Directed Research Project Reading: *An Introduction to Manifolds* by Loring W. Tu

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This document is structured chapter-wise as follows: A summarizing flowchart (sometimes with additional remarks) pertaining to links between various concepts discussed in a single chapter, followed by original solutions to specifically instructional chapter problems and exercises. Also note that some distinct chapters that link well together have been combined into one. Happy reading!

Contents

1 Euclidean Spaces

1.1 Smooth Functions on a Euclidean Space

$$\boxed{\begin{array}{l} f \;\; is\ C^\infty \\ at\ p \in \mathbb{R}^n \end{array}} \quad \Longleftarrow \quad \boxed{\begin{array}{l} f\ is\ real - analytic \\ in\ some\ nbhd.\ U \\ containing\ p \in \mathbb{R}^n \end{array}}$$

Figure 1: .

The converse only holds partially as in *Lemma* 1.4 from the book that follows:

Lemma 1.4 (Taylor's theorem with remainder). *Let f be a C^∞ function on an open subset $U \subset \mathbb{R}^n$* that is star-shaped with respect to a point $p = (p^1, \dots\dots, p^n)$ in U. Then there are functions $g_1(x), \dots\dots, g_n(x) \in C^\infty(U)$ such that

$$f(x) = f(p) + \sum_{i=1}^{n}(x^i - p^i)g_i(x), \quad g_i(p) = \frac{\partial f}{\partial x^i}(p).$$

Problem 1.6* Taylor's theorem with remainder to order 2
Solution: Let $f \in C^\infty(\mathbb{R}^2)$. Then by choosing $p = (0,0)$ in the lemma above we have:

$$f(x,y) = f(0,0) + xg_1(x,y) + yg_2(x,y),$$

where $g_1(x,y), g_2(x,y) \in C^\infty(\mathbb{R}^2)$ and $g_1(0,0) = \dfrac{\partial f}{\partial x}(0,0)$ and $g_2(0,0) = \dfrac{\partial f}{\partial y}(0,0)$. Thus we can apply the same remainder theorem on g_1 and g_2 to get:

$$\begin{aligned} f(x,y) &= f(0,0) + x(g_1(0,0) + xh_1(x,y) + yh_2(x,y)) + y(g_2(0,0) + xl_1(x,y) + yl_2(x,y)) \\ \Rightarrow f(x,y) &= f(0,0) + x\frac{\partial f}{\partial x}(0,0) + y\frac{\partial f}{\partial y}(0,0) + x^2 g_{11}(x,y) + xyg_{12}(x,y) + y^2 g_{22}(x,y), \end{aligned}$$

where $h_1(x,y) = g_{11}(x,y)$, $l_2(x,y) = g_{22}(x,y)$ and $h_2(x,y) + l_1(x,y) = g_{12}(x,y)$ are all C^∞ over $\mathbb{R}^n$.

Problem 1.7* A function with a removable singularity
Solution: Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be the C^∞ function given by $h(t,u) = (t, tu)$. We also have from the previous problem:

$$\begin{aligned} g(t,u) &= \frac{1}{t}\Big(t^2 g_{11}(t,tu) + t^2 u g_{12}(t,tu) + t^2 u^2 g_{22}(t,tu)\Big) \\ &= tg_{11}(t,tu) + tug_{12}(t,tu) + tu^2 g_{22}(t,tu) = k(t,u), \end{aligned}$$

for $t \neq 0$. As $g_{ij}(t, tu) = (g_{ij} \circ h)(t, u)$ is a composition of C^∞ functions, it is also $C^\infty \Rightarrow k(t, u)$ is C^∞. Also, as $k(0, u) = g(0, u) = 0$, we can see that $k(t, u) = g(t, u)$ for all $(t, u) \in \mathbb{R}^2$. Thus $g(t, u)$ is C^∞ as well.

1.2 Tangent Vectors in $\mathbb{R}^n$ as Derivations

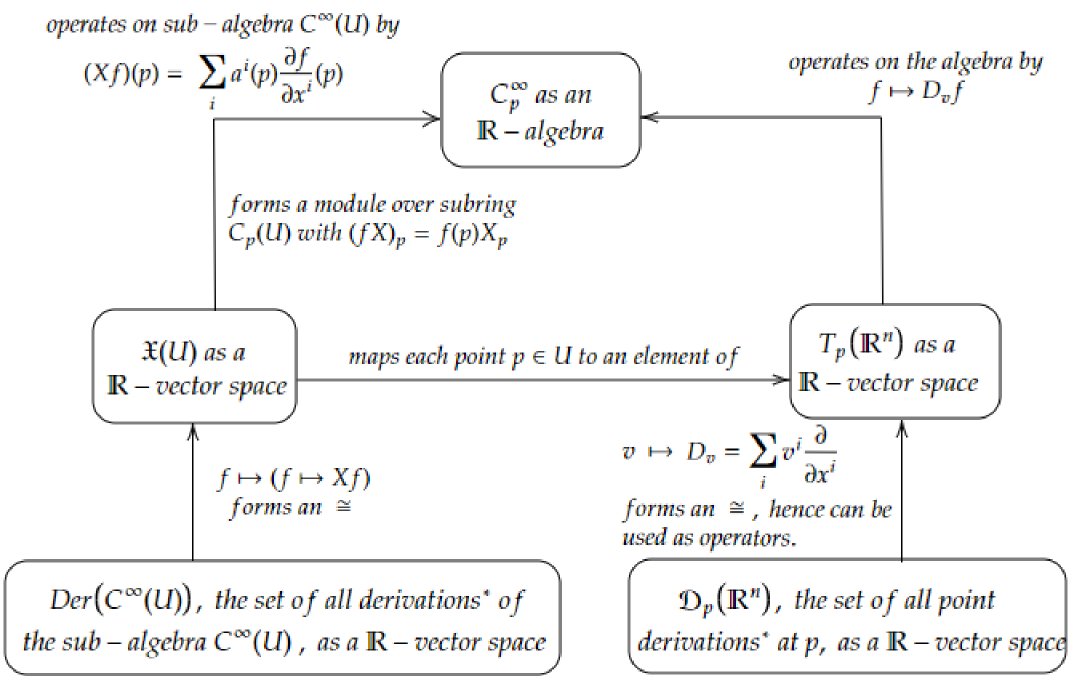


Figure 2: *The general idea of a derivation is that of a linear operator that satisfies some form of the *Leibniz rule* that we are familiar with for derivatives. Also, strictly speaking, we consider the elements of $C^\infty(U)$ modulo the equivalence relation $(f, U) \sim (g, V) \Leftrightarrow f_{|U \cap V} = g_{|U \cap V}$.

Problem 2.4 Product of derivations

Solution: Assume D_1 and D_2 are derivations over an algebra A over a field K. Now consider:

$$\begin{aligned}(D_1 \circ D_2)(ab) = D_1(D_2(ab)) &\overset{\text{leib.rule}}{=} D_1((D_2 a)b + a(D_2 b)) \overset{\text{lin.}}{=} D_1((D_2 a)b) + D_1(a(D_2 b)) \\ &= (D_1 \circ D_2)(a)b + (D_2 a)(D_1 b) + (D_1 a)(D_2 b) + a(D_1 \circ D_2)(b),\end{aligned}$$

where we can see clearly that the *Leibniz rule* is not satisfied. On the other hand, we can see that:

$$\begin{aligned}(D_1 \circ D_2 - D_2 \circ D_1)(ab) &= (D_1 \circ D_2)(a)b + (D_2 a)(D_1 b) + (D_1 a)(D_2 b) + a(D_1 \circ D_2)(b) \\ &\quad -((D_2 \circ D_1)(a)b + (D_1 a)(D_2 b) + (D_2 a)(D_1 b) + a(D_2 \circ D_1)(b)) \\ &= (D_1 \circ D_2 - D_2 \circ D_1)(a)b + a(D_1 \circ D_2 - D_2 \circ D_1)(b),\end{aligned}$$

thus $(D_1 \circ D_2 - D_2 \circ D_1)$ satisfies the *Leibniz rule* and is a derivation.

