Predicates and Quantifiers

Course Code: CSC 1204 Course Title: Discrete Mathematics

Dept. of Computer Science
Faculty of Science and Technology

Lecturer No:	4	Week No:	2	Semester:	Summer 2021-22
Lecturer:	*Israt Jahan Mouri (israt.mouri@aiub.edu)*				

Lecture Outline

1.3 Predicates and Quantifiers

- Predicates
- Quantifiers
- Universal Quantifier, ∀
- Existential Quantifier, ∃
- Precedence of Quantifiers
- Negating Quantified Expressions
- Translating from English into Logical Expressions

Objectives and Outcomes

- **Objectives**: To understand predicates , universal and existential quantifiers, how to translate English statements into logical expressions
- **Outcomes**: Students are expected to be able to explain predicate logic, be able to find out the truth value of universal and existential quantifications, be able to translate English statements into logical expressions using predicates, quantifiers and logical connectives.

Predicates

- ***Predicate***: A property that the subject of the statement can have.
- Example: " x > 3 "

  x: variable

  >3: predicate

- We can denote the statement "x is greater than 3" by *P*(x), where *P* denotes the predicate "is greater than" and x is the variable. The statement *P*(x) is also said to be the value of the propositional function *P* at x.
  - P(x): x>3
  - The value of the propositional function *P* at x
- **Note**: *Once a value has been assigned to the variable x, the statement P(x) becomes a proposition and has a truth value (either TRUE or FALSE)*

Predicates

- A predicate is a sentence that contains a finite number of variables and becomes a proposition when specific values are substituted for the variables.
- A ***predicate*, or propositional function**, is a function that takes some variable(s) as arguments and returns True or False.
- A **PREDICATE** is symbolized by a **CAPITAL LETTER** and the ***variable(s)*** by ***small letter(s).***
- The sentence "*x* is a bachelor" is symbolized as *P(x)*,

  where *x* is a variable. When concrete values are substitutedin place of *x*, a proposition results(with a truth value, either True or False). P(*x*) is also called a *propositional function* , because each choice of *x* produces a proposition *P*(x) that is either true or false.

Example 1

- Let $P(x)$ denote the statement “x>3”.
  What are the truth values of $P(4)$ and $P(2)$?

- **Solution**: Given ==> $P(x)$: “x>3”
- We obtain the statement $P(4)$ by setting $x = 4$ in the statement “x>3”. Hence $P(4)$, which is the statement “**4>3**”, is **true**.
- However, $P(2)$ which is the statement “**2>3**”, is **false**.

Example 2

- Let, *A*(x) : "Computer x is under attack by an intruder". Suppose that of the computers on campus, only C1 and C7 are currently under attack by intruders. What are the truth values of *A* (C1), *A*(C3), *A*(C7)?

- **Solution**:

- *A*(C1): "Computer C1 is under attack by an intruder" is true
- *A*(C7): "Computer C7 is under attack by an intruder" is true
- *A*(C3): "Computer C3 is under attack by an intruder" is false

Why ? Because C3 is not in the list of computers that are attacked by intruders.

Multivariable Predicates

- *Multivariable* Predicates ==> Predicates that have more than one variable.
- For example, $Q(x, y)$: "$x = y + 3$" ,

  where x and y are variables and Q is the predicate.

- Note: When values are assigned to the variables x and y, the statement $Q(x, y)$ has a truth value.

Example 3

Let $Q(x, y)$ denote the statement "$x = y + 3$".

What are the truth values of the propositions $Q(1,2)$ and $Q(3,0)$?

- **Solution**:

- To obtain $Q(1,2)$, set $x=1$ and $y=2$ in the statement $Q(x,y)$.

  Therefore, $Q(1,2)$: "**$1 = 2 + 3$**" is **false**

  Similarly, $Q(3,0)$: "**$3 = 0 + 3$**" is **true**

Examples of Propositional Functions

- Let "$x + y = z$" be denoted by $R(x, y, z)$ and U (for all three variables) be the integers. Find these truth values:

  R(2,-1,5)

  Solution: F

  R(3,4,7)

  Solution: T

  R(x, 3, z)

  Solution: Not a Proposition

- Now let "$x - y = z$" be denoted by $Q(x, y, z)$, with U as the integers. Find these truth values:

  Q(2,-1,3)

  Solution: T

  Q(3,4,7)

  Solution: F

  Q(x, 3, z)

  Solution: Not a Proposition

Compound Expressions

- Connectives from propositional logic carry over to predicate logic.
- If $P(x)$ denotes "$x > 0$," find these truth values:

  $P(3) \vee P(-1)$ **Solution**: T

  $P(3) \wedge P(-1)$ **Solution**: F

  $P(3) \rightarrow P(-1)$ **Solution**: F

  $P(3) \rightarrow \neg P(-1)$ **Solution**: T

- Expressions with variables are not propositions and therefore do not have truth values. For example,

  $P(3) \wedge P(y)$

  $P(x) \rightarrow P(y)$

- When used with quantifiers (to be introduced next), these expressions (propositional functions) become propositions.

