

Def<sup>g</sup>- A curve  $\gamma : [a, b] \rightarrow \mathbb{R}^2$  is called

closed if  $\gamma(a) = \gamma(b)$  and  $\gamma'(a) = \gamma'(b)$ .

Simple if it is one to one

Convex

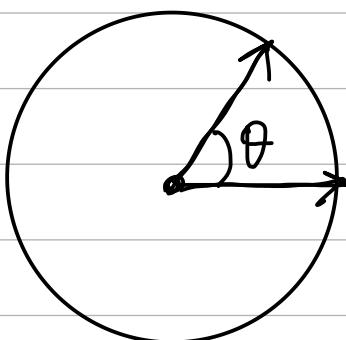
Simple closed curve that lie  
on one side of their tangent lines.

Recall.  $t(s) = \langle \cos(\theta(s)), \sin(\theta(s)) \rangle$

$$\theta' = \kappa \quad \left[ \theta(s) = \theta_0 + \int_0^s \kappa(t) dt \right]$$

Regard:  $t : [a, b] \rightarrow S^1$

$$\theta(b) - \theta(a) = 2k\pi$$



Prop: If  $f: [a, b] \rightarrow S^1$ , cont. then  $\exists a$   
 $f(a) = f(b)$

continuous  $\theta: [a, b] \rightarrow \mathbb{R}$  s.t

$$f(t) = \langle \cos(\theta(t)), \sin(\theta(t)) \rangle \quad \forall t \in [a, b]$$

$\theta(t)$  is unique upto adding a multiple of  $2\pi$ .

i.e

if  $\varphi(t)$  is any other polar angle function then

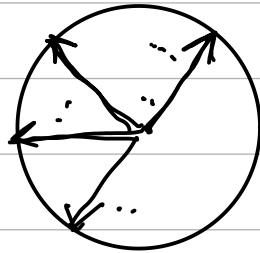
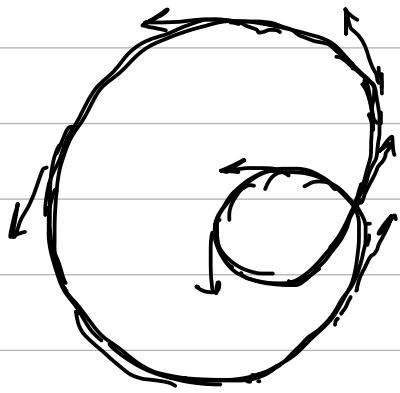
$$\varphi(t) = \theta(t) + 2k\pi$$

$$\deg(f) = \frac{\theta(b) - \theta(a)}{2\pi}$$

no. of times the domain is wrapped

c.c around the circle

$\deg(\vec{t})$  rotation index of the curve.



Rotation index = 2.

Ex! Do it for a circle with different parametrizations.



$$ds = \frac{1}{K} d\theta \Rightarrow d\theta = K ds$$

$$\int_a^b d\theta = \int_a^b K(s) ds$$

$\theta(b) - \theta(a) = \text{Total curvature}$



$2\pi (\text{rotation index}) = \text{Total curvature}$



Prop:- Let  $f_1, f_2 : [a, b] \rightarrow S^1$  be cont. funcs

with  $f_1(a) = f_1(b)$  and  $f_2(a) = f_2(b)$ . If  $f_1$  and  $f_2$  have different degrees that

$$f_1(t_0) = -f_2(t_0) \text{ for some } t_0 \in [a, b].$$

Proof-

Let  $\theta_1, \theta_2 : [a, b] \rightarrow \mathbb{R}$  be the angle functions for  $f_1$  &  $f_2$ . Consider

$$\delta(t) = \theta_2(t) - \theta_1(t)$$

Now

$$|\delta(b) - \delta(a)| = |\theta_2(b) - \theta_2(a) - (\theta_1(b) - \theta_1(a))| \\ \geq 2\pi$$

as degrees are different

Since  $\delta(t)$  has a net-change of atleast  $2\pi$ , there must be an odd integer multiple of  $\pi$  between  $\delta(a)$  and  $\delta(b)$ .

