Symplectic Geometry and Toric Varieties

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Abstract

In this expository work, I have currently explored the fundamentals of symplectic geometry, and also classified their local behavior through symplectic equivalences given by the Darboux and Moser Theorems. Moreover, I have also talked about the role of closely related contact structures and associated Reeb vector fields. Finally, I end it by discussing the natural connections between symplectic and complex geometry, with notions of Dolbeault cohomolgy and even psuedo-holomorphic curves being touched upon. It may be noted that as of now, there has not been much exploration into the second part of this thesis: toric varieties/manifolds. This is because I wanted to cover all of the fundamentals of symplectic geometry in depth, even if some were relatively unrelated to the study of toric manifolds. This will enable me to explore other directions within this exciting field in the future. That said, as far as this thesis is concerned, I will now be moving on to study Hamiltonian actions and moment maps, until eventually breaking upon the grand correspondence between Delzant Polytopes and toric varieties.

Contents

Chapter 1

Symplectic Structure

1.1 Symplectic Vector Spaces

Let V be an n-dimensional vector space over $\mathbb{R}$ with basis $\{w_1, w_2, \dots, w_n\}$.

Definition 1.1 *A bilinear form $\Omega : V \times V \to \mathbb{R}$ that satisfies $\Omega(v,u) = -\Omega(u,v)$ for all $u, v \in V$ is called skew-symmetric.*

We know that every bilinear form L can be represented in the form of a matrix $\mathfrak{L}$ with $L(u,v) = u^T \mathfrak{L} v$ and $[\mathfrak{L}]_{ij} = L(w_i, w_j)$ considering the basis above. It is easy to see that every skew-symmetric bilinear form must have a skew-symmetric matrix representation. Moreover, we also have the following important result:

Theorem 1.2 *Let Ω be a skew-symmetric bilinear form on a real vector space V. Then there exists a basis in which the matrix for Ω is of the form:*

$$\begin{pmatrix} 0 & \mathrm{Id} & 0 & \dots & 0 \\ -\mathrm{Id} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

More precisely, the basis is given by $\{e_1, e_2, \dots, e_m, f_1, f_2, \dots, f_m, u_1, u_2, \dots, u_k\}$ where $\Omega(e_i, f_j) = \delta_{ij}$, $\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0$ for all i, j, and $\Omega(u_i, v) = 0$ for all $v \in V$.

Let us consider an example. Let $M = \mathbb{R}^6$ be a smooth 6-dimensional manifold and let T_pM be the tangent space at $p \in M$. Let $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ be a 2-form on

M. Then $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is a skew-symmetric bilinear form on the 6-dimensional real vector space T_pM (bilinear forms of this form will be the main object of interest for us moving ahead). It is easy to see that in the basis $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}\}$ of T_pM, the matrix representation for this 2-form is:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

It can also be seen that the basis $\{\frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}), \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}), \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}), \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}), \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}\}$ also gives the same matrix. So the basis in the theorem above is not unique.

Now consider the linear map induced by Ω given as $\tilde{\Omega} : V \to V^*$ with $\tilde{\Omega}(v) = \Omega(v, ;)$.

Definition 1.3 *Let (V, Ω) be a pair consisting of a real vector space V and an associated skew-symmetric bilinear form Ω. If ker$(\tilde{\Omega}) = \{0\}$, Ω is called non-degenerate and (V, Ω) is called a symplectic vector space.*

Note that this condition is equivalent to span$\{u_1, u_2, \ldots, u_k\} = \{0\}$, which implies that V must be even-dimensional. Clearly, (T_pM, ω_p) was not symplectic, but (T_pN, ω_p) with $N = \mathbb{R}^4$ is. Moreover, non-degenerate bilinear forms have matrix representations of the form:

$$\begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$$

There is also a natural notion of equivalence among symplectic vector spaces: *Two symplectic vector spaces (V_1, Ω_1) and (V_2, Ω_2) are called symplectomorphic if there exists a vector space isomorphism $\phi : V_1 \to V_2$ that preserves the symplectic structure.* By preservation of symplectic structure it is meant that the pullback $\phi^*\Omega_2 = \Omega_1$.

Now we are finally in a position to define the symplectic manifold.

