PHY 100 Home Lab Experiment 2: Kepler's Laws and Orbits

Abdullah Ahmed (2024-10-0035)
Instructor: Kaneez Amna
Lahore University of Management Sciences
Physics Department
(Dated: April 24, 2021)

Abstract: Kepler's Three Laws of Planetary Motion explain phenomenon related to the orbital motion of planets, and form the basis for any modern day planetary study. The report will aim to prove some of these laws, use them to simulate the orbital motion of the Moon around the Earth, and talk about certain hypothetical situations regarding prediction of asteroid trajectories.

I. KEPLER'S FIRST LAW AND INTRODUCING ELLIPSES

Ellipses are defined by the loci of points whose sum of the distances r_1 and r_2 from it's respective foci F_1 and F_2 has a certain constant value (example drawn below). The lengths labelled a and b are the *semi-major* and *semi-minor* axes respectively. The distance between the center and any of the foci is labelled as c.

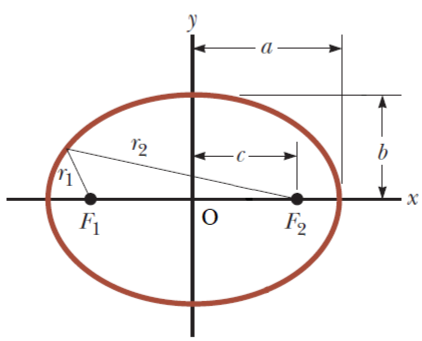


Fig 1.1 A labelled ellipse

Kepler's First Law states that, in contrast to what was thought at his time, planetary orbits that are bound to the gravitational force center are not generally circular, but instead, they are elliptical. It also states that the Sun was only situated at one focus of the ellipse, while the other one didn't hold any significance. The circular orbit was a special case of *Kepler's First law*, where the foci coincided at the center.
The general equation of the ellipse centered around the origin in Cartesian Coordinates is given by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

We will now derive some properties regarding the ellipse that will be useful when applying them to elliptically orbiting planetary bodies.

Firstly, we define a quantity ϵ that is to be referred to as the *eccentricity* of an ellipse, and is equal to $\sqrt{1 - \frac{b^2}{a^2}}$. It follows that:

$$a^2(1 - \epsilon^2) = b^2 \qquad (1)$$

Next we shall define two points of special interest on the ellipse, known as the *perihelion* and *aphelion*. Choosing one focus of interest (usually the one with the central body), say F_1, the *perihelion* is the point closest to F_1 on the ellipse, while the *aphelion* is the point farthest away from F_1 on the ellipse.

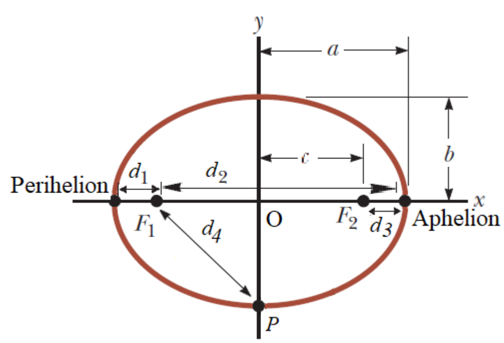


Fig 1.2 *Perihelion* and *Aphelion* of an ellipse

Analyzing the above ellipse at the *Aphelion*:

$$d_2 + d_3 = constant = (a - c) + (a + c) = 2a \qquad (2)$$

. Now considering point P, we see that PF_1=PF_2 since $\triangle F_1PF_2$ is isosceles. Combining this with (2):

$$d_4 = a \qquad (3)$$

Now considering $\triangle F_1PO$, that is a right angled triangle, it follows by the *Pythagorean Theorem* and (3):

$$a^2 = b^2 + c^2 \qquad (4)$$

Substituting (1) into (4):

$$a^2 = a^2(1 - \epsilon^2) + c^2$$

$$c = a\epsilon \qquad (5)$$

It follows from (5) that:

$$d_1 = a(1-\epsilon)$$

$$d_2 = a(1+\epsilon)$$

Therefore, after combining with (1) and (2):

$$b^2 = d_1 d_2 \qquad (6)$$

$$a = \frac{d_1 + d_2}{2} \qquad (7)$$

. These two results will turn out to be imperative when discussing Kepler's Third Law.

