APPENDIX A

Partitioned Matrices and Kronecker Products

Partitioned Matrices

It is sometimes useful to partition the elements of a matrix into MN submatrices as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \cdots & \mathbf{A}_{1N} \\ \vdots & & \vdots \\ \mathbf{A}_{M1} & \cdots & \mathbf{A}_{MN} \end{bmatrix}.$$

This is a **partitioned matrix**. The subscript of the submatrices are defined in the same fashion as those for the elements of a matrix. For example, we might write

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix},$$

with

$$\mathbf{A}_{11} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}, \quad \mathbf{A}_{12} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}, \quad \mathbf{A}_{21} = [3 \quad 6], \quad \mathbf{A}_{22} = 9.$$

A common special case is where $M = N$ and the off-diagonal blocks ($\mathbf{A}_{mn}$ for $m \neq n$) are all zero matrices:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & & \\ & \ddots & \\ & & \mathbf{A}_{MM} \end{bmatrix}.$$

This is a **block diagonal matrix**.

Addition and Multiplication of Partitioned Matrices

Matrix addition and multiplication extend to partitioned matrices. Therefore,

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} \mathbf{A}_{11}+\mathbf{B}_{11} & \cdots & \mathbf{A}_{1N}+\mathbf{B}_{1N} \\ \vdots & & \vdots \\ \mathbf{A}_{M1}+\mathbf{B}_{M1} & \cdots & \mathbf{A}_{MN}+\mathbf{B}_{MN} \end{bmatrix}, \tag{A.1}$$

$$\mathbf{AB} = \begin{bmatrix} \mathbf{A}_{11} & \cdots & \mathbf{A}_{1N} \\ \vdots & & \vdots \\ \mathbf{A}_{M1} & \cdots & \mathbf{A}_{MN} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} & \cdots & \mathbf{B}_{1L} \\ \vdots & & \vdots \\ \mathbf{B}_{N1} & \cdots & \mathbf{B}_{NL} \end{bmatrix} = \begin{bmatrix} \sum_{n=1}^{N} \mathbf{A}_{1n}\mathbf{B}_{n1} & \cdots & \sum_{n=1}^{N} \mathbf{A}_{1n}\mathbf{B}_{nL} \\ \vdots & & \vdots \\ \sum_{n=1}^{N} \mathbf{A}_{Mn}\mathbf{B}_{n1} & \cdots & \sum_{n=1}^{N} \mathbf{A}_{Mn}\mathbf{B}_{nL} \end{bmatrix}. \tag{A.2}$$

In all these expressions, the matrices must be conformable for the operations involved. With respect to addition, the dimension of $\mathbf{A}_{mn}$ and $\mathbf{B}_{mn}$ must be the same for all m $(= 1, 2, \ldots, M)$ and n $(= 1, 2, \ldots, N)$. For multiplication, the number of columns in $\mathbf{A}_{mn}$ must equal the number of rows in $\mathbf{B}_{n\ell}$ for all m $(= 1, 2, \ldots, M)$, n $(= 1, 2, \ldots, N)$, and ℓ $(= 1, 2, \ldots, L)$. A special case of multiplication is when $\mathbf{B}$ is a **stacked vector c**:

$$\mathbf{Ac} = \begin{bmatrix} \mathbf{A}_{11} & \cdots & \mathbf{A}_{1N} \\ \vdots & & \vdots \\ \mathbf{A}_{M1} & \cdots & \mathbf{A}_{MN} \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_N \end{bmatrix} = \begin{bmatrix} \sum_{n=1}^{N} \mathbf{A}_{1n}\mathbf{c}_n \\ \vdots \\ \sum_{n=1}^{N} \mathbf{A}_{Mn}\mathbf{c}_n \end{bmatrix}. \tag{A.3}$$

Several cases frequently encountered are of the form

$$\begin{bmatrix} \mathbf{A}_{11} & & \\ & \ddots & \\ & & \mathbf{A}_{MM} \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_M \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{c}_1 \\ \vdots \\ \mathbf{A}_{MM}\mathbf{c}_M \end{bmatrix}, \tag{A.4}$$

$$\begin{bmatrix} \mathbf{A}_{11} & & \\ & \ddots & \\ & & \mathbf{A}_{MM} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} & & \\ & \ddots & \\ & & \mathbf{B}_{MM} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_{11} & & \\ & \ddots & \\ & & \mathbf{A}_{MM}\mathbf{B}_{MM} \end{bmatrix}, \tag{A.5}$$