1.3 The Exterior Algebra of Multicovectors

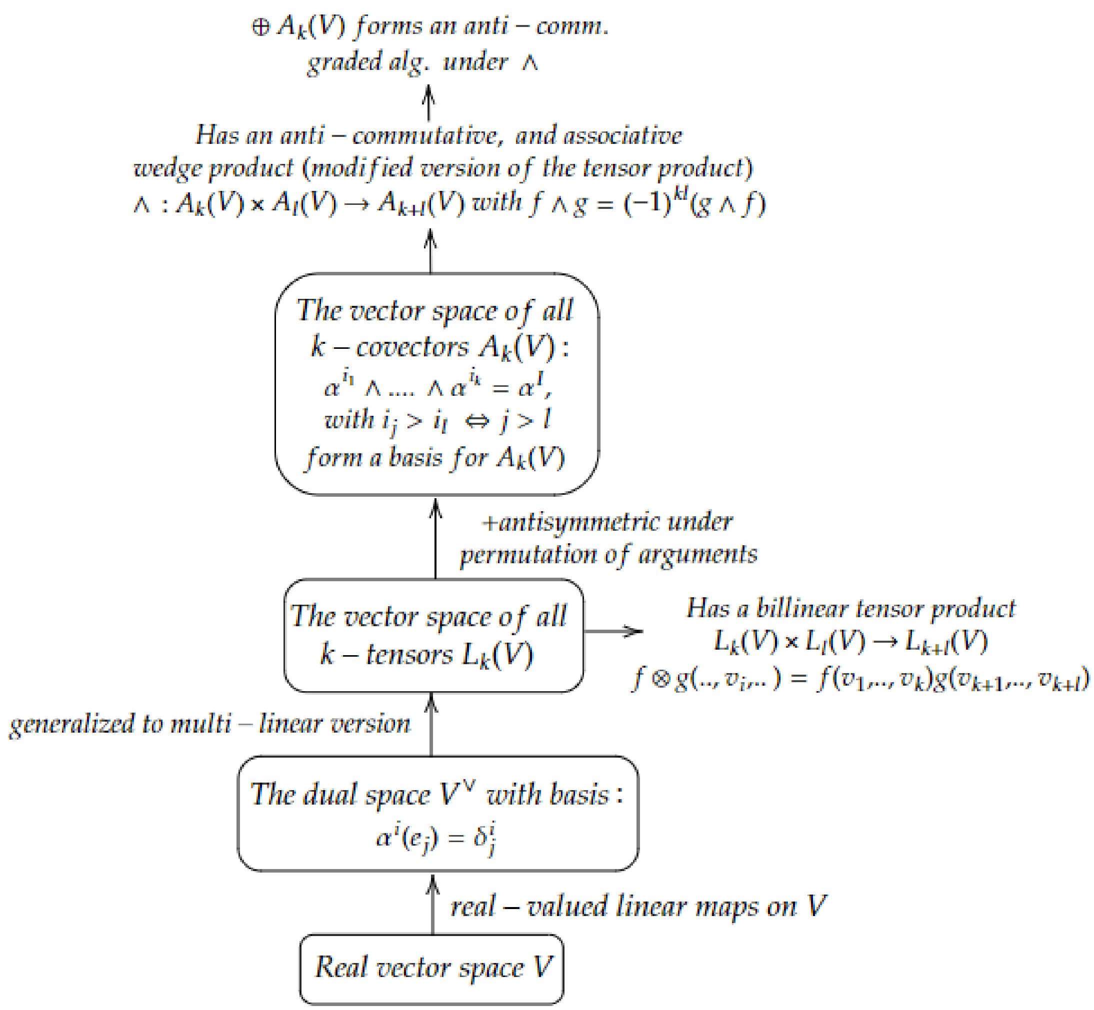


Figure 3: .

Problem 3.4 A characterization of alternating k-tensors
Solution: Assume we have an alternating k-tensor f on a vector space $V \Rightarrow \sigma f = \text{sgn}(\sigma) f$ for all $\sigma \in S_k$. Let $\sigma = (i \;\; i+1)$Then we have:

$$f(..., v_{i+1}, v_i, ...) = \sigma f(..., v_i, v_{i+1}, ...) = \text{sgn}(\sigma) f(..., v_i, v_{i+1}, ...) = -f(..., v_i, v_{i+1}, ...).$$

Now assume f changes sign whenever two successive arguments are interchanged. Consider an arbitrary permutation $\tau \in S_k$ and the corresponding ordered tuple $(\tau(1), ..., \tau(k))$. Starting from 1$= \tau(i_1)$ for some i_1, we swap its place with successive members to the left in the tuple above, until it reaches the first index. We then do the same for the next integer 2 up until $k-1$, such that the tuple is sorted by the end of this process.

In this way, as each swap consists of a smaller integer being swapped with a larger one to the left of it, we can see that the total number of successive swaps made is precisely equal to the number of *inversions*, thus giving us:

$$\tau f = f(v_{\tau(1)}, ..., v_{\tau(k)}) = (-1)^{\text{no. of inv.}} f(v_1, ..., v_k) = \operatorname{sgn}(\tau) f(v_1, ..., v_k)$$

Problem 3.10* Linear independence of covectors

Solution: Assume we are given that $\alpha^1 \wedge .... \wedge \alpha^k \neq 0$. Assume now on the contrary that $\{\alpha^i\}$ is a linearly dependent set in $V^\vee \Rightarrow \exists j$ such that $\alpha^j = \sum_{i\neq j}^k c_i^j \alpha^i$ for some $c_i^j \in \mathbb{R}$. Thus we have:

$$\alpha^1 \wedge ... \wedge \alpha^j \wedge ... \wedge \alpha^k = \alpha^1 \wedge ...(\sum_{i\neq j}^k c_i^j \alpha^i) ... \wedge \alpha^k \overset{\text{lin.}}{=} \sum_{i\neq j}^k c_i^j (\alpha^1 \wedge ...\alpha^i ... \wedge \alpha^k\ ) = 0,$$

as for all terms in the sum above, at least one other superscript is equal to i, and $\alpha^i \wedge \alpha^i = 0$ by anti-commutativity of the wedge product. This is a contradiction, and thus $\{\alpha^i\}$ is a linearly independent set.
Now assume that the set $\{\alpha^i\}$ is linearly independent$\Rightarrow$ it can be extended to a basis $\alpha^1, ..., \alpha^n$ of $V^\vee$ where $n = \dim(V)$. Thus $\alpha^1 \wedge .... \wedge \alpha^k$ is a basis vector for $A_k(V)$ and hence cannot be zero.

Problem 3.11* Exterior multiplication

Solution: Let α be a 1-covector and γ be a k-covector.
Assume $\gamma = \alpha \wedge \beta$ for some $(k-1)$-covector $\beta \Rightarrow \alpha \wedge \gamma = \alpha \wedge \alpha \wedge \beta = 0$.
Now assume $\alpha \wedge \gamma = 0$. Denote α by β_1, and extend it to a basis $\beta_1, ..., \beta_n$ of $V^\vee$ of dimension $n \Rightarrow \beta^I$ forms a basis for $A_k(V)$ where I varies over all strictly increasing multi-indices of length $k \Rightarrow \gamma = \sum_J c_J \beta^J$ for some $c_J \in \mathbb{R}$. The sum can be split up as follows:

$$\gamma = \sum_{1\in J} c_J \beta^J + \sum_{1\notin J} c_J \beta^J$$
$$\Rightarrow \alpha \wedge \gamma = \sum_{1\in J} c_J(\alpha \wedge \beta^J) + \sum_{1\notin J} c_J(\alpha \wedge \beta^J) = \sum_{1\notin J} c_J(\alpha \wedge \beta^J) = 0 \qquad (1)$$

As $\alpha \wedge \beta^J$ are linearly independent in $A_{k+1}(V)$ for $1 \notin J$, (1) $\Rightarrow c_J = 0$ for all J such that $1 \notin J \Rightarrow \gamma = \sum_{1\in J} c_J \beta^J = \alpha \wedge \beta$ for some $(k-1)$-covector β.

1.4 Differential Forms on $\mathbb{R}^n$

This section provides enough knowledge to be able to outline the framework of the *de Rahm complex* of U, i.e, forming an exact sequence using the exterior derivative on $\Omega^k(U)$ with the *kernel* consisting of *closed* forms and the *image* consisting of *exact* forms. This leads to the formation of the *de Rahm Cohomology*.

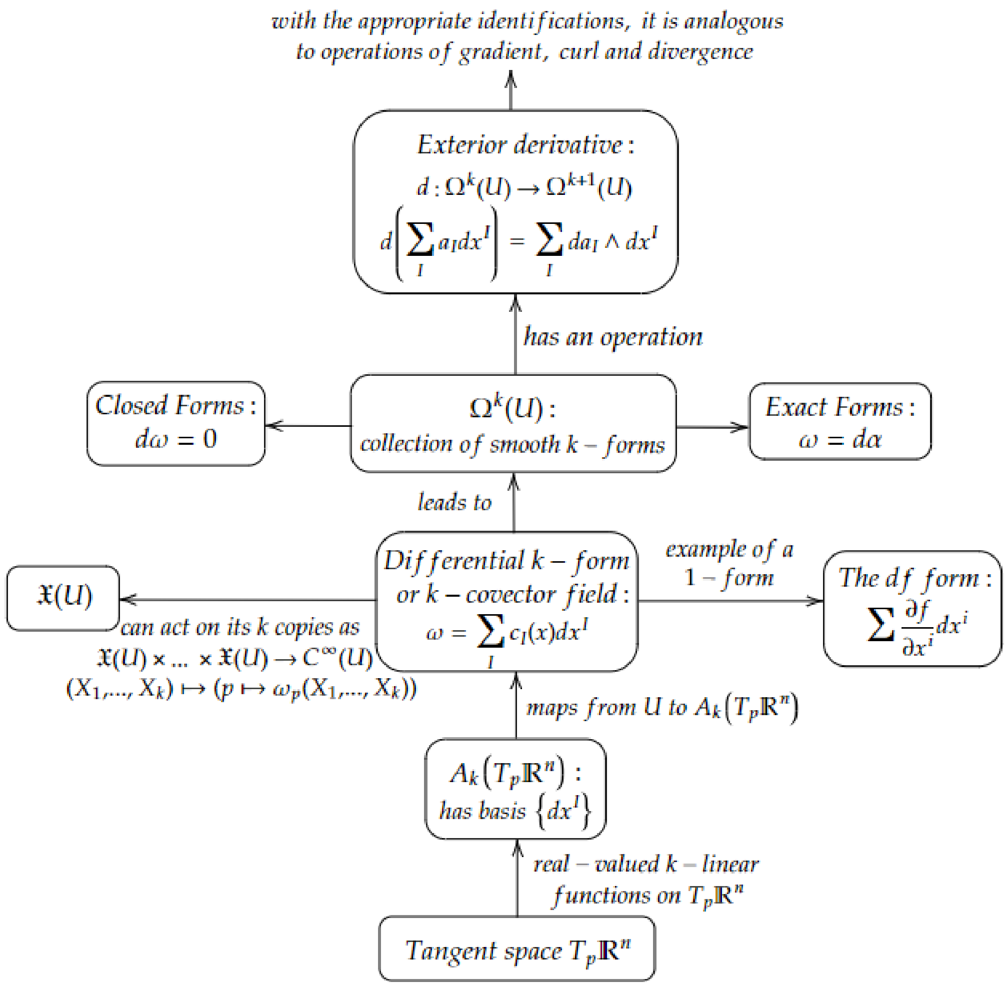


Figure 4: .

Probelem 4.7 Commutator for derivations and anti-derivations
Solution:

$$(D_1 \circ D_2)(ab) = D_1((D_2 a)b + (-1)^{km_2} a D_2 b)$$
$$= (D_1 \circ D_2)(a)b + (-1)^{m_1(k+m_2)}(D_2 a)(D_1 b)$$
$$+(-1)^{km_2}((D_1 a)(D_2 b) + (-1)^{km_1} a(D_1 \circ D_2)(b))$$

Similarly, we can calculate $D_2 \circ D_1(ab)$, by switching D_1 with D_2 and m_1 with m_2, and finally get after obvious manipulations:

$$[D_1, D_2](ab) = (D_1 \circ D_2 - (-1)^{m_1 m_2} D_2 \circ D_1)(ab)$$
$$= [D_1, D_2](a)b - (-1)^{k(m_1+m_2)} a[D_1, D_2]b,$$

implying that $[D_1, D_2]$ is a superderivation of degree $m_1 + m_2$.

