

# Symplectic Geometry and Toric Varieties



Abdullah Ahmed

Syed Babar Ali School of Science and Engineering

Lahore University of Management Sciences

Mathematics Department

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# Abstract

In this expository work, I have currently explored the fundamentals of symplectic geometry, and also classified their local behavior through symplectic equivalences given by the Darboux and Moser Theorems. Moreover, I have also talked about the role of closely related contact structures and associated Reeb vector fields. Finally, I end it by discussing the natural connections between symplectic and complex geometry, with notions of Dolbeault cohomology and even pseudo-holomorphic curves being touched upon. It may be noted that as of now, there has not been much exploration into the second part of this thesis: toric varieties/manifolds. This is because I wanted to cover all of the fundamentals of symplectic geometry in depth, even if some were relatively unrelated to the study of toric manifolds. This will enable me to explore other directions within this exciting field in the future. That said, as far as this thesis is concerned, I will now be moving on to study Hamiltonian actions and moment maps, until eventually breaking upon the grand correspondence between Delzant Polytopes and toric varieties.

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# Chapter 1

## Symplectic Structure

### 1.1 Symplectic Vector Spaces

Let  $V$  be an  $n$ -dimensional vector space over  $\mathbb{R}$  with basis  $\{w_1, w_2, \dots, w_n\}$ .

**Definition 1.1** *A bilinear form  $\Omega : V \times V \rightarrow \mathbb{R}$  that satisfies  $\Omega(v, u) = -\Omega(u, v)$  for all  $u, v \in V$  is called skew-symmetric.*

We know that every bilinear form  $L$  can be represented in the form of a matrix  $\mathfrak{L}$  with  $L(u, v) = u^T \mathfrak{L} v$  and  $[\mathfrak{L}]_{ij} = L(w_i, w_j)$  considering the basis above. It is easy to see that every skew-symmetric bilinear form must have a skew-symmetric matrix representation. Moreover, we also have the following important result:

**Theorem 1.2** *Let  $\Omega$  be a skew-symmetric bilinear form on a real vector space  $V$ . Then there exists a basis in which the matrix for  $\Omega$  is of the form:*

$$\begin{pmatrix} 0 & \text{Id} & 0 & \dots & 0 \\ -\text{Id} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

*More precisely, the basis is given by  $\{e_1, e_2, \dots, e_m, f_1, f_2, \dots, f_m, u_1, u_2, \dots, u_k\}$  where  $\Omega(e_i, f_j) = \delta_{ij}$ ,  $\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0$  for all  $i, j$ , and  $\Omega(u_i, v) = 0$  for all  $v \in V$ .*

Let us consider an example. Let  $M = \mathbb{R}^6$  be a smooth 6-dimensional manifold and let  $T_p M$  be the tangent space at  $p \in M$ . Let  $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$  be a 2-form on

$M$ . Then  $\omega_p : T_p M \times T_p M \rightarrow \mathbb{R}$  is a skew-symmetric bilinear form on the 6-dimensional real vector space  $T_p M$  (bilinear forms of this form will be the main object of interest for us moving ahead). It is easy to see that in the basis  $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}\}$  of  $T_p M$ , the matrix representation for this 2-form is:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

It can also be seen that the basis  $\{\frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}), \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}), \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}), \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}), \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}\}$  also gives the same matrix. So the basis in the theorem above is not unique.

Now consider the linear map induced by  $\Omega$  given as  $\tilde{\Omega} : V \rightarrow V^*$  with  $\tilde{\Omega}(v) = \Omega(v, \cdot)$ .

**Definition 1.3** Let  $(V, \Omega)$  be a pair consisting of a real vector space  $V$  and an associated skew-symmetric bilinear form  $\Omega$ . If  $\ker(\tilde{\Omega}) = \{0\}$ ,  $\Omega$  is called non-degenerate and  $(V, \Omega)$  is called a symplectic vector space.

Note that this condition is equivalent to  $\text{span}\{u_1, u_2, \dots, u_k\} = \{0\}$ , which implies that  $V$  must be even-dimensional. Clearly,  $(T_p M, \omega_p)$  was not symplectic, but  $(T_p N, \omega_p)$  with  $N = \mathbb{R}^4$  is. Moreover, non-degenerate bilinear forms have matrix representations of the form:

$$\begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix}$$

There is also a natural notion of equivalence among symplectic vector spaces: *Two symplectic vector spaces  $(V_1, \Omega_1)$  and  $(V_2, \Omega_2)$  are called symplectomorphic if there exists a vector space isomorphism  $\phi : V_1 \rightarrow V_2$  that preserves the symplectic structure.* By preservation of symplectic structure it is meant that the pullback  $\phi^* \Omega_2 = \Omega_1$ .

Now we are finally in a position to define the symplectic manifold.

## 1.2 Symplectic Manifolds

**Definition 1.4** A differential 2-form on a manifold  $M$  is called a symplectic form iff it is closed and  $\omega_p$  is a non-degenerate bilinear form on  $T_p M$  for all  $p \in M$ .

