant top forms on a compact, connected Lie group
Suppose G is a compact, connected Lie group of dimension n with Lie algebra $\mathfrak{g}$.

Solution(a): Given a left-invariant $n-$ form on G, we aim to show that $r_a^*\omega$ is also left-invariant for any $a \in G$. Take any point $p \in M$ and set of n vectors $X_i \in T_pM$:

$$\begin{aligned}
(l_g^*(r_a^*\omega))_p(X_1, ...X_n) &= (r_a^*\omega)_{gp}(l_{g_*}X_1, ..., l_{g_*}X_n) \\
&= \omega_{gpa}(r_{a_*}(l_{g_*}X_1), ..., r_{a_*}(l_{g_*}X_n)) \\
&= \omega_{gpa}((r_a \circ l_g)_*X_1, ..., (r_a \circ l_g)_*X_n) \\
&= \omega_{gpa}((l_g \circ r_a)_*X_1, ..., (l_g \circ r_a)_*X_n) \\
&= \omega_{gpa}(l_{g_*}(r_{a_*}X_1), ..., l_{g_*}(r_{a_*}X_n)) \\
&= (r_a^*(l_g^*\omega))_p(X_1, ..., X_n) \\
&= (r_a^*\omega)_p(X_1, ..., X_n).
\end{aligned}$$

Solution(b): As $\dim\Omega^n(G)^G = 1$, we can see that $r_a^*\omega = f(a)\omega$. Now consider $r_{ab}^*\omega = f(ab)\omega$. We can see that as $r_{ab} = r_b \circ r_a \Rightarrow r_{ab}^* = r_b^* \circ r_a^*$. Thus $r_{ab}^*\omega = f(b)f(a)\omega \Rightarrow f(ab) = f(a)f(b)$. Thus $f : G \to \mathbb{R}^\times$ is a group homomorphism.

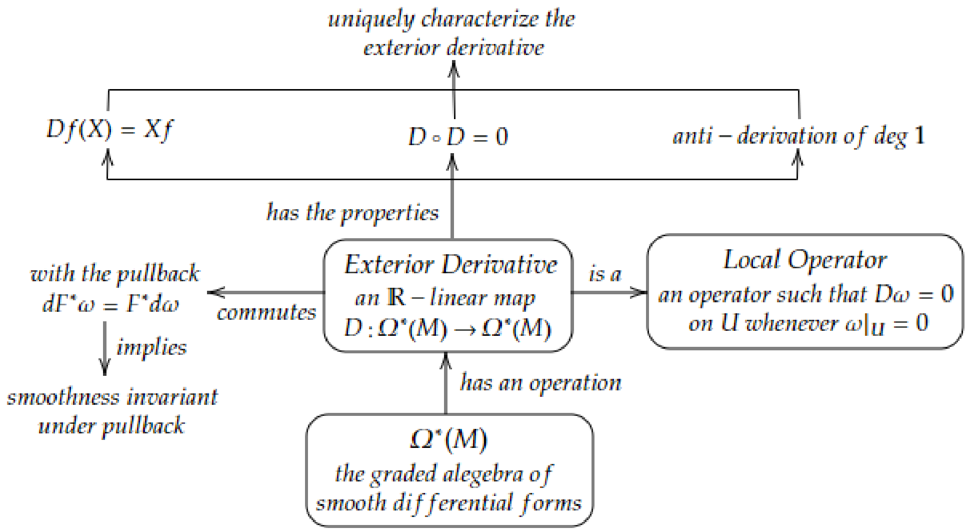
Solution(c): First note that $f(a)\omega_e = (r_a^*\omega)_e = r_a^*(\omega_a) = r_a^*l_{a^{-1}}^*(\omega_e) = (r_a \circ l_{a^{-1}})^*(\omega_e) = c_{a^{-1}}^*(\omega_e)$, where $c_{a^{-1}}$ is the conjugation map. Let $e \in G$ belong to a chart $(U, x^1, ..., x^n)$, giving T_eG the basis $\{\partial_i\}$. As $c_{a^{-1}*} : T_eG \to T_eG$ is linear map, let it be represented by the matrix $[a_{ij}]$ using the basis $\{\partial_i\}$. Furthermore let $\omega_e = rdx^1|_e \wedge .... \wedge dx^n|_e$. Thus we have $f(a)\omega_e(b_1\partial_1, ..., b_n\partial_n) = f(a)r\Pi_i(b_i\partial_i)_i = f(a)r\Pi_ib_i$. On the other hand we have:

$$c_{a^{-1}}^*(\omega_e)(b_1\partial_1, ..., b_n\partial_n) = \omega_e(c_{a^{-1}*}(b_1\partial_1), ...c_{a^{-1}*}(b_n\partial_n)) = r\Pi_i(c_{a^{-1}*}(b_i\partial_i))_i = r\Pi_i(a_{ii}b_i).$$

Comparing both sides of the starting equation, we get $f(a) = \Pi_ia_{ii}$. Thus we now define a map $h : GL(\mathfrak{g}) \to \mathbb{R}$ given by $h([a_{ij}]) = \Pi_ia_{ii}$. Clearly h is C^∞. Now consider the C^∞ composition $h \circ \mathrm{Ad} \circ i : G \to \mathbb{R}$. After smoothly restricting the codomain to the regular submanifold $\mathbb{R}^\times$, one can see that $f = h \circ \mathrm{Ad} \circ i$, and hence is also C^∞.

Solution(d): $f(G)$ is the image of a connected compact set, and hence should be connected and compact in $\mathbb{R}^\times$. In this case, it is equivalent to have $f(G)$ be a closed and bounded interval, that is, of the form $[a, b] - \{0\}$. Moreover, it should be multiplicatively closed and closed under inverses as f is a group homomorphism. Firstly, we can see that for any $b > 1$ or $b < 0$, we have $b^2 > b \Rightarrow b^2 \notin [a, b]$, thus $0 \leq b \leq 1$. As $1 \in f(G) \Rightarrow b = 1$. Now assume $a \neq b \Rightarrow$ there exists some $0 < r < 1$ in $f(G)$. But $\frac{1}{r} > 1 \Rightarrow \frac{1}{r} \in f(G)$. This is a contradiction, hence $a = b$ and $f(G) - 1$. Thus $r_a^*\omega = \omega$ for all $a \in G$.

5.3 The Exterior Derivative



Problem 19.4 Pullback of a restriction
Let $F : N \to M$ be a C^∞ map of manifolds, U an open subset of M, and $F|_{F^{-1}(U)} : F^{-1}(U) \to U$ the restriction of F to $F^{-1}(U)$. Prove that if $\omega \in \Omega^k(M)$, then:

$$\left(F|_{F^{-1}(U)}\right)^*(\omega|_U) = (F^*\omega)|_{F^{-1}(U)}.$$

*Solution:*We have:

$$(F|_{F^{-1}(U)})^*(\omega|_U) = (F|_{F^{-1}(U)})^*(i_U^*\omega) = (i \circ F|_{F^{-1}(U)})^*(\omega).$$

On the other hand:

$$(F^*\omega)|_{F^{-1}(U)} = i^*_{F^{-1}(U)}(F^*\omega) = (F \circ i_{F^{-1}(U)})^*\omega.$$

As $i \circ F|_{F^{-1}(U)} \equiv F \circ i_{F^{-1}(U)}$, we have $(F|_{F^{-1}(U)})^*(\omega|_U) = (F^*\omega)|_{F^{-1}(U)}$.

Problem 19.6 Local Operators
An operator $L : \Omega^*(M) \to \Omega^*(M)$ is support-decreasing if supp $L(\omega) \subseteq$ supp ω *for every k-form $\omega \in \Omega^*(M)$ for all $k \geq 0$. Show that an operator on $\Omega^*(M)$ is local if and only if it is support-decreasing.*

Solution: ($\Rightarrow$) Assume an operator $L : \Omega^*(M) \to \Omega^*(M)$ is a local operator. Consider any $\omega \in \Omega^*(M)$. Let $p \in \text{supp } \omega^c \Rightarrow \omega_p = 0$. As supp ω^c is an open nbhd. of $p \Rightarrow L(\omega) \equiv 0$ in supp ω^c containing p. Thus $p \notin \text{supp } L(\omega) \Rightarrow \text{supp } L(\omega) \subseteq \text{supp } \omega$.
($\Leftarrow$) Assume the operator $L : \Omega^*(M) \to \Omega^*(M)$ is support-decreasing. Take any $\omega \in \Omega^*(M)$ that is identically zero in an open set $U \Rightarrow U \subseteq \text{supp } \omega^c \subseteq (\text{supp } L(\omega))^c$.

Thus $L(\omega) \equiv 0$ on U as well. Thus L is a local operator.

Problem 19.7 Derivations of C^∞ function are local operators
Prove that a derivation of $C^\infty(M)$ is a local operator on $C^\infty(M)$.

Solution: Consider any $\omega \in \Omega^*(M)$ that vanishes on some open set U. Take any point $p \in U$ and a corresponding C^∞ bump function f at p supported in U . Clearly $f\omega \equiv 0$. Thus we get:

$$0 = D(f\omega) = Df \wedge \omega + f \wedge (D\omega).$$

Evaluating both sides at p, we get $(D\omega)_p = 0$. Thus $D\omega \equiv 0$ on U.

Problem 19.8 Nondegenerate 2-forms
(a)Prove that on $\mathbb{C}^n$ with real coordinates $x_1, y_1, ..., x_n, y_n$, the 2-form $\omega = \sum_{j=1}^n dx^j \wedge dy^j$ is nondegenerate.
(b)Prove that is λ is the Liouville of the total space T^*M of the cotangent bundle, then $d\lambda$ is a nondegenerate 2-form.

Solution(a): With $\omega = \sum_{j=1}^n dx^j \wedge dy^j$, consider the following:

$$\begin{aligned}
\omega \wedge .... \wedge \omega(\text{n times}) &= \sum_{i_1,..,i_n}^{n} dx^{i_1} \wedge dy^{i_1} \wedge ...... \wedge dx^{i_n} \wedge dy^{i_n}. \\
&= \sum_{i_1 \neq i_2 .... \neq i_n}^{n} dx^{i_1} \wedge dy^{i_1} \wedge ...... \wedge dx^{i_n} \wedge dy^{i_n} \\
&= \sum_{\sigma \in S_n} dx^{\sigma(1)} \wedge dy^{\sigma(1)} \wedge ...... \wedge dx^{\sigma(n)} \wedge dy^{\sigma(n)}.
\end{aligned}$$

Now we can see that any element of the sum above is invariant under the following switching of successive pairs of 2-forms: $dx^{\sigma(1)} \wedge dy^{\sigma(1)} \wedge .... \wedge dx^{\sigma(k)} \wedge dy^{\sigma(k)} \wedge dx^{\sigma(k+1)} \wedge dy^{\sigma(k+1)} .... \wedge dx^{\sigma(n)} \wedge dy^{\sigma(n)} = dx^{\sigma(1)} \wedge dy^{\sigma(1)} \wedge .... \wedge dx^{\sigma(k+1)} \wedge dy^{\sigma(k+1)} \wedge dx^{\sigma(k)} \wedge dy^{\sigma(k)} .... \wedge dx^{\sigma(n)} \wedge dy^{\sigma(n)}$. Thus each element in the sum above is equivalent, and hence:

$$\omega \wedge .... \wedge \omega(\text{n times}) = n!(dx^1 \wedge dy^1 \wedge ..... \wedge dx^n \wedge dy^n) \neq 0.$$

Thus ω is non-degenerate.
Solution(b): Recall the Liouville form $\lambda_{\omega_p} = \sum_i (f_i \circ \pi)(\omega_p) d\tilde{x}^i$ where $\omega_p = \sum_i f_i(p) dx^i$ locally in a chart $(U, x^1, .., x^n)$ around $p \Rightarrow$ the corresponding coordinates around ω_p are $\tilde{x}^i = x^i \circ \pi, ...\tilde{x}^n, f_1, ..., f^n$. We can see that:

$$d\lambda = \sum_i d(f_i \circ \pi) \wedge d\tilde{x}^i = \sum_i \sum_j \left(\frac{\partial f_i}{\partial \tilde{x}_j} d\tilde{x}^j + \frac{\partial f_i}{\partial c_j} df^j \right) \wedge d\tilde{x}^i = \sum_i dc^i \wedge d\tilde{x}^i.$$

This is exactly analogous to the differential form in (a), and hence, is non-degenerate.