2 Manifolds

2.1 Smoothness on Manifolds

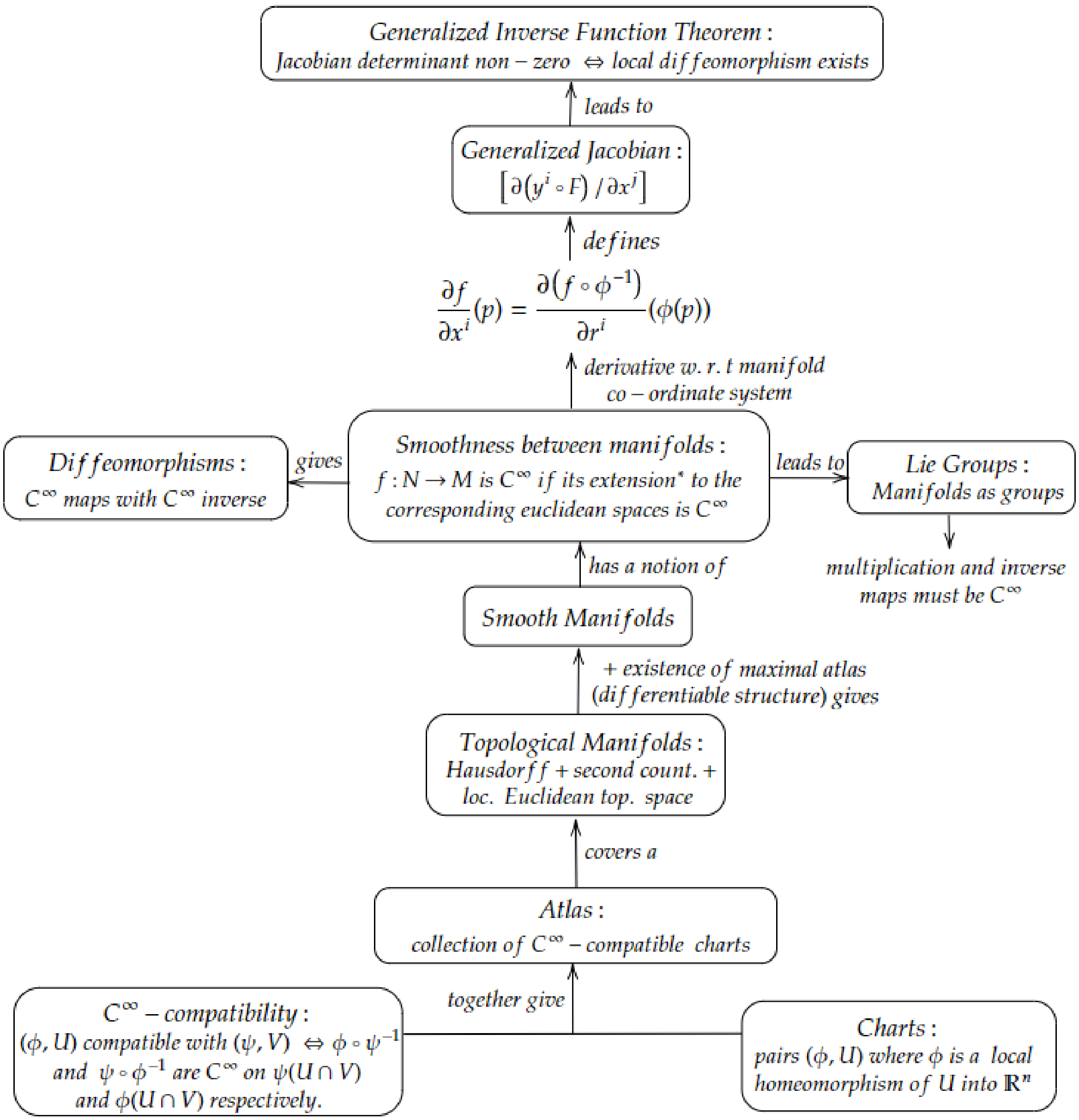


Figure 5: *Extension here alludes to a common idea of having the manifold locally inherit the smoothness from its associated euclidean space through its atlas. Given any point p and corresponding charts in the differentiable structure (ϕ, U) and (ψ, V) with $p \in U \subseteq N$ and $f(p) \in V \subseteq M$, we can *extend* f to the map $\psi \circ f \circ \phi^{-1} : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$, with $A = \phi(f^{-1}(V) \cap U)$.

As mentioned above, ideas of differentiability in the general setting of manifolds is *inherited* from their associated euclidean spaces-a relatively intuitive way to go about introducing differential structure on them. As is known, there exists a stronger notion of differentiability for complex-valued functions, i.e, *complex differentiability.* One could tweak the definition to impose that the charts be *holomorphic* instead, have the manifold inherit the notions of complex differentiability. One can assume that this would produce nicer results as compared to smooth manifolds, as is the case when results in complex analysis are compared to real analysis. As it turns out, this is true and this special class of manifolds are known as *complex manifolds*, and they have a very deep and active theory associated with them.

Problem 5.4* Existence of a coordinate neighbourhood
Solution: Let $\mathfrak{A} = \{(U_\alpha, \phi_\alpha)\}$ be the maximal atlas of the smooth manifold M. Let U be an open set containing a point p. As $p \in M \Rightarrow \exists\,(U_\alpha, \phi_\alpha) \in \mathfrak{A}$ such that $p \in U_\alpha$. Then we have that $p \in V = U \cap U_\alpha \subseteq U$ is open. Clearly the restriction $\phi_\alpha|_V$ is local homeomorphism $\Rightarrow$ $(V, \phi_\alpha|_V)$ is a chart. As (U_α, ϕ_α) is C^∞-compatible with all other charts in $\mathfrak{A}$, so is $(V, \phi_\alpha|_V)$ as $V \subseteq U_\alpha \Rightarrow (V, \phi_\alpha|_V) \in \mathfrak{A}$ with $p \in V \subseteq U$.

Proposition 5.18 An atlas for a product manifold
Proof: Let M and N be m and n dimensional manifolds respectively, with atlases $\mathfrak{A}_M = \{(U_\alpha, \phi_\alpha)\}$ and $\mathfrak{A}_N = \{(V_i, \psi_i)\}$. We claim that $\mathfrak{A} = \{(U_\alpha \times V_i, \phi_\alpha \times \psi_i : U_\alpha \times V_i \to \mathbb{R}^m \times \mathbb{R}^n)\}$ is an atlas for $N \times M$, which is hence a $n + m$ dimensional manifold.
Clearly as $\mathfrak{A}$ covers $M \times N$, we must only check C^∞-compatibility. Let $(U_\alpha \times V_i, \phi_\alpha \times \psi_i)$ and $(U_\beta \times V_j, \phi_\beta \times \psi_j)$ be two arbitrary charts in $\mathfrak{A}$. We can see that:

$$(\phi_\alpha \times \psi_i) \circ (\phi_\beta \times \psi_j)^{-1} = (\phi_\alpha \circ \phi_\beta^{-1}) \times (\psi_i \circ \psi_j^{-1}),$$

is C^∞ on $(U_\alpha \times V_i) \cap (U_\beta \times V_j) = (U_\alpha \cap U_\beta) \times (V_i \cap V_j)$ as each component is C^∞ due to C^∞-compatibility among $\mathfrak{A}_M$ and $\mathfrak{A}_N$. A similar argument can be made for $(\phi_\beta \times \psi_j) \circ (\phi_\alpha \times \psi_i)^{-1}$. Hence $\mathfrak{A}$ is an atlas.

Problem 6.1 Differentiable structures on $\mathbb{R}$
Solution: Assume we have two differentiable structures on $\mathbb{R}$-$\mathbb{R}_1$ and $\mathbb{R}_2$-generated by the maximal atlas of $(\mathbb{R}, \phi = id$ and $(\mathbb{R}, \psi(x) = x^{1/3}$ respectively. We can see that as $\phi^{-1} \circ \psi = \psi$ is not a C^∞ map from $\mathbb{R} \to \mathbb{R}$, the two differentiable structures are distinct. But we can check that $f : \mathbb{R}_1 \to \mathbb{R}_2$ defined by $f(x) = x^3$ is a diffeomorphism between the two:

$$\psi \circ f \circ \phi^{-1} = id \text{ is } C^\infty \text{ on } \phi(f^{-1}(\mathbb{R}) \cap \mathbb{R}) = \mathbb{R} \text{ and,}$$
$$\phi \circ f^{-1} \circ \psi^{-1} = id \text{ is } C^\infty \text{ on } \psi(f^{-1}(\mathbb{R}) \cap \mathbb{R}) = \mathbb{R}.$$

Exercise 6.18 Smoothness of a map to a Cartesian product
Solution: Let $f = (f_1, f_2) : N \to M_1 \times M_2$, and $\mathfrak{A}_{M_1} = \{(U_\alpha, \phi_\alpha)\}$, $\mathfrak{A}_{M_2} = \{(V_\alpha, \psi_\alpha)\}$, and $\mathfrak{A}_N = \{(W_\alpha, \chi_\alpha)\}$ be atlases for M_1, M_2 and N respectively. Correspondingly, by

Proposition 5.18 above, $\mathfrak{A} = \{(U_\alpha \times V_\alpha, \phi_\alpha \times \psi_\alpha)$ is an atlas for $M_1 \times M_2$.
Assume f is $C^\infty \Rightarrow f \circ \pi_i = f_i$ is C^∞ as a composition of C^∞ maps.
Now assume f_1 and f_2 are both C^∞. Then we can see that for any $p \in W_\alpha$ and $f(p) \in U_\alpha \times V_\alpha$:

$$(\phi_\alpha \times \psi_\alpha) \circ f \circ \chi^{-1} = (\phi_\alpha \circ f_1 \circ \chi^{-1}) \times (\psi_\alpha \circ f_2 \circ \chi^{-1}),$$

is C^∞ at p as its component maps are C^∞.

2.2 Quotients and Real Projective Plane

Not much was to be discussed here apart from what is already known from an introductory course in Topology and Algebraic Geometry.

