MATH 407: GENERAL TOPOLOGY
KNOTS AND LINKS
Presentation

Date: Mon Tue Wed Thu Fri Sat

2024-10-0035 Abdullah Ahmed

Introduction:

Knot theory was born in the nineteenth century, with famous mathematicians like Carl Friedrich Gauss and John Listing being among the first to work on the field of study. A Knot is treated essentially as a topological object, and hence, like with any other, we are primarily concerned with its classification. As such, we will introduce a notion of equivalence among Knots and then follow it by a discussion on certain invariants that will help us classify them.

1 What are Knots and Links?

Def 1.1: A Knot is any embedding $f: S^1 \to \mathbb{R}^3$. A Link, on the other hand, is an embedding of a finite collection of circles S^1 into $\mathbb{R}^3$. Each embedded circle is known as a component of the link.

Roughly speaking two Knots or two links are equivalent if one can be continuously deformed into the other. This means no cutting or crossing of the Knot or link is allowed. We Define this rigourously now.

2 Isotopy:

Def 2.1: A homotopy $F: X \times I \to Y$ is called an isotopy if $F\big|_{X \times \{t\}}$ is a homeomorphism for all $t \in I$.

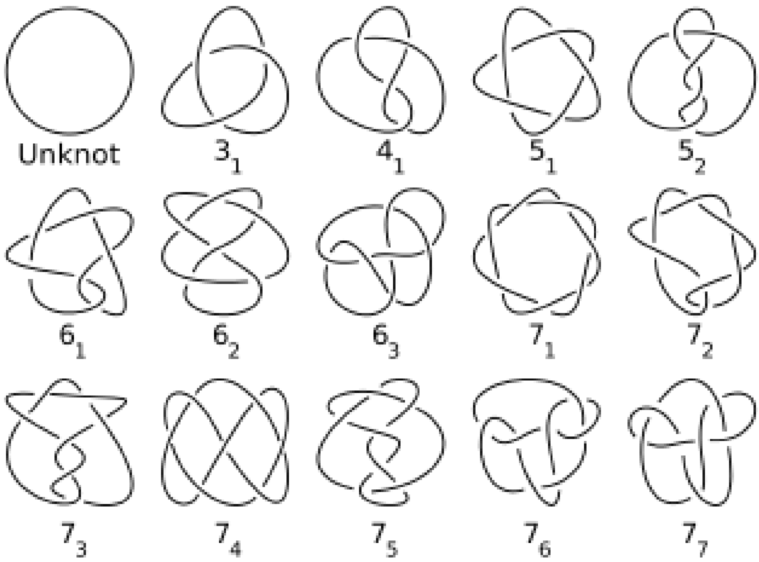


Fig 1.1: Examples of some famous knots.

Thus one can intrepret this as an isotopy being a deformation of a space X over time.

Def. 2.2 Given embeddings $f: Y \to X$ and $g: Y \to X$, f and g are known as <u>ambient isotopic</u> if there exists an isotopy $F: X \times I \to X$ s.t $F(x, 0) = x \ \forall x \in X$ and $F(f(y), 1) = g(y)$ $\forall y \in Y$. The space X is known as the <u>ambient space</u> and F as an <u>ambient isotopy</u>.

EXAMPLE 12.1. Define $F : \mathbb{R}^2 \times I \to \mathbb{R}^2$ by $F(x, t) = (t + 1)x$. When $t = 0$, F is just the identity map on $\mathbb{R}^2$. But as t grows from 0 to 1, each vector in $\mathbb{R}^2$ beginning at the origin is stretched in length until the end of the isotopy, when they are all twice as long as they originally were. (See Figure 12.2.)

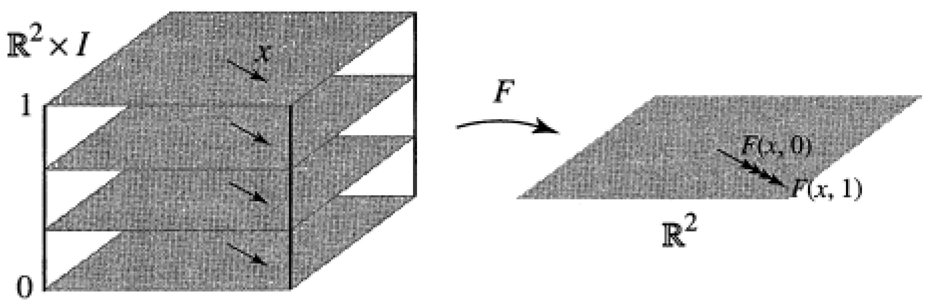


Fig 2.1: An example of an isotopy.

