Chapter 6

Auction Theory for the New Economy

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New Economy Handbook

Auctions occupy a conspicuous place in the commerce of the new economy. For items ranging from inexpensive collectibles sold on the Internet to billion-dollar spectrum licenses for mobile telephony, auctions are being used increasingly to discover price and determine allocations. This chapter provides an elementary introduction to auction theory. The first part reviews single-item auctions, including the sealed-bid auctions used frequently for procurement and the ascending auctions used widely by both traditional auction houses (e.g., Sotheby's and Christie's) and on-line bazaars (e.g., eBay). The second part discusses the more recent innovations related to auctions of multiple items, describing the basis for auctions in the telecommunications, energy, and environmental sectors and, perhaps, pointing to the shape of things to come.

Clock auction A type of dynamic auction in which the auctioneer announces prices, bidders respond with quantities desired at the announced prices, and the auctioneer iteratively adjusts the prices until the market clears.

Combinatorial auction Another term used for an auction with *package bidding*.

Dutch auction As used in auction theory, a format for auctioning a single item. The auctioneer starts at a high price and announces successively lower prices. The first bidder to bid wins the item and pays the current price at the time he bids. The same term is often used in the financial press to refer to the *uniform-price auction* for multiple items.

Efficiency As used in auction theory, the objective of maximizing the gains from trade, i.e., putting items in the hands who value them the most.

English auction A format for auctioning a single item. Bidders dynamically submit successively higher bids for the item. The final bidder wins the item and pays the amount of her final bid.

First-price auction A format for auctioning a single item. Bidders simultaneously submit sealed bids for the item. The highest bidder wins the item and pays the amount of his bid.

Package bidding In an auction for multiple items, this allows bids that each comprise a "package" (i.e., a set of items) and an associated payment. A bid is interpretable as an all-or-nothing offer for the specified package at the associated payment; there is no requirement that the bidder be willing to purchase a part of the package for a part of the payment.

Pay-as-bid auction A format for auctioning multiple identical items. Bidders simultaneously submit sealed bids comprising inverse demand curves. Each bidder wins the quantity demanded at the clearing price and pays the amount that she bid for each unit won.

Revenue maximization The objective of maximizing the seller's expected revenues. An auction that attains this objective is often referred to as an "optimal auction."

Second-price auction A format for auctioning a single item. Bidders simultaneously submit sealed bids for the item. The highest bidder wins the item and pays the amount bid by the second highest bidder.

Simultaneous ascending auction A format for auctioning multiple items, commonly used for telecommunications spectrum licenses. The items are auctioned simultaneously, and the auction does not conclude for any individual item until it concludes for all items. Each bid comprises a single item and an associated price. Bidding is constrained by a minimum bid increment and by an activity rule that limits a bidder's new bids based on his past bidding activity. The auction concludes when no new bids are submitted, and the standing high bids are then deemed to be winning bids.

Uniform-price auction A format for auctioning multiple identical items. Bidders simultaneously submit sealed bids comprising inverse demand curves. Each bidder wins the quantity demanded at the clearing price and pays the clearing price for each unit won.

Vickrey auction In the context of auctioning a single item, this is another term for the *second-price auction*. In the context of auctioning multiple identical items, this is an auction format in which bidders simultaneously submit sealed bids comprising inverse demand curves. Each bidder wins the quantity demanded at the clearing price and pays the opportunity cost (relative to the bids submitted) for each unit won.

Vickrey–Clarke–Groves (VCG) mechanism A format for auctioning multiple items. Bidders simultaneously submit sealed bids comprising their valuations for all sets of items. The auctioneer then determines an efficient assignment of the items based on the bids. Payments are determined so as to allow each bidder a payoff equaling the incremental surplus that she brings to the auction.

Winner's curse The insight that winning an item in an auction may confer "bad news" to the winner on account that it indicates that the opposing bidders possessed adverse information about the item's value. Sometimes, the connotation in using the term is that the winner may have ignored this informational effect in his bidding, possibly leading the winner to have paid more than the item is worth.

I. INTRODUCTION

In recent years, there has been a vast increase in the prominence and variety of auctions in the economy. Whereas off-line auctions have a history dating back to Babylonia in the fifth century BC, the advent of on-line auctions in the 1990s has greatly accelerated the growth of auctions for selling and buying.

This chapter will attempt to provide an elementary introduction to the economic theory of auctions. The first part of the chapter (Sections II and III) focuses on single-item auctions. In the old economy, this conjures visions of art auctions at Sotheby's and Christie's, as well as traditional sealed-bid procurement auctions by governmental authorities. In the new economy, single-item auctions are often associated with eBay and other Internet bazaars. We will review the first-price and second-price sealed-bid auctions, as well as the dynamic English and Dutch auctions. Following the auctions literature, our emphasis will be on identifying the equilibria of the various auction formats and comparing their outcomes regarding revenues and economic efficiency.

The second part of the chapter (Sections IV–VI) turns the focus to auctions of multiple items. Some of the oldest of these are the auctions of government bonds, such as the U.S. Treasury auction. Some of the greatest innovations have occurred in the auctions of telecommunications licenses during the period from 1994 to the present, such as the U.S. Federal Communications Commission spectrum auctions and the European UMTS (3G) auctions. We will review three different approaches that may be taken to simultaneously auctioning multiple items. First, we consider multiunit, sealed-bid auction formats, including the pay-as-bid and uniform-price auctions that are generally used for selling government securities. Second, we consider "clock auctions," in which the auctioneer announces prices for the various goods, bidders respond with quantities desired at these prices, and the auctioneer updates the prices in a dynamic process until the market clears. Third, we consider multiple-item generalizations of the English auction, including the simultaneous ascending auction (in which bidders iteratively bid prices for the various goods) and ascending auctions with package bidding (in which bidders may propose payments for sets of items). In each, we will offer a rationale for the approach, outline some of the theoretical results that are known, and suggest real-world applications.

The scope and detail of the present chapter are necessarily quite limited. For deeper and more comprehensive treatments of auctions, two notable books by Krishna (2002) and by Milgrom (2003) are especially recommended to readers. Earlier survey articles by McAfee and McMillan (1997) and by Wilson (1992) also provide excellent treatments of the literature on single-item auctions. A compendium by Klemperer (2000) brings together many of the best articles on auction theory.

Almost all of what is being said in the current chapter holds whether the subject matter is new economy or old economy, and whether the implementation of the auction is on-line or at a traditional auction house. At the same time, the abundance of (and differing logistics of) auctions on the Internet has produced some diversity and experimentation, even in staid auction formats such as the English auction. Some of the variations, in turn, have generated interesting on-line phenomena not typically present at Sotheby's or Christie's. For example, the rule on eBay has been that the highest bid received before a fixed closing time wins the auction, giving rise to the phenomenon of "sniping," in which many bidders submit their bids near the very last second of the auction. By way of contrast, the rule in Amazon auctions was that a new high bid extended the closing time by 5 minutes, significantly reducing the incentive for sniping (Roth and Ockenfels, 2002). Meanwhile, all of the Federal Communications Commission auctions included activity rules requiring bidders to place meaningful bids in early rounds of the auction in order to retain the right to bid late in the auction, rendering sniping essentially impossible. The analysis in this chapter largely abstracts away from such fine details of the auction, but of course in practice these details may be important determinants of performance.

In any case, it should be emphasized that the preferred medium today for implementing new auctions is on the Internet. Electronic bidding gives the auction designer greater flexibility and control in designing an efficient auction process, it vastly reduces the participation costs of bidders, and it greatly diminishes the tangible expenses associated with running an auction. Thus, whereas the subject matter of this article is really "auctions" and not the "new economy," these two largely unrelated terms often find themselves thoroughly intertwined.

II. SEALED-BID AUCTIONS FOR SINGLE ITEMS

The first two formats considered for auctioning single items are both sealed-bid auctions. In the language of game theory, each of these is a static game. Bidders submit their sealed bids in advance of a deadline, without knowledge of any of their opponents' bids. After the deadline, the auctioneer unseals the bids and determines a winner:

1. **First-price auction.** Bidders simultaneously submit sealed bids for the item. The highest bidder wins the item and pays the amount of his bid.

2. **Second-price auction.** Bidders simultaneously submit sealed bids for the item. The highest bidder wins the item and pays the amount bid by the second highest bidder.

Note that the preceding auction formats (and, indeed, all of the auctions described in this chapter) have been described for a regular auction in which the auctioneer offers items for sale and the bidders are buyers. Each can easily be restated for a "reverse auction" (i.e., procurement auction), in which the auctioneer solicits the purchase of items and the bidders are sellers. For example, in a second-price reverse auction, the lowest bidder is chosen to provide the item and is paid the amount bid by the second lowest bidder.

A. Rationale for Studying the Second-Price Auction

It is clearly evident why we study the first-price, English and Dutch auctions. The first-price auction is the standard procedure used in sealed-bid auctions for a single item. The English auction (discussed at length in Section III) is the format used for centuries at Sotheby's and Christie's, as well as that used overwhelmingly by Internet auctioneers such as eBay. The Dutch auction (also treated in Section III), though less common, is used in flower auctions in the Netherlands and in fish auctions in various locations around the globe.

However, a common reaction when the student is introduced to the second-price auction is skepticism or disbelief. Why would anybody design an auction procedure with the perverse feature that the winner need pay only the second highest price? Where do we actually observe this auction format in use? And, if the seller only required the winner to pay the second highest price, would not the seller be unnecessarily throwing away revenues?

The clearest answers to these questions are obtained by considering an English auction with proxy bidding. Instead of bidders submitting their bids directly, the bidding is done indirectly by computerized "proxy agents," who bid on behalf of the bidders. We may then consider the game in which bidders provide (secret) bid limits to their proxy agents. Suppose that bidder I instructs his proxy to bid up to $500 for the item, bidder II instructs her proxy to bid up to $400, and bidder III instructs her proxy to bid up to $300. If the auction proceeds in $1 bid increments, then bidder II's final bid will be $399 or $400, and bidder I will win the item with a bid of $400 or $401. If we ignore bid increments, the bidder who selects the highest bid limit wins the item and pays the amount of the *second highest* bidder's bid limit. Thus, the second-price auction may be viewed as an exact representation of an English auction with proxy bidding. Moreover, given the increasing prevalence of the English auction format in recent years, it seems unlikely that there is any substantial sense in which the seller is unnecessarily throwing away revenues by selecting an auction format with a second-price character rather than a first-price character. (Furthermore, see the treatment of revenue equivalence in Section II.F.)

