Symplectic Geometry and Toric Varieties

Abdullah Ahmed

Syed Babar Ali School of Science and Engineering

Lahore University of Management Sciences

Mathematics Department

A semesterly report submitted

in partial fulfillment of

the requirements for a

Bachelor's Thesis in Mathematics

Last Updated: 2nd February 2024

Currently in progress

Abstract (Rough)

In this expository work, I have currently explored the fundamentals of symplectic geometry, and also classified their local behavior through symplectic equivalences given by the Darboux and Moser Theorems. Moreover, I have also talked about the role of closely related contact structures and associated Reeb vector fields. Finally, I end it by discussing the natural connections between symplectic and complex geometry, with notions of Dolbeault cohomolgy and even psuedo-holomorphic curves being touched upon. It may be noted that as of now, there has not been much exploration into the second part of this thesis: toric varieties/manifolds. This is because I wanted to cover all of the fundamentals of symplectic geometry in depth, even if some were relatively unrelated to the study of toric manifolds. This will enable me to explore other directions within this exciting field in the future. That said, as far as this thesis is concerned, I will now be moving on to study Hamiltonian actions and moment maps, until eventually breaking upon the grand correspondence between Delzant Polytopes and toric varieties.

Contents

Chapter 1

Symplectic Structure

1.1 Symplectic Vector Spaces

Let V be an n-dimensional vector space over $\mathbb{R}$ with basis $\{w_1, w_2, \ldots, w_n\}$.

Definition 1.1.1. *A bilinear form $\Omega : V \times V \to \mathbb{R}$ that satisfies $\Omega(v, u) = -\Omega(u, v)$ for all $u, v \in V$ is called skew-symmetric.*

We know that every bilinear form L can be represented in the form of a matrix $\mathfrak{L}$ with $L(u, v) = u^T \mathfrak{L} v$ and $[\mathfrak{L}]_{ij} = L(w_i, w_j)$ considering the basis above. It is easy to see that every skew-symmetric bilinear form must have a skew-symmetric matrix representation. Moreover, we also have the following important result:

Theorem 1.1.2. *Let Ω be a skew-symmetric bilinear form on a real vector space V. Then there exists a basis in which the matrix for Ω is of the form:*

$$\begin{pmatrix} 0 & Id & 0 & \ldots & 0 \\ -Id & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

More precisely, the basis is given by $\{e_1, e_2, \ldots, e_m, f_1, f_2, \ldots, f_m, u_1, u_2, \ldots, u_k\}$ where $\Omega(e_i, f_j) = \delta_{ij}$, $\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0$ for all i, j, and $\Omega(u_i, v) = 0$ for all $v \in V$.

Example 1.1.3. Let M be a smooth 6-dimensional manifold and let T_pM be the tangent space at $p \in M$. Let $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ be a 2-form on M. Then $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is a skew-symmetric bilinear form on the 6-dimensional real vector space T_pM. It is easy to see that in the basis $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}\}$ of T_pM, the matrix representation for this 2-form is:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

It can also be seen that the basis $\{\frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}), \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}), \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}), \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}), \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}\}$ also gives the same matrix. So the basis in the theorem above is not unique, as expected. △

Now consider the linear map $\tilde{\Omega} : V \to V^*$ induced by Ω defined as: $\tilde{\Omega}(v) = \Omega(v, .)$.

Definition 1.1.4. *Let (V, Ω) be a pair consisting of a real vector space V and an associated skew-symmetric bilinear form Ω. If ker$(\tilde{\Omega}) = \{0\}$, Ω is called* **non-degenerate** *and (V, Ω) is called a symplectic vector space.*

Note that this condition is equivalent to span$\{u_1, u_2, \ldots, u_k\} = \{0\}$, where u_i's are the basis elements given in Theorem 1.1.2. This implies that V must be even-dimensional. Clearly, (T_pM, ω_p) was not symplectic, but (T_pN, ω_p) with $N = \{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_4}\}$. Moreover, non-degenerate bilinear forms have matrix representations of the form:

$$\begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$$

There is also a natural notion of equivalence among symplectic vector spaces:

Definition 1.1.5. *Two symplectic vector spaces (V_1, Ω_1) and (V_2, Ω_2) are called symplectomorphic if there exists a vector space isomorphism $\phi : V_1 \to V_2$ that preserves the symplectic structure.*

By preservation of symplectic structure it is meant that the pullback $\phi^*\Omega_2 = \Omega_1$. Now we are finally in a position to define the symplectic manifold.

1.2 Symplectic Manifolds

Definition 1.2.1. *A differential 2-form on a manifold M is called a symplectic form if and only if it is closed and ω_p is a non-degenerate bilinear form on T_pM for all $p \in M$.*

Definition 1.2.2. *A pair (M, ω) is called a symplectic manifold if ω is a symplectic form on M.*

Since $\dim T_pM = \dim M$, symplectic manifold must be even-dimensional. It can be observed that a 2-form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ with coordinates $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ is a symplectic form on $\mathbb{R}^{2n}$. Moreover, we can also replace the condition of non-degeneracy with an equivalent and more informative one:

Lemma 1.2.3. *A differential 2-form $\omega \in \Omega^2(M)$ is non-degenerate if and only if $\dim M = 2n$ and ω^n is non-vanishing, that is, ω^n is a volume form.*

Proof. First note that by Theorem 1.1.2, for $\dim T_pM = 2n$, there exists a basis $\{e_1, ...e_m, f_1, ..., f_m, u_1, ..., u_{2n-2m}\}$ for T_pM such that $\omega_p(u_i, ;) = 0$, $\omega_p(e_i, f_j) = \delta_{ij}$ and $\omega_p(e_i, e_j) = \omega_p(f_i, f_j) = 0$ for all i, j. This gives us that $\omega_p = \sum_{i=1}^n e_i^* \wedge f_i^*$ where e_i^*, f_i^* are part of corresponding dual basis for T_p^*M.
Now assume ω_p is non-degenerate for all $p \in M$. Then we have that $\dim M = 2n$ and $m = n$ from before:

$$\begin{aligned}\underbrace{\omega_p \wedge .... \wedge \omega_p}_{n-\text{times}} &= \sum_{i_1,..,i_n}^{n} e^{i_1} \wedge f^{i_1} \wedge ...... \wedge e^{i_n} \wedge f^{i_n}. \\ &= \sum_{i_1 \neq i_2 .... \neq i_n}^{n} e^{i_1} \wedge f^{i_1} \wedge ...... \wedge e^{i_n} \wedge f^{i_n} \\ &= \sum_{\sigma \in S_n} e^{\sigma(1)} \wedge f^{\sigma(1)} \wedge ...... \wedge e^{\sigma(n)} \wedge f^{\sigma(n)}.\end{aligned}$$

Now we can see that any element of the sum above is invariant under the following switching of successive pairs of 2-forms: $e^{\sigma(1)} \wedge f^{\sigma(1)} \wedge .... \wedge e^{\sigma(k)} \wedge f^{\sigma(k)} \wedge e^{\sigma(k+1)} \wedge f^{\sigma(k+1)} .... \wedge e^{\sigma(n)} \wedge f^{\sigma(n)} = e^{\sigma(1)} \wedge f^{\sigma(1)} \wedge .... \wedge e^{\sigma(k+1)} \wedge f^{\sigma(k+1)} \wedge e^{\sigma(k)} \wedge f^{\sigma(k)} .... \wedge e^{\sigma(n)} \wedge f^{\sigma(n)}$. Thus each element in the sum above is equivalent, and hence:

$$\underbrace{\omega_p \wedge .... \wedge \omega_p}_{n-\text{times})} = n!(e^1 \wedge f^1 \wedge ..... \wedge e^n \wedge f^n) \neq 0,$$

at any point $p \in M$. Thus ω^n is a volume form on M.
Now assume on the other hand that ω_p is degenerate for some $p \in M$, where $\dim M = 2n$. Then we have that $2n - 2m \geq 2$, and thus $n \geq m + 1$. We have, by a similar argument to before, $\omega_p^m = m!(e^1 \wedge f^1 \wedge ..... \wedge e^m \wedge f^m) \Rightarrow \omega_p^{m+1} = 0$ by anti-commutativity. Thus $\omega_p^n = 0$ and ω^n is not a volume form. □

Our notion for equivalent symplectic vector spaces carries over to symplectic manifolds:

Definition 1.2.4. *Two symplectic manifolds* (M_1, ω_1) *and* (M_2, ω_2) *are symplectomorphic if there exists a diffeomorphism* $\phi : M_1 \to M_2$ *with* $\phi^*\omega_2 = \omega_1$. The map ϕ is called a *symplectomorphism.*

One can ask a bunch of questions now: When does a symplectic form exist on an even-dimensional manifold? And if it does, when is it unique up to a symplectomorphism? That is, when is (M, ω_1) symplectomorphic to (M, ω_2)? Finally, does there at least exist some local commonality among all symplectic structures? As we will see, some of these questions have been fully answered, while others remain open problems to this day. For now, let us explore the existence of symplectic structures in different instances.

