# POWER SERIES

In previous lectures we studied series with constant terms. In this section/lecture we shall consider series whose terms involve variables.

**Definition**

A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots\cdots$$

where $c_0, c_1, c_2, \ldots$ are constants and $x$ is a variable is called as **power series in $x$**.

For example.

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots\cdots$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

More generally

**Definition**

A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots\cdots$$

where $c_n$'s are co-efficients (consts), $a$ is a constant and $x$ is a variable, is called **power series in $(x-a)$**.

For example

$\boxed{a=1}$

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{n+1} = 1 + \frac{x-1}{2} + \frac{(x-1)^2}{3} + \frac{(x-1)^3}{4} + \cdots$$

$\boxed{a=-3}$

$$\sum_{n=0}^{\infty} \frac{(-1)^k (x+3)^k}{k!} = 1 - (x+3) + \frac{(x+3)^2}{2!} - \frac{(x+3)^3}{3!} + \cdots$$

If a numerical value is substituted for $x$ in a power series $\sum c_n (x-a)^n$, then the resulting series of constants may converge or diverge.

In the discussion of power series convergence is still a major question. The convergence of the power series will depend upon the values of $x$ that we put into the series. The problem of determing those values of $x$ for which a given power series converges is addressed by following theorem.

## Theorem

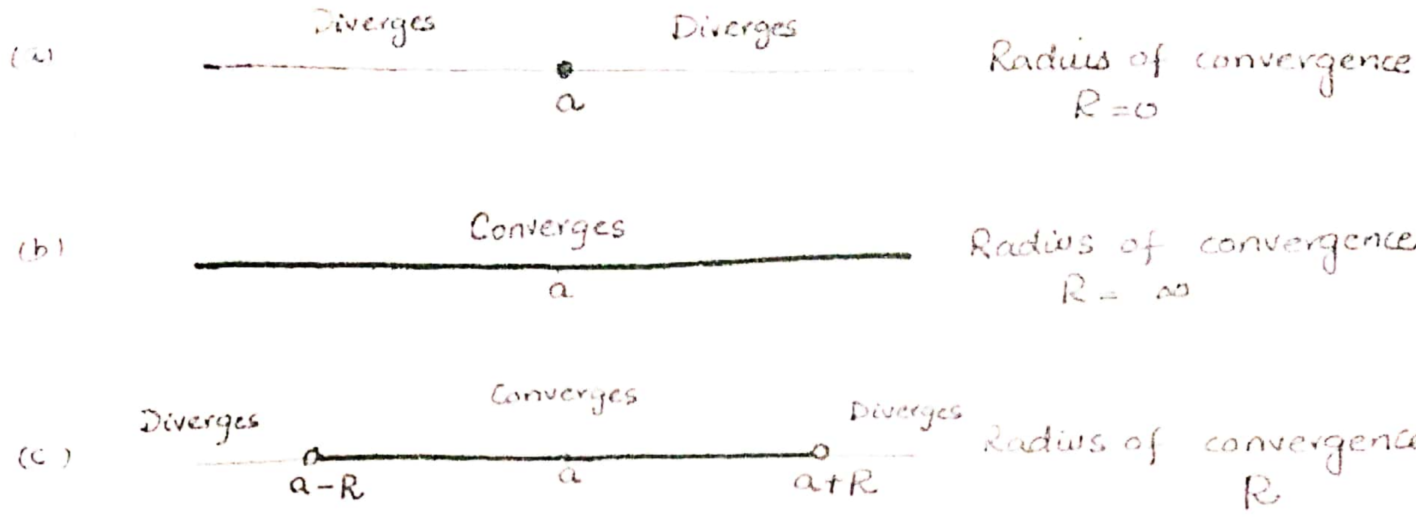
For a power series $\sum c_n (x-a)^n$ exactly one statement is true:

a) The series converges only for $x=a$.

b) The series converges absolutely (and hence converges) for all values of $x$.

c) The series converges absolutely (and hence converges) for all $x$ in some finite open interval $(a-R, a+R)$ and diverges if $x < a-R$ or $x > a+R$. At either of the points $(x=a-R)$ or $(x=a+R)$ the series may converge absolutely, conditionally or diverges.