**Definition 1.5** A pair  $(M, \omega)$  is called a symplectic manifold if  $\omega$  is a symplectic form on  $M$ .

As  $\dim T_p M = \dim M$ , symplectic manifolds must be even-dimensional. Note that our example from before  $(\mathbb{R}^4, \omega)$  is a symplectic manifold. Moreover, if we define a general 2-form  $\omega = \sum_{i=1}^n dx_i \wedge dy_i$  on  $\mathbb{R}^{2n}$  with coordinates  $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$ , then  $(\mathbb{R}^{2n}, \omega)$  is a symplectic manifold.

In fact, the condition of non-degeneracy above can be replaced with an equivalent and more informative one:

**Lemma 1.6** A differential 2-form  $\omega \in \Omega^2(M)$  is non-degenerate if and only if  $\dim M = 2n$  is even, and  $\omega^n$  is non-vanishing, that is,  $\omega^n$  is a volume form.

*Proof.* First note that by theorem 1.2, for  $\dim T_p M = 2n$ , there exists a basis  $\{e_1, \dots, e_m, f_1, \dots, f_m, u_1, \dots, u_{2n-2m}\}$  for  $T_p M$  such that  $\omega_p(u_i, \cdot) = 0$ ,  $\omega_p(e_i, f_j) = \delta_{ij}$  and  $\omega_p(e_i, e_j) = \omega_p(f_i, f_j) = 0$  for all  $i, j$ . This gives us that  $\omega_p = \sum_{i=1}^n e_i^* \wedge f_i^*$  where  $e_i^*, f_i^*$  are part of corresponding dual basis for  $T_p^* M$ .

Now assume  $\omega_p$  is non-degenerate for all  $p \in M$ . Then we have that  $\dim M = 2n$  and  $m = n$  from before:

$$\begin{aligned} \omega_p \wedge \dots \wedge \omega_p (n \text{ times}) &= \sum_{i_1, \dots, i_n} e^{i_1} \wedge f^{i_1} \wedge \dots \wedge e^{i_n} \wedge f^{i_n}. \\ &= \sum_{i_1 \neq i_2 \dots \neq i_n} e^{i_1} \wedge f^{i_1} \wedge \dots \wedge e^{i_n} \wedge f^{i_n} \\ &= \sum_{\sigma \in S_n} e^{\sigma(1)} \wedge f^{\sigma(1)} \wedge \dots \wedge e^{\sigma(n)} \wedge f^{\sigma(n)}. \end{aligned}$$

Now we can see that any element of the sum above is invariant under the following switching of successive pairs of 2-forms:  $e^{\sigma(1)} \wedge f^{\sigma(1)} \wedge \dots \wedge e^{\sigma(k)} \wedge f^{\sigma(k)} \wedge e^{\sigma(k+1)} \wedge f^{\sigma(k+1)} \dots \wedge e^{\sigma(n)} \wedge f^{\sigma(n)} = e^{\sigma(1)} \wedge f^{\sigma(1)} \wedge \dots \wedge e^{\sigma(k+1)} \wedge f^{\sigma(k+1)} \wedge e^{\sigma(k)} \wedge f^{\sigma(k)} \dots \wedge e^{\sigma(n)} \wedge f^{\sigma(n)}$ . Thus

each element in the sum above is equivalent, and hence:

$$\omega_p \wedge \dots \wedge \omega_p (n \text{ times}) = n!(e^1 \wedge f^1 \wedge \dots \wedge e^n \wedge f^n) \neq 0,$$

at any point. Thus  $\omega^n$  is a volume form on  $M$ .

Now assume on the other hand that  $\omega_p$  is degenerate for some  $p \in M$ , where  $\dim M = 2n$ . Then we have that  $2n - 2m \geq 2$ , and thus  $n \geq m + 1$ . We have, by a similar argument to before,  $\omega_p^m = m!(e^1 \wedge f^1 \wedge \dots \wedge e^m \wedge f^m) \Rightarrow \omega_p^{m+1} = 0$  by anti-commutativity. Thus  $\omega_p^n = 0$  and  $\omega^n$  is not a volume form. ■

Our notion for equivalent symplectic vector spaces carries over to symplectic manifolds: *Two symplectic manifolds  $(M_1, \omega_1)$  and  $(M_2, \omega_2)$  are symplectomorphic if there exists a diffeomorphism  $\phi : M_1 \rightarrow M_2$  with  $\phi^*\omega_2 = \omega_1$ .* The map  $\phi$  is called a *symplectomorphism*.

One can ask a bunch of questions now: When does a symplectic form exist on an even-dimensional manifold? And if it does, when is it unique up to a symplectomorphism? That is, when is  $(M, \omega_1)$  symplectomorphic to  $(M, \omega_2)$ ? Finally, does there at least exist some local commonality among all symplectic structures? As we will see, some of these questions have been fully answered, while others remain open problems to this day. For now, let us explore the existence of symplectic structures in different instances.