Example 1.2.5. Consider the Mobius strip. Assume it has a symplectic form ω. Then we must have ω is a volume form, which then implies that the Mobius strip is orientable. This is a contradiction, and so the Mobius strip must not support a symplectic structure in the first place. $\triangle$

This gives us a more general result.

Proposition 1.2.6. *Every* $2n$*-dimensional symplectic manifold* (M, ω) *is orientable with orientation form* ω^n.

Is this a sufficient condition? Symplectic geometry would be very boring if it was: consider the orientable surface S^{2n}. Assume there exists a symplectic form ω on S^{2n}. Then ω^n is a volume form on S^{2n}. We can see that if this form is exact, we can apply Stoke's Theorem to get:

$$\mathrm{vol}(S^{2n}) = \int_{S^{2n}} \omega^n = \int_{\partial S^{2n}} \alpha = 0,$$

where ω^n is assumed to be exact and equal to $d\alpha$. This is contradictory as S^{2n} is compact and hence $\mathrm{vol}(S^{2n}) > 0$. Thus ω^n is not exact, which implies that ω is also not exact (if $\omega = d\alpha$, then $\omega^n = (d\alpha)^n = d(\alpha \wedge (d\alpha)^{n-1})$ is exact). Thus the de Rham cohomology class $[\omega] \neq 0$. If $n > 1$, this is contradictory to the fact that $H^2(S^{2n}) = 0$. Thus for $n > 1$, such an ω must not exist in the first place, and S^{2n} must not support a symplectic structure. That said, it is not difficult to see that S^2 supports a symplectic structure with its canonical area form being a symplectic form ($\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$).

Although we have seen smooth manifolds where no symplectic structure exists, there always exists an associated manifold with an intrinsic symplectic structure: the cotangent bundle. Recall that the cotangent bundle T^*M of an n-dimensional manifold M has local coordinates defined by $(x, \eta) \in T^*M \mapsto (x_1, ..., x_n, \eta_1, ..., \eta_n) \in \mathbb{R}^{2n}$. Define intrinsically the **Liouville** 1-form α on T^*M by $\alpha_{(x,\eta)} = \pi^*\eta$, where π is the canonical projection of T^*M onto M. Clearly, this definition is coordinate-independent. Furthermore, define the closed 2-form $\omega = -d\alpha$. Easy calculation shows that in any coordinate chart $(U, x_1, ..., x_n, \eta_1, ...\eta_n)$, $\alpha = \sum_{i=1}^n \eta_i dx_i$ and $\omega = \sum_{i=1}^n dx_i \wedge d\eta_i$. Through an identical calculation to that of Lemma 1.2.3, we have that ω is symplectic and (T^*M, ω) is a symplectic manifold.

It is clear that T^*M is a symplectic manifold intrinsically linked to the differential structure of M. Thus, it is no surprise that diffeomorphic spaces have symplectomorphic cotangent bundles:

Theorem 1.2.7. *Let $f : M_1 \to M_2$ be a diffeomorphism. Then there exists a symplectomorphism $g : (T^*M_1, \omega_1) \to (T^*M_2, \omega_2)$.*

Proof. Define $g = \bar{f} : (T^*M_1, \omega_1) \to (T^*M_2, \omega_2)$ by $\bar{f}(x, \eta) = (f(x), (f^{-1})^*\eta)$. First we show that this is a diffeomorphism. It is clear that $\bar{f}$ is bijective due to the bijectivity of f and the pullback f^*. Now consider any $p \in U = (x_1, ..., x_n, \eta_1, ..., \eta_n)$ and $\bar{f}(p) \in V = (y_1, ..., y_n, \bar{\eta}_1, ..., \bar{\eta}_n)$ To argue the smoothness of $\bar{f}$ at p, we have that one of the component functions f is smooth by definition, and for $(f^{-1})^*$ we have:

$$\begin{aligned}(f^{-1})^*\eta = \sum_{i=1}^n (\eta_i \circ f^{-1}) d(x_i \circ f^{-1}) &= \sum_{i,j}^n (\eta_i \circ f^{-1}) \frac{\partial(x_i \circ f^{-1})}{\partial y_j} dy_j \\ &= \sum_{j=1}^n \left(\sum_{i=1}^n (\eta_i \circ f^{-1}) \frac{\partial(x_i \circ f^{-1})}{\partial y_j} \right) dy_j.\end{aligned}$$

So $\eta_i(p) \mapsto \eta_i(p) \frac{\partial(x_i \circ f^{-1})}{\partial y_j}(f(p))$, which is clearly a smooth mapping. The same can be argued for $\bar{f}^{-1} = (f^{-1}, f^*)$, and thus $\bar{f}$ is a diffeomorphism. Finally, it is also not too hard to see that as $\pi_2 \circ \bar{f} = f \circ \pi_1$, and given the intrinsic definition of the Liouville form α above, $\bar{f}^*\alpha_2 = \alpha_1$ and hence $\bar{f}^*\omega_2 = \omega_1$. [2] $\square$

We can in fact show a partial converse:

Proposition 1.2.8. *Any symplectomorphism $g : (T^*M, \omega) \to (T^*M, \omega)$ that preserves α must be the lift of some diffeomorphism.*

Proof. Let $p = (x, \eta)$ and $g(p) = (y, \zeta)$. As $g^*\alpha = \alpha$, we have that at p, $g^*\alpha_{g(p)} = \alpha_p$ and hence $g^*(\pi^*\zeta_y) = \pi^*\eta_x$. Then $(\pi \circ g)^*\zeta_y = \pi^*\eta_x$ and $(\pi \circ g)^*(\lambda\zeta)_y = \pi^*(\lambda\eta)_x$ for any $\lambda \in \mathbb{R}$. This implies $g^*(\alpha_{(y,\lambda\zeta)}) = \alpha_{(x,\lambda\eta)}$. As g is bijective, we know that there exists $q = (z, \beta) \in T^*M$ such that $g(q) = (y, \lambda\zeta)$ and hence $g^*(\alpha_{(y,\lambda\zeta)}) = \alpha_{(z,\beta)}$. Given the explicit local form of α from before, it is obvious that $(z, \beta) = (x, \lambda\eta)$. Hence $g(x, \lambda\eta) =$

$(y, \lambda\zeta)$. Specifically taking $\lambda = 0$, we have that $g(x, \eta) = (y, \zeta) \Rightarrow g(x, 0) = (y, 0)$. This implies $\pi(g(x, \eta)) = y$ for all $\eta \in T_x^*M$. Thus we can simply define a diffeomorphism $h : M \to M_0 \subset T^*M$ with $h(x) = (x, 0)$, and have $f : M \to M$ be a diffeomorphism such that $f = h^{-1} \circ g|_{M_0} \circ h$. Clearly $\pi \circ g = f \circ \pi$. Taking p as before, we can see that the lift of f gives $\bar{f}(p) = (f(x), \gamma) = (y, \gamma)$ where $f^*\gamma_y = \eta_x \Rightarrow (f \circ \pi)^*\gamma_y = \pi^*(f^*\gamma_y) = \pi^*\eta_x = (\pi \circ g)^*\zeta_y$. Using the relation between f and g, we get $(\pi \circ g)^*\gamma_y = (\pi \circ g)^*\zeta_y$. As $\pi \circ g$ is a subjective submersion, we must have $\gamma_y = \zeta_y$ and hence $g = \bar{f}$. $\square$

Now although we have made some progress on our questions from before, we are still far from addressing the general question of symplectomorphic manifolds. To address this, we will first take a detour to understand some important sub-structures of symplectic manifolds.

1.3 Lagrangian Submanifolds

Recall that a submanifold X of M is defined to be a manifold with an immersive homeomorphism $i : X \to M$ where $i(X)$ is closed in M.