Problem 7.5* Orbit space of a continuous group action
Solution: Assume a group G acts on S continuously, i.e, the map $\bullet : S \times G \to S$ is continuous.
Consider any open set $U \subseteq S$. We must show that $\pi(U)$ is open in the quotient topology on $S/G \Leftrightarrow \pi^{-1}(\pi(U))$ is open in S. For any $[p] \in \pi(U)$, $\pi^{-1}([p]) = \bullet^{-1}(\{p\}) \Rightarrow \pi^{-1}(\pi(U)) = \bullet^{-1}(U)$ which is open.

Problem 7.6 Quotient of $\mathbb{R}$ by $2\pi\mathbb{Z}$
Solution: We know that $\mathbb{R}/2\pi\mathbb{Z}$ is homeomorphic to S^1, which is a smooth manifold with some atlas $\mathfrak{A} = \{(U_\alpha, \phi_\alpha)\}$. Let f be a homeomorphism from $\mathbb{R}/2\pi\mathbb{Z}$ to S^1. Then we claim $\mathfrak{A}' = \{(f^{-1}(U_\alpha), \phi_\alpha \circ f)\}$ is an atlas for $\mathbb{R}/2\pi\mathbb{Z}$. It is is easy to see that $\mathfrak{A}'$ covers $\mathbb{R}/2\pi\mathbb{Z}$ and that $\phi_\alpha \circ f$ are local homeomorphisms. We can also see that for any two charts $(f^{-1}(U_\alpha), \phi_\alpha \circ f), (f^{-1}(U_\beta), \phi_\beta \circ f) \in \mathfrak{A}'$:

$$(\phi_\alpha \circ f) \circ (\phi_\beta \circ f)^{-1} = (\phi_\alpha \circ f) \circ (f^{-1} \circ \phi_\beta^{-1}) = \phi_\alpha \circ \phi_\beta^{-1},$$

is C^∞ on $(\phi_\beta \circ f)(f^{-1}(U_\alpha) \cap f^{-1}(U_\beta)) = \phi(U_\alpha \cap U_\beta)$, by C^∞-compatibility in $\mathfrak{A}$. The same holds for $(\phi_\beta \circ f) \circ (\phi_\alpha \circ f)^{-1}$. Thus the two arbitrary charts are C^∞-compatible, and $\mathbb{R}/2\pi\mathbb{Z}$ is a smooth manifold. (Note that the argument is general enough to hold for any two homeomorphic spaces.)

3 The Tangent Space

3.1 The Tangent Space

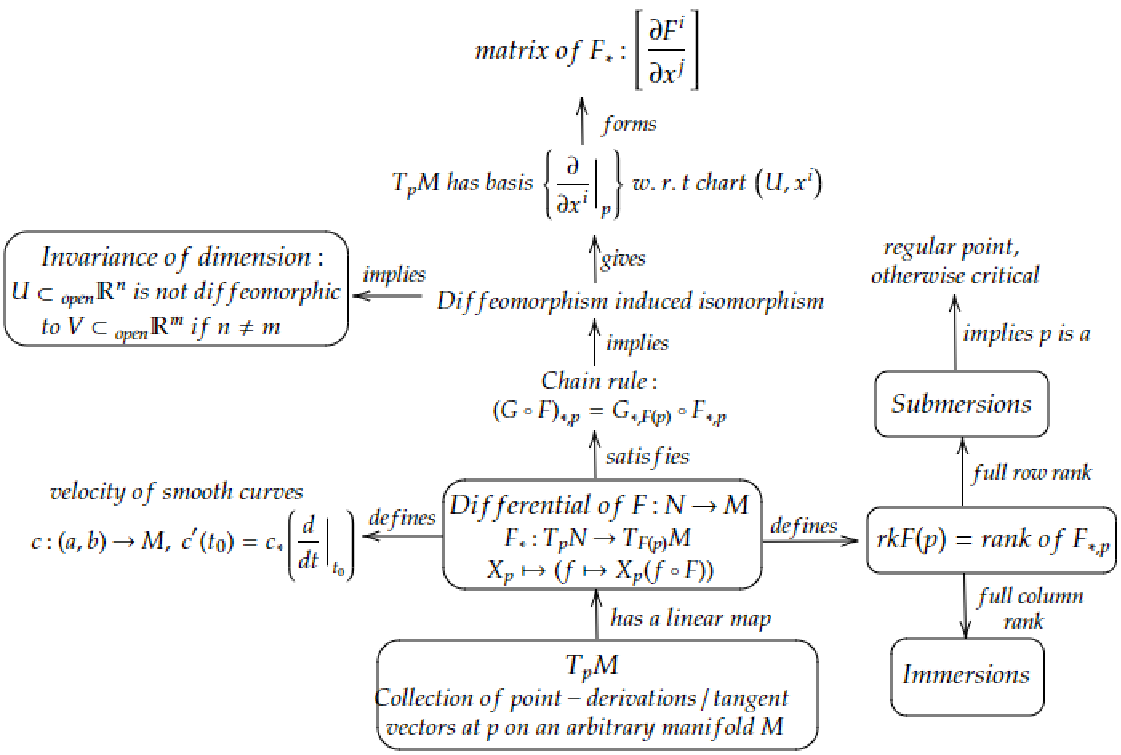


Figure 6: .

Problem 8.1* Differential of a map:
Solution: As $\frac{\partial}{\partial u}\Big|_{F(p)}$, $\frac{\partial}{\partial v}\Big|_{F(p)}$, and $\frac{\partial}{\partial w}\Big|_{F(p)}$ form a basis for $T_{F(p)}\mathbb{R}^3$, we have:

$$F_*(\partial/\partial x|_p) = a_u(\partial/\partial u|_{F(p)}) + a_v(\partial/\partial u|_{F(p)}) + a_w(\partial/\partial u|_{F(p)}).$$

Acting both sides on u, v, and w we get:

$$\partial/\partial x|_p(u \circ F) = 1 = a_u$$
$$\partial/\partial x|_p(v \circ F) = \partial/\partial x|_p(y) = 0 = a_v$$
$$\partial/\partial x|_p(w \circ F) = \partial/\partial x|_p(xy) = y = a_w$$

Thus $F_*(\partial/\partial x|_p) = (\partial/\partial u|_{F(p)}) + y(\partial/\partial u|_{F(p)})$.

Proposition 8.15 (Velocity of a curve in local coordinates)
Proof: Let $c : (a, b) \to M$ be a smooth curve, with $(U, x^1, ..., x^m)$ as a coordinate chart

around $c(t)$. Then we have:

$$c'(t) = c_*\left(\frac{d}{dt}\bigg|_t\right) = \sum_{i=1}^{m} a^i \frac{\partial}{\partial x^i}\bigg|_{c(t)}.$$

Acting both sides on x^j:

$$\frac{d}{dt}\bigg|_t (x^j \circ c) = \dot{c}(t) = \sum_{i=1}^{m} a^i \delta_i^j = a^j$$
$$\Rightarrow c'(t) = \sum_{i=1}^{m} \dot{c}^i(t) \frac{\partial}{\partial x^i}\bigg|_{c(t)}$$

Problem 8.7* Tangent space to a product

Solution: Consider the following coordinate charts: $(U, x^1, ..., x^m)$ in M and $(V, y^1, ..., y^n)$ in N, where $p \in U$ and $q \in V$. The corresponding coordinate chart in $M \times N$ is $(U \times V, \tilde{x}^1, ..., \tilde{x}^m, \tilde{y}^1, ..., \tilde{y}^m)$, where $\tilde{x}^i = x^i \circ \pi_1$ and $\widetilde{y^i} = y^i \circ \pi_2 \Rightarrow$ the basis for $T_p(M \times N)$ is $\left\{\frac{\partial}{\partial \tilde{x}^1}\bigg|_{(p,q)}, ..., \frac{\partial}{\partial \tilde{x}^m}\bigg|_{(p,q)}, \frac{\partial}{\partial \tilde{y}^1}\bigg|_{(p,q)}, ..., \frac{\partial}{\partial \tilde{y}^n}\bigg|_{(p,q)}\right\}$. On the other hand, the basis for $T_pM \times T_pN$ is $\left\{\left(\frac{\partial}{\partial x^1}\bigg|_p, \mathbf{0}\bigg|_q\right), ..., \left(\frac{\partial}{\partial x^m}\bigg|_p, \mathbf{0}\bigg|_q\right), \left(\mathbf{0}\bigg|_p, \frac{\partial}{\partial y^1}\bigg|_q\right), ..., \left(\mathbf{0}\bigg|_p, \frac{\partial}{\partial y^n}\bigg|_q\right)\right\}$. We show that (π_{1*}, π_{2*}) maps $\frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,q)} \mapsto \left(\frac{\partial}{\partial x^i}\bigg|_p, \mathbf{0}\bigg|_q\right)$ and $\frac{\partial}{\partial \tilde{y}^i}\bigg|_{(p,q)} \mapsto \left(\mathbf{0}\bigg|_p, \frac{\partial}{\partial y^i}\bigg|_q\right)$, hence mapping a basis to a basis, and thus being an isomorphism:

$$(\pi_{1*}, \pi_{2*})\left(\frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,q)}\right) = \sum_{i=1}^{m} a^i \left(\frac{\partial}{\partial x^i}\bigg|_p, \mathbf{0}\bigg|_q\right) + \sum_{i=1}^{n} b^i \left(\mathbf{0}\bigg|_p, \frac{\partial}{\partial y^i}\bigg|_q\right)$$
$$\Rightarrow \left(\pi_{1*}\left(\frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,q)}\right), \pi_{2*}\left(\frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,q)}\right)\right) = \sum_{i=1}^{m} a^i \left(\frac{\partial}{\partial x^i}\bigg|_p, \mathbf{0}\bigg|_q\right) + \sum_{i=1}^{n} b^i \left(\mathbf{0}\bigg|_p, \frac{\partial}{\partial y^i}\bigg|_q\right),$$

where we can compare entry wise and act on the coordinate functions:

$$\pi_{1*}\left(\frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,q)}\right)(x^j) = \sum_{i=1}^{m} a^i \frac{\partial}{\partial x^i}\bigg|_p (x^j)$$
$$\Rightarrow \frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,q)} (x^j \circ \pi_1) = \sum_{i=1}^{m} a^i \delta_i^j = a^j$$
$$\Rightarrow \frac{\partial}{\partial r^i}\bigg|_{\phi(p,q)} (x^j \circ \pi_1 \circ (\phi^{-1} = (\tilde{x}^1, ..., \tilde{x}^m, \tilde{y}^1, ..., \tilde{y}^n)^{-1}) = \frac{\partial}{\partial r^i}\bigg|_{\phi(p,q)} (r^j) = \delta_i^j = a^j.$$

Similarly, one can find that $b^i = 0 \Rightarrow (\pi_{1*}, \pi_{2*})\left(\frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,q)}\right) = \left(\frac{\partial}{\partial x^i}\bigg|_p, \mathbf{0}\bigg|_q\right)$. The same arguments can be applied to basis elements of the form $\left.\frac{\partial}{\partial \tilde{y}^i}\right|_{(p,q)}$ to show $(\pi_{1*}, \pi_{2*})\left(\left.\frac{\partial}{\partial \tilde{y}^i}\right|_{(p,q)}\right) = \left(\left.\mathbf{0}\right|_p, \left.\frac{\partial}{\partial y^i}\right|_q,\right)$.

