Appendix B

Rudiments of Matrix Algebra

This appendix offers the essentials of matrix algebra required to understand Appendix C and some of the material in Chapter 18. The discussion is nonrigorous, and no proofs are given. For proofs and further details, the reader may consult the references.

B.1 Definitions

Matrix

A matrix is a rectangular array of numbers or elements arranged in rows and columns. More precisely, a matrix of **order,** or **dimension,** M by N (written as $M \times N$) is a set of $M \times N$ elements arranged in M rows and N columns. Thus, letting boldface letters denote matrices, an $(M \times N)$ matrix $\mathbf{A}$ may be expressed as

$$\mathbf{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{M1} & a_{M2} & a_{M3} & \cdots & a_{MN} \end{bmatrix}$$

where a_{ij} is the element appearing in the ith row and the jth column of $\mathbf{A}$ and where $[a_{ij}]$ is a shorthand expression for the matrix $\mathbf{A}$ whose typical element is a_{ij}. The order, or dimension, of a matrix—that is, the number of rows and columns—is often written underneath the matrix for easy reference.

$$\underset{2\times 3}{\mathbf{A}} = \begin{bmatrix} 2 & 3 & 5 \\ 6 & 1 & 3 \end{bmatrix} \qquad \underset{3\times 3}{\mathbf{B}} = \begin{bmatrix} 1 & 5 & 7 \\ -1 & 0 & 4 \\ 8 & 9 & 11 \end{bmatrix}$$

Scalar

A scalar is a single (real) number. Alternatively, a scalar is a 1×1 matrix.

Column Vector

A matrix consisting of M rows and only one column is called a **column vector.** Letting the boldface lowercase letters denote vectors, an example of a column vector is

$$\underset{4\times 1}{\mathbf{x}} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 9 \end{bmatrix}$$

Row Vector

A matrix consisting of only one row and N columns is called a **row vector.**

$$\underset{1\times 4}{\mathbf{x}} = [1 \quad 2 \quad 5 \quad -4] \qquad \underset{1\times 5}{\mathbf{y}} = [0 \quad 5 \quad -9 \quad 6 \quad 10]$$

Transposition

The transpose of an $M \times N$ matrix $\mathbf{A}$, denoted by $\mathbf{A}'$ (read as $\mathbf{A}$ prime or $\mathbf{A}$ transpose) is an $N \times M$ matrix obtained by interchanging the rows and columns of $\mathbf{A}$; that is, the ith row of $\mathbf{A}$ becomes the ith column of $\mathbf{A}'$. For example,

$$\underset{3\times 2}{\mathbf{A}} = \begin{bmatrix} 4 & 5 \\ 3 & 1 \\ 5 & 0 \end{bmatrix} \qquad \underset{2\times 3}{\mathbf{A}'} = \begin{bmatrix} 4 & 3 & 5 \\ 5 & 1 & 0 \end{bmatrix}$$

Since a vector is a special type of matrix, the transpose of a row vector is a column vector and the transpose of a column vector is a row vector. Thus

$$\mathbf{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}' = [4 \quad 5 \quad 6]$$

We shall follow the convention of indicating the row vectors by primes.

Submatrix

Given any $M \times N$ matrix $\mathbf{A}$, if all but r rows and s columns of $\mathbf{A}$ are deleted, the resulting matrix of order $r \times s$ is called a **submatrix** of $\mathbf{A}$. Thus, if

$$\underset{3\times 3}{\mathbf{A}} = \begin{bmatrix} 3 & 5 & 7 \\ 8 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

and we delete the third row and the third column of $\mathbf{A}$, we obtain

$$\underset{2\times 2}{\mathbf{B}} = \begin{bmatrix} 3 & 5 \\ 8 & 2 \end{bmatrix}$$

which is a submatrix of $\mathbf{A}$ whose order is 2×2.