Consider the Mobius strip. Assume it has a symplectic form  $\omega$ . Then we must have  $\omega$  is a volume form, which then implies that the Mobius strip is orientable. This is a contradiction, and so the Mobius strip must not support a symplectic structure in the first place. This gives us a general result:

**Proposition 1.7** *Every  $2n$ -dimensional symplectic manifold  $(M, \omega)$  is orientable with orientation form  $\omega^n$ .*

Is this a sufficient condition? Symplectic geometry would be very boring if it was: consider the orientable surface  $S^{2n}$ . Assume there exists a symplectic form  $\omega$  on  $S^{2n}$ . Then  $\omega^n$  is a volume form on  $S^{2n}$ . We can see that if this form is exact, we can apply Stoke's Theorem to get:

$$\text{vol}(S^{2n}) = \int_{S^{2n}} \omega^n = \int_{\partial S^{2n}} \alpha = 0,$$

where  $\omega^n$  is assumed to be exact and equal to  $d\alpha$ . This is contradictory as  $S^{2n}$  is compact and hence  $\text{vol}(S^{2n}) > 0$ . Thus  $\omega^n$  is not exact, which implies that  $\omega$  is also not exact (if  $\omega = d\alpha$ , then  $\omega^n = (d\alpha)^n = d(\alpha \wedge d\alpha^{n-1})$  is exact). Thus the de Rham cohomology class  $[\omega] \neq 0$ . If  $n > 1$ , this is contradictory to the fact that  $H^2(S^{2n}) = 0$ . Thus for  $n > 1$ , such an  $\omega$  must not exist in the first place, and  $S^{2n}$  must not support a symplectic structure. That said, it is not difficult to see that  $S^2$  supports a symplectic structure with its canonical area form being a symplectic form ( $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ ).

Although we have seen smooth manifolds where no symplectic structure exists, there always exists an associated manifold with an intrinsic symplectic structure: the cotangent bundle. Recall that the cotangent bundle  $T^*M$  of an  $n$ -dimensional manifold  $M$  has local coordinates defined by  $(x, \eta) \in T^*M \mapsto (x_1, \dots, x_n, \eta_1, \dots, \eta_n) \in \mathbb{R}^{2n}$ . Define intrinsically the **Liouville** 1-form  $\alpha$  on  $T^*M$  by  $\alpha_{(x, \eta)} = \pi^*\eta$ . Clearly, this definition is coordinate-independent. Furthermore, define the closed 2-form  $\omega = -d\alpha$ . Easy calculation shows that in any coordinate chart  $(U, x_1, \dots, x_n, \eta_1, \dots, \eta_n)$ ,  $\alpha = \sum_{i=1}^n \eta_i dx_i$  and  $\omega = \sum_{i=1}^n dx_i \wedge d\eta_i$ . Through an identical calculation to that of Lemma 1.6, we have that  $\omega$  is symplectic and  $(T^*M, \omega)$  is a symplectic manifold.

It is clear that  $T^*M$  is a symplectic manifold intrinsically linked to the differential structure of  $M$ . Thus, it is no surprise that diffeomorphic spaces have symplectomorphic cotangent bundles:

**Theorem 1.8** *Let  $f : M_1 \rightarrow M_2$  be a diffeomorphism. Then there exists a symplectomorphism  $g : (T^*M_1, \omega_1) \rightarrow (T^*M_2, \omega_2)$ .*

*Proof.* Define  $g = \bar{f} : (T^*M_1, \omega_1) \rightarrow (T^*M_2, \omega_2)$  by  $\bar{f}(x, \eta) = (f(x), (f^{-1})^*\eta)$ . First we show that this is a diffeomorphism. It is clear that  $\bar{f}$  is bijective due to the bijectivity of  $f$  and the pullback  $f^*$ . Now consider any  $p \in U = (x_1, \dots, x_n, \eta_1, \dots, \eta_n)$  and  $\bar{f}(p) \in V = (y_1, \dots, y_n, \bar{\eta}_1, \dots, \bar{\eta}_n)$ . To argue the smoothness of  $\bar{f}$  at  $p$ , we have that one of the component functions  $f$  is smooth by definition, and for  $(f^{-1})^*$  we have:

$$\begin{aligned} (f^{-1})^*\eta &= \sum_{i=1}^n (\eta_i \circ f^{-1}) d(x_i \circ f^{-1}) = \sum_{i,j} (\eta_i \circ f^{-1}) \frac{\partial(x_i \circ f^{-1})}{\partial y_j} dy_j \\ &= \sum_{j=1}^n \left( \sum_{i=1}^n (\eta_i \circ f^{-1}) \frac{\partial(x_i \circ f^{-1})}{\partial y_j} \right) dy_j. \end{aligned}$$

So  $\eta_i(p) \mapsto \eta_i(p) \frac{\partial(x_i \circ f^{-1})}{\partial y_j}(f(p))$ , which is clearly a smooth mapping. The same can be argued for  $\bar{f}^{-1} = (f^{-1}, f^*)$ , and thus  $\bar{f}$  is a diffeomorphism. Finally, it is also not too



hard to see that as  $\pi_2 \circ \bar{f} = f \circ \pi_1$ , and given the intrinsic definition of the Liouville form  $\alpha$  above,  $\bar{f}^* \alpha_2 = \alpha_1$  and hence  $\bar{f}^* \omega_2 = \omega_1$ . [1] ■

We can in fact show a partial converse: *Any symplectomorphism  $g : (T^*M, \omega) \rightarrow (T^*M, \omega)$  that preserves  $\alpha$  must be the lift of some diffeomorphism.*