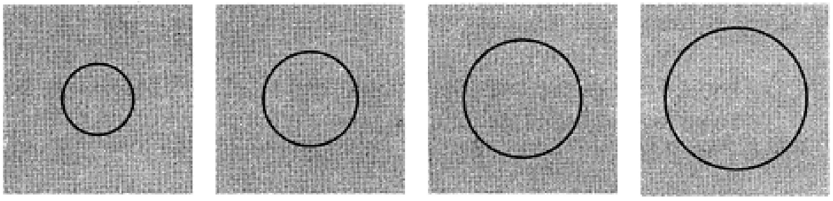


Fig 2.2: An ambient isotopy doubling the radius of the circle, a trivial knot.

This gives the notion of deforming the background space X itself such that the two embeddings of Y into X become the same, i.e, a homeomorphic copy of Y ~~be~~ deforms into another. This is precisely the notion of equivalence we will use for Knots.

Def: 2.3 Two Knots $f, g : S^1 \to \mathbb{R}^3$ are equivalent if they are ambient isotopic. A collection of equivalent Knots is called a Knot type.

Note that the existence of an isotopy ensures the deformation is homeomorphic at each step, and hence no cutting or ~~crossing~~ passing ~~over~~ of the Knot ~~over~~ through itself can't occur.

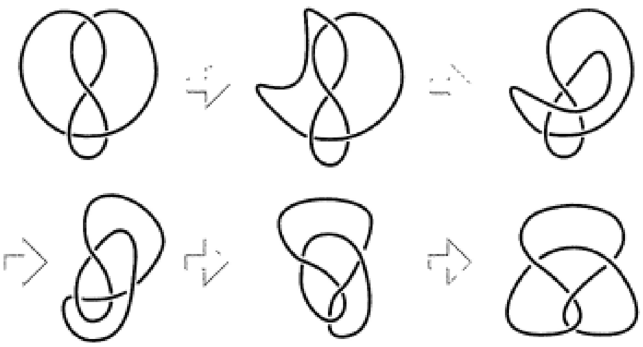


Fig 2.3: An ambient isotopy deforming a knot into another equivalent knot.

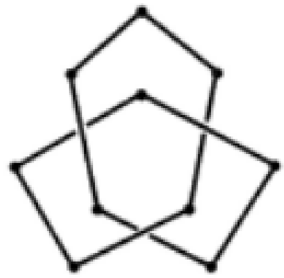


Fig 3.1: A Polygonal Knot.

3 Polygonal Knots and triangular moves:

For simplicity, we assume our knots are polygonal, i.e, made up of a finite number of edges and vertices. Note that as we can estimate any smooth knot with a large number of vertices and edges, this assumption is useful.

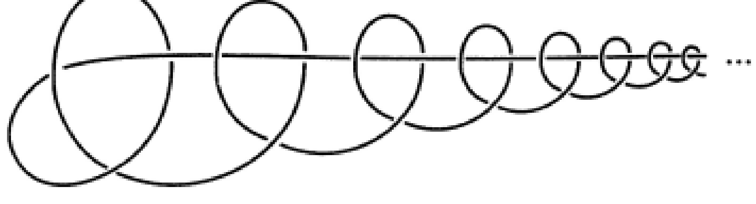
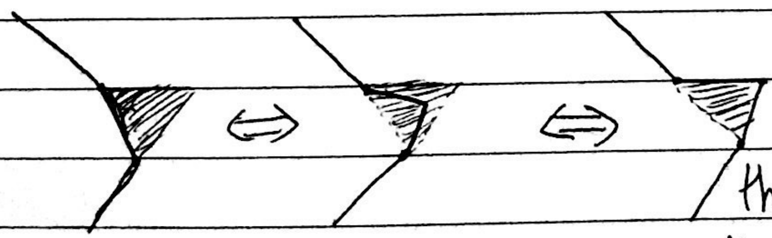


Fig 3.2: A wildly embedded knot.