1.2 Symplectic Manifolds

Definition 1.4 *A differential 2-form on a manifold M is called a symplectic form iff it is closed and ω_p is a non-degenerate bilinear form on T_pM for all $p \in M$.*

Definition 1.5 *A pair (M, ω) is called a symplectic manifold if ω is a symplectic form on M.*

As dimTpM=dim$M$, symplectic manifolds must be even-dimensional. Note that our example from before $(\mathbb{R}^4, \omega)$ is a symplectic manifold. Moreover, if we define a general 2-form $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$ with coordinates $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$, then $(\mathbb{R}^{2n}, \omega)$ is a symplectic manifold.

In fact, the condition of non-degeneracy above can be replaced with an equivalent and more informative one:

Lemma 1.6 *A differential 2-form $\omega \in \Omega^2(M)$ is non-degenerate if and only if dimM=2n is even, and ω^n is non-vanishing, that is, ω^n is a volume form.*
Proof. First note that by theorem 1.2, for dim$T_pM = 2n$, there exists a basis $\{e_1, ...e_m, f_1, ..., f_m, u_1, ..., u_{2n-2m}\}$ for T_pM such that $\omega_p(u_i, ;) = 0$, $\omega_p(e_i, f_j) = \delta_{ij}$ and $\omega_p(e_i, e_j) = \omega_p(f_i, f_j) = 0$ for all i, j. This gives us that $\omega_p = \sum_{i=1}^{n} e_i^* \wedge f_i^*$ where e_i^*, f_i^* are part of corresponding dual basis for T_p^*M.
Now assume ω_p is non-degenerate for all $p \in M$. Then we have that dim$M = 2n$ and $m = n$ from before:

$$\begin{aligned}
\omega_p \wedge .... \wedge \omega_p(\text{n times}) &= \sum_{i_1,..,i_n}^{n} e^{i_1} \wedge f^{i_1} \wedge ...... \wedge e^{i_n} \wedge f^{i_n}. \\
&= \sum_{i_1 \neq i_2 .... \neq i_n}^{n} e^{i_1} \wedge f^{i_1} \wedge ...... \wedge e^{i_n} \wedge f^{i_n} \\
&= \sum_{\sigma \in S_n} e^{\sigma(1)} \wedge f^{\sigma(1)} \wedge ...... \wedge e^{\sigma(n)} \wedge f^{\sigma(n)}.
\end{aligned}$$

Now we can see that any element of the sum above is invariant under the following switching of successive pairs of 2-forms: $e^{\sigma(1)} \wedge f^{\sigma(1)} \wedge .... \wedge e^{\sigma(k)} \wedge f^{\sigma(k)} \wedge e^{\sigma(k+1)} \wedge f^{\sigma(k+1)} .... \wedge e^{\sigma(n)} \wedge f^{\sigma(n)} = e^{\sigma(1)} \wedge f^{\sigma(1)} \wedge .... \wedge e^{\sigma(k+1)} \wedge f^{\sigma(k+1)} \wedge e^{\sigma(k)} \wedge f^{\sigma(k)} .... \wedge e^{\sigma(n)} \wedge f^{\sigma(n)}$. Thus each element in the sum above is equivalent, and hence:

$$\omega_p \wedge .... \wedge \omega_p(\text{n times}) = n!(e^1 \wedge f^1 \wedge ..... \wedge e^n \wedge f^n) \neq 0,$$

at any point. Thus ω^n is a volume form on M.
Now assume on the other hand that ω_p is degenerate for some $p \in M$, where $\dim M = 2n$. Then we have that $2n - 2m \geq 2$, and thus $n \geq m + 1$. We have, by a similar argument to before, $\omega_p^m = m!(e^1 \wedge f^1 \wedge ..... \wedge e^m \wedge f^m) \Rightarrow \omega_p^{m+1} = 0$ by anti-commutativity. Thus $\omega_p^n = 0$ and ω^n is not a volume form. ■

Our notion for equivalent symplectic vector spaces carries over to symplectic manifolds: *Two symplectic manifolds* (M_1, ω_1) *and* (M_2, ω_2) *are symplectomorphic if there exists a diffeomorphism* $\phi : M_1 \to M_2$ *with* $\phi^*\omega_2 = \omega_1$. The map ϕ is called a *symplectomorphism.*

One can ask a bunch of questions now: When does a symplectic form exist on an even-dimensional manifold? And if it does, when is it unique up to a symplectomorphism? That is, when is (M, ω_1) symplectomorphic to (M, ω_2)? Finally, does there at least exist some local commonality among all symplectic structures? As we will see, some of these questions have been fully answered, while others remain open problems to this day. For now, let us explore the existence of symplectic structures in different instances.

