MATH 309: Introduction to Analysis II

Equivalence of the Completeness of $\mathbb{R}$ and the Bolzano-Weierstrass Theorem

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2024-10-0035

November 19, 2022

Bolzano-Weierstrass Theorem $\Rightarrow$ Completeness of $\mathbb{R}$

We must first prove that the Bolzano-Weierstrass Theorem $\Rightarrow$ Archimedian property. This is an essential result that is usually proven using Completeness of $\mathbb{R}$. To do this, assume on the contrary that $\mathbb{N}$ is bounded. Then the sequence $x_n = n$ is a bounded sequence. By BWT, there exists a subsequence $x_{n_k} = n_k$ that is convergent to some $r \in \mathbb{R}$. Note that $n_{k+1} \geq n_k + 1$ for all $k \in \mathbb{N}$ (this property can be deduced from the order on $\mathbb{N}$ and without assuming the Archimidean property). Take $\epsilon = 0.5$. Then there exists a $k' \in \mathbb{N}$ such that $n_k \in (r - \epsilon, r + \epsilon)$ for all $k > k'$. This implies that $n_{k'+2} - n_{k'+1} < 2\epsilon = 1 \Rightarrow n_{k'+2} < n_{k'+1} + 1$. This is in contradiction to the property from before. Thus $\mathbb{N}$ is unbounded. Moreover, this gives us that $\dfrac{1}{2^n} \to 0$ and that $\sum_{i=1}^{\infty} \dfrac{1}{2^i} = 1$ as usual. We will now proceed with the main proof.

Consider any set $A \subseteq \mathbb{R}$ that is bounded $\Rightarrow$ there exists $a \in \mathbb{R}$ such that a is not an upper bound of A, but $a + 1$ is. It is assumed that A does not have a maximal element; otherwise, $\sup(A) = \max(A) \in \mathbb{R}$ trivially. Consider the construction of the following sequence:

$$x_n = \begin{cases} a & \text{if } n = 0 \\ x_{n-1} + \frac{1}{2n} & \text{if } x_{n-1} \text{ is not an upper bound of } A \\ x_{n-1} - \frac{1}{2n} & \text{if } x_{n-1} \text{ is an upper bound of } A \end{cases}$$

Firstly, we shall prove that $\{x_n\}$ is a Cauchy sequence. Consider any $\varepsilon > 0 \Rightarrow$ there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{2^n} < \varepsilon$ for all $n \geq n_0 \Rightarrow \sum_{l=n+1}^{\infty} 2^{-l} = \frac{1}{2^n} < \varepsilon$ for all $n > n_0$. We can see that $|x_n - x_m| \leq \sum_{l=n+1}^{m} 2^{-l} < \sum_{l=n+1}^{\infty} 2^{-l} < \varepsilon$, where $m > n > n_0 \Rightarrow \{x_n\}$ is a Cauchy sequence.

By the assumption of the Bolzano-Weierstrass Theorem, x_n is convergent, and $\{x_n\} \to r$ for some $r \in \mathbb{R}$.

Proposition 1: If $\{x_n\}$ is strictly increasing, then $\sup(A) = (a + 1) \in \mathbb{R}$. If not, then there does not exist $m \in \mathbb{N}$ such that $\{x_n\}$ is strictly increasing for all $n > m$.

Proof: Assume the sequence is strictly increasing $\Rightarrow x_n$ is not an upper bound of A for all $n \in \mathbb{N}$, and thus $x_n = a + \sum_{l=1}^{n} 2^{-l} \Rightarrow x_n \to (a + 1)$. We already have that $(a + 1)$ is an upper bound of A by assumption. Clearly now, for any $\varepsilon > 0$, there exists $\bar{n} \in \mathbb{N}$ such that $(a + 1) - x_n < \varepsilon$ for all $n > \bar{n}$, where x_n are not upper bounds of $A \Rightarrow [(a + 1) - \varepsilon]$ is not an upper bound of A for all $\varepsilon > 0 \Rightarrow \sup(A) = (a + 1) \in \mathbb{R}$.