It also prevents the existence of some wildly embedded Knots. as above.
As such we can assume further that our ambient isotopy is also piece-wise linear, i.e, are a composition of deformations on each edge.
Such ambient isotopies can be realized completely by what are known as triangle moves; of which there are two:

1) Replace an edge of the Knot with two other edges, such that the three edges bound a triangle that intersects the Knot along the first edge only.

2) Replace two adjacent edges of the Knot by one edge, such that the three edges bound a triangle that intersects the Knot only in the first two edges.



It is clear that these moves can be realized as ambient isotopies, deforming the Knot continuously across the triangle.

Thus these triangular moves preserve the Knot type.

4 Knot Projections and Diagrams:

Def: 4.1 Given any Knot in $\mathbb{R}^3$, it can be projected on to a plane to give a closed planar curve. Given any projection of a Knot, it is called a regular projection if:

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1) There are no n-points for $n \geq 3$ (No point in the projection corresponds to more than two points on the Knot in $\mathbb{R}^3$)

2) There are only finitely many double points (2-points) in the projection.

3) No double point corresponds to a vertex of the Knot.

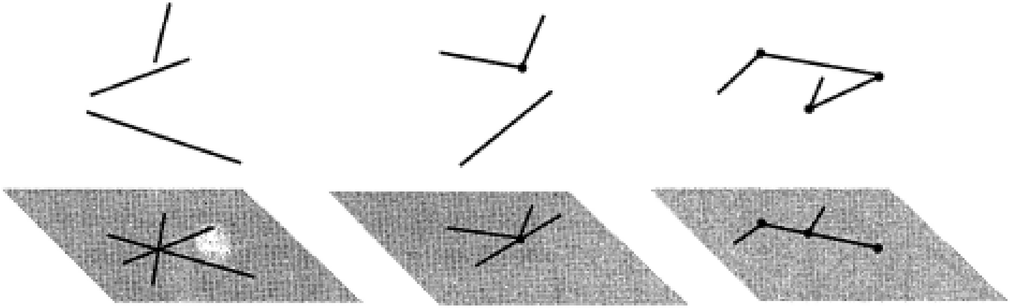


Fig 4.1: Examples of non-regular projections.

How do we know such a projection exists? Roughly speaking, any n-point ($n \geq 2$) or any vertex coinciding ~~with a knot~~ in a double point, can be resolved by rotating or translating the knot ~~...~~ by an arbitrarily small amount, such that the knot itself does not fundamentally change. Thus any projection can be made regular.

<u>Note:</u> One ~~...~~ can very easily see that a regular projection produces a <u>topological graph</u> in the plane, with vertices and double points corresponding to vertices on the graph.

Finally in order to obtain information regarding the height of different parts of the knot ~~...~~ at double points, these can be indicated on projections to produce <u>knot projections</u> or <u>diagrams</u>. These double points are also known as <u>crossings</u>

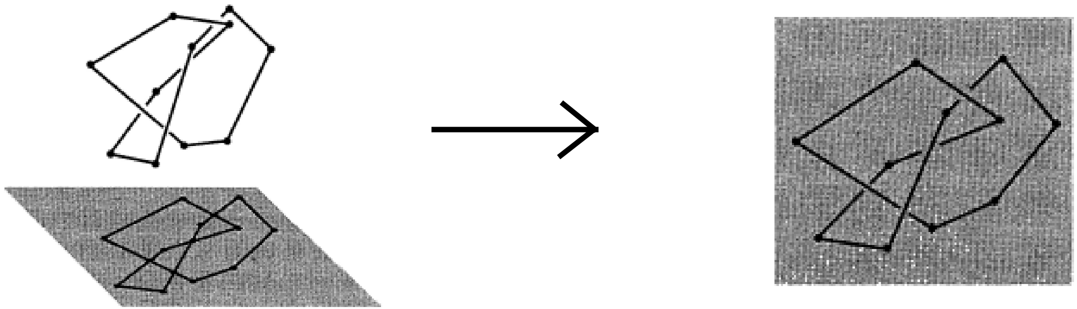


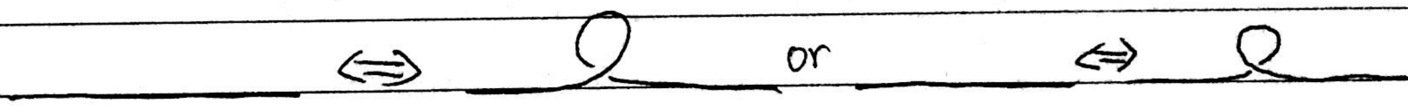
Fig 4.2: Formation of a knot projection diagram.