Consider the Mobius strip. Assume it has a symplectic form ω. Then we must have ω is a volume form, which then implies that the Mobius strip is orientable. This is a contradiction, and so the Mobius strip must not support a symplectic structure in the first place. This gives us a general result:

Proposition 1.7 *Every* $2n$*-dimensional symplectic manifold* (M, ω) *is orientable with orientation form* ω^n*.*

Is this a sufficient condition? Symplectic geometry would be very boring if it was: consider the orientable surface S^{2n}. Assume there exists a symplectic form ω on S^{2n}. Then ω^n is a volume form on S^{2n}. We can see that if this form is exact, we can apply Stoke's Theorem to get:

$$vol(S^{2n}) = \int_{S^{2n}} \omega^n = \int_{\partial S^{2n}} \alpha = 0,$$

where ω^n is assumed to be exact and equal to $d\alpha$. This is contradictory as S^{2n} is compact and hence $vol(S^{2n}) > 0$. Thus ω^n is not exact, which implies that ω is also not exact (if $\omega = d\alpha$, then $\omega^n = (d\alpha)^n = d(\alpha \wedge d\alpha^{n-1})$ is exact). Thus the de Rham cohomology class $[\omega] \neq 0$. If $n > 1$, this is contradictory to the fact that $H^2(S^{2n}) = 0$. Thus for $n > 1$, such an ω must not exist in the first place, and S^{2n} must not support a symplectic structure. That said, it is not difficult to see that S^2 supports a symplectic structure with its canonical area form being a symplectic form ($\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$).

Although we have seen smooth manifolds where no symplectic structure exists, there always exists an associated manifold with an intrinsic symplectic structure: the contangent bundle. Recall that the contangent bundle T^*M of an n-dimensional manifold M has local coordinates defined by $(x, \eta) \in T^*M \mapsto (x_1, ..., x_n, \eta_1, ..., \eta_n) \in \mathbb{R}^{2n}$. Define intrinsically the **Liouville** 1-form α on T^*M by $\alpha_{(x,\eta)} = \pi^*\eta$. Clearly, this definition is coordinate-independent. Furthermore, define the closed 2-form $\omega = -d\alpha$. Easy calculation shows that in any coordinate chart $(U, x_1, ..., x_n, \eta_1, ...\eta_n)$, $\alpha = \sum_{i=1}^n \eta_i dx_i$ and $\omega = \sum_{i=1}^n dx_i \wedge d\eta_i$. Through an identical calculation to that of Lemma 1.6, we have that ω is symplectic and (T^*M, ω) is a symplectic manifold.

It is clear that T^*M is a symplectic manifold intrinsically linked to the differential structure of M. Thus, it is no surprise that diffeomorphic spaces have symplectomorphic cotangent bundles:

Theorem 1.8 *Let $f : M_1 \to M_2$ be a diffeomorphism. Then there exists a symplectomorphism $g : (T^*M_1, \omega_1) \to (T^*M_2, \omega_2)$.*
Proof. Define $g = \bar{f} : (T^*M_1, \omega_1) \to (T^*M_2, \omega_2)$ by $\bar{f}(x, \eta) = (f(x), (f^{-1})^*\eta)$. First we show that this is a diffeomorphism. It is clear that $\bar{f}$ is bijective due to the bijectivity of f and the pullback f^*. Now consider any $p \in U = (x_1, ..., x_n, \eta_1, ..., \eta_n)$ and $\bar{f}(p) \in V = (y_1, ..., y_n, \bar{\eta}_1, ..., \bar{\eta}_n)$ To argue the smoothness of $\bar{f}$ at p, we have that one of the component functions f is smooth by definition, and for $(f^{-1})^*$ we have:

$$\begin{aligned}(f^{-1})^*\eta = \sum_{i=1}^n (\eta_i \circ f^{-1}) d(x_i \circ f^{-1}) &= \sum_{i,j}^n (\eta_i \circ f^{-1}) \frac{\partial(x_i \circ f^{-1})}{\partial y_j} dy_j \\ &= \sum_{j=1}^n \left(\sum_{i=1}^n (\eta_i \circ f^{-1}) \frac{\partial(x_i \circ f^{-1})}{\partial y_j} \right) dy_j.\end{aligned}$$