Problem 19.11 A C^∞ nowhere-vanishing form on a smooth hypersurface
(a) Construct a C^∞ nowhere-vanishing 1-form on M= $f^{-1}(0)$ where 0 is a regular value of a C^∞ function on $\mathbb{R}^2$.
(b) Repeat the same with f being a function on $\mathbb{R}^3$.
(c) Generalize this to a function f on $\mathbb{R}^{n+1}$.

Solution(a): We have:

$$f(x,y)=0$$
$$\Rightarrow df=0 \Rightarrow f_x dx + f_y dy = 0$$

Using this we can construct the following one-form:

$$\omega = \begin{cases} \dfrac{dy}{f_x} & f_x \neq 0 \\ -\dfrac{dx}{f_y} & f_y \neq 0. \end{cases}$$

This is well-defined as f_x and f_y never vanish simultaneously due to f being smooth, and for f_x and f_y both non-zero, $\dfrac{dy}{f_x} = -\dfrac{dx}{f_y}$. Clearly, by definition, ω is a nowhere vanishing 1-form.
Solution(b): Similarly, for a smooth hypersurface $g(x,y,z)=0$ we have $g_x dx + g_y dy + g_z dz = 0$. Now wedge both sides with dz to get $g_x dz \wedge dx - g_y dy \wedge dz = 0$. Similarly, by wedging by dx and dy, we get $g_y dx \wedge dy - g_z dz \wedge dx = 0$ and $-g_x dx \wedge dy + g_z dy \wedge dz = 0$. Now we can define the following 2-form:

$$\omega = \begin{cases} \dfrac{dx \wedge dy}{f_z} & f_z \neq 0 \\ \dfrac{dy \wedge dz}{f_x} & f_x \neq 0 \\ \dfrac{dz \wedge dx}{f_y} & f_y \neq 0. \end{cases}$$

This is also well-defined as g_x, g_y, and g_z are never simultaneously zero as g is smooth, and for any pair of f_x, f_y or f_z that are simultaneously non-zero, we have that $\dfrac{dx \wedge dy}{f_z} = \dfrac{dy \wedge dz}{f_x} = \dfrac{dz \wedge dx}{f_y}$. Clearly ω is nowhere vanishing.
Solution(c): Assume we have a smooth hypersurface given by $f(x^1, ..., x^{n+1}) = 0$ in $\mathbb{R}^{n+1}$. Then by taking the exterior derivative on both sides, we get $\sum_i f_{x^i} dx^i = 0$. Now

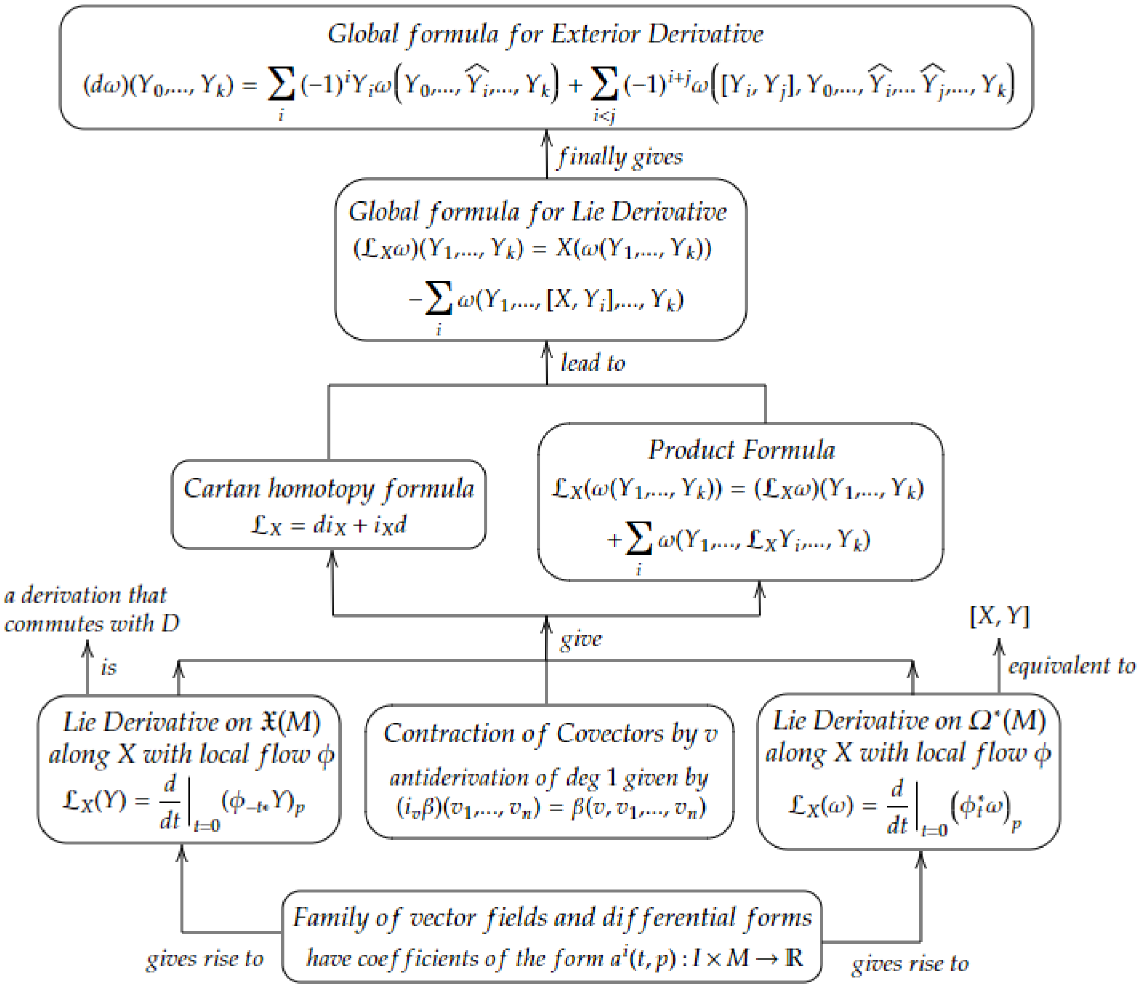
define the following $n-$form:

$$\omega = \begin{cases} \dfrac{dx^2 \wedge ..... \wedge dx^n}{f_{x^1}} & f_{x^1} \neq 0 \\ . \\ . \\ . \\ (-1)^{i-1}\dfrac{dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ...dx^n}{f_{x^i}} & f_{x^i} \neq 0 \\ . \\ . \\ . \\ (-1)^{n-1}\dfrac{dx^1 \wedge ..... \wedge dx^{n-1}}{f_{x^n}} & f_{x^n} \neq 0, \end{cases}$$

To argue that this is a well-defined form, first note that not all of f_{x^i} can be zero simultaneously, due to f being smooth. Now we claim that for any pairs f_{x^i} and f_{x^j} being non-zero simultaneously, $(-1)^{i-1}\dfrac{dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ...dx^n}{f_{x^i}} = (-1)^{j-1}\dfrac{dx^1 \wedge ... \wedge \widehat{dx^j} \wedge ...dx^n}{f_{x^j}}$, thus making ω well-defined. Other than that, it is easy to see that ω is a nowhere vanishing differential form. Now consider taking the wedge product with $dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge \widehat{dx^j} \wedge ... \wedge dx^n$ on both sides of $\sum_i f_{x^i}dx^i = 0$, for arbitrary $i < j$. We get:

$$\begin{aligned} f_{x^i}... \wedge \widehat{dx^i} \wedge ... \wedge \widehat{dx^j} \wedge ... \wedge dx^n \wedge dx^i + f_{x^j}... \wedge \widehat{dx^i} \wedge ... \wedge \widehat{dx^j} \wedge ... \wedge dx^n \wedge dx^j = 0 \\ \Rightarrow (-1)^{n-i-1}f_{x^i}dx^1 \wedge ... \wedge \widehat{dx^j} \wedge ... \wedge dx^n + (-1)^{n-j}f_{x^j}dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^n = 0 \\ \Rightarrow (-1)^{n-i-1}f_{x^i}dx^1 \wedge ... \wedge \widehat{dx^j} \wedge ... \wedge dx^n = (-1)^{n-j-1}f_{x^j}dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^n \\ \Rightarrow (-1)^j(1/f_{x^j})dx^1 \wedge ... \wedge \widehat{dx^j} \wedge ... \wedge dx^n = (-1)^i(1/f_{x^i})dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^n \\ \Rightarrow (-1)^{i-1}\frac{dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ...dx^n}{f_{x^i}} = (-1)^{j-1}\frac{dx^1 \wedge ... \wedge \widehat{dx^j} \wedge ...dx^n}{f_{x^j}}. \end{aligned}$$

5.4 The Lie Derivative and Interior Multiplication



Problem 20.3* Derivative of a smooth family of vector fields
Show that the definition of the derivative of a smooth family of vector fields on M is independent of the chart $(U, x_1, \dots, x_n)$ containing p.

Solution: Consider an arbitrary smooth family of vector fields $X_t \in \mathfrak{X}(M)$. Consider a point $p \in M$ and two charts $(U, x^1, \dots, x^n)$ and $(V, y^1, \dots, y^n)$ containing p. Then $X_t(p) = \sum_i a^i(t,p)\frac{\partial}{\partial x^i} = \sum_i b^i(t,p)\frac{\partial}{\partial y^i}$, where $b_i(t,p) = \sum_j a^j(t,p)\frac{\partial y^i}{\partial x^j}(p)$. Now the derivative of X_t w.r.t the parameter t is defined by the derivative of the coefficient functions. Thus we have:

$$
\begin{aligned}
\frac{d}{dt}X_t = \sum_i \frac{d}{dt}b^i(t,p)\frac{\partial}{\partial y^i} &= \sum_i \frac{d}{dt}\left(\sum_j a^j(t,p)\frac{\partial y^i}{\partial x^j}\right)\frac{\partial}{\partial y^i} \\
&= \frac{d}{dt}\left(\sum_i \left(\sum_j a^j(t,p)\frac{\partial y^i}{\partial x^j}\right)\frac{\partial}{\partial y^i}\right) \\
&= \frac{d}{dt}\left(\sum_i a^i(t,p)\frac{\partial}{\partial x^i}\right) \\
&= \sum_i \frac{d}{dt}a^i(t,p)\frac{\partial}{\partial x^i}
\end{aligned}
$$

Thus the derivative is independent of the choice of local coordinates.

Problem 20.6* Global formula for the exterior derivative
Assume $k \geq 1$. For a smooth k-form ω and smooth vector fields $Y_0, Y_1, ..., Y_k$ on a manifold M:

$$
(d\omega)(Y_0, ..., Y_k) = \sum_{i=0}^{k}(-1)^i Y_i \omega(Y_0, ..., \widehat{Y_i}, ..., Y_k) + \sum_{0\leq i<j\leq k}(-1)^{i+j}\omega([Y_i, Y_j], Y_0, ..., \widehat{Y_i}, ..., \widehat{Y_j}, ..., Y_k),
$$

Solution: It is clear that the given formula holds for k=1 by Proposition 20.13. Now assume it holds for all $r < k$, where $r \in \mathbb{N}$. For $r = k$ we have:

$$
(d\omega)(Y_0, ..., Y_k) = (i_{Y_0}d\omega)(Y_1, ..., Y_k) = (\mathfrak{L}_{Y_0}\omega)(Y_1, ..., Y_k) - (di_{Y_0}\omega)(Y_1, ..., Y_k),
$$

by the Cartan homotopy formula. Now using the global defintion of the Lie derivatives along with the assumption that the above formula holds for $r = k - 1$:

$$
\begin{aligned}
(d\omega)(Y_0, ..., Y_k) =& Y_0(\omega)(Y_1, ..., Y_k) - \sum_{i=1}^{r}\omega(Y_1, ..., [Y_0, Y_i], ..., Y_k) \\
&- \sum_{i=1}^{r}(-1)^{i-1}Y_i(i_{Y_0}\omega)(Y_1, .., \widehat{Y_i}, ..., \widehat{Y_j}, ..., Y_k) \\
&- \sum_{1\leq i<j\leq k}(-1)^{(i-1)+(j-1)}(i_{Y_0}\omega)([Y_i, Y_j], Y_1, ..., \widehat{Y_i}, ..., \widehat{Y_j}, ..., Y_k)
\end{aligned}
$$

This gives us:

$$\begin{aligned}\Rightarrow (d\omega)(Y_0,...,Y_k) =& Y_0(\omega)(Y_1,...,Y_k) - \sum_{i=1}^{r}\omega(Y_1,...,[Y_0,Y_i],...,Y_k)\\ &+ \sum_{i=1}^{r}(-1)^i Y_i\omega(Y_0,Y_1,..,\widehat{Y_i},...,\widehat{Y_j},...,Y_k)\\ &- \sum_{1\le i<j\le k}(-1)^{i+j}\omega(Y_0,[Y_i,Y_j],...,\widehat{Y_i},...,\widehat{Y_j},...,Y_k).\end{aligned}$$

Now note that we can combine the first term into the second sum. Furthermore, we reindex the first sum by j:

$$\begin{aligned}\Rightarrow (d\omega)(Y_0,...,Y_k) =& \sum_{i=0}^{r}(-1)^i Y_i\omega(Y_0,Y_1,..,\widehat{Y_i},...,\widehat{Y_j},...,Y_k) + \sum_{j=1}^{r}(-1)^j\omega([Y_0,Y_j],Y_1,,...,Y_k)\\ &+ \sum_{1\le i<j\le k}(-1)^{i+j}\omega([Y_i,Y_j],Y_0,...,\widehat{Y_i},...,\widehat{Y_j},...,Y_k)\\ =& \sum_{i=0}^{k}(-1)^i Y_i\omega(Y_0,...,\widehat{Y_i},...,Y_k) + \sum_{0\le i<j\le k}(-1)^{i+j}\omega([Y_i,Y_j],Y_0,...,\widehat{Y_i},...,\widehat{Y_j},...,Y_k)\end{aligned}$$

Thus the global formula holds for all $r \in \mathbb{N}$.