5 Planar isotopies and Reidemeister moves:

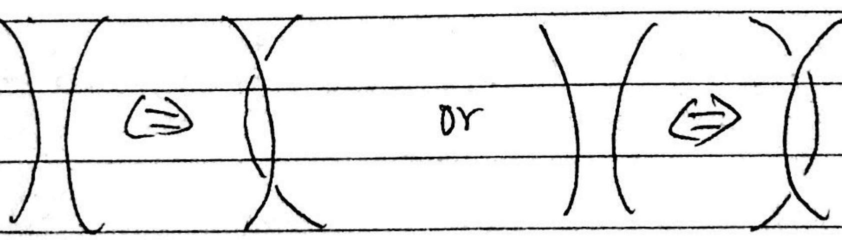
A planar isotopy is a piece-wise linear isotopy of the plane. These can be used to deform one knot projection into another through which one can construct a corresponding ambient isotopy of $\mathbb{R}^3$ between the knots themselves. But the converse is not true necessarily. This is because an ambient isotopy of $\mathbb{R}^3$ can cause a projection to become non-regular, by producing an extra crossing, but this is not possible through any planar isotopy that must be a homeomorphism at every instant.

This implies that our projections are still lacking. Isotopies of the projection do not completely characterize ~~all~~ ambient isotopies in $\mathbb{R}^3$. To resolve this issue, we introduce another form of deformation on knot projections, known as Reidemeister moves:

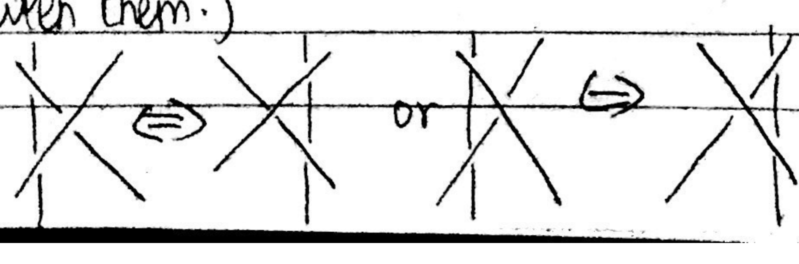
Type 1: These moves allow one to put in or take out any kink:



Type 2: These moves allow one to slide ~~over or under~~ a strand over or under its adjacent strand:



Type 3: These moves allow one to slide a strand ~~past~~ past a crossing (either from below the two strands that make the crossing, above them, or between them.)



It is easy to see these moves do not change the Knot type as they can easily be imagined as ambient isotopies in $\mathbb{R}^3$.

6 Reidemeister theorem:

Th: 6.1 Two Knots are equivalent if and only if, there is a finite sequence of planar isotopies and Reidemeister moves taking a Knot projection of one to a Knot projection of ~~another~~ the other.

Proof: We have already seen that planar isotopies correspond to ambient isotopies, but the converse was not true. This is where Reidemeister moves come in to allow for deformations that do not ~~pre~~ preserve regularity of Knot projections.

($\Leftarrow$) Suppose we have ~~that~~ that two Knots ~~are~~ have Knot projections that are related by a finite sequence of planar isotopies and Reidemeister moves $\Rightarrow$ As both correspond to ambient isotopies, the Knots must be equivalent.

($\Rightarrow$) Now assume two Knots are equivalent $\Rightarrow$ there exists an ambient isotopy between them. The ambient isotopies correspond to planar isotopies as long as the projection remains regular. We can show that there are three possibilities when it does not:

Case 1: An n-point ($n \geq 3$) is produced. This can ~~only~~ happen by the movement of an edge over a crossing
$\Rightarrow$ This is a type 3 Reidemeister move (Figure below through (c)).