B. The Private Values Model

A seller wishes to allocate a single unit of a good or service among n bidders ($i = 1, \ldots, n$). The bidders bid simultaneously and independently as in a non-cooperative static game. Bidder i's payoff from receiving the item in return for the payment y is given by $v_i - y$ (whereas bidder i's payoff from not winning the item is normalized to zero). Each bidder i's valuation, v_i, for the item is private information. Bidder i knows v_i at the time he submits his bid. Meanwhile, the opposing bidders $j \neq i$ view v_i as a random variable whose realization is unknown, but which is drawn according to the known joint distribution function $\hat{F}(v_1, \ldots, v_i, \ldots, v_n)$.

This model is referred to as the *private values* model, because each bidder's valuation depends only on her own—and not the other bidders'—information. (By contrast, in a *pure common values* model, $v_i = v_j$, for all i, $j = 1, \ldots n$; in an *interdependent values* model, bidder i's valuation is allowed to be a function of $v_{-i} = \{v_j\}_{j \neq i}$ as well as of v_i.) Under private values, some especially simple and elegant results hold, particularly for the second-price auction.

Two additional assumptions are frequently made. First, we generally assume that bidders are *risk neutral* in evaluating their payoffs under uncertainty. That is, each bidder seeks merely to maximize the mathematical expectation of his payoff. For example, if bidder i believes that he has a probability p of winning the item for a payment of y and a probability $(1 - p)$ of winning nothing and paying nothing, then his expected payoff equals $p(v_i - y)$. Second, we often assume *independence* of the private information. That is, the random variable v_i ($i = 1, \ldots, n$) is often assumed to be statistically independent: the joint distribution function, $\hat{F}(v_1, \ldots, v_n)$, is given by the product $\Pi_{i=1}^{n} F_i(v_i)$ of separate distribution functions $F_i(\cdot)$ for each of the v_i. However, both the risk-neutrality and independence assumptions are unnecessary for solving the second-price auction, which we turn to next.

C. Solution of the Second-Price Auction

Sincere bidding is a Nash equilibrium of the second-price sealed-bid auction, under private values. That is, if each bidder i submits the bid $b_i = v_i$, then there is no incentive for any bidder to unilaterally deviate. Moreover, sincere bidding is the unique outcome of the elimination of weakly dominant strategies, making the sincere bidding equilibrium an especially compelling outcome.

Let us compare bidder i's payoff from the sincere bid of $b_i = v_i$, with her payoff from instead bidding $b_i' < v_i$ ("shading" her bid) or $b_i'' > v_i$ ("inflating" her bid). We will see that the bidder always receives at least as great a payoff from sincere bidding as she receives from shading or inflating her bid, for all possible combinations of her opponents' bids. Thus, sincere bidding is optimal, regardless of the bids or bidding strategies of opposing bidders.

To see this, it will be sufficient to focus on $\hat{b}_{-i} = \max_{j \neq i}\{b_j\}$, the highest among the opponents' bids. The payoffs resulting from bidder i bidding sincerely and from shading or inflating his bids are shown in the respective columns of Table 1. Compare bidder i's payoff from shading his bid to $b_i' < v_i$ with his payoff from bidding sincerely. If the highest bid among opposing bidders is less than b_i' or greater than v_i, this change has no effect on bidder i's payoff; in the former case bidder i wins either way, and in the latter case bidder i loses either way. However, in the event that the highest bid, $\hat{b}_{-i}$, among opposing bidders is between b_i' and v_i, the change makes a difference: if bidder i bids v_i, he wins the auction and thereby achieves a payoff of $v_i - \hat{b}_{-i} > 0$, whereas, if bidder i bids b_i', he loses the auction and receives a payoff of zero. Thus, a bid of $b_i = v_i$ weakly dominates bidding $b_i' < v_i$. Similarly, compare bidder i's payoff from inflating his bid to $b_i'' > v_i$ with his payoff from bidding sincerely. If the highest bid among opposing bidders is less than v_i or greater than b_i'', this change has no effect on bidder

Table 1

Sincere Bidding is a Dominant Strategy in the Second-Price Auction

	And bidder i shades his bid: $b_i' < v_i$	And bidder i bids sincerely: $b_i = v_i$	And bidder i inflates his bid: $b_i'' > v_i$
Payoff if $\hat{b}_{-i} \le b_i'$	$v_i - \hat{b}_{-i}$	$v_i - \hat{b}_{-i}$	$v_i - \hat{b}_{-i}$
Payoff if $b_i' < \hat{b}_{-i} \le v_i$	0	$v_i - \hat{b}_{-i}$	$v_i - \hat{b}_{-i}$
Payoff if $v_i < \hat{b}_{-i} \le b_i''$	0	0	$v_i - \hat{b}_{-i}$
Payoff if $\hat{b}_{-i} > b_i''$	0	0	0

i's payoff; in the former case bidder i wins either way, and in the latter case bidder i loses either way. However, in the event that the highest bid, $\hat{b}_{-i}$, among opposing bidders is between v_i and b_i'', the change makes a difference: if bidder i bids v_i, he loses the auction and receives a payoff of zero, whereas if bidder i bids b_i'', he wins the auction and thereby achieves a payoff of $v_i - \hat{b}_{-i}$. Unfortunately, $\hat{b}_{-i}$ then exceeds v_i, and so $v_i - \hat{b}_{-i} < 0$. Thus, a bid of $b_i = v_i$ weakly dominates bidding $b_i'' > v_i$.

Theorem 1. *Assume that the bidders have pure private values. Then sincere bidding is the weakly dominant strategy for every bidder in the second-price sealed-bid auction.*

Note that the preceding argument in no way uses the risk-neutrality or independence assumptions, nor does it use the additional symmetry assumptions introduced in Section II.E. Another way of viewing this is that sincere bidding is an *ex post* equilibrium of the second-price auction, in the sense that the sincere bidding strategy would remain optimal even if the bidder were to learn her opponents' bids before she needed to move. Indeed, one of the strengths of the result that sincere bidding is a Nash equilibrium in undominated strategies is that it basically relies only upon the private values assumption and is otherwise extremely robust to the specification of the model.

D. INCENTIVE COMPATIBILITY IN ANY SEALED-BID AUCTION FORMAT

Consider any equilibrium of *any* sealed-bid auction format in the private values model. Given that bidder i's valuation is private information, observe that there is nothing to force bidder i to bid according to her true valuation v_i rather than according to some other valuation w_i. As a result, the equilibrium must have a structure that gives bidder i the incentive to bid according to her true valuation. This requirement is known as *incentive compatibility*.

In the following derivation, we will make both the risk-neutrality and independence assumptions. Given the independence assumption on valuations, we let $F_i(\cdot)$ denote the distribution function of v_i, and we assume that the support of $F_i(\cdot)$ is an interval $[\underline{v}_i, \bar{v}_i]$. In this notation, $\underline{v}_i$ and $\bar{v}_i$ are the lowest and highest possible valuations, respectively, of bidder i.

Let $\Pi_i(v_i)$ denote bidder i's expected payoff, let $P_i(v_i)$ denote bidder i's probability of winning the item, and let $Q_i(v_i)$ denote bidder i's expected payment in this equilibrium when his valuation is v_i. The reader should note that $Q_i(v_i)$ is intended here to refer to the unconditional expected payment—*not* the expected payment conditional on winning the item—of bidder *i*. Given the risk-neutrality assumption on bidder *i*, $\Pi_i(v_i)$ is given by

$$\Pi_i(v_i) = P_i(v_i)v_i - Q_i(v_i). \tag{1}$$

Next, we pursue the observation that there is nothing forcing bidder *i* to bid according to his true valuation v_i rather than according to some other valuation w_i. Define $\pi_i(w_i, v_i)$ to be bidder *i*'s expected payoff from employing the bidding strategy of a bidder with valuation w_i when his true valuation is v_i. Observe that

$$\pi_i(w_i, v_i) = P_i(w_i)v_i - Q_i(w_i), \tag{2}$$

since bidder *i*'s probability of winning and expected payment depend exclusively on his bid, not on his true valuation. (However, his utility from winning the item depends on his true valuation.) One of the fundamental points developed early in the economic analysis of auctions is that bidder *i* will voluntarily choose to bid according to his true valuation only if his expected payoff is greater than that from bidding according to another valuation w_i, that is:

$$\Pi_i(v_i) \geq \pi_i(w_i, v_i), \text{ for all } v_i, w_i \in [\underline{v}_i, \overline{v}_i] \text{ and all } i = 1, \ldots, n. \tag{3}$$

Inequality (3), referred to as the *incentive-compatibility constraint*, has very strong implications.

Next, note that $\Pi_i(v_i) = \pi_i(v_i, v_i) = \max_{w_i} \pi_i(w_i, v_i)$. It is straightforward to see that $\Pi_i(\cdot)$ is monotonically nondecreasing and continuous. Consequently, it is differentiable almost everywhere and equals the integral of its derivative. If we apply the Envelope Theorem at any v_i where $\Pi_i(\cdot)$ is differentiable, it yields

$$\frac{d\Pi_i(v_i)}{dv_i} = \left.\frac{\partial \pi_i(w_i, v_i)}{\partial v_i}\right|_{w_i = v_i} = P_i(w_i)|_{w_i = v_i} = P_i(v_i). \tag{4}$$

By integrating Eq. (4), we have

$$\Pi_i(v_i) = \Pi_i(\underline{v}_i) + \int_{\underline{v}_i}^{v_i} P_i(x)dx, \quad \text{for all } v_i \in [\underline{v}_i, \overline{v}_i] \text{ and all } i = 1, \ldots, n. \tag{5}$$

E. Solution of the First-Price Auction

The first-price sealed-bid auction requires two symmetry assumptions in order to yield a fairly simple solution. First, we assume *symmetric bidders*, in the sense that the joint distribution function $\hat{F}(v_1, \ldots, v_i, \ldots, v_n)$ governing the bidders' valuations is a symmetric function of its arguments. This assumption and the associated notation are simplest to state if independence is assumed. In this case, we write $F_i(\cdot)$ for the distribution function of each v_i; symmetry is the assumption that $F_i = F$ for all $i = 1, \ldots, n$ or, in other words, the assumption that the various v_i's are identically distributed, as well as independent, random variables. However, a similar derivation with only slightly more cumbersome notation is possible if the bidders are symmetric but the v_i's exhibit statistical dependencies [see the assumption of affiliation defined in Section III.C and Theorem 14 of Milgrom and Weber (1982)]. We write $[\underline{v}, \bar{v}]$ for the support of $F(\cdot)$. In addition, we assume that $F(\cdot)$ is a continuous function, so that there are no mass points in the probability distributions of the bidders' valuations.