Problem 20.7 $\mathfrak{F}$-Linearity and the Lie Derivative
Prove $\mathfrak{L}_{fX}\omega = df \wedge i_X\omega + f\mathfrak{L}_X\omega$.

Solution: Using the Cartan homotopy formula we have:

$$\begin{aligned}\mathfrak{L}_{fX}\omega = di_{fX}\omega + i_{fX}d\omega &= d(fi_X\omega) + fi_X(d\omega)\\ &= df \wedge i_X\omega + fd(i_X\omega) + fi_X(d\omega) = df \wedge i_X\omega + f\mathfrak{L}_X\omega.\end{aligned}$$

Problem 20.8 Bracket of the Lie Derivative and Interior Multiplication
Show $\mathfrak{L}_X i_Y - i_Y\mathfrak{L}_X = i_{[X,Y]}$.

Solution: Using the global formula for Lie derivatives, we get for arbitrary $\omega \in \Omega^k(M)$ and $Y, Y_1, ..., Y_{k-1} \in \mathfrak{X}(M)$:

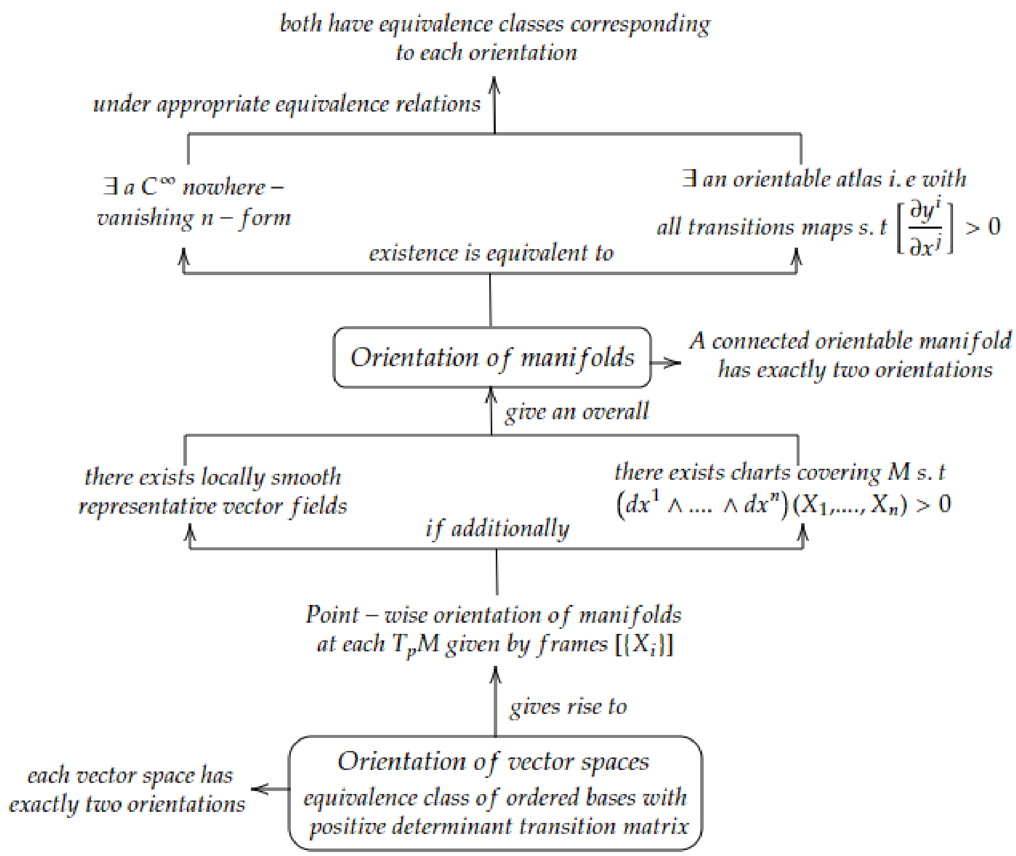
$$\begin{aligned}\mathfrak{L}_X i_Y(\omega)(Y_1,...,Y_{k-1}) - i_Y\mathfrak{L}_X(\omega)(Y_1,...,Y_{k-1}) =& X(i_Y\omega(Y_1,...,Y_k))\\ &- \sum_{i=1}^{k-1} i_Y\omega(Y_1,...,[X,Y_i],...,Y_{k-1})\\ &- \mathfrak{L}_X(\omega)(Y,Y_1,...,Y_{k-1}).\end{aligned}$$

Further expanding:

$$
\begin{aligned}
[\mathfrak{L}_X, i_Y](\omega)(Y_1, ...Y_{k-1}) =& X(\omega(Y, Y_1, ..., Y_k)) - \sum_{i=1}^{k-1} \omega(Y, Y_1, ..., [X, Y_i], ..., Y_{k-1}) \\
& - X(\omega(Y, Y_1, ..., Y_k)) + \sum_{i=1}^{k-1} \omega(Y, Y_1, ..., [X, Y_i], ..., Y_{k-1}) \\
& + \omega([X, Y], Y_1, ..., Y_{k-1}) \\
& = i_{[X,Y]}\omega(Y_1, ..., Y_{k-1}).
\end{aligned}
$$

6 Integration

6.1 Orientations



Problem 21.1* Locally constant map on a connected space
Show that a locally constant map $f : S \to Y$ on a nonempty connected space S is constant.

Solution: Consider any arbitrary $y \in Y$. Let $s \in f^{-1}(y)$. Then there exists a nbhd. U of s such that $f(U) = f(s) = y \Rightarrow U \subseteq f^{-1}(y)$. Thus $f^{-1}(y)$ is open$\Rightarrow S = \cup_{y \in Y} f^{-1}(y)$ is a decomposition of S into disjoint open sets. As S is connected, we must have that only one of these sets is non-empty $\Rightarrow S = f^{-1}(y)$ for some $y \in Y$. Thus f is constant.

Problem 21.2 Continuity of pointwise orientations
Prove that a pointwise orientation $[(X_1, ..., X_n)]$ on a manifold M is continuous if and only if every point $p \in M$ has a coordinate neighborhood $(U, \phi) = (U, x_1, ..., x_n)$ such that for all $q \in U$, the differential $\phi_{*,q} : T_qM \to T_{f(q)}\mathbb{R}^n \equiv \mathbb{R}^n$ carries the orientation of T_qM to the standard orientation of $\mathbb{R}^n$ in the following sense: $\phi_* X_{1,q}, ..., \phi_* X_{n,q} \sim (\partial/\partial r_1, ..., \partial/\partial r)$.

Solution: ($\Rightarrow$) Assume the a pointwise orientaton $[(X_1, ..., X_n)]$ on a manifold M

is continuous. Consider any arbitrary $p \in M$. By Lemma 21.4, there a exists a chart $(U, \phi) = (U, x^1, ...x^n)$ around p such that $(dx^1 \wedge .... \wedge dx^n)(X_1, ..., X_n) > 0$. Letting $X_i = \sum_j a^i_j(q) \frac{\partial}{\partial x^j}\Big|_q$ in U, the previous statement implies $det[a^i_j(q)] > 0$ for all $q \in U$. Now considering the differential $\phi_{*,q} : T_qM \to \mathbb{R}^n$, we have trivially that the matrix of $\phi_{*,q}$ is the identity matrix$\to$ $\phi_*(X_{i,q}) = \sum_j a^i_j(q) \frac{\partial}{\partial r^j}$. Thus the transition matrix between $(\phi_* X_{1,q}, ..., \phi_* X_{n,q})$ and $\left(\frac{\partial}{\partial r^1}, ..., \frac{\partial}{\partial r^n}\right)$ is $[a^i_j]$, that has positive determinant. Thus $(\phi_* X_{1,q}, ..., \phi_* X_{n,q}) \sim \left(\frac{\partial}{\partial r^1}, ..., \frac{\partial}{\partial r^n}\right)$. The exact reverse argument gives $(\Leftarrow)$.

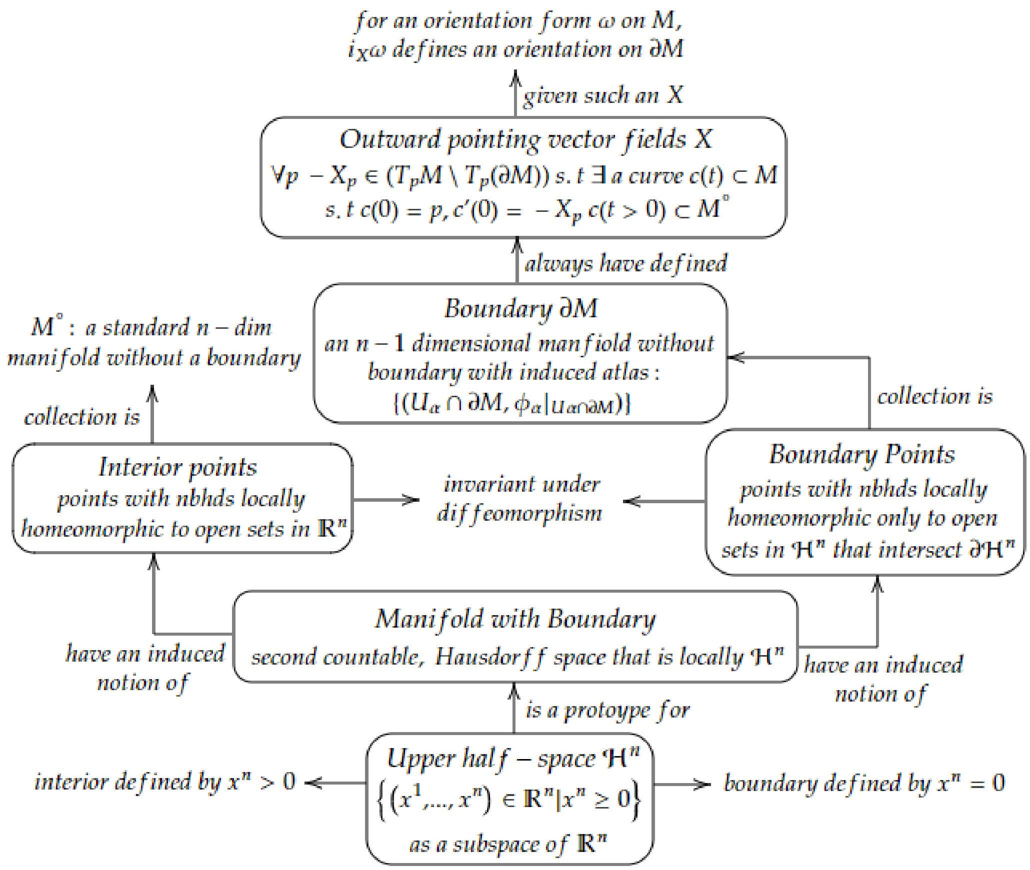
Problem 21.8 Orientability of a parallelizable manifold
Show that a parallelizable manifold is orientable.

Solution: Referring to Problem 16.8, every parallelizable manifold has a smooth frame $X_1, ..., X_n$, thus implying that $[(X_1, ..., X_n)]$ is an orientation. Thus every parallelizable manifold is orientable. Note that this also implies that every Lie group, as it is parallelizable, is also orientable.