So $\eta_i(p) \mapsto \eta_i(p) \frac{\partial(x_i \circ f^{-1})}{\partial y_j}(f(p))$, which is clearly a smooth mapping. The same can be argued for $\bar{f}^{-1} = (f^{-1}, f^*)$, and thus $\bar{f}$ is a diffeomorphism. Finally, it is also not too hard to see that as $\pi_2 \circ \bar{f} = f \circ \pi_1$, and given the intrinsic definition of the Liouville form α above, $\bar{f}^*\alpha_2 = \alpha_1$ and hence $\bar{f}^*\omega_2 = \omega_1$.[1] ■

We can in fact show a partial converse: *Any symplectomorphism* $g : (T^*M, \omega) \to (T^*M, \omega)$ *that preserves* α *must be the lift of some diffeomorphism.*
Proof. Let $p = (x, \eta)$ and $g(p) = (y, \zeta)$. As $g^*\alpha = \alpha$, we have that at p, $g^*\alpha_{g(p)} = \alpha_p$ and hence $g^*(\pi^*\zeta_y) = \pi^*\eta_x$. Then $(\pi \circ g)^*\zeta_y = \pi^*\eta_x$ and $(\pi \circ g)^*(\lambda\zeta)_y = \pi^*(\lambda\eta)_x$ for any $\lambda \in \mathbb{R}$. This implies $g^*(\alpha_{(y,\lambda\zeta)}) = \alpha_{(x,\lambda\eta)}$. As g is bijective, we know that there exists $q = (z, \beta) \in T^*M$ such that $g(q) = (y, \lambda\zeta)$ and hence $g^*(\alpha_{(y,\lambda\zeta)}) = \alpha_{(z,\beta)}$. Given the explicit local form of α from before, it is obvious that $(z, \beta) = (x, \lambda\eta)$. Hence $g(x, \lambda\eta) = (y, \lambda\zeta)$. Specifically taking $\lambda = 0$, we have that $g(x, \eta) = (y, \zeta) \Rightarrow g(x, 0) = g(y, 0)$. This implies $\pi(g(x, \eta)) = y$ for all $\eta \in T_x^*M$. Thus we can simply define a diffeomorphism $h : M \to M_0 \subset T^*M$ with $h(x) = (x, 0)$, and have $f : M \to M$ be a diffeomorphism such that $f = h^{-1} \circ g|_{M_0} \circ h$. Clearly $\pi \circ g = f \circ \pi$. Taking p as before, we can see that the lift of f gives $\bar{f}(p) = (f(x), \gamma) = (y, \gamma)$ where $f^*\gamma_y = \eta_x \Rightarrow (f \circ \pi)^*\gamma_y = \pi^*(f^*\gamma_y) = \pi^*\eta_x = (\pi \circ g)^*\zeta_y$. Using the relation between f and g, we get $(\pi \circ g)^*\gamma_y = (\pi \circ g)^*\zeta_y$. As $\pi \circ g$ is a subjective submersion, we must have $\gamma_y = \zeta_y$ and hence $g = \bar{f}$. ■

Now although we have made some progress on our questions from before, we are still far from addressing the general question of symplectomorphic manifolds. To address this, we will first take a detour to understand some important sub-structures of symplectic manifolds.

1.3 Lagrangian Submanifolds

Recall that a submanifold X of M is defined to be a manifold with an immersive homeomorphism $i : X \to M$ where $i(X)$ is closed in M.