3.2 Submanifolds

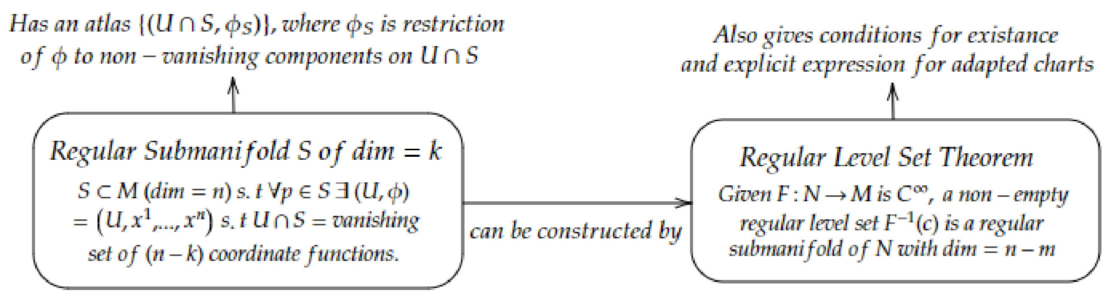


Figure 7: .

9.4* Regular Submanifolds
Solution: Consider any $p \in S \Rightarrow \exists$ a neighbourhood U of p s.t one of the coordinates, either x or y is a C^∞ function of the other. Assume WLOG that $x = p(y)$, which gives us that $\exists$ a chart $V \subset U, x = p(y), y)$. Consider the function $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ s.t $F(x, y) = (x - p(y), y)$. We can see that the $det\left[\frac{\partial F^i}{\partial x^j}(q)\right] = 1 \neq 0$ for all $q \in U \Rightarrow F$ is a diffeomorphism and $(U, x - p(y), y)$ is a chart around p s.t $U \cap S = Z(x - p(y))$. Thus, as p was arbitrary, S is a regular submanifold of codimension 1.

Problem 9.6 Euler's formula
Solution: Assume $F \in S[x_0, x_1, ..., x_n]$ is a homogenous polynomial of degree k. Then $F = \sum_\alpha c_\alpha x^\alpha$, where α are multi-indices depicting different monomials of degree k (for example for $\alpha = (1, 0, 3) = x_0^1 x_1^0 x_2^3$ is a monomial of degree 4 in $S[x_0, x_1, x_2]$). Then we have:

$$\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i} = \sum_{i=0}^{n} x_i \frac{\partial \sum_\alpha c_\alpha x^\alpha}{\partial x_i} = \sum_\alpha c_\alpha \sum_{i=0}^{n} x_i \frac{\partial x^\alpha}{\partial x_i} = \sum_\alpha c_\alpha \sum_{i=0}^{n} \alpha_i x^\alpha = \sum_\alpha k c_\alpha x^\alpha = kF.$$

Problem 9.7 Smooth projective hypersurfaces
Solution: Assume we have a projective hypersurface defined by $F^{-1}(0) = Z(F(x_0, x_1, x_2)) \subset \mathbb{RP}^2$. We need to show that given $\left(\frac{\partial F}{\partial x_i}\right)$ are not simultaneously zero on $F^{-1}(0)$, $F^{-1}(0)$ is a regular level set and hence the projective hypersurface is smooth. For this, we need to calculate the Jacobian determinant of the function $F : \mathbb{RP}^n \to \mathbb{R}$ for all $p \in F^{-1}(0)$. Given the standard atlas on $\mathbb{RP}^n$, assume WLOG that $p \in F^{-1}(0) \cap U_0$ and that the respective chart is $(U_0, \phi) = \left(U_0, x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0}\right)$ (Note that $\phi^{-1}(x, y) = [1 : x : y]$).

Then we have:

$$\frac{\partial}{\partial x}\bigg|_p F = \frac{\partial}{\partial r^1}\bigg|_{\phi(p)} (F \circ \phi^{-1}) = \frac{\partial F}{\partial x_1}(p).$$

Similary one can find that $\frac{\partial}{\partial y}\big|_p F = \frac{\partial F}{\partial x_2}(p)$. Now assume on the contrary that the Jacobian determinant is zero, i.e, $\frac{\partial F}{\partial x_1}(p) = \frac{\partial F}{\partial x_2}(p) = 0$. Then by Problem 9.6, we can see that as $x_0 \neq 0$ and $F(p) = 0 \Rightarrow \frac{\partial F}{\partial x_0}(p) = 0$, a contradiction. Hence the Jacobian determinant must be non-zero $\Rightarrow p$ is a regular point. One can similarly argue this for $p \in F^{-1}(0) \cap U_1$ or $F^{-1}(0) \cap U_2$, to show that $F^{-1}(0)$ is a regular level set. Hence the projective hypersurface is smooth.

3.3 Categories and Functors

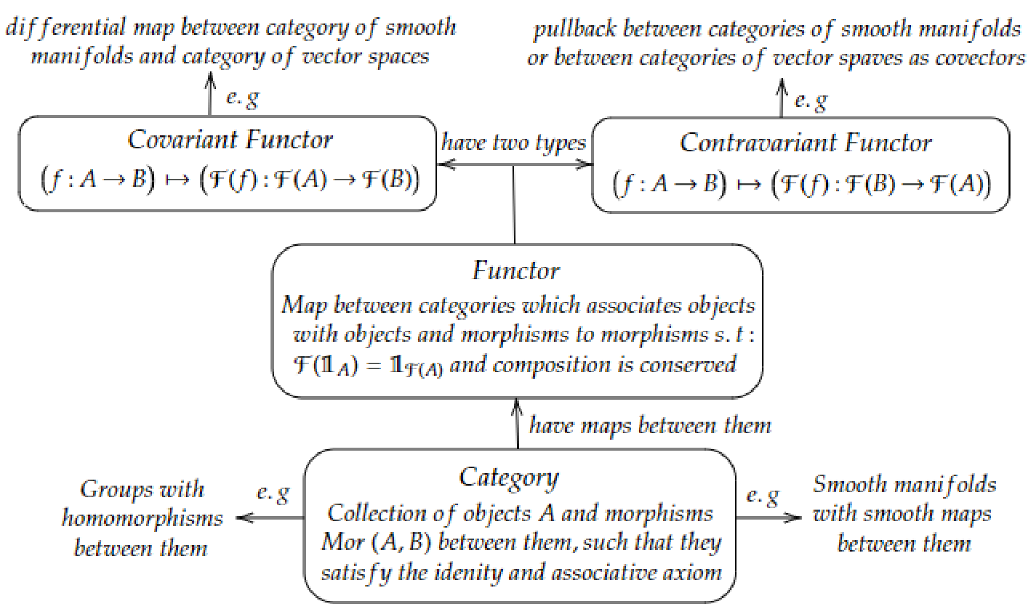


Figure 8: .

Problem 10.1 Differential of the inverse map

Solution: Assume we have a diffeomorphism $F : N \to M$ and that $p \in N$. Then we have the differential maps $F_{*,p} : T_pN \to T_{F(p)}M$ and $(F^{-1})_{*,F(p)} : T_{F(p)}M \to T_pN$ such that:

$$((F^{-1})_{*,F(p)} \circ F_{*,p})(X_p)f = (F_{*,p}(X_p))(f \circ F^{-1}) = X_p(f \circ F^{-1} \circ F) = X_p(f),$$

for arbitrary $X_p \in T_pN$ and $f \in C^\infty(N)$. Thus $((F^{-1})_{*,F(p)} \circ F_{*,p})(X_p) = X_p \Rightarrow (F^{-1})_{*,F(p)} \circ F_{*,p} = \mathbb{1}_{T_pN}$. Similarly, it can be shown that $F_{*p} \circ (F^{-1})_{*,F(p)} = \mathbb{1}_{T_{F(p)}N}$.

Thus $(F^{-1})_{*,F(p)} = (F_{*,p})^{-1}$ is an isomorphism of vector spaces T_pN and $T_{F(p)}M$.

Proposition 10.3 (Isomorphism under a functor)
Proof: Let $\mathfrak{F} : \mathfrak{C} \to \mathfrak{D}$ be a covariant functor from a category $\mathfrak{C}$ to category $\mathfrak{D}$. Assume $f \in Mor(A, B)$ is an isomorphism in $\mathfrak{C} \Rightarrow \exists\, g \in Mor(B, A)$ s.t $g \circ f = \mathbb{1}_A$ and $f \circ g = \mathbb{1}_B$. Now $\mathfrak{F}(f) : \mathfrak{F}(A) \to \mathfrak{F}(B)$ and $\mathfrak{F}(g) : \mathfrak{F}(B) \to \mathfrak{F}(A)$ are morphisms in $\mathfrak{D}$. Then we have:

$$\mathfrak{F}(f) \circ \mathfrak{F}(g) = \mathfrak{F}(f \circ g) = \mathfrak{F}(\mathbb{1}_B) = \mathbb{1}_{\mathfrak{F}(B)},$$

and similarly $\mathfrak{F}(g) \circ \mathfrak{F}(f) = \mathbb{1}_{\mathfrak{F}(A)}$. Thus $\mathfrak{F}(f)$ is an isomorphism in $\mathfrak{D}$. One can almost identically prove this for the case where $\mathfrak{F}$ is a contravariant functor.