Problem 21.9 Orientability of the total space of the tangent bundle
Prove that the total space TM of the tangent bundle is always orientable.

Solution: Consider any two overlapping charts on the total space $(TU, \tilde{\phi}) = (TU, x^1 \circ \pi, ..., x^n \circ \pi, c^1, ..., c^n)$ and $(TV, \tilde{\psi}) = (TV, y^1 \circ \pi, ..., y^n \circ \pi, b^1, ..., b^n)$ which correspond to the charts on the base space $(U, \phi) = (U, x^1, ..., x^n)$ and $(V, \psi) = (V, y^1, ...y^n)$ and corresponding coefficient functions c^i and b^i respectively. Referring to Problem 12.2, we can see that the jacobian determinant of the transition map between the two charts on TM is $\left[\frac{\partial y^i}{\partial x^j}\right]^2$, which is everywhere positive (cannot be 0 as the transition map is a diffeomorphism). Thus TM is always orientable.

6.2 Manifolds with Boundary



Problem 22.3* Inward-pointing vectors at the boundary
Let M be a manifold with boundary and let $p \in \partial M$. Show that $X_p \in T_pM$ is inward-pointing if and only if in any coordinate chart $(U, x_1, ..., x_n)$ centered at p, the coefficient of $(\partial/\partial x_n)_p$ in X_p is positive.

Solution: ($\Rightarrow$) Assume $X_p \in T_pM$ is an inward pointing vector and $(U, x^1, ..., x^n)$ is a chart centered around p. Then we have that there exists an $\epsilon > 0$ and curve $c(t)$ in M such that $c(0) = p$, $c'(0) = X_p$, and $c((0, \epsilon)) \subseteq M^\circ$. As $X_p = \sum_i a^i \left.\dfrac{\partial}{\partial x^i}\right|_p$, we can see that $a_n = X_p(x_n) = c'(0)(x_n) = \left.\dfrac{d}{dt}\right|_0 (x_n \circ c)$. We know that $(x_n \circ c) : \mathbb{R} \to \mathbb{R}$ is such that $(x_n \circ c)(0) = x_n(p) = 0$, as $p \in \partial M$. Moreover, as $c((0, \epsilon)) \subseteq M^\circ \Rightarrow (x_n \circ c)((0, \epsilon)) \subseteq \mathcal{H}^\circ$. Thus $(x_n \circ c)(t) > 0$ for $t \in (0, \epsilon) \Rightarrow a_n = \left.\dfrac{d}{dt}\right|_0 (x_n \circ c) > 0$.
($\Leftarrow$) Assume $X_p \in T_pM$ is such that for every chart $(U, x^1, ..., x^n)$ centered around $p \in \partial M$, the coefficient of $(\partial/\partial x^n)_p = a_n$ is positive. Correspondingly, there exists a chart $(U \cap \partial M, x^1|_{U \cap \partial M}, ..., x^{n-1}|_{U \cap \partial M})$ around p in ∂M. Thus for any $Y \in T_p(\partial M)$, $Y(x_n|_{U \cap \partial M}) = 0$ as $x_n \equiv 0$ on ∂M. But on the other hand $X_p(x_n) = a_n > 0$. Thus $X_p \notin T_p(\partial M)$. Moreover, we can construct a curve $c : (-\alpha, \alpha) :\to M$ for $\alpha > 0$ such that

$c(0) = p$ and $c'(0) = X_p$. We can see that $0 < a_n = X_p(x_n) = c'(0)(x_n) = \left.\frac{d}{dt}\right|_0 (x_n \circ c)$. Thus there exists an $0 < \epsilon < \alpha$ such that $(x_n \circ c)(\delta) - (x_n \circ c)(0) = (x_n \circ c)(\delta) > 0$ for $0 < \delta < \epsilon$. Thus $(x_n \circ c)((0, \epsilon)) \subseteq \mathcal{H}^\circ \Rightarrow c((0, \epsilon)) \subseteq M^\circ$. Thus restricting c to $[0, \epsilon)$, we can see that X_p is an inward pointing vector.

Problem 22.5 Boundary orientation

Let M be an oriented manifold with boundary, ω an orientation form for M, and X a C^∞ outward-pointing vector field along ∂M.
(a) If τ is another orientation form on M, then $\tau = f\omega$ for a C^∞ everywhere-positive function f on M. Show that $i_X\tau = f i_X\omega$ and therefore, $i_X\tau \sim i_X\omega$ on ∂M.
(b) Prove that if Y is another C^∞ outward-pointing vector field along ∂M, then $i_X\omega \sim i_Y\omega$ on ∂M.

Solution(a): It is trivial to see that $i_X\tau = i_X f\omega = f i_X\omega$.
Solution(b): Assume $i_X\omega$ corresponds to the orientation of ∂M given by the frame $[(X_1, ..., X_{n-1})]$, that is, $i_X\omega(X_1, ..., X_{n-1}) = \omega(X, X_1, ..., X_{n-1}) > 0$. Consider any point p with the centeralized chart $(U, x^1, ...x^n)$. Take $\omega = f dx^1 \wedge ... \wedge dx^n$ for a nowhere-vanishing smooth function f in U. We then have $\omega(X, X_1, ..., X_n)(p) = f(p)dx^1 \wedge ... \wedge dx^n(X, X_1, ..., X_n)(p) = (-1)^{n-1} f dx^1 \wedge .... \wedge dx^n(X_1, ..., X_{n-1}, X)(p)$. As X is an outward pointing vector field, we know the coefficient of $(\partial/\partial x^n)_p$ in X_p is negative, say $a(p)$. Furthermore, as $X_{i_p} \in T_p\partial M$, $X_{i_p}(x_n) = 0$ for all i. Thus we can finally see that $i_X\omega(X_1, ..., X_{n-1})(p) = a(p)f(p)(-1)^{n-1}dx^1 \wedge .... \wedge dx^{n-1}(X_1, ..., X_{n-1})$. Similarly, we can see that for another outward pointing vector field Y, $i_Y\omega(X_1, ..., X_{n-1}) = b(p)f(p)(-1)^{n-1}dx^1 \wedge .... \wedge dx^{n-1}(X_1, ..., X_{n-1})$ for another negative function $b(p)$ that is the coefficient of $(\partial/\partial x^n)_p$ in Y_p. Thus we can see that $i_Y\omega(X_1, ..., X_{n-1})$ has the same parity as $i_X\omega(X_1, ..., X_n)$, and thus $i_Y\omega \sim i_X\omega$.

Problem 22.9 Boundary orientation on a sphere

Orient the unit sphere S^n in $\mathbb{R}^{n+1}$ as the boundary of the closed unit ball. Show that an orientation form on S^n is:

$$\sum_{i=1}^{n+1} (-1)^{i-1} x^i dx^1 \wedge .... \wedge \widehat{dx^i} \wedge .... \wedge dx^{n+1}.$$

Solution: Consider the standard orientation in $\mathbb{R}^{n+1}$ being restricted to the closed unit ball. This is given by the orientation form $\omega = dx^1 \wedge .... \wedge dx^{n+1}$. Given the outward pointing vector field $X = \sum x^i \partial/\partial x^i$, the boundary orientation on S^n is then given by $i_X\omega$. First noting that $i_X dx_i = X(x_i) = x^i$:

$$\begin{aligned} i_X\omega &= i_X(dx^1 \wedge .... \wedge dx^{n+1}) = x^i dx^2 \wedge .... \wedge dx^{n+1} - dx^1 \wedge (i_X(dx^2 \wedge .... \wedge dx^{n+1})) \\ &= x^i dx^2 \wedge .... \wedge dx^{n+1} - x^2 dx^1 \wedge dx^3 \wedge .... \wedge dx^{n+1} + dx^1 \wedge dx^2 \wedge (i_X(dx^3 \wedge .... \wedge dx^{n+1})), \end{aligned}$$

where is is easy to see that successive applications of the antiderivation property gives us:

$$i_X\omega = \sum_{i=1}^{n+1}(-1)^{i-1}x^i dx^1 \wedge .... \wedge \widehat{dx^i} \wedge .... \wedge dx^{n+1}.$$

Problem 22.10 Orientation on the upper hemisphere of a sphere
Let U be the upper hemisphere of S^n with $x_{n+1} > 0$ with coordinates $x_1, ..., x_n$.
(a) Find an orientation form for U.
(b) Show that the projection map $\pi : U \to \mathbb{R}$ is orientation-preserving if and only if n is even.

Solution(a): We already established an orientation form ω on the sphere S^n in the previous problem. The corresponding orientation on its open subset U will be given by the restriction of ω onto U. This will be done by setting $x^{n+1} = \sqrt{1 - \sum_{i=1}^n (x^i)^2}$ in ω. Note that $dx^{n+1} = \sum_i \dfrac{-x^i}{x^{n+1}} dx^i$. Substituting this into ω:

$$\begin{aligned}
\omega|_U &= \sum_{i=1}^{n}(-1)^{i-1}x^i dx^1 \wedge .... \wedge \widehat{dx^i} \wedge .... \wedge dx^n \wedge \left(\sum_j \frac{-x^j}{x^{n+1}}dx^j\right) + (-1)^n x^{n+1} dx^1 \wedge .... \wedge dx^n \\
&= \sum_{i=1}^{n}(-1)^n \frac{(x^j)^2}{x^{n+1}} dx^1 \wedge .... \wedge dx^n + (-1)^n x^{n+1} dx^1 \wedge .... \wedge dx^n \\
&= (-1)^n \frac{1}{x^{n+1}} dx^1 \wedge .... \wedge dx^n.
\end{aligned}$$

Solution(b):

$$\begin{aligned}
\pi \text{ is orientation-preserving } &\Leftrightarrow [\pi^*(dx^1 \wedge ... \wedge dx^n)] \sim [\omega] \\
&\Leftrightarrow [d(\pi^* x^1) \wedge ... \wedge d(\pi^* x^n)] \sim [\omega] \\
&\Leftrightarrow [dx^1 \wedge ... \wedge dx^n] \sim \left[(-1)^n \frac{1}{x^{n+1}} dx^1 \wedge .... \wedge dx^n\right] \\
&\Leftrightarrow (-1)^n \frac{1}{x^{n+1}} > 0 \text{ on } U \Leftrightarrow n \text{ is even.}
\end{aligned}$$

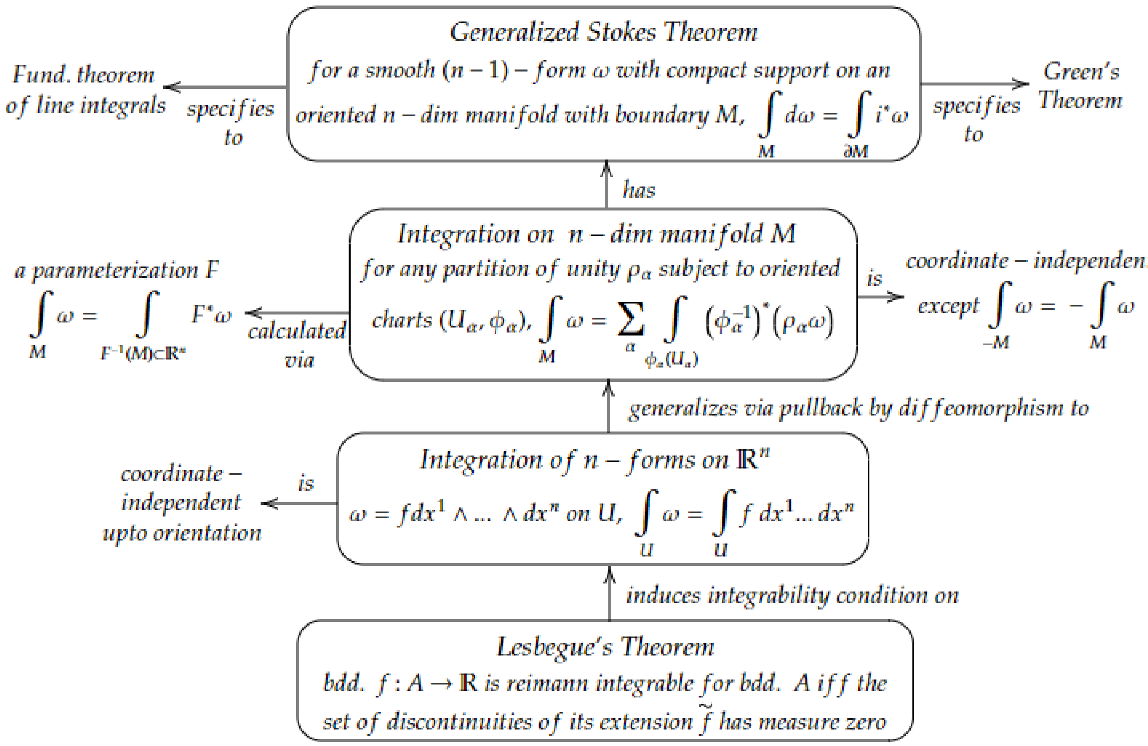
Problem 22.11 Antipodal map on a sphere and the orientability of $\mathbb{R}P^n$
The antipodal map a: $S^n \to S^n$ is orientation if and only if n is odd. Thus show that $\mathbb{R}P^n$ is orientable for odd n.

