ALTERNATING SERIES AND ABSOLUTE CONVERGENCE

The tests for convergence that we have discussed thus far can be applied only to positive term Series. We now consider infinite series that contain both positive and negative terms. One of the simplest, and most useful, series of this type is an alternating series, in which terms are alternately positive and negative.

It is customary to express an alternating series in one of the forms

$$a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n-1} a_n + \cdots$$

or

$$-a_1 + a_2 - a_3 + a_4 - \cdots + (-1)^n a_n + \cdots$$

with $a_k > 0$ for every k. The next theorem provides the main test for convergence of these series. For convenience we consider $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$.

ALTERNATING SERIES TEST

The alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n-1} a_n + \cdots$$

is convergent if following two conditions are satisfied.

(i) $a_k \geq a_{k+1} > 0$ for every k

(ii) $\lim_{n \to \infty} a_n = 0$

Example

Determine whether the alternating series converges or diverges.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{4n^2-3}$$

$$a_n = \frac{2n}{4n^2-3}$$

For proving ① $a_k - a_{k+1} \geqslant 0$

Thus, if $a_n = \frac{2n}{4n^2-3}$, then for every positive integer k,

$$a_k - a_{k+1} = \frac{2k}{4k^2-3} - \frac{2(k+1)}{4(k+1)^2-3}$$

$$= \frac{8k^2+8k+6}{(4k^2-3)(4k^2+8k+1)} \geqslant 0$$

For checking ②

$$\lim_{n\to\infty} \frac{2n}{4n^2-3} = 0$$

Thus the alternating series converges.

DEFINITION

A series $\sum a_n$ is absolutely convergent if the series

$$\sum |a_n| = |a_1| + |a_2| + \cdots + |a_n| + \cdots$$

is convergent.

Note that if $\sum a_n$ is a positive - term series, then $|a_n| = a_n$, and in this case, absolute convergence is the same as convergence.

Example.

Prove that the following alternating series s absolutely convergent.

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots + (-1)^n \frac{1}{n^2} + \cdots$$

Solution.

Taking the absolute value of each term gives us

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} + \cdots$$

which is a convergent p-series. Hence by above definition the alternating series is absolutely convergent.

Example

The alternating harmonic series is

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + (-1)^{n-1} \frac{1}{n} + \dots$$

Show that the series is

(a) convergent (b) not absolutely convergent.

(a) Conditions (i) and (ii) of the alternating series test are satisfied, because

$$\frac{1}{k} > \frac{1}{k+1} \text{ for every } k \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} = 0$$

Hence, the alternating harmonic series is convergent.

(b) To examine the series for absolute convergence, we will apply definition for absolute convergence.

$$\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{1}{n} \right| = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots$$

This series is the divergent harmonic series.

$\Rightarrow$ the alternating harmonic series is not absolutely convergent.

eries that are convergent but not absolutely convergent, uch as the alternating harmonic series above, are given special name,

Definition

A series $\sum a_n$ is conditionally convergent if $\sum a_n$ is convergent and $\sum |a_n|$ is divergent.

Theorem

If a series $\sum a_n$ is absolutely convergent, then $\sum a_n$ is convergent.

Example

Let $\sum a_n$ be the series

$$\frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} - \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} - \frac{1}{2^7} - \frac{1}{2^8} + \cdots$$

where the signs of the terms vary in pairs as indicated and where $|a_n| = 1/2^n$. Determine whether $\sum a_n$ converges or diverges.

Solution.

The series is neither alternating, nor geometric nor positive-term, so none of the earlier tests can be applied. Let us consider the series of absolute values:

$$\sum |a_n| = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots$$

This series is geometric, with $r = \frac{1}{2}$ and since $\frac{1}{2} < 1$, it is convergent. As the given series is absolutely convergent and hence it is convergent.

Example:

Determine whether the following series is convergent or divergent:

$$\sin 1 + \frac{\sin 2}{2^2} + \frac{\sin 3}{3^2} + \cdots + \frac{\sin n}{n^2} + \cdots$$

Solution

The series contains both positive and negative terms but it is not an alternating series, because, for example the first three terms are positive and next three are negative. The series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$$

Since $-1 < \sin n < 1$

$$|\sin n| < 1$$

$$\Rightarrow \frac{|\sin n|}{n^2} < \frac{1}{n^2}$$

the series of absolute values $\sum_{n=1}^{\infty} \left| \sin n/n^2 \right|$ is dominated by the convergent p-series $\sum (1/n^2)$ and hence is convergent

Thus the given series is absolutely convergent and therefore is convergent.

* An arbitrary series may be classified in exactly one of the following ways :

(i) absolutely convergent (ii) conditionally convergent.

(iii) divergent

Ratio & Root test for Absolute Convergence

Let $\sum a_n$ be a series of non zero terms, and suppose

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

, If $L < 1$, the series is absolutely convergent

, If $L > 1$, the series is divergent

, If $L = 1$, apply a different test; the series may be absolutely convergent, conditionally convergent, or divergent.

* We can also state a root test for absolute convergence.

Example.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2+4}{2^n}$$

Using ratio test, we obtain

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)^2+4}{2^{n+1}} \cdot \frac{2^n}{n^2+4} \right|$$

$$= \lim_{n \to \infty} \frac{1}{2} \left(\frac{n^2+2n+5}{n^2+4} \right) = \frac{1}{2} < 1$$

absolutely convergent.

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