*Proof.* Let  $p = (x, \eta)$  and  $g(p) = (y, \zeta)$ . As  $g^* \alpha = \alpha$ , we have that at  $p$ ,  $g^* \alpha_{g(p)} = \alpha_p$  and hence  $g^*(\pi^* \zeta_y) = \pi^* \eta_x$ . Then  $(\pi \circ g)^* \zeta_y = \pi^* \eta_x$  and  $(\pi \circ g)^*(\lambda \zeta)_y = \pi^*(\lambda \eta)_x$  for any  $\lambda \in \mathbb{R}$ . This implies  $g^*(\alpha_{(y, \lambda \zeta)}) = \alpha_{(x, \lambda \eta)}$ . As  $g$  is bijective, we know that there exists  $q = (z, \beta) \in T^*M$  such that  $g(q) = (y, \lambda \zeta)$  and hence  $g^*(\alpha_{(y, \lambda \zeta)}) = \alpha_{(z, \beta)}$ . Given the explicit local form of  $\alpha$  from before, it is obvious that  $(z, \beta) = (x, \lambda \eta)$ . Hence  $g(x, \lambda \eta) = (y, \lambda \zeta)$ . Specifically taking  $\lambda = 0$ , we have that  $g(x, \eta) = (y, \zeta) \Rightarrow g(x, 0) = g(y, 0)$ . This implies  $\pi(g(x, \eta)) = y$  for all  $\eta \in T_x^*M$ . Thus we can simply define a diffeomorphism  $h : M \rightarrow M_0 \subset T^*M$  with  $h(x) = (x, 0)$ , and have  $f : M \rightarrow M$  be a diffeomorphism such that  $f = h^{-1} \circ g|_{M_0} \circ h$ . Clearly  $\pi \circ g = f \circ \pi$ . Taking  $p$  as before, we can see that the lift of  $f$  gives  $\bar{f}(p) = (f(x), \gamma) = (y, \gamma)$  where  $f^* \gamma_y = \eta_x \Rightarrow (f \circ \pi)^* \gamma_y = \pi^*(f^* \gamma_y) = \pi^* \eta_x = (\pi \circ g)^* \zeta_y$ . Using the relation between  $f$  and  $g$ , we get  $(\pi \circ g)^* \gamma_y = (\pi \circ g)^* \zeta_y$ . As  $\pi \circ g$  is a subjective submersion, we must have  $\gamma_y = \zeta_y$  and hence  $g = \bar{f}$ . ■

Now although we have made some progress on our questions from before, we are still far from addressing the general question of symplectomorphic manifolds. To address this, we will first take a detour to understand some important sub-structures of symplectic manifolds.

### 1.3 Lagrangian Submanifolds

Recall that a submanifold  $X$  of  $M$  is defined to be a manifold with an immersive homeomorphism  $i : X \rightarrow M$  where  $i(X)$  is closed in  $M$ .

**Definition 1.9** *Let  $(M, \omega)$  be a  $2n$ -dimensional symplectic manifold. Then a submanifold  $Y$  of dimension  $n$  with  $i^* \omega = \omega|_{T_p Y} \equiv 0$  is called a Lagrangian submanifold of  $X$ .*

A simple example of a Lagrangian submanifold is the *zero section* [1] of  $T^*M$  given by  $M_0 = \{(x, \eta) \in T^*M \mid \eta = 0 \text{ in } T_x^*M\}$ . This is clearly an  $n$ -dimensional submanifold of  $T^*M$  defined by  $\eta_i = 0$  for all  $i$ . Moreover,  $\alpha|_{T_p M_0} \equiv 0 \Rightarrow \omega|_{T_p M_0} \equiv 0$ , and so  $M_0$  is a Lagrangian submanifold of  $T^*M$ . One can in fact prove a more general result stating that for a 1-form  $\mu$  on  $M$ , the section  $M_\mu = \{(x, \mu_x) \in T^*M\}$  is a Lagrangian submanifold of

$T^*M$  if and only if  $\mu$  is closed[1]. This gives a whole family of Lagrangian submanifolds that can be *generated* by functions  $f \in C^\infty(M)$  as  $M_{df}$ .

We now unravel the connection between our question about symplectomorphisms and Lagrangian submanifolds. Consider a diffeomorphism  $\phi : (M_1, \omega_1) \rightarrow (M_2, \omega_2)$  where  $\dim M_1 = \dim M_2 = 2n$ . It induces a map  $\psi : M_1 \rightarrow M_1 \times M_2$  given by  $\psi(p) = (p, \phi(p))$ . The image of  $\psi$  is the graph of the function  $\phi$ , and is a closed embedding of  $M_1$  into  $M_1 \times M_2$ . This gives us that  $M_1$  is a  $2n$ -dimensional submanifold of  $M_1 \times M_2$ . We now impose on  $M_1 \times M_2$  a symplectic structure given by the *twisted* closed 2-form  $\bar{\omega} = \pi_1^* \omega_1 - \pi_2^* \omega_2$ , where  $\pi_i$  is projection onto  $M_i$ . One can easily check that this is indeed a symplectic form. Considering  $\psi^* \bar{\omega} = (\pi_1 \circ \psi)^* \omega_1 - (\pi_2 \circ \psi)^* \omega_2 = \omega_1 - \phi^* \omega_2$ , we can see that  $\phi$  is a symplectomorphism if and only if  $\psi^* \bar{\omega} = 0$  and  $\psi(M_1)$  is a Lagrangian submanifold of  $(M_1 \times M_2, \bar{\omega})$ . This gives us a new approach to the problem of finding symplectomorphisms!