II. PROOF OF KEPLER'S SECOND AND THIRD LAWS

If the system to be considered consists only of the orbiting planet and the Sun (or, generally speaking, any central body), it is easy to see that the only force present in the system is the gravitational attraction between the two bodies, which is an internal force in the system. This implies that since there is no external force on the system, energy must be conserved. Furthermore, as the internal force is a central force that always acts along the radial direction, it induces no torque as $|\vec{F_g} \times \vec{r}| = 0$. Thus, *angular momentum* (denoted by $\vec{L}$) must also be conserved.

Kepler's Second Law states that in an elliptical orbit, the orbiting body sweeps out equal amounts of area in equal time. This statement easily follows from conservation of angular momentum. Consider the following diagram: The planet with mass m orbiting the Sun with

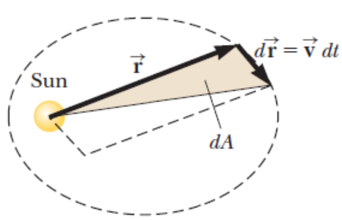


Fig 2.1 Area swept by planet is half of the outlined parallelogram

mass M sweeps an area dA in a time interval dt. For an infinitesimal time interval, $\vec{dr}$ can be considered a straight line. Hence, the triangular area dA can be taken as half of the area of the parallelogram defined by sides $\vec{r}$ and $\vec{dr}$. It follows thus:

$$dA = \frac{1}{2}|\vec{r} \times \vec{dr}|$$

$$dA = \frac{1}{2}|\vec{r} \times \vec{v}dt|$$

$$\frac{dA}{dt} = \frac{1}{2}|\vec{r} \times \vec{v}|$$

$$\frac{dA}{dt} = \frac{1}{2m}|\vec{r} \times \vec{p}|$$

$$\frac{dA}{dt} = \frac{\vec{L}}{2m} = constant \qquad (8)$$

This proves *Kepler's Second Law.*

Kepler's Third Law is perhaps the most useful law as far as practical applicability is concerned. It states that $T^2 \propto a^3$, where T is the *time period* of the elliptical orbit, and a is the *semi-major axis*. We shall use energy and angular momentum conservation, along with previously derived quantities to prove this mathematically.

Consider the conserved quantities at the *aphelion* and *perihelion* as in Fig 1.2. We have:

$$L = mv_1 d_1 = mv_2 d_2 \qquad (9)$$

$$E = \frac{mv_1^2}{2} - \frac{GmM}{d_1} = \frac{mv_2^2}{2} - \frac{GmM}{d_2} \qquad (10)$$

Rearranging (10) and dividing by m eventually gives us:

$$\frac{v_1^2 - v_2^2}{2} = GM(\frac{1}{d_1} - \frac{1}{d_2})$$

$$\frac{(v_1 - v_2)(v_1 + v_2)}{2} = GM(\frac{1}{d_1} - \frac{1}{d_2}) \qquad (11)$$

Now using the value of v_1 and v_2 as $\dfrac{L}{md_1}$ and $\dfrac{L}{md_2}$ respectively from (9) in (11):

$$(\frac{1}{2})(\frac{L^2}{m^2})(\frac{1}{d_1} - \frac{1}{d_2})(\frac{1}{d_1} + \frac{1}{d_2}) = GM(\frac{1}{d_1} - \frac{1}{d_2})$$

$$(\frac{L^2}{2m^2})(\frac{1}{d_1} + \frac{1}{d_2}) = GM$$

$$(\frac{L^2}{2m^2}) = GM(\frac{d_1 d_2}{d_1 + d_2})$$

$$(\frac{L^2}{4m^2}) = (\frac{GM}{2})(\frac{d_1 d_2}{d_1 + d_2}) \qquad (12)$$

Note that the LHS is nothing but $(\dfrac{dA}{dt})^2$ from (8), and using the fact that the total area of an ellipse is known to be πab, we get from (12):

$$(\frac{dA}{dt})^2 = (\frac{GM}{2})(\frac{d_1 d_2}{d_1 + d_2})$$

$$(\frac{\pi ab}{T})^2 = (\frac{GM}{2})(\frac{d_1 d_2}{d_1 + d_2})$$

$$T^2 = (\frac{2(\pi ab)^2}{GM})(\frac{d_1 + d_2}{d_1 d_2}) \quad (13)$$

where T is the *time period* of the elliptical orbit. Finally, we can use the relations derived before in (6) and (7) to see that:

$$\frac{2a}{b^2} = \frac{d_1 + d_2}{d_1 d_2} \quad (14)$$

Substituting (14) into (13):

$$T^2 = (\frac{2(\pi ab)^2}{GM})(\frac{2a}{b^2})$$

$$T^2 = (\frac{4\pi^2 a^3}{GM})$$

This proves *Kepler's Third Law.*

III. MOON'S ORBIT AROUND THE EARTH

The trajectory of any body in an orbit is defined by its *total energy* that has two components: the *kinetic energy* and the *potential energy.* As the *potential energy* is strictly dependent on inherent parameters like the masses of the bodies interacting, the initial *kinetic energy* when entering the orbit is the more significant factor in deciding whether the orbit will be *circular, elliptical, hyperbolic* or *parabolic.* A total energy $E < 0$ implies a *bound orbit*, while an *unbound orbit* is characterized by $E \geq 0$. Assuming a circular orbit for the moon with mass M_m around the earth with mass M_E would require using the equation for *centripetal motion*, that is:

$$F_g = \frac{M_m v_0^2}{r}$$

$$\frac{GM_E M_m}{r^2} = \frac{M_m v_0^2}{r}$$

$$v_0 = \sqrt{\frac{GM_E}{r}} \approx 1018ms^{-1} \quad (15)$$

Running a simulation for the orbit at this initial speed gives Fig 3(a) (next page). It assumes the Earth to be at the *origin* and the Moon to start at a point horizontally beside it.
As we can see, its gives a perfectly circular orbit inline with the the theory. As we increase the initial velocity, it is expected that the orbit becomes more skewed, that is, elliptical. This is because the *total energy* of the orbit changes, thus changing the shape of the orbit. This can be seen from Fig 3(b) and Fig 3(c) with speeds slightly higher and lower respectively.
It is also possible for the object to reach what is known as the *escape velocity*, where the *kinetic energy* becomes higher than the *potential energy*, and thus the *total energy* is positive. This characterizes an *unbound orbit*, where the object only enters into the orbit momentarily before moving away and never re-entering. For example, the *escape velocity* of the Moon in Earth's orbit can be calculated as follows:

$$K.E + P.E > 0$$

$$\frac{M_m v^2}{2} - \frac{GM_E M_m}{r} > 0$$

$$\frac{M_m v^2}{2} > \frac{GM_E M_m}{r}$$

$$v > \sqrt{\frac{2GM_E}{r}}$$

$$v > \sqrt{2}v_0 \approx 1.41v_0 \approx 1440ms^{-1}$$

The unbound orbit at $v = 1.45v_0$ is clearly indicated in Fig 3(d).

IV. ENERGY DISTRIBUTION IN ELLIPTICAL ORBITS

Although we have touched upon the fact that the *total energy* in an elliptical orbit must be conserved as there is no external force acting on the system, we will now delve into how different forms of energy change during an orbit.
Using the same conservation relations as in (10), we get that:

$$2E = \frac{mv_1^2}{2} - \frac{GmM}{d_1} + \frac{mv_2^2}{2} - \frac{GmM}{d_2}$$

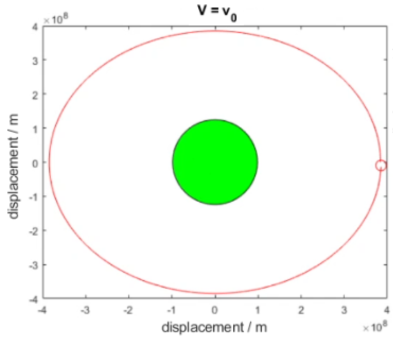
Using again v_1 and v_2 as $\frac{L}{md_1}$ and $\frac{L}{md_2}$ respectively from (9):

$$2E = (\frac{L^2}{2m})(\frac{1}{d_1^2} + \frac{1}{d_2^2}) - GmM(\frac{1}{d_1} + \frac{1}{d_2})$$

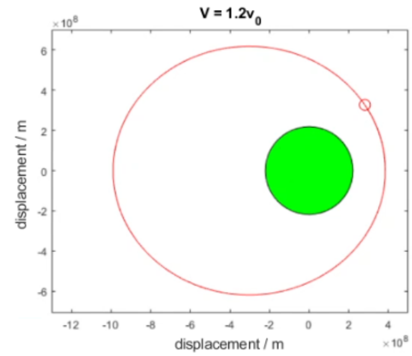
$$2E = (\frac{L^2}{2m})(\frac{d_1^2 + d_2^2}{d_1^2 d_2^2}) - GmM(\frac{d_1 + d_2}{d_1 d_2})$$

$$2E = (\frac{L^2}{2m})(\frac{(d_1 + d_2)^2 - 2d_1 d_2}{d_1^2 d_2^2}) - GmM(\frac{d_1 + d_2}{d_1 d_2})$$