Definition 1.3.1. *Let (M, ω) be a 2n-dimensional symplectic manifold. Then a submanifold Y of dimension n with $i^*\omega = \omega|_{T_pY} \equiv 0$ is called a Lagrangian submanifold.*

Example 1.3.2. A simple example of a Lagrangian submanifold is the *zero section* (see [2]) of T^*M given by $M_0 = \{(x, \eta) \in T^*M | \, \eta = 0 \text{ in } T_x^*M\}$. This is clearly an n-dimensional submanifold of T^*M defined by $\eta_i = 0$ for all i. Moreover, $\alpha|_{T_pM_0} \equiv 0 \Rightarrow \omega|_{T_pM_0} \equiv 0$, and so M_0 is a Lagrangian submanifold of T^*M. One can in fact prove a more general result stating that for a 1-form μ on M, the section $M_\mu = \{(x, \mu_x) \in T^*M\}$ is a Lagrangian submanifold of T^*M if and only if μ is closed, see [2]. This gives a whole family of Lagrangian submanifolds that can be *generated* by functions $f \in C^\infty(M)$ as M_{df}. $\triangle$

We now unravel the connection between our question about symplectomorphisms and Lagrangian submanifolds. Consider a diffeomorphism $\phi : (M_1, \omega_1) \to (M_2, \omega_2)$ where $\dim M_1 = \dim M_2 = 2n$. It induces a map $\psi : M_1 \to M_1 \times M_2$ given by $\psi(p) = (p, \phi(p))$. The image of ψ is the graph of the function ϕ, and is a closed embedding of M_1 into $M_1 \times M_2$. This gives us that M_1 is a $2n$-dimensional submanifold of $M_1 \times M_2$. We now impose on $M_1 \times M_2$ a symplectic structure given by the *twisted* closed 2-form $\bar{\omega} = \pi_1^*\omega_1 - \pi_2^*\omega_2$, where π_i is projection onto M_i. One can easily check that this is indeed a symplectic form. Considering $\psi^*\bar{\omega} = (\pi_1 \circ \psi)^*\omega_1 - (\pi_2 \circ \psi)^*\omega_2 = \omega_1 - \phi^*\omega_2$, we can see that ϕ is a symplectomorphism if and only if $\psi^*\bar{\omega} = 0$ and $\psi(M_1)$ is a Lagrangian submanifold of $(M_1 \times M_2, \bar{\omega})$. This gives us a new approach to the problem of finding symplectomorphisms! see for more details,[2].

Let us try to apply this. Consider $M_1 = (T^*X_1, \omega_1)$ and $M_2 = (T^*X_2, \omega_2)$ for some manifolds X_1 and X_2. We shall now work backwards; that is, look for Lagrangian submanifolds of $(M_1 \times M_2 \cong T^*(X_1 \times X_2), \bar{\omega})$. We know that it is easy to find Lagrangian submanifolds of $(M_1 \times M_2 \cong T^*(X_1 \times X_2), \omega = \pi_1^*\omega_1 + \pi_2^*\omega_2)$, where ω is just the canonical 2-form for the cotangent bundle $T^*(X_1 \times X_2)$. Assume we have such a Lagrangian submanifold Y. Then defining $\sigma : M_1 \times M_2 \to M_1 \times M_2$ as $\sigma(x, y, \eta_1, \eta_2) = (x, y, \eta_1, -\eta_2)$, we can see that $\sigma(Y)$ is a Lagrangian submanifold of $(M_1 \times M_2, \bar{\omega})$. So if we start off with $Y = (M_1 \times M_2)_{df}$ for some $f \in C^\infty(X_1 \times X_2)$, we get $\sigma(Y) = (x, y, d_x f, -d_y f)$. And so, if this is the graph of a diffeomorphism $\phi : M_1 \to M_2$, then ϕ must be a symplectomorphism. Thus, $\phi(x, \eta) = (y, \zeta)$ must satisfy the following *Hamilton* equations:

$$\eta_i = \frac{\partial f}{\partial x_i}(x, y)$$
$$\zeta_i = -\frac{\partial f}{\partial y_i}(x, y).$$

By the implicit function theorem, for a solution $y = \phi_1(x, \eta)$ to exist for the first differential equation, we must have $\left[\dfrac{\partial^2 f}{\partial y_j \partial x_i}\right]_{i,j}$ is invertible locally.

Example 1.3.3. Consider a Riemannian manifold (X, g) that is *geodesically convex* and *geodesically complete*. This implies that for every $(x, v) \in TX$ there exists a unique minimizing geodesic of constant velocity v starting at x given as $\exp(x, v) : \mathbb{R} \to X$. If we define $f : X \times X \to \mathbb{R}$ as $f(x, y) = -\frac{1}{2}d(x, y)^2$, we can show that the symplectomorphism generated by f can be identified (through g) with the *geodesic flow* on X (the endomorphism $(x, v) \mapsto \exp(x, v)(1)$ of TX). Firstly, recall that the identification between TX and T^*X is given as $(x, v) \leftrightarrow g_x(v, ;)$. Thus the Hamilton equations modify to:

$$g_x(v, ;) = d_x f$$
$$g_y(w, ;) = -d_y f,$$

where we aim to find (y, w) as function of (x, v). If we act both sides of the first equation on $\frac{d\exp(x,v)}{dt}(0) = v$, we get:

$$g_x(v, v) = d_x f(v) = \frac{d}{dt}(f(\exp(x, v)(t), y))(0)$$
$$= -\frac{1}{2}\frac{d}{dt}(d(\exp(x, v)(t), y)^2)(0).$$

We can see that if we take $y = \exp(x,v)(1)$ we get by definition of the Riemannian distance that $d(\exp(x,v)(t), \exp(x,v)(1)) = \int_t^1 \sqrt{g_x\left(\frac{d\exp(x,v)}{dt}, \frac{d\exp(x,v)}{dt}\right)} = \int_t^1 \sqrt{g_x(v,v)} = \sqrt{g_x(v,v)}(1-t)$. Plugging this into the equation above confirms that this is indeed the unique solution to the first Hamilton equation. Moreover, if we act $\frac{d\exp(x,v)}{dt}(1) = v$ on both sides of the second equation, we get:

$$\begin{aligned} g_x(w,v) = -d_y f(v) &= \frac{d}{dt}(f(\exp(x,v)(t), \exp(x,v)(0)))(1) \\ &= \frac{1}{2}\frac{d}{dt}(d(\exp(x,v)(t), \exp(x,v)(0))^2)(1). \end{aligned}$$

Through a similar calculation as before, we get $d(\exp(x,v)(t), \exp(x,v)(0)) = -t\sqrt{g_x(v,v)}$, and plugging this into the equation above gives $g_x(w,v) = g_x(v,v)$. Moreover, if we consider any orthogonal $\bar{v} \in T_yX$ to v, we get $g_x(w,\bar{v}) = \frac{1}{2}\frac{d}{dt}(d(\exp(y,\bar{v})(t), \exp(x,v)(0))^2)(0)$. As $\exp(x,v)$ is a minimizing geodesic, we have that $d(\exp(y,\bar{v})(t), \exp(x,v)(0))$ is minimum when the two geodesics intersect at $t = 0$. Thus the derivative expression above is 0 and $g(w,\bar{v}) = 0$ for all orthogonal $\bar{v} \in T_yX$ to v. This implies $g(w, ;) = g(v, ;)$. By non-degeneracy of g, the identification we made before is bijective, and hence $w = v = \exp(x,v)(1)$. And so, the symplectomorphism f can be identified with the endomorphism $(x,v) \mapsto \exp(x,v)(1)$ of TX. △

Now that we have some familiarity with the construction and finding of symplectomorphisms, we can start to explore questions about the existence of various kinds of equivalences between symplectic structures.

1.4 Symplectic Equivalences

We first recall few definitions and ideas.

