Propositional Logic (cont.)

Course Code: CSC 1204 Course Title: Discrete Mathematics

Dept. of Computer Science
Faculty of Science and Technology

Lecturer No:	**2**	**Week No:**	**1**	**Semester:**	**Summer 21-22**
Lecturer:	*Israt Jahan Mouri (israt.mouri@aiub.edu)*				

Lecture Outline

1.1 Propositional Logic

- Logic
- Propositional Logic
- Propositions
- Propositional Variables
- Compound Propositions
- Logical Operators
- Truth Value
- Truth Tables of Compound Propositions
- Conditional Statements
- Logic and Bit Operations

* We have already covered

Objectives and Outcomes

- **Objectives**: To understand how to construct a truth table for a compound proposition, to understand the conditional statement $p \to q$ and different equivalent expressions of $p \to q$, to understand bit operations.

- **Outcomes**: Students are expected to be able construct a truth table for a given compound proposition, be able to explain the conditional statement $p \to q$ and it's equivalent expressions, be able to perform Bit Operations.

Conditional Statements

- Let p and q be propositions.
- The *conditional statement* $p \rightarrow q$ is the proposition "if p, then q."
- The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise.
- In the conditional statement $p \rightarrow q$, ***p*** *is called* ***hypothesis*** *and* ***q*** *is called* ***conclusion***.
- This one is the English usage of "if, then" or "implies".
- The connective $\rightarrow$ is called the 'conditional connective'.
- A **conditional statement** is **also called** an **implication**.

Truth Table for Conditional Statement

TABLE 5 The Truth Table for the Conditional Statement $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Implication

- If p and q are propositions, then $p \rightarrow q$ is a *conditional statement* or *implication* which is read as "if p, then q" and has this truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \rightarrow q$ denotes "If I am at home then it is raining."
- In $p \rightarrow q$, p is the *hypothesis* (*antecedent* or *premise*) and q is the *conclusion* (or *consequence*).

Implication

- In $p \to q$ there does not need to be any connection between the antecedent or the consequent. The "meaning" of $p \to q$ depends only on the truth values of p and q.
- These implications are perfectly fine but would not be used in ordinary English.
  - "If the moon is made of green cheese, then I have more money than Bill Gates. "
  - "If the moon is made of green cheese, then I'm on welfare."
  - "If 1 + 1 = 3, then your grandma wears combat boots."

Implication

- One way to view the logical conditional is to think of an obligation or contract.
  - "If I am elected, then I will lower taxes."
  - "If you get 100% on the final, then you will get an A."
- If the politician is elected and does not lower taxes, then the voters can say that he or she has broken the campaign pledge. Something similar holds for the professor. This corresponds to the case where p is true and q is false.

Equivalent Expression of $p \rightarrow q$

In terms of words, the proposition **$p \rightarrow q$ also reads**:

a. **if p, then q,** ***or*** **if p, q**
b. **p implies q**
c. **p is a sufficient condition for q ,** ***or*** **a sufficient condition for q is p**
d. **q is a necessary condition for p,** ***or*** **a necessary condition for p is q**
e. **p only if q**
f. **q if p ,** ***or*** **q, if p**
g. **q whenever p**
h. **q when p**
i. **q unless $\neg$p**

Remember!

- The hypothesis expresses a sufficient condition
- The conclusion expresses a necessary condition
- "***but***" is a logical synonym for "***and***"
- "***when***" / "***whenever***" means the same as "***if***"
- The **hypothesis** is the clause following "**if**"
- The ***conclusion*** is the clause following "***then***"
- "***only if***" clause is the **conclusion**
- **If *hypothesis*, then *conclusion***

Example 7 (page 7)

Let p : "Maria learns discrete mathematics" and

q : "Maria will find a good job."

Express the statement **p → q** as a statement in English.

Solution:

- *"**If** Maria learns discrete mathematics, **then** she will find a good job."*

There are many other ways to express this conditional statement in English.

- *"Maria will find a good job **when** she learns discrete mathematics"*
- *"**For** Maria to get a good job, **it is sufficient for her** to learn discrete mathematics".*
- *"Maria will find a good job **unless** **she does not** learn discrete mathematics."*

Exercise 19

- Write each of these statements in the form **"if p, then q"** in English.

a) It snows **whenever** the wind blows from the northeast.