Problem 10.7 Pullback in the top dimension
Solution: Given a linear transformation $L : V \to V$ where V is a vector space of dimension n, the pullback $L^* : A_n(V) \to A_n(V)$ is defined as $L^*f(v_1, ..., v_n) = f(L(v_1), ..., L(v_n))$. It is easy to see that given f is alternating, $f(v_1, ..., v_n) = 0$ if the set $\{v_1, ..., v_n\}$ is linearly dependent: assume $v_k = \sum_{i \neq k} c^i v_i$ for some k. Then we have:

$$f(v_1, ..., v_k, ..., v_n) = f(v_1, ..., \sum_{i \neq k} c^i v_i, ..., v_n) = \sum_{i \neq k} c^i f(v_1, ..., v_i, ...v_n) = 0$$

We also know that if $\{v_i\}$ is linearly dependent, then so is $\{L(v_i)\}$. Thus $f(v_1, ..., v_n) = 0 = L^*f(v_1, ..., v_n) = Det(L)L^*f(v_1, ..., v_n)$ trivially holds. Now we assume that the set $\{v_i\}$ is strictly linearly independent $\Rightarrow$ it is a basis for V. Assume the matrix for L in this basis $\Rightarrow L(v_i)$ are the columns of this matrix. We know that the determinant function is also a $n-$covector, and that:

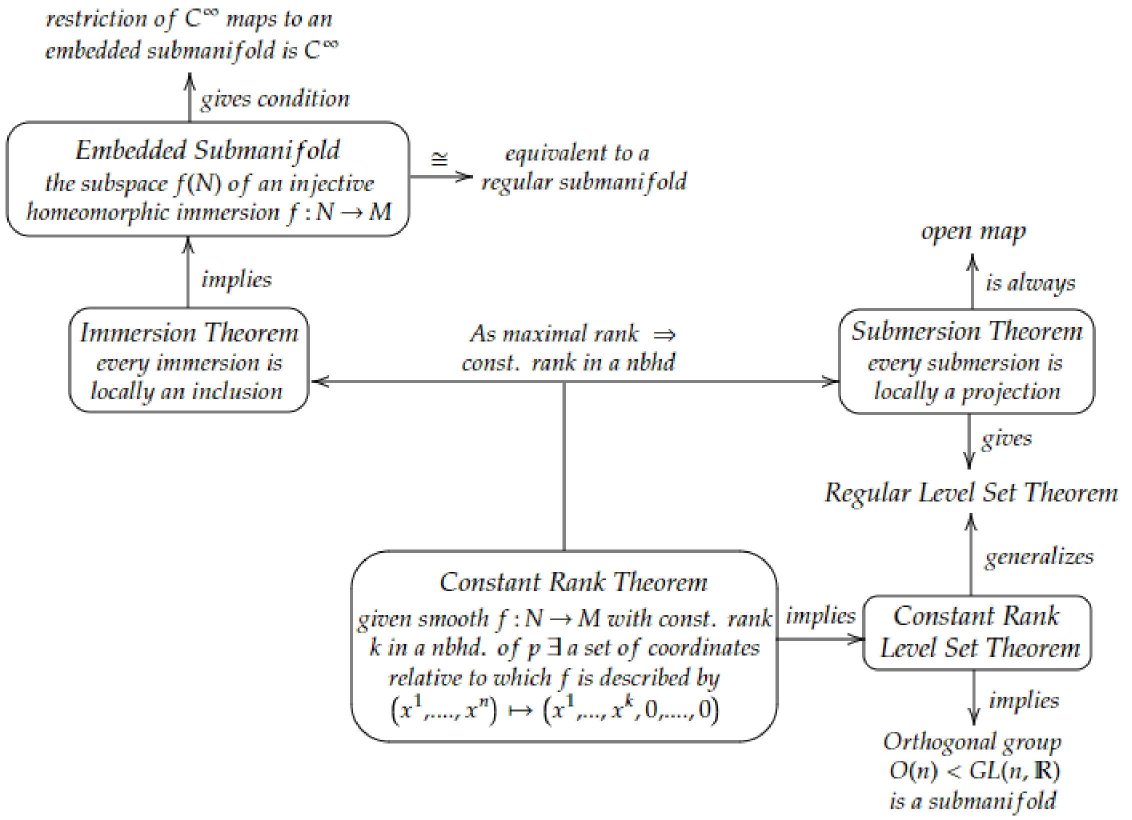
$$d = Det(L) = Det(L(v_1), ..., L(v_n)) = d \times Det(v_1, ..., v_n),$$

where $Det(v_1, ..., v_n) = Det(\mathbb{I}) = 1$. Note that the factoring out of d in the last step above follows from the well known *row operations*. These operations are based only on the alternating property of the determinant function, and hence can just as easily be generalized to any arbitrary $k-$covector, hence giving:

$$L^*f(v_1, ..., v_n) = f(L(v_1), ..., L(v_n)) = d \times f(v_1, ..., v_n),$$

where $d = Det(L)$.

3.4 The Rank of a Smooth Map



Problem 11.3* Critical points of a smooth map on a compact manifold
Solution: Consider a map $f : N \to \mathbb{R}^m$ where N is a compact manifold. Assume f has no critical points, that is, it is a submersion$\Rightarrow$ f is an open map. Then $f(N)$ is an open subset of $\mathbb{R}^m$. Also, as N is compact and f in continuous, $f(N)$ is compact in $\mathbb{R}^m \Rightarrow f(N)$ is closed in $\mathbb{R}^m$. Also, as $\mathbb{R}^m$ is not compact itself, $f(N) \neq \mathbb{R}^m$. Thus, $\mathbb{R}^m = f(N) \cup f(N)^c$ forms a separation of $\mathbb{R}^m \Rightarrow$ this is a contradiction as $\mathbb{R}^m$ is connected. Thus f must have a critical point.

Problem 11.4 Differential of an inclusion map
Solution: Assume we have the coordinate map $\phi(a,b,c) = (u,v)$, with $u(a,b,c) = a$ and $v(a,b,c) = b$ on the upper hemisphere of $S^2 \subset \mathbb{R}^3$. Then we have for $i_* : T_pS^2 \to T_p\mathbb{R}^3$ at $p = (a,b,c)$:

$$i_*\left(\frac{\partial}{\partial u}\bigg|_p\right) = \alpha^1\left(\frac{\partial}{\partial x}\bigg|_p\right) + \beta^1\left(\frac{\partial}{\partial y}\bigg|_p\right) + \gamma^1\left(\frac{\partial}{\partial z}\bigg|_p\right)$$

$$\Rightarrow i_*\left(\frac{\partial}{\partial u}\bigg|_p\right)(x) = \alpha^1\left(\frac{\partial}{\partial x}\bigg|_p\right)(x) + \beta^1\left(\frac{\partial}{\partial y}\bigg|_p\right)(x) + \gamma^1\left(\frac{\partial}{\partial z}\bigg|_p\right)(x)$$

$$\Rightarrow \left(\frac{\partial}{\partial u}\bigg|_p\right)(x \circ i) = \left(\frac{\partial}{\partial r^1}\bigg|_{(a,b)}\right)(x \circ i \circ \phi^{-1}) = \left(\frac{\partial}{\partial r^1}\bigg|_{(a,b)}\right)(r^1) = 1 = \alpha^1.$$

Similarly:

$$i_*\left(\frac{\partial}{\partial u}\Big|_p\right)(y) = \alpha^1\left(\frac{\partial}{\partial x}\Big|_p\right)(y) + \beta^1\left(\frac{\partial}{\partial y}\Big|_p\right)(y) + \gamma^1\left(\frac{\partial}{\partial z}\Big|_p\right)(y)$$
$$\Rightarrow \left(\frac{\partial}{\partial u}\Big|_p\right)(y\circ i) = \left(\frac{\partial}{\partial r^1}\Big|_{(a,b)}\right)(y\circ i\circ\phi^{-1}) = \left(\frac{\partial}{\partial r^1}\Big|_{(a,b)}\right)(r^2) = 0 = \beta^1,$$

and:

$$i_*\left(\frac{\partial}{\partial u}\Big|_p\right)(z) = \alpha^1\left(\frac{\partial}{\partial x}\Big|_p\right)(z) + \beta^1\left(\frac{\partial}{\partial y}\Big|_p\right)(z) + \gamma^1\left(\frac{\partial}{\partial z}\Big|_p\right)(z)$$
$$\Rightarrow \left(\frac{\partial}{\partial u}\Big|_p\right)(z\circ i) = \left(\frac{\partial}{\partial r^1}\Big|_{(a,b)}\right)(z\circ i\circ\phi^{-1}) = \left(\frac{\partial}{\partial r^1}\Big|_{(a,b)}\right)(\sqrt{1-(r^1)^2-(r^2)^2}) = -\frac{a}{c} = \gamma^1.$$

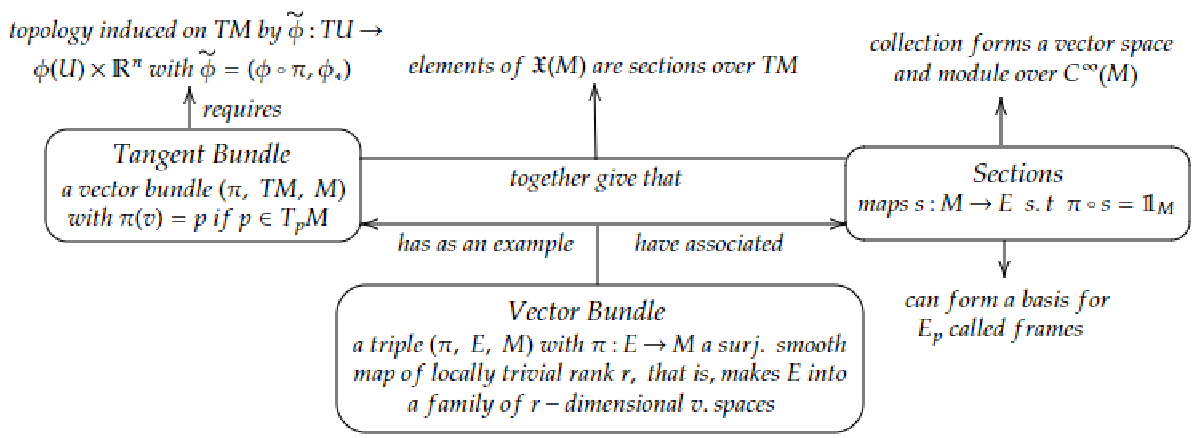
Similarly, one can see that:

$$i_*\left(\frac{\partial}{\partial v}\Big|_p\right) = \alpha^2\left(\frac{\partial}{\partial x}\Big|_p\right) + \beta^2\left(\frac{\partial}{\partial y}\Big|_p\right) + \gamma^2\left(\frac{\partial}{\partial z}\Big|_p\right),$$

with $\alpha^2 = 0$, $\beta^2 = 1$, $\gamma^2 = -\dfrac{b}{c}$.