Definition 1.4.1. *An isotopy is a map $\rho : M \times \mathbb{R} \to M$ such that $\rho_t = \rho(;,t) : M \to M$ is a diffeomorphism for all t, and $\rho_0 = id_M$.*

Every isoptopy gives an associated time-dependent vector field v_t given by $v_t = \frac{d\rho_t}{dt} \circ \rho_t^{-1}$. For a compact manifold, the converse is also true: for any time-dependent vector field v_t, there exists an isotopy satisfying the differential equation $\frac{d\rho_t}{dt} = v_t \circ \rho_t$, with initial condition $\rho_0 = id_M$. By Picard–Lindelof theorem, this also exists locally for non-compact manifolds. This allows us to define the *Lie derivative* along v_t as a map $\mathfrak{L}_{v_t} : \Omega^k(M) \to \Omega^k(M)$ given by $\mathfrak{L}_{v_t}\omega = \frac{d}{dt}(\rho_t^*\omega)|_{t=0}$, where ρ is the local isotopy associated with v_t. The Lie derivative satisfies the *Cartan magic formula:* $\mathfrak{L}_{v_t}\omega = i_{v_t}d\omega + di_{v_t}\omega$, and as $\rho_{h+t} = \rho_t \circ \rho_h$ (by uniqueness of solution to the differential equation from before), it also satisfies $\frac{d}{dt}\rho_t^*\omega|_{t=\tau} = \rho_\tau^*\mathfrak{L}_{v_t}\omega$. By the chain rule, this gives $\frac{d}{dt}\rho_t^*\omega_t = \rho_t^*(\mathfrak{L}_{v_t}\omega_t + \frac{d\omega_t}{dt})$.

Also recall that the *normal space* to a k-dimensional submanifold $S \subset M$ at $p \in S$ is defined as $n-k$ dimensional quotient vector space $N_pS = T_pM/T_pS$. This gives us the *normal bundle* NS, an n-dimensional manifold. It is not hard to see that S embeds into NS as the zero-section as well. In fact, we also have the following well-known result that states that these embeddings are locally the same:

Theorem 1.4.2. (Tubular neighborhood theorem) *Let $S \subset M$ be a submanifold of M. There exists a neighborhood U_0 of S in NS, and a neighborhood U of S in M, and a diffeomorphism $\phi : U_0 \to U$ with $\phi|_S = id_S$. Moreover, U_0 in convex in each N_pS.*

This gives us that S is a deformation retract of U and hence the inclusion induces an isomorphism of de Rham cohomologies. Now we return to the question from before, but slightly altered: *Does there exist a symplectomorphism between (M, ω_0) and (M, ω_1) that is homotopic to the identity?* The reason we have added this extra condition is that it restricts our problem to the specific case where $[\omega_0] = [\omega_1]$. Naturally, this problem is easier.

Define $S_c = \{\text{symplectic forms } \omega \text{ on } M \text{ with } [\omega] = c\}$ endowed with the C^∞ topology. Assume $\omega_0, \omega_1 \in S_c$, and that there exists a smooth straight line path connecting both of them in S_c, i.e, $\omega_t = t\omega_0 + (1-t)\omega_1$ is symplectic for all $t \in [0,1]$. Moreover, as $\omega_0 - \omega_1 = d\mu$ for some 1-form μ on M, we have $\omega_t = \omega_1 + td\mu \in S_c$ for all t. We claim that this is a sufficient condition for the existence of a symplectomorphism that is homotopic to the identity.

Proof. Before we prove that the existence of a straight line path is sufficient, we note in hindsight that if it is true, then (M, ω_t) is symplectomorphic to (M, ω_0) for all t by reparameterizing the path ω_t. All of these implications can be satisfied if there exists an isotopy ρ with $\rho_t^*\omega_t = \omega_0$. Looking for the existence of such an isotopy is exactly what is known as *Moser's trick*. If we additionally assume that M is compact, we can instead look for the associated vector field v_t, and integrate to get ρ_t. We have the following condition on v_t as in [2]:

$$\frac{d}{dt}(\rho_t^*\omega_t) = 0 = \rho_t^*\left(\mathfrak{L}_{v_t}\omega_t + \frac{d\omega_t}{dt}\right) \Leftrightarrow di_{v_t}\omega_t + d\mu = 0.$$

We can find a solution for this easily by solving for v_t such that $i_{v_t}\omega_t + \mu = 0$ (as ω_t is nondegenerate). Thus, we obtain the isotopy we wanted. □

One can generalize this argument to any $\omega_0, \omega_1 \in S_c$ that lie in the same path-connected component:

Theorem 1.4.3. (Moser Theorem) *Let M be compact manifold with symplectic forms ω_0 and ω_1 connected by a smooth family of symplectic forms ω_t, with $0 \leq t \leq 1$, such that $[\omega_t] = c$ is constant. Then there exists an isotopy ρ of M with $\rho_t^* \omega_t = \omega_0$*

Let's apply this.

Proposition 1.4.4. *Let M a compact and oriented 2-manifold. Then each 2-cohomology class has a unique symplectic representative.*

Proof. Consider an orientable and compact 2-dimensional manifold M with areas forms ω_1 and ω_2. Clearly, these are symplectic forms. Assume further that $[\omega_1] = [\omega_2]$. First, let us show that ω_1 and ω_2 determine the same orientation. Assume they do not. Then $\omega_1 = -f\omega_2$ for some smooth $f > 0$. Moreover, we have $\int_M \omega_1 - \omega_2 = \int_M (1+f)\omega_1 > 0$. This is a contradiction as $\omega_1 - \omega_2 = d\mu$ is exact, and hence by Stoke's theorem $\int_M \omega_1 - \omega_2 = 0$. Thus ω_1 and ω_2 must determine the same orientation, and $\omega_1 = f\omega_2$ for some smooth $f > 0$. Now consider the smooth straight line path $\omega_t = t\omega_1 + (1-t)\omega_2 = (1 - t + ft)\omega_2$. As $t \in [0,1]$, ω_t can only vanish at some point p if $f(p) = (t-1)/t \leq 0$. This is a contradiction. Hence ω_t is symplectic for all t. By Moser Theorem, the symplectic structure induced by ω_1 and ω_2 is the same. $\square$

Moser Theorem also allows us to find a symplectic equivalence if two different symplectic forms agree on a compact submanifold. Recall the tubular neighborhood theorem−if X to be a compact submanifold of M, then there is a neighborhood U of X in M such that X is a deformation retract of U. If ω_1 and ω_2 are two symplectic forms that agree on X, i.e, $i^*(\omega_1 - \omega_2)|_X = 0$, then $(\omega_1 - \omega_2)|_U = d\mu$ is exact by homotopy invariance of the de Rham cohomology. This allows one to define a smooth family $\omega_t = \omega_2 + td\mu$ that vanishes on X. After some shrinking of neighborhoods, and applying Moser's Trick again, we have another theorem by Moser:

Theorem 1.4.5. (Relative Moser Theorem) *Let X be a compact submanifold of a symplectic manifold M with two symplectic forms ω_1 and ω_2 agreeing on X. Then there exists neighborhoods U and V of X in M, and a diffeomorphism $\phi : U \to V$ with $\phi^*\omega_2 = \omega_1$ and $\phi|_X = id_X$.*

If we take $X = p$, and use the canonical symplectic basis for the T_pM, we finally get the grand theorem classifying symplectic structures locally:

Theorem 1.4.6. (Darboux Theorem) *Let (M, ω) be a symplectic manifold, and let p be any point in M. Then we can find a coordinate system $(U, x_1, ..., x_n, y_1, ...y_n)$ centered at p such that on U:*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

If X is a compact Lagrangian submanifold of both (M, ω_1) and (M, ω_2), i.e, $i^*\omega_1 = i^*\omega_2 = 0$, then relative Moser applies identically to give what is known as the *Weinstein Lagrangian Neighborhood Theorem*. This theorem is imperative in helping us classify Lagrangian structures locally. But first, we prove a small result.

Lemma 1.4.7. *Let (V, Ω) be symplectic vector space with a lagrangian subspace U. Then there exists a canonical isomorphism between V/U and U^*.*

Proof. Define $\tilde{\Omega} : V/U \to U^*$ given by $\tilde{\Omega}([v]) = \Omega(v, ;)$. This is well-defined as for any $v \oplus u$, $\Omega(v \oplus u, ;) = \Omega(v, ;) + \Omega(u, ;) = \Omega(v, ;)$ as $\Omega|_U = 0$. Moreover, it is a non-degenerate pairing, and so the map above is an isomorphism. $\square$

Note that this allows us to canonically identify NX and T^*X. Now recall that every manifold X is a Lagrangian submanifold of (T^*X, ω_0) as the zero section. Assume X is also a Lagrangian submanifold of (M, ω). By using the tubular neighborhood theorem with the identification $NX \cong T^*X$, followed by Weinstein's Lagrangian neighborhood theorem, we obtain neighborhoods U of X in (T^*X, ω_0), V of X in M, and a composite symplectomorphism $\phi : (U, \omega_0) \to (V, \omega)$ that is identity on X[2]. This implies that every Lagrangian embedding is symplectically locally equivalent to the embedding of a manifold as the zero-section of its cotangent space! This classification is called the *Weinstein tubular neighborhood theorem.*This result has important applications.

