THE RATIO AND ROOT TESTS

For the integral test to be applied to a positive term series $\sum a_n$ with $a_n = f(n)$, the terms must be decreasing and we must be able to integrate $f(x)$. These conditions often rule out series that involve factorials and other complicated expressions. In this lecture, we examine two tests that can be used to help determine convergence or divergence when other tests are not applicable.

RATIO TEST

Let $\sum a_n$ be a positive term series, and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$$

(i) If $L < 1$, the series is convergent.

(ii) If $L > 1$ or $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$, the series is divergent.

(iii) If $L = 1$, apply a different test.

Example

Determine the convergence or divergence of:

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}.$$

Applying the ratio test gives us

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}$$

$$= \lim_{n \to \infty} \frac{(n+1)^{n+1}}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{n^n}$$

$$= \lim_{n \to \infty} \frac{\cancel{(n+1)}(n+1)^n}{\cancel{(n+1)}\, n^n}$$

$$= \lim_{n \to \infty} \frac{(n+1)^n}{n^n}$$

$$= \lim_{n \to \infty} \left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

Since $e > 1$, the series diverges.

ILLUSTRATION

Series $\sum a_n$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$	Suggestion.
$\sum_{n=1}^{\infty} \frac{2n^2+3n+4}{5n^5-7n^3+n}$	1	Show converge by using Limit co test with $b_n = 1/n$
$\sum_{n=1}^{\infty} \frac{\ln n}{n}$	1	Show divergence by using the inte test.

ROOT TEST

Let $\sum a_n$ be a positive-term series, and suppose that $\lim_{n\to\infty} (a_n)^{1/n} = L$.

(i) If $L < 1$, the series is convergent.

(ii) If $L > 1$ or $\lim_{n\to\infty} (a_n)^{1/n} = \infty$ the series is divergent.

(iii) If $L = 1$, apply a different test.

xample.

$$\sum_{n=2}^{\infty} \frac{5^{n+1}}{(\ln n)^n}$$

$$\sum_{n=2}^{\infty} \frac{5 \cdot 5^n}{(\ln n)^n}.$$

sing root test:

$$\lim_{n \to \infty} \left(\frac{5 \cdot 5^n}{(\ln n)^n} \right)^{1/n} = \lim_{n \to \infty} \frac{5^{1/n}\, 5}{\ln n}$$

$$= \frac{5}{\infty} = 0 <$$

$\Rightarrow$ the given series converges.