# Probability and Statstics

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# Textbooks

- **Probability & Statistics for Engineers & Scientists**, Ninth Edition, Ronald E. Walpole, Raymond H. Myer
- **Elementary Statistics: Picturing the World,** $6^{th}$ Edition, Ron Larson and Betsy Farber
- **Elementary Statistics**, $13^{th}$ Edition, Mario F. Triola

**Three things cannot be long hidden: the sun, the moon, and the truth**

**Buddha**

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- **Probability & Statistics for Engineers & Scientists**, Ninth Edition, Ronald E. Walpole, Raymond H. Myer

- **Elementary Statistics: Picturing the World, $6^{th}$ Edition, Ron Larson and Betsy Farber**

- **Elementary Statistics**, **$13^{th}$ Edition, Mario F. Triola**

# Reference books

- **Probability and Statistical Inference, Ninth Edition,** Robert V. Hogg, Elliot A. Tanis, Dale L. Zimmerman
- **Probability Demystified**, Allan G. Bluman
- **Schaum's Outline of Probability,** Second Edition, Seymour Lipschutz, Marc Lipson
- **Python for Probability, Statistics, and Machine Learning,** José Unpingco
- **Practical Statistics for Data Scientists: 50 Essential Concepts,** Peter Bruce and Andrew Bruce
- **Think Stats: Probability and Statistics for Programmers,** Allen Downey

# References

Readings for these lecture notes:

**Probability & Statistics for Engineers & Scientists**, Ninth Edition, Ronald E. Walpole, Raymond H. Myer

**Elementary Statistics**, 10th Edition, Mario F. Triola

http://mathworld.wolfram.com/CircularPermutation.html

https://www.zero-factorial.com/whatis.html

http://mathworld.wolfram.com/CircularPermutation.html

http://doubleroot.in/lessons/permutations-combinations/circular-permutations-examples/#.WbeY04-cHIU

http://www.onlinemathlearning.com/combinations.html

**These notes contain material from the above resources.**

# Counting Sample Points

- In many cases, we shall be able to solve a probability problem by counting the number of points in the **sample space without actually listing each element**.

- The **fundamental principle of counting**, often referred to as the **multiplication rule**, is stated in Rule 2.1.

**Rule 2.1:** If an operation can be performed in $n_1$ **ways**, and if for **each of these ways a second operation** can be performed in $n_2$ **ways**, then **the two operations** can be performed together in $n_1 n_2$ **ways**.

- **Example :** How many **sample points** are there in the **sample space** when a **pair of dice** is thrown once?

**Solution :**

- The first die gives us, **$n_1 = 6$ ways**.

- For each of these **6 ways**, the **second die** gives us, **$n_2 = 6$ ways**.

- Therefore, the pair of dice give us **$n_1 n_2 = (6)(6) = 36$ possible ways**.

**Example :** If a **22-member club** needs to elect a **chair** and a **treasurer**, how many **different ways** can these **two** to be elected?

**Solution :**

- For the **chair position**, we have $\mathbf{n_1}$ = **22 ways**

- For the **treasurer** position, for each of those **21 possibilities,** we have $\mathbf{n_2}$ = **21ways**

- Total number of ways = $\mathbf{n_1 \times n_2 = 22 \times 21 = 462}$

- **Rule 2.2:** If an operation can be performed in $n_1$ ways, and **if for each of these a second operation** can be performed in $n_2$ **ways**, and for each of the **first two a third operation** can be performed in $n_3$ **ways**, and so forth, then the sequence of $k$ operations can be performed in $n_1 n_2 \cdots n_k$ **ways**.

- **Example:** Sam is going to assemble a computer by himself. He has the choice of chips from **two brands**, a **hard drive** from **four**, **memory** from **three**, and an **accessory bundle** from **five** local stores. How many **different ways** can Sam order the parts?

**Solution :**

$n_1$ = 2 **(No of brands)**

$n_2$ = 4 **(No of hard drives)**

$n_3$ = 3 **(No of memory sticks)**

$n_4$ = 5 **(No of accessory bundles)**

- Total number of ways = $n_1 \times n_2 \times n_3 \times n_4 = 2 \times 4 \times 3 \times 5 = 120$

# Permutation

**Permutation:** A **permutation** is an arrangement of all or part of a set of objects.

OR

An arrangement of a set of ***n* objects** in a **given order** is called a ***permutation*** of the objects (taken all at a time).

**Example:** Consider the three **letters *a, b,*** and ***c***.

The possible **permutations** are ***abc, acb, bac, bca, cab,*** and ***cba***.