Assume now that it is not strictly increasing, and on the contrary, there exists $m \in \mathbb{N}_{>1}$ such that $\{x_n\}$ is strictly increasing for all $n > m$. Thus, we have there exists $\tilde{m} \in \mathbb{N}$ such that $\{x_n\}$ is strictly increasing for all $n > \tilde{m}$ and $x_{\tilde{m}-1} > x_{\tilde{m}}$. That is, there is an index where the sequence starts to become strictly increasing. This can be argued as $m < \infty$ by assumption.

This gives us that $x_{\tilde{m}-1}$ is an upper bound, and that $x_{\tilde{m}} = x_{\tilde{m}-1} - \dfrac{1}{2^{\tilde{m}}}$. Also, as $x_{\tilde{m}-1}$ is not equal to $\sup(A)$ by assumption, we have that there exists an upper bound $\tilde{a}$ such that $x_{\tilde{m}} < \tilde{a} < x_{\tilde{m}-1} \Rightarrow x_{\tilde{m}-1} - \dfrac{1}{2^{\tilde{m}}} < \tilde{a} < x_{\tilde{m}-1}$. As the sequence is now strictly increasing $\Rightarrow x_n = x_{\tilde{m}} + \sum_{l=\tilde{m}+1}^{n} 2^{-l}$ where $n > m \Rightarrow x_n \to \left(x_{\tilde{m}} + \dfrac{1}{2^{\tilde{m}}}\right) = x_{\tilde{m}-1} \Rightarrow$ there exists $\tilde{n} \in \mathbb{N}_{>\tilde{m}}$ such that $\tilde{a} < x_{\tilde{n}} < x_{\tilde{m}-1}$, and $x_{\tilde{n}}$ is not an upper bound as the sequence is now strictly increasing. This is a contradiction as $\tilde{a}$ is an upper bound. Thus, the sequence never becomes strictly increasing.

Proposition 2: There does not exist $m \in \mathbb{N}$ such that $\{x_n\}$ is strictly decreasing for all $n > m$.

Proof: Assume on the contrary such an m exists $\Rightarrow$ there exists $\tilde{m} \in \mathbb{N}$ such that $\{x_n\}$ is strictly decreasing for all $n > \tilde{m}$ and $x_{\tilde{m}-1} < x_{\tilde{m}} \Rightarrow x_{\tilde{m}-1}$ is not an upper bound, and $x_{\tilde{m}} = x_{\tilde{m}-1} + \frac{1}{2^{\tilde{m}}} \Rightarrow$ there exists $\tilde{a} \in A$ such that $x_{\tilde{m}} > \tilde{a} > x_{\tilde{m}-1}$.
As the sequence is strictly decreasing now, we can use a similar argument to the one in Proposition 1 $\Rightarrow x_n \to x_{\tilde{m}-1} \Rightarrow$ there exists $\tilde{n} \in \mathbb{N}_{>\tilde{m}}$ such that $a > x_{\tilde{n}} > x_{\tilde{m}-1}$, and $x_{\tilde{n}}$ is an upper bound as the sequence is strictly decreasing now. This is a contradiction as $\tilde{a} \in A$. Hence, the sequence never becomes strictly decreasing.

Proposition 3: If x_m is an upper bound $\Rightarrow x_n < x_m$ for all $n > m$.

Proof: Clearly $x_{m+1} = x_m - \frac{1}{2m+1}$. Now we also have $x_{(m+1)+k} \leq x_{m+1} + \sum_{l=m+2}^{m+1+k} 2^{-l} < x_{m+1} + \sum_{l=m+2}^{\infty} 2^{-l} = x_{m+1} + \frac{1}{2m+1} = x_m \Rightarrow x_n < x_m$ for all $n > m$.

Proposition 4: If x_m is not an upper bound $\Rightarrow x_n > x_m$ for all $n > m$.

Proof: The proof follows from an almost identical argument to Proposition 3.

Now, using Propositions 1 and 2, we can build the following subsequences:

$$x_{n_k} = \text{subsequence of all upper bounds in } x_n$$

$$x_{n_l} = \text{subsequence of all non-upper bounds in } x_n$$

as we can always find elements of $\{x_n\}$ that are upper bounds or non-upper bounds due to $\{x_n\}$ never becoming a strictly increasing or a strictly decreasing sequence. Both of these are also respectively strictly decreasing and strictly increasing subsequences by Propositions 3 and 4. As $x_n \to r \Rightarrow x_{n_k}, x_{n_l} \to r$.

