Homework 5

Due date: **Nov 13, Tuesday,** class time.

1. [30pt] Write a computer program to solve linear convection equation (i.e., wave equation)

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{1}$$

for periodic domain of $x \in [0, 1]$ and $a = 1$ using
(a) Euler scheme: $u_i^{(n+1)} = u_i^{(n)} - c(u_i^{(n)} - u_{i-1}^{(n)})$
(b) Lax-Wendroff scheme: $u_i^{(n+1)} = u_i^{(n)} - \frac{c}{2}(u_{i+1}^{(n)} - u_{i-1}^{(n)}) + \frac{c^2}{2}(u_{i+1}^{(n)} - 2u_i^{(n)} + u_{i-1}^{(n)})$
(c) 3rd-order Runge-Kutta central scheme:

$$\begin{aligned}
u_i^{(1)} &= u_i^{(n)} - \frac{c}{2}(u_{i+1}^{(n)} - u_{i-1}^{(n)}) \\
u_i^{(2)} &= \frac{3}{4}u_i^{(n)} + \frac{1}{4}u_i^{(1)} - \frac{c}{8}(u_{i+1}^{(1)} - u_{i-1}^{(1)}) \\
u_i^{(n+1)} &= \frac{1}{3}u_i^{(n)} + \frac{2}{3}u_i^{(2)} - \frac{c}{3}(u_{i+1}^{(2)} - u_{i-1}^{(2)})
\end{aligned}$$

where $c = a\Delta t/\Delta x$.

Use the following initial condition

$$u(t = 0, x) = \sin(2\pi x) \tag{2}$$

and periodic boundary condition

$$u(t, x + 1) = u(t, x). \tag{3}$$

It can be shown that the exact solution is $u(t, x) = \sin(2\pi(x - at))$.

(i) Using $N = 100$ grid points (i.e., $\Delta x = 0.01$), run each code with $c = 0.5$ (i.e., $\Delta t = 0.005$) and compare your numerical solutions with the exact solution after one period (i.e., $t = 1$) and 10 periods (i.e., $t = 10$). Discuss your observations. Note that exact solution for 1 period or 10 periods is exactly same for the initial condition due to the periodic nature of the problem. Please plot the exact solution and all these three numerical solutions (with labels different color lines) on the same figure to reduce the size of your document. You can produce one figure at $t = 1$ and another one at $t = 10$.

(ii) Repeat the same analysis written in (i) for $c = 1.0$ (i.e., $\Delta t = 0.01$). Discuss your observations. Explain why the simple Euler scheme behaves good for the $c = 1$ case?

2. [20pt] **Develop** the implicit compact Pade (ICP) scheme for the heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{4}$$

and establish the corresponding tridiagonal system of equations. **Show that** the following ICP scheme is second order accurate in time and fourth order accurate in space. Note that this is the scheme we have developed in the lecture on October 19th.

$$a_i u_{i-1}^{(n+1)} + b_i u_i^{(n+1)} + c_i u_{i+1}^{(n+1)} = r_i^{(n)} \tag{5}$$

where

$$a_i = \frac{12}{\Delta x^2} - \frac{2}{\alpha \Delta t} \tag{6}$$

$$b_i = -\frac{24}{\Delta x^2} - \frac{20}{\alpha \Delta t} \tag{7}$$

$$c_i = \frac{12}{\Delta x^2} - \frac{2}{\alpha \Delta t} \tag{8}$$

$$r_i = -\frac{2}{\alpha \Delta t}(u_{i+1}^{(n)} + 10u_i^{(n)} + u_{i-1}^{(n)}) - \frac{12}{\Delta x^2}(u_{i+1}^{(n)} - 2u_i^{(n)} + u_{i-1}^{(n)}) \tag{9}$$

where Δt is the time step and Δx is the equidistant mesh size.

3. [50pt] Write a computer program to solve unsteady heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{10}$$

for $x \in [-1, 1]$ and $\alpha = 1/\pi^2$ using
(a) Forward Time Central Space (FTCS) scheme (explicit):

$$\frac{u_i^{(n+1)} - u_i^{(n)}}{\Delta t} = \alpha \frac{u_{i+1}^{(n)} - 2u_i^{(n)} + u_{i-1}^{(n)}}{\Delta x^2} \tag{11}$$

(b) Backward Time Central Space (BTCS) scheme (implicit):

$$\frac{u_i^{(n+1)} - u_i^{(n)}}{\Delta t} = \alpha \frac{u_{i+1}^{(n+1)} - 2u_i^{(n+1)} + u_{i-1}^{(n+1)}}{\Delta x^2} \tag{12}$$

(c) Crank-Nicolson (CN) scheme (implicit):

$$\frac{u_i^{(n+1)} - u_i^{(n)}}{\Delta t} = \frac{\alpha}{2} \frac{u_{i+1}^{(n+1)} - 2u_i^{(n+1)} + u_{i-1}^{(n+1)}}{\Delta x^2} + \frac{\alpha}{2} \frac{u_{i+1}^{(n)} - 2u_i^{(n)} + u_{i-1}^{(n)}}{\Delta x^2} \tag{13}$$

(d) The implicit compact Pade (ICP) scheme developed in the problem 2.

Use the following initial condition

$$u(t = 0, x) = -\sin(\pi x), \tag{14}$$

and boundary conditions

$$u(t, x = -1) = 0, \quad u(t, x = 1) = 0. \tag{15}$$

Run your programs within the time interval $t \in [0, 1])$ using the following numerical parameters

Case 1: $\Delta t = 0.01$ and $\Delta x = 0.025$
Case 2: $\Delta t = 0.0025$ and $\Delta x = 0.025$
Case 3: $\Delta t = 0.001$ and $\Delta x = 0.025$

Compare your solutions with the analytical solution given by

$$u(t, x) = -\sin(\pi x) e^{-t}. \tag{16}$$

Plot your solutions at $t = 1$ for each cases (i.e., please prepare two different figures for each case and include numerical solutions of different schemes and the exact solution in the same figure).

Compute absolute error at $t = 1$ and plot for each case (i.e., x_i versus $error(x_i)$ where you should put x_i at abscissa and put $error(x_i)$ at ordinate). Absolute error can be written as

$$error(x_i) = |u_i^{exact} - u_i^{numerics}|. \tag{17}$$

Discuss your observations including order of accuracy and numerical stability conditions for each scheme. We know that explicit FTCS scheme given by (a) has some stability restriction and implicit schemes should be unconditionally stable. Have you verified these conditions in your computations? Also, we know that reducing the Δt, generally we should get more accurate results. Have you seen any non-intuitive behaviour? Using Taylor series expansion (i.e., modified equation analysis), find the truncation error for the numerical scheme given by (a) and explain this non-intuitive behaviour.