Probability and Statstics

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Textbook

- **Probability & Statistics for Engineers & Scientists**, Ninth Edition, Ronald E. Walpole, Raymond H. Myer

Reference books

- **Probability Demystified**, Allan G. Bluman
- **Schaum's Outline of Probability and Statistics**
- **MATLAB Primer**, Seventh Edition
- **MATLAB Demystified** by McMahon, David

References

Readings for these lecture notes:

- **Schaum's Outline of Probability, Second Edition (Schaum's Outlines)** by by Seymour Lipschutz, Marc Lipson
- **Probability & Statistics for Engineers & Scientists**, Ninth Edition, Ronald E. Walpole, Raymond H. Myer
- https://wordwatchtowers.wordpress.com/2009/12/21/underestimate-or-overestimate/
- Elementary Statistics, Tenth Edition, Mario F. Triola
- http://www.sjsu.edu/faculty/gerstman/

These notes contain material from the above resources.

Populations and Samples

- The totality of observations with which we are concerned, whether their number be finite or infinite, constitutes what we call a **population**.

- A **population** consists of the totality of the observations with which we are concerned.

- A **sample** is a subset of a population.

Bias

Any **sampling procedure** that produces inferences that consistently **overestimate** or consistently **underestimate** some characteristic of the population is said to be **biased**.

To eliminate any **possibility of bias** in the sampling procedure, it is desirable to choose a **random sample** in the sense that the observations are made **independently** and at **random**.

Overestimate vs. Underestimate

- **Overestimate** means ‘to form too high an estimate of’

- **Underestimate** means to estimate that something is smaller or less important than it actually is

Parameter vs. Statistic

- **Statistical inference** involves drawing conclusions about **characteristics of populations**.

- Among these characteristics are constants which are called **population parameters**. Two important parameters are the **population mean** and the **population variance**.

- Any function of the random variables constituting a **random sample** is called a **statistic**.

Sampling Distribution [1]

- The probability distribution of a statistic is called a **sampling distribution**.

- The field of **statistical inference** is basically concerned with **generalizations** and **predictions**.

- For example, we might claim, based on the opinions of several people interviewed on the street, that in a forthcoming election **60% of the eligible voters** in the city of Detroit favor a certain candidate. In this case, **we are dealing with a random sample of opinions from a very large finite population**.

Sampling Distribution [2]

- As a second illustration we might state that the **average cost** to build a residence in Charleston, South Carolina, is between **$330,000 and $335,000**, based on the **estimates of 3 contractors** selected at random from the 30 now building in this city. The population being sampled here is **again finite** but **very small**.

Sampling Distribution [3]

- Finally, let us consider a **soft-drink machine** designed to dispense, on average, **240 milliliters** per drink. A company official who computes the mean of **40** drinks obtains $\bar{x} = 236$ milliliters and, on the basis of this value, decides that the machine is still dispensing drinks with an average content of $\mu = 240$ milliliters. The **40** drinks represent a sample from the **infinite population** of possible drinks that will be dispensed by this machine.

Inference about the Population from Sample Information [1]

- In each of the examples above, we computed a **statistic** from a **sample selected** from the **population**, and from this **statistic** we made various statements concerning the values of **population parameters** that **may or may not be true**.
- The company official made the decision that the soft-drink machine dispenses drinks with an average content of **240 milliliters**, even though the sample mean was **236 milliliters**, because he knows from sampling theory that, if **$\mu = 240$** milliliters, such a sample value could easily occur.

Inference about the Population from Sample Information [2]

- In fact, if he ran similar tests, say every hour, he would expect the values of the statistic **$\bar{x}$ to fluctuate** above and below **μ = 240** milliliters. Only when the value of $\bar{x}$ is substantially different from 240 milliliters will the company official initiate action to adjust the machine.

- Since a statistic is a **random variable** that depends only on the observed sample, it must have a **probability distribution**.

- The probability distribution of a **statistic** is called a **sampling distribution**.

Sampling Distribution of a Statistic

The **sampling distribution of a statistic** (such as a **sample proportion** or **sample mean**) is the distribution of all values of the statistic when **all possible samples** of the **same size *n*** are taken from the same population.

Sampling Distribution of the Mean

The **sampling distribution of the mean** is the distribution of **sample means**, with **all samples** having the **same sample size *n*** taken from the **same population**.

Sampling Distribution of the Proportion

- The **sampling distribution of the proportion** is the distribution of **sample proportions**, with all samples having the **same sample size *n*** taken from the **same population**.