Case 2: A vertex over another point is produced. This can occur in two ways. ~~(Figure belo~~

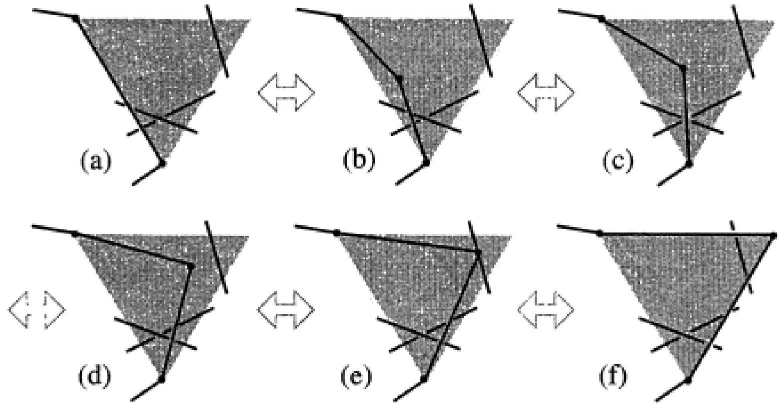


Fig 6.1(a): Examples of non-regular projections being produced from ambient isotopies.

Subcase 1: The ~~ver~~ edges of the vertex are both not adjacent to the edge on which the point is contained ⇒ This is a type 2 Reidemeister move. (Figure above through (e))

Subcase 2: At least one of the ~~The~~ edges of the vertex are adjacent to the edge on which the point is contained ⇒ This will be kink being produced ⇒ This is a type 1 Reidemeister move. (Figure below)

Thus all ambient isotopies can be realized as a finite composition of planar isotopies and Reidemeister moves. Thus we can get from one knot projection to the others' through these only.

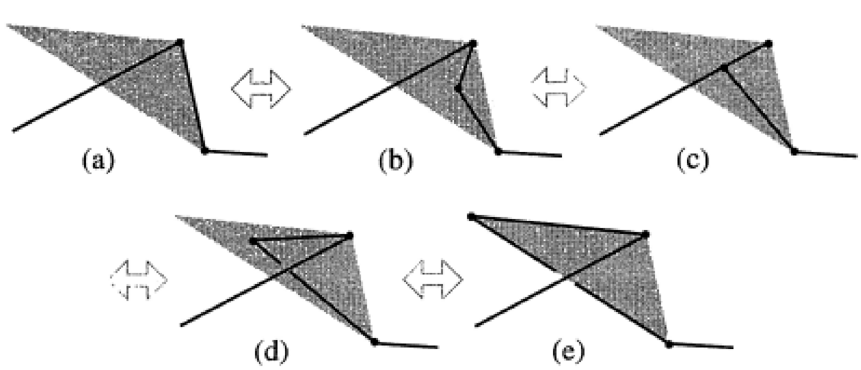


Fig 6.1(b)

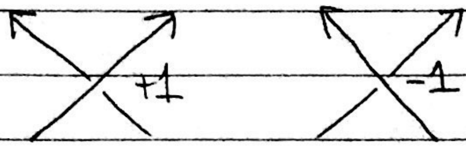
7 ## Invariants:

Now that we have established a much more accessible criteria for checking knot equivalences, we can introduce the notion of an invariant. ~~An~~

An invariant is a property of a knot that does not change through ambient isotopies. Thus, it can be used to differentiate between knots or links that are not equivalent. Let us start with a relatively simple invariant of links.

8 Linking number:

Assume L to be some two-component link. On each component one can choose a direction to travel along the component on. This is known as choosing an <u>orientation</u>. Label each crossing in the following way, and denote the label of a crossing c as $L(c)$.



Note: These are knot projections

Def. 8.1 The linking number of an oriented component link L is:

$$lk(L) = \frac{1}{2}\sum_{c} L(c),$$ the sum being over all crossings involving both components.

Theorem:

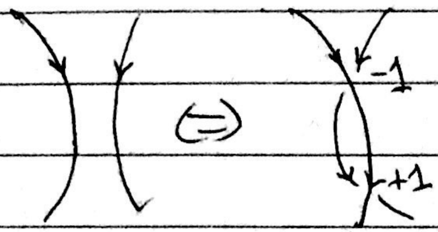
Th: 8.2 If two oriented links are equivalent, then all of their knot projections have the same linking number

Proof: It is enough to show that the linking number is not affected by planar isotopies or Reidemeister moves.

Planar isotopies do not change any crossing, and hence does not change the linking number.

A type 1 Reidemeister move only changes crossings of a knot with itself, and hence does not affect the linking number.