Problem 11.5 One-to-one immersion of a compact manifold
Solution: Assume $f : N \to M$ is a one-to-one immersion (hence continuous) with N as a compact manifold. Then the induced map $\tilde{f} : N \to f(N)$ is continuous and bijective. Now take any closed set $C \subset N \Rightarrow C$ is compact $\Rightarrow \tilde{f}(C)$ is compact in $f(N)$. As $f(N)$ is Hausdorff under the subspace topology, $\tilde{f}(C)$ is closed in $f(N)$. Thus $\tilde{f}^{-1}$ is continuous, and $\tilde{f}$ is a homeomorphism of N with its image $f(N)$. Thus f is an embedding.

3.5 The Tangent Bundle



Problem 12.2 Transition functions for the total space of the tangent bundle
Solution: Consider two overlapping coordinate charts on a manifold M given by $(U, x^1, ..., x^n) = (U, \phi)$ and $(V, x^1, ..., x^n) = (U, \psi)$. They induce coordinate charts $(TU, \tilde{\phi})$ and $(TV, \tilde{\psi})$ on TM. This gives us the transition function $F = \tilde{\psi} \circ \tilde{\phi}^{-1}$ with $(x^1(p), ..., x^n(p), a^1, ..., a^n) \mapsto (y^1(p), ..., y^n(p), b^1, ..., b^n)$ where $\sum_i a^i \frac{\partial}{\partial x^i} = \sum_i b^i \frac{\partial}{\partial y^i}$. Not that if we apply y^j to both sides of this relation, we get:

$$\sum_i a^i \frac{\partial y^j}{\partial x^i} = b^j \tag{1}$$

Now we can get to calculating the Jacobian matrix of this transition functions. Firstly, note that by definition, $\frac{\partial F^i}{\partial r^j} = 0$ for $1 \leq i \leq n$ and $n+1 \leq j \leq 2n$, or for $n+1 \leq i \leq 2n$ and $1 \leq j \leq n$, i.e, b^i only depend on a^i and y^i only depend on x^i. We can see that for $n + 1 \leq i, j \leq 2n$, (1) gives us that:

$$\frac{\partial F^i}{\partial r^j} = \frac{\partial b^i}{\partial a^j} = \frac{\partial y^i}{\partial x^j}.$$

Furthermore, for $1 \leq i, j \leq n$:

$$\frac{\partial F^i}{\partial r^j} = \frac{\partial (y^i \circ \phi^{-1})}{\partial r^j} = \frac{\partial y^i}{\partial x^j}.$$

Thus the Jacobian matrix is the following block matrix: $\begin{pmatrix} \left[\frac{\partial y^i}{\partial x^j}\right] & 0 \\ 0 & \left[\frac{\partial y^i}{\partial x^j}\right] \end{pmatrix}.$

It is not hard to see that the Jacobian determinant is then $(\det[\partial y^i/\partial x^j])^2$.

Proposition 12.9(ii) Smoothness of scalar multiplication
Solution: Assume s is a C^∞ section of a C^∞ vector bundle $\pi : E \to M$, and f is a C^∞ real-valued function on M. We want to show that $(fs) : M \to E$ defined as $(fs)(p) = f(p)s(p)$ is a C^∞ section of E. Firstly, it is clear that it is indeed a section of E, as E_p is closed under scalar multiplication for all p. Now consider any point $p \in M$, and a corresponding trivializing open set $\pi(p) \in U$ with trivialization $\phi \Rightarrow \exists (V, \psi)$ with $p \in V \subseteq \pi^{-1}(U)$. As s is $C^\infty \Rightarrow ((\psi \times \mathbb{1}) \circ \phi \circ s \circ \psi^{-1})$ is C^∞. We can see that:

$$((\psi \times \mathbb{1}) \circ \phi \circ s \circ \psi^{-1})(\psi(p)) = (p, c^1, ..., c^n),$$

for some C^∞ functions c^i. Now if we instead consider $((\psi \times \mathbb{1}) \circ \phi \circ (fs) \circ \psi^{-1})$, as ϕ is $\mathbb{R}$-linear:

$$((\psi \times \mathbb{1}) \circ \phi \circ s \circ \psi^{-1})(\psi(p)) = (p, f(p)c^1, ..., f(p)c^n),$$

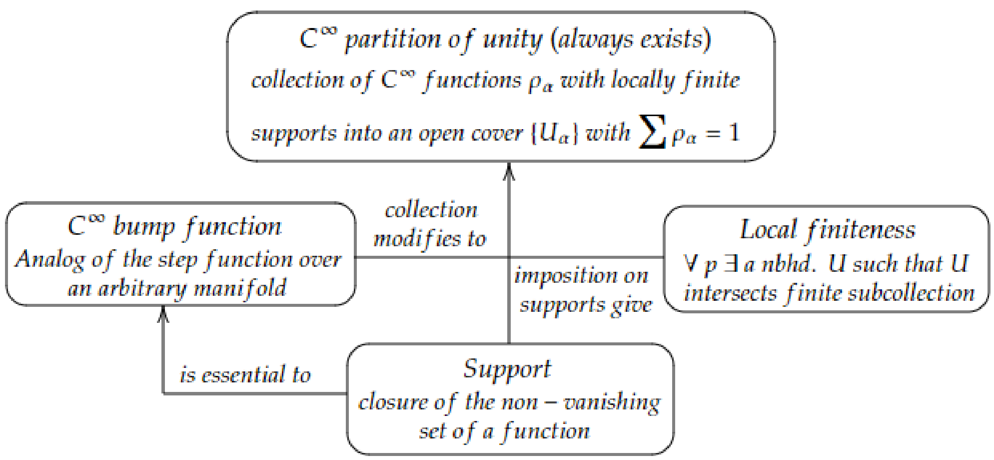
which is also C^∞ as the product (fc^i) is C^∞ for all i. Thus (fs) is a C^∞ section on E.

Problem 12.4 Coefficients relative to a smooth frame
Solution: Let $\pi : E \to M$ be a C^∞ vector bundle and $s^1, ..., s^r$ be a C^∞ frame for E over an open set U and $t_1, ..., t_r$ be the frame of trivialization. Then for any $e \in \pi^{-1}(U)$, $e = \sum_{j=1}^r b^j(e)t_j(p) = \sum_{j=1}^r c^j(e)s_j(p)$. We show that $t_j : \pi^{-1}(U) \to \mathbb{R}$ is C^∞, and hence so are s_j. Consider the following action of the trivialization ϕ of E over U:

$$\phi(e) = \phi(\sum_{j=1}^r b^j(e)t_j(p)) \overset{\text{lin.}}{=} \sum_{j=1}^r b^j(e)\phi(t_j(p)) = \sum_{j=1}^r b^j(e)(\pi(e), e_j) = (\pi(e), b^1(e), ..., b^r(e)),$$

where b^j are components of ϕ. As ϕ is C^∞, so are the b^j. Now we can take $s_i(p) = \sum_j d_i^j t_j(p) \Rightarrow \sum_j b^j(e)t_j(p) = \sum_{i,j} c^i(e)d_i^j t_j(p) \Rightarrow b^j(e) = \sum_i c^i(e)d_i^j$. As the matrix $[d_i^j]$ consists of C^∞ functions, its inverse multiplied by the column vector b^j will give that c^i are also C^∞.

3.6 Bump Functions and Partitions of Unity



The idea of a C^∞ partition is that it gives an explicit connection between the local and global properties of a smooth manifold. One can understand this somewhat intuitively considering that ρ_α are 'local' functions (vanish outside U_α) with the global property of their sum being consistenly equal to 1.

Lemma 13.5* Support of a finite sum
Proof: Define $F = \sum_i \rho_i$. Consider any $p \in F^{-1}(\mathbb{R}^\times)$. This implies that $\rho_j(p) \neq 0$ for some $j \Rightarrow p \in \rho_j^{-1}(\mathbb{R}^\times)$. Thus $F^{-1}(\mathbb{R}^\times) \subseteq \bigcup_i \rho_i^{-1}(\mathbb{R}^\times) \Rightarrow \mathbf{cl}(F^{-1}(\mathbb{R}^\times)) \subseteq \mathbf{cl}(\bigcup_i \rho_i^{-1}(\mathbb{R}^\times)) = \bigcup_i \mathbf{cl}(\rho_i^{-1}(\mathbb{R}^\times)) \Rightarrow \text{supp}(\sum_i \rho_i) \subseteq \bigcup_i \text{supp}(\rho_i)$.

Problem 13.2* Locally finite family and compact set
Solution: For every $p \in K$, take a neighbourhood U_p that intersects finitely many of

$\{A_\alpha\}$. Then $\{U_p\}_{p\in K}$ forms an open cover of K. As K is compact$\Rightarrow$ there exists a finite subcover $\{U_i\}_{i=1}^n$. Then $\bigcup_i U_i$ is neighbourhood of K that intersects finitely many $\{A_\alpha\}$.