Quantifiers

- **Quantification: Two Categories –**
- Universal quantification: A predicate is true for **every element** in the domain
- Existential quantification: There is **one or more elements** in the domain for which a predicate is true

- Domain /domain of discourse/universe of discourse:

→The values a variable in a *propositional function* may take.

Quantifiers

1. Universal Quantifier: $\forall$ is called the universal quantifier.
"$\forall$" reads "for **A**ll"

2. Existential Quantifier: $\exists$ is called the existential quantifier.
"$\exists$" reads "there **E**xists"

The Universal Quantifier

Definition: The *universal quantification* of *P(x)* is the statement *"P(x)* for all values of x in the domain".

- The notation $\forall x\, P(x)$ denotes the universal quantification of *P(x)*.
- We read $\boldsymbol{\forall x\, P(x)}$ **as** "for all *x P(x)*" or "for every *x P(x)*"
- An element for which *P*(x) is false is called a **counterexample** of $\boldsymbol{\forall x\, P(x)}$

The Universal Quantifier

- "$\forall x\, P(x)$" is true when ***every instance*** of x makes $P(x)$ true when plugged in

- Like taking **conjunction** over the entire universe:
  $\forall x\, P(x) \equiv P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \ldots \wedge P(x_n)$

Example 8

Let $P(x)$ be the statement "$x + 1 > x$"
What is the **truth value** of the quantification $\forall x\, P(x)$, where the domain consists of all real numbers?

Solution: Because $P(x)$ is true for all real numbers x, the truth value of the quantification $\forall x\, P(x)$ is **true**.

Note: If we add 1 to any real number x, that number is always bigger than x

Example 9

- Let $Q(x)$ be the statement "$x<2$". What is the truth value of the quantification $\forall x\, Q(x)$, where the domain consists of all real numbers?

- **Solution**: $Q(x)$ is not true for every real number x, because, for instance, $Q(3)$ is false. That is, $x = 3$ is a counterexample for the statement $\forall x\, Q(x)$.

  Thus, the truth value of $\forall x\, Q(x)$ is false.

(Modified) Example 10

Let P(x) be the statement "$x^2>0$". What is the truth value of the quantification $\forall x\, P(x)$, where the universe of discourse consists of all integers?

Solution: P(x) is not true for all integers.
We can give a counter example. We see that x = 0 is a counterexample, because $x^2 = 0$ when x = 0, so that x^2 is not greater than 0 when x = 0.
Therefore, **truth value of $\forall x\, P(x)$ is false**.

Example 11

What is the truth value of $\forall x\, P(x)$, where $P(x)$ is the statement "$x^2<10$" and the domain consists of the positive integers not exceeding 4?

Solution: The statement $\forall x\, P(x)$ is the same as the conjunction **$P(1) \wedge P(2) \wedge P(3) \wedge P(4)$,** because the domain consists of the integers 1, 2, 3, and 4.
Because P(4), which is the statement "$4^2<10$", is false, it follows that **$\forall x\, P(x)$ is false.**

Another Example

What is the truth value of $\forall x\, P(x)$, where P(x) is the statement "$x^2<10$" and the domain consists of the positive integers less than 4?

Solution: The statement $\forall x\, P(x)$ is the same as the conjunction **$P(1) \wedge P(2) \wedge P(3)$,** because the domain consists of the integers 1, 2, 3.
So, the truth value of $\forall x\, P(x)$ **is true.**

How?----------**See Below**------------------------------------

P(1): "$1^2<10$", is true

P(2): "$2^2<10$" is true

P(3): "$3^2<10$" is true

$$\mathbf{P(1) \wedge P(2) \wedge P(3) \equiv T \wedge T \wedge T \equiv T}$$

The Existential Quantifier

- **Definition**: The *existential quantification* of $P(x)$ is the proposition "There exists an element x in the domain such that $P(x)$".
- We denote the existential quantification of $P(x)$ by $\exists x\, P(x)$
- Existential quantification $\exists x\, P(x)$ is read as:
  - "There is an x such that $P(x)$", or
  - "There is at least one x such that P(x)", or
  - "for some x $P(x)$"

The Existential Quantifier

- "$\exists x\, P(x)$" is true when *an instance* can be found which when plugged in for *x*, makes $P(x)$ true.