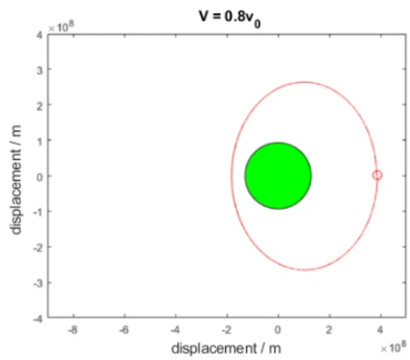
$$2E = (\frac{L^2}{2m})(\frac{(d_1 + d_2)^2}{d_1^2 d_2^2} - \frac{2}{d_1 d_2}) - GmM(\frac{d_1 + d_2}{d_1 d_2})$$



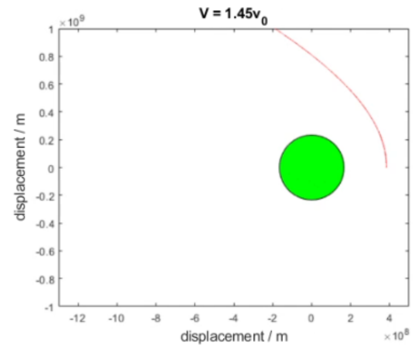
(a) Moon's orbit with speed v_0



(b) Moon's orbit with speed $1.2v_0$



(c) Moon's orbit with speed $0.8v_0$



(d) Moon's *unbound* orbit with speed $1.45v_0$

Fig 3 Simulation of Moon's orbit

Now substituting from (12) for the angular momentum term, and from (6) and (7) in a single step:

$$2E = GMm(\frac{b^2}{2a})((\frac{2a}{b^2})^2 - \frac{2}{b^2}) - GmM(\frac{2a}{b^2})$$

$$E = \frac{-GMm}{2a} \qquad (16)$$

Now that the *total energy* of the system has been calculated, it is easy to compute the distribution of *kinetic energy* and *potential energy* at say, position (x, y) where the focus containing the central body is at $(0, 0)$.

$$P.E = \frac{-GMm}{r}$$

$$K.E = E - P.E$$

where $r = \sqrt{x^2 + y^2}$.
Analyzing the formulas allows to confirm our intuition that as the planet comes closer to the central body, it speeds up as its $P.E$ decreases and $K.E$ increases. Similarly, as it moves farther away from the central body, it slows down as its $P.E$ increases and $K.E$ decreases.

V. HALLEY'S COMET EXAMPLE

Although *conservation of energy* allows us to calculate values of velocity for the orbiting body at different parts of the orbit, it is usually more convenient to utilize *conservation of angular momentum.* A great example to illustrate the ease of calculation is *Halley's Comet*, which elliptically orbits the Sun with a time period of *75 years*. Consider we know that it has a velocity of $5.45 \times 10^4 ms^-1$ at its *perihelion*, which is $8.78 \times 10^{10} m$ away from the Sun. We can use conservation of angular momentum to calculate it's velocity at the *aphelion*, which is $5.28 \times 10^{12} m$ away:

$$\vec{L_p} = |\vec{r_p} \times \vec{p_p}| = \vec{L_a} = |\vec{r_a} \times \vec{p_a}|$$

$$m|\vec{r_p}||\vec{v_p}|sin(\theta_p) = m|\vec{r_a}||\vec{v_a}|sin(\theta_a)$$

Since the *radial vector* and *velocity vector* are perpendicular to each other at the *aphelion* and *perihelion*, we get:

$$|\vec{r_p}||\vec{v_p}| = |\vec{r_a}||\vec{v_a}|$$

$$8.78 \times 10^{10} \times 5.45 \times 10^4 = 5.28 \times 10^{12} v_a$$

$$v_a \approx 906 ms^-1$$

VI. ASTEROID TRAJECTORY PREDICTION

Let us consider the following scenario: An asteroid at a perpendicular distance $3R_E(R_E = \text{radius of the Earth})$ from the center of the earth was detected far away (negligible P.E assumed) with a certain velocity $\vec{v_1}$. Is it possible to calculate the minimum value v_{min} of $|\vec{v_1}|$ such that it does not hit the earth? It is easy to see that at such a value, the trajectory of the meteorite will be as such that it would just pass the center of the earth at a distance R_E as shown in Fig 6.1.