Since  $\delta$  is cont  $\Rightarrow$  (IVT)  $\exists t_0 \in [a, b]$  s.t

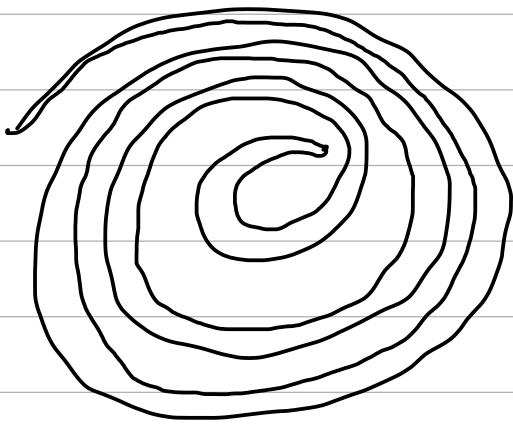
$$\delta(t_0) = (2k+1)\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \theta_2(t_0) = \theta_1(b) + (2k+1)\pi$$

$$\Rightarrow f_2(t_0) = -f_1(t_0).$$

### Hopf's Theorem

let  $\gamma: [a, b] \rightarrow \mathbb{R}^2$  be a simple closed curve. The rotation index of  $\gamma$  is either 1 or -1.



Proof:-

let  $\gamma: [a, b] \rightarrow \mathbb{R}^2$  be a simple closed plane curve and let  $C$  be

its trace.

let  $p \in C$  be a point s.t  $C$  is entirely on one side of the tangent line  $L$ .

Assume WLOG,  $\gamma$  is parametrized by its arc length and  $\gamma(a) = p$ .

Consider the triangle

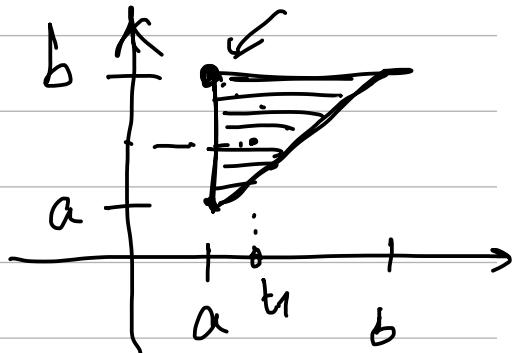
$$T = \{(t_1, t_2) : a \leq t_1 \leq t_2 \leq b\}$$

Define

$\alpha: T \rightarrow S^1$  as

$$\alpha(t_1, t_2) = \begin{cases} \gamma'(t_1) & t_1 = t_2 \\ \end{cases}$$

$$\begin{cases} \frac{\gamma(t_2) - \gamma(t_1)}{\|\gamma(t_2) - \gamma(t_1)\|} & t_1 \neq t_2 \text{ and } (t_1, t_2) \neq (a, b) \\ \gamma' - \gamma'(a) & (t_1, t_2) = (a, b). \end{cases}$$



For most inputs  $\gamma(t_1, t_2)$  is just a unit vector pointing in the direction from  $\gamma(t_1)$  to  $\gamma(t_2)$ .

But ensure  $\gamma'$  is continuous.