B.2 Types of Matrices

Square Matrix

A matrix that has the same number of rows as columns is called a **square matrix.**

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 & 5 & 8 \\ 7 & 3 & 1 \\ 4 & 5 & 0 \end{bmatrix}$$

Diagonal Matrix

A square matrix with at least one nonzero element on the main diagonal (running from the upper-left-hand corner to the lower-right-hand corner) and zeros elsewhere is called a **diagonal matrix.**

$$\underset{2\times 2}{\mathbf{A}} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad \underset{3\times 3}{\mathbf{B}} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Scalar Matrix

A diagonal matrix whose diagonal elements are all equal is called a **scalar matrix.** An example is the variance-covariance matrix of the population disturbance of the classical linear regression model given in Equation (C.2.3), namely,

$$\text{var-cov}(\mathbf{u}) = \begin{bmatrix} \sigma^2 & 0 & 0 & 0 & 0 \\ 0 & \sigma^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma^2 \end{bmatrix}$$

Identity, or Unit, Matrix

A diagonal matrix whose diagonal elements are all 1 is called an **identity,** or **unit, matrix** and is denoted by $\mathbf{I}$. It is a special kind of scalar matrix.

$$\underset{3\times 3}{\mathbf{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \underset{4\times 4}{\mathbf{I}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Symmetric Matrix

A square matrix whose elements above the main diagonal are mirror images of the elements below the main diagonal is called a **symmetric matrix.** Alternatively, a symmetric matrix is such that its transpose is equal to itself; that is, $\mathbf{A} = \mathbf{A}'$. That is, the element a_{ij} of $\mathbf{A}$ is equal to the element a_{ji} of $\mathbf{A}'$. An example is the variance-covariance matrix given in Equation (C.2.2). Another example is the correlation matrix given in (C.5.1).

Null Matrix

A matrix whose elements are all zero is called a **null matrix** and is denoted by $\mathbf{0}$.

Null Vector

A row or column vector whose elements are all zero is called a **null vector** and is also denoted by $\mathbf{0}$.

Equal Matrices

Two matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be equal if they are of the same order and their corresponding elements are equal; that is, $a_{ij} = b_{ij}$ for all i and j. For example, the matrices

$$\underset{3\times 3}{\mathbf{A}} = \begin{bmatrix} 3 & 4 & 5 \\ 0 & -1 & 2 \\ 5 & 1 & 3 \end{bmatrix} \quad \text{and} \quad \underset{3\times 3}{\mathbf{B}} = \begin{bmatrix} 3 & 4 & 5 \\ 0 & -1 & 2 \\ 5 & 1 & 3 \end{bmatrix}$$

are equal; that is $\mathbf{A} = \mathbf{B}$.

B.3 Matrix Operations

Matrix Addition

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$. If $\mathbf{A}$ and $\mathbf{B}$ are of the same order, we define matrix addition as

$$\mathbf{A} + \mathbf{B} = \mathbf{C}$$

where $\mathbf{C}$ is of the same order as $\mathbf{A}$ and $\mathbf{B}$ and is obtained as $c_{ij} = a_{ij} + b_{ij}$ for all i and j; that is, $\mathbf{C}$ is obtained by adding the corresponding elements of $\mathbf{A}$ and $\mathbf{B}$. If such addition can be effected, $\mathbf{A}$ and $\mathbf{B}$ are said to be *conformable* for addition. For example, if

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & -1 & 3 \\ -2 & 0 & 1 & 5 \end{bmatrix}$$

and $\mathbf{C} = \mathbf{A} + \mathbf{B}$, then

$$\mathbf{C} = \begin{bmatrix} 3 & 3 & 3 & 8 \\ 4 & 7 & 9 & 14 \end{bmatrix}$$

Matrix Subtraction

Matrix subtraction follows the same principle as matrix addition except that $\mathbf{C} = \mathbf{A} - \mathbf{B}$; that is, we subtract the elements of $\mathbf{B}$ from the corresponding elements of $\mathbf{A}$ to obtain $\mathbf{C}$, provided $\mathbf{A}$ and $\mathbf{B}$ are of the same order.