Second, we restrict attention to *symmetric*, *monotonically increasing equilibria* in pure strategies. The assumed symmetry of bidders opens the possibility for the existence of a symmetric equilibrium, but there is also the logical possibility of asymmetric equilibria in symmetric games. Any pure-strategy equilibrium can be characterized by the bid functions $\{B_i(\cdot)\}_{i=1}^{n}$, which give bidder i's bid $B_i(v_i)$ when her valuation is v_i. Our assumption is that $B_i = B$ for all $i = 1, \ldots, n$, where $B(\cdot)$ is a strictly increasing function.

We proceed with the derivation of a solution as follows. Observe that in any symmetric, monotonically increasing equilibrium, bidder i beats bidder j if and only if $B(v_j) < B(v_i)$ and, hence, if and only if $v_j < v_i$. (We can ignore the event $v_j = v_i$; this is a zero-probability event because we have assumed that the distribution of valuations has no mass points.) Consequently, bidder i wins the item if and only $v_j < v_i$ for all $j \neq i$. Because the $\{v_j\}_{j \neq i}$ are i.i.d. random variables, bidder i has probability $F(v_i)^{n-1}$ of winning the auction when her valuation is v_i. We write $P_i(v_i) = F(v_i)^{n-1}$ for all $v_i \in [\underline{v}, \bar{v}]$ and all $i = 1, \ldots, n$.

Moreover, in a first-price auction, the bidder's payoff equals $v_i - B(v_i)$ if she wins the auction, whereas it equals zero if she loses. Consequently, her expected payoff equals

$$\Pi_i(v_i) = P_i(v_i)[v_i - B(v_i)] = F(v_i)^{n-1}[v_i - B(v_i)]. \tag{6}$$

Observe from Eq. (6) that, if $v_i = \underline{v}$, bidder i's probability of winning equals zero and, hence, $\Pi_i(\underline{v}) = 0$. Substitution of this fact and $P_i(v_i) = F(v_i)^{n-1}$ into Eq. (5) yields

$$\Pi_i(v_i) = \int_{\underline{v}}^{v_i} F(x)^{n-1}\, dx \text{ for all } v_i \in [\underline{v}, \bar{v}] \text{ and all } i = 1, \ldots, n. \tag{7}$$

Combining of Eq. (6) with Eq. (7) and solving for $B(\cdot)$ yields

$$B(v_i) = v_i - \frac{\Pi_i(v_i)}{F(v_i)^{n-1}} = v_i - \frac{\int_{\underline{v}}^{v_i} F(x)^{n-1}\,dx}{F(v_i)^{n-1}}. \tag{8}$$

Equation (8) provides us with a symmetric equilibrium in pure strategies of the first-price sealed-bid auction. Moreover, differentiation of Eq. (8) with respect to v_i demonstrates that $B'(v_i) > 0$, verifying that the equilibrium exhibits the posited strict monotonicity.

Theorem 2. *Assume that the bidders' valuations are independent and identically distributed, and assume that the bidders are risk neutral. Then Eq.* (8) *provides the unique symmetric Bayesian–Nash equilibrium of the first-price sealed-bid auction.*

The best known illustrative example of Theorem 2 may be found when $F(v) = v$, that is, when each bidder's valuation is independently distributed according to the uniform distribution on [0, 1]. In this case, Eq. (8) simplifies to

$$B(v_i) = \left[\frac{n-1}{n}\right] v_i. \tag{9}$$

Equation (9) has the interpretation that each bidder significantly shades her bid when there are relatively few bidders participating in the first-price auction. For example, with only two bidders, the equilibrium has each bidder only bidding half of her valuation. However, as the number of bidders n increases toward infinity, each bidder converges to bidding her true valuation.

F. Revenue Equivalence

One of the classic results of auction theory is revenue equivalence, which provides a set of assumptions under which the sellers' and buyers' expected payoffs are guaranteed to be the same under different auction formats. One of the most important applications of revenue equivalence is that the solution to the second-price auction in Section II.C and the solution to the first-price auction in Section II.E give the seller identical expected revenues.

Theorem 3 (The Revenue Equivalence Theorem; Myerson, 1981). *Assume that the random variables representing the bidders' valuations satisfy independence, and assume that bidders are risk neutral. Consider any two auction formats satisfying each of the following two properties: (i) each auction format assigns the item(s) to the same bidders for every realization* *of the random variables and (ii) each auction format gives the same expected payoff to the lowest valuation type, $\underline{v}_i$, of each bidder i. Then each bidder earns the same expected payoff from each of the two auction formats, and consequently the seller earns the same expected revenues from each of the two auction formats.*

For an auction of a single item, this conclusion follows directly from Eq. (5). Recall that we derived this equation for any equilibrium of *any* sealed-bid auction format. If for every realization of the random variables each of two auction formats assigns the item to the same bidder, then each bidder's probability, $P_i(\cdot)$, of winning is the same under the two auction formats. If, in addition, $\Pi_i(\underline{v}_i)$ is the same under the two auction formats, then Eq. (5) implies that the entire function $\Pi_i(\cdot)$ is the same under the two auction formats. Because this holds for every bidder *i*, and because the expected gains from trade are the same under the two auction formats, it follows from an accounting identity that the seller's expected revenues are also the same under the two auction formats.

Under the symmetry assumptions that we made in Section II.E, the Revenue Equivalence Theorem applies to the first-price and second-price auctions. We have already argued that, in a symmetric, monotonically increasing equilibrium of the first-price auction, the highest bid corresponds to the highest valuation, and so the item is assigned efficiently for every realization of the random variables. Moreover, when $v_i = \underline{v}$, the expected payoff of bidder *i* equals zero. Now note that identical observations hold for the dominant-strategy outcome of the second-price auction. By the Revenue Equivalence Theorem, all bidders (before learning their opponents' valuations) and the seller are indifferent between using a first-price or second-price auction.

III. DYNAMIC AUCTIONS FOR SINGLE ITEMS

The next two formats considered for auctioning single items are dynamic auctions: participants bid sequentially over time and, potentially, learn something about their opponents' bids during the course of the auction. In the first dynamic auction considered the price *ascends*, and in the second dynamic auction considered the price *descends*.

1. **English auction.** Bidders dynamically submit successively higher bids for the item. The final bidder wins the item and pays the amount of his final bid.
2. **Dutch auction.** The auctioneer starts at a high price and announces successively lower prices, until some bidder expresses her

willingness to purchase the item by bidding. The first bidder to bid wins the item and pays the current price at the time she bids.

Note that, as in Section II, each of these auction formats has been described for a regular auction in which the auctioneer offers items for sale, but they can easily be restated for a "reverse auction." For example, in an English reverse auction, the bids would descend rather than ascend, whereas in a Dutch reverse auction, the auctioneer would announce successively higher prices.

A. Solution of the Dutch Auction

An observation by Vickrey (1961) is that the Dutch auction is strategically equivalent to the first-price sealed-bid auction. The Dutch auction operates by the auctioneer announcing successively lower prices until some bidder expresses his willingness to purchase the item. To see the equivalence between the Dutch and first-price auctions, consider what a strategy b_i by bidder i in the Dutch auction really means: "If no other bidder bids for the item at any price higher than \$$b_i$, then I am willing to step in and purchase it at \$$b_i$." Just as in the first-price sealed-bid auction, the bidder i who selects the highest strategy b_i in the Dutch auction wins the item and pays the amount b_i. Furthermore, although the Dutch auction is explicitly dynamic, there is nothing that can happen while it is running that would lead any bidder to change his strategy. If strategy b_i was a best response for bidder i evaluated at the starting price p_0, then b_i remains a best response evaluated at any price $p < p_0$, assuming that no other bidder has bid at a price between p_0 and p. Meanwhile, if another bidder has already bid, then there is nothing that bidder i can do; the Dutch auction is over. Hence, an equilibrium of the first-price sealed-bid auction, as calculated in Eq. (8), also constitutes an equilibrium of the Dutch auction and vice versa. Moreover, the same conclusion holds for any possible information structure in the model.

The only way that the two auction formats could have different outcomes is if the dynamic nature of the Dutch auction enabled a bidder to learn anything during the course of the bidding that would lead him to change his strategy choice. However, one of only two things may be learned in the Dutch auction before the price drops to b_i. One possibility is that the bidder learns that none of his opponents were willing to purchase at any price greater than \$$b_i$. But this is what the strategy choice of b_i in the Dutch auction was premised on. The other possibility is that the bidder learns that an opponent was willing to purchase at more than \$$b_i$. But in the latter case, it is already too late for the bidder to do anything about it; the opponent has won.

B. Solution of the English Auction

By way of contrast, some meaningful learning and/or strategic interaction potentially occurs during the course of an English auction, so the outcome is potentially different from the outcome of the second-price sealed-bid auction. To discuss this, let us see two ways of formally modeling the English auction.

The first way of modeling the English auction is as a "clock auction": the auctioneer starts at a low price and announces successively higher prices. At every price, each bidder is asked to indicate her willingness to purchase the item. The price continues to rise so long as two or more bidders indicate interest. The auction concludes at the first time that fewer than two bidders indicate interest, and the item is awarded at the final price.

A second way of modeling the English auction is as a game in which bidders themselves name prices. The auction rules set a discrete grid of allowable bid prices, for example, integer numbers of dollars. Bidders are permitted to successively place bids, subject to the constraints that the bid amounts are taken from the discrete grid of allowable prices and that each bid amount is greater than all previous bid amounts. The auction concludes when no bidder is willing to place an additional bid, and the item is awarded to the final bidder at her final bid amount.

Whereas the second way of modeling the English auction comes closer to matching the empirical description of a Sotheby's or an eBay auction, the first way of modeling yields simpler arguments and cleaner results. If bidders name prices, then all of our equilibrium results are approximate results (subject to the amount of the bid increment, i.e., the width of the discrete grid of allowable prices) or become limiting results as the bid increment goes to zero. By contrast, in the clock auction, if the auctioneer raises prices continuously, then exact equilibrium results are produced.

