Lecture 3

Theorem

Suppose that the sequences $\{a_n\}$ and $\{b_n\}$ converge to limit L_1 and L_2 respectively, and c is a constant then

$$\lim_{n \to \infty} c = c$$

$$\lim_{n \to \infty} c\, a_n = c \lim_{n \to \infty} a_n = c L_1$$

$$\lim_{n \to \infty} (a_n \pm b_n) = L_1 \pm L_2$$

$$\lim_{n \to \infty} (a_n b_n) = L_1 L_2$$

$$\lim_{n \to \infty} \left(\frac{a_n}{b_n}\right) = \frac{L_1}{L_2} \quad , L_2 \neq 0$$

→ that the algebraic techniques used to find limits of the form $\lim_{x \to \infty}$ can also be used for limits of form

$\lim_{n \to \infty}$.

etermine whether the sequence converges or diverges.
it converges, find the limit.

$$a_n = \frac{n}{2n+1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n+1}$$

$$= \frac{1}{2}$$

$\Rightarrow$ sequence $\left\{\frac{n}{2n+1}\right\}$ converges to $\frac{1}{2}$.

$$a_n = \frac{n-2}{n^2+2n+1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n-2}{n^2+2n+1}$$

$$= 0$$

Show that $a_n = \sqrt[n]{n}$ is a convergent sequence.

$$a_n = (n)^{1/n}$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} n^{1/n} \qquad (\infty^0)$$

$$\lim_{n \to \infty} \ln a_n = \lim_{n \to \infty} \{\ln n^{1/n}\}$$

$$\lim_{n \to \infty} \ln a_n = \lim_{n \to \infty} 1/n \ \ln n$$

$$\lim_{n \to \infty} \ln a_n = \lim_{n \to \infty} \frac{\ln n}{n} \qquad \left(\frac{\infty}{\infty}\right)$$

{(we have used the result for solving) $\lim_{x \to \infty} \frac{\ln x}{x}$ $\frac{\infty}{\infty}$

L'Hospital rule.

$\lim_{x \to \infty} \frac{1/x}{1} = 0$ }

$$\Rightarrow \quad = \lim_{n \to \infty} \frac{\ln n}{n} \approx 0$$

$$\lim_{n \to \infty} e^{\ln a_n} = e^0$$

$$\lim_{n \to \infty} a_n = 1$$

$\Rightarrow$ $a_n = \sqrt[n]{n}$ converges to 1.

DEFINITION

A sequence $\{a_n\}$ is bounded from above if $\exists$ a number M such that $a_n \leq M$ $\forall\, n$. The number M is an upper bound for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the least upper bound, for $\{a_n\}$.

$1, 2, 3, \ldots\ldots, n, \ldots$ has no upper bound.

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\ldots, \frac{n}{n+1}, \ldots$ is bounded above by $M=1$.

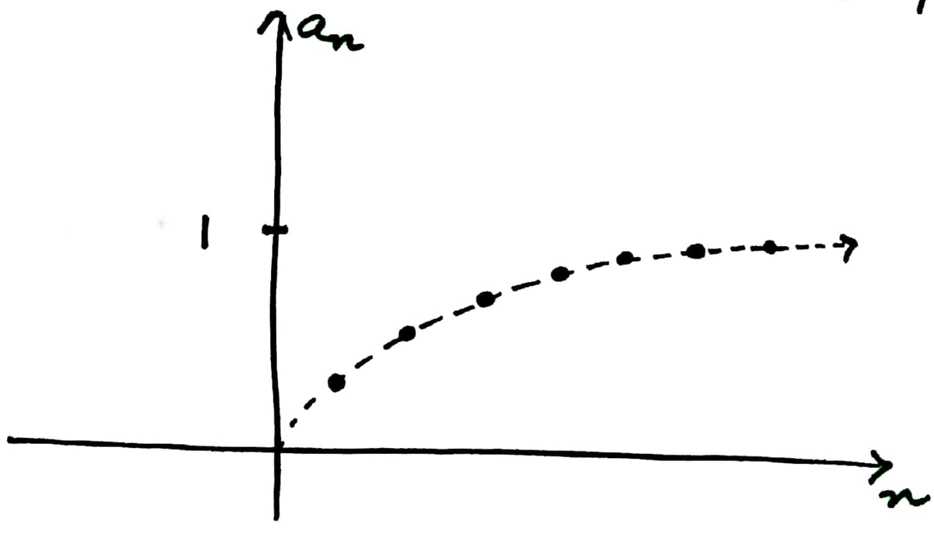
'1' is least upper bound.

as no number less than '1' is an upper bound for the sequence.

Theorem

A non decreasing sequence of real numbers converges iff it is bounded from above. If a non decreasing sequence converges, it converges to it's least upper bound.

Limited Growth model



Determine if the sequence is non decreasing and if it is bounded from above.

$$a_n = \frac{3n+1}{n+1}$$

For non decreasing sequence.

$$\boxed{a_n \leq a_{n+1}}$$

$$\boxed{a_n - a_{n+1} \leq 0}$$

$$\frac{a_n - a_{n+1}}{a_n} \leq 0$$

$$1 - \frac{a_{n+1}}{a_n} \leq 0$$

$$-\frac{a_{n+1}}{a_n} \leq -1$$

$$\boxed{\frac{a_{n+1}}{a_n} \geq 1}$$

$$a_n = \frac{3n+1}{n+1}$$

$$a_{n+1} = \frac{3(n+1)+1}{(n+1)+1}$$

is for non-decreasing sequence

$$a_n - a_{n+1} \leq 0$$

$$\frac{3n+1}{n+1} - \frac{3n+4}{n+2} \leq 0$$

$$(3n+1)(n+2) - (3n+4)(n+1) \leq 0$$

$$(3n^2 + 6n + n + 2) - (3n^2 + 3n + 4n + 4) \leq 0$$

$$2 - 4 \leq 0$$

$-2 \leq 0$ which is true

› Given sequence is non-decreasing.

r bounded from above

$$\lim_{n \to \infty} \frac{3n+1}{n+1} = 3$$

$\Rightarrow$ sequence $\frac{3n+1}{n+1}$ is bounded from above.

$$a_n = \frac{4^{n+1} + 3^n}{4^n}$$

$$= \frac{4^{n+1}}{4^n} + \frac{3^n}{4^n}$$

$$= \frac{4^n 4}{4^n} + (3/4)^n$$

$$a_n = 4 + (3/4)^n$$

We will check whether sequence is non increasing or not.

non increasing sequence $\Rightarrow$ $a_n \geq a_{n+1}$

$$a_n - a_{n+1} \geq 0$$

$$\left[4 + (3/4)^n\right] - \left[4 + (3/4)^{n+1}\right] \geq 0$$

$$(3/4)^n - (3/4)^{n+1} \geq 0$$

$$(3/4)^n \left[1 - 3/4\right] \geq 0$$

$$1/4\,(3/4)^n \geq 0$$

which is true. $\Rightarrow$ Given sequence is :

To check whether it is bounded from below or not

$$\lim_{n \to \infty} \left(4 + \left(\frac{3}{4}\right)^n\right) = 4 + \lim_{n \to \infty} \left(\frac{3}{4}\right)^n$$

$$= 4 + \lim_{n \to \infty} \frac{1^n}{\left(\frac{4}{3}\right)^n}$$

$$= 4$$