$\gamma$  is a loop  $\Rightarrow$

$$\lim_{t \rightarrow b-a} \frac{\gamma(t) - \gamma(a)}{b-a-t} = - \lim_{\substack{t \rightarrow b-a \\ t < (b-a)}} \frac{\gamma(b-a) - \gamma(t)}{(b-a) - t}$$

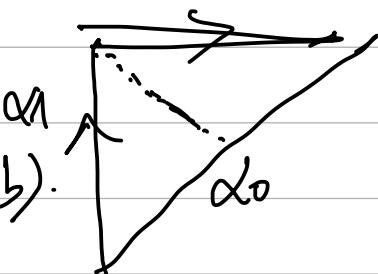
$$\left. \begin{aligned} \gamma(a) &= \gamma(a+b-a) = \gamma(b) \\ \end{aligned} \right\} = -\gamma'(b-a) = -\gamma'(a)$$

Let  $\alpha_0: [0,1] \rightarrow T$  be a parametrization

of the line segment from  $(a,a)$  to  $(b,b)$ .

$$\alpha_1: [0,1] \rightarrow T$$

$$(a,a) \text{ to } (a,b) \rightarrow (a,b) \rightarrow (b,b)$$



$$\alpha_0(t) = (a - at + bt, a - at + bt) \quad \left( \frac{a+b}{2}, \frac{a+b}{2} \right)$$

$$\alpha'_1(t) = \begin{cases} (a - 2at + 2bt, a - 2at + 2bt) & y - \frac{a+b}{2} = \frac{a+b}{2} - b \\ (2a - 2at + 2bt - b, b) & \frac{a+b}{2} - a \end{cases}$$

$$(1-2t)(a, a) + 2t(a, b) \\ (2-2t)(a, b) + (2t-1)(b, b)$$

$$y - \frac{a}{2} - \frac{b}{2} = -1 \left( x - \frac{a}{2} - \frac{b}{2} \right)$$

$$y - \frac{a}{2} - \frac{b}{2} = -x + \frac{a}{2} + \frac{b}{2}$$

$$1 - 2\left(t - \frac{1}{2}\right) + 2\left(t - \frac{1}{2}\right)$$

$$y = -x + a + b$$

$$t \in [0, 1]$$

$$\alpha_s(t) \stackrel{?}{=}$$

$$(1-t)(a, a) + t(b, b)$$

$$\alpha_s(t) = (a - as + bs, a - as + bs)$$

$$\alpha_s(t) = \{$$

$$\alpha_s(t) = (1-s)\alpha_0(t) + s\alpha_1(t)$$

$(s, t) \mapsto \alpha_s(t)$  is a cont. func

$$[0, 1] \times [0, 1] \rightarrow T$$

For each  $s \in [0, 1]$ , let  $D(s)$  be the degree of

$$D = \text{M} \circ \alpha_s : [0, 1] \rightarrow S^1$$

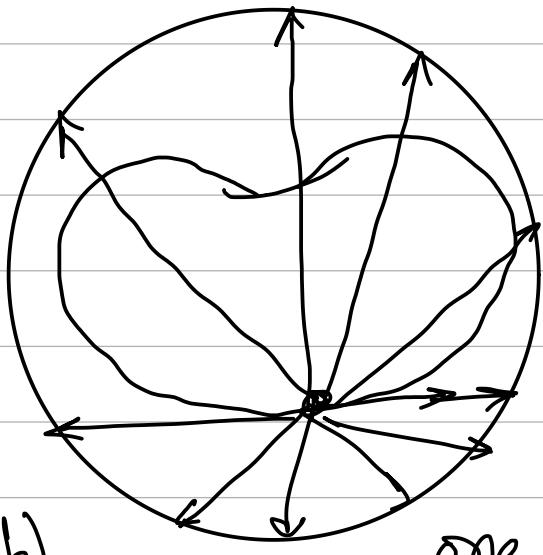
$D$  cont + integer valued  $\Rightarrow D$  must be constant

$$D(0) = \text{M} \circ \alpha_0$$

= deg (unit tangent of  $\gamma$ )

= rotation index of  $\gamma$

$$D(0) = D(1) = \text{M} \circ \alpha_1$$



$(a,a)$  to  $(a,b)$   $\longrightarrow$  one Semicircle (above)

$(a,b)$  to  $(b,b)$   $\longrightarrow$  one Semi-circle (below)

$$D(1) = 1 \quad \text{or} \quad -1$$



"Four vertex theorem"