It follows that the set of values for which a power series in $(x-a)$ converges is always an interval centered at $x=a$; we call this the **interval of convergence**




In part (a) the interval of convergence reduces to the single point $x=a$, we can say that series has radius of convergence $R=0$.

In part (b) the interval of convergence is infinite, we can say that the series has radius of convergence $R=\infty$.

In part (c) the interval of convergence extends between $a-R$ and $a+R$, we can say that the series has radius of convergence $R$.

The main tool for finding the interval of convergence of a power series is the ratio test for absolute convergence.

Example

Find interval of convergence & radius of convergenc of the power series.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} x^n$$

Let $u_n = \frac{x^n}{\sqrt{n}}$. Using ratio test

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{\sqrt{n}}{\sqrt{n+1}} x \right|$$

$$= \lim_{n \to \infty} \left| \frac{\sqrt{n}}{\sqrt{n+1}} \right| |x| = |x|$$

It follows from the ratio test that the power series is absolutely convergent if $|x|<1$, i.e $-1<x<1$.
The numbers 1 & $-1$ must be investigated separately by substitution in the power series.

If we put $x=1$,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} (1)^n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$$

which is a divergent p-series with $p=\frac{1}{2}$.

If we put $x=-1$,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} (-1)^n = -1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \cdots + (-1)^n \frac{1}{\sqrt{n}} + \cdots$$

which converges by alternating series test.

Thus the power series converges if $-1 \leq x < 1$
or we can say Interval of Convergence is $[-1, 1)$.
Radius of Convergence is $R=1$

Example

$$\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n!} (x-4)^n \qquad U_n = (-1)^n \frac{3^n}{n!} (x-4)^n$$

Find the interval of convergence.

Using Ratio test for absolute convergence.

$$\lim_{n\to\infty} \left| \frac{U_{n+1}}{U_n} \right| = \lim_{n\to\infty} \left| \frac{3^{n+1} (x-4)^{n+1}}{(n+1)!} \times \frac{n!}{3^n (x-4)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{3 (x-4)}{n+1} \right|$$

$$= 3 \lim_{n\to\infty} \frac{1}{n+1} |x-4|$$

$$= 0 < 1 \qquad \forall x$$

$\Rightarrow$ Interval of convergence is $(-\infty, \infty)$.

Find the radius of convergence & interval of convergence

$$\sum_{n=1}^{\infty} \frac{(\ln n)(x-3)^n}{n}$$

$$U_n = \frac{(\ln n)(x-3)^n}{n} , \quad U_{n+1} = \frac{\ln(n+1)\,(x-3)^{n+1}}{n+1}$$

Using Ratio test for absolute convergence

$$\lim_{n \to \infty} \left| \frac{U_{n+1}}{U_n} \right| = \lim_{n \to \infty} \left| \frac{\ln(n+1)(x-3)^{n+1}}{n+1} \cdot \frac{n}{\ln n\,(x-3)^n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{\ln(n+1)\ (x-3)\ n}{(\ln n)\ \ n+1} \right|$$

$$= |x-3| \lim_{n \to \infty} \left| \frac{n(\ln(n+1))}{(n+1)\ln n} \right|$$

$$= |x-3| \lim_{n \to \infty} \left| \frac{\ln(n+1)}{\ln n} \right| . \lim_{n \to \infty} \left| \frac{n}{n+1} \right|$$

$$= |x-3| \left[ \quad \frac{\infty}{\infty} \text{ form} \right] (1)$$

$$= |x-3| \left[ \lim_{n \to \infty} \left| \frac{1/(n+1)}{1/n} \right| \right] = |x-3| \lim_{n \to \infty} \left| \frac{n}{n+1} \right|$$

$$= |x-3| < 1$$

$$-1 < x-3 < 1$$

$$2 < x < 4$$

At $x = 2$

$$\sum_{n=1}^{\infty} \frac{\ln n \, (-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$$

It is divergent series

At $x=4$

$$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$ It is divergent series

Interval of convergence $(2, 4)$

Radius of convergence $R=1$