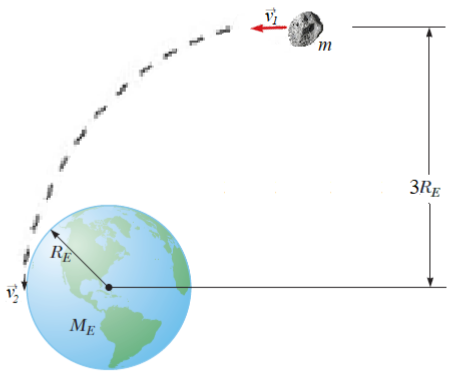


Fig 6.1 Meteorite Trajectory

Now that we know the trajectory, we will use *conservation of angular momentum* and *energy* at the *initial* and *impact* positions to find v_{min}:

$$|\vec{L_1}| = mv_{min}(3R_E) = |\vec{L_2}| = mv_2(R_E)$$

$$3v_{min} = v_2 \qquad (17)$$

We also have from *conservation of energy*:

$$(\frac{1}{2})(mv_{min}^2) = (\frac{1}{2})(mv_2^2) - \frac{GM_Em}{R_E}$$

Using (17) in above, dividing by m throughout, and rearranging:

$$(\frac{1}{2})(v_{min}^2 - (3v_{min})^2) = -\frac{GM_E}{R_E}$$

$$-4v_{min}^2 = -\frac{GM_E}{R_E}$$

$$v_{min} = \sqrt{(\frac{GM_E}{4R_E})} \approx 3951ms^{-1}$$

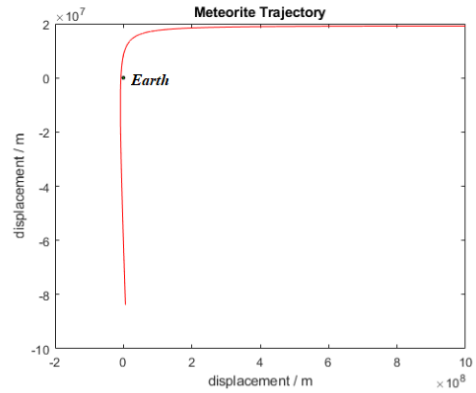
Total energy of the system at $v_1 = v_{min}$:

$$E = K.E + P.E$$

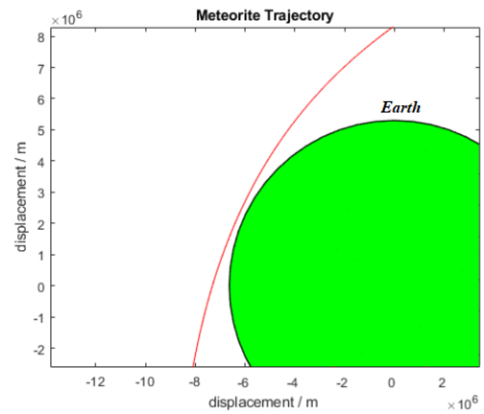
$$E = \frac{1}{2}m(\sqrt{(\frac{GM_E}{4R_E})}^2)$$

$$E = \frac{GM_Em}{8R_E} > 0 \qquad (18)$$

Hence the orbit will be *unbound*, and have a hyperbolic shape. The simulation in 6.2 confirms the hyperbolic trajectory at $v_1 = v_{min}$.



(a) Meteorite hyperbolic trajectory



(b) Zoomed in to show close impact

Fig 6.2 Meteorite Simulation

VII. LIMITATIONS

It must be clarified that it is *not* the case that every orbiting system has a *fixed* central body. By *Newton's Third Law*, the central body *must* be experiencing an equal gravitational force that is opposite in direction to the force the orbiting body experiences. As such, the system is not a *one body problem.* Nevertheless, assuming a fixed central body is a good approximation when they are much heavier than the orbiting bodies. When the approximation is not justified, we have to consider the *reduced mass* for the *two body problem*, which is given by:

$$\mu = \frac{Mm}{M+m}$$

where the both bodies orbit the *barycenter* of the system which is given by their *center of mass.*