Symplectic topology and Toric Varieties

Part II: Hamiltonian geometry and Delzant's Theorem

Abdullah Ahmed[1]

[1]Department of Mathematics
Lahore University of Management Sciences, Pakistan
Supervisor: Dr. Shaheen Nazir

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- The Lie bracket $W = [X, Y]$ of two vector fields X, Y is the unique vector field W such that $W_p f = X_p Y f - Y_p X f$ for all p and $f \in C^\infty(M)$.

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A vector field X_H on a symplectic manifold (M, ω) is called Hamiltonian if $i_X \omega = dH$ is exact. H is then called the Hamiltonian function of X.

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- Example: The exactness condition above reduce to Hamilton's equations on the trajectory of a particle moving in standard phase space $(\mathbb{R}^{2n}, \sum_i dq_i \wedge dp_i)$, with $H : \mathbb{R}^{2n} \to \mathbb{R}$ as the energy function.

Hamiltonian actions

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- Let S^1 act on S^2 as $\rho_\theta(\alpha, h) = (\alpha + \theta, h)$. The corresponding vector field is the Hamiltonian vector field $X_h = \frac{\partial}{\partial \theta}$.

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- How are Lie algebras and Lie groups related to each other?

Left-invariant vector fields

Definition

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- Example: The Lie algebra associated with $G = U(n)$ is $\mathfrak{g} = i\mathfrak{H}$, the space of skew-hermitian matrices.

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- These representations can be interpreted as a G-action on its Lie algebra $\mathfrak{g}$, and allow generalization the theory of Hamiltonian actions.

Moment Maps

Definition

A symplectic action $\psi : G \to Sympl(M, \omega)$ is Hamiltonian if there exists a map $\mu : M \to \mathfrak{g}^*$ satisfying:
(i) For each $X \in \mathfrak{g}$, define $\mu^X : p \mapsto \mu_p(X) \in \mathbb{R}$, and let $\tilde{X}$ be the vector field generated on M by the action of the integral curve of X through e in G: $\{exp(tX) | t \in \mathbb{R}\}$. Then $d\mu^X = i_{\tilde{X}}\omega$.
(ii) $\mu \circ \psi_g = Ad_g^* \circ \mu$ for all $g \in G$, i.e, μ is equivariant.

- (M, ω, G, μ) is called a *Hamiltonian G-space*, and μ is called the *moment map*.

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- Example: $U(n)$ acts on $\mathbb{C}^n$ with moment map $\mu(z)(X) = \frac{i}{2} z^* X z$.

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- Useful note: For compact and connected Lie groups, $H^k_{\text{de Rham}}(M;\mathbb{R}) = H^k(\mathfrak{g};\mathbb{R})$.

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- Example: For the usual rotation action of S^1 on $\mathbb{C}^{n+1}$, $\mu^{-1}(0) = S^{2n+1}$, and $\mu^{-1}(0)/G \cong \mathbb{CP}^n$, with ω_{red} being exactly the Fubini-Study form.

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- More generally, shifted moment maps allow for reduction across other levels $\mu^{-1}(\zeta)$ as long as they are preserved under the action.

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Theorem (Atiyah, Guilleman-Sternberg)

Let (M, ω) be a compact and connected symplectic manifold with a Hamiltonian $\mathbb{T}^m$ action on it with moment map μ. Then the levels of μ are connected, and $\mu(M)$ is the convex hull of the image of fixed points of the action.

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- An observation: Dimension of the face + Dimension of the stabilizer$=m$.