Problem 13.3 Smooth Urysohn Lemma

Solution(a): Given two disjoint closed sets A and B of a manifold M, consider the open cover $\{M - A, M - B\}$. By Theorem 13.7, there exists a C^∞ partition of unity $\{\rho_{M-A}, \rho_{M-B}\}$ subordinate to the open cover. Thus as $\text{supp}(\rho_{M-B}) \subseteq (M - B) \Rightarrow \rho_{M-B}(B) = 0$. Similarly $\rho_{M-A}(A) = 0$. Moreover, as $\rho_{M-A} + \rho_{M-B} = 1 \Rightarrow \rho M - B(A) = \{1\}$. Thus ρ_{M-B} is a C^∞ function that is identically 1 on A and identically 0 on B.

Solution(b): Let A be a closed subset of M and U be an open set containing A. Then $\{U, M - A\}$ is an open cover of $M \Rightarrow$ by Theorem 13.7, there exists a C^∞ partition $\{\rho_U, \rho_{M-A}\}$ subordinate to the open cover. As $\rho_{M-A}(A) = \{0\}$ and $\rho_U + \rho_{M-A} = 1 \Rightarrow$, $\rho_U(A) = \{1\}$. Thus ρ_U is a C^∞ function that is identically 1 on A with $\text{supp}(\rho_U) \subseteq U$ by definition.

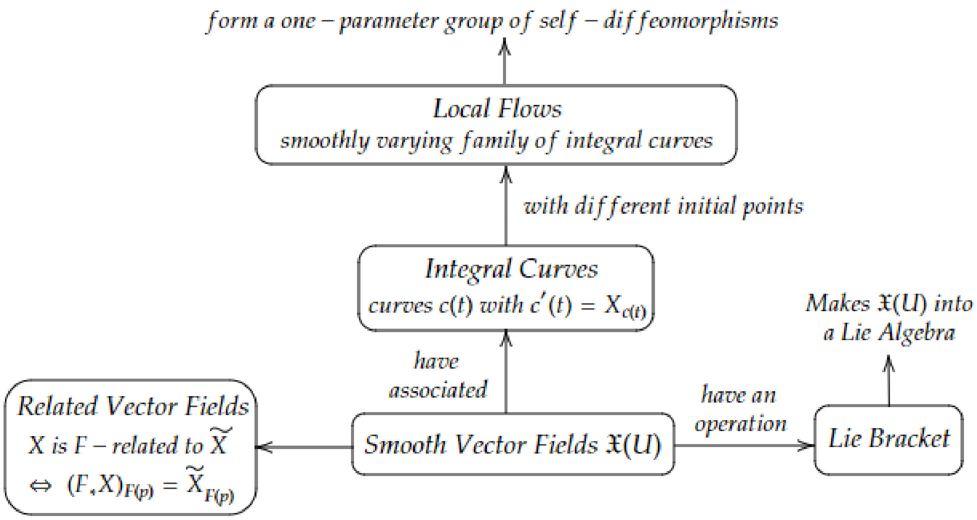
Problem 13.4 Support of the pullback of a function

Solution: Let $F : N \to M$ and $h : M \to \mathbb{R}$ be C^∞. We can see that $(F^*h)^{-1}(\mathbb{R}^\times) = F^{-1}(h^{-1}(\mathbb{R}^\times)) \subseteq F^{-1}(\mathbf{cl}(h^{-1}(\mathbb{R}^\times))) = F^{-1}(\text{supp}(h)) \Rightarrow \mathbf{cl}(F^{-1}(h^{-1}(\mathbb{R}^\times))) \subseteq \mathbf{cl}(F^{-1}(\text{supp}(h))) = F^{-1}(\text{supp}(h))$ as $F^{-1}(\text{supp}(h))$ is the continuous preimage of a closed set, and hence closed.

Problem 13.6 Pullback of a partition of unity

Solution: Take $\{\rho_\alpha\}$ to be a partition of unity on a manifold M subordinate to an open cover $\{U_\alpha\}$ and $F : N \to M$ be C^∞. Consider the collection $\{\text{supp}(F^*\rho_\alpha)\}$. We know from Problem 13.4 that $\text{supp}(F^*\rho_\alpha) \subseteq F^{-1}(\text{supp}(\rho_\alpha))$. Thus if we are able to show that the collection $\{F^{-1}(\text{supp}(\rho_\alpha))\}$ is locally finite, so is $\{\text{supp}(F^*\rho_\alpha)\}$. Consider any $p \in N \Rightarrow F(p) \in M$. As $\{\text{supp}(\rho_\alpha)\}$ is locally finite, there exists a neighbourhood U of $F(p)$ such that U intersects only finitely many supports $\Rightarrow F^{-1}(U)$ is neighbourhood of p that intersects finitely many $F^{-1}(\text{supp}(\rho_\alpha)) \Rightarrow \{F^{-1}(\text{supp}(\rho_\alpha))\}$ is locally finite and so is $\{\text{supp}(F^*\rho_\alpha)\}$. It is trivial to see that $F^*\rho_\alpha$ are also non-negative functions, and as $\sum_\alpha \rho_\alpha = 1$, this implies that $\sum_\alpha (F^*\rho_\alpha) = \sum_\alpha (\rho_\alpha \circ F) = 1$. Finally, we can see that $\text{supp}(F^*\rho_\alpha) \subseteq F^{-1}(\text{supp}(\rho_\alpha)) \subseteq F^{-1}(U_\alpha)$ from the initial assumption about $\{\rho_\alpha\}$. Thus, $\{F^*\rho_\alpha\}$ forms a partition of unity on N subordinate to the open cover $\{F^{-1}(U_\alpha)\}$.

3.7 Vector Fields



Problem 14.2 Vector field on an odd sphere

Solution: The gradient vector on S^{2n-1} is defined as $v = (2x^1, ..., 2x^n, 2y^1, ..., 2y^n)$ at a point $p = (x^1, ..., x^n, y^1, ..., y^n)$. Representing the vector field X at point p as the vector $X_p = (-y^1, ..., -y^n, x^1, ..., x^n)$, we can see that $\sum_i v^i X_p^i = \sum_i(-y^i x^i + y^i x^i) = 0$. Thus X_p is tangent to S^{2n-1}, and hence, as p was arbitrary, X defines vector field on S^{2n-1}. We can also see that it is non-vanishing, as for it to vanish $y^i = x^i = 0$ for all $i \Rightarrow \sum_i (x^i)^2 + (y^i)^2 = 0 \neq 1$, which is a contradiction as the point must lie on S^{2n-1}. To check the smoothness of the vector field, consider the following atlas on S^{2n-1}: $\{(U_{i1} = \{p \in S^{2n-1} : x^i > 0\}, \pi_{x^i}), (U_{i2} = \{p \in S^{2n-1} : x^i < 0\}, \pi_{x^i}), (V_{i1} = \{p \in S^{2n-1} : y^i > 0\}, \pi_{y^i}), (V_{i2} = \{p \in S^{2n-1} : y^i < 0\}, \pi_{y^i})\}_{i=1}^n$. Consider now any point $p \in S^{2n-1}$. WLOG assume $p \in U_{11}$ with chart $\pi_{x_1} = (x^2, ..., x^n, y^1, ...y^n)$. The vector field in terms of these coordinates is then $X = x^1 \dfrac{\partial}{\partial y^1} + \sum_{i=2}^n (-y^i \dfrac{\partial}{\partial x^i} + x^i \dfrac{\partial}{\partial y^i})$. Here we must note that as $x^1 \circ \pi_{x^1}^{-1}(x^2, ..., x^n, y^1, ..., y^n) = \sqrt{(x^2)^2 + ... + (x^n)^2 + (y^1)^2 + ... + (y^n)^2)}$ is C^∞, x^1 is C^∞ at p. The other coefficients are trivially smooth, giving that the X is a smooth vector field on S^{2n-1}.

Problem 14.4 Integral curves in the plane

Solution: Consider the vector field $X_{(x,y)} = x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y}$ on $\mathbb{R}^2$. Then its integral curve $c(t) = (x(t), y(t))$ is given by the following differential equations: $\dot{x}(t) = x$ and $\dot{y}(t) = -y$. These are standard ODEs with solutions $x(t) = e^t + c$ and $y(t) = e^{-t} + d$ for some constants c, d. Taking our initial point to be $p = (x_0, y_0)$, this gives $c = x_0 - 1$ and $d = y_0 - 1$.

Problem 14.13 Vector field under a diffeomorphism

Solution: Let $F : N \to M$ be a diffeomorphism, and g be a C^∞ function and X be a C^∞ vector field on N. Consider now two vector fields: $F_*(gX)$ and $(g \circ F^{-1})F_*X$. We prove these are equal by acting them on an arbitrary C^∞ function f on M at $F(p)$:

$$(F_*(gX))_{F(p)}f = (gX)_p(f \circ F) = g(p)X_p(f \circ F).$$
$$((g \circ F^{-1})F_*X)_{F(p)}f = (g \circ F^{-1})(F(p))(F_*X)_{F(p)}f = g(p)X_p(f \circ F).$$

Problem 14.14 Lie bracket under a diffeomorphism
Solution: Let $F : N \to M$ be a C^∞ diffeomorphism, with X and Y as C^∞ vector fields on N. Consider the followng related vectors fields :$F_*[X, Y]$ and $[F_*X, F_*Y]$. Let p be an arbitrary point in N contained in a chart $(U, x^1, ...x^n)$, and $X = \sum_i a_i(p)\frac{\partial}{\partial x^i}$, $Y = \sum_i b_i(p)\frac{\partial}{\partial x^i}$. We prove the following are equal by acting them on an arbitrary C^∞ function f on M at $F(p)$:

$$\begin{aligned}
(F_*[X, Y])_{F(p)}f &= [X, Y]_p(f \circ F) = (X_pY - Y_pX)(f \circ F) \\
[F_*X, F_*Y]_{F(p)}f &= (F_*X)_{F(p)}(F_*Y)f - (F_*Y)_{F(p)}(F_*X)f \\
&= X_p(((F_*Y)f) \circ F) - Y_p(((F_*X)f) \circ F) \\
&= X_p((F_*Y)_F f) - Y_p((F_*X)_F f) \\
&= X_p(Y(f \circ F)) - Y_p(X(f \circ F)) \\
&= (X_pY - Y_pX)(f \circ F).
\end{aligned}$$

4 Lie Groups and Lie Algebras

4.1 Lie Groups