Scalar Multiplication

To multiply a matrix $\mathbf{A}$ by a scalar λ (a real number), we multiply each element of the matrix by λ:

$$\lambda\mathbf{A} = [\lambda a_{ij}]$$

For example, if $\lambda = 2$ and

$$\mathbf{A} = \begin{bmatrix} -3 & 5 \\ 8 & 7 \end{bmatrix}$$

then

$$\lambda\mathbf{A} = \begin{bmatrix} -6 & 10 \\ 16 & 14 \end{bmatrix}$$

Matrix Multiplication

Let $\mathbf{A}$ be $M \times N$ and $\mathbf{B}$ be $N \times P$. Then the product $\mathbf{AB}$ (in that order) is defined to be a new matrix $\mathbf{C}$ of order $M \times P$ such that

$$c_{ij} = \sum_{k=1}^{N} a_{ik} b_{kj} \qquad \begin{matrix} i = 1, 2, \ldots, M \\ j = 1, 2, \ldots, P \end{matrix}$$

That is, the element in the ith row and the jth column of $\mathbf{C}$ is obtained by multiplying the elements of the ith row of $\mathbf{A}$ by the corresponding elements of the jth column of $\mathbf{B}$ and summing over all terms; this is known as the *row by column* rule of multiplication. Thus, to obtain c_{11}, the element in the first row and the first column of $\mathbf{C}$, we multiply the elements in the first row of $\mathbf{A}$ by the corresponding elements in the first column of $\mathbf{B}$ and sum over all terms. Similarly, to obtain c_{12}, we multiply the elements in the first row of $\mathbf{A}$ by the corresponding elements in the second column of $\mathbf{B}$ and sum over all terms, and so on.

Note that for multiplication to exist, matrices $\mathbf{A}$ and $\mathbf{B}$ must be conformable with respect to multiplication; that is, the number of columns in $\mathbf{A}$ must be equal to the number of rows in $\mathbf{B}$. If, for example,

$$\underset{2\times 3}{\mathbf{A}} = \begin{bmatrix} 3 & 4 & 7 \\ 5 & 6 & 1 \end{bmatrix} \quad \text{and} \quad \underset{3\times 2}{\mathbf{B}} = \begin{bmatrix} 2 & 1 \\ 3 & 5 \\ 6 & 2 \end{bmatrix}$$

$$\begin{aligned} \mathbf{AB} = \underset{2\times 2}{\mathbf{C}} &= \begin{bmatrix} (3\times 2) + (4\times 3) + (7\times 6) & (3\times 1) + (4\times 5) + (7\times 2) \\ (5\times 2) + (6\times 3) + (1\times 6) & (5\times 1) + (6\times 5) + (1\times 2) \end{bmatrix} \\ &= \begin{bmatrix} 60 & 37 \\ 34 & 37 \end{bmatrix} \end{aligned}$$

But if

$$\underset{2\times 3}{\mathbf{A}} = \begin{bmatrix} 3 & 4 & 7 \\ 5 & 6 & 1 \end{bmatrix} \quad \text{and} \quad \underset{2\times 2}{\mathbf{B}} = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$$

the product **AB** is not defined since **A** and **B** are not conformable with respect to multiplication.

Properties of Matrix Multiplication

1. Matrix multiplication is not necessarily *commutative;* that is, in general, $\mathbf{AB} \neq \mathbf{BA}$. Therefore, the order in which the matrices are multiplied is very important. **AB** means that **A** is *postmultiplied* by **B** or **B** is *premultiplied* by **A**.
2. Even if **AB** and **BA** exist, the resulting matrices may not be of the same order. Thus, if **A** is $M \times N$ and **B** is $N \times M$, **AB** is $M \times M$ whereas **BA** is $N \times N$, hence of different order.
3. Even if **A** and **B** are both square matrices, so that **AB** and **BA** are both defined, the resulting matrices will not be necessarily equal. For example, if

$$\mathbf{A} = \begin{bmatrix} 4 & 7 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 5 \\ 6 & 8 \end{bmatrix}$$

then

$$\mathbf{AB} = \begin{bmatrix} 46 & 76 \\ 15 & 31 \end{bmatrix} \quad \text{and} \quad \mathbf{BA} = \begin{bmatrix} 19 & 17 \\ 48 & 58 \end{bmatrix}$$

and $\mathbf{AB} \neq \mathbf{BA}$. An example of $\mathbf{AB} = \mathbf{BA}$ is when both **A** and **B** are identity matrices.