In addition, the clock-auction formulation has two other advantages. First, when bidders name prices, the bidders may engage in strategic jump bidding or otherwise attempt to manipulate the pace of activity in the auction, complicating the analysis. Second, when bidders name prices, one is never sure how many opponents remain "in": an active bidder may repeatedly place bids, or she may sit on the sidelines and allow other bidders to bid up the price. By contrast, in the clock auction, the fact that the auctioneer names the prices eliminates all possibilities for bidders to jump bid or otherwise influence the pace. And if the rules include "irrevocable exit," i.e., once a bidder indicates that she is no longer interested in the item she is not permitted to reenter the bidding, then the number of bidders actively remaining in the bidding is always known. Due to these simplicity advantages, we shall use the continuous clock-auction formulation of the English auction in the remainder of this section and in Section III.C.

With pure private values, as already hinted at in Section II.A, the outcome of the English auction is the same as the outcome of the second-price sealed-bid auction. A bidder's strategy designates the price at which she will drop out of the auction (assuming that at least one opponent still remains). Each bidder finds it a best response to stay in the auction so long as the price is less than her valuation and to drop out of the auction as soon as the price is greater than her valuation. Hence, the outcome of the English auction is for each bidder to set her strategy equal to her true valuation, the same outcome as in the second-price sealed-bid auction.

However, matters become more complicated in the case of interdependent valuations, when each bidder's valuation depends not only on her own information, v_i, but also on the opposing bidders' information, v_{-i}. We turn to this case next.

C. The Winner's Curse and Revenue Ranking under Interdependent Values

Another of the classic results of auction theory is the revenue ranking of the various auction formats when bidders' valuations are interdependent. Whenever a bidder's valuation depends on the opposing bidders' information in an increasing way, the possibility of a *winner's curse* is introduced. Winning an item in an auction may confer "bad news" to a bidder, in the sense that it indicates that his opponents possessed adverse information about the item's value (leading the opponents to bid sufficiently low that the given bidder won). The potential for falling victim to the winner's curse may induce restrained bidding, curtailing the seller's revenues. In turn, some auction formats may then produce relatively higher revenues than others, depending on the extent to which they mitigate the winner's curse and make it safe for bidders to bid more aggressively.

The basic intuition used in comparing auction formats is that, to the extent that the winner's payment depends on the opposing bidders' information, the winner's curse is mitigated. This intuition is sometimes described as the "linkage principle." Thus, the second-price sealed-bid auction may yield higher expected revenues than the first-price sealed-bid auction: the price paid by the winner of a second-price auction depends on the information possessed by the highest losing bidder, whereas the price paid by the winner of a first-price auction depends exclusively on his own information. Moreover, the English auction may yield higher expected revenues than the second-price sealed-bid auction: the price paid by the winner of an English auction may depend on the information possessed by *all* of the losing bidders (who are observed as they drop out), whereas the price paid by the winner of a second-price sealed-bid auction depends only on the information of the *highest* losing bidder.

These conclusions rely on an assumption known as "affiliation," which intuitively means something very close to "nonnegative correlation." More precisely, let $v = (v_1, \ldots, v_n)$ and $v' = (v'_1, \ldots, v'_n)$ be possible realizations of the n bidders' random variables, and let $f(\cdot)$ denote the joint density function. Let $v \vee v'$ denote the component-wise maximum of v and v', and let $v \wedge v'$ denote the component-wise minimum. We say that the random variables are *affiliated* if

$$f(v \vee v')\, f(v \wedge v') \geq f(v)\, f(v'), \quad \text{for all } v, v' \in [\underline{v}, \overline{v}]^n. \tag{10}$$

Note that independent random variables are included (as a boundary case) in the definition of affiliated random variables: independence is the case where the affiliation inequality [Eq. (10)] is satisfied with equality. However, at the same time, affiliation provides that two high realizations or two low realizations of the random variables are at least as likely as one high and one low realization, etc., meaning something close to nonnegative correlation.

These conclusions also rely on a monotonicity assumption: each bidder's valuation depends in a (weakly) increasing way on both his own and the opposing bidders' information. In addition, we make some of the other assumptions from before. Each bidder is risk neutral in evaluating his payoff under uncertainty. Furthermore, we make the two symmetry assumptions of Section II.E: bidders are symmetric, in the sense that the joint distribution governing the bidders' information is a symmetric function of its arguments, and we restrict attention to symmetric, monotonically increasing equilibria in pure strategies.

Theorem 4 (Milgrom and Weber, 1982). *Assume that the bidders' information is given by symmetric, affiliated random variables and that the mapping from information to valuations is symmetric and monotonic. Also assume that bidders are risk neutral. Then the first-price sealed-bid auction, the second-price sealed-bid auction, and the English auction for a single item all possess symmetric, monotonic equilibria. However, whereas these* *equilibria are all efficient, they may be ranked by revenues: the English auction yields the greatest expected revenues, the second-price sealed-bid auction yields expected revenues less than or equal to those of the English auction, and, in turn, the first-price sealed-bid auction yields expected revenues less than or equal to those of the second-price sealed-bid auction.*

IV. SEALED-BID AUCTIONS FOR MULTIPLE ITEMS

A. Sealed-Bid, Multiunit Auction Formats for Homogeneous Goods

There are three principal sealed-bid, multiunit auction formats for M homogeneous goods. The defining characteristic of a homogeneous good is that each of the M individual items is identical (or a close substitute). Consequently, bids can be expressed in terms of prices that the bidder is willing to pay for a first, second, etc. unit of the homogeneous good, without indicating the identity of the particular good that is desired. Treatment of goods as homogeneous has the effect of dramatically simplifying the description of the bids that are submitted and the overall auction procedure. This simplification is especially appropriate in treating subject matter such as financial securities or energy products. Any two $10,000 U.S. government bonds with the same interest rate and the same maturity *are* identical, just as any two megawatts of electricity provided at the same location on the electrical grid at the same time *are* identical. This simplification has also been made at times in treating other subject matter where the different items are not literally identical but nevertheless are close substitutes. For example, the auctioneer will sometimes treat two adjacent telecommunications licenses comprising equal amounts of bandwidth as being identical, despite the fact that they are very slightly different.

In each of the three principal sealed-bid auction formats for homogeneous goods, a bid comprises an inverse demand curve, i.e., a (weakly) decreasing function $p_i(q)$ for $q \in [0, M]$, representing the price offered by bidder i for a first, second, etc. unit of the good. (Note that this notation may be used to treat situations where the good is perfectly divisible, as well as situations where the good is offered in discrete quantities.) The bidders submit sealed bids in advance of a deadline. After the deadline, the auctioneer unseals the bids and aggregates them, determining the clearing price at which demand equals supply. Each bidder wins the quantity demanded at the clearing price, but his payment varies according to the particular auction format.

1. **Pay-as-bid auction.**[1] Bidders simultaneously submit sealed bids comprising inverse demand curves. Each bidder wins the quantity demanded at the clearing price and pays the amount that he bid for each unit won.
2. **Uniform-price auction.**[2] Bidders simultaneously submit sealed bids comprising inverse demand curves. Each bidder wins the quantity demanded at the clearing price and pays the clearing price for each unit won.
3. **Multiunit Vickrey auction.** Bidders simultaneously submit sealed bids comprising inverse demand curves. Each bidder wins the quantity demanded at the clearing price and pays the opportunity cost (relative to the bids submitted) for each unit won.

Note that (as before, for single-item formats) each of these auction formats has been described for a regular auction in which the auctioneer offers items for sale and the bidders are buyers, and each format can easily be restated for a "reverse auction."

Sealed-bid, multiunit auction formats are best known in the financial sector for their longtime and widespread use in the sale of central government securities. For example, a survey of OECD countries in 1992 found that Australia, Canada, Denmark, France, Germany, Italy, Japan, New Zealand, the United Kingdom and, of course, the United States then used sealed-bid auctions for selling at least some of their government securities. The pay-as-bid auction is the traditional format used for government securities, and it was used for all U.S. Treasury bills until the early 1990s. Beginning in 1993, the United States began an "experiment" of using the uniform-price auction for 2- and 5-year government notes, and beginning in 1998, the United States switched entirely to the uniform-price auction for all issues. Meanwhile, the multiunit Vickrey auction was introduced in Vickrey's classic 1961 publication.

B. Equilibrium of the Pay-as-Bid Auction

The pay-as-bid auction is the traditional format used for selling government securities. It is straightforward to see that the pay-as-bid auction can

[1] Pay-as-bid auctions are also known as "discriminatory auctions" or "multiple-price auctions."

[2] Uniform-price auctions are often referred to in the financial press as "Dutch auctions" (generating some confusion with respect to our own standard usage from the auction theory literature). They are also known as "nondiscriminatory auctions," "competitive auctions," or "single-price auctions."

be correctly viewed as a multiunit generalization of the first-price auction. However, it is quite difficult to calculate a Nash equilibrium of the pay-as-bid auction, unless the underlying economic environment is such that an efficient equilibrium exists. In turn, to guarantee the existence of an efficient equilibrium, *three* symmetry assumptions need to be made together. First, as when we analyzed the first-price auction, we assume that bidders are symmetric, in the sense that the joint distribution governing the bidders' information is symmetric with respect to the bidders. Second, we also need to assume that the bidders regard every unit of the good as symmetric: that is, each bidder i has a constant marginal valuation for every quantity $q_i \in [0, \lambda_i]$, where λ_i is a capacity limitation on the quantity of units that bidder i can consume. Third, the bidders must be symmetric in their capacity limitations: $\lambda_i = \lambda$ for all bidders i. Given these three symmetry assumptions, the pay-as-bid auction has a solution very similar to the first-price auction for a single item. However, without these three symmetry assumptions, the pay-as-bid auction inherits an undesirable inefficiency property: absent the three symmetry assumptions, all Nash equilibria of the pay-as-bid auction will generally be inefficient. For example, the following theorem shows that, with symmetry across units (i.e., constant marginal valuation) assumed, an equilibrium can exist that attains full efficiency only if the other two symmetry assumptions are also satisfied.