Let's do an example. Consider  $M_1 = (T^*X_1, \omega_1)$  and  $M_2 = (T^*X_2, \omega_2)$  for some manifolds  $X_1$  and  $X_2$ . We shall now work backwards; that is, look for Lagrangian submanifolds of  $(M_1 \times M_2 \cong T^*(X_1 \times X_2), \bar{\omega})$ . We know that it is easy to find Lagrangian submanifolds of  $(M_1 \times M_2 \cong T^*(X_1 \times X_2), \omega = \pi_1^* \omega_1 + \pi_2^* \omega_2)$ , where  $\omega$  is just the canonical 2-form for the cotangent bundle  $T^*(X_1 \times X_2)$ . Assume we have such a Lagrangian submanifold  $Y$ . Then defining  $\sigma : M_1 \times M_2 \rightarrow M_1 \times M_2$  as  $\sigma(x, y, \eta_1, \eta_2) = (x, y, \eta_1, -\eta_2)$ , we can see that  $\sigma(Y)$  is a Lagrangian submanifold of  $(M_1 \times M_2, \bar{\omega})$ [1]. So if we start off with  $Y = (M_1 \times M_2)_{df}$  for some  $f \in C^\infty(X_1 \times X_2)$ , we get  $\sigma(Y) = (x, y, d_x f, -d_y f)$ . And so, if this is the graph of a diffeomorphism  $\phi : M_1 \rightarrow M_2$ , then  $\phi$  must be a symplectomorphism. Thus,  $\phi(x, \eta) = (y, \zeta)$  must satisfy the following *Hamilton* equations:

$$\begin{aligned}\eta_i &= \frac{\partial f}{\partial x_i}(x, y) \\ \zeta_i &= -\frac{\partial f}{\partial y_i}(x, y).\end{aligned}$$

By the implicit function theorem, for a solution  $y = \phi_1(x, \eta)$  to exist for the first differential equation, we must have  $\left[ \frac{\partial^2 f}{\partial y_j \partial x_i} \right]_{i,j}$  is invertible locally. Let us now apply this.

Consider a Riemannian manifold  $(X, g)$  that is *geodesically convex* and *geodesically complete*. This implies that for every  $(x, v) \in TX$  there exists a unique minimizing geodesic of constant velocity  $v$  starting at  $x$  given as  $\exp(x, v) : \mathbb{R} \rightarrow X$ . If we define  $f : X \times X \rightarrow \mathbb{R}$  as  $f(x, y) = -\frac{1}{2}d(x, y)^2$ , we can show that the symplectomorphism generated by  $f$  can

be identified (through  $g$ ) with the *geodesic flow* on  $X$  (the endomorphism  $(x, v) \mapsto \exp(x, v)(1)$  of  $TX$ ). Firstly, recall that the identification between  $TX$  and  $T^*X$  is given as  $(x, v) \leftrightarrow g_x(v, ;)$ . Thus the Hamilton equations modify to:

$$\begin{aligned} g_x(v, ;) &= d_x f \\ g_y(w, ;) &= -d_y f, \end{aligned}$$

where we aim to find  $(y, w)$  as function of  $(x, v)$ . If we act both sides of the first equation on  $\frac{d\exp(x, v)}{dt}(0) = v$ , we get:

$$\begin{aligned} g_x(v, v) &= d_x f(v) = \frac{d}{dt}(f(\exp(x, v)(t), y))(0) \\ &= -\frac{1}{2} \frac{d}{dt}(d(\exp(x, v)(t), y)^2)(0). \end{aligned}$$

We can see that if we take  $y = \exp(x, v)(1)$  we get by definition of the Riemannian distance that  $d(\exp(x, v)(t), \exp(x, v)(1)) = \int_t^1 \sqrt{g_x\left(\frac{d\exp(x, v)}{dt}, \frac{d\exp(x, v)}{dt}\right)} = \int_t^1 \sqrt{g_x(v, v)} = \sqrt{g_x(v, v)}(1 - t)$ . Plugging this into the equation above confirms that this is indeed the unique solution to the first Hamilton equation. Moreover, if we act  $\frac{d\exp(x, v)}{dt}(1) = v$  on both sides of the second equation, we get:

$$\begin{aligned} g_x(w, v) &= -d_y f(v) = \frac{d}{dt}(f(\exp(x, v)(t), \exp(x, v)(0)))(1) \\ &= \frac{1}{2} \frac{d}{dt}(d(\exp(x, v)(t), \exp(x, v)(0))^2)(1). \end{aligned}$$

Through a similar calculation as before, we get  $d(\exp(x, v)(t), \exp(x, v)(0)) = -t\sqrt{g_x(v, v)}$ , and plugging this into the equation above gives  $g_x(w, v) = g_x(v, v)$ . Moreover, if we consider any orthogonal  $\bar{v} \in T_y X$  to  $v$ , we get  $g_x(w, \bar{v}) = \frac{1}{2} \frac{d}{dt}(d(\exp(y, \bar{v})(t), \exp(x, v)(0))^2)(0)$ . As  $\exp(x, v)$  is a minimizing geodesic, we have that  $d(\exp(y, \bar{v})(t), \exp(x, v)(0))$  is minimum when the two geodesics intersect at  $t = 0$ . Thus the derivative expression above is 0 and  $g(w, \bar{v}) = 0$  for all orthogonal  $\bar{v} \in T_y X$  to  $v$ . This implies  $g(w, ;) = g(v, ;)$ . By non-degeneracy of  $g$ , the identification we made before is bijective, and hence  $w = v = \exp(x, v)(1)$ . And so, the symplectomorphism  $f$  can be identified with the endomorphism  $(x, v) \mapsto \exp(x, v)(1)$  of  $TX$ .