4. A row vector postmultiplied by a column vector is a scalar. Thus, consider the ordinary least-squares residuals $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n$. Letting **u** be a column vector and $\mathbf{u}'$ be a row vector, we have

$$\begin{aligned} \hat{\mathbf{u}}'\hat{\mathbf{u}} &= [\hat{u}_1 \quad \hat{u}_2 \quad \hat{u}_3 \quad \cdots \quad \hat{u}_n] \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \\ \vdots \\ \hat{u}_n \end{bmatrix} \\ &= \hat{u}_1^2 + \hat{u}_2^2 + \hat{u}_3^2 + \cdots + \hat{u}_n'^2 \\ &= \sum \hat{u}_i^2 \qquad \text{a scalar [see Eq. (C.3.5)]} \end{aligned}$$

5. A column vector postmultiplied by a row vector is a matrix. As an example, consider the population disturbances of the classical linear regression model, namely, $u_1, u_2, \ldots, u_n$. Letting **u** be a column vector and $\mathbf{u}'$ a row vector, we obtain

$$\begin{aligned} \mathbf{uu}' &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} [u_1 \quad u_2 \quad u_3 \quad \cdots \quad u_n] \\ &= \begin{bmatrix} u_1^2 & u_1u_2 & u_1u_3 & \cdots & u_1u_n \\ u_2u_1 & u_2^2 & u_2u_3 & \cdots & u_2u_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ u_nu_1 & u_nu_2 & u_nu_3 & \cdots & u_n^2 \end{bmatrix} \end{aligned}$$

which is a matrix of order $n \times n$. Note that the preceding matrix is symmetrical.

6. A matrix postmultiplied by a column vector is a column vector.
7. A row vector postmultiplied by a matrix is a row vector.
8. Matrix multiplication is *associative;* that is, $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$, where $\mathbf{A}$ is $M \times N$, $\mathbf{B}$ is $N \times P$, and $\mathbf{C}$ is $P \times K$.
9. Matrix multiplication is distributive with respect to addition; that is, $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ and $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$.

Matrix Transposition

We have already defined the process of matrix transposition as interchanging the rows and the columns of a matrix (or a vector). We now state some of the properties of transposition.

1. The transpose of a transposed matrix is the original matrix itself. Thus, $(\mathbf{A}')' = \mathbf{A}$.
2. If $\mathbf{A}$ and $\mathbf{B}$ are conformable for addition, then $\mathbf{C} = \mathbf{A} + \mathbf{B}$ and $\mathbf{C}' = (\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$. That is, the transpose of the sum of two matrices is the sum of their transposes.
3. If $\mathbf{AB}$ is defined, then $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$. That is, the transpose of the product of two matrices is the product of their transposes in the reverse order. This can be generalized: $(\mathbf{ABCD})' = \mathbf{D}'\mathbf{C}'\mathbf{B}'\mathbf{A}'$.
4. The transpose of an identity matrix $\mathbf{I}$ is the identity matrix itself; that is $\mathbf{I}' = \mathbf{I}$.
5. The transpose of a scalar is the scalar itself. Thus, if λ is a scalar, $\lambda' = \lambda$.
6. The transpose of $(\lambda\mathbf{A})'$ is $\lambda\mathbf{A}'$ where λ is a scalar. [*Note:* $(\lambda\mathbf{A})' = \mathbf{A}'\lambda' = \mathbf{A}'\lambda = \lambda\mathbf{A}'$.]
7. If $\mathbf{A}$ is a square matrix such that $\mathbf{A} = \mathbf{A}'$, then $\mathbf{A}$ is a symmetric matrix. (See the definition of symmetric matrix given in Section B.2.)