The Central Limit Theorem

Central Limit Theorem: If $\bar{X}$ is the mean of a random **sample of size *n*** taken from a **population** with **mean *μ*** and **finite variance *σ*²**, then the limiting form of

the distribution of $Z = \dfrac{\bar{X} - \mu}{\sigma / \sqrt{n}}$,

as $n \rightarrow \infty$, is the **standard normal distribution** $n(z; 0, 1)$.

The Central Limit Theorem

- The normal approximation for $\bar{X}$ will generally be good if ***n* ≥ 30**, provided the population distribution is not **terribly skewed**.
- If ***n* < 30**, the approximation is good only if the population is **not too different** from a **normal distribution**.
- As stated above, if the **population is known to be normal**, the **sampling distribution of $\bar{X}$** will follow a **normal distribution** exactly, no matter how **small the size of the samples**.

The Central Limit Theorem

- The sample size ***n* = 30** is a guideline to use for the **Central Limit Theorem**.

- However, as the statement of the theorem implies, the presumption of normality on the distribution **of** $\bar{X}$ becomes more accurate **as *n* grows larger**.

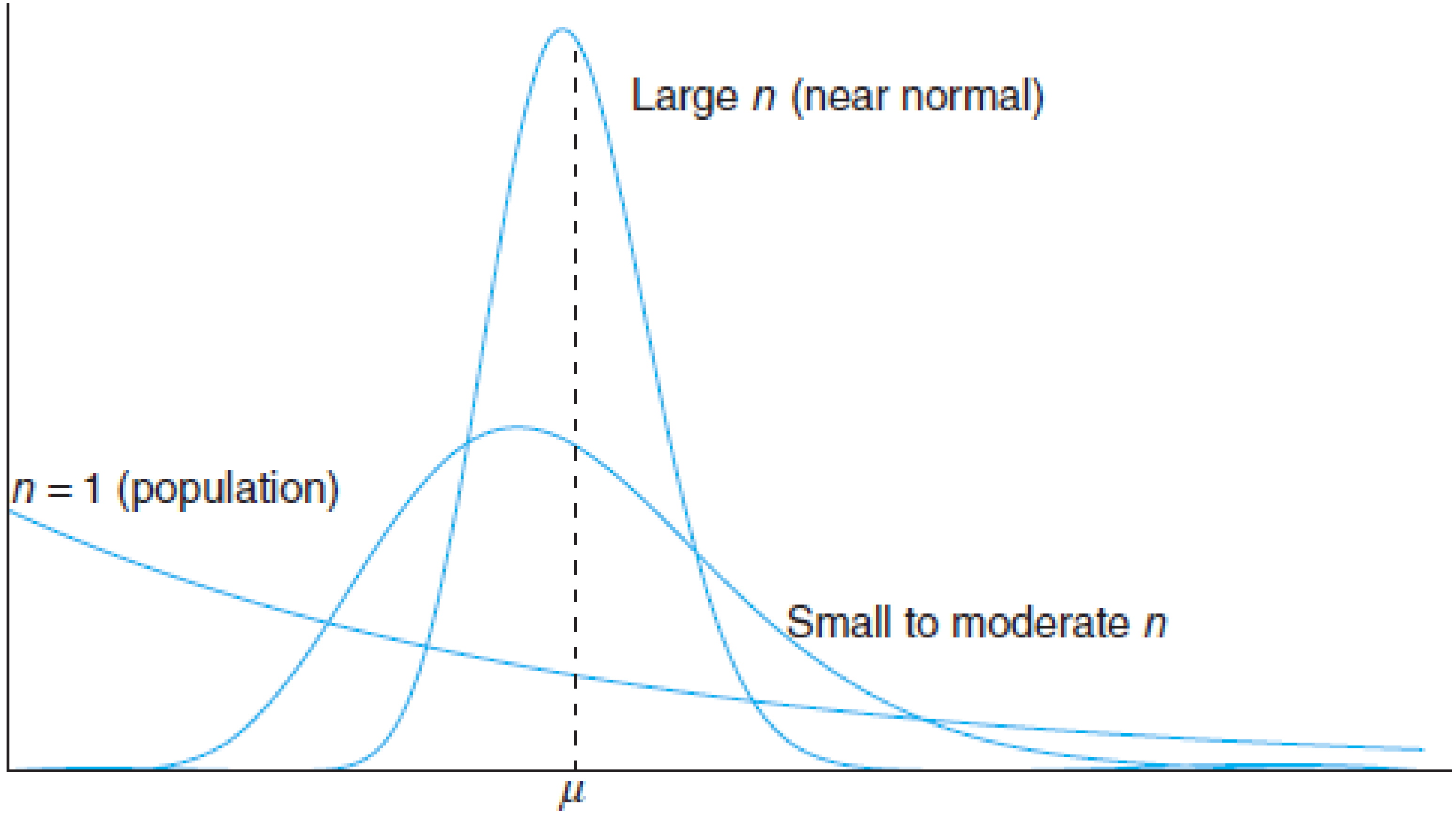


Illustration of the Central Limit Theorem (distribution of $\bar{X}$ for n = 1, moderate n, and large n).