Consider now a type 2 Reidemeister move, where a strand is moved under another component's ~~crossing~~ strand:
(or vice versa)



Adding the labels of the two new crossings does not change the linking number as $+1 - 1 = 0$

Consider now finally a type 3 Reidemeister move, where a strand is moved over a crossing. It is easy to see here that no extra ~~new~~ crossings are found ~~are~~ reversed.

Thus we have that the linking number is invariant under ambient isotopies $\Rightarrow$ Links with different linking number cannot be equivalent.

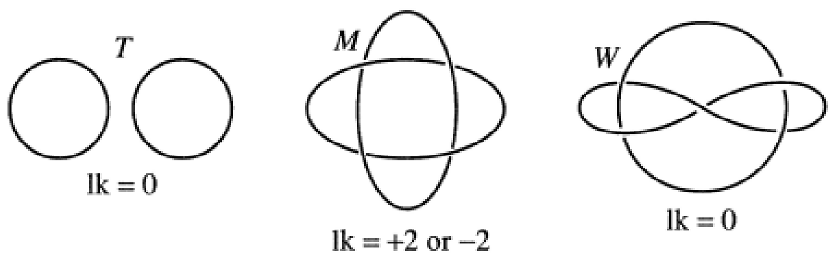


Fig 8.1: We can only say that the second knot is necessarily distinct from the other two as it has a different linking number. Note that the first and last knots are not equivalent.

9 **Polynomial construction in Knot theory:**

The aim is to construct more invariants that will allow to distinguish between different types of Knots. One such ~~invariant~~ invariant involves the construction of a polynomial associated with each Knot/Link type. To begin with, we will not assume this polynomial is the same for each Knot projection, and it will be Laurent and assumed to be in 3 variables A, B, C. We denote the polynomial of a Knot or link P by $\langle P \rangle$. We impose some rules ~~on~~ ~~what~~ we would like the bracket polynomial to satisfy:

1) $\langle O \rangle = 1$, where O is the trivial Knot.
2) $\langle P \cup O \rangle = C\langle P \rangle$, where this implies that the polynomial of a Knot ~~with~~ union a trivial Knot, is C times the polynomial of the initial Knot.

3) $\langle \times \rangle = A\langle)(\rangle + B\langle \asymp \rangle$, where this implies that the polynomial of a Knot can be written as a combination of polynomials associated with Knots with one fewer crossing. This is specifically called the skein relation. Note that this indicates that you have contribution from two Knots; one with the crossing cut horizontally and one with the crossing cut vertically.

~~Nowe~~ Now we impose that these polynomials must be invariant under Reidemeister moves: For type 2 Reidemeister moves:

$$\begin{aligned}
\langle)(\rangle &= \cancel{A\langle} \; A\langle \asymp \rangle + B\langle)(\rangle \\
&= A[A\langle \asymp \rangle + B\langle \asymp \rangle] + B[A\langle)(\rangle + B\langle \asymp \rangle] \\
&= AA\langle \asymp \rangle + ABC\langle \asymp \rangle + BA\langle)(\rangle + BB\langle \asymp \rangle \\
&= [A^2 + ABC + B^2]\langle \asymp \rangle + BA\langle)(\rangle.
\end{aligned}$$

We want $\langle)(\rangle = \langle)(\rangle \Rightarrow B = A^{-1},\ C = -A^2 - A^{-2}$. Thus the rules can be restated as:

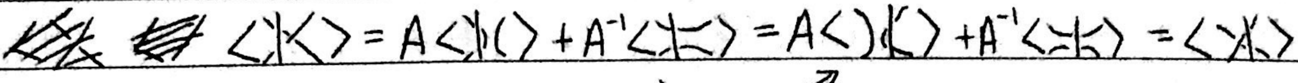
Rule 1: $\langle O \rangle = 1$

Rule 2: $\langle P \cup O \rangle = C\langle P \rangle = (-A^2 - A^{-2})\langle P \rangle$

Rule 3: $\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$

These can now be used to find the bracket polynomials for some Knots, only in the variable A.