Proposition 5: r is an upper bound of A.

Proof: Assume it is not $\Rightarrow$ there exists $a_0 \in A$ such that $a_0 > r$. Take $\varepsilon = a_0 - r$. By convergence of x_{n_k}, there exists $k_0 \in \mathbb{N}$ such that $x_{n_k} - r < a_0 - r$ for all $k > k_0 \Rightarrow x_{n_k} < a_0$. This is a contradiction as x_{n_k} is an upper bound of A. Hence, r is an upper bound of A.

Now consider any $\varepsilon > 0$. We have that there exists $l_0 \in \mathbb{N}$ such that $r - x_{n_l} < \varepsilon$ for all $l > l_0 \Rightarrow x_{n_l} > r - \varepsilon$. Thus as x_{n_l} is not an upper bound of A, $r - \varepsilon$ is not either for all $\varepsilon > 0$. Combining this with Proposition 5 that r is an upper bound of A, we finally have that $r = \sup(A) \in \mathbb{R}$. Thus, $\mathbb{R}$ is complete.

Completeness of $\mathbb{R} \Rightarrow$ Bolzano-Weierstrass Theorem

Proof: Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}$ such that $|x_n| < M$ for all $n \in \mathbb{N}$. This implies $\sup\{x_n\} = r \in \mathbb{R}$ exists.

Case 1: $r \neq x_n$ for all $n \in \mathbb{N}$.

Consider the construction of the following subsequence: Take $x_{n_1} = x_k$ for any $k \in \mathbb{N}$. Set $\delta_1 = \dfrac{\min\{|x_i - r|\}}{2}$ where $1 \leq i \leq n_1$. We know that $\delta_1 > 0$ as $r > x_n$ for all $n \in \mathbb{N}$ by assumption. Furthermore, as $r = \sup\{x_n\}$, there exists $\tilde{n} \in \mathbb{N}$ such that $|x_{\tilde{n}} - r| < \delta_1$. Clearly, by construction, $\bar{n} > n_1$. Set $x_{n_2} = x_{\tilde{n}}$. Set $\delta_2 = \dfrac{\min\{|x_i - r|\}}{2} \leq \delta_1/2$ where $1 \leq i \leq n_2$, and repeat to build a subsequence x_{n_k}. We can see that $|x_{n_k} - r| < \delta_{k-1}$ for all $k \in \mathbb{N}$. But $\delta_k \leq \dfrac{\delta_{k-1}}{2}$ for all $k \in \mathbb{N} \Rightarrow \delta_k \to 0 \Rightarrow x_{n_k} \to r$.

Case 2: $r = x_m$ for some $m \in \mathbb{N}$.

Sub-Case 1: there exists $\tilde{n} \in \mathbb{N}$ such that $\sup\{x_n : n > \tilde{n}\} \notin \{x_n : n > \tilde{n}\}$.

Repeat Case 1 for the sequence x_n where $n > \tilde{n}$ to find a subsequence x_{n_k} that converges to $\sup\{x_n : n > \tilde{n}\} \in \mathbb{R}$ as $\{x_n : n > \tilde{n}\}$ is also bounded.

Sub-Case 2: For all $\tilde{n} \in \mathbb{N}$, $\sup\{x_n : n > \tilde{n}\} \in \{x_n : n > \tilde{n}\}$.

Consider the following subsequence: Take $x_{n_1} = x_m$. Set $x_{n_2} = \sup\{x_n : n > n_1(= m)\}$. Thus $x_{n_2} \in \{x_n : n > n_1\}$. Set $x_{n_3} = \sup\{x_n : n > n_2\}$. Thus $x_{n_3} \in \{x_n\, n > n_2\}$. Similarly, $x_{n_k} = \sup\{x_n : n > n_{k-1}\}$, where $x_{n_k} \in \{x_n\, n > n_{k-1}\}$. This is a monotonically non-increasing subsequence in $\{x_n\}$ which is bounded below. Thus x_{n_k} is convergent by the Monotone Convergence Theorem.

Thus, there always exists a subsequence x_{n_k} of x_n that converges.■