The Central Limit Theorem

When selecting a **simple random sample** from a **population** with **mean** and **standard deviation**, it is essential to know these principles:

1. If ***n* > 30**, then the **sample means** have a distribution that can be **approximated by a normal distribution** with **mean μ** and **standard deviation** $\boldsymbol{\sigma / \sqrt{n}}$ (This guideline is commonly used, regardless of the distribution of the original population.)

2. If ***n* ≤ 30** and the **original population** has a **normal distribution**, then the **sample means** have a normal distribution with mean **μ** and standard deviation $\boldsymbol{\sigma / \sqrt{n}}$

The Central Limit Theorem

3. If ***n* ≤ 30** but the **original population** does not have **a normal distribution**, then the methods of this section do not apply.

- Try to keep this big picture in mind: As we sample from a population, we want to know the behavior of the sample means.

- The ***central limit theorem*** tells us that if the **sample size is large enough**, the distribution of **sample means** can be **approximated by a *normal distribution***, even if the **original population is not** normally distributed.

The Central Limit Theorem and the Sampling Distribution of $\overline{x}$

Given:

1. The **random variable *x*** has a **distribution (which may or may not be normal)** with mean **μ** and standard deviation **σ**.

2. **Simple random samples** all of the **same size *n* are selected from the population**. (The samples are selected so that all possible samples of size *n* have the same chance of being selected.)

The Central Limit Theorem and the Sampling Distribution of $\bar{x}$

Conclusions:

1. The distribution of **sample means** $\bar{x}$ will, as the sample size increases, approach a ***normal* distribution**.
2. The **mean of all sample means** is the population mean (That is, the normal distribution from Conclusion 1 has mean **μ**).
3. The **standard deviation** of all **sample means** is **σ** $/\sqrt{n}$ (That is, the normal distribution from Conclusion 1 has standard deviation **σ** $/\sqrt{n}$)

The Central Limit Theorem and the Sampling Distribution of $\bar{x}$

Practical Rules Commonly Used

If the original population is not itself normally distributed, here is a common guideline:

1. For samples of **size *n* greater than 30**, the distribution of the **sample means** can be **approximated reasonably well** by a **normal distribution**.(There are **exceptions**, such as **populations** with very **nonnormal** distributions requiring sample **sizes larger than 30**, but such exceptions **are relatively rare**.) **The approximation gets better as the sample size *n* becomes larger**.

The Central Limit Theorem and the Sampling Distribution of $\overline{x}$

Practical Rules Commonly Used

2. If the **original population** is itself **normally distributed**, then the **sample means** will be **normally distributed** for ***any* sample size *n*** (**not just the values of *n* larger than 30**).

Notation for Sampling Distribution of $\bar{x}$

The **central limit theorem** involves **two different distributions**: the distribution of the **original population** and the distribution of the s**ample means**.

- We use the symbols and to denote the mean **μ** and standard deviation **σ** of the original population, but we use the following **new notation** for the **mean** and **standard deviation** of the **distribution of sample means.**

$$\mu_{\bar{X}} = \mu \text{ and}$$

$$\sigma_{\bar{X}} = \sigma / \sqrt{n}$$

Example Water Taxi Safety

We noted that some passengers died when a water taxi sank in Baltimore's Inner Harbor. **Men are typically heaver than women and children, so when loading a water taxi, let's assume a worst-case scenario in which all passengers are men**. Based on data from the National Health and Nutrition Examination Survey, assume that weights of men are normally distributed with a **mean** of **172 lb** and a **standard deviation of 29** lb.

Example Water Taxi Safety cont.

a. Find the probability that if an individual man is randomly selected, his weight will be **greater than 175 lb**.

b. Find the probability that **20** randomly selected men will have a **mean that is greater than 175 lb** (so that their total weight exceeds the safe capacity of 3500 lb).