Example 1.4.8. Consider $\mathrm{Sympl}(M, \omega) = \{f : M \xrightarrow{\cong} M | f^*\omega = \omega\}$ in the C^1 topology. Take $f \in \mathrm{Sympl}(M, \omega)$ that is in a sufficiently small neighborhood of $id \in \mathrm{Sympl}(M, \omega)$. We have that the graphs $\Gamma_f, \Gamma_{id} \cong M$ are lagrangian submanifolds of $(M \times M, \bar{\omega})$. By Weinstein's tubular neighborhood theorem, as f is sufficiently C^1 close to id, its graph is symplectically embedded into T^*M as the image of a smooth section $M_\mu = \{(p, \mu_p) \in T^*M\}$[2], where μ is closed and C^1-close to the zero form. On the other hand, if we take μ to be a closed 1-form that is C^1-close to the zero form, its section M_u will lie in a sufficiently small neighborhood of the zero section in U_2. Thus ϕ^{-1} is defined on M_μ. As μ is C^1-close to the zero form, $\phi^{-1}(M_\mu)$ also smoothly intersects $\{p\} \times M$ once for all p at $g(p)$. Thus, g is a diffeomorphism of M and $M_\mu \cong \Gamma_g$. As μ is closed, g is a symplectomorphism. This gives us that sufficiently small C^1 neighborhood of id in

$\mathrm{Sympl}(M,\omega)$ is homeomorphic to a similar neighborhood of the zero form in the space of closed 1-forms on M. $\triangle$

Example 1.4.9. Consider a compact symplectic manifold M with vanishing de Rham cohomology of degree 1. Take any $f \in \mathrm{Sympl}(M,\omega)$ that is C^1-close to identity. Then, by the observation above, it is generated by $h \in C^\infty(M)$ through the recipe described in section 1.3. Now, as h is defined on a compact manifold, and hence achieves a maximum and minimum, it must have at least two critical points where $dh_p = 0$. Via the zero-section embedding discussed before, we have that Γ_f intersects Γ_{id} at two points[2]. Thus f must have two fixed points. $\triangle$

Example 1.4.10. Similar to above, we can now also say something about Lagrangian intersections. Assume we have two compact Lagrangian submanifolds X and Y with vanishing de Rham cohomology of degree 1. Assume further that they are C^1-close, i.e, there exists a diffeomorphism $f : X \to Y$ that is C^1-close to the inclusion $i : X \to M$. Then, by Weinstein's tubular neighborhood theorem, there exist neighborhoods U_1 of X in M, U_2 of the zero-section of T^*X, and symplectomorphism ϕ between the two. We can assume $Y \subset U_1$. Now, lifting the diffeomorphism f, we can embed Y into U_2 as the section of a closed 1-form on X. We can see that μ must vanish for at least two points as X is compact and $H^1_{deRham}(X) = 0$. Via ϕ^{-1}, this corresponds to X and Y intersecting in at least two points. $\triangle$

We have come a long way now in understanding the local behavior of symplectic structures. That said, this is just the tip of the iceberg as far as contemporary research in this area is concerned. Moreover, many questions remain open problems to this day, and have even led to developments of new theories. One such problem is along the themes of the examples above about fixed points of symplectomorphisms called *Arnold Conjecture*, and how it led to the development of *Morse theory* in the form of *Floer homology.* As we cannot hope to cover all of the above in such depth, we shall, for now, take a detour to explore another important idea in symplectic geometry, and that is *contact structure.*

Chapter 2

Contact Structure

2.1 Contact Manifolds

Definition 2.1.1. *A contact element on a manifold M is a pair (p, H_p) where $p \in M$ and $H_p \subset T_pM$ is a tangent hyperplane.*

If we define a field of contact elements $H : p \mapsto H_p$, it is not hard to see that H is locally defined by the kernel of a 1-form α.

Definition 2.1.2. *A contact structure on M is a smooth field of tangent hyperplanes $H \subset TM$ such that, for any locally defining 1-form α, $d\alpha|_H$ is symplectic. The pair (M, H) is called a contact manifold and α is called a local contact form.*

It is easy to see that for a contact manifold, $\dim H_p = \dim(\ker\alpha_p) = 2n$ for all p, and $T_pM = \ker\alpha_p \oplus \ker d\alpha_p$ (where α is a locally defining 1-form) is $2n+1$-dimensional. As will be seen moving ahead, contact structures are often considered to be the 'odd' counterpart of symplectic structures. As a first glimpse into this, we show that the symplectic condition above can be replaced by a more similar one we saw in symplectic manifolds:

Proposition 2.1.3. *Let H be a field of hyperplanes on a manifold. Then H is a contact structure if and only if $\alpha \wedge (d\alpha)^n$ is non-vanishing for any locally defining 1-form α.*

Proof. Assume α is locally defining 1-form for $p \in M$. We can decompose T_pM and choose a basis $\{e_1, ...e_{2n+1}\}$ such that $H_p = \text{span}(e_2, ..., e_{2n+1})$. We can see that:

$$\begin{aligned}
\alpha \wedge (d\alpha)^n(e_1, ..., e_{2n+1}) \neq 0 &\Leftrightarrow \alpha_p(e_1)(d\alpha_p)^n(e_2, ..., e_{2n+1}) \neq 0 \\
&\Leftrightarrow (d\alpha_p)^n(e_2, ..., e_{2n+1}) \neq 0 \\
&\Leftrightarrow (d\alpha_p)^n|_{H_p} \neq 0 \\
&\Leftrightarrow (d\alpha)^n|_H \text{ is symplectic.}
\end{aligned}$$

Note that if α is a global form, then $\alpha \wedge (d\alpha)^n$ is a volume form on M, and hence M is orientable. And it is not hard to see that if TM/H is an orientable quotient line bundle with orientation form $\bar{\alpha}$, then α (the corresponding form on M) is a global contact form. $\square$

Drawing another analogy to symplectic structures, a *contactomorphism* is defined as a diffeomorphism ϕ such that $\phi_* H = H$. We can also define a canonical vector field R for contact structures satisfying $i_R d\alpha = 0$ and $i_R \alpha = 1$ for any locally defining 1-form α. It is called the *Reeb vector field*, and given the decomposition of the tangent space from before, we can see that it is unique up to multiplication by a non-vanishing smooth function. Moreover, the flow of the Reeb vector field is a family of contactomorphisms preserving the contact form. Let us look at some examples now to solidify our understanding.

Example 2.1.4. Consider $\mathbb{R}^{2n+1}$ with coordinates $(x_1, .., x_n, y_1, ..., y_n, z)$. Let $\alpha = \sum_{i=1}^n (x_i dy_i) + dz$. Then as $d\alpha = \sum_{i=1}^n (dx_i \wedge dy_i)$, we can see that $\alpha \wedge (d\alpha)^n = dx_1 \wedge ... \wedge dx_n \wedge ... dy_1 \wedge dy_n \wedge dz$ is a volume form. Hence $\mathbb{R}^{2n+1}$ admits a contact structure for $n \geq 0$. The corresponding field of hyperplanes is given by $\ker\alpha_p$ as $H_{\tilde{p}} = \{\sum_{i=1}^n (a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial y_i}) - \sum_{i=1}^n (b_i \tilde{x}_i) \frac{\partial}{\partial z} | a_i, b_i \in \mathbb{R}\}$. The corresponding Reeb vector field is given by $\frac{\partial}{\partial z}$, and its flow is given by translation in the z-direction: $\rho_t(\tilde{p}) = \tilde{p} + t(0, 0, ..., 1)$. It is clear that this preserves the contact form. In fact, any translational diffeomorphism by a vector of the form $(0, ..., 0, \bar{y}_1, \bar{y}_2, ..., \bar{z})$ is contactomorphism. $\triangle$

Just like the cotangent bundle T^*M is a canonical symplectic structure associated with any manifold M, we can also define a canonical contact structure associated with M. Define the following equivalence relation on ($T^*M\backslash$ zero section) given by $(p, \zeta) \sim (p, \zeta') \Leftrightarrow \zeta = \lambda\zeta'$ for some $\lambda \in \mathbb{R}\backslash\{0\}$. We shall show that $\mathbb{P}^*M = (T^*M\backslash$ zero section$)/\sim$ has a contact structure.