Matrix Inversion

An inverse of a square matrix $\mathbf{A}$, denoted by $\mathbf{A}^{-1}$ (read $\mathbf{A}$ inverse), if it exists, is a unique square matrix such that

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

where $\mathbf{I}$ is an identity matrix whose order is the same as that of $\mathbf{A}$. For example

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \qquad \mathbf{A}^{-1} = \begin{bmatrix} -1 & \frac{1}{2} \\ \frac{6}{8} & -\frac{1}{4} \end{bmatrix} \qquad \mathbf{AA}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

We shall see how $\mathbf{A}^{-1}$ is computed after we study the topic of determinants. In the meantime, note these properties of the inverse.

1. $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$; that is, the inverse of the product of two matrices is the product of their inverses in the reverse order.
2. $(\mathbf{A}^{-1})' = (\mathbf{A}')^{-1}$; that is, the transpose of $\mathbf{A}$ inverse is the inverse of $\mathbf{A}$ transpose.

B.4 Determinants

To every square matrix, $\mathbf{A}$, there corresponds a number (scalar) known as the determinant of the matrix, which is denoted by det $\mathbf{A}$ or by the symbol $|\mathbf{A}|$, where $|\ |$ means "the determinant of." Note that a matrix per se has no numerical value, but the determinant of a matrix is a number.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -7 \\ 2 & 5 & 0 \\ 3 & 8 & 6 \end{bmatrix} \qquad |\mathbf{A}| = \begin{vmatrix} 1 & 3 & -7 \\ 2 & 5 & 0 \\ 3 & 8 & 6 \end{vmatrix}$$

The $|\mathbf{A}|$ in this example is called a determinant of order 3 because it is associated with a matrix of order 3×3.

Evaluation of a Determinant

The process of finding the value of a determinant is known as the *evaluation, expansion,* or *reduction* of the determinant. This is done by manipulating the entries of the matrix in a well-defined manner.

Evaluation of a 2×2 Determinant

If

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

its determinant is evaluated as follows:

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

which is obtained by cross-multiplying the elements on the main diagonal and subtracting from it the cross-multiplication of the elements on the other diagonal of matrix **A**, as indicated by the arrows.

Evaluation of a 3×3 Determinant

If

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then

$$|\mathbf{A}| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

A careful examination of the evaluation of a 3×3 determinant shows:

1. Each term in the expansion of the determinant contains one and only one element from each row and each column.
2. The number of elements in each term is the same as the number of rows (or columns) in the matrix. Thus, a 2×2 determinant has two elements in each term of its expansion, a 3×3 determinant has three elements in each term of its expansion, and so on.
3. The terms in the expansion alternate in sign from $+$ to $-$.
4. A 2×2 determinant has two terms in its expansion, and a 3×3 determinant has six terms in its expansion. The general rule is: The determinant of order $N \times N$ has $N! = N(N-1)(N-2)\cdots 3 \cdot 2 \cdot 1$ terms in its expansion, where $N!$ is read "N factorial." Following this rule, a determinant of order 5×5 will have $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ terms in its expansion.[1]

Properties of Determinants

1. A matrix whose determinantal value is zero is called a **singular matrix,** whereas a matrix with a nonzero determinant is called a **nonsingular matrix.** The inverse of a matrix as defined before does not exist for a singular matrix.

[1]To evaluate the determinant of an $N \times N$ matrix, **A**, see the references

2. If all the elements of any row of **A** are zero, its determinant is zero. Thus,

$$|\mathbf{A}| = \begin{vmatrix} 0 & 0 & 0 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{vmatrix} = 0$$

3. $|\mathbf{A}'| = |\mathbf{A}|$; that is, the determinants of **A** and **A** transpose are the same.
4. Interchanging any two rows or any two columns of a matrix **A** changes the sign of $|\mathbf{A}|$.