μ = 172 and **σ = 29**

a) $Z = \frac{x - \mu}{\sigma} = Z = \frac{175 - 172}{29} =$ **0.10**

P (X > 175) = P (Z > 0.10) = 1 - P (Z < 0.10) = 1 – 0. 5398

P (X > 175) = **0.4602 ans**

b) $Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{175 - 172}{29/\sqrt{20}} = \frac{3}{6.4846} =$ **0.46**

P ($\bar{x}$ > 175) = P (z > 0.46) = 1- P ($z \le$ 0.46)

= 1 – 0.6772

P ($\bar{x}$ > 175) = **0.3228 ans**

Basics of inference[1]

- **Statistical inference** is the act of **generalizing** from a **sample to a population** with calculated degree of certainty. The two forms of statistical inference are **estimation** and **hypothesis testing**.

- A statistical **population** represents the set of all possible values for a variable. In practice, we do not study the entire population.

Basics of inference[2]

- Instead, we use data in a **sample** to shed light on the wider population.

- The term **parameter** is used to refer to a numerical **characteristic** of a **population**. Examples of parameters include the **population mean (μ)** and the population **standard deviation (σ).**

Basics of inference[3]

- A numerical characteristic of the **sample** is a statistic.

- We introduce a particular type of **statistic** called an **estimate**. The **sample mean** $\bar{x}$ is the natural estimator of **population mean μ**. Sample standard **deviation s** is the natural estimator of population **standard deviation σ**.

- The parameter is a fixed **constant**. In contrast, the **estimator varies** from **sample to sample**.

Basics of inference[4]

	Parameter	Estimators
Source	Population	Sample
Value known?	No	Yes (calculate)
Notation	Greek (μ)	Roman ($\overline{x}$)
Vary from sample to sample	No	Yes
Error-prone	No	Yes

Sampling distribution of a mean (SDM)

- If we had the opportunity to take repeated samples from the same population, samples means ($\bar{x}$s) would vary from sample to sample and form a **sampling distribution means (SDM)**.

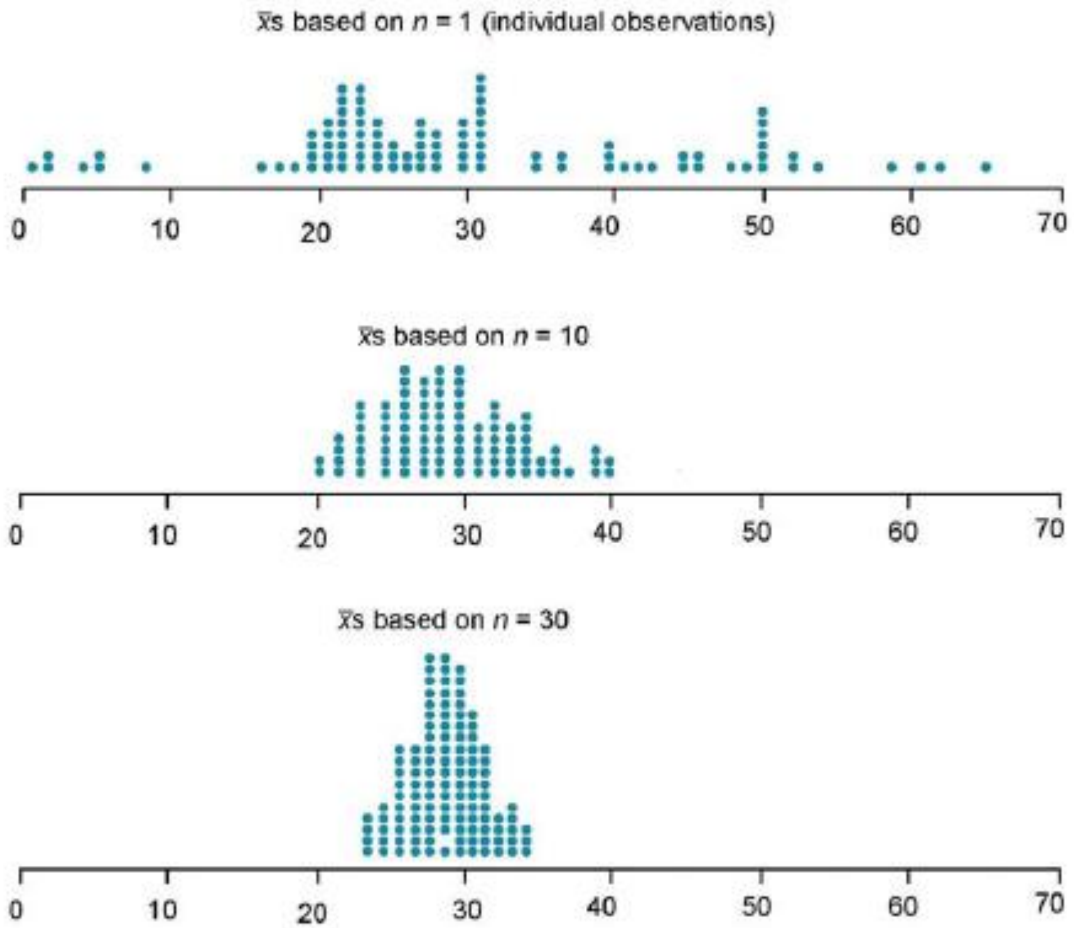
- Let's run a simulation experiment. Our simulation will be based on sampling a population of **N = 600** age values. The population mean **age $\mu = 29.5$.** The population standard deviation **$\sigma = 13.6$**