Let $\mathfrak{C} = \{(p, \chi_p) | \chi_p$ is a hyperplane in $T_pM\}$, the collection of all contact elements in M. It can be naturally identified with $\mathbb{P}^*M$ under the well-defined map $\phi(p, [\zeta]) = (p, \ker\zeta)$. This also allows us to import the smooth structure of $\mathbb{P}^*M$ (given by $M \times \mathbb{RP}^{n-1}$) to $\mathfrak{C}$ to make it a $2n-1$-dimensional manifold. Now define a natural field of hyperplanes on $\mathfrak{C}$ given by $\mathfrak{H}_{(p,\chi_p)} = (\pi_*)^{-1}\chi_p$ where π is the projection of $\mathfrak{C}$ onto M. Under the differential of the map (now a diffeomorphism) ϕ from before, we get a field of hyperplanes $\mathbb{H}$ on $\mathbb{P}^*M$. In fact, we can show that $\mathbb{H}_{(p,[\zeta])} = \ker(\tilde{\pi}^*\zeta)$, where $\tilde{\pi}$ is the projection of $\mathbb{P}^*M$ onto M. This is not hard to see: $\ker\tilde{\pi}^*\zeta = (\tilde{\pi}_*)^{-1}\ker\zeta = ((\pi \circ \phi)_*)^{-1}\ker\zeta = (\phi_*)^{-1}((\pi_*)^{-1}(\ker\zeta)) = \mathbb{H}_{(p,[\zeta])}$ by definition. Not that the defining 1-form above is very similar to the canonical 1-from α on the cotangent bundle. This we expand on now.

Let us first clarify the coordinate structure on $\mathbb{P}^*M$. Let $(p, \chi_p) \in \mathfrak{C}$. Choose the coordinates $(x_1, ..., x_n)$ in a neighborhood U of $p \in M$ to be such that $\chi_p = \text{span}(\frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n})$. Now import these coordinates to $\mathbb{P}^*M$ via ϕ^{-1}. Given its definition, at $\phi^{-1}(p, \chi_p)$, at least one $\zeta_i \neq 0$. Given the fact that $\frac{\partial}{\partial x_1} \notin \ker\chi_p$, we can see that $\zeta_1 \neq 0$. Then $\zeta_i \neq 0$ in some neighborhood U of p as well. Thus, similar to what we do in projective space, we can scale our coordinates such that $\zeta_1 = 1$ in U and so $(x_1, ..., x_n, \zeta_2, ..., \zeta_n)$ play the role of our coordinates in U. If we define $\alpha = dx_1 + \sum_{i=2}^n \zeta_i dx_i$ in U, we can see that as $\alpha_{(p,[\zeta])} = \pi^*\zeta_p$ in some smaller neighborhood contained in U, α is a locally defining 1-form for $\mathbb{H}$. Moreover $d\alpha = \sum_{i=2}^n d\zeta_i dx_i$, and so $(d\alpha)^n = (n-1)! d\zeta_2 \wedge dx_2 \wedge \ldots d\zeta_n \wedge dx_n$. Given that $\mathbb{H}_(p, [\zeta]) = \text{span}(\frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial \zeta_1}, ..., \frac{\partial}{\partial \zeta_n})$, we can see that $d\alpha$ is nondegenerate on it. Hence α is a contact form, and $(\mathbb{P}^*M, \mathbb{H})$ is contact a manifold. By the identification above $(\mathfrak{C}, \mathfrak{H})$ is also a contact manifold, and it is canonical.

Example 2.1.5. Let $M = \mathbb{R}^3$. Then $T^*M \cong \mathbb{R}^6$ and $\mathfrak{C} \cong \mathbb{P}^*M \cong \mathbb{R}^3 \times \mathbb{RP}^2$ is the canonical contact manifold. △

Example 2.1.6. Let $M = S^1 \times S^1 \cong \mathbb{T}^2$. Then $T^*M \cong \mathbb{T}^2 \times \mathbb{R}^2$, and $\mathbb{P}^*M \cong \mathbb{T}^2 \times \mathbb{RP}^1$ is the canonical contact manifold. △

Now one could ask if there exists a local classification for contact structures similar to symplectic manifolds. To show this, we first define a technique that helps connect contact and symplectic structures directly: *symplectization.*

Theorem 2.1.7. *Let (M,H) be a $(2n-1)$-dimenisonal contact manifold with contact form α. Define $\tilde{M} = M \times \mathbb{R}$, with projection $\pi : \tilde{M} \to M$ and τ the additional coordinate on $\mathbb{R}$. Then $\omega = d(e^\tau \pi^*\alpha)$ is symplectic on $\tilde{M}$, and $(\tilde{M}, \omega)$ is called the symplectization of (M, H).*

Proof. We have:

$$\begin{aligned} \omega &= d(e^\tau \pi^*\alpha) = e^\tau (d\tau \wedge \pi^*\alpha + \pi^*(d\alpha)) \\ \Rightarrow \omega^n &= e^{n\tau}(nd\tau \wedge \pi^*\alpha \wedge \pi^*(d\alpha)^{n-1} + \pi^*(d\alpha)^n) \\ &= ne^{n\tau}(d\tau \wedge \pi^*(\alpha \wedge (d\alpha)^{n-1})). \end{aligned}$$

As $\alpha \wedge (d\alpha)^{n-1}$ is a volume form on M, it is not hard to see that ω^n above is also a volume form on $\tilde{M}$. Hence $(\tilde{M}, \omega)$ is a symplectic manifold. □

Now we prove the contact version of Darboux's Theorem.

Theorem 2.1.8. *Let (M,H) be a contact manifold and $p \in M$. Then there exists a neighborhood U of p with coordinates $(x_1, ..., x_n, y_1, ..., y_n, z)$ on which $\alpha = \sum_{i=1}^n x_i dy_i + dz$.*

Proof. Symplectize (M, H) and identify $M \cong M \times \{0\} \subset M \times \mathbb{R} = \tilde{M}$. As $d(e^\tau \pi^* \alpha)$ is symplectic on $\tilde{M}$, there exists a neighborhood $\tilde{U}$ of $(p, 0) \in \tilde{M}$ such that $d(e^\tau \pi^* \alpha) = \sum_{i=1}^n dx_i \wedge dy_i + d\tau \wedge dz = d(\sum_{i=1}^n x_i dy_i + \tau dz)$. Thus, by local exactness of closed forms, there exists a smaller neighborhood in $\tilde{U}$ such that $e^\tau \pi^* \alpha = \sum_{i=1}^n x_i dy_i + \tau dz + d\beta$ for some $v \in C^\infty(\tilde{M})$. It is clear from here that as there is no $d\tau$ term on the left side, β does not depend on τ. Moreover, restricting these forms to M where $\pi = id$ and $\tau = 0$, we get $\alpha = \sum_{i=1}^n x_i dy_i + d\beta$. We only need to now show that $(x_1, ..., x_n, y_1, ..., y_n, \beta)$ forms a coordinate system on $\pi(\tilde{U})$ containing p. It is sufficient to check that $\frac{\partial \beta}{\partial z} \neq 0$ for the Jacobian to be invertible. If we have otherwise, then $\frac{\partial}{\partial z} \in \ker\alpha$. But then we have that $d\alpha$ is degenerate on $\ker\alpha$, which contradicts it being a contact form. Hence $\frac{\partial \beta}{\partial z} \neq 0$, and we are done.[1] $\square$

We have now started to explore how symplectic manifolds have natural relationships to other structures (for instance, contact structure). Moving ahead, we shall continue in this direction, and now discuss the link between symplectic and complex geometry. This is usually the starting point for how modern-day symplectic geometry is studied.

Chapter 3

Almost Complex Structure

3.1 Complex Structure

Definition 3.1.1. *A complex structure on a vector space V is a linear map $J : V \to V$ with $J^2 = -Id$.*

J above plays a role that is similar to multiplication by the complex number i, and hence gives the structure of a complex vector space to V. To establish a connection between symplectic and complex structures, we introduce the following notion of compatibility between the two: *A complex structure J on a symplectic vector space (V, Ω) is considered compatible if $G_J(u, v) := \Omega(u, Jv)$ is a positive inner product on V.* This essentially boils down to checking two conditions:

Lemma 3.1.2. J is Ω-compatible if and only if $\Omega(Ju, Jv) = \Omega(u, v)$ and $\Omega(u, Ju) > 0$ for all $u \neq 0$.