EXAMPLE 1 If

$$\mathbf{A} = \begin{bmatrix} 6 & 9 \\ -1 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -1 & 4 \\ 6 & 9 \end{bmatrix}$$

where **B** is obtained by interchanging the rows of **A**, then

$$|\mathbf{A}| = 24 - (-9) \quad \text{and} \quad |\mathbf{B}| = -9 - (24)$$
$$= 33 \qquad\qquad\qquad = -33$$

5. If every element of a row or a column of **A** is multiplied by a scalar λ, then $|\mathbf{A}|$ is multiplied by λ.

EXAMPLE 2 If

$$\lambda = 5 \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 5 & -8 \\ 2 & 4 \end{bmatrix}$$

and we multiply the first row of **A** by 5 to obtain

$$\mathbf{B} = \begin{bmatrix} 25 & -40 \\ 2 & 4 \end{bmatrix}$$

it can be seen that $|\mathbf{A}| = 36$ and $|\mathbf{B}| = 180$, which is 5 $|\mathbf{A}|$.

6. If two rows or columns of a matrix are identical, its determinant is zero.
7. If one row or a column of a matrix is a multiple of another row or column of that matrix, its determinant is zero. Thus, if

$$\mathbf{A} = \begin{bmatrix} 4 & 8 \\ 2 & 4 \end{bmatrix}$$

where the first row of **A** is twice its second row, $|\mathbf{A}| = 0$. More generally, if any row (column) of a matrix is a linear combination of other rows (columns), its determinant is zero.

8. $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$; that is, the determinant of the product of two matrices is the product of their (individual) determinants.

Rank of a Matrix

The rank of a matrix is the order of the largest square submatrix whose determinant is not zero.

EXAMPLE 3

$$\mathbf{A} = \begin{bmatrix} 3 & 6 & 6 \\ 0 & 4 & 5 \\ 3 & 2 & 1 \end{bmatrix}$$

It can be seen that $|\mathbf{A}| = 0$. In other words, **A** is a singular matrix. Hence although its order is 3×3, its rank is less than 3. Actually, it is 2, because we can find a 2×2 submatrix whose determinant is not zero. For example, if we delete the first row and the first column of **A**, we obtain

$$\mathbf{B} = \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$$

whose determinant is -6, which is nonzero. Hence the rank of **A** is 2. As noted previously, the inverse of a singular matrix does not exist. Therefore, for an $N \times N$ matrix **A**, its rank must be N for its inverse to exist; if it is less than N, **A** is singular.

Minor

If the ith row and jth column of an $N \times N$ matrix **A** are deleted, the determinant of the resulting submatrix is called the **minor** of the element a_{ij} (the element at the intersection of the ith row and the jth column) and is denoted by $|\mathbf{M}_{ij}|$.

EXAMPLE 4

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The minor of a_{11} is

$$|\mathbf{M_{11}}| = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

Similarly, the minor of a_{21} is

$$|\mathbf{M_{21}}| = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}a_{33} - a_{13}a_{32}$$

The minors of other elements of **A** can be found similarly.

Cofactor

The cofactor of the element a_{ij} of an $N \times N$ matrix **A**, denoted by c_{ij}, is defined as

$$c_{ij} = (-1)^{i+j}|\mathbf{M_{ij}}|$$

In other words, a cofactor is a *signed* minor, the sign being positive if $i + j$ is even and being negative if $i + j$ is odd. Thus, the cofactor of the element a_{11} of the 3×3 matrix **A** given previously is $a_{22}a_{33} - a_{23}a_{32}$, whereas the cofactor of the element a_{21} is $-(a_{12}a_{33} - a_{13}a_{32})$ since the sum of the subscripts 2 and 1 is 3, which is an odd number.

Cofactor Matrix

Replacing the elements a_{ij} of a matrix **A** by their cofactors, we obtain a matrix known as the **cofactor matrix** of **A**, denoted by (cof **A**).

