Lecture # 4

Positive term series

In this section, we consider only positive-term series - that is series $\sum a_n$ such that $a_n > 0$ for every n. Although this approach may appear to be very specialized, positive-term series are the foundation for all of our future work with seri

The next theorem shows that to eastablish convergence or divergence of a positive term series, it is sufficient to determine whether the sequence of partial sum $\{S_n\}$ is bounded.

Theorem

If $\sum a_n$ is a positive-term series and if there exists a number M such that

$$S_n = a_1 + a_2 + \cdots + a_n < M$$

for every n, then the series converges and has a sum $S \leq M$. If no such M exists, the series diverges.

Integral test:

If $\sum a_n$ is a series, let $f(n) = a_n$ and let f be a function obtained by replacing n with x. If f is positive-valued, continuous and decreasing for every real number $x \geq 1$, then the series $\sum a_n$

(i) converges if $\int_1^{\infty} f(x)\, dx$ converges

(ii) diverges if $\int_1^{\infty} f(x)\, dx$ diverges

Example 1

Use the integral test to prove that the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

diverges.

Solution

Since $a_n = \frac{1}{n}$, we let $f(n) = \frac{1}{n}$. Replacing n by x gives us $f(x) = 1/x$. Because f is positive valued, continuous, and decreasing for $x \geq 1$, we can apply the integral test

$$\int_1^{\infty} \frac{1}{x}\, dx = \lim_{t \to \infty} \int_1^{t} \frac{1}{x}\, dx = \lim_{t \to \infty} \left| \ln x \right|_1^t$$

$$= \lim_{t \to \infty} [\ln t - \ln 1] = \infty$$

Example 2

Determine whether the infinite series $\sum ne^{-n^2}$ converges or diverges.

Solution

Since $a_n = ne^{-n^2}$, we let $f(n) = ne^{-n^2}$

$$f(x) = xe^{-x^2}$$

If $x \geq 1$, then f is positive valued and continuous. The first derivative may be used to determine whether f is decreasing. Since

$$f'(x) = e^{-x^2} - 2x^2e^{-x^2} = e^{-x^2}(1-2x^2) < 0$$

f is decreasing on $[1, \infty)$. We therefore apply the integral test as follows:

$$\int_1^\infty xe^{-x^2}\,dx = \lim_{t\to\infty} -\frac{1}{2}\int_1^t -(2x)e^{-x^2}\,dx = \lim_{t\to\infty}\left[-\tfrac{1}{2}e^{-x^2}\right]_1^t$$

$$= -\tfrac{1}{2}\lim_{t\to\infty} e^{-t^2} - e^{-1}.$$

$$= -\tfrac{1}{2}\lim_{t\to\infty}\left(\frac{1}{e^{t^2}} - \frac{1}{e}\right) = \tfrac{1}{2e}.$$

Hence the series converges.

An integral test may also be used if the function f satisfies the conditions for every $x \geq k$ for some positive integer k. In this case, we merely replace the integral by $\int_k^\infty f(x)\,dx$. This corresponds to deleting the first $k-1$ terms of the series.

A p-series, or a hyperharmonic series, is a series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

where p is a positive real number.

__Theorem:__

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

(i) converges if $p > 1$

(ii) diverges if $p \leq 1$

Example

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $\qquad$ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

The next theorem allows us to use known Convergent (divergent) series to establish the convergence (divergence) of other series.

Basic Comparison test:

Let $\sum a_n$ and $\sum b_n$ be positive-term series.

(i) If $\sum b_n$ converges and $a_n \leq b_n$ for every positive integer n, then $\sum a_n$ converges.

(ii) If $\sum b_n$ diverges and $a_n \geq b_n$ for every positive integer n, then $\sum a_n$ diverges.

Example 3

Determine whether the series converges or diver

(a) $$\sum_{n=1}^{\infty} \frac{1}{2+5^n}$$

For every $n \geq 1$

$$\frac{1}{2+5^n} < \frac{1}{5^n} = \left(\frac{1}{5}\right)^n$$

Since $\sum (1/5)^n$ is a convergent geometric ser the given series converges, by the Basic Comparison T

(b) $\sum_{n=2}^{\infty} \frac{3}{\sqrt{n}-1}$

The p-series $\sum 1/\sqrt{n}$ diverges, and hence so does the series obtained by disregarding the first term $1/\sqrt{1}$.

If $n \geq 2$, then

$$\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$$

hence

$$\frac{3}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$$

It follows from the basic comparison test that given series diverges.

When we use a basic comparison test, we must first decide on a suitable series $\sum b_n$ and then prove that either $a_n \leq b_n$ or $a_n \geq b_n$ for every n greater than some positive integer k. This proof can be very difficult if a_n is a complicated expression. The following comparison test is often easier to apply, because after deciding on $\sum b_n$, we need only take a limit of the quotient a_n/b_n as $n \to \infty$.

Limit Comparison test.

Let $\sum a_n$ and $\sum b_n$ be positive term series. If there is a positive real number c such that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$$

then either both series diverge or both series converge.

* illustration

a_n	Deleting terms of least magnitude	Choice of b_n
$\dfrac{3n+1}{4n^3+n^2-2}$	$\dfrac{3n}{4n^3} = \dfrac{3}{4n^2}$	$\dfrac{1}{n^2}$
$\dfrac{5}{\sqrt{n^2+2n+7}}$	$\dfrac{5}{\sqrt{n^2}} = \dfrac{5}{n}$	$\dfrac{1}{n}$
$\dfrac{\sqrt[3]{n^2+4}}{6n^2-n-1}$	$\dfrac{\sqrt[3]{n^2}}{6n^2} = \dfrac{n^{2/3}}{6n^2} = \dfrac{1}{6n^{4/3}}$	$\dfrac{1}{n^{4/3}}$

Example 4

Determine whether the series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+1}}$

$$a_n = \frac{1}{\sqrt[3]{n^2+1}}$$

let $b_n = \dfrac{1}{\sqrt[3]{n^2}}$

applying limit comparison test.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{2/3}}{(n^2+1)^{1/3}}$$

$$= \lim_{n\to\infty} \left(\frac{n^2}{n^2+1}\right)^{1/3}$$

$$= \left(\lim_{n\to\infty} \frac{n^2}{n^2+1}\right)^{1/3}$$

$$= 1^{1/3} = 1 > 0$$

Since b_n diverges (by divergent p-series test)

so does $\sum a_n$.

It is important to note that we cannot use $b_n = \dfrac{1}{\sqrt[3]{n^2}}$ with basic comparison test. because $a_n < b_n$ instead of $a_n \geq b_n$.

(b) $$\sum_{n=1}^{\infty} \frac{3n^2+5n}{2^n(n^2+1)}$$

The n^{th} term of the series is

$$a_n = \frac{3n^2+5n}{2^n n^2 + 2^n}$$

$$b_n = \frac{1}{2^n}$$

Applying the limit comparison test

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3n^2+5n}{2^n(n^2+1)} \cdot \frac{2^n}{1}$$

$$= 3 > 0$$

As $\sum b_n = \sum \frac{1}{2^n}$ is a convergent geometric series

the series $\sum a_n$ is also convergent.