Proof. First note that J is bijective. As such, we can take (Ju, v) to an arbitrary element of $V \times V$. Then we have:

$$G_J(Ju, v) = G_J(v, Ju) \Leftrightarrow \Omega(Ju, Jv) = \Omega(v, -u) = \Omega(u, v)$$

and $G_J(u, u) > 0$ for all $u \neq 0 \Leftrightarrow \Omega(u, Ju) > 0$ for all $u \neq 0$. □

Now the question arises as to whether such a compatible complex structure J exists for every symplectic vector space (V, Ω). It is not hard to see that it does: Take $e_1, ..., e_n, f_1, ..., f_n$ to be the canonical symplectic basis, and define $J(e_i) = f_i$ and $J(f_i) = -e_i$. This is Ω-compatible. In fact, given the first property in Lemma 3.1.2, the converse is true as well: *If J is Ω-compatible, then there exists a symplectic basis of the form $e_1, ..., e_n, Je_1, ..., Je_n$.* Given any complex structure J, this allows us to construct an Ω

such that J is Ω-compatible.

There is also a canonical (basis-independent) method of constructing a Ω-compatible complex structure J. Let G be any positive inner product on (V, Ω). Given that $\Omega(u, ;)$ and $G(u, ;)$ are isomorphism from V to V^*, there exists an operator A on V such that $\Omega(u, v) = G(Au, v)$. One can then perform polar decomposition on $A = \sqrt{AA^*}J$, where J turns out to be our desired complex structure.[2] The canonical nature of this construction implies that for smooth family of symplectic vector spaces (V_t, Ω_t), we may construct a corresponding smooth family of Ω_t-compatible complex structures J_t.

We now explore the connections between complex and symplectic structures on vector spaces. Let J be a compatible complex structure on a symplectic vector space (V, Ω). Let $e_1, ..., e_n, Je_1, ..., Je_n$ be the symplectic basis mentioned above. Considering V as vector space over $\mathbb{C}$ through J, we can see that this new vector space $V^{\mathbb{C}}$ has basis $e_1, ..., e_n$. Now let L be the matrix representation of an element of $GL(n, \mathbb{C})$. Let us separate the real and 'imaginary' parts of $L = X + J \circ Y (= X + iY)$ where $X, Y \in GL(n, \mathbb{R})$. This allows us to interpret L as an element of $GL(2n, \mathbb{R})$ acting on V as the original $\mathbb{R}$-vector space as follows: We have that $L(e_i) = X(e_i) + J(Y(e_i))$ and $L(J(e_i)) = J(L(e_i)) = J(X(e_i)) - Y(e_i)$, and so can be represented by the real $2n \times 2n$ matrix: $\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$. Note that a matrix of this form is orthogonal if and only if correspondingly $X + iY$ is unitary. This interpretation of $GL(n, \mathbb{C})$ allows us to study interactions between complex and symplectic structures.

Proposition 3.1.3. *The intersection of any two of Sp(2n), O(2n), GL(n,$\mathbb{C}$), is U(n) (identified as an element of GL(2n,$\mathbb{R}$) as above).*

Proof. It is clear from the last remark above that $GL(n, \mathbb{C}) \cap O(2n) = U(n)$. Let us compute the intersection of $Sp(2n)$ and $O(2n)$. Recall that $Sp(2n)$ is the set of symplectomorphisms of the standard symplectic euclidean space, and $O(2n)$ is the set of matrices L such that $L^{-1} = L^T$. In the symplectic basis, a matrix $A \in Sp(2n)$ must be such that $A^T \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix} A = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$. If A is additionally an orthogonal matrix, then we have $A^{-1} \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix} A = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$, which implies that the columns of A form an orthonormal basis $v_1, ..., v_n, w_1, ..., w_n$ such that $\begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix} v_i = w_i$ and

$\begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix} w_i = -v_i$, that is A is of the form $\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$. Moreover as $A^T A = Id$, we have $XX^T + YY^T = Id$ and $XY^T - YX^T = 0$. This gives us that $(X+iY)(X+iY)^\dagger = Id$. Hence A is unitary. The reverse inclusion follows similarly.
For a matrix $A \in Sp(2n) \cap GL(n, \mathbb{C})$, it is of the form $\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$ and $A^T \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix} A = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix} = A^{-1} \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix} A$. As $\begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$ is invertible, we get $AA^T = Id$. The reverse inclusion holds similarly. $\square$

3.2 Almost Complex Manifolds

Definition 3.2.1. *An* ***almost complex structure*** *on a manifold M is smooth field J of complex structures on the tangent spaces such that $J_x : T_xM \to T_xM$ and $J_x^2 = -Id$ for all $x \in M$. The pair (M, J) is called an* ***almost complex manifold.***

As was the case in symplectic vector spaces, we analogously have that an almost complex structure J on a symplectic manifold (M, ω) is considered *compatible* if g with $g_x : T_xM \times T_xM \to \mathbb{R}$ with $g(u, v) = \omega_x(u, J_x v)$ is a riemannian metric on M. After we have chosen any riemannian metric (which always exists), the existence of a compatible almost complex structure for any symplectic manifold follows from the canonical recipe discussed before. Its smoothness is also obvious due to coordinate independence.

By the recipe above, changing the choice of the riemannian metric may change the resulting compatible almost complex structure. How different can they be? The following proposition answers the question.

Theorem 3.2.2. *The set of all almost compatible complex structures on a symplectic manifold (M, ω) is path-connected.*

Proof. Consider any two almost compatible complex structures J_1 and J_2, with respective riemannian metrics g_1, g_2. Then $g_t = tg_1 + (1-t)g_2$ for $t \in [0, 1]$ is a family of riemannian metrics. Applying the canonical recipe of polar decomposition to (ω, g_t) gives a smooth family of almost compatible complex structures joining J_1 and J_2.[2] $\square$

At this point we have three structures defined on a symplectic manifold: a riemannian metric g, a symplectic form ω, and an almost complex structure J. It is easy to see that we can create any one using the other two: $g(u, v) = \omega(u, Jv)$, $\omega(u, v) = g(Ju, v)$, and $J(u) = \tilde{g}^{-}(\tilde{\omega}(u))$ where $\tilde{g}(u) = g(u, ;)$ and $\tilde{\omega}(u) = \omega(u, ;)$. Any triple (g, ω, J) that are related as above are considered a **compatible triple**. This remarkable three-way compatibility also allows us to show another result mirroring the above:

Theorem 3.2.3. *The set of all symplectic structures compatible with an almost complex manifold (M, J) is path-connected.*

Proof. Taking any two compatible symplectic forms ω_0 and ω_1, and corresponding riemannian metrics g_0 and g_1, define $\omega_t(u, v) = tg_0(Ju, v) + (1 - t)g_1(Ju, v)$. $\square$

We can further expand on the study of almost complex manifolds by considering the natural complexification of its tangent space $TM \otimes \mathbb{C}$, and extending J linearly as $J(v \otimes c) = (Jv) \otimes c$. This allows J to take on eigenvalues $+i, -i$, and partition the tangent space into the respective eigenspaces $T_{1,0} \oplus T_{0,1}$. It is not hard to see that $T_{1,0} = \{v \otimes 1 - Jv \otimes i | v \in TM\}$ and $T_{0,1} = \{v \otimes 1 + Jv \otimes i | v \in TM\}$. The elements of $T_{1,0}$ and $T_{0,1}$ are respectively called *J-holomorphic tangent vectors* and *J-antiholomorphic tangent vectors*. Similarly, the cotangent bundle can be complexified and split as $T^*M \otimes \mathbb{C} \cong T^{1,0} \oplus T^{0,1} = (T_{1,0})^* \oplus (T_{0,1})^*$. It is not hard to see again that $T^{1,0} = \{\zeta \otimes 1 - (\zeta \circ J) \otimes i | \zeta \in T^*M\}$ and $T^{0,1} = \{\zeta \otimes 1 + (\zeta \circ J) \otimes i | \zeta \in T^*M\}$.