Definition 1.9 *Let* (M, ω) *be a 2n-dimensional symplectic manifold. Then a submanifold* Y *of dimension* n *with* $i^*\omega = \omega|_{T_pY} \equiv 0$ *is called a Lagrangian submanifold of* X*.*

A simple example of a Lagrangian submanifold is the *zero section*[1] of T^*M given by $M_0 = \{(x, \eta) \in T^*M |\, \eta = 0 \text{ in } T_x^*M\}$. This is clearly an n-dimensional submanifold of T^*M defined by $\eta_i = 0$ for all i. Moreover, $\alpha|_{T_pM_0} \equiv 0 \Rightarrow \omega|_{T_pM_0} \equiv 0$, and so M_0 is a Lagrangian submanifold of T^*M. One can in fact prove a more general result stating that for a 1-form μ on M, the section $M_\mu = \{(x, \mu_x) \in T^*M\}$ is a Lagrangian submanifold of

T^*M if and only if μ is closed[1]. This gives a whole family of Lagrangian submanifolds that can be *generated* by functions $f \in C^\infty(M)$ as M_{df}.

We now unravel the connection between our question about symplectomorphisms and Lagrangian submanifolds. Consider a diffeomorphism $\phi : (M_1, \omega_1) \to (M_2, \omega_2)$ where $\dim M_1 = \dim M_2 = 2n$. It induces a map $\psi : M_1 \to M_1 \times M_2$ given by $\psi(p) = (p, \phi(p))$. The image of ψ is the graph of the function ϕ, and is a closed embedding of M_1 into $M_1 \times M_2$. This gives us that M_1 is a $2n$-dimensional submanifold of $M_1 \times M_2$. We now impose on $M_1 \times M_2$ a symplectic structure given by the *twisted* closed 2-form $\bar{\omega} = \pi_1^*\omega_1 - \pi_2^*\omega_2$, where π_i is projection onto M_i. One can easily check that this is indeed a symplectic form. Considering $\psi^*\bar{\omega} = (\pi_1 \circ \psi)^*\omega_1 - (\pi_2 \circ \psi)^*\omega_2 = \omega_1 - \phi^*\omega_2$, we can see that ϕ is a symplectomorphism if and only if $\psi^*\bar{\omega} = 0$ and $\psi(M_1)$ is a Lagrangian submanifold of $(M_1 \times M_2, \bar{\omega})$. This gives us a new approach to the problem of finding symplectomorphisms!

Let's do an example. Consider $M_1 = (T^*X_1, \omega_1)$ and $M_2 = (T^*X_2, \omega_2)$ for some manifolds X_1 and X_2. We shall now work backwards; that is, look for Lagrangian submanifolds of $(M_1 \times M_2 \cong T^*(X_1 \times X_2), \bar{\omega})$. We know that it is easy to find Lagrangian submanifolds of $(M_1 \times M_2 \cong T^*(X_1 \times X_2), \omega = \pi_1^*\omega_1 + \pi_2^*\omega_2)$, where ω is just the canonical 2-form for the cotangent bundle $T^*(X_1 \times X_2)$. Assume we have such a Lagrangian submanifold Y. Then defining $\sigma : M_1 \times M_2 \to M_1 \times M_2$ as $\sigma(x, y, \eta_1, \eta_2) = (x, y, \eta_1, -\eta_2)$, we can see that $\sigma(Y)$ is a Lagrangian submanifold of $(M_1 \times M_2, \bar{\omega})$[1]. So if we start off with $Y = (M_1 \times M_2)_{df}$ for some $f \in C^\infty(X_1 \times X_2)$, we get $\sigma(Y) = (x, y, d_x f, -d_y f)$. And so, if this is the graph of a diffeomorphism $\phi : M_1 \to M_2$, then ϕ must be a symplectomorphism. Thus, $\phi(x, \eta) = (y, \zeta)$ must satisfy the following *Hamilton* equations:

$$\begin{aligned} \eta_i &= \frac{\partial f}{\partial x_i}(x, y) \\ \zeta_i &= -\frac{\partial f}{\partial y_i}(x, y). \end{aligned}$$

By the implicit function theorem, for a solution $y = \phi_1(x, \eta)$ to exist for the first differential equation, we must have $\left[\dfrac{\partial^2 f}{\partial y_j \partial x_i}\right]_{i,j}$ is invertible locally. Let us now apply this.

