## Definition 1

### Infinite series

An infinite series (series) is an expression of the form

$$a_1 + a_2 + \cdots + a_n + \cdots,$$

or in summation notation

$$\sum_{n=1}^{\infty} a_n \quad , \quad \text{or} \quad \sum a_n$$

Each number $a_k$ is the term of the series and $a_n$ is the $n^{th}$ term.

~~Remember there is a confusi~~

" Remember that a series is an expression that represents an infinite sum of numbers. A sequence is a collection of numbers thats are in one-to-one correspondence with the positive integers.

The sequence of partial sums in the next definition is a special type of sequence that we obtain by using the terms of a series.

## Definition 2

(i) The $k^{th}$ partial sum $S_K$ of the series $\sum a_n$ is

$$S_K = a_1 + a_2 + \cdots + a_K$$

(ii) The sequence of partial sums of the series $\sum a_n$ is

$$S_1, S_2, S_3, \ldots, S_n, \ldots$$

where

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_4 = a_1 + a_2 + a_3 + a_4$$

$$\vdots$$

Thus $S_{1000}$ is the sum of first one thousand terms of $\sum a_n$. If the sequence $\{S_n\}$ has a limi[t] $S$, we call $S$ the sum of the series $\sum a_n$.

## Definition 3

A series is convergent if its sequence of partial sums $\{S_n\}$ converges – that is, if

$$\lim_{n \to \infty} S_n = S \qquad S \in \mathbb{R}.$$

The limit $S$ is the sum of series $\sum a_n$ and we write

$$S = a_1 + a_2 + \cdots + a_n + \cdots$$

The series $\sum a_n$ is divergent if $\{S_n\}$ diverges. A divergent series has no sum.

## Example

Suppose we know that the sum of the first $n$ terms of the series $\sum_{n=1}^{\infty} a_n$ is

$$S_n = a_1 + a_2 + \cdots + a_n = \frac{2n}{3n+5}$$

Then the sum of the series is the limit of the sequence $\{S_n\}$:

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{2n}{3n+5} = 2/3$$

In above example we were given an expression for the sum of the first $n$ terms, but it's usually not easy to find such an expression. In next example, however, we look at a famous series for which we can find an explicit formula for $S_n$.

An important example of an infinite series is the geometric series.

$$a + ar + ar^2 + \cdots\cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \quad a \neq 0$$

If $r = 1$, then $S_n = a + a + \cdots + a = na \longrightarrow \pm\infty$.
Since $\lim_{n \to \infty} S_n$ doesn't exist the geometric series diverges in this case.

If $r \neq 1$, we have

$$S_n = a + ar + ar^2 + \cdots\cdots + ar^{n-1}$$

$$rS_n = ar + ar^2 + \cdots\cdots + ar^{n-1} + ar^n$$

Subtracting these equations, we get

$$S_n - rS_n = a - ar^n$$

$$\boxed{S_n = \frac{a(1-r^n)}{1-r}}$$

If $-1 < r < 1$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}.$$

$\Rightarrow$ when $|r| < 1$ the geometric series is convergent and its sum is $\frac{a}{1-r}$.

If $r \leq -1$ or $r > 1$, geometric series again diverges.

## Example

Find the sum of the geometric series.

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots\cdots$$

## Solution

$a = 5 \qquad r = -2/3$

As $|r| < 1$ the series is convergent

and its sum is

$$\frac{a}{1-r} = \frac{5}{1 + 2/3} = \frac{5}{5/3} = 3.$$

## Example

Is the series $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ convergent or divergent?

$$\sum_{n=1}^{\infty} 2^{2n} 3^{1-n} = \sum_{n=1}^{\infty} 2^{3n-n} 3^{1-n}$$

$$= \sum_{n=1}^{\infty} 2^{3n} 2^{-n} 3 \, 3^{-n}$$

$$= \sum_{n=1}^{\infty} (2\cdot 3)^{-n} \, 3 \, 2^{3n}$$

$$= \sum_{n=1}^{\infty} 3 \frac{2^{3n}}{6^{n}} = \sum_{n=1}^{\infty} 3 \frac{8^{n}}{6^{n}} = \sum_{n=1}^{\infty} 3 \frac{\cancel{2} 4}{\cancel{2} 3}$$

$$= \sum_{n=1}^{\infty} 3 \left(\frac{4}{3}\right)^{n}$$

$$= \sum_{n-1=0}^{\infty} 3 \left(\frac{4}{3}\right)^{n-1}$$

$$= \sum_{n=2}^{\infty} 3 \left(\frac{4}{3}\right)^{n-1}$$

As $r > 1$

$\Rightarrow$ divergent.