Sampling distribution of a mean (SDM)

- Imagine taking repeated samples, each of ***n* = 10**. Do this **100 times**.

- In one experiment, it just so happened that the first $\bar{x}$ was **36.4**, the second $\bar{x}$ was **30.2**, and the third $\bar{x}$ was **24.6**

Sampling distribution of $\bar{x}$



Statistical Inference

- **Statistical inference** consists of those methods by which one makes **inferences or generalizations** about a **population**.

- The trend today is to distinguish between the **classical method** of estimating a **population parameter**, whereby inferences are based strictly on information obtained from a **random sample** selected from the **population**.

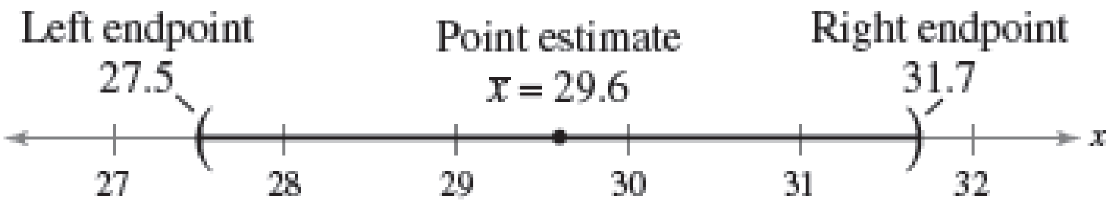
- **Statistical inference** may be divided into two major areas: **estimation** and **tests of hypotheses**.

Point Estimate

A **point estimate** is a single value estimate for a population parameter. The most unbiased point estimate of the population mean μ is the sample mean $\bar{x}$.

Interval Estimate

An **interval estimate** is an interval, or range of values, used to estimate a population parameter.



Interval Estimate

To form an **interval estimate**, use the **point estimate** as the center of the interval, and then add and subtract a **margin of error**.

Level of Confidence

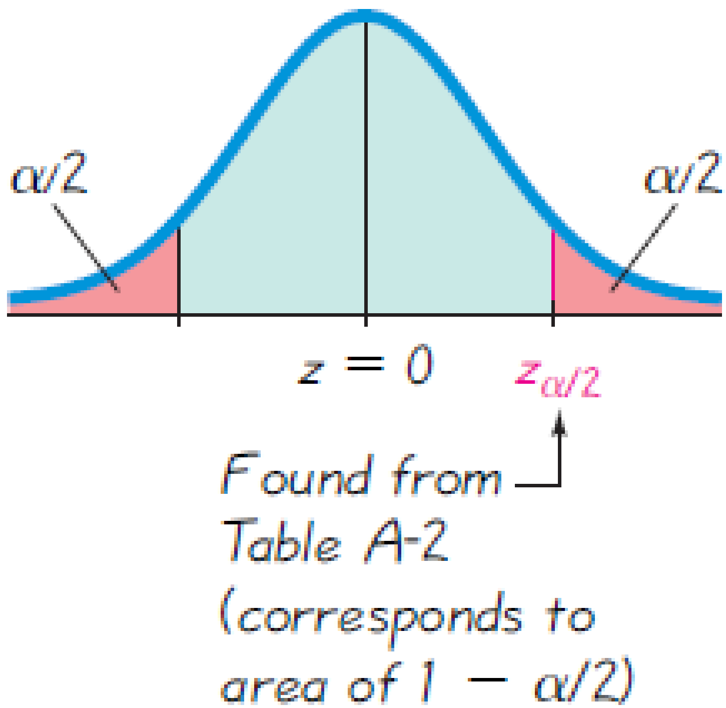
The **level of confidence** ***c* or *1 - α*** is the probability that the **interval estimate** contains the population parameter, assuming that the estimation process is **repeated a large number of times**.

Critical Values [1]

- A **critical value** is the number on the borderline separating sample statistics that **are likely** to occur from those that are **unlikely to occur.**

- The number $z_{\alpha/2}$ is a **critical value** that is a ***z* score** with the property that it separates an area of $\alpha/2$ in the right tail of the standard normal distribution.