THEOREM 5 (Ausubel and Cramton, 2002). *Suppose that bidders have independent pure private values for multiple units. Suppose that each bidder i has a constant marginal valuation for every quantity $q_i \in [0, \lambda_i]$ and a zero marginal valuation for every quantity $q_i > \lambda_i$, where λ_i is a capacity limitation. If bidders' valuations are identically distributed and their capacities, λ_i, are equal, then an efficient Bayesian–Nash equilibrium of the pay-as-bid auction exists. However, if bidders' valuations are not identically distributed or if their capacities are unequal, then generically an efficient equilibrium of the pay-as-bid auction does not exist.*

C. EQUILIBRIUM OF THE UNIFORM-PRICE AUCTION

Another critique of the pay-as-bid auction has been that it may discourage participation by less informed bidders. For example, suppose that some bidders in the Treasury auction possess superior information about the likely market-clearing price, whereas other bidders possess inferior information. If the less informed bidders bid according to their expectations of the market-clearing price, then there will be some states of the world in which the less informed bidders will win a share of the Treasury bills being auctioned, but they will pay a higher price than the more informed bidders who win in the same auction. Anticipating this problem, the less informed bidder will bid less aggressively or may refrain from participating in the auction at all.

Based on this intuition, Milton Friedman proposed in 1959 that the U.S. Treasury switch from a pay-as-bid auction to a uniform-price auction. Bidders who bid more than what turns out to be the market-clearing price would pay not their own bids but the market-clearing price. Consequently, less informed bidders would not run the risk of winning units at higher prices than those paid by more informed bidders, potentially increasing participation in the auction.

Not exactly overnight, the U.S. Treasury shifted toward the uniform-price format. Beginning in 1992, it began to experiment with the uniform-price rule for selling 2-year and 5-year notes, while retaining the pay-as-bid rule for other maturity securities. Then, in November 1998, it switched entirely to the uniform-price auction.

Unfortunately, despite the superficial similarity of the uniform-price auction both to a competitive mechanism and to the second-price auction of a single item, all of the equilibria of the uniform-price auction are inefficient.

Theorem 6 (Ausubel and Cramton, 2002). (a) *Suppose that bidders have independent pure private values that nontrivially depend on their private information, and suppose that bidders exhibit continuous diminishing marginal valuations for multiple units. Then an efficient equilibrium of the uniform-price auction does not exist.*

(b) *Suppose that bidders have independent pure private values for multiple units. Suppose that each bidder i has a constant marginal valuation for every quantity $q_i \in [0, \lambda_i]$ and a zero marginal valuation for every quantity $q_i > \lambda_i$, where λ_i is a capacity limitation. If bidders' capacities, λ_i, are equal and if the supply is an integer multiple of λ_i, then an efficient Bayesian–Nash equilibrium of the uniform-price auction exists. However, if the bidders' capacities are unequal or if the supply is not an integer multiple of λ_i, then an efficient equilibrium of the uniform-price auction does not exist.*

Observe that the exception to the inefficiency result of part (b) of Theorem 6 is equivalent to the situation where there is an integer quantity M of identical units available and the bidders all have unit demands—the one environment where the uniform-price auction is fully efficient.

The intuition for part (a) of Theorem 6 comes from a close examination of optimal bidding strategy in an auction in which all winners pay the

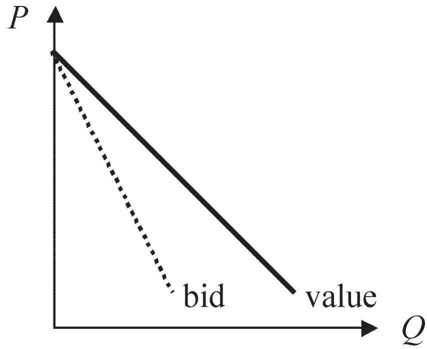

Figure 1 Demand reduction in the uniform-price auction.

amount of the highest losing bid.[3] Suppose that the item being auctioned can be bought in integer quantities, and suppose that a given bidder chooses to bid for k units. Exactly as in the second-price auction for a single item, it is weakly dominant for the bidder to bid her true valuation for the *first* unit on which she bids; in any state of the world in which the bidder i's first bid determines the price paid in the auction, bidder i wins zero units. However, as soon as we look at the bidder's bid for a *second* unit, the truthful bidding property is lost. Bidder i runs the risk that her second bid will determine the price she pays for his first unit, giving bidder i the incentive to shade her bid. *Demand reduction*, as this bid shading is known, becomes increasingly important as the number of units increases; the bid for a kth unit could potentially determine (and increase) the price for all $(k - 1)$ units won, providing a larger incentive to depress one's bids. The typical qualitative shape of a bidder's optimal bid function in the uniform-price auction is displayed in Fig. 1. Observe that the bid coincides with true value for the initial unit of quantity, but it is increasingly shaded below value for each subsequent unit of quantity.

In turn, the differing amounts of demand reduction by different bidders at the clearing price generate inefficiency in the auction outcome. As argued in the previous paragraph, the bid curve intersects the marginal-value curve, for each bidder, at $Q = 0$, but the degree of shading increases in quantity

[3] With discrete units, the uniform-price auction may be specified so that winners pay either the amount of the highest losing bid or the amount of the lowest winning bid. In the actual Treasury auctions, the rule is the lowest winning bid; however, the analysis is simpler (and the degree of bid shading is lower) when the rule is the highest losing bid. In Treasury auctions, the number of units sold is so large that the difference is empirically immaterial, and with divisible units and continuous demands the rule can simply be specified as the market-clearing price.

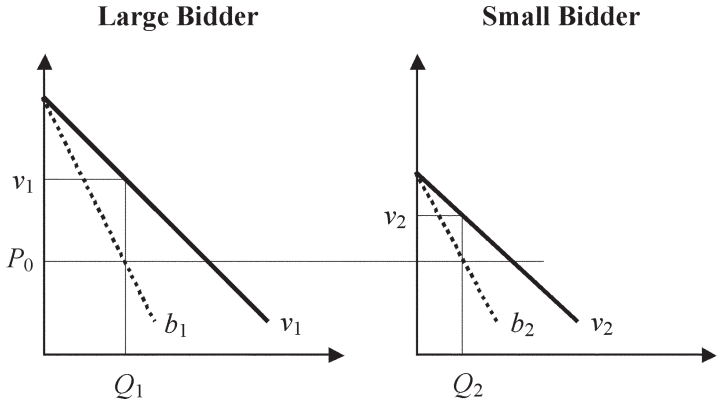

Figure 2 Inefficiency in the uniform-price auction [reproduced with permission from Ausubel and Cramton (2002)].

for all $Q > 0$. The implications of this are depicted in Fig. 2, which is borrowed from Ausubel and Cramton (2002). The auction outcome equates the *bid* curves of the two bidders; indeed, at the clearing price, P_0, the large bidder is assigned a quantity of Q_1 and the smaller bidder is awarded a quantity of Q_2, together satisfying $b_1(Q_1) = P_0 = b_2(Q_2)$. However, efficiency would require us to equate the bidders' marginal values, *not* their bids. As seen in Fig. 2, the large bidder is shading more than the small bidder at the clearing price P_0, that is, $v_1(Q_1) - b_1(Q_1) > v_2(Q_2) - b_2(Q_2)$. Consequently, $v_1(Q_1) > v_2(Q_2)$, meaning that the large bidder wins (inefficiently) too little and the small bidder wins (inefficiently) too much. In summary, the uniform-price auction gives the large bidder an incentive to make room for her smaller rival.

To put it differently, an efficient equilibrium requires that there be a consistent mapping from marginal values to bids; otherwise there is no way for the auction to assign the goods to the bidders with the highest marginal values. Thus, we reach a contradiction to efficiency.

D. Equilibrium of the Vickrey Auction

By comparing Sections II.C and IV.C, we see that the uniform-price auction should not be viewed as the multiunit generalization of the second-price sealed-bid auction for a single item. In the second-price auction, sincere bidding is a dominant strategy; by contrast, in the uniform-price auction, it is optimal to shade all bids other than the first.

The correct multiunit generalization of the second-price auction is given by the (multiunit) Vickrey auction. As in the pay-as-bid auction and the uniform-price auction, bidders simultaneously submit sealed bids comprising inverse demand curves, and each bidder wins the quantity demanded at the clearing price. However, rather than paying the price he bid or the clearing price for each unit, a winning bidder pays the *opportunity cost* for each unit won. The computation is illustrated by the example given in Table 2.

In the example of Table 2, five identical items are auctioned to two bidders in a sealed-bid auction. Bidder 1 submits bids of 56, 47, 44, 37, and 33, and bidder 2 submits bids of 53, 47, 42, 38, and 35. By ranking the bids in descending order, we see that 56, 53, 47, 47, and 44 are the winning bids and that bidder 1 and bidder 2 win three and two units, respectively. These are, of course, the respective payments in the pay-as-bid auction. If the uniform-price auction is specified in terms of the highest rejected bid, then the payments in the uniform-price auction are all 42. However, the *opportunity cost* (with respect to the bids) of providing a third unit to bidder 1 is 42, the opportunity cost of providing a second unit to bidder 1 is 38, and the opportunity cost of providing a first unit to bidder 1 is 35, because these are the (losing) bids by bidder 2 that would win the units in lieu of bidder 1. Similarly, the opportunity cost of providing a second unit to bidder 2 is 37 and the opportunity cost of providing a first unit to bidder 2 is 33, because these are the (losing) bids by bidder 1 that would win the units in lieu of

Table 2

Example Illustrating Rules of Pay-as-Bid, Uniform-Price, and Vickrey Auctions

Bidder	Bid (5 units available)	Payment in pay-as-bid auction	Payment in uniform-price auction	Payment in Vickrey auction
Bidder 1				
1st unit	56	56	42	42
2nd unit	47	47	42	38
3rd unit	44	44	42	35
4th unit	37			
5th unit	33			
Bidder 2				
1st unit	53	53	42	37
2nd unit	47	47	42	33
3rd unit	42			
4th unit	38			
5th unit	35			

bidder 2. Stated more generally, in the multiunit Vickrey auction, if a bidder wins K units, he pays the Kth highest rejected bid by any of his opponents for his first unit, the $(K-1)$st highest rejected bid by any of his opponents for his second unit, . . . , and the highest rejected bid by any of his opponents for his Kth unit.

The reason why bidders have the incentive to shade their bids in the pay-as-bid auction and uniform-price auction is that, at least some of the time, their payments are determined by the amounts of their own bids. For example, observe that, in Table 2, the payment by bidder 2 for his second unit is determined by his own second bid in the pay-as-bid auction and by his own third bid in the uniform-price auction. However, in the multiunit Vickrey auction, the payment is determined solely with respect to opponents' bids, and so the dominant-strategy result from the second-price sealed-bid auction generalizes.