# Permutation

- **Definition** For any **non-negative integer** $n$, $n!$, called **"$n$ factorial,"** is defined as $n! = n(n-1) \cdots (2)(1)$, with special case $0! = 1$.

- **Theorem 2.1:** The number of permutations of **$n$ objects** is $n!$.

- **Example** The number of permutations of the **four letters** $a$, $b$, $c$, and $d$ will be $4! = 24$.

# Why 0! one?

The idea of the factorial (in simple terms) is used to compute the number of permutations (combinations) of arranging a set of **n numbers**.

| n | Number of permutations (n!) | Visual examples |
|---|---|---|
| 1 | 1 | {1} |
| 2 | 2 | {1, 2}, {2, 1} |
| 3 | 6 | {1, 2, 3}, {1, 3, 2}, {2, 1, 3}, {2, 3, 1}, {3, 1, 2}, {3, 2, 1} |
| ⋮ | ⋮ | ⋮ |
| 0 | 1 | {} |

# Why 0! one?

$$n! = n \times (n - 1)!$$

$$\Rightarrow (n - 1)! = \frac{n!}{n}$$

Substitute 1, we get

$$\Rightarrow (1 - 1)! = \frac{1!}{1}$$

$$\Rightarrow 0! = 1$$

# Permutations Rule (When items are all different)

- **Theorem 2.2:** The number of permutations of **n distinct objects** taken **r** at a **time** is $_nP_r = \frac{n!}{(n-r)!}$, where $r \leq n$

# Permutations Rule (When items are all different)

**1.** There are ***n different* items available**. (This rule **does not apply** if **some of the items are identical** to others.)

objects will not repeat in a single combination.

**2.** We **select *r*** of the ***n* items (without replacement).**

like: aac.bbc will not be there in permutaions.

**3.** We consider **rearrangements** of the **same items** to be **different sequences**. (The permutation of ***ABC*** is different from ***CBA*** and is counted separately)

# Permutations Rule (When items are all different)

- If the preceding **requirements are satisfied**, the number of **permutations (or sequences)** of ***r* items** selected from ***n* different** available items **(without replacement)** is $_nP_r = \frac{n!}{(n-r)!}$, where $r \leq n$.

**Example :** In one year, **three awards (research, teaching, and service)** will be given to a class of **25** graduate students in a statistics department. If each student can receive at **most one award**, how many possible selections are there?

**Solution :** Since the awards are **distinguishable**, it is a permutation problem. The total number of sample points is

$$ {}_{25}P_3 = \frac{25!}{(25-3)!} = \frac{25!}{22!} = (25)(24)(23) = 13,\ 800. $$

**Example Clinical Trial of New Drug** When testing a new drug, Phase I involves **only 8 volunteers**, and the objective is to assess the drug's safety. To be very cautious, you plan to treat the **8 subjects in sequence**, so that any particularly adverse effect can allow for stopping the treatments before any other subjects are treated. If **10 volunteers** are available and **8 of them are to be selected**, how **many different sequences of 8 subjects** are possible?

**Solution** We have ***n* = 10 different subjects** available, and we plan to **select *r* = 8** of them **without replacement**. The number of different sequences of arrangements is found as shown

$$_{10}P_8 = \frac{10!}{(10-8)!} = \frac{10!}{2!} = 1{,}814{,}400$$

**Example** Find the number of ways of forming four-digit codes in which no digit is repeated.

In permutation, no element is repeated like there is no (aab or bcb).

**Solution**

To form a four-digit code with no repeating digits, you need to select 4 digits from a group of 10,

so **n = 10** and **r = 4**.

$$_{10}P_4 = \frac{10!}{(10-4)!} = \frac{10!}{6!} = 5040$$

So, there are **5040** possible four-digit codes that do not have repeating digits.

**Example** How many 4-digit numbers are there with no digit repeated?

**Solution**

Total number of ways = $9 \times 9 \times 8 \times 7$

$= 4536$

**Example Forty-three** race cars started the **2013 Daytona 500**. How many ways can the cars finish first, second, and third?