Critical Values[2]



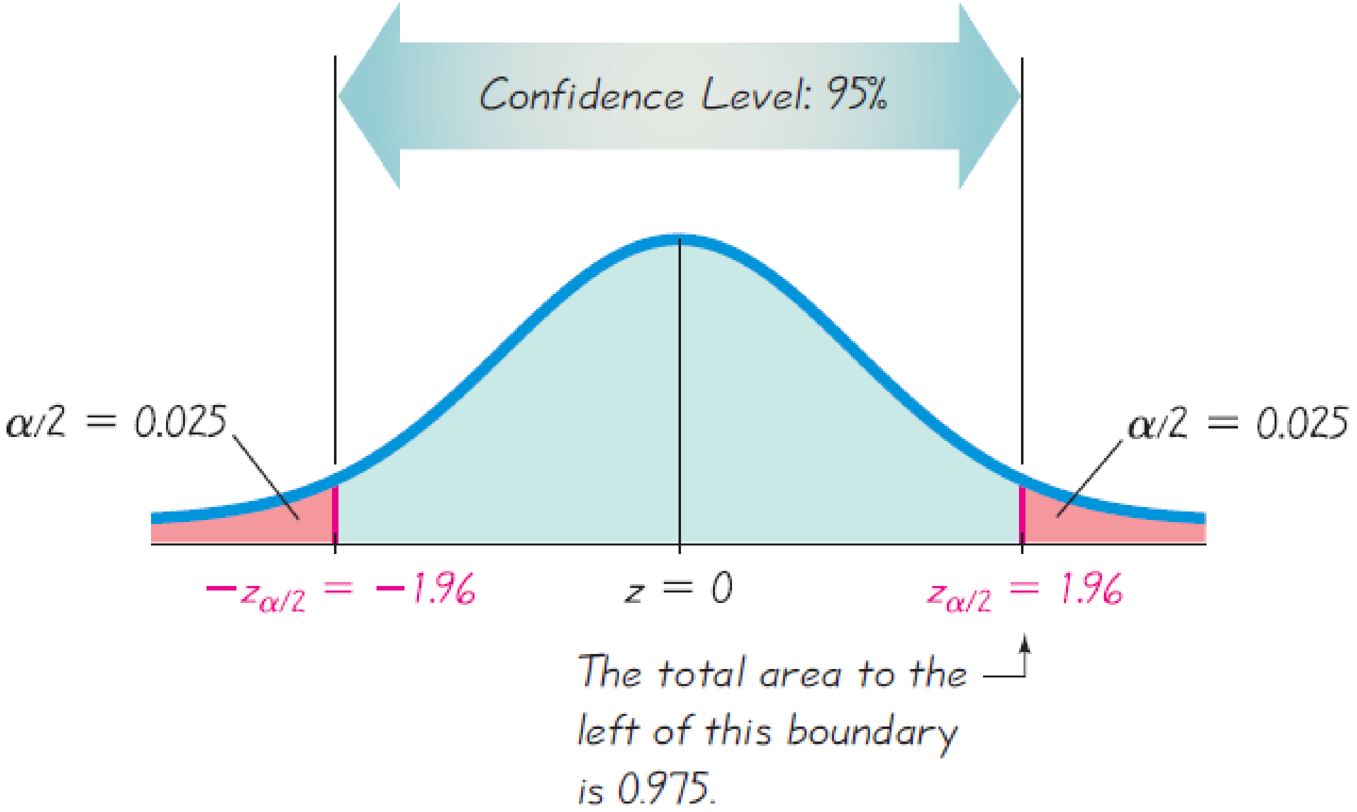
Critical Value $z_{\alpha/2}$ in the Standard Normal Distribution

Example Finding a Critical Value Find the critical value $z_{\alpha/2}$ corresponding to a 95% confidence level.

Solution

When α = 0.05

$z_{\alpha/2} = z_{0.0250} = 1.96$ $\quad \because 1 - \alpha/2 = 1 - 0.250 = 0.9750$



Confidence Level $(1 - \alpha)$	α	Critical values, $z_{\alpha/2}$
90%	0.10	1.645
95%	0.05	1.96
99%	0.01	2.575

Margin of Error

Given a level of confidence ***$1 - \alpha$***, the **margin of error *E*** (sometimes also called the **maximum error of estimate** or error tolerance) is the **greatest possible distance** between the **point estimate** and the **value of the parameter it is estimating**

$$E = Z_{\alpha}\frac{\sigma}{\sqrt{n}}$$ Margin of error for **μ** (***σ*** known)

when these conditions are met.