THEOREM 7 (Vickrey, 1961). *Assume that the bidders have pure private values and (weakly) diminishing marginal values. Then sincere bidding is the weakly dominant strategy for every bidder in the multiunit Vickrey auction, yielding an efficient outcome.*

The reader may remain a bit puzzled as to why a revenue-maximizing seller would ever utilize the Vickrey auction. After all, for any given set of bids by the bidders, the payments are highest in the pay-as-bid auction, next highest in the uniform-price auction, and lowest in the Vickrey auction (see Table 2). Of course, what is omitted from this naive observation is that the bids do *not* remain fixed as the auction format is varied. Rather, for a given set of bidder valuations, the bids should be expected to be the highest in the Vickrey auction (because that is the only format without bid shading) and the lowest in the pay-as-bid auction (because the winner's payment is affected by his own bid, for sure). Thus, the Vickrey auction yields the seller the lowest function of the highest bids, whereas the pay-as-bid auction yields the seller the highest function of the lowest bids, suggesting that the revenue ranking is inherently ambiguous.

When ranking the auctions on the basis of allocative efficiency achieved, there is relatively less ambiguity. Due to Theorem 7, higher bids in the Vickrey auction always correspond to higher values, and so the items are always assigned to the bidders who value them the most. However, considerable ambiguity remains as to whether the pay-as-bid auction or uniform-price auction yields greater efficiency, as well as revenues.

THEOREM 8 (Ausubel and Cramton, 2002). *Assume that the bidders have pure private values. Then the revenue ranking of the pay-as-bid auction, the uniform-price auction, and the Vickrey auction is ambiguous: it is possi-*

ble to construct classes of examples for which any of the three formats yields the highest revenues. The efficiency of the Vickrey auction is unambiguously the greatest, whereas the efficiency ranking of the pay-as-bid auction and the uniform-price auction is again ambiguous.

E. The Vickrey–Clarke–Groves (VCG) Mechanism

The Vickrey (1961) auction of Section IV.D was generalized to the case of heterogeneous items by Clarke (1971) and Groves (1973). Let N be any arbitrary finite set of items to be sold, and let L be the set of bidders. An *allocation* is a vector $x \equiv (x_\ell)_{\ell \in L}$, where each component $x_\ell \subset N$ denotes the set of items assigned to bidder $\ell \in L$. An allocation x is said to be *feasible* if it assigns each item to at most one bidder, i.e., if $x_k \cap x_\ell = \varnothing$ whenever $k \neq \ell$ $(k, \ell \in L)$.

Let X denote the set of all feasible allocations. If $v_\ell(x_\ell)$ denotes bidder ℓ's valuation for the set x_ℓ of items, then an *efficient allocation* x^* is a solution to

$$x^* \in \arg\max_{x \in X} \left\{ \sum_{\ell \in L} v_\ell(x_\ell) \right\}, \tag{11}$$

and the maximum social surplus available given the set L of bidders is

$$w(L) = \max_{x \in X} \left\{ \sum_{\ell \in L} v_\ell(x_\ell) \right\} = \sum_{\ell \in L} v_\ell(x_\ell^*). \tag{12}$$

In the *Vickrey–Clarke–Groves (VCG) mechanism*, each bidder $\ell \in L$ submits a sealed bid comprising a report, $v_\ell(\cdot)$, of her valuations for all subsets of N. After all bids are submitted, the auctioneer determines an efficient allocation x^* of the items, i.e., a solution to Eq. (11). Whereas bidder ℓ thus receives the subset $x_\ell \subset N$, she does not pay her bid $v_\ell(x_\ell)$. Rather, her payment $y_\ell \in \mathbb{R}$ is selected to satisfy

$$v_\ell(x_\ell) - y_\ell = w(L) - w(L \setminus \ell), \text{ where } w(L \setminus \ell) = \max_{x \in X} \left\{ \sum_{k \in L \setminus \ell} v_k(x_k) \right\}. \tag{13}$$

Thus, bidder ℓ is allowed a payoff equaling the *incremental surplus* that she brings to the auction, i.e., the (maximized) social surplus when she is present at the auction minus the (maximized) social surplus when she is absent. As in the Vickrey auction, a bidder's payment equals the opportunity cost of assigning the items to the bidder.

Applied to a setting with a single item, observe that the Vickrey–Clarke–Groves mechanism reduces to the second-price sealed-bid auction. Applied to a setting with multiple identical items and nonincreasing marginal valuations, the Vickrey–Clarke–Groves mechanism reduces to the (multiunit) Vickrey auction. The dominance properties of these special cases extend to the setting with heterogeneous items.

THEOREM 9 [Clarke (1971) and Groves (1973)]. *Assume that the bidders have pure private values for an arbitrary set of items. Then sincere bidding is a weakly dominant strategy for every bidder in the Vickrey–Clarke–Groves mechanism, yielding an efficient outcome.*

V. CLOCK AUCTIONS FOR MULTIPLE ITEMS

In Section IV, we saw a number of sealed-bid approaches for auctions of multiple items. However, in auctions of single items in the real world—especially in on-line auctions—there appears to be a preference for the English auction, that is, a preference for dynamic auctions over sealed-bid auctions. Thus, one of the important focuses of research has been on generalizing the English auction to multiple items.

This section and the next will focus on perhaps the two most important strands of development for dynamic auctions. They are important developments conceptually, in that each approach has yielded significant theoretical results on efficiency. Moreover, they are important developments empirically, because each approach has already begun to be used in empirical applications.

Thinking about the alternative ways that we often model the English auction can identify both strands of the development. In one approach, the auctioneer announces prices, and bidders' responses are limited to the reporting of quantities desired at the announced prices until a closing is reached. Such "clock auction" approaches will be the topic of this section. In a second approach, bidders submit bids comprising both prices and quantities in an iterative fashion until no further bids are offered. Approaches along the latter lines will be the topic of Section VI.

A. UNIFORM-PRICE ASCENDING-CLOCK AUCTIONS

The starting point for modern "clock-auction" designs is a hypothetical trading institution from classical economics literature, often used as a device or thought experiment for understanding convergence to a general equilibrium. A fictitious auctioneer (sometimes referred to as the *Walrasian auctioneer*) announces a price vector, p, and bidders respond by reporting the quantity vectors that they wish to transact at these prices. The auctioneer then calculates the excess demand and increases or decreases each component of the price vector according as the excess demand is positive or negative (*Walrasian tâtonnement*). This iterative process continues until a price vector is reached at which excess demand is zero, and trades occur only at the final price vector.

In the new economy, the Walrasian auctioneer has been reconceptualized as the basis for a real trading institution. Instead of a fictional auctioneer serving as a metaphor for a market-clearing process, a real auctioneer announces prices and accepts bids of quantities. In one of the earliest proposals, the U.S. Department of the Treasury, the Securities and Exchange Commission, and the Board of Governors of the Federal Reserve System jointly authored the *Joint Report on the Government Securities Market* in 1992. The *Joint Report*, discussing the case of homogeneous goods, proposed replacing the sealed-bid Treasury auction with a uniform-price ascending-clock auction. This proposal was never implemented. However, in the early 2000s, dynamic clock auctions with on-line bidding were implemented in several large real-world applications.

Electricité de France (EDF), the dominant power producer in France and the world's largest electricity group, now runs quarterly ascending-clock auctions of generation capacity. The EDF Generation Capacity Auctions, which began in September 2001, are intended to auction off 6000 megawatts of capacity (about 10% of France's electricity supply) by the end of 2003. In a typical auction, base-load electricity contracts and peak-load electricity contracts, of five different durations each, are sold simultaneously in an ascending-clock auction. The auctioneer announces a vector of prices, bidders respond with vectors of quantities, and the price is adjusted iteratively so as to obtain market clearing in both base-load and peak-load electricity contracts.

The UK government allocated £215 million in incentive payments for greenhouse gas emission reductions by a descending-clock auction. [The clock descended because this was a procurement auction (or "reverse auction"): the government sought to purchase emission reductions at minimum cost.] In the UK Emissions Trading Scheme Auction, the world's first auction for greenhouse gas reduction, in March 2002, the price started at £100 per metric ton of emissions, and the price was allowed to descend until a point was reached where the budgeted incentive monies were exactly spent. At the final price of £53.37 per metric ton, the 34 winning bidders collectively took on the obligation to reduce their annual greenhouse gas emissions below their 1998–2000 levels by 4 million metric tons over 5 years.

Observe that the uniform-price ascending-clock auction is a dynamic version of the uniform-price sealed-bid auction of Section IV.C. This is clearest in an auction for a homogeneous good. In the ascending auction, the bidder is iteratively queried for the quantity desired at a variety of prices for the good, whereas in the sealed-bid auction, the bidder is asked to submit, on a one-time basis, a demand curve specifying the quantities at all possible prices. In each, a single clearing price is determined, and bidders pay the clearing price for every unit they win. The one important difference is that, in the ascending auction, bidders receive repeated feedback as to the opposing bidders' demands at the various prices.

In general, ascending-clock auction formats for multiple items could be expected to offer several decisive advantages over the corresponding sealed-bid auction format. First, the insight from single-item auctions that the continuous feedback about other bidders' valuations would ameliorate the winner's curse and lead to more aggressive bidding might be expected to carry over to the multiunit environment. Second, ascending-clock auctions, better than sealed-bid auctions, allow bidders to maintain the privacy of their valuations for the items being sold. Bidders never need to submit any indications of interest at any prices beyond the auction's clearing price. Third, when there are two or more types of items, auctioning them simultaneously enables bidders to submit bids based on the substitution possibilities or complementarities among the items at various price vectors. At the same time, the iterative nature of the auction economizes on the amount of information submitted: demands do not need to be submitted for all price vectors, but only for price vectors reached along the convergence path to equilibrium.

However, as was argued in Section IV.C, the uniform-price sealed-bid auction suffers from demand reduction and inefficiency. The same theoretical logic applies to the uniform-price ascending-clock auction. Indeed, the problem of bidders optimally reducing their quantities bid well below their true demands can become much worse in the dynamic version of the auction [see Ausubel and Schwartz (1999)]. As a practical matter, demand reduction may not undermine the outcome of an auction where there is a strong degree of competition for every item being sold. However, if one or more of the bidders have significant market power, it may become important to use an auction format that is not susceptible to demand reduction.