Now that we have some familiarity with the construction and finding of symplectomorphisms, we can start to explore questions about the existence of various kinds of equivalences between symplectic structures.

## 1.4 Symplectic Equivalences

We first recall a few definitions and ideas.

**Definition 1.10** *An isotopy is a map  $\rho : M \times \mathbb{R} \rightarrow M$  such that  $\rho_t = \rho(\cdot, t) : M \rightarrow M$  is a diffeomorphism for all  $t$ , and  $\rho_0 = id_M$ .*

Every isotopy gives an associated time-dependent vector field  $v_t$  given by  $v_t = \frac{d\rho_t}{dt} \circ \rho_t^{-1}$ . For a compact manifold, the converse is also true: for any time-dependent vector field  $v_t$ , there exists an isotopy satisfying the differential equation  $\frac{d\rho_t}{dt} = v_t \circ \rho_t$ , with initial condition  $\rho_0 = id_M$ . By Picard–Lindelof theorem, this also exists locally for non-compact manifolds. This allows us to define the *Lie derivative* along  $v_t$  as a map  $\mathfrak{L}_{v_t} : \Omega^k(M) \rightarrow \Omega^k(M)$  given by  $\mathfrak{L}_{v_t}\omega = \frac{d}{dt}(\rho_t^*\omega)|_{t=0}$ , where  $\rho$  is the local isotopy associated with  $v_t$ . The Lie derivative satisfies the *Cartan magic formula*:  $\mathfrak{L}_{v_t}\omega = i_{v_t}d\omega + di_{v_t}\omega$ , and as  $\rho_{h+t} = \rho_t \circ \rho_h$  (by uniqueness of solution to the differential equation from before), it also satisfies  $\frac{d}{dt}\rho_t^*\omega|_{t=\tau} = \rho_\tau^*\mathfrak{L}_{v_t}\omega$ . By the chain rule, this gives  $\frac{d}{dt}\rho_t^*\omega_t = \rho_t^*(\mathfrak{L}_{v_t}\omega_t + \frac{d\omega_t}{dt})$ .

Moreover, recall that the *normal space* to a  $k$ -dimensional submanifold  $S \subset M$  at  $p \in S$  is defined as  $n - k$  dimensional quotient vector space  $N_pS = T_pM/T_pS$ . This gives us the *normal bundle*  $NS$ , an  $n$ -dimensional manifold. It is not hard to see that  $S$  embeds into  $NS$  as the zero-section as well. In fact, the following well-known result gives us that these embeddings are locally the same:

**Theorem 1.11 (Tubular neighborhood Theorem)** *There exists a neighborhood  $U_0$  of  $S$  in  $NS$ , and a neighborhood  $U$  of  $S$  in  $M$ , and a diffeomorphism  $\phi : U_0 \rightarrow U$  with  $\phi|_S = id_S$ . Moreover,  $U_0$  is convex in each  $N_pS$ .*

This gives us that  $S$  is a deformation retract of  $U$  and hence the inclusion induces an isomorphism of de Rham cohomologies. Now we return to the question from before, but slightly altered: *Does there exist a symplectomorphism between  $(M, \omega_0)$  and  $(M, \omega_1)$  that is homotopic to the identity?* The reason we have added this extra condition is that it restricts our problem to the specific case where  $[\omega_0] = [\omega_1]$ . Naturally, this problem is easier.

Define  $S_c = \{\text{symplectic forms } \omega \text{ on } M \text{ with } [\omega] = c\}$  endowed with the  $C^\infty$  topology. Assume  $\omega_0, \omega_1 \in S_c$ , and that there exists a smooth straight line path connecting both of them in  $S_c$ , i.e.  $\omega_t = t\omega_0 + (1-t)\omega_1$  is symplectic for all  $t \in [0, 1]$ , and as  $\omega_0 - \omega_1 = d\mu$  for some 1-form  $\mu$  on  $M$ , we have  $\omega_t = \omega_1 + td\mu \in S_c$  for all  $t$ . We claim that this is a

sufficient condition for the existence of a symplectomorphism that is homotopic to the identity.

Note that if the above is true, then  $(M, \omega_t)$  is symplectomorphic to  $(M, \omega_0)$  for all  $t$  by reparameterizing the path  $\omega_t$ . All of these implications can be satisfied if there exists an isotopy  $\rho$  with  $\rho_t^* \omega_t = \omega_0$ . Looking for the existence of such an isotopy is exactly what is known as *Moser's trick*. If we additionally assume that  $M$  is compact, we can instead look for the associated vector field  $v_t$ , and integrate to get  $\rho_t$ . We have the following condition on  $v_t$ [1]:

$$\frac{d}{dt}(\rho_t^* \omega_t) = 0 = \rho_t^* \left( \mathfrak{L}_{v_t} \omega_t + \frac{d\omega_t}{dt} \right) \Leftrightarrow di_{v_t} \omega_t + d\mu = 0.$$

We can find a solution for this easily by solving for  $v_t$  such that  $i_{v_t} \omega_t + \mu = 0$  (as  $\omega_t$  is nondegenerate). Thus, we obtain the isotopy we wanted. One can generalize this argument to any  $\omega_0, \omega_1 \in S_c$  that lie in the same path-connected component:

**Theorem 1.12 (Moser Theorem)** *Let  $M$  be compact manifold with symplectic forms  $\omega_0$  and  $\omega_1$  connected by a smooth family of symplectic forms  $\omega_t$ , with  $0 \leq t \leq 1$ , such that  $[\omega_t] = c$  is constant. Then there exists an isotopy  $\rho$  of  $M$  with  $\rho_t^* \omega_t = \omega_0$ .*

Let's apply this to a specific example.