Adjoint Matrix

The adjoint matrix, written as (adj **A**), is the transpose of the cofactor matrix; that is, (adj **A**) = (cof **A**)′.

B.5 Finding the Inverse of a Square Matrix

If $\mathbf{A}$ is square and nonsingular (that is, $|\mathbf{A}| \neq 0$), its inverse $\mathbf{A}^{-1}$ can be found as follows:

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}(\text{adj } \mathbf{A})$$

The steps involved in the computation are as follows:

1. Find the determinant of $\mathbf{A}$. If it is nonzero, proceed to step 2.
2. Replace each element a_{ij} of $\mathbf{A}$ by its cofactor to obtain the cofactor matrix.
3. Transpose the cofactor matrix to obtain the adjoint matrix.
4. Divide each element of the adjoint matrix by $|\mathbf{A}|$.

EXAMPLE 5

Find the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 4 \\ 2 & 1 & 3 \end{bmatrix}$$

Step 1. We first find the determinant of the matrix. Applying the rules of expanding a 3×3 determinant given previously, we obtain $|\mathbf{A}| = -24$.

Step 2. We now obtain the cofactor matrix, say, $\mathbf{C}$:

$$\mathbf{C} = \begin{bmatrix} \begin{vmatrix} 7 & 4 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 5 & 7 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 7 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 5 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 5 & 7 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 17 & -7 & -9 \\ -3 & -3 & 3 \\ -13 & 11 & -3 \end{bmatrix}$$

Step 3. Transposing the preceding cofactor matrix, we obtain the following adjoint matrix:

$$(\text{adj } \mathbf{A}) = \begin{bmatrix} 17 & -3 & -13 \\ -7 & -3 & 11 \\ -9 & 3 & -3 \end{bmatrix}$$

Step 4. We now divide the elements of (adj $\mathbf{A}$) by the determinantal value of -24 to obtain

$$\mathbf{A}^{-1} = -\frac{1}{24}\begin{bmatrix} 17 & -3 & -13 \\ -7 & -3 & 11 \\ -9 & 3 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{17}{24} & \frac{3}{24} & \frac{13}{24} \\ \frac{7}{24} & \frac{3}{24} & -\frac{11}{24} \\ \frac{9}{24} & -\frac{3}{24} & \frac{3}{24} \end{bmatrix}$$

It can be readily verified that

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is an identity matrix. The reader should verify that for the illustrative example given in Appendix C (see Section C.10) the inverse of the $\mathbf{X}'\mathbf{X}$ matrix is as shown in Eq. (C.10.5).

B.6 Matrix Differentiation

To follow the material in Appendix CA, Section CA.2, we need some rules regarding matrix differentiation.

RULE 1

If $\mathbf{a}' = [a_1 \quad a_2 \quad \cdots \quad a_n]$ is a row vector of numbers, and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is a column vector of the variables $x_1, x_2, \ldots, x_n$, then

$$\frac{\partial(\mathbf{a}'\mathbf{x})}{\partial \mathbf{x}} = \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

RULE 2

Consider the matrix $\mathbf{x}'\mathbf{A}\mathbf{x}$ such that

$$\mathbf{x}'\mathbf{A}\mathbf{x} = [x_1 \quad x_2 \quad \cdots \quad x_n] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Then

$$\frac{\partial(\mathbf{x}'\mathbf{A}\mathbf{x})}{\partial \mathbf{x}} = 2\mathbf{A}\mathbf{x}$$

which is a column vector of n elements, or

$$\frac{\partial(\mathbf{x}'\mathbf{A}\mathbf{x})}{\partial \mathbf{x}} = 2\mathbf{x}'\mathbf{A}$$

which is a row vector of n elements.

References

Chiang, Alpha C., *Fundamental Methods of Mathematical Economics,* 3d ed., McGraw-Hill, New York, 1984, chapters 4 and 5. This is an elementary discussion.

Hadley, G., *Linear Algebra,* Addison-Wesley, Reading, Mass., 1961. This is an advanced discussion.