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A Delzant polytope $\triangle$ in $\mathbb{R}^n$ is a convex polytope that is:
(i) simple, i.e, has n edges meeting each vertex;
(ii) rational, i.e, each edge meeting at a vertex p is of the form $p + tu_i$ where $u_i \in \mathbb{Z}^n$;
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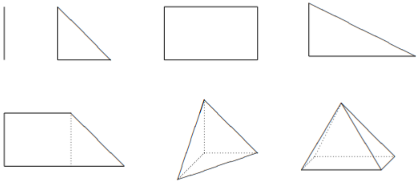

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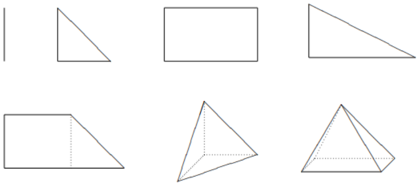

- Which of the above are Delzant polytopes?
- Any Delzant polytope can be written as the intersection of half-spaces $\triangle = \{x \in \mathbb{R}^n | \langle x, v_i \rangle \leq \lambda_i\}$ where $\lambda_i \in \mathbb{R}$ and v_i are outward pointing vectors (usually chosen to be primitive) to each *facet*.

Delzant's Theorem

Theorem (Delzant, 1988)

There is a one-to-one correspondence between symplectic toric manifolds and Delzant polytopes:

$$(M^{2n}, \omega, \mathbb{T}^n, \mu) \leftrightarrow \mu(M).$$

- We shall sketch the proof of the surjectivity.

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- We shall sketch the proof of the surjectivity.
- Motivation: Every n-dimensional Delzant polytope with d facets can be formed by the intersection of an affine plane with the negative orthant $\mathbb{R}^d_-$.
- Idea: For an n-dimensional Delzant polytope with d facets, we show that there is symplectic quotient of $(\mathbb{C}^d, \omega_0, \mathbb{T}^d, \mu)$ with action $(e^{2\pi i t_1}, ..., e^{2\pi i t_d}) \centerdot (z_1, ..., z_d) = (e^{2\pi i t_1} z_1, ..., e^{2\pi i t_d} z_d)$ and moment map $\phi(z_1, ..., z_d) = -\pi(|z_1|^2, ..., |z_d|^2) + (\lambda_1, ..., \lambda_d)$ that does the job.

The Delzant Construction

- Let $\triangle = \{x \in (\mathbb{R}^n)^* | \langle x, v_i \rangle \leq \lambda_i, \ i = 1, ..., d\}$ and $\{e_i\}$ be standard basis of $\mathbb{R}^d$.

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- $Z = (i^* \circ \phi)^{-1}(0)$ is a $(d + n)$-dimensional compact submanifold of $\mathbb{C}^d$ and N acts freely on Z.
- Using Marsden-Weinstein-Meyer reduction, we obtain a symplectic quotient $(M_\triangle, \omega_\triangle)$, which is a Hamiltonian $\mathbb{T}^n$ space.

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- $(i^* \circ \phi)^{-1}(0) = S^2_{\frac{a}{\pi}}$.
- $(M_\triangle, \omega_\triangle) = (\mathbb{CP}^1, \omega_{FS})$, has a standard toric structure under rotational action of $\mathbb{T}^2/N = S^1$.

Example

- Consider the Delzant polytope $\triangle = [0, a] \subset \mathbb{R}^*$ with $n = 1$ and $d = 2$, $v_1 = 1$ and $v_2 = -1$.
- $\triangle$ is defined by $\langle x, v_1 \rangle \leq 0$ and $\langle x, v_2 \rangle \leq a$.
- $\pi(e_1) = 1$ and $\pi(e_2) = -1 \Rightarrow$ker $\pi =$span$(e_1 + e_2) \Rightarrow N \cong S^1$.
- $i^*(x_1, x_2) = x_1 + x_2$ and $i^* \circ \phi(z_1, z_2) = -\pi(|z_1|^2 + |z_2|^2) + a$.
- $(i^* \circ \phi)^{-1}(0) = S^2_{\frac{a}{\pi}}$.
- $(M_\triangle, \omega_\triangle) = (\mathbb{CP}^1, \omega_{FS})$, has a standard toric structure under rotational action of $\mathbb{T}^2/N = S^1$.
- Neat observation: Consider a face F of $\phi(Z)$ of dimension $n - r$ dimension of the stabilizer of a point $z \in Z$ with image in F has dimension r.

Thank you![2][1][3]

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