**Proposition 1.13** *Let  $M$  a compact and oriented 2-manifold. Then each 2-cohomology class has a unique symplectic representative.*

*Proof.* Consider an orientable and compact 2-dimensional manifold  $M$  with areas forms  $\omega_1$  and  $\omega_2$ . Clearly, these are symplectic forms. Assume further that  $[\omega_1] = [\omega_2]$ . First, let us show that  $\omega_1$  and  $\omega_2$  determine the same orientation. Assume they do not. Then  $\omega_1 = -f\omega_2$  for some smooth  $f > 0$ . Moreover, we have  $\int_M \omega_1 - \omega_2 = \int_M (1 + f)\omega_1 > 0$ . This is a contradiction as  $\omega_1 - \omega_2 = d\mu$  is exact, and hence by Stoke's theorem  $\int_M \omega_1 - \omega_2 = 0$ . Thus  $\omega_1$  and  $\omega_2$  must determine the same orientation, and  $\omega_1 = f\omega_2$  for some smooth  $f > 0$ . Now consider the smooth straight line path  $\omega_t = t\omega_1 + (1 - t)\omega_2 = (1 - t + ft)\omega_2$ . As  $t \in [0, 1]$ ,  $\omega_t$  can only vanish at some point  $p$  if  $f(p) = (t - 1)/t \leq 0$ . This is a contradiction. Hence  $\omega_t$  is symplectic for all  $t$ . By Moser Theorem, the symplectic structure induced by  $\omega_1$  and  $\omega_2$  is the same. ■

Moser Theorem also allows us to find a symplectic equivalence if two different symplectic forms agree on a compact submanifold. Recall the tubular neighborhood theorem. If we consider  $X$  to be a compact submanifold of  $M$ , then there is a neighborhood  $U$  of  $X$  such

that  $X$  is a deformation retract of  $U$ . If  $\omega_1$  and  $\omega_2$  are two symplectic forms that agree on  $X$ , i.e.,  $i^*(\omega_1 - \omega_2)|_X = 0$ , then  $(\omega_1 - \omega_2)|_U = d\mu$  is exact by homotopy invariance of the de Rham cohomology. This allows one to define a smooth family  $\omega_t = \omega_2 + td\mu$  that vanishes on  $X$ . After some shrinking of neighborhoods, and applying Moser's Trick again[1], we have another theorem by Moser:

**Theorem 1.14 (Relative Moser Theorem)** *Let  $X$  be a compact submanifold of a symplectic manifold  $M$  with two symplectic forms  $\omega_1$  and  $\omega_2$  agreeing on  $X$ . Then there exists neighborhoods  $U$  and  $V$  of  $X$  in  $M$ , and a diffeomorphism  $\phi : U \rightarrow V$  with  $\phi^*\omega_2 = \omega_1$  and  $\phi|_X = id_X$ .*

If we take  $X = p$ , and use the canonical symplectic basis for the  $T_pM$ , we finally get the grand theorem classifying symplectic structures locally:

**Theorem 1.15 (Darboux Theorem)** *Let  $(M, \omega)$  be a symplectic manifold, and let  $p$  be any point in  $M$ . Then we can find a coordinate system  $(U, x_1, \dots, x_n, y_1, \dots, y_n)$  centered at  $p$  such that on  $U$ :*

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i.$$

If  $X$  is compact Lagrangian submanifold of both  $(M, \omega_1)$  and  $(M, \omega_2)$ , i.e.,  $i^*\omega_1 = i^*\omega_2 = 0$ , then relative Moser applies identically to give what is known as the *Weinstein Lagrangian Neighborhood Theorem*. This theorem is imperative in helping us classify Lagrangian structures locally. But first, we prove a small result.

**Lemma 1.16** *Let  $(V, \Omega)$  be symplectic vector space with a lagrangian subspace  $U$ . Then there exists a canonical isomorphism between  $V/U$  and  $U^*$ .*

*Proof.* Define  $\tilde{\Omega} : V/U \rightarrow U^*$  given by  $\tilde{\Omega}([v]) = \Omega(v, ;)$ . This is well-defined as for any  $v \oplus u$ ,  $\Omega(v \oplus u, ; ) = \Omega(v, ; ) + \Omega(u, ; ) = \Omega(v, ; )$  as  $\Omega|_U = 0$ . Moreover, it is a non-degenerate pairing, and so the map above is an isomorphism. ■

Note that this allows us to canonically identify  $NX$  and  $T^*X$ . Now recall that every manifold  $X$  is a Lagrangian submanifold of  $(T^*X, \omega_0)$  as the zero section. Assume  $X$  is also a Lagrangian submanifold of  $(M, \omega)$ . By using the tubular neighborhood theorem with the identification  $NX \cong T^*X$ , followed by Weinstein's Lagrangian neighborhood theorem, we obtain neighborhoods  $U$  of  $X$  in  $(T^*X, \omega_0)$ ,  $V$  of  $X$  in  $M$ , and a composite

symplectomorphism  $\phi : (U, \omega_0) \rightarrow (V, \omega)$  that is identity on  $X$ . This implies that every Lagrangian embedding is symplectically locally equivalent to the embedding of a manifold as the zero-section of its cotangent space! This classification is called the *Weinstein tubular neighborhood theorem*.