Solution:

$$
\begin{aligned}
a \text{ is orientation-preserving } &\Leftrightarrow [a^*\omega] \sim [\omega] \\
&\Leftrightarrow \left[a^*\left(\sum_{i=1}^{n+1}(-1)^{i-1}x^i dx^1 \wedge .... \wedge \widehat{dx^i} \wedge .... \wedge dx^{n+1}\right)\right] \sim [\omega] \\
&\Leftrightarrow \left[\left(\sum_{i=1}^{n+1}(-1)^{i-1}(a^*x^i) d(a^*x^1) \wedge .... \wedge \widehat{d(a^*x^i)} \wedge .... \wedge d(a^*x^{n+1})\right)\right] \sim [\omega] \\
&\Leftrightarrow \left[(-1)^{n+1}\left(\sum_{i=1}^{n+1}(-1)^{i-1}x^i dx^1 \wedge .... \wedge \widehat{dx^i} \wedge .... \wedge dx^{n+1}\right)\right] \sim [\omega] \\
&\Leftrightarrow [(-1)^{n+1}\omega] \sim [\omega] \Leftrightarrow n \text{ is odd.}
\end{aligned}
$$

We can now see that as $\mathbb{R}P^n = S^n / \sim_a$, the orientation ω induces a consistent orientation on $\mathbb{R}P^n$ for odd n.

6.3 Integration on Manifolds



Problem 23.3* Integral under a diffeomorphism
Suppose N and M are connected, oriented n-manifolds and $F : N \to M$ is a diffeomorphism. Prove that for any $\omega \in \Omega_c^k(M)$

$$\int_N F^*\omega = \pm \int_M \omega,$$

where the sign depends on whether F is orientation reversing or orientation preserving.

Solution: Let $\{U_\alpha, \phi_\alpha\}$ be an oriented atlas on M, with a partition of unity $\{\rho_\alpha\}$ subject to U_α. Correspondingly $\{(F^{-1}(U_\alpha), F^*\phi_\alpha)\}$ forms an oriented atlas on N. Depending on whether F is orientation-preserving or orientation-reversing, the orientation given by this induced atlas is the same or opposite to the orientation of N respectively. Similarly, we also have an induced partition of unity on N given by $\{F^*\rho_\alpha\}$ subject to

$F^{-1}(U_\alpha)$. Thus we have:

$$\int_N F^*\omega = \pm\sum_\alpha \int_{F^{-1}(U_\alpha)} (F^*\rho_\alpha)F^*\omega = \pm\sum_\alpha \int_{\phi(U_\alpha)} ((F^*\phi_\alpha)^{-1})^*((F^*\rho_\alpha)F^*\omega)$$
$$= \pm\sum_\alpha \int_{\phi(U_\alpha)} (\phi_\alpha^{-1})^* \circ (F^{-1})^*((F^*\rho_\alpha)F^*\omega)$$
$$= \pm\sum_\alpha \int_{\phi(U\alpha)} (\phi_\alpha^{-1})^*(\rho_\alpha\omega) = \pm\int_M \omega.$$

Problem 23.4* Stoke's Theorem
Prove Stokes's theorem for $\mathbb{R}^n$ and for $\mathcal{H}^n$.

Solution: First we prove the case for $\mathbb{R}^n$, considering the standard coordinates $x^1, ..., x^n$ and standard orientation. Take an $n-1$ form $\omega = \sum_i f_i dx^1\wedge...\wedge\widehat{dx^i}\wedge...\wedge dx^n$ with compact support in $\mathbb{R}^n$. Then we can choose $a > 0$ large enough such that $[-a, a]^n$ contains the supports of f_i in its interior. Now computing $d\omega = \sum_i(-1)^{i-1}(f_i)_{x^i} dx^1 \wedge .... \wedge dx^n$. Now we have:

$$\int_{\mathbb{R}^n} d\omega = \sum_i (-1)^{i-1} \int_{\mathbb{R}^n} (f_i)_{x^i} dx^1...dx^n = \sum_i (-1)^{i-1} \int_{\mathbb{R}^{n-1}} \int_{-a}^{a} (f_i)_{x^i} dx^i dx^1...\widehat{dx^i}...dx^n$$
$$= \sum_i (-1)^{i-1} \int_{\mathbb{R}^{n-1}} (f_i)]_{x^i=-a}^{a} dx^1...\widehat{dx^i}...dx^n = 0.$$

On the other hand, as the boundary of $\mathbb{R}^n$ is empty, $\int_{\partial\mathbb{R}^n} \omega = 0$ as well. Thus Stoke's Theorem holds for $\mathbb{R}^n$.
For $\mathcal{H}^n$, the steps are similar except that we choose an $a > 0$ large enough such that $[-a, a]^{n-1} \times [0, a]$ contains the support of f_i in its interior. Now we have:

$$\int_{\mathbb{R}^n} d\omega = \sum_i (-1)^{i-1} \int_{\mathbb{R}^n} (f_i)_{x^i} dx^1...dx^n$$
$$= \sum_{i=1}^{n-1} (-1)^{i-1} \int_{\mathbb{R}^{n-1}} \int_{-a}^{a} (f_i)_{x^i} dx^i dx^1...\widehat{dx^i}...dx^n + (-1)^{n-1} \int_{\mathbb{R}^{n-1}} \int_0^a (f_n)_{x^n} dx^n dx^1...dx^{n-1}$$
$$= (-1)^n \int_{\mathbb{R}^{n-1}} f_n(x^1, .., x^{n-1}, 0) dx^1...dx^{n-1}.$$

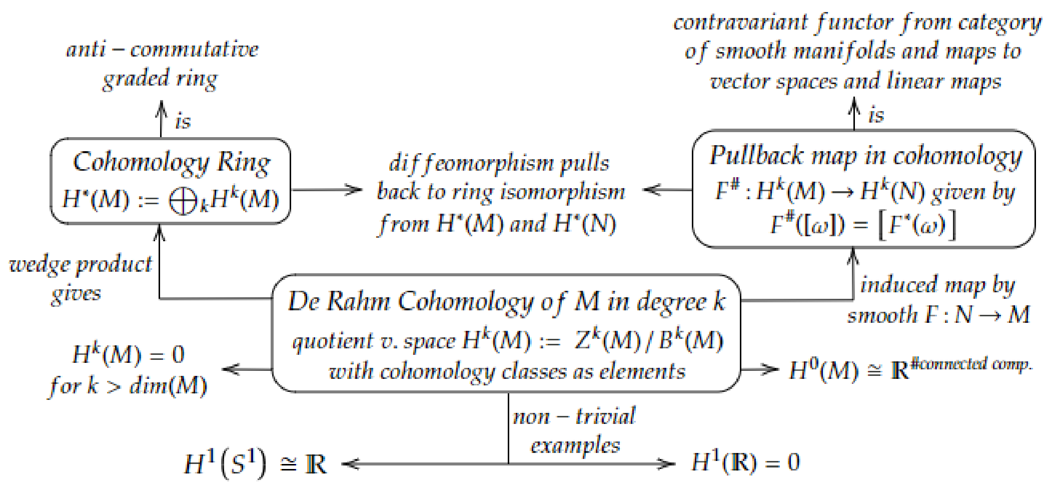
On the other hand, we can see that $\partial\mathcal{H}^n$ is defined by $x^n = 0$, and hence has $dx^n = 0$. This gives that $\omega|_{\partial\mathcal{H}^n} = f_n(x^1, .., x^{n-1}, 0)dx^1 \wedge ... \wedge dx^{n-1}$. Furthermore, the orientation on $\partial\mathcal{H}^n$ is given by $i_{-\partial/\partial x^n}(dx^1 \wedge ... \wedge dx^n) = (-1)^n dx^1 \wedge ... \wedge dx^{n-1}$. Thus:

$$\int_{\partial\mathcal{H}^n} \omega = (-1)^n \int_{\mathbb{R}^{n-1}} f_n(x^1, .., x^{n-1}, 0) dx^1...dx^{n-1}$$

as well. Thus Stoke's Theorem holds for $\mathcal{H}^n$.

7 De Rham Theory

7.1 De Rham Cohomology



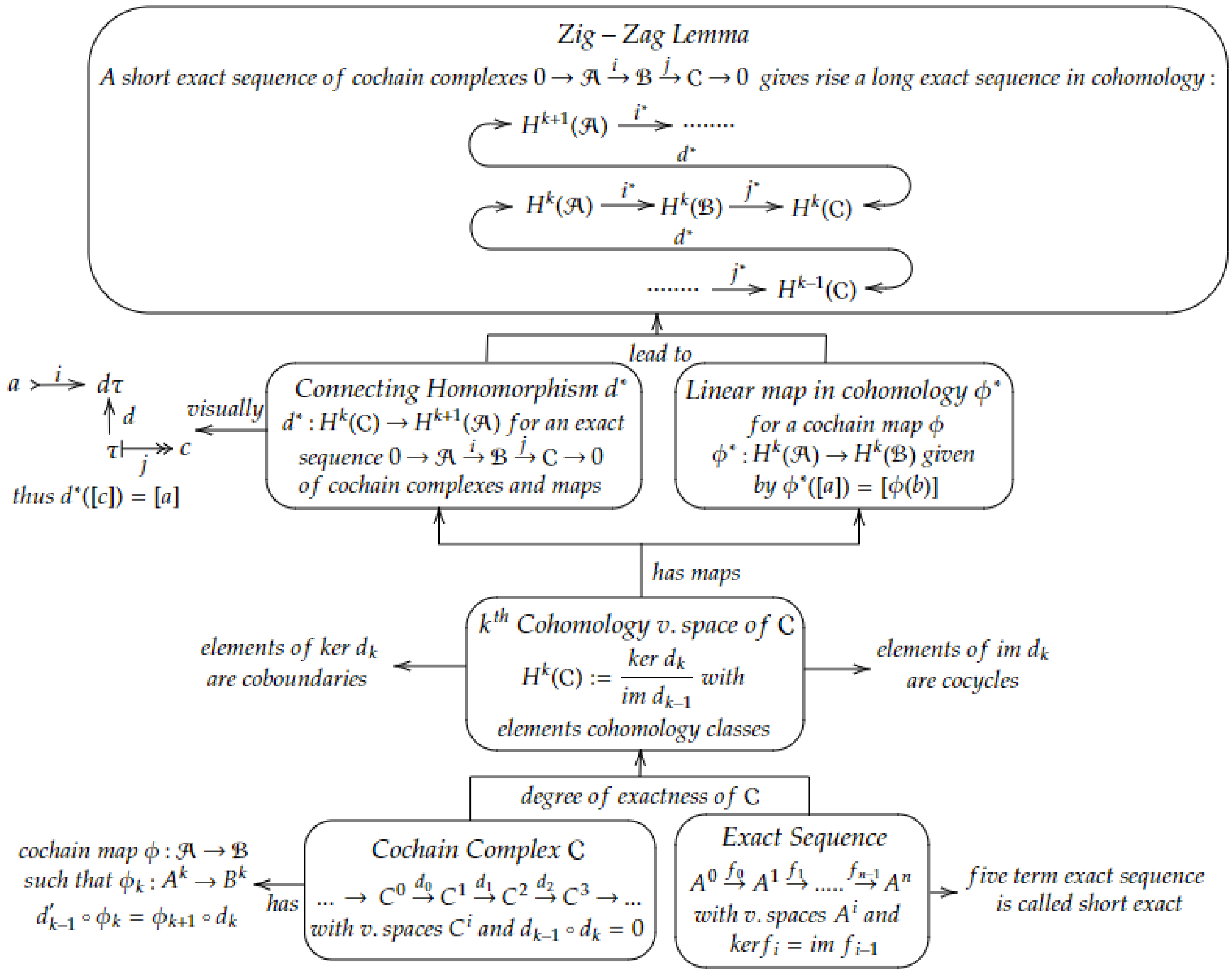
Problem 24.1 Nowhere-vanishing 1-forms
Prove that a nowhere-vanishing 1-form on a compact manifold cannot be exact.