Consider a Riemannian manifold (X, g) that is *geodesically convex* and *geodesically complete*. This implies that for every $(x, v) \in TX$ there exists a unique minimizing geodesic of constant velocity v starting at x given as $\exp(x, v) : \mathbb{R} \to X$. If we define $f : X \times X \to \mathbb{R}$ as $f(x, y) = -\frac{1}{2}d(x, y)^2$, we can show that the symplectomorphism generated by f can be identified (through g) with the *geodesic flow* on X (the endomorphism $(x,v) \mapsto \exp(x,v)(1)$ of TX). Firstly, recall that the identification between TX and T^*X is given as $(x,v) \leftrightarrow g_x(v,;)$. Thus the Hamilton equations modify to:

$$g_x(v,;) = d_x f$$
$$g_y(w,;) = -d_y f,$$

where we aim to find (y,w) as function of (x,v). If we act both sides of the first equation on $\frac{d\exp(x,v)}{dt}(0) = v$, we get:

$$g_x(v,v) = d_x f(v) = \frac{d}{dt}(f(\exp(x,v)(t),y))(0)$$
$$= -\frac{1}{2}\frac{d}{dt}(d(\exp(x,v)(t),y)^2)(0).$$

We can see that if we take $y = \exp(x,v)(1)$ we get by definition of the Riemannian distance that $d(\exp(x,v)(t),\exp(x,v)(1)) = \int_t^1 \sqrt{g_x\left(\frac{d\exp(x,v)}{dt},\frac{d\exp(x,v)}{dt}\right)} = \int_t^1 \sqrt{g_x(v,v)} = \sqrt{g_x(v,v)}(1-t)$. Plugging this into the equation above confirms that this is indeed the unique solution to the first Hamilton equation. Moreover, if we act $\frac{d\exp(x,v)}{dt}(1) = v$ on both sides of the second equation, we get:

$$g_x(w,v) = -d_y f(v) = \frac{d}{dt}(f(\exp(x,v)(t),\exp(x,v)(0)))(1)$$
$$= \frac{1}{2}\frac{d}{dt}(d(\exp(x,v)(t),\exp(x,v)(0))^2)(1).$$

Through a similar calculation as before, we get $d(\exp(x,v)(t),\exp(x,v)(0)) = -t\sqrt{g_x(v,v)}$, and plugging this into the equation above gives $g_x(w,v) = g_x(v,v)$. Moreover, if we consider any orthogonal $\bar{v} \in T_yX$ to v, we get $g_x(w,\bar{v}) = \frac{1}{2}\frac{d}{dt}(d(\exp(y,\bar{v})(t),\exp(x,v)(0))^2)(0)$. As $\exp(x,v)$ is a minimizing geodesic, we have that $d(\exp(y,\bar{v})(t),\exp(x,v)(0))$ is minimum when the two geodesics intersect at $t=0$. Thus the derivative expression above is 0 and $g(w,\bar{v}) = 0$ for all orthogonal $\bar{v} \in T_yX$ to v. This implies $g(w,;) = g(v,;)$. By non-degeneracy of g, the identification we made before is bijective, and hence $w = v = \exp(x,v)(1)$. And so, the symplectomorphism f can be identified with the endomorphism $(x,v) \mapsto \exp(x,v)(1)$ of TX.

Now that we have some familiarity with the construction and finding of symplectomorphisms, we can start to explore questions about the existence of various kinds of equivalences between symplectic structures.

1.4 Symplectic Equivalences

We first recall a few definitions and ideas.

Definition 1.10 *An isotopy is a map $\rho : M \times \mathbb{R} \to M$ such that $\rho_t = \rho(;, t) : M \to M$ is a diffeomorphism for all t, and $\rho_0 = id_M$.*

Every isoptopy gives an associated time-dependent vector field v_t given by $v_t = \frac{d\rho_t}{dt} \circ \rho_t^{-1}$. For a compact manifold, the converse is also true: for any time-dependent vector field v_t, there exists an isotopy satisfying the differential equation $\frac{d\rho_t}{dt} = v_t \circ \rho_t$, with initial condition $\rho_0 = id_M$. By Picard–Lindelof theorem, this also exists locally for non-compact manifolds. This allows us to define the *Lie derivative* along v_t as a map $\mathfrak{L}_{v_t} : \Omega^k(M) \to \Omega^k(M)$ given by $\mathfrak{L}_{v_t}\omega = \frac{d}{dt}(\rho_t^*\omega)|_{t=0}$, where ρ is the local isotopy associated with v_t. The Lie derivative satisfies the *Cartan magic formula:* $\mathfrak{L}_{v_t}\omega = i_{v_t}d\omega + di_{v_t}\omega$, and as $\rho_{h+t} = \rho_t \circ \rho_h$ (by uniqueness of solution to the differential equation from before), it also satisfies $\frac{d}{dt}\rho_t^*\omega|_{t=\tau} = \rho_\tau^*\mathfrak{L}_{v_t}\omega$. By the chain rule, this gives $\frac{d}{dt}\rho_t^*\omega_t = \rho_t^*(\mathfrak{L}_{v_t}\omega_t + \frac{d\omega_t}{dt})$.