1. The sample is **random**.

2. At least one of the following is true: The population is **normally distributed** or ***n*** $\geq$**30**.

Estimation

A candidate for public office may wish to **estimate** the true **proportion** of voters favoring him by obtaining opinions from a **random sample of 100** eligible voters.

The fraction of voters in the sample favoring the candidate could be used as an estimate of the **true proportion in the population** of voters.

A knowledge of the **sampling distribution** of a **proportion** enables one to **establish the degree of accuracy** of such an **estimate**. This problem falls in the **area of estimation**.

Tests of Hypotheses

A floor wax is more scuff-resistant than brand *B* floor wax. He or she might hypothesize that **brand *A* is better than brand** *B* and, after proper testing, accept or reject this hypothesis.

In this example, we do not attempt to **estimate a parameter**, but instead we try to arrive at a correct decision about a **prestated hypothesis**.

Once again we are dependent on **sampling theory** and the use of data to provide us with some measure of accuracy for our decision.

Point Estimate [1]

A **point estimate** of some population parameter $\boldsymbol{\theta}$ is a single value of a statistic $\widehat{\boldsymbol{\theta}}$.

For example, the value $\bar{\mathbf{x}}$ of the statistic $\bar{\mathbf{X}}$, computed from a sample of size n, is a point estimate of the population parameter μ. Similarly, $\hat{\mathbf{p}}$ **= x/n** is **a point estimate** of the **true proportion p** for a binomial experiment.

An estimator is not expected to estimate the population parameter **without error**. We do not expect $\bar{\mathbf{X}}$ to estimate $\boldsymbol{\mu}$ exactly, but we certainly hope that **it is not far off**.

Point Estimate[2]

For a particular sample, it is possible to obtain a closer estimate of **μ** by using the sample median **$\tilde{X}$** as an estimator. Consider, for instance, a sample consisting of the values **2, 5, and 11** from a population whose **mean is 4** but is supposedly unknown.

We would estimate μ to **be $\bar{x} = 6$**, using the sample mean as our estimate, or **$\tilde{x} = 5$**, using the sample median as our estimate. In this case, the estimator **$\tilde{X}$** produces an estimate closer to the true parameter than does the estimator **$\bar{X}$**.

Point Estimate[3]

On the other hand, if our random sample contains the values **2, 6, and 7**, then $\tilde{\mathbf{x}}$ **= 6** and $\bar{\mathbf{x}}$ **= 5**, so $\bar{\mathbf{x}}$ is the better estimator.

Not knowing the true value of **μ**, we must decide in advance whether to use $\bar{\mathbf{X}}$ or $\tilde{\mathbf{X}}$ as our estimator.

Unbiased Estimator

A statistic $\hat{\theta}$ is said to be an **unbiased estimator** of the parameter $\boldsymbol{\theta}$ if $\boldsymbol{\mu_{\hat{\theta}} = E(\hat{\theta}) = \theta}$

Unbiased Estimator

Example 1: $E(\bar{x}) = \mu$, so $\bar{x}$ is an **unbiased estimator** of μ

Example 2: $E(s^2) = \sigma^2$, so s^2 is an **unbiased estimator** of σ^2

Variance of a Point Estimator [1]

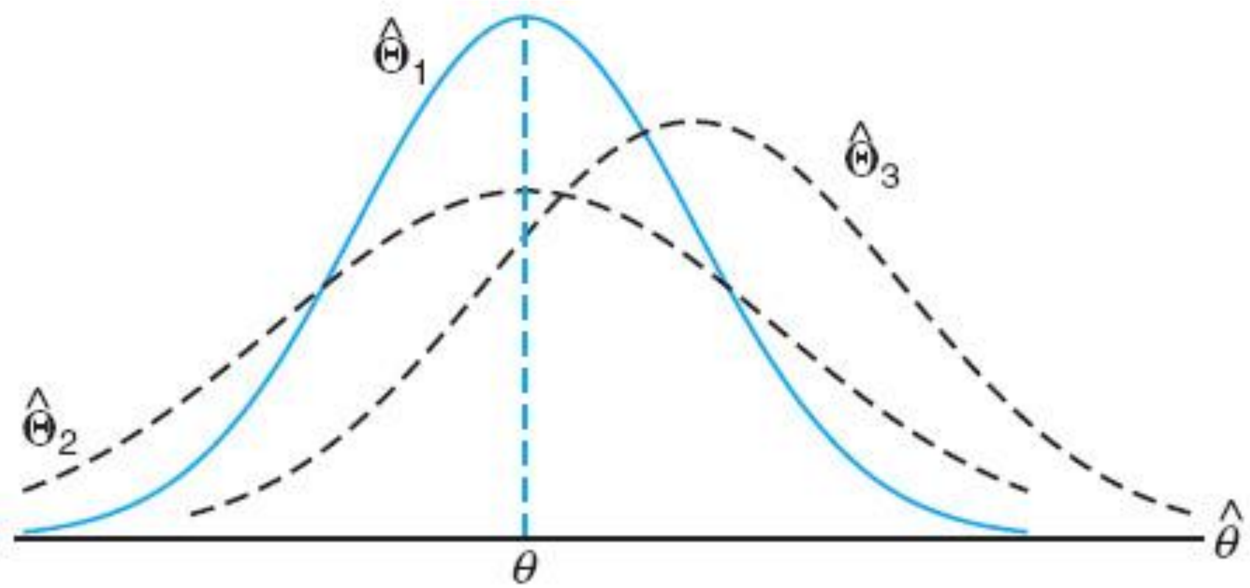
- If $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are two unbiased estimators of the same population parameter θ, we want to choose the **estimator whose sampling distribution** has the **smaller variance**.