B. Efficient Ascending Auctions

Ausubel (2002a) proposed an efficient ascending auction design for homogeneous goods, which utilizes the same general structure as the auction format of Section V.A, but has a different payment rule that eliminates the incentives for demand reduction. As in the dynamic version of the uniform-price auction, the price is allowed to adjust iteratively until demand equals supply, and bidders are allotted the quantities that they bid at the final clearing price. However, we have seen that uniform pricing leads to low prices and inefficient outcomes. The Ausubel auction takes a different approach to determining the bidders' payments, which in principle should dynamically replicate the Vickrey auction outcome and, thus, yield full allocative efficiency.

The efficient ascending auction is most easily described with an example. Suppose that five identical items are offered to four bidders. Let the auction begin with the auctioneer announcing a price of 10 and the bidders responding with the following quantities:

Price	Bidder A	Bidder B	Bidder C	Bidder D
10	3	2	2	2

Then, at the initial price of 10, the aggregate demand of 3 + 2 + 2 + 2 = 9 exceeds the available supply of 5. The auctioneer allows the price to increase continuously. Now suppose that the bidders continue to bid the same quantities until the price reaches 25, at which point bidder D drops out of the auction, yielding:

Price	Bidder A	Bidder B	Bidder C	Bidder D
25	3	2	2	0

At the price of 25, the aggregate demand of 3 + 2 + 2 + 0 = 7 continues to exceed the available supply of 5. However, let us examine this situation carefully from bidder A's perspective. Bidder A's opponents collectively demand only 2 + 2 + 0 = 4 units, whereas 5 units are available. It may now be said that bidder A has *clinched* winning 1 unit: whatever happens now (provided that bidders B–D bid monotonically), bidder A is certain to win at least 1 unit. The auction rules take the fact of "clinching" literally and award 1 unit to bidder A at the clinching price of 25.

Because there is still excess demand, the auctioneer continues to adjust the price upward. Suppose that the next change in bidders' demands occurs at a price of 30, at which point bidder B reduces his demand from 2 to 1, yielding:

Price	Bidder A	Bidder B	Bidder C	Bidder D
30	3	1	2	0

At the price of 30, the aggregate demand of 3 + 1 + 2 + 0 = 6 continues to exceed the available supply of 5. However, again, examine this situation carefully from bidder A's perspective. Bidder A's opponents collectively demand only 1 + 2 + 0 = 3 units, whereas 5 units are available. It may now be said that bidder A has clinched winning 2 units: whatever happens now (provided that bidders B–D bid monotonically), bidder A is certain to win at least 2 units. The auction rules continue to take the fact of clinching literally and award a second unit to bidder A at the new clinching price of 30. By the same token, let us examine this situation carefully from bidder C's perspective. Bidder C's opponents collectively demand only 3 + 1 + 0 = 4 units, whereas 5 units are available. It may therefore be said that bidder C has clinched winning 1 unit: whatever happens now (provided that bidders A, B, and D bid monotonically), bidder C is certain to win at least 1 unit. The auction rules take the fact of clinching literally and award a unit to bidder C at the clinching price of 30.

Because there is still excess demand, the auctioneer continues to adjust the price upward. Suppose that the final change in bidders' demands occurs at a price of 35, at which point bidder B reduces his demand from 1 to 0, yielding:

Price	Bidder A	Bidder B	Bidder C	Bidder D
35	3	0	2	0

At the price of 35, the aggregate demand of 3 + 0 + 2 + 0 = 5 is brought equal to the supply of 5. Thus, the market-clearing price is 35. Bidder A, who had already received a first unit at 25 and a second unit at 30, wins a third unit at 35. Bidder C, who had already received a first unit at 30, wins a second unit at 35. In summary, we have the following auction outcome:

	Bidder A	Bidder B	Bidder C	Bidder D
Units won	3	0	2	0
Payments	25 + 30 + 35	0	30 + 35	0

Thus, bidders are awarded the quantities that they bid at the final price, but bidders pay according to the prices at which they "clinch" the various units. We have the following result:

THEOREM 10 (Ausubel, 2002a). *Assume that the bidders have pure private values and (weakly) diminishing marginal values. Then sincere bidding by every bidder is an equilibrium of the Ausubel auction. Moreover, in a suitable discrete specification of the game under incomplete information,* *sincere bidding is the unique outcome of iterated elimination of weakly dominated strategies.*

One way of explaining why sincere bidding is incentive compatible is that the equilibrium outcome of the dynamic Ausubel auction effectively replicates the outcome of the static Vickrey auction. For example, in the preceding example, bidder A received 3 units; the third unit was awarded at the highest rejected bid of 35, the second unit was awarded at the second highest rejected bid of 30, and the first unit was awarded at the third highest rejected bid of 25.

An efficient clock auction is extended from the case of identical objects to an environment with multiple types of commodities in Ausubel (2002b).

VI. GENERALIZATIONS OF THE ENGLISH AUCTION FOR HETEROGENEOUS ITEMS

In this section, we will consider another promising approach to generalizing the English auction for multiple items. Recall that, in Section V, we examined "clock-auction" designs, in which the auctioneer announced prices and bidders' responses were limited to the reporting of quantities desired at the announced prices. Another approach is to allow bidders to submit bids comprising both prices and quantities in an iterative fashion, until no further bids are offered. Theory and practice have pursued this avenue in each of two ways: the *simultaneous ascending auction*, in which independent bids are submitted for each item in the auction, and the *ascending auction with package bidding*, in which a bid may consist of a set of items together with an all-or-nothing price for the set.

A. Simultaneous Ascending Auctions

The simultaneous ascending auction, proposed in comments to the U.S. Federal Communications Commission (FCC) by academics Paul Milgrom, Robert Wilson, and Preston McAfee, has been used in auctions on six continents allocating more than $100 billion worth of spectrum licenses. Some of the best known applications of the simultaneous ascending auction include the following: the Nationwide Narrowband Auction (July 1994), the first use of the simultaneous ascending auction; the PCS A/B Auction (December 1994–March 1995), the first large-scale auction of mobile telephone licenses, which raised $7 billion; the UK UMTS Auction (March–April 2000), which raised £22.5 billion; and the German UMTS

Auction (July–August 2000), which raised 50 billion euro. Some accounts of the rationale behind and introduction of the simultaneous ascending auction include Cramton (1995), McMillan (1994), and Milgrom (2000, 2003).

In the simultaneous ascending auction, a collection of items is put up for sale simultaneously in a fashion similar to the English auction, but the auction does not conclude for any single item until it concludes for all of the items. In each of a sequence of rounds, bidders may submit new bids for one or more items, where a bid is a pair consisting of an item together with an associated price. The price must be at least the previous high bid plus a minimum bid increment that is specified by the auctioneer. After each round, the results of the bidding are announced: at a minimum, the new standing high bids for each item are revealed, and often all of the submitted bids (losing, as well as winning) and the identities of the high bidders are disclosed. The auction continues until a round is completed in which no new bids are submitted, and then the high bidders win the items at their respective high bids.

As such, the simultaneous ascending auction is a modern version of the "silent auction" that is frequently used in fundraisers by charitable institutions. The silent auction also operates by putting a collection of items up for sale simultaneously and inviting bidders to submit successively higher bids for each. However, in a typical silent auction, a fixed closing time such as 9:00 PM is announced to bidders. In that event, the off-line equivalent of "sniping" is apt to occur. There is little incentive for serious bidding before say 8:55 PM, and then a heated exchange of successive increases may ensue in the last 5 minutes of the auction. The tendency of bidders to wait as "snakes in the grass" until the end of the auction thwarts one of the major reasons for conducting a dynamic auction in the first place. Bidders are unable to receive any significant feedback about their opponents' valuations and incorporate it into their own bids during the course of the auction.

Thus, the clever innovation of the simultaneous ascending auction is the introduction of *activity rules* into the auction design. Activity rules are bidding constraints that require bidders to place meaningful bids in early rounds of the auction in order to retain the right to bid late in the auction. A typical application of activity rules can be found in the Nationwide Narrowband Auction. In this July 1994 auction of 10 paging licenses, the activity rule was a monotonicity constraint concerning the number of licenses on which a bidder could be active in a given round. A bidder was said to be "active" on a license if either she placed a new bid (equal to at least the standing high bid plus the minimum bid increment) in the current round, or she was the standing high bidder on the license from the previous round. Each bidder established an initial eligibility of up to three, based on her application to enter the auction and a dollar deposit. Suppose that a given bidder's initial eligibility was three. Then in the initial round, the bidder could place bids on at most three licenses. In each successive round, the bidder could place new bids such that the sum of standing high bids plus new bids equaled at most three. However, if in any round the sum of standing high bids plus new bids was reduced to a number $A < 3$, then A would become the new limit on activity in all subsequent rounds of the auction.

The rules of the *simultaneous ascending auction* may be summarized as follows: Items are offered simultaneously, and the auction does not conclude for any individual item until it concludes for all items. Bidders submit bids in a sequence of rounds. Each bid comprises a single item and an associated price. Bidding is constrained by a minimum bid increment and by an activity rule that limits a bidder's bidding activity in the current round based on her past bidding activity. The auction concludes when a round elapses in which no new bids are submitted, and the standing high bids are then deemed to be winning bids. Payments equal the amounts of the winning bids.

Empirically, the simultaneous ascending auction has performed very well in spectrum auctions. In the various U.S. spectrum auctions, the fact that licenses for different geographic regions were sold simultaneously in the same auction appears to have enabled bidders to assemble coherent geographic blocks, realizing many of the synergies associated with owning adjacent licenses. In several of the U.S. and European spectrum auctions, the price seems to have successfully been driven so high that the winning bidders later regretted their purchases, suggesting that the auction format was also successful in obtaining full value for the seller.

Theoretically, the simultaneous ascending auction yields full efficiency when bidders have unit demands (i.e., in situations where the items are similar and each bidder can purchase at most one item). However, note that the fact that independent bids are submitted for each item should lead to arbitrage of the various prices and, hence, lend a uniform-price character to the auction outcome. Consequently, when bidders have multiunit demands, the same critique that was applied to the uniform-price sealed-bid (Section IV.C) and ascending-clock (Section V.A) auctions would also predict demand reduction and inefficient outcomes of the simultaneous ascending auction. The next section argues that the incentive for demand reduction can be avoided if bidders, instead of needing to place independent bids for each item, are permitted to place bids on packages of items.