- Like taking **disjunction** over the entire domain

  $\exists x\, P(x) \equiv P(x_1) \vee P(x_2) \vee P(x_3) \vee \ldots P(x_n)$

Example 14

- Let P(x) denote the statement “x>3”. What is the **truth value** of the quantification $\exists x\, P(x)$, where the domain consists of all real numbers?

- **Solution**: Because “x>3” is sometimes true –for instance, when $x = 4$, the existential quantification of P(x) , which is $\exists x\ P(x)$, **is true**.

Example 15

- Let $Q(x)$ denote the statement "$x = x+1$". What is the truth value of the quantification $\exists x\, Q(x)$, where the domain consists of all real numbers?

- **Solution**: Because $Q(x)$ is false for every real number x, the existential quantification of $Q(x)$, which is $\exists x\, Q(x)$, is **false**.

- Note: If we add 1 to any real number x, that number will NEVER be equal to x, it will be always 1 bigger than x

Example 16

- What is the truth value of $\exists x\, P(x)$, where P(x) is the statement "$x^2>10$" and the universe of discourse consists of the positive integers not exceeding 4?

- **Solution**: Because the domain is { 1, 2, 3, 4}, the proposition $\exists x$ $P(x)$ is the same as the **disjunction**

  $P(1) \vee P(2) \vee P(3) \vee P(4)$.

  Because $P(4)$, which is the statement "$4^2>10$" , is true, it follows that truth value of **$\exists$x P(x) is true**.

Class Work

1. Let $P(x)$ denote the statement "$x>0$". What is the truth value of the quantification $\exists x\, P(x)$" , where the domain consists of integers?
2. Let $P(x)$ denote the statement "$x>0$". What is the truth value of the quantification $\forall x\, P(x)$" , where the domain consists of non-negative integers?
3. Let $P(x)$ denote the statement "$x>0$". What is the truth value of the quantification $\exists x\, P(x)$" , where the domain consists of negative integers?
4. Let $P(x)$ denote the statement "$x<2$". What is the truth value of the quantification $\exists x\, P(x)$" , where the domain consists of all prime numbers?
5. Let $P(x)$ denote the statement "$x \le 2$". What is the truth value of the quantification $\exists x\, P(x)$" , where the domain consists of all prime numbers?

Answers

1. True
2. False
3. False
4. False
5. True

Universal & Existential Quantifiers: When True? When False?

Table 1: Quantifiers		
Statements	***When true?***	***When false?***
$\forall x\ P(x)$	P(x) is true for every x.	There is an x for which P(x) is false.
$\exists x\ P(x)$	There is an x for which P(x) is true.	P(x) is false for every x.

Precedence of Quantifiers

- The **quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators** from propositional calculus.
- For example, $\forall x\, P(x) \vee Q(x)$ is the disjunction of $\forall x\, P(x)$ and $Q(x)$.

  In other words, it means $(\forall x\, P(x)) \vee Q(x)$ rather than $\forall x\, (\, P(x)) \vee Q(x)\,)$

Negating Quantified Expressions: De Morgan's Laws for Quantifiers

- The rules for negations for quantifiers are called De Morgan's laws for quantifiers
- Recall De Morgan's identities/Laws:
  - Negation of Conjunction: $\neg(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \equiv (\neg p_1 \vee \neg p_2 \vee \ldots \vee \neg p_n)$
  - Negation of Disjunction: $\neg(p_1 \vee p_2 \vee \ldots \vee p_n) \equiv (\neg p_1 \wedge \neg p_2 \wedge \ldots \wedge \neg p_n)$

- Since the quantifiers are the same as taking a bunch of **AND**'s ($\forall$) or **OR**'s ($\exists$), we have:
- Universal Negation: $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$
- Existential Negation: $\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$

Translating from English to Logic

Example 1: Translate the following sentence into predicate logic:
"Every student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

If *U* is all students in this class, define a propositional function J(x) denoting "x has taken a course in Java" and translate as $\forall x\, J(x)$.

Translating from English to Logic

Example 2: Translate the following sentence into predicate logic:
“Some student in this class has taken a course in Java.”

Solution:

First decide on the domain *U*.

If *U* is all students in this class, translate as

$$\exists x\, J(x)$$

Equivalences in Predicate Logic

- Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value
  - for every predicate substituted into these statements and
  - for every domain of discourse used for the variables in the expressions.
- The notation $S \equiv T$ indicates that S and T are logically equivalent.
- **Example**: $\forall x \neg\neg S(x) \equiv \forall x\, S(x)$

Negating Quantified Expressions

- Consider $\forall x\, J(x)$
  - "Every student in your class has taken a course in Java."
  - Here $J(x)$ is "x has taken a course in Java" and
  - the domain is students in your class.
- Negating the original statement gives "It is not the case that every student in your class has taken Java." This implies that "There is a student in your class who has not taken Java."