Find sum of the series $\sum_{n=0}^{\infty} x^n$, when $|x| < 1$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

This is a geometric series with $a = 1$ and $r = x$. Since $|r| = |x| < 1$, it converges and it (4) gives

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

**Example**

Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

This is not a geometric series, so we go back to the definition of a convergent series and compute partial sums.

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$$

We can simplify this expression, if we use the partial fraction decomposition.

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

Thus we have.

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left( \frac{1}{i} - \frac{1}{i+1} \right)$$

$$= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= 1-\frac{1}{n+1}$$

and so $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty}\left(1-\frac{1}{n+1}\right) = 1$

Therefore the given series is convergent and

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$$

Notice that the terms cancel in pairs. This is an example of a telescoping sum: Because of all the cancellations the sum collapses (like a pirates collapsing telescope) into just two terms-

**EXAMPLE ■ 2** Given the series

$$\sum_{n=1}^{\infty}(-1)^{n-1} = 1 + (-1) + 1 + (-1) + \cdots + (-1)^{n-1} + \cdots,$$

**(a)** find $S_1$, $S_2$, $S_3$, $S_4$, $S_5$, and $S_6$

**(b)** find $S_n$

**(c)** show that the series diverges

**SOLUTION**

**(a)** By Definition (8.12),

$$S_1 = 1, \quad S_2 = 0, \quad S_3 = 1, \quad S_4 = 0, \quad S_5 = 1, \quad \text{and} \quad S_6 = 0.$$

**(b)** We can write $S_n$ as follows:

$$S_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

**(c)** Since the sequence of partial sums $\{S_n\}$ oscillates between 1 and 0, it follows that $\lim_{n\to\infty} S_n$ does not exist. Hence, the series diverges.

## Important Results

① If a series $\sum a_n$ is convergent, then $\lim_{n \to \infty} a_n = 0$

$\Rightarrow$ If a series converges, then the limit of it's $n^{th}$ term $a_n$ as $n \to \infty$ is 0. The converse may not be true that if if $\lim_{n \to \infty} a_n = 0 \not\Rightarrow \sum a_n$ is convergent.

The harmonic series is an illustration of a divergent series $\sum a_n$ for which $\lim_{n \to \infty} a_n = 0$.

Harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

(i) If $\lim_{n \to \infty} a_n \neq 0 \Rightarrow \sum a_n$ is divergent.

(ii) If $\lim_{n \to \infty} a_n = 0$ then further investigation is necessary to determine about convergence & divergence

Example

| Series | $n^{th}$ term test | Conclusion |
| --- | --- | --- |
| $\sum_{n=1}^{\infty} \frac{n}{2n+1}$ | $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$ | Diverges |
| $\sum_{n=1}^{\infty} \frac{1}{n^2}$ | $\lim_{n\to\infty} \frac{1}{n^2} = 0$ | Further investigation is required |
| $\sum_{n=1}^{\infty} \frac{e^n}{n}$ | $\lim_{n\to\infty} \frac{e^n}{n} = \infty$ | Diverges. |

(2) For any positive integer k, the series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots \quad \text{and}$$

$$\sum_{n=k+1}^{\infty} a_n = a_{k+1} + a_{k+2} + \cdots$$

either both converge or both diverge.

Example

$$\frac{1}{3.4} + \frac{1}{4.5} + \cdots + \frac{1}{(n+2)(n+3)} + \cdots$$

is a series obtained by deleting the first two terms of convergent telescoping series $\sum \frac{1}{n(n+1)}$

Hence by theorem the given series converges.

(3) If $\sum a_n$ & $\sum b_n$ are convergent series with sums $A$ and $B$, respectively then

(i) $\sum (a_n + b_n)$ Converges and has sum $A+B$

(ii) $\sum c\, a_n$ converges and has sum $cA$.

(4) If $\sum a_n$ is a convergent series and $\sum b_n$ is divergent, then series $\sum (a_n+b_n)$ is divergent.

Example

$$\sum_{n=1}^{\infty} \left( \frac{1}{5^n} + \frac{1}{n} \right)$$