Moreover, recall that the *normal space* to a k-dimensional submanifold $S \subset M$ at $p \in S$ is defined as $n-k$ dimensional quotient vector space $N_pS = T_pM/T_pS$. This gives us the *normal bundle* NS, an n-dimensional manifold. It is not hard to see that S embeds into NS as the zero-section as well. In fact, the following well-known result gives us that these embeddings are locally the same:

Theorem 1.11 (Tubular neighborhood Theorem) *There exists a neighborhood U_0 of S in NS, and a neighborhood U of S in M, and a diffeomorphism $\phi : U_0 \to U$ with $\phi|_S = id_S$. Moreover, U_0 in convex in each N_pS.*

This gives us that S is a deformation retract of U and hence the inclusion induces an isomorphism of de Rham cohomologies. Now we return to the question from before, but slightly altered: *Does there exist a symplectomorphism between (M, ω_0) and (M, ω_1) that is homotopic to the identity?* The reason we have added this extra condition is that it restricts our problem to the specific case where $[\omega_0] = [\omega_1]$. Naturally, this problem is easier.

Define $S_c = \{\text{symplectic forms } \omega \text{ on } M \text{ with } [\omega] = c\}$. Assume $\omega_0, \omega_1 \in S_c$, and that there exists a straight line path connecting both of them in S_c, i.e, $\omega_t = t\omega_0 + (1-t)\omega_1$ is symplectic for all $t \in [0, 1]$, and as $\omega_0 - \omega_1 = d\mu$ for some 1-form μ on M, we have $\omega_t = \omega_1 + td\mu \in S_c$ for all t. We claim that this is a sufficient condition for the existence

of a symplectomorphism that is homotopic to the identity.
Note that if the above is true, then (M, ω_t) is symplectomorphic to (M, ω_0) for all t by reparameterizing the path ω_t. All of these implications can be satisfied if there exists an isotopy ρ with $\rho_t^*\omega_t = \omega_0$. Looking for the existence of such an isotopy is exactly what is known as *Moser's trick*. If we additionally assume that M is compact, we can instead look for the associated vector field v_t, and integrate to get ρ_t. We have the following condition on v_t[1]:

$$\frac{d}{dt}(\rho_t^*\omega_t) = 0 = \rho_t^*\left(\mathfrak{L}_{v_t}\omega_t + \frac{d\omega_t}{dt}\right) \Leftrightarrow di_{v_t}\omega_t + d\mu = 0.$$

We can find a solution for this easily by solving for v_t such that $i_{v_t}\omega_t + \mu = 0$ (as ω_t is nondegenerate). Thus, we obtain the isotopy we wanted. One can generalize this argument to any $\omega_0, \omega_1 \in S_c$ that lie in the same path-connected component:

Theorem 1.12 (Moser Theorem) *Let M be compact manifold with symplectic forms ω_0 and ω_1 connected by a smooth family of symplectic forms ω_t, with $0 \leq t \leq 1$, such that $[\omega_t] = c$ is constant. Then there exists an isotopy ρ of M with $\rho_t^*\omega_t = \omega_0$.*

Let's apply this to a specific example.