- Hence, if $\sigma^2_{\widehat{\theta_1}} < \sigma^2_{\widehat{\theta_2}}$, we say that $\widehat{\theta_1}$ is a **more efficient estimator** of θ than $\widehat{\theta}_2$.

Variance of a Point Estimator [2]

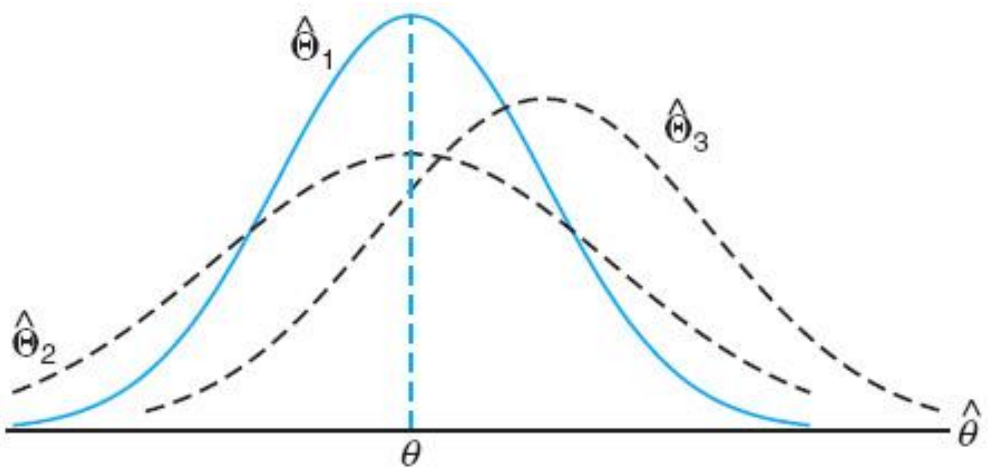
- If we consider all possible unbiased estimators of some parameter θ, the one with the smallest variance is called the **most efficient estimator** of **θ**.

simply means whose height is more, it is more efficient as more height => smaller sigma



Variance of a Point Estimator [3]

The figure illustrates the sampling distributions of three different estimators, $\widehat{\boldsymbol{\theta}_1}$, $\widehat{\boldsymbol{\theta}_2}$, and $\widehat{\boldsymbol{\theta}_3}$, all estimating $\boldsymbol{\theta}$. It is clear that only $\widehat{\boldsymbol{\theta}_1}$ **and** $\widehat{\boldsymbol{\theta}_2}$ are **unbiased**, since their distributions are centered at $\boldsymbol{\theta}$. The estimator $\widehat{\boldsymbol{\theta}_1}$ has a smaller variance than $\widehat{\boldsymbol{\theta}_2}$ and is therefore more efficient. Hence, our choice for an estimator of $\boldsymbol{\theta}$, among the three considered, would be $\widehat{\boldsymbol{\theta}_1}$.



Variance of a Point Estimator [2]

For normal populations, one can show that both $\overline{X}$ **and** $\widetilde{X}$ are **unbiased estimators** of the population mean μ, but the variance of $\overline{X}$ is smaller than the variance of $\widetilde{X}$.

Thus, both estimates $\overline{x}$ and $\widetilde{x}$ will, on average, equal the population mean μ, but $\overline{x}$ is likely to be closer to μ for a given sample, and thus $\overline{X}$ is **more efficient** than $\widetilde{X}$

(in general)