**Solution**

You need to select three race cars from a group of 43, so n = 43 and r = 3. Because the order is important, the number of ways the cars can finish first, second, and third is

$$_{43}P_3 = \frac{43!}{(43-3)!} = \frac{10!}{6!} = 74{,}046.$$

**Theorem 2.3:** The number of **permutations** of ***n* objects** arranged in a circle is **($n$ – 1)!.**

**Example** In how many ways can **6 people** be seated at a **round table**?

**Solution**

**Here n = 6** (total number of people)

The total number of ways = $(6 – 1)! = 120$

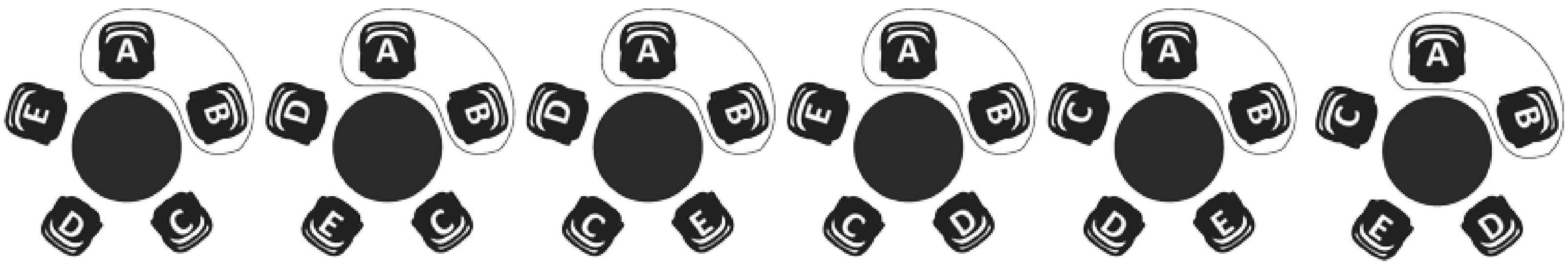
**Example** Find the number of ways in which **5 people A, B, C, D, E** can be seated at **a round table**, such that

a. **A and B** must always sit together.

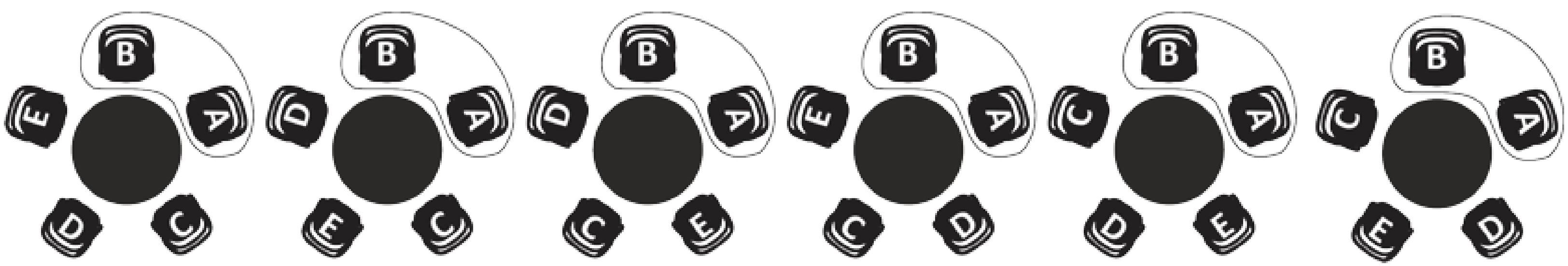
b. **C and D** must not sit together.

Find seated together and subtract from total.

- **Solution a.** If we wish to seat **A and B** together in all arrangements, we can consider these **two as one unit**, along with 3 **others**. So effectively we've to arrange **4 people** in a circle. The number of ways = **(4 – 1)! = 6**



But in each of these arrangements, **A** and **B** can themselves interchange places in **2 ways**.



Therefore, the total number of ways will be $\mathbf{6 \times 2 = 12}$.

# 5 people A, B, C, D, E

- **b.** The total number of ways will be **(5 – 1)!** or **24**.

- Similar to **a.** above, the number of cases in which **C** and **D** are seated together, will be **12**.

- Therefore the required number of ways = **24 – 12 = 12.**

# Permutations when repetition is allowed

- So far we have considered **permutations** of **distinct objects**. That is, all the objects were completely different or distinguishable.

- Obviously, if the letters ***b* and *c*** are both equal to ***x***, then the **6 permutations** of the **letters *a*, *b*, and *c*** become ***axx*, *axx***, ***xax*, *xax*, *xxa*,** and ***xxa***, of which only **3** are distinct.

- Therefore, with **3 letters**, **2 being the same**, we have **3!/2! = 3** distinct permutations.

# Permutations Rule (When some items are identical to others)

- **Theorem 2.4:** The number of **distinct permutations** of ***n*** **things** of which $n_1$ are of one kind, $n_2$ of a second kind, . . . , $n_k$ of a $k^{th}$ kind is

$$\frac{n!}{n_1!n_2! \cdots nk!}$$

*where* $n_1 + n_2 + n_3 + \ldots + n_k = n$.