Solution: An nowhere-vanishing exact 1-form α is of the form $\alpha = df$ for some smooth function f on M. As M is compact $\Rightarrow f(M)$ attains a maximum in $\mathbb{R}$. Let this be denoted by $f(p)$ where $p \in M$ is contained in some chart $(U, x^1, ..., x^n)$. As p is also a local maxima, by Problem 8.10, p must be a critical point of $f \Rightarrow \dfrac{\partial f}{\partial x^i} = 0$ for all i. Thus $df_p = \alpha_p = 0$. This is a contradiction as α was nowhere-vanishing.

Problem 24.2 Cohomology in degree zero
Suppose a manifold M has infinitely many connected components. Compute its de Rham cohomology vector space $H^0(M)$ in degree 0.

Solution: As M is second countable, it admits a countable basis $\{B_\alpha\}$. Furthermore, as f is locally connected (due to being locally euclidean), each connected component of M is open. Let $\mathcal{C}$ denote the set of connected components. To each $C \in \mathcal{C}$, we can associate a basis set B_α contained in C. This is an injective map from $\mathcal{C}$ to $\{B_\alpha\}$ as each connected component is disjoint. Thus $\mathcal{C}$ is at most countable.
Now as there are no non-zero exact forms of degree 0, we have $H^0(M) \cong Z^0(M)$, that is, the vector space of all closed 0-forms on M. A 0-form on M is a C^∞ function f on M such that $df = 0$. This implies that f is locally constant on M, and constant on each connected component. Thus f is defined by its value on each connected map, and this induces an isomorphism between $Z^0(M)$ and $\mathbb{R}^\infty$. Thus $H^0(M) \cong \mathbb{R}^\infty$.

7.2 The Long Exact Sequence in Cohomology



Exercise 25.5 Connecting homomorphism
Show that the connecting homomorphism: $d^* : H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A})$ is well-defined.

Solution: The connecting homomorphism is given by the following diagram:

$$\begin{tikzcd}
a \arrow[r, tail, "i"] & db \\
 & b \arrow[u, "d"'] \arrow[r, mapsto, two heads, "j"'] & c
\end{tikzcd}$$

where $b \in B^k$ is any element such that $j(b) = c$, and $a \in A^{k+1}$ is uniquely defined by $i(a) = db$. This gives $d^*([c]) = [a]$.
Now assume $\exists \bar{b}$ such that $j(\bar{b}) = c$. Then we also have the following diagram:

$$\begin{tikzcd}
\bar{a} \arrow[r, tail, "i"] & d\bar{b} & \\
& \bar{b} \arrow[u, "d"] \arrow[r, mapsto, two heads, "j"] & c
\end{tikzcd}$$

We aim to show that $[a] = [\bar{a}]$. We have that $j(b - \bar{b}) = 0$, thus $(b - \bar{b}) \in ker\, j$. By exactness, we have that $(b - \bar{b}) \in \operatorname{im} i$. Thus there exists an $e \in A^k$ such that $i(e) = b - \bar{b}$. Thus $i(de) = d(i(e)) = d(b - \bar{b})$. This gives us the following commutative diagram:

$$\begin{tikzcd}
de \arrow[r, tail, "i"] & d(b - \bar{b}) & \\
e \arrow[u, "d"] \arrow[r, tail, "i"] & (b - \bar{b}) \arrow[u, "d"] \arrow[r, mapsto, two heads, "j"] & 0
\end{tikzcd}$$

But we also have that $i(a - \bar{a}) = d(b - \bar{b}) \Rightarrow$ as i is injective, $de = a - \bar{a} \Rightarrow [a] = [\bar{a}]$.

Now we check whether for a different representative of $[c]$ of the form $c + d\tau$ gives the same output $[a]$. We have the following diagram representing $d^*([c + d\tau]) = [a']$:

$$\begin{tikzcd}
a' \arrow[r, tail, "i"] & db' & \\
& b' \arrow[u, "d"] \arrow[r, mapsto, two heads, "j"] & c + d\tau
\end{tikzcd}$$

Thus $j(b - b') = d\tau$ for some $\tau \in C^{k-1}$. As j is surjective, $\exists g \in B^{k-1}$ such that $j(g) = \tau$. This gives us the following diagram for $d^*([d\tau]) = [a - a']$:

$$\begin{tikzcd}
a - a' \arrow[r, tail, "i"] & d(b - b') & \\
& (b - b') \arrow[u, "d"] \arrow[r, mapsto, two heads, "j"] & d\tau \\
& g \arrow[r, mapsto, two heads, "j"'] & \tau \arrow[u, "d"]
\end{tikzcd}$$

Thus we will have $j(dg) = d(j(g)) = d\tau$. Now, we can instead consider the computation of $d^*([d\tau])$ using dg instead of $(b - b')$. This gives us $d^*([d\tau]) = [l]$ for some $l \in A^{k+1}$ such that:

$$
\begin{tikzcd}
l \arrow[r, tail, "i"] & 0 & \\
 & dg \arrow[u, "d"'] \arrow[r, maps to, two heads, "j"] & d\tau \\
 & g \arrow[u, "d"] \arrow[r, maps to, two heads, "j"'] & \tau \arrow[u, "d"']
\end{tikzcd}
$$

As i is injective $\Rightarrow l = 0$. As proven before, the choice of dg should not change the result as compared to before, thus $[a - a'] = [0] \Rightarrow [a] = [a']$. Thus the connecting homomorphism is well-defined.

Problem 25.3 Long exact cohomology sequence
Prove the exactness of the sequence in the Zig-Zag lemma at $H^k(\mathcal{A})$ and $H^k(\mathcal{B})$.

Solution: First we check exactness at $H^k(\mathcal{B})$. Consider any $[i(a)] \in \text{im } i^*$. Then $j^*([(i(a)]) = [j(i(a))] = [0]$ as $\text{im } i = ker\, j$. Thus $\text{im } i^* \subseteq ker\, j^*$. Now take any $[b] \in ker\, j^* \Rightarrow j(b) = d\tau$ for some $\tau \in C^{k-1}$. As j is surjective $\Rightarrow \exists b' \in B^{k-1}$ such that $j(b') = \tau$. Furthermore, $j(db') = d(j(b')) = d\tau$. This gives us the following diagram:

$$
\begin{tikzcd}
db' \arrow[r, maps to, two heads, "j"] & d\tau \\
b' \arrow[u, "d"] \arrow[r, maps to, two heads, "j"'] & \tau \arrow[u, "d"']
\end{tikzcd}
$$

This gives us that $j(b - db') = 0 \Rightarrow (b - db') \in ker\, j = \text{im } i$. Thus $\exists a \in A^k$ such that $i(a) = b - db'$. Furthermore $i(da) = d(i(a)) = db - ddb' = db = 0$ as b is a cocycle. Thus we have $i^*([a]) = [b]$. Thus $[b] \in \text{im } i^* \Rightarrow \text{im } i^* = ker\, j$.

Now we check exactness at $H^k(\mathcal{A})$. Consider any $d^*([c]) = [a] \in \text{im } d^*$ for any $[c] \in H^{k-1}(C)$. This is given by:

$$
\begin{tikzcd}
a \arrow[r, tail, "i"] & d\tau & \\
 & \tau \arrow[u, "d"'] \arrow[r, maps to, two heads, "j"'] & c
\end{tikzcd}
$$

Clearly $i^*([a]) = [i(a)] = [d\tau] = [0]$. Thus $[a] \in ker\, i^*$ and $\text{im } d^* \subseteq ker\, i^*$. Furthermore, take an arbitrary $[a] \in ker\, i^* \Rightarrow i(a) = d^\tau$ for some $\tau \in B^{k-1}$. This trivially gives us:

$$\begin{tikzcd}
a \arrow[r, tail, "i"] & d\tau & \\
& \tau \arrow[u, "d"'] \arrow[r, mapsto, two heads, "j"'] & j(\tau)
\end{tikzcd}$$

As $d(j(\tau)) = j(d\tau) = j(i(a)) = 0$, $j(\tau)$ is a cocyle. Hence $d^*([j(\tau)]) = [a]$, and hence $ker\, i^* = \text{im } d^*$.

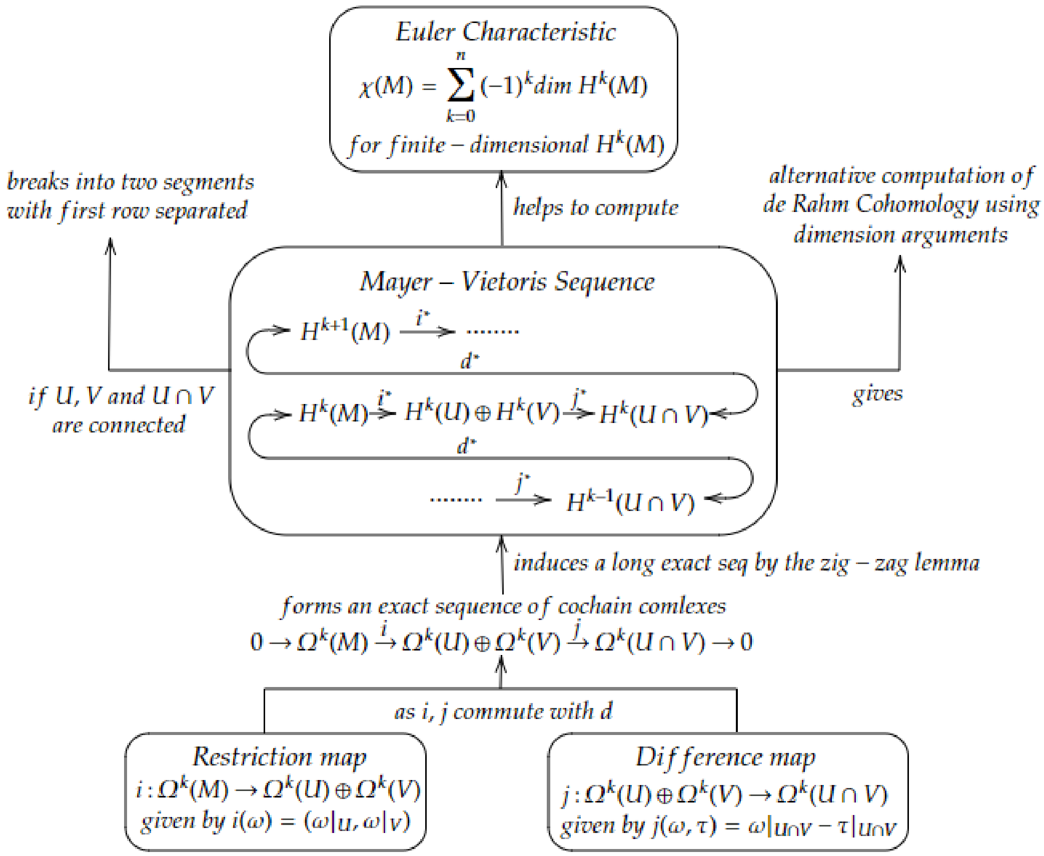
Problem 25.4* The snake lemma
Prove the Snake lemma.

Solution: The commutative diagram can be extended into an almost identical extension given by:

$$\begin{tikzcd}
& 0 & 0 & 0 & \\
0 \arrow[r] & A^1 \arrow[u] \arrow[r] & B^1 \arrow[u] \arrow[r] & C^1 \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & A^0 \arrow[u, "\alpha"] \arrow[r] & B^0 \arrow[u, "\beta"] \arrow[r] & C^0 \arrow[u, "\gamma"] \arrow[r] & 0 \\
& 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

We can see that this is clearly a short exact cochain complex given by $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$. Furthermore, we have that $H^0(\mathcal{A}) := \dfrac{ker\,\alpha}{\text{im } 0} = ker\,\alpha$, and $H^1(\mathcal{A}) := \dfrac{ker\, 0}{\text{im } \alpha} = \dfrac{A}{\text{im } \alpha} = coker\,\alpha$. Similarly $H^0(\mathcal{B}) = ker\,\beta$, $H^1(\mathcal{B}) = coker\,\beta$, $H^0(\mathcal{C}) = ker\,\gamma$, $H^1(\mathcal{C}) = coker\,\gamma$. Now applying the zig-zag lemma, we get the induced long exact sequence in the snake lemma.