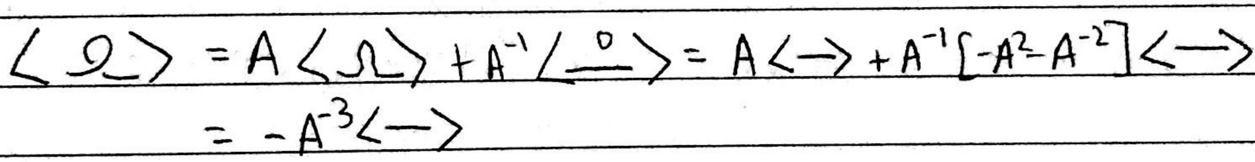
Let us now look at the effects of other Reidemeister moves. For type 3 Reidemeister moves:



intermediate step

Where the intermediate step follows from a type 2 Reidemeister move and hence leaves the polynomial unchanged. Thus this implies our definition of the bracket polynomial is invariant under type 3 Reidemeister moves.

Let us now look at the effect of a type 1 Reidemeister move:



Thus the invariance does not hold. We can see that some extra factor of A must be taken into account for. Thus we move to now modifying our polynomial.

10 Kauffman X polynomial:

Def: 10.1 Given a projection P of an oriented link, the writhe of P, denoted by $w(P)$, is the sum of the labels at all crossings.

Def: 10.2 The Kauffman X polynomial of a projection P of an oriented link is defined to be the polynomial $X(P) = (-A^3)^{w(P)}\langle P\rangle$.

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Th: 10.3 The Kauffman X polynomial is an invariant for Knots and ~~links~~ oriented Links.

Proof: The reason this holds only for oriented Links only, is because the writhe depends on the orientation of the components of a Link, and hence the Kauffman X polynomial cannot be an invariant for general links

As before, we know type2 Reidemeister moves only deal with crossings between different knots, that is, those that contribute to the crossing number, which we know is invariant under these moves. Similarly, type 3 Reidemeister moves do not affect the writhe either. This implies ~~the~~ the Kauffman X polynomial is invariant under these two moves

In a type 1 Reidemeister move, a kink is removed, ~~lead~~ causing the writhe to decrease by 1, which implies a factor of $-A^3$ is multiplied with the polynomial. But this exactly cancels out the $-A^{-3}$ factor from before, leaving the polynomial unchanged. Similarly, doing the reverse move increases the writhe by 1 and this time the new factor of $-A^{-3}$ cancels out the $-A^3$ factor, leaving the polynomial unchanged $(\langle \text{———} \rangle = -A^3 \langle \quad \rangle)$.

(Same can be confirmed for a kink with the opoosite crossing.)

Thus the Kauffman X polynomial is an invariant for Knots and oriented Links.

Note: The Jones polynomial is equivalent to this, replacing A with $t^{-1/4}$.

11 Date: The unKnotting algorithm:

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Th: 11.1 Given a Knot projection, there is a subset of crossings that can be ~~switch~~ switched to obtain a diagram of the trivial Knot.

Proof: Start at a ~~po~~ non-crossing point on P, the Knot projection. Choosing a direction of travel, change every undercrossing you meet to an over-crossing. Do not change the crossing when encountered the second time. This, we claim, produces the trivial Knot as one makes their way back to the starting point.

Let the modified Knot projection be P'. We show it is the projection of a trivial Knot in $\mathbb{R}^3$. Assume P' lies in the x-y plane. For each $(x,y) \in P'$, choose $(x,y,z) \in \mathbb{R}^3$ s.t z decreases continuously as we move along our ~~starting~~ chosen direction from the starting point, until the last crossing. After that, it increases to the starting point in $\mathbb{R}^3$ as it gets closer to the starting point in P'. Clearly the projection of this Knot, K, onto the x-y plane gives the same points as P', and gives an overcrossing at every crossing as we encounter them in the direction chosen, by virtue of z decreasing continuously. Thus it does indeed have Knot projection P'.

We can see that each segment of the Knot K, from the starting point to the last crossing and then from the last ~~segm~~ crossing back to the first point, can ~~be~~ be each deformed into half-circles to make S^1 (because z continuously decreases and then increases, hence there is no intertwining).

Explicit argument: For each (x,y,z), map it under f to $(0,0,z)$. For each segment of the Knot, it is homeomorphically mapped to a straight line, both of which can be joined to make a circle.