B. Ascending Auctions with Package Bidding

In an ascending auction with package bidding, also frequently referred to as a *combinatorial auction*, a bid is a pair comprising a package (i.e., a subset of the set of all items in the auction) and an associated price. Observe that the Vickrey–Clarke–Groves mechanism of Section IV.E is itself one of the earliest proposed package auctions, albeit a static auction. Rassenti *et al*. (1982) are often credited with the first experimental study of package auctions. They studied the problem of allocating airport time slots, a natural application for package auctions given that landing and takeoff slots are strong complements. Banks *et al*. (1989) conducted an early and influential study of ascending package auctions. They defined several alternative sets of rules for ascending package auctions, developed some theoretical results, and conducted an experimental study. Other important contributions include Bernheim and Whinston (1986), Kwasnica *et al*. (2002), and Parkes and Ungar (2000).

In a package auction, a bid is interpretable as an all-or-nothing offer for the specified package at the specified price; there is no requirement that the bidder be willing to purchase a part of the package for a part of the price. Some of the motivation for a package auction comes from what is known in the literature as the *exposure problem*. For example, consider a spectrum auction in which licenses for an eastern region and a western region are offered, with a bidder who would obtain synergies from owning the two licenses together. Let us suppose that the bidder has a $1 billion valuation for the eastern license by itself, a $1 billion valuation for the western license by itself, but a $3 billion valuation for the combination of the eastern and western licenses together. In a simultaneous ascending auction, the bidder would need to place independent bids for the two regions. If the bidder believed that the auction was near its conclusion, he might be willing to place bids of $1.2 billion for the eastern license and $1.2 billion for the western license—greater than his stand-alone values for the respective licenses (but together less than his value for the combination). However, in doing so, the bidder would run the risk that the price on the eastern license might rise to $2 billion at the same time that his bid for the western license remained standing as the high bid. In that scenario, the bidder is sorely disappointed: his $1.2 billion offer for the western license remains binding, while any purchase of the combination of two licenses would cost at least $3.2 billion. The bidder would pay at least $0.2 billion more than his valuation for licenses, and, anticipating this risk, he may refrain from bidding more than his stand-alone values. This may prevent synergies from being realized. Moreover, when synergies are present, there is no way to avoid this possibility in an auction where all bids are independent, except by making the bids nonbinding, which would serve to undermine the entire auction process.

The difficulty with independent bids in this example is that the bidder is unable to make the western bid contingent on the eastern bid (and vice versa). By contrast, with package bidding, the bidder is permitted to propose a payment for a set of items, without in any way offering part of the proposed payment for part of the set. For example, the bidder can offer \$2.4 billion for the {eastern, western} combination, without incurring the risk of winning either of the single licenses for greater than his stand-alone value. This would seem to encourage bidding reflective of bidder valuations, as well as to encourage efficiency in the auction outcome.

To the extent that bidders value some of the items in the auction as substitutes, then it may also become important that any two bids by the same bidder are treated as *mutually exclusive*. Let us modify the previous example by assuming that two equivalent eastern licenses (denoted by east 1 and east 2) and two equivalent western licenses (denoted by west 1 and west 2) are offered, and that bidders find more than one license covering a given region to be redundant. The bidder whose preferences we have specified might then like to place \$2.4 billion bids on each of the following four packages: {east 1, west 1}, {east 2, west 2}, {east 1, west 2}, and {east 2, west 1}. Unfortunately, without the mutual-exclusivity requirement, the bidder runs the risk that his first and second package bids (or his third and fourth package bids) may *both* be accepted, winning him all four licenses for \$4.8 billion (when his value for both regions has been assumed to be just \$3 billion). This difficulty is avoided if the auction rules treat the four bids as mutually exclusive, allowing the auctioneer to accept at most one of them. Observe that a rule of mutual exclusivity is *fully expressive* in the sense that it enables the bidder to express any arbitrary preferences. For example, if the bidder wishes to authorize the possibility that two of his package bids are both accepted, he can effectively opt out of the mutual exclusivity by submitting a fifth package bid comprising {east 1, east 2, west 1, west 2} and an associated payment.

Together, the exposure problem and the possibility of substitute goods motivate an ascending auction procedure in which bidders can iteratively submit arbitrary package bids that are treated as mutually exclusive. Following notation similar to that in Section IV.E, let there be a finite set N of items to be sold. A *package* $S \subset N$ is a subset of the set of all items. A *package bid* is a pair (S, y) comprising a package S and an associated payment y.

Let L be the set of bidders. An *allocation* is a vector $x \equiv (x_\ell)_{\ell \in L}$, where each component $x_\ell \subset N$ denotes the package assigned to bidder $\ell \in L$. An allocation x is said to be *feasible* if it assigns each item to at most one bidder, i.e., if $x_k \cap x_\ell = \emptyset$ whenever $k \neq \ell$ ($k, \ell \in L$). Let X denote the set of all feasible allocations. In addition, let $B^t{}_\ell(S)$ denote the highest bid that bidder ℓ has submitted in the auction by time t for package S (or zero if bidder ℓ has submitted no bids for package S). The *provisional winning allocation*, x^{*t}, maximizes bid revenues at time t over all feasible allocations:

$$x^{*t} \in \arg\max_{x \in X} \left\{ \sum_{\ell \in L} B_\ell^t(x_\ell) \right\}. \tag{14}$$

Finally, the *provisional winning bids* $(x_\ell^{*t}, B_\ell^t\ (x_\ell^{*t}))_{\ell \in L}$ are simply the bids associated with the provisional winning allocation. Note that Eq. (14) implicitly incorporates mutual exclusivity into the selection of the provisional winning bids.

Ausubel and Milgrom (2002) define two ascending package auction formats:

1. **Ascending package auction.** Bidders submit package bids in a sequence of bidding rounds. Each bid must be greater than the bidder's prior bids for the same package by at least a specified minimum bid increment. After each round, the auctioneer announces the provisional winning bids. The auction concludes when a round elapses in which no new bids are submitted. Payments then equal the amounts of the winning package bids.
2. **Ascending proxy auction.** Each bidder enters valuations for the various packages into a *proxy bidder*. The proxy bidders then bid on behalf of the bidders in an ascending package auction in which the minimum bid increment is taken arbitrarily close to zero.

The latter auction format may be viewed both as a proposed new format that greatly speeds the progress of the auction and as a modeling device for obtaining results about the former auction. Ausubel and Milgrom are able to establish efficiency results and a partial equilibrium characterization for the latter auction format. A bidder is said to bid *sincerely* if he submits his true valuation, $v(S)$, to his proxy bidder for every package S, and a bidder is said to bid *semisincerely* if he submits his true valuation less a positive constant, $v(S)–c$, to his proxy (where the same constant c is used for all packages S with values of at least c). In any coalitional form game with transferable utility, the *core* is the set of all payoff allocations that are feasible and upon which no coalition of players can improve. The following result refers to the coalitional form game corresponding to the package economy: the value of any coalition that includes the seller is the total value associated with an efficient allocation among the buyers in the coalition, and the value of any coalition without the seller equals zero. Thus, in interpreting the following result, note in particular that any outcome in the core attains full efficiency.

Theorem 11 (Ausubel and Milgrom, 2002). *In the ascending proxy auction, given any reported preferences, the final payoff allocation is in the core relative to the reported preferences. For any payoff vector π that is a bidder-Pareto-optimal point in the core, a Nash equilibrium of the ascending proxy auction with associated payoff vector π exists. Conversely, for any Nash equilibrium in semisincere strategies at which losing bidders bid sincerely, the associated payoff vector is a bidder-Pareto-optimal point in the core.*

Furthermore, Ausubel and Milgrom show that the set of all economic environments essentially dichotomizes into two cases. First, if all bidders have substitute preference for all goods, then there is a single point in the core that Pareto dominates all other points in the core, and it equals the payoff vector from the Vickrey–Clarke–Groves mechanism. Thus, in this case, the outcome of the ascending proxy auction coincides with the outcome of the VCG mechanism. Second, if at least one bidder has nonsubstitute preferences, then an additive preference profile exists for the remaining bidders such that there is more than one bidder-Pareto-optimal point in the core. In this second case, the VCG payoff vector is *not* an element of the core, and Ausubel and Milgrom argue that the low revenues of the VCG mechanism then become problematic.

VII. CONCLUSION

In auctions of single items, such as the auction of an impressionist painting at Sotheby's or a used car on eBay, the classic English auction will remain the favored means of auction. For high-stakes auctions of multiple types of commodities, each available in multiple units, the ascending-clock auction is the preferred auction format, and so clock auctions are likely to be observed with increasing frequency in the future. They serve the goals of efficiency and revenue maximization by systematically conveying market information to the auction participants in real time. And, important to many applications, ascending-clock auctions are extremely fast: auctions of several types of commodities can be completed within hours, and auctions of homogeneous goods can be completed in perhaps minutes. This rapidity of the auction process minimizes bidders' participation costs and is especially important in applications such as energy markets, where the value of the underlying product may itself be continually changing.

When many heterogeneous (but related) items are offered, the ascending auction with package bidding appears to be the preferred auction format. The available results on core payoffs suggest a high level of efficiency in outcomes even when the goods are not substitutes. Extensions of ascending package auctions are particularly appropriate for procurement auctions, where different suppliers are likely to offer products with slightly different characteristics, and the buyer should be permitted to express her relative preferences for the various characteristics. Two substantial challenges remain in the effective implementation of package auctions. First, if the procedure uses reasonably small bid increments and if it allows a non-provisionally winning bidder to remain active in the auction by raising her bids on relatively few (of the many) packages, then the auction has the possibility of taking an exceedingly long time to conclude. However, developments toward integrating proxy bidding into the auction design should much improve the speed of package auctions. Second, in order for package auctions to yield full efficiency, bidders in principle may need to express their valuations for every subset of items. With N items, the number of subsets equals 2^N, quickly requiring the bidders to provide a potentially prohibitive amount of information. Are there compact ways for bidders to express the vast array of valuations? Can well-designed interfaces or computer agents assist the bidders in developing the necessary information, especially in situations where the bidders may themselves be unsure of their own valuations? These questions are important and are attracting growing research interest today. The answers may determine whether package auctions will be able to proliferate in the new economy in coming years in the same way that more traditional auction formats have flourished on-line to date. For further discussion, see the following chapter in this volume.

ACKNOWLEDGMENTS

I am indebted to Peter Cramton, Paul Milgrom, and Jesse Schwartz for their implicit contributions. The generous support of National Science Foundation Grants SES-01-12906 and IIS-02-05489 is gratefully acknowledged.

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