7.3 The Mayer-Vietoris Sequence



Exercise 26.3 Smooth extension of a function
Let $\{\rho U, \rho V\}$ be a partition of unity subordinate to the open cover $\{U, V\}$. Define $f_V : V \to \mathbb{R}$ by:

$$f_V(x) = \begin{cases} \rho_U(x) f(x) & \text{for } x \in U \cap V \\ 0 & \text{for } x \in V - (U \cap V) \end{cases}$$

Prove that f_V is C^∞ on V.

Solution: Consider any $p \in V$. If $p \in U \cap V$, then there exists a chart around p contained in $U \cap V$ over which both ρ_U and f are smooth. Thus $\rho_U f$ is also smooth at p. If $p \in V - (U \cap V)$ then there exists an open set W containing p such that $W \cap \text{supp}\rho_U = \emptyset$. Thus there exists a chart (Y, ϕ) around p contained in W such that $f_V|_Y = 0$ is smooth. thus f_V is smooth on V.

Problem 26.1 Short exact Mayer-Vietoris sequence
Prove the exactness of this sequence at the first two terms:

$$0 \to \Omega^k(M) \xrightarrow{i} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{j} \Omega^k(U \cap V) \to 0.$$

Solution: First we check exactness at $\Omega^k(M)$. For this we need to show that i is injective. Assume $i(\omega) = (i_U^*\omega, i_V^*\omega) = (\omega|_U, \omega|_V) = (0,0)$. As $U \cup V = M$, this clearly means $\omega = 0$ on M. Thus i is injective.
Now we check exactness at $\Omega^k(U) \oplus \Omega^k(V)$. Consider any $i(\omega) = (\omega|_U, \omega|_V) \in \text{im } i$. Then $j(i(\omega)) = (\omega|_U)|_{U\cap V} - (\omega|_V)|_{U\cap V} = \omega|_{U\cap V} - \omega|_{U\cap V} = 0$. Thus im $i \subseteq ker\, j$. Now take any $(\omega, \tau) \in ker\, j \Rightarrow \omega|_{U\cap V} = \tau|_{U\cap V}$. Consider a $k-$ form σ on M well-defined by $\sigma|_U = \omega$ and $\sigma|_V = \tau$. For any $p \in M$, we have that $p \in U$ or $p \in V$. In either case, there exists a chart (W, ϕ) around p contained in U or V. Then $\sigma|_W = \omega|_W$ or $\sigma|_W = \tau|_W$. In both cases, σ is C^∞ at p. Thus $\sigma \in \Omega^k(M)$. Clearly $i(\sigma) = (\omega, \tau)$, thus $(\omega, \tau) \in \text{im } i$. Thus im $i = ker\, j$.

Problem 26.2 Alternating sum of dimensions
Let

$$0 \to A^0 \xrightarrow{d_0} A^1 \xrightarrow{d_1} A^2 \xrightarrow{d_2} .... \to A^m \to 0$$

be an exact sequence of finite-dimensional vector spaces. Show that

$$\sum_{k=0}^{m} (-1)^k \text{dim } A^k = 0$$

Solution: We have at each step dim A^k = dim $ker\, d_k$ + dim im d_k = dim im d_{k-1} + dim im d_k by exactness. This gives:

$$\sum_{k=0}^{m} (-1)^k \text{dim } A^k = \sum_{k=0}^{m} (-1)^k (\text{dim im } d_{k-1} + \text{dim im } d_k) = \text{dim im } d_{-1} + (-1)^m \text{dim im } d_m,$$

where the last step follows from the sum being a telescoping series. As d_{-1} and d_m are the zero maps, the sum equals 0.

Exercise 26.5 Euler characteristics in terms of an open cover
Prove $\chi(M) - (\chi(U) + \chi(V)) + \chi(U \cap V) = 0$.

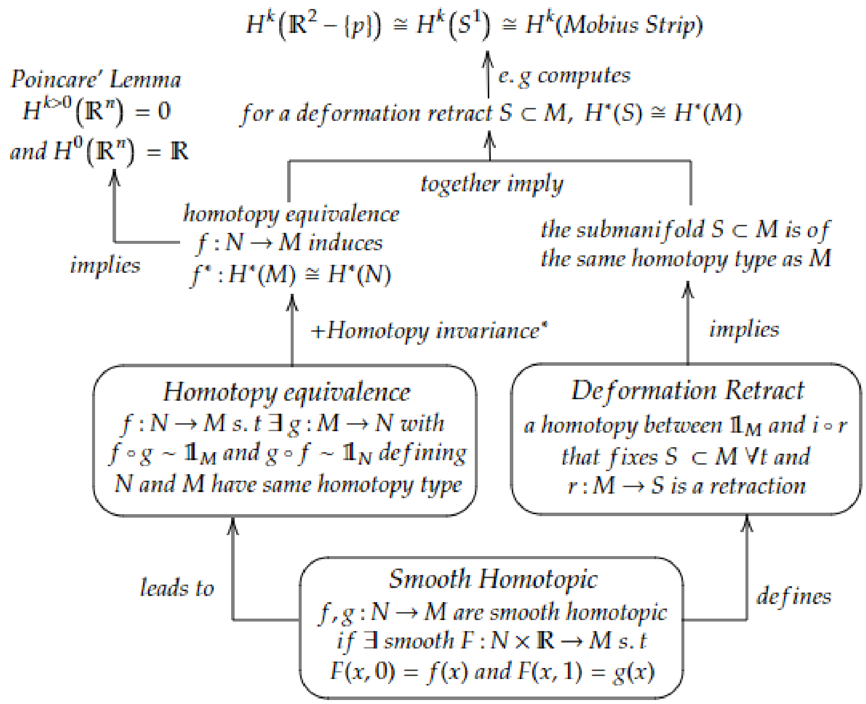
Solution: Given that the Mayer-Vietoris sequence is exact, we can apply Problem 26.2:

$$\sum_{k=0}^{n} (-1)^{3k} \text{dim } H^k(M) + (-1)^{3k+1} (\text{dim } H^k(U) + \text{dim } H^k(V)) + (-1)^{3k+2} \text{dim } H^k(U \cap V) = 0$$

$$\Rightarrow \sum_{k=0}^{n} (-1)^k (\text{dim } H^k(M) - (\text{dim } H^k(U) + \text{dim } H^k(V)) + \text{dim } H^k(U \cap V)) = 0$$

$$\Rightarrow \chi(M) - (\chi(U) + \chi(V)) + \chi(U \cap V) = 0$$

7.4 Homotopy Invariance



*Will be discussed in the last chapter.

Problem 27.1 Homotopy equivalence
Let M, N, and P be manifolds. Prove that if M and N are homotopy equivalent and N and P are homotopy equivalent, then M and P are homotopy equivalent.

Solution: As M and N are homotopy equivalent, and N and P are homotopy equivalent, there exists homotopy equivalences $f : M \to N$ and $g : N \to P$ with $h : N \to M$ and $k : P \to N$ as their homotopic inverses respectively. Thus $(h \circ f) \sim \mathbb{1}_M$, $(f \circ h) \sim \mathbb{1}_N$, $(k \circ g) \sim \mathbb{1}_N$ and $(g \circ k) \sim \mathbb{1}_P$. We want to show that $g \circ f : M \to P$ is a homotopy equivalence with $h \circ k : P \to M$ as a homotopic inverse.
To show that $(h \circ k) \circ (g \circ f) \sim \mathbb{1}_M$, we show first that $(h \circ k) \circ (g \circ f) \sim (h \circ f)$. Then by transitivity of the homotopic relation, our result follows. Define the following homotopy $F : M \times \mathbb{R} \Rightarrow M$ given by $F = h \circ G \circ (f \times \mathbb{1})$, where G is a homotopy between $(k \circ g)$ and $\mathbb{1}_N$. Clearly F is smooth, and we can check that that under F:

$$(x, 0) \xrightarrow{f \times \mathbb{1}} (f(x), 0) \xrightarrow{G} (k \circ g)(f(x)) \xrightarrow{h} (h \circ k \circ g \circ f)(x)$$
$$(x, 1) \xrightarrow{f \times \mathbb{1}} (f(x), 1) \xrightarrow{G} f(x) \xrightarrow{h} (h \circ f)(x)$$

Thus F is a homotopy between $(h \circ k) \circ (g \circ f)$ and $h \circ f$.
Similarly, we can show that $(g \circ f) \circ (h \circ k) \sim (g \circ k) \sim \mathbb{1}_P$. For this we construct the

smooth homotopy $H : P \times \mathbb{R} \to P$ given by $F = g \circ K \circ (k \times \mathbb{1})$, where K is a homotopy between $(f \circ h)$ and $\mathbb{1}_N$. Now under H:

$$(x, 0) \xrightarrow{k\times\mathbb{1}} (k(x), 0) \xrightarrow{K} (f \circ h)(f(x)) \xrightarrow{g} (g \circ f \circ h \circ k)(x)$$
$$(x, 1) \xrightarrow{k\times\mathbb{1}} (k(x), 1) \xrightarrow{G} k(x) \xrightarrow{h} (g \circ k)(x)$$

Thus H is a homotopy between $(g \circ f) \circ (h \circ k)$ and $(g \circ k)$. This proves that M and P are homotopy equivalent.

Problem 27.2 Contractibility and path-connectedness
Show that a contractible manifold is path-connected.

Solution: Assume M is contractible. Choose any two points $p, q \in M$. We know that M and $\{p\}$ have the same homotopy type. Thus the constant map $f : M \to \{p\}$ is a homotopy equivalence with inclusion i as a homotopic inverse. Thus there exists a smooth homotopy between $i \circ f$ and $\mathbb{1}_M$ given by $F(x, t) : M \times \mathbb{R} \to M$ such that $F(x, 0) = p$ and $F(x, 1) = x$ for all $x \in M$. Restricting $F|_{x=q} : \mathbb{R} \to M$ is such that $F|_{x=q}(0) = p$ and $F|_{x=q}(1) = q$. Restricting this further to the closed interval $[0, 1]$ we obtain a continuous path from p to q. As these points were arbitrary, M is path-connected.

Problem 27.3 Deformation retract from a cylinder to a circle
Show that the circle $S^1 \times \{0\}$ is a deformation retract of the cylinder $S^1 \times \mathbb{R}$.

Solution: Consider the smooth map $F : (S^1 \times \mathbb{R}) \times \mathbb{R} \to S^1 \times \mathbb{R}$ given by $F((p, z), t) = (p, (1 - t)z)$. We can see that $F((p, z), 0) = (p, z)$, and that $F((p, 0), t) = (p, 0) \in S^1 \times \{0\}$ for all t. Furthermore $F((p, z), 1) = (p, 0)$ is a retraction from $S^1 \times \mathbb{R}$ to $S^1 \times \{0\}$. Thus the circle $S^1 \times \{0\}$ is a deformation retract of the cylinder $S^1 \times \mathbb{R}$.

7.5 Computation of the de Rham Cohomology

This chapter mainly involved computation of the de Rham Cohomology vector space of the torus as $H^1(\sum) = \mathbb{R}^2$ and $H^2(\sum) = \mathbb{R}$. The following imperative result is also mentioned:

Lemma 28.3. *Suppose p is in a compact oriented surface M without boundary, and $i : C \to M - \{p\}$ is the inclusion of a small circle around the puncture. Then the restriction map:*

$$i^* : H^1(M - \{p\}) \to H^1(C)$$

is the zero map.
This allows the computation of the de Rham Cohomology of $\sum_2$ to give $H^1(\sum_2) = \mathbb{R}^4$ and $H^2(\sum_2) = \mathbb{R}$.

Problem 28.1 Real Projective Plane
Compute the cohomology of the real projective plane $\mathbb{R}P^2$.