Symbolically $\neg\forall x\, J(x)$ and $\exists x\, \neg J(x)$ are equivalent

Negating Quantified Expressions

- Now Consider $\exists x\, J(x)$

  "There is a student in this class who has taken a course in Java."

  Where $J(x)$ is "x has taken a course in Java."

- Negating the original statement gives "It is not the case that there is a student in this class who has taken Java." This implies that "Every student in this class has not taken Java"

Symbolically $\neg\exists x\, J(x)$ and $\forall x\, \neg J(x)$ are equivalent

De Morgan's Laws for Quantifiers

- The rules for negating quantifiers are:

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

- The reasoning in the table shows that:

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

- These are important. You will use these.

Translating *from* **English** *into* **Logical Expressions**

Example 23 (p.40): Express the statement **"Every student in the class has studied calculus"** using predicates and quantifiers.

Solution: First, we rewrite the statement so that we can clearly identify the appropriate quantifiers to use. Doing so, we obtain:
"For every student in the class, that student has studied calculus".
Next we introduce a variable x so that our statement becomes –
"For every student x in the class, x has studied calculus"
Continuing, we introduce the predicate $C(x)$, which is the statement "x has studied calculus"
Consequently, if the **universe of discourse** for x consists of the **students in the class**, we can translate our statement as $\forall x\, C(x)$.

Example 23 (p.40)

Note: There are other correct approaches; different domains of discourse and other predicates can be used. **For example**, If we **change** the **domain** to consists of **all people**, we need to express our statement as

"For every person x, if person x is a student in this class then x has studied calculus."

If **S(*x*)** represents the statement that **person *x* is in this class**, our statement can be expressed as $\forall x\, S(x) \rightarrow C(x)$.

***Note**: For the second way (when the **domain is all people**), we always want to use **conditional statements** with **universal quantifiers** and **conjunctions** with **existential quantifiers**.*

Extra Example

- Express the statement "Someone in your school has studied calculus" using predicates and quantifiers. Let the **domain** consists of **all people.**
- **Solution:**

  Let, **S(x)** be the propositional functions " *x* is in your school" and

  C(*x*) be the propositional function "*x* has studied calculus".

  $\exists x\ (\ S(x) \land C(x)\ )$

- **Note that** if the **domain** consists of the **students in your school**, then we can write $\exists x\ C(x)$

Exercise 9 (p.43)

- Let *P(x)* be the statement "*x* can speak Russian" and let Q*(x)* be the statement "*x* knows the computer language C++", Express each of these sentences in terms of *P(x), Q (x),* quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.

a) There is a student at your school who can speak Russian and who knows C++.

b) There is a student at your school who can speak Russian but who doesn't know C++.

c) Every student at your school either can speak Russian or knows C++.

d) No student at your school can speak Russian or knows C++

Answers

a) $\exists x\, (p(x) \wedge Q(x))$

b) $\exists x\, (p(x) \wedge \neg Q(x))$

c) $\forall x\, (P(x) \vee Q(x))$

d) $\forall x \neg (P(x) \vee Q(x))$

Exercise 25 (p. 44)

Translate each of the statements into logical expressions using predicates, quantifiers, and logical connectives.
Let the **domain** be **all people**

a) No one is perfect.
b) Not everyone is perfect.
c) All your friends are perfect.
d) At least one of your friends is perfect.
e) Everyone is your friend and is perfect.
f) Not everybody is your friend or someone is not perfect.

Solution

Let *P(x)* be "*x* is perfect"; let *F(x)* be "*x* is your friend".

a) $\forall x \neg P(x)$
b) $\neg \forall x P(x)$
c) $\forall x (F(x) \rightarrow P (x))$
d) $\exists x (F(x) \wedge P(x))$
e) $\forall x (F(x) \wedge P(x))$ *or* $(\forall x (F(x)) \wedge (\forall x P(x))$
f) $(\neg\forall x (F(x)) \vee (\exists x \neg P(x))$

Practice @ Home

- Relevant Odd-Numbered Exercises first
- Then Even-Numbered Exercises

Books

1. *Discrete Mathematics and its applications with combinatorics and graph theory (7th edition)* by Kenneth H. Rosen [Indian Adaptation by KAMALA KRITHIVASAN], published by McGraw-Hill.

2. A textbook of Discrete Mathematics by Swapan Kumar Sarkar.

References

1. Discrete Mathematics, *Richard Johnsonbaugh*, Pearson education, Inc.
2. Discrete Mathematical Structures, *Bernard Kolman*, *Robert C. Busby*, *Sharon Ross,* Prentice-Hall, Inc.
3. *SCHAUM'S outlines Discrete Mathematics(2nd edition)*, by *Seymour Lipschutz*, *Marc Lipson*