Ans. **If** the wind blows from the northeast, **then** it snows

b) The apple trees will bloom **if** it stays warm for a week.

Ans. **If** it stays warm for a week, **then** the apple trees will bloom

Exercise 19

c) That the Pistons win the championship **implies** that they beat the Lakers.

Ans. **If** the Pistons win the championship, **then** they beat the Lakers.

d) It is **necessary** to walk 8 miles to get to the top of Long's Peak.

Ans. **If** you get to the top of Long's Peak, **then** you must have walked eight miles.

Exercise 19

e) To get tenure as a professor, it is **sufficient** to be world-famous.

Ans. **If** you are world-famous, **then** you will get tenure as a professor.

f) **If** you drive more than 400 miles, you will need to buy gasoline.

Ans. **If** you drive more than 400 miles, **then** you will need to buy gasoline.

Exercise 19

g) Your guarantee is good **only if** you bought your CD player less than 90 days ago.

Ans. **If** your guarantee is good, **then** you must have bought your CD player less than 90 days ago.

h) Jan will go swimming **unless** the water is too cold.

Ans. **If** the water is not too cold, **then** Jan will go swimming.

Converse, Contrapositive, and Inverse

- We can form some new conditional statements starting with a conditional statement $\boldsymbol{p \rightarrow q}$

- The **Converse of** $\boldsymbol{p \rightarrow q}$ is the proposition $\boldsymbol{q \rightarrow p}$
- The ***Contrapositive* of** $\boldsymbol{p \rightarrow q}$ is the proposition $\boldsymbol{\neg q \rightarrow \neg p}$
- The ***Inverse of*** $\boldsymbol{p \rightarrow q}$ is the proposition $\boldsymbol{\neg p \rightarrow \neg q}$

Examples of Converse, Contrapositive and Inverse

- ***Converse***: $p \to q$ ==> $\boldsymbol{q \to p}$

Example: "If it is noon, then I am hungry."

Converse: "If I am hungry, then it is noon."

- ***Contrapositive***: $p \to q$ ==> $\boldsymbol{\neg q \to \neg p}$

Example: "If it is noon, then I am hungry."

Contrapositive: "If I am not hungry, then it is not noon."

- ***Inverse***: $p \to q$ ==> $\boldsymbol{\neg p \to \neg q}$

Example: "If it is noon, then I am hungry."

Inverse: "If it is not noon, then I am not hungry."

Bi-Conditional

- Let p and q be propositions.
- The *bi-conditional statement* $\boldsymbol{p \leftrightarrow q}$ is the proposition **"*p* if and only if *q*."**
- Bi-conditional statements are also called "*bi-implications*"

- Question : Which operator is the opposite of $\leftrightarrow$?
- *Answer:* $\leftrightarrow$ has exactly the opposite truth table as $\oplus$.

Bi-Conditional

- Some alternative ways "*p* if and only if *q*" is expressed in English:

  - ***p* is necessary and sufficient for *q***
  - **if *p* then *q* , and conversely**
  - ***p* iff *q***

Truth Table for Bi-conditional $p \leftrightarrow q$

- If p and q are propositions, then we can form the *biconditional* proposition $p \leftrightarrow q$, read as "p if and only if q ." The biconditional $p \leftrightarrow q$ denotes the proposition with this truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- If p denotes "I am at home." and q denotes "It is raining." then $p \leftrightarrow q$ denotes "I am at home if and only if it is raining."

Example of Bi-conditional statement

- **Example 10 (Page 9):** Let *p* be the statement " You can take the flight" and *q* be the statement " You buy a ticket". What is the statement for $p \leftrightarrow q$ **?**

- **Solution:**

"You can take the flight if and only if you buy a ticket"

How to Construct a Truth Table for a Compound Proposition?

- At first look at the number of propositions (e.g. *p, q, r*) in the given compound proposition.
- There will be 2^n **number of rows** in the truth table, where *n* is the number of propositions in the compound proposition.
- Draw the table. In the first row, write down the name of propositions (e.g. *p, q, r*) starting from left/first column.
- Construct the truth table step by step.