This result above has important applications. For instance, it allows us to characterize the tangent space at identity of the group of symplectomorphisms  $\text{Symp}(M, \omega) = \{f : M \xrightarrow{\cong} M \mid f^*\omega = \omega\}$  (The term *tangent space* is used loosely here for now—consider it to be the behavior of the group in a small neighborhood). Let us look at  $\text{Symp}(M, \omega)$  in the  $C^1$  topology. Consider  $f \in \text{Symp}(M, \omega)$  that is in a sufficiently small neighborhood of  $id \in \text{Symp}(M, \omega)$ . We have that the graphs  $\Gamma_f, \Gamma_{id} \cong M$  are lagrangian submanifolds of  $(M \times M, \bar{\omega})$ . By Weinstein's tubular neighborhood theorem, there exist neighborhoods  $U_1$  of  $\Gamma_{id}$  in  $M \times M$ ,  $U_2$  of the zero section in  $T^*M$ , and a symplectomorphism  $\phi$  between the two. Given that  $f$  is sufficiently  $C^1$  close to  $id$ , its graph is also symplectically embedded into  $T^*M$  as the image of a smooth section  $M_\mu = \{(p, \mu_p) \in T^*M\}$ , where  $\mu$  is closed and  $C^1$ -close to the zero form. On the other hand, if we take  $\mu$  to be a closed 1-form that is  $C^1$ -close to the zero form, its section  $M_\mu$  will lie in a sufficiently small neighborhood of the zero section in  $U_2$ . Thus  $\phi^{-1}$  is defined on  $M_\mu$ . As  $\mu$  is  $C^1$ -close to the zero form,  $\phi^{-1}(M_\mu)$  also smoothly intersects  $\{p\} \times M$  once for all  $p$  at  $g(p)$ . Thus,  $g$  is a diffeomorphism of  $M$  and  $M_\mu \cong \Gamma_g$ . As  $\mu$  is closed,  $g$  is a symplectomorphism. This gives us that sufficiently small  $C^1$  neighborhood of  $id$  in  $\text{Symp}(M, \omega)$  is homeomorphic to a similar neighborhood of the zero form in the space of closed 1-forms on  $M$ .

This has implications in a very active area of symplectic geometry research, which pertains to fixed points of symplectomorphisms. Consider a compact symplectic manifold  $M$  with vanishing de Rham cohomology of degree 1. Take any  $f \in \text{Symp}(M, \omega)$  that is  $C^1$ -close to identity. Then, by the observation above, it is generated by  $h \in C^\infty(M)$  through the recipe described in section 1.3. Now, as  $h$  is defined on a compact manifold, and hence achieves a maximum and minimum, it must have at least two critical points where  $dh_p = 0$ . Via the zero-section embedding discussed before, we have that  $\Gamma_f$  intersects  $\Gamma_{id}$  at two points. Thus  $f$  must have two fixed points.

Similarly, we can now also say something about Lagrangian intersections. Assume we have two compact Lagrangian submanifolds  $X$  and  $Y$  with vanishing de Rham cohomology of degree 1. Assume further that they are  $C^1$ -close, i.e, there exists a diffeomorphism  $f : X \rightarrow Y$  that is  $C^1$ -close to the inclusion  $i : X \rightarrow M$ . Then, by Weinstein's tubular neighborhood theorem, there exist neighborhoods  $U_1$  of  $X$  in  $M$ ,  $U_2$  of the zero-section of

$T^*X$ , and symplectomorphism  $\phi$  between the two. We can assume  $Y \subset U_1$ . Now, lifting the diffeomorphism  $f$ , we can embed  $Y$  into  $U_2$  as the section of a closed 1-form on  $X$ . We can see that  $\mu$  must vanish for at least two points as  $X$  is compact and  $H_{deRham}^1(X) = 0$ . Via  $\phi^{-1}$ , this corresponds to  $X$  and  $Y$  intersecting in at least two points.

We have come a long way now in understanding the local behavior of symplectic structures. That said, this is just the tip of the iceberg as far as contemporary research in this area is concerned. Moreover, many questions remain open problems to this day, and have even led to developments of new theories. One such example is *Arnold Conjecture* regarding fixed points of symplectomorphisms, and how it led to the development of *Morse theory* in the form of *Floer homology*. As we cannot hope to cover all of the above in such depth, we shall, for now, take a detour to explore another important idea in symplectic geometry, and that is *contact structure*.



# Chapter 2

## Contact Structure

### 2.1 Contact Manifolds

# Bibliography

- [1] Annas Cannas de Silva. *Lectures on Symplectic Geometry*, volume 1764 of *Lecture Notes in Mathematics*. Springer-Verlag, 2001.