$$\begin{bmatrix} \mathbf{A}'_{11} & & \\ & \ddots & \\ & & \mathbf{A}'_{MM} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} & \cdots & \mathbf{B}_{1M} \\ \vdots & & \vdots \\ \mathbf{B}_{M1} & \cdots & \mathbf{B}_{MM} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & & \\ & \ddots & \\ & & \mathbf{A}_{MM} \end{bmatrix} = \begin{bmatrix} \mathbf{A}'_{11}\mathbf{B}_{11}\mathbf{A}_{11} & \cdots & \mathbf{A}'_{11}\mathbf{B}_{1M}\mathbf{A}_{MM} \\ \vdots & & \vdots \\ \mathbf{A}'_{MM}\mathbf{B}_{M1}\mathbf{A}_{11} & \cdots & \mathbf{A}'_{MM}\mathbf{B}_{MM}\mathbf{A}_{MM} \end{bmatrix}, \tag{A.6}$$

$$\begin{bmatrix} \mathbf{A}'_{11} & & \\ & \ddots & \\ & & \mathbf{A}'_{MM} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} & \cdots & \mathbf{B}_{1M} \\ \vdots & & \vdots \\ \mathbf{B}_{M1} & \cdots & \mathbf{B}_{MM} \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_M \end{bmatrix} = \begin{bmatrix} \mathbf{A}'_{11}\mathbf{B}_{11}\mathbf{c}_1 + \cdots + \mathbf{A}'_{11}\mathbf{B}_{1M}\mathbf{c}_M \\ \vdots \\ \mathbf{A}'_{MM}\mathbf{B}_{M1}\mathbf{c}_1 + \cdots + \mathbf{A}'_{MM}\mathbf{B}_{MM}\mathbf{c}_M \end{bmatrix}, \tag{A.7}$$

$$\begin{bmatrix} \mathbf{A}'_{11} & \cdots & \mathbf{A}'_{MM} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} & \cdots & \mathbf{B}_{1M} \\ \vdots & & \vdots \\ \mathbf{B}_{M1} & \cdots & \mathbf{B}_{MM} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} \\ \vdots \\ \mathbf{A}_{MM} \end{bmatrix} = \sum_{m=1}^{M}\sum_{h=1}^{M} \mathbf{A}'_{mm}\mathbf{B}_{mh}\mathbf{A}_{hh}, \tag{A.8}$$

where $\mathbf{c}_m$ $(m = 1, 2, \ldots, M)$ are column vectors.

Inverting Partitioned Matrices

The inverse of a block diagonal matrix is

$$\begin{bmatrix} \mathbf{A}_{11} & & \\ & \ddots & \\ & & \mathbf{A}_{MM} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}_{11}^{-1} & & \\ & \ddots & \\ & & \mathbf{A}_{MM}^{-1} \end{bmatrix}, \tag{A.9}$$

provided $\mathbf{A}_{mm}$ $(m = 1, 2, \ldots, M)$ are invertible. This can be verified by direct multiplication.

For the general 2×2 partitioned matrix, one form of the **partitioned inverse** is

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}_{11}^{-1} + \mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{F}_2\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{F}_2 \\ -\mathbf{F}_2\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{F}_2 \end{bmatrix}, \tag{A.10}$$

where $\mathbf{F}_2 = (\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12})^{-1}$. This can be verified easily by multiplying $\mathbf{A}$ by the inverse. In view of the symmetry of the calculation, the upper left block can also be written as

$$\mathbf{F}_1 = (\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21})^{-1}.$$

Kronecker Products

For general matrices $\mathbf{A}$ $(M \times N)$ and $\mathbf{B}$ $(K \times L)$, the **Kronecker product** is defined as

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1N}\mathbf{B} \\ \vdots & & \vdots \\ a_{M1}\mathbf{B} & \cdots & a_{MN}\mathbf{B} \end{bmatrix}. \tag{A.11}$$

This is an $MK \times NL$ matrix. A special case is when both $\mathbf{A}$ and $\mathbf{B}$ are vectors:

$$\mathbf{a} \otimes \mathbf{b} = \begin{bmatrix} a_1 \\ \vdots \\ a_M \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix} = \begin{bmatrix} a_1\mathbf{b} \\ \vdots \\ a_M\mathbf{b} \end{bmatrix}. \tag{A.12}$$

It is straightforward but cumbersome to verify the following:

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}, \tag{A.13}$$

provided $\mathbf{A}$ and $\mathbf{C}$ are conformable and $\mathbf{B}$ and $\mathbf{D}$ are conformable, and

$$(\mathbf{A} \otimes \mathbf{B})' = \mathbf{A}' \otimes \mathbf{B}', \tag{A.14}$$

$$(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}, \tag{A.15}$$

provided $\mathbf{A}$ and $\mathbf{B}$ are invertible.