Truth Tables For Compound Propositions

- Construction of a truth table:
- Rows
  - Need a row for every possible combination of values for the atomic propositions.
- Columns
  - Need a column for the compound proposition (usually at far right)
  - Need a column for the truth value of each expression that occurs in the compound proposition as it is built up.
    - This includes the atomic propositions

Example Truth Table

Construct a truth table for $p \vee q \to \neg r$

p	q	r	¬r	p ∨ q	p ∨ q → ¬r
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Equivalent Propositions

- Two propositions are e*quivalent* if they always have the same truth value.

- **Example**: Show using a truth table that the conditional is equivalent to the contrapositive.

Solution:

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \to \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Using a Truth Table to Show Non-Equivalence

Example: Show using truth tables that neither the converse nor inverse of an implication are not equivalent to the implication.

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg p \rightarrow \neg q$	$q \rightarrow p$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Example: Construct a truth table for

$(p \oplus q) \oplus r$

p	q	r	$p \oplus q$	$(p \oplus q) \oplus r$
T	T	T	F	T
T	T	F	F	F
T	F	T	T	F
T	F	F	T	T
F	T	T	T	F
F	T	F	T	T
F	F	T	F	T
F	F	F	F	F

Precedence of Logical Operators

- Negation operator is applied before all other logical operators
- Conjunction operator takes precedence over disjunction operator
- Conditional and bi-conditional operators have lower precedence
- Parentheses are used whenever necessary

Precedence of Logical Operators

TABLE 8
Precedence of Logical Operators.

Operator	*Precedence*
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Logic and Bit Operations

- **bit** ==> ***b***inary dig***it***
- Boolean variable: either **true** or **false**
  - Can be represented by a bit
- **Bit String**: A *bit string* is a sequence of zero or more bits. The *length* of the string is the number of bits in the string.
- Example 20(p.15): 101010011 is a bit string of length *nine*

Bit Operations

Computer **bit operations correspond to the logical connectives.** By replacing true by a one and false by a zero in the truth tables for the operators ∧, ∨, and ⊕, the tables shown in Table 9 for the corresponding bit operations are obtained. We will also use the notation *OR, AND, and XOR for the operators ∨,∧, and ⊕ respectively, as is done in various programming languages.*

Table for Bit Operations

TABLE 9 **Table for the Bit Operators *OR*, *AND*, and *XOR*.**

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

Bit string and bit operation

DEFINITION 7 A *bit string* is a sequence of zero or more bits. The *length* of this string is the number of bits in the string.

EXAMPLE 12 101010011 is a bit string of length nine. ◀

We can extend bit operations to bit strings. We define the **bitwise** ***OR***, **bitwise** ***AND***, and **bitwise** ***XOR*** of two strings of the same length to be the strings that have as their bits the *OR*, *AND*, and *XOR* of the corresponding bits in the two strings, respectively. We use the symbols $\vee$, $\wedge$, and $\oplus$ to represent the bitwise *OR*, bitwise *AND*, and bitwise *XOR* operations, respectively. We illustrate bitwise operations on bit strings with Example 13.

EXAMPLE 13 Find the bitwise *OR*, bitwise *AND*, and bitwise *XOR* of the bit strings 01 1011 0110 and 11 0001 1101. (Here, and throughout this book, bit strings will be split into blocks of four bits to make them easier to read.)

Solution: The bitwise *OR*, bitwise *AND*, and bitwise *XOR* of these strings are obtained by taking the *OR*, *AND*, and *XOR* of the corresponding bits, respectively. This gives us

$$\begin{array}{ll} 01\ 1011\ 0110 & \\ \underline{11\ 0001\ 1101} & \\ 11\ 1011\ 1111 & \text{bitwise } OR \\ 01\ 0001\ 0100 & \text{bitwise } AND \\ 10\ 1010\ 1011 & \text{bitwise } XOR \end{array}$$

◀

Books

- *Discrete Mathematics and its applications with combinatorics and graph theory (7th edition)* by Kenneth H. Rosen [Indian Adaptation by KAMALA KRITHIVASAN], published by McGraw-Hill

References

1. Discrete Mathematics, *Richard Johnsonbaugh*, Pearson education, Inc.
2. Discrete Mathematical Structures, *Bernard Kolman*, *Robert C. Busby*, *Sharon Ross,* Prentice-Hall, Inc.
3. *SCHAUM'S outlines Discrete Mathematics(2nd edition)*, by *Seymour Lipschutz*, *Marc Lipson*