# Permutations Rule (When some items are identical to others)

**Requirements**

**1.** There are ***n* items available**, and some items are **identical** to others.

**2.** We select all of the ***n* items (without replacement).**

**3.** We consider **rearrangements of distinct items** to be **different sequences**.

If the preceding requirements are satisfied, and if there are $n_1$ **alike**, $n_2$ **alike**, . . . , $n_k$ **alike**, the number

$$\frac{n!}{n_1!n_2! \cdot \cdot \cdot nk!}$$

**Example** A building contractor is planning to develop a subdivision. The subdivision is to consist of **6 one-story houses**, **4 two-story houses**, and **2 split-level houses**. In how many **distinguishable** ways can the houses be arranged?

**Solution**

The total number of arrangements = $\frac{12!}{6!\,4!\,2!}$

= 13,860

distinguishable ways

**Example** Calculate the number of distinguishable permutations of the letters **AAAABBC**.

**Solution**

The total number of arrangements = $\dfrac{7!}{4!\,2!\,1!}$

= 105

distinguishable ways

❑**Example** In a college football training session, the defensive coordinator needs to have **10 players** standing in a row. Among these **10 players**, there are **1 freshman**, **2 sophomores**, **4 juniors**, and **3 seniors**. How many **different ways** can they be arranged in a row if only their **class level** will be distinguished?

- **Solution**

**$n = 10$** (Total number of players)

**$n_1 = 1$** (Total number of freshman)

**$n_2 = 2$** (Total number of sophomores)

**$n_3 = 4$** (Total number of juniors)

**$n_4 = 3$** (Total number of seniors)

The total number of arrangements $= \dfrac{10!}{1!\,2!\,4!\,3!}$

$= 12,600.$

**Theorem 2.5:** The number of ways of **partitioning** a set of ***n*** **objects** into ***r*** **cells** with **$n_1$ elements** in the first cell, **$n_2$ elements** in the second, and so forth, is

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\,n_2! \cdots n_r!}$$

where $n_1 + n_2 + \cdots + n_r = n$

**Example :** In how many ways can **7 graduate** students be assigned to **1 triple** and **2 double** hotel rooms during a conference?

***Solution :***

$$\frac{n!}{n_1!n_2! \cdot \cdot \cdot nk_!}$$

**Here n = 7**

$n_1 = 3$

$n_2 = 2$

$n_3 = 2$

The total number of possible partitions would be

$$\frac{7!}{3!\ 2!\ 2!} = 210$$

**Theorem 2.6:** The number of combinations of ***n* distinct objects** taken ***r*** at a time is

$$ {}_nC_r = \frac{n!}{r!(n-r)!}, \text{ where } r \leq n. $$

# Combinations Rule

**Requirements**

**1.** There are ***n different*** items available.

**2.** We select ***r*** of the ***n* items** (**without replacement**).

**3.** We consider rearrangements of the **same items** to **be the same**. (The combination ***ABC*** is the same as ***CBA***.)

If the preceding requirements are satisfied, the number of **combinations** of ***r* items** selected from ***n* different items** is

$$_{n}C_{r} = \frac{n!}{r!(n-r)!}$$

# Combination vs. Permutations

- The **2-permutations** of the letters **A, B, C, and D** are*:*

  **AB, AC, AD, BA, BC, BD, CA, CB, CD, DA, DB, DC.**

- The combinations of two out of these four letters are:

  **AB, AC, AD, BC, BD, CD.**

(Since the elements of a combination are unordered, **BA** is not viewed as being distinct from **AB**.)

- **Example:** In a lottery, each ticket has **5 one-digit numbers 0-9** on it.

a) You win if your ticket has the **digits in any order.** What are your changes of winning?

b) You would win only if your ticket has the digits in **the required order.** What are your chances of winning?

Solution:

There are 10 digits to be taken 5 at a time.

a) The number of ways of **selecting 5** tickets **from 10** is

$$ {}_{10}C_5 = \frac{10!}{5!(10-5)!} = \frac{10 \times 9 \times 8 \times 7 \times 6}{5!} = 252 $$

∴The chances of winning are **1 out of 252 (0.0040 or 0.3968 %**

- b) Since the order matters, we should use permutation instead of combination.

$$_{10}P_5 = \frac{10!}{(10-5)!} = \frac{10!}{5!} = 10 \times 9 \times 8 \times 7 \times 6 = \mathbf{30240}$$

∴ The chances of winning are 1 out of 30240 (**0.0000330668** or **0.0033 %**)