Solution: Consider $\mathbb{R}P^2$ as D^2 with anti-podal points identified with each other. Consider the open cover $\{U, V\}$ where U consists of $\mathbb{R}P^2$ with the center point $p \in D^2$ removed, and V is an open disk containing p. We can see that U deformation retracts to S^1 with anti-podal points identified, which is just $\mathbb{R}P^1 \cong S^1$. Thus $U \sim S^1$. Similarly, V is contractible, but $U \cap V \sim S^1$. Taking into account that $\mathbb{R}P^2$ is connected, this gives us the following Mayer-Vietoris sequence:

$$0 \to \mathbb{R} \xrightarrow{\alpha_1} \mathbb{R}^2 \xrightarrow{\alpha_1} \mathbb{R} \xrightarrow{\alpha_3} H^1(\mathbb{R}P^2) \xrightarrow{\alpha_4} \mathbb{R} \xrightarrow{i^*} \mathbb{R} \xrightarrow{\alpha_5} H^2(\mathbb{R}P^2) \to 0 \to .....$$

Firstly, it should be noted that i^* is restriction from $H^1(U)$ to $H^1(U \cap V)$. We can show that i^* is an isomorphism by showing that $i : U \cap V \to U$ is a homotopy equivalence. Define the map $j : U \to U \cap V$ by $j(r, \theta) = (b, \theta)$ for some b less than the radius of V. Note that this map is well-defined as for points on the boundary of D^2 at $r = 1$, $j(1, \theta) = (b, 2\theta) = (b, 2\theta + 2\pi) = j(1, \theta + \pi)$. It is easy to see that $i \circ j \sim \mathbb{1}_U$ and $j \circ i \sim \mathbb{1}_{U \cap V}$.
Now that i^* is an isomorphism, we must have that $\alpha_5 = 0 \Rightarrow H^2(\mathbb{RP}^2) = 0$. Moreover as i^* is injective, $\alpha_4 = 0$. Thus we can decompose the above sequence into:

$$0 \to \mathbb{R} \xrightarrow{\alpha_1} \mathbb{R}^2 \xrightarrow{\alpha_1} \mathbb{R} \xrightarrow{\alpha_3} H^1(\mathbb{R}P^2) \to 0$$

Using Problem 26.2, we get that $1 - 2 + 1 - \dim\ H^1(\mathbb{R}P^2) = 0 \Rightarrow H^1(\mathbb{R}P^2) = 0$.

Problem 28.2 The $n-$sphere
Compute the cohomology of the sphere S^n.

Solution: Let us first compute the cohomology of S^2. Take a point $p \in S^2$, and consider the open cover $\{U = S^2/\{p\}, V\}$, where V is an open disk around p and hence is contractible. Furthermore, $U \cap V \sim S^1$, and $U = S^2/\{p\} \cong \mathbb{R}^2$. Now as U, V and $U \cap V$ are all connected, we may begin the Mayer-Vietoris sequence from $H^1(S^2)$ as follows:

$$0 \to H^1(S^2) \xrightarrow{\beta} H^1(U) \oplus H^1(V) \xrightarrow{\alpha} H^1(S^1) \xrightarrow{d^*} H^2(S^2) \xrightarrow{\gamma} H^2(U) \oplus H^2(V) \to ....$$

As $H^k(U) = H^k(V) = 0$ for $k > 0$, and β is injective $\Rightarrow H^1(S^2) = 0$. Furthermore, as $\alpha = 0 \Rightarrow d^*$ is injective. Also, as $\gamma = 0 \Rightarrow d^*$ is also surjective and thus an isomorphism. Thus $H^2(S^2) \cong \mathbb{R}$, and $H^k(S^2) = 0$ for all $k > 2$.
This leads us to make the following hypthesis: $H^k(S^n) = \mathbb{R}$ for $k = 0, n$ and is 0 otherwise. We will prove this inductively. Clearly it holds for $k = 0, 1, 2$. Assume it holds for all $n < k$. For S^k, consider a point $p \in S^k$ and an open cover $\{S^n/\{p\} \cong \mathbb{R}^n, V\}$, where V is an open disk $\text{int}(D^n)$ containing p. Thus we have that V is contractible and $U \cap V \sim S^{n-1}$. As U, V and $U \cap V$ are all connected, we may start the Mayer-Vietoris sequence from $H^1(S^k)$. Now as $H^l(S^{n-1}) = 0$ for $l < n - 1$, and $H^l(U) = H^l(V) = 0$ by the inductive hypothesis, each row of the exact sequence can be separated into a exact sequence:

$$0 \to H^l(S^k) \to 0,$$

where $l < n$. This trivially implies that $H^l(S^k) = 0$ for $l < k$. For $l = k$, we have the exact sequence:

$$0 \to H^{k-1}(S^{k-1}) \xrightarrow{d^*} H^k(S^k) \to 0 \to ...$$

Clearly, by exactness, d^* is an isomorphism and thus $H^k(S^k) \cong H^{k-1}(S^{k-1}) \cong \mathbb{R}$.

Problem 28.3 Cohomology of a multiply punctured plane
Let $p_1, ..., p_n$ be distinct points in $\mathbb{R}^2$. Compute the de Rham cohomology of $\mathbb{R}^2 - \{p_1, ..., p_n\}$.

Solution: Consider the doubly punctured plane $\mathbb{R}^2 - \{p, q\}$. Cover $\mathbb{R}^2 - \{p\}$ with $\{U = \mathbb{R}^2 - \{p, q\}, V\}$, where V is an open disc around q. Furthermore, $U \cap V \sim S^1 \sim U \cap V$, and V is contractible. Clearly U, V and $U \cap V$ are all connected, and hence we can start the Mayer-Vietoris sequence from $H^1(\mathbb{R}^2 - \{p\}) \cong H^1(S^1) \cong \mathbb{R}$ to get:

$$0 \to \mathbb{R} \xrightarrow{\beta} H^1(U) \xrightarrow{\alpha} \mathbb{R} \xrightarrow{d^*} 0 \to H^2(U) \to 0 \to ...$$

where every row for $k > 1$ becomes $0 \to H^k(U) \to 0$. Thus we have that $H^k(U) = 0$ for $k > 1$. For $k = 1$, we can see that as β is injective, $\dim \mathit{ker}\, \alpha = 1$. Furthermore as α is surjective, dim im $\alpha = 1$. Thus by the rank-nullity theorem, $\dim H^1(U) = 2$.

Thus $H^1(\mathbb{R}^2 - \{p, q\}) \cong \mathbb{R}^2$.
This leads us to form the following hypothesis: $H^1(\mathbb{R}^2 - \{p_1, ..., p_n\}) \cong \mathbb{R}^n$, and $H^k(\mathbb{R}^2 - \{p_1, ..., p_n\}) = 0$ for $k > 1$ (it is obvious that $H^0(\mathbb{R}^2 - \{p_1, ..., p_n\}) \cong \mathbb{R}$ by connectedness). We prove this inductively. This clearly holds for $n = 1, 2$ as before. Assume it holds for all $n < k$. Now for $n = k$, let $\mathbb{R}^2 - \{p_1, ..., p_{k-1}\}$ be covered by $\{\mathbb{R}^2 - \{p_1, ..., p_k\}, V\}$ where V is an open disc around p_k. Again, V is contractible and $U \cap V \sim S^1$. Furthermore, as U, V, $U \cap V$ are all connected, we can start the Mayer-Vietoris sequence as:

$$0 \to H^1(\mathbb{R}^2 - \{p_1, ..., p_{k-1}\}) \cong \mathbb{R}^{k-1} \xrightarrow{\beta} H^1(\mathbb{R}^2 - \{p_1, ..., p_k\}) \xrightarrow{\alpha} H^1(S^1) \cong \mathbb{R} \to 0 \to ...$$

where it is clear that $H^k(\mathbb{R}^2 - \{p_1, ..., p_k\}) = 0$ for $k > 1$. Furthermore, as β is injective, $\dim ker\, \alpha = k-1$, while dim im $\alpha = 1$ by surjectivity. Thus $H^1(\mathbb{R}^2 = \{p_1, ..., p_k\} \cong \mathbb{R}^k$.

Problem 28.4 Cohomology of a surface of genus g
Compute the cohomology vector space of a compact orientable surface $\sum_g$ of genus g.

Solution: We hypothesize that $H^k(\sum_n) = 0$ for $k > 2$, $H^0(\sum_n) = H^2(\sum_n) = \mathbb{R}$ and $H^1(\sum_n) = \mathbb{R}^{2n}$. We prove this inductively. This clearly holds for $n = 1, 2$. Assume it holds for all $n < k$. Now consider covering $\{U = \sum_{k-1} - \{p\}, V\}$ of $\sum_{k-1}$, where $p \in \sum_{k-1}$ and V is an open disk around it. Note that V is contractible, and that $U \cap V \sim S^1$. Furthermore, as U, V and $U \cap V$ are all connected, we can start the Mayer-Vietoris sequences as follows:

$$0 \to \mathbb{R}^{2k-2} \xrightarrow{\alpha} H^1(U) \xrightarrow{i^*} \mathbb{R} \xrightarrow{d^*} \mathbb{R} \xrightarrow{\gamma} H^2(U) \to 0 \to 0 \to H^3(U) \to 0 \to ...$$

As i^* is restriction onto S^1, we know that i^* is the zero map $\Rightarrow \alpha$ is surjective as well as injective. Thus $H^1(U) \cong \mathbb{R}^{2k-2}$. Furthermore, d^* is injective, and hence a surjection as well$\Rightarrow \gamma$ is the zero map. This implies $H^2(U) = 0$. We can also see that trivially, $H^l(U) = 0$ for all $l > 2$ as well.
Now we proceed to calculate the cohomology of $\sum_k$. Cover it with $\{U, V\}$ where U consists of one genus of the torus, and V covers the rest $k - 1$ genuses. Note that this way $U \cap V \sim S^1$ and $U \sim \sum_1 - \{p\}$, and $V \sim \sum_{k-1} - \{q\}$. Furthermore, as U, V, and $U \cap V$ are all connected, we can start the Mayer-Vietoris sequence as:

$$0 \to H^1(\sum{}_k) \xrightarrow{\alpha} \mathbb{R}^{2k-2} \oplus \mathbb{R}^2 \xrightarrow{i^*} \mathbb{R} \xrightarrow{\beta} H^2(\sum{}_k) \to 0 \to 0 \to H^3(\sum{}_k) \to 0 \to ...$$

As i^* is the zero map, α is an isomorphism and hence $H^1(\sum{}_k) \cong \mathbb{R}^{2k}$. Furthermore β is an isomorphism too, giving $H^2(\sum{}_k) \cong \mathbb{R}$. We can trvially see that $H^l(\sum{}_k) = 0$ for $l > 2$.

7.6 Proof of Homotopy Invariance

Homotopy Axiom for de Rahm Cohomology

$f, g : M \to N$ *are smoothly homotopic* $\Rightarrow f^{\#} = g^{\#}$

$\uparrow$ *finally proves*

induced maps in cohomology are equal $i_0^{\#} = i_1^{\#}$

$\uparrow$ *implies*

$d \circ K + K \circ d = i_1^* - i_0^*$ *with*

$i_j : M \to M \times \mathbb{R}$ *s.t* $i_j = (\mathbb{1}, j)$

$\uparrow$ *is a cochain homotopy*

$K : \Omega^*(M \times \mathbb{R}) \to \Omega^{*-1}(M)$

with $K(f\pi^*\eta) = 0$ *and* $K(f dt \wedge \pi^*\eta) = \eta \int_0^1 f dt$

and linearity over locally finite sums

$\uparrow$ *implies the uniqueness and well-definedness of*

Every smooth form on $M \times \mathbb{R}$ *is a locally finite sum of the form:*

$$\sum_\alpha f_\alpha(x,t)\pi^*\eta_\alpha + \sum_\beta g_\beta(x,t) dt \wedge \pi^*\zeta_\beta$$

where f_α *and* g_β *are smooth on* $M \times \mathbb{R}$ *and* η_α *and* ζ_β *is a smooth form on* M

Problem 29.2 Linearity of pullback over locally finite sums
Let $h : N \to M$ *be a* C^∞ *map, and* $\sum \omega_\alpha$ *a locally finite sum of* C^∞ *k-forms on M. Prove that* $h^*(\sum \omega_\alpha) = \sum h^*\omega_\alpha$.

Solution: Consider any point $p \in N$. Then $h(p)$ contains a nbhd. V such that $\sum \omega_\alpha$ is a finite sum over V. As h is $C^\infty \Rightarrow h^{-1}(V)$ is an open nbhd. of p. Now we have:

$$\begin{aligned}
\left(h^*\left(\sum \omega_\alpha\right)\right)\Big|_{h^{-1}(V)} &= h^*\left(\left(\sum \omega_\alpha\right)\Big|_V\right) \\
&= \sum (h^*\omega_\alpha|_V) \\
&= \sum ((h^*\omega_\alpha)|_{h^{-1}(V)}) \\
&= \left(\sum \left(h^*\omega_\alpha\right)\right)\Big|_{h^{-1}(V)}.
\end{aligned}$$

Thus as every point $p \in M$ has a nbhd. where $h^*(\sum \omega_\alpha) = \sum h^*\omega_\alpha$, the equality holds over all of N.

8 Acknowledgements and Final Note

There were also some extra chapters in the appendix of the book, for instance on the Symplectic Group, which I did study but were not included in this report.

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References

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