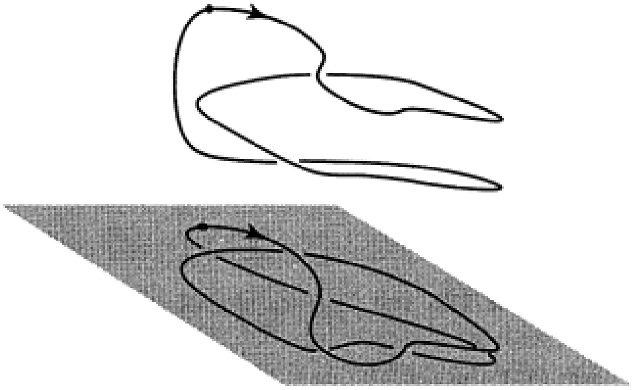

Fig 11.1: An example of an unknotted knot formed throught the algorithm.

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Note: The minimum number of crossing changes, across all Knot projections, needed to make the Knot trivial is Known as the <u>unknotting number</u>.

12 Chiral ~~Knot~~ and Ampichiral Knots:

Def: 12.1 A Knot that is equivolent to its mirror image is said to be amphichiral. Otherwise, the Knot is said to be chiral.

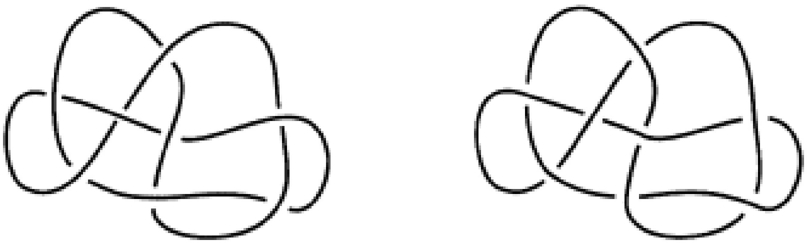

Fig 12.1: A knot and its mirror image.

Theorem:

Th: 12.2 The Kauffman X polynomial of the mirror image of a Knot K is obtained by replacing A with A^{-1} in the Kauffman X polynomial of ~~the~~ K.

Proof: Firstly, it is important to note that the mirror image is independent of the plane of reflection. This follows from the fact that any translation or rotation of a reflecting plane, is equivalent to the ~~inverse~~ same translation or rotation of the mirror Knot itself. As this does not change the Knot type, ~~the~~ reflection across the new plane produces an equivalent mirror ~~ima~~ image. Thus we can most simply consider reflection across the plane of a Knot projection, which trivially produces a Knot with a Knot projection with overcrossings and undercrossings switched. Let P' now be the mirror image projection of Knot Projection ~~P~~ P. Now to check whether the Kauffman X polynomial changes accordingly, we need to consider the effect of switching overcrossings with undercrossings. We can easily see that as labels switch sign at crossings, ~~the~~ $w(P') = -w(P)$. Also, we can see that

Checking the 3 rules for the bracket polynomial for the effect of ~~swit~~ switching the crossings is sufficient to argue the same relabeling for any general bracket polynomial.

The first rule trivially satisfies the ~~swit~~ ~~switching~~ switching as there are no terms of A or A^{-1} present. (And the mirror image is the same)

The second rule is already symmetric in A and A^{-1}, hence will satisfy the switching.

The third rule is ~~4~~

$$\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle \quad \ldots \quad (1)$$

~~Swit~~ switching the overcrossing to become an under crossing, we get $\langle \times \rangle$. But this is just a rotation of 90° of the LHS in (1). Thus we get, rotating throughout by 90°:

$$\langle \times \rangle = A\langle \asymp \rangle + A^{-1}\langle)(\rangle$$
$$= A^{-1}\langle)(\rangle + A\langle \asymp \rangle$$

which is the same as (1), but with A^{-1} substituted in place for A.

Thus, as the rules satisfy the condition that A^{-1} is substituted for A if overcrossings are ~~swit~~ switched with undercrossings, the same substitution will occur for any arbitrary bracket polynomial.

$$\Rightarrow X(P'(A)) = (-A^3)^{w(P')}\langle P'(A)\rangle = (-A^3)^{-w(P)}\langle P'(A)\rangle$$
$$= \left(-(A^{-1})\right)^{w(P)}\langle P(A^{-1})\rangle$$
$$= X(P(A^{-1}))$$

Thus the Kauffman polynomial for the mirror image of a knot will be the Kauffman polynomial for the original knot with A^{-1} substituted in place for A.