Proposition 1.13 *Let M a compact and oriented 2-manifold. Then each 2-cohomology class has a unique symplectic representative.*
Proof. Consider an orientable and compact 2-dimensional manifold M with areas forms ω_1 and ω_2. Clearly, these are symplectic forms. Assume further that $[\omega_1] = [\omega_2]$. First, let us show that ω_1 and ω_2 determine the same orientation. Assume they do not. Then $\omega_1 = -f\omega_2$ for some smooth $f > 0$. Moreover, we have $\int_M \omega_1 - \omega_2 = \int_M (1+f)\omega_1 > 0$. This is a contradiction as $\omega_1 - \omega_2 = d\mu$ is exact, and hence by Stoke's theorem $\int_M \omega_1 - \omega_2 = 0$. Thus ω_1 and ω_2 must determine the same orientation, and $\omega_1 = f\omega_2$ for some smooth $f > 0$. Now consider the smooth straight line path $\omega_t = t\omega_1 + (1-t)\omega_2 = (1 - t + ft)\omega_2$. As $t \in [0, 1]$, ω_t can only vanish at some point p if $f(p) = (t-1)/t \leq 0$. This is a contradiction. Hence ω_t is symplectic for all t. By Moser Theorem, the symplectic structure induced by ω_1 and ω_2 is the same. ■

Moser Theorem also allows us to find a symplectic equivalence if two different symplectic forms agree on a compact submanifold. Recall the tubular neighborhood theorem. If we consider X to be a compact submanifold of M, then there is a neighborhood U of X such that X is a deformation retract of U. If ω_1 and ω_2 are two symplectic forms that agree on X, i.e, $i^*(\omega_1 - \omega_2)|_X = 0$, then $(\omega_1 - \omega_2)|_U = d\mu$ is exact by homotopy invariance of the de Rham cohomology. This allows one to define a smooth family $\omega_t = \omega_2 + td\mu$ that vanishes on X. After some shrinking of neighborhoods, and applying Moser's Trick again[1], we have another theorem by Moser:

Theorem 1.14 (Relative Moser Theorem) *Let X be a compact submanifold of a symplectic manifold M with two symplectic forms ω_1 and ω_2 agreeing on X. Then there exists neighborhoods U and V of X in M, and a diffeomorphism $\phi : U \to V$ with $\phi^*\omega_2 = \omega_1$ and $\phi|_X = id_X$.*

If we take $X = p$, and use the canonical symplectic basis for the T_pM, we finally get the grand theorem classifying symplectic structures locally:

Theorem 1.15 (Darboux Theorem) *Let (M, ω) be a symplectic manifold, and let p be any point in M. Then we can find a coordinate system $(U, x_1, ..., x_n, y_1, ...y_n)$ centered at p such that on U:*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

If X is compact Lagrangian submanifold of both (M, ω_1) and (M, ω_2), i.e, $i^*\omega_1 = i^*\omega_2 = 0$, then relative Moser applies identically to give what is known as the *Weinstein Lagrangian Neighborhood Theorem*. This theorem is imperative in helping us classify Lagrangian structures locally. But first, we prove a small result.

Lemma 1.16 *Let (V, Ω) be symplectic vector space with a lagrangian subspace U. Then there exists a canonical isomorphism between V/U and U^*.*
Proof. Define $\tilde{\Omega} : V/U \to U^*$ given by $\tilde{\Omega}([v]) = \Omega(v, ;)$. This is well-defined as for any $v \oplus u$, $\Omega(v \oplus u, ;) = \Omega(v, ;) + \Omega(u, ;) = \Omega(v, ;)$ as $\Omega|_U = 0$. Moreover, it is a non-degenerate pairing, and so the map above is an isomorphism. ∎

Note that this allows us to canonically identify NX and T^*X. Now recall that every manifold X can be embedded into (T^*X, ω_0) as a Lagrangian submanifold. Assume X is Lagrangian submanifold of (M, ω). By using the tubular neighborhood theorem with the identification $NX \cong T^*X$, followed by Weinstein's Lagrangian neighborhood theorem, we obtain neighborhoods U of X in (T^*X, ω_0), V of X in M, and a composite symplectomorphism $\phi : (U, \omega_0) \to (V, \omega)$ that is identity on X. This implies that every Lagrangian embedding is symplectically locally equivalent to the embedding of a manifold as the zero-section of its cotangent space! This classification is called the *Weinstein tubular neighborhood theorem*.

This result above has important applications.

Chapter 2

Contact Structure

2.1 Contact Manifolds

Bibliography

[1] Annas Cannas de Silva. *Lectures on Symplectic Geometry*, volume 1764 of *Lecture Notes in Mathematics*. Springer-Verlag, 2001.