"Every simple closed convex plane curve has atleast 4 vertices"

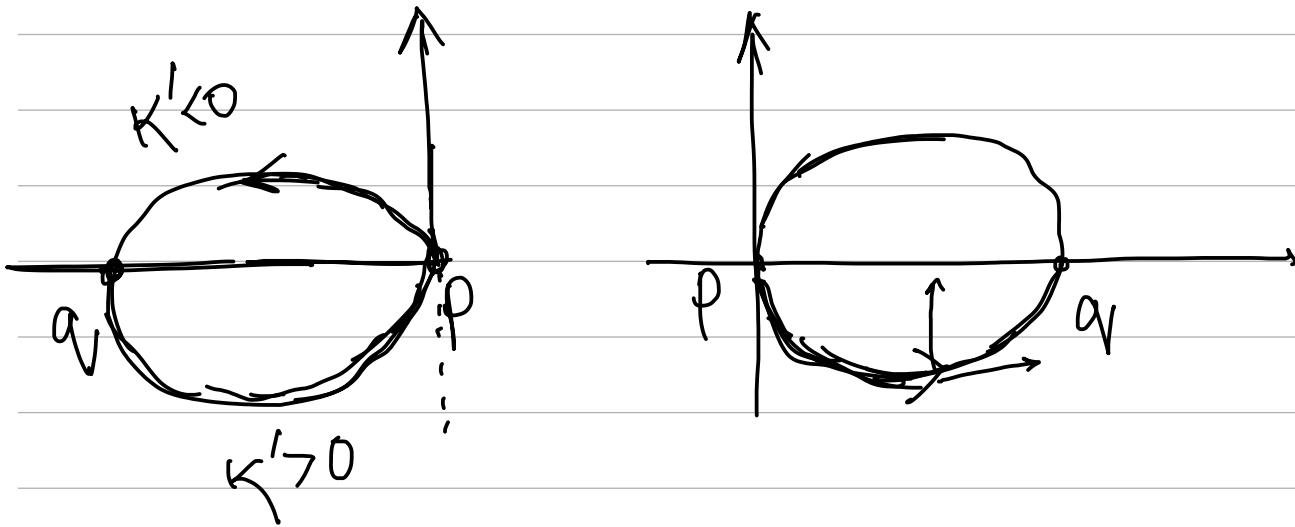
Def:- A point  $r(t)$  on the trace is called a vertex if signed curvature is  $\max \{\min \text{ at } r(t)\}$ .

Proof. let  $C$  be the trace.

If  $\kappa_s = C$  then infinitely many vertices.

Assume vertices of  $\gamma$  are at  $p \in C(\kappa_s^{\max})$  and  $q \in C(\kappa_s^{\min})$

let  $L$  be a line through  $p$  and  $q$ .



choose arc-length parametrization

$$\gamma: [0, L] \rightarrow \mathbb{R}^2 \text{ s.t } \gamma(0) = p = \gamma(L)$$

and  $\gamma(\alpha) = L$  for some  $\alpha \in (0, L)$ .

Notice that  $k'_s \leq 0$  on  $(0, a)$ . ( $\max \rightarrow \min$ )

$k'_s \geq 0$  on  $(a, L)$ .

WLOG, Assume p is origin and L is x-axis.

Take  $\gamma = (x, y)$ , y changes sign only at a.

In this case

$k'(t)y(t)$  does not change

sign on  $[0, L]$

$$\Rightarrow \int_0^L y(t) k'(t) dt \neq 0.$$

but

$$y(t)k(t) \Big|_0^L - \int_0^L y'(t)k(t)dt$$

$$\gamma = (x, y) \Rightarrow \gamma' = (x', y')$$

$$\gamma'' = (x'', y'') = K(-y', x')$$

$$-\int_0^L x'' dt$$

$$= -$$