Now we can define the set of *complex-valued forms* $\Omega^k(M; \mathbb{C})$. Define $\Lambda^{l,m} = \Lambda^l(T^{1,0}) \wedge \Lambda^m(T^{0,1})$, and then define $\Lambda^k(T^*M \otimes \mathbb{C}) = \oplus_{l+m=k}\Lambda^{l,m}$. Then define the set of forms of type (l, m) as $\Omega^{l,m}(M; \mathbb{C})$ =sections of $\Lambda^{l,m}$, and $\Omega^k(M; \mathbb{C}) = \oplus_{l+m=k}\Omega^{l,m}(M; \mathbb{C})$ = set of sections of $\Lambda^k(T^*M \otimes \mathbb{C})$. The exterior derivative $d : \Omega^k(M; \mathbb{C}) \to \Omega^{k+1}(M; \mathbb{C})$ as a map can have different projections onto different form types. Most notably used are $\partial = \pi^{l+1,m} \circ d : \Omega^{l,m}(M; \mathbb{C}) \to \Omega^{l+1,m}(M; \mathbb{C})$ and $\bar{\partial} = \pi^{l,m+1} \circ d : \Omega^{l,m}(M; \mathbb{C}) \to \Omega^{l,m+1}(M; \mathbb{C})$. Note that $d = \partial + \bar{\partial}$ on smooth real-valued functions $\Omega^{0,0}(M; \mathbb{C})$.

All this said, we still have not explicitly unraveled any connections to **complex manifolds**, i.e, manifolds that are locally biholomorphic to $\mathbb{C}^n$. In fact, we shall now see that every complex manifold is almost complex, allowing us to utilize all that we have discussed above.

3.3 Complex Manifolds

Definition 3.3.1. *A **complex manifold** of dimension n is topological manifold M equipped with a complete complex atlas $\mathfrak{A} = \{(U_\alpha \subset M, V_\alpha \subset \mathbb{C}^n, \phi_\alpha : U\alpha \to V_\alpha)\}$ where $\cup_\alpha U_\alpha = M$ and ϕ_α are biholomorphisms. Moreover, the transition maps are also biholomorphic.*

Theorem 3.3.2. *Every complex manifold has a canonical almost complex structure.*

Proof. Let M be an n-dimensional complex manifold with complex coordinates $(z_1, ..., z_n)$. Writing each $z_j = x_j + iy_j$, we get $(x_1, ..., x_n, y_1, ..., y_n)$ as real coordinates of M. Thus the tangent space at any point $p \in M$ has a real-basis $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_n}\}$. We can locally define a complex structure $J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}$ and $J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i}$. The fact that this is well-defined on all of M boils down to utilizing the *Cauchy-Riemann Equations* on the biholomorphic transition maps between intersecting coordinate charts.[2] $\square$

How does the differential structure we defined on almost complex manifolds before look like for complex manifolds? First notice that the complexification of the tangent space, in this case, is just given by $T_pM \otimes \mathbb{C} = \mathbb{C}\text{-span}\{\frac{\partial}{\partial x_j}|_p, \frac{\partial}{\partial y_j}|_p\}$. Solving for the eigenspaces of J, we get $T_{1,0} = \mathbb{C}\text{-span}\{\frac{\partial}{\partial x_j}|_p - i\frac{\partial}{\partial y_j}|_p\}$ and $T_{0,1} = \mathbb{C}\text{-span}\{\frac{\partial}{\partial x_j}|_p + i\frac{\partial}{\partial y_j}|_p\}$. Thus the corresponding complexified tangent bundle $T^{1,0}$ is given by $\mathbb{C}$-span $\{dx_j + idy_j\}$ and $T^{0,1} = \mathbb{C}$-span $\{dx_j - idy_j\}$. Letting $\frac{\partial}{\partial z_j}|_p = \frac{\partial}{\partial x_j}|_p - i\frac{\partial}{\partial y_j}|_p$, $\frac{\partial}{\partial \bar{z}_j}|_p = \frac{\partial}{\partial x_j}|_p + i\frac{\partial}{\partial y_j}|_p$, $dz_j = dx_j + idy_j$ and $\bar{d}z_j = dx_j - idy_j$, we get $T_{1,0} = \mathbb{C}\text{-span}\{\frac{\partial}{\partial z_j}\}$, $T_{0,1} = \mathbb{C}\text{-span}\{\frac{\partial}{\partial \bar{z}_j}\}$, $T^{1,0} = \mathbb{C}\text{-span}\{dz_j\}$ with $T^{0,1} = \mathbb{C}\text{-span}\{\bar{d}z_j\}$. The latter gives us a suitable basis to deduce that the general (l,m)-form looks like $\sum_{|J|=l,|L|=m} b_{J,K} dz_J \wedge d\bar{z}_K$ where J and K are multi-indices. In fact, the existence of such a basis implies that $d = \partial + \bar{\partial}$ on all forms on complex manifolds, where $\partial\beta = \sum_j \frac{\partial\beta}{\partial z_j} dz_j$ and $\bar{\partial}\beta = \sum_j \frac{\partial\beta}{\partial \bar{z}_j} dz_j$.[2]

Is the converse true? That is, if $d = \partial + \bar{\partial}$ on all complex-valued forms on an almost complex manifold, is it necessarily complex? If it is, such a J is called *integrable.* It was proven in 1957 in the *Newlander-Nirenberg theorem* that this converse is true. In fact, they introduced a much more tangible condition as well: the *Nijenhuis tensor* $\mathfrak{N}(v,w) = [Jv, Jw] - J[v, Jw] - J[Jv, w] - [v, w]$ is identically zero if and only if J is integrable. We prove one side of this assertion.

Proposition 3.3.3. *Assume M is a complex manifold and J is its canonical complex structure. Then $\mathfrak{N}$ vanishes everywhere.*

Proof. Consider the coordinates $(z_1, ..., z_n)$ with $z_j = x_j + iy_j$. Then $dz_j|_p = dx_j|_p + idy_j|_p$, and the real and imaginary parts form a basis for the $2n$-dimensional real vector space T_p^*M. Moreover, note that dz_j is J-holomorphic $(dz_j \circ J = idz_j)$ as $dz_j \in T^{1,0}$. If we evaluate $dz_j(\mathfrak{N}(v,w))$ we get:

$$\begin{aligned}
dz_j(\mathfrak{N}(v,w)) &= [Jv,Jw]z_j - J[v,Jw]z_j - J[Jv,w]z_j - [v,w]z_j \\
&= Jv(Jwz_j) - Jw(Jvz_j) - i(v(Jwz_j) - Jw(vz_j)) \\
&\quad - i(Jv(wz_j) - w(Jvz_j)) - (v(wz_j) - w(vz_j)) \\
&= iJv(wz_j) - iJw(vz_j) + v(wz_j) + iJw(vz_j) - iJv(wz_j) - w(vz_j) - v(wz_j) + w(vz_j) \\
&= 0,
\end{aligned}$$

where we have used the complex linearity of the map $f \mapsto vf$ to pull out the factors of i. As this holds for all j and points $p \in M$, and dz_j is a basis for T_p^*M, we must have $\mathfrak{N} \equiv 0$. □

Example 3.3.4. Assume the map $v \mapsto [v,w]$ is complex linear ($[Jv,w] = J[v,w]$). Then $\mathfrak{N}(v,w) = [Jv,Jw] - [Jv,Jw] - [-v,w] - [v,w] \equiv 0$, and J is integrable. △

Example 3.3.5. Assume M is a surface with an almost complex structure J. Then $\{v_p, J_p v_p\}$ forms a basis for any $v_p \in T_pM/\{0\}$. Consider then $\mathfrak{N}(v,v), \mathfrak{N}(Jv,Jv), \mathfrak{N}(v,Jv)$. All of them equal 0. Hence $\mathfrak{N} \equiv 0$, and J is integrable. Thus as any orientable surface is symplectic, and hence admits an almost complex structure, we can conclude that every orientable surface is a complex manifold. △

Note that in the proof of Proposition 3.3.3, the key point was the existence of *J-holomorphic functions* z_j whose differentials satisfied $dz_j \circ J = idz_j$. These naturally occur for complex manifolds, but are rare for almost complex manifolds. This is where we come to a crossroads. In studying symplectic manifolds with even non-integrable almost complex structures, the modern-day technique that was introduced by *Mikhael Gromov* utilizes *pseudo-holomorphic curves* that satisfy $df \circ i = J \circ df$. On the other, there is the study of symplectic manifolds that permit a complex manifold structure. Such a special class of manifolds, called *Kähler* manifolds, have a very rich contemporary study that we delve into next.

Bibliography

[1] V.I Arnold. *Mathematical Methods of Classical Mechanics.* Springer-Verlag, 1974.

[2] Annas Cannas de Silva. *Lectures on Symplectic Geometry*, volume 1764 of *Lecture Notes in Mathematics.* Springer-Verlag, 2001.