IE 203 PS 2 – Solutions

Q1. Set Packing Problem

Solution: Integer Programming Model

Let x_j be a binary decision variable for each contract $j \in \{1, 2, 3, 4, 5\}$:

$$x_j = \begin{cases} 1 & \text{if Contract } j \text{ is accepted} \\ 0 & \text{otherwise} \end{cases}$$

Combining the above, the full Set Packing model is:

$$\begin{aligned}
\text{Maximize} \quad & Z = 200x_1 + 250x_2 + 180x_3 + 300x_4 + 220x_5 & \\
\text{Subject to} \quad & x_1 + x_3 \leq 1 & \text{(Monday Constraint)} \\
& x_1 + x_2 \leq 1 & \text{(Tuesday Constraint)} \\
& x_2 + x_4 \leq 1 & \text{(Wednesday Constraint)} \\
& x_3 + x_5 \leq 1 & \text{(Thursday Constraint)} \\
& x_4 + x_5 \leq 1 & \text{(Friday Constraint)} \\
& x_j \in \{0, 1\} & \forall j \in \{1, \ldots, 5\}
\end{aligned}$$

Q2. LP Relaxation of the 0–1 Knapsack Problem

Solution

Part a) Original IP Model

Let $x_i \in \{0,1\}$ be a binary variable: $x_i = 1$ if item i is packed, 0 otherwise (where $i \in \{A,B,C,D,E,F\}$).

$$\begin{aligned} \text{Maximize} \quad & Z = 12x_A + 14x_B + 6x_C + 8x_D + 7x_E + 6x_F \\ \text{Subject to} \quad & 5x_A + 7x_B + 2x_C + 6x_D + 3x_E + 4x_F \le 15 && \text{(Capacity)} \\ & x_i \in \{0,1\}, && \forall\, i \in \{A,B,C,D,E,F\} \end{aligned}$$

Part b) LP Relaxation

The LP relaxation replaces the binary constraint $x_i \in \{0,1\}$ with $0 \le x_i \le 1$:

$$\begin{aligned} \text{Maximize} \quad & Z = 12x_A + 14x_B + 6x_C + 8x_D + 7x_E + 6x_F \\ \text{Subject to} \quad & 5x_A + 7x_B + 2x_C + 6x_D + 3x_E + 4x_F \le 15 && \text{(Capacity)} \\ & 0 \le x_i \le 1, && \forall\, i \in \{A,B,C,D,E,F\} \end{aligned}$$

Part c) Solving the LP Relaxation

Method: Greedy by Value-to-Weight Ratio.

In the LP relaxation, x_i can be any fraction in $[0,1]$. This means we can pack a *fraction* of an item. The optimal strategy is: sort items by their value-per-kilogram ratio v_i/w_i in decreasing order, and greedily fill the knapsack. If an item fits completely, set $x_i = 1$; when the knapsack is nearly full and the next item does not fit entirely, pack only the fractional portion that fills the remaining capacity exactly.

First, compute and rank the value-to-weight ratios:

Item	Weight (kg)	Value ($)	Ratio v_i/w_i
C	2	6	3.00 (1st)
A	5	12	2.40 (2nd)
E	3	7	2.33 (3rd)
B	7	14	2.00 (4th)
F	4	6	1.50 (5th)
D	6	8	1.33 (6th)

Greedy fill (capacity = 15 kg):

Step	Item	Weight taken	Remaining capacity	x_i
1	C	2 kg (of 2 kg)	$15 - 2 = 13$	1
2	A	5 kg (of 5 kg)	$13 - 5 = 8$	1
3	E	3 kg (of 3 kg)	$8 - 3 = 5$	1
4	B	5 kg (of 7 kg)	$5 - 5 = 0$	5/7
5	F	–	–	0
6	D	–	–	0

Optimal LP solution:

$$x_C = 1, \quad x_A = 1, \quad x_E = 1, \quad x_B = \tfrac{5}{7}, \quad x_F = 0, \quad x_D = 0$$

Objective value:

$$Z^* = 6(1) + 12(1) + 7(1) + 14\left(\tfrac{5}{7}\right) + 6(0) + 8(0) = 6 + 12 + 7 + 10 = 35$$

Why is this optimal?

The ratio v_i/w_i tells us how much value we gain per kilogram of capacity spent on item i. With a single capacity constraint and fractional variables, it is always better to fill each kilogram of capacity with the item that gives the most value per kilogram. More formally: suppose at some step we instead used one kilogram for a lower-ratio item j rather than a higher-ratio item k (with $v_k/w_k > v_j/w_j$). Swapping that kilogram to item k would strictly increase the objective. Therefore, any deviation from greedy order can only decrease the objective value, and the greedy fractional solution is optimal for the LP relaxation.

Q3. Convexity of a Polyhedron

Part a) Convexity of P

By definition, a set P is convex if for any two points $x, y \in P$ and any $\lambda \in [0,1]$, the point $z = \lambda x + (1-\lambda)y$ is also in P.

Let $x, y \in P$. This means they satisfy the linear constraints (e.g., $-2x_1 + x_2 \leq 1$). Consider the first constraint for the combined point z:

$$-2z_1 + z_2 = -2(\lambda x_1 + (1-\lambda)y_1) + (\lambda x_2 + (1-\lambda)y_2)$$

Rearranging by λ:

$$= \lambda(-2x_1 + x_2) + (1-\lambda)(-2y_1 + y_2)$$

Since $x, y \in P$, we know $-2x_1 + x_2 \leq 1$ and $-2y_1 + y_2 \leq 1$. Thus:

$$\leq \lambda(1) + (1-\lambda)(1) = 1$$

The constraint holds for z. This logic applies to all linear inequalities. Thus, P is convex.

Part b) General Proof

Let a polyhedron be defined as $S = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. Let $u, v \in S$ and $\lambda \in [0,1]$. By definition, $Au \leq b$ and $Av \leq b$. We check if $z = \lambda u + (1-\lambda)v$ is in S:

$$Az = A(\lambda u + (1-\lambda)v) = \lambda(Au) + (1-\lambda)(Av)$$

Since $Au \leq b$, $Av \leq b$, and $\lambda \geq 0, (1-\lambda) \geq 0$:

$$Az \leq \lambda b + (1-\lambda)b = b$$

Thus $z \in S$, and the polyhedron is convex.

Part c) Extreme Points and Directions

1. Extreme Points (Vertices)

An extreme point is found by setting two constraints to equality and verifying the candidate satisfies all remaining constraints. We introduce shorthand labels for the five constraints (for use in this solution only):

$$\text{(C1)}: -2x_1 + x_2 \leq 1, \quad \text{(C2)}: x_1 - x_2 \leq 1, \quad \text{(C3)}: x_1 + x_2 \geq 4, \quad \text{(N1)}: x_1 \geq 0, \quad \text{(N2)}: x_2 \geq 0.$$

There are $\binom{5}{2} = 10$ candidate pairs. For each pair we solve the two equalities, then check whether all five constraints hold:

Pair	**Solution**	**Check remaining constraints**
C1 & C2	$-2x_1 + x_2 = 1,\ x_1 - x_2 = 1.$ Adding: $-x_1 = 2 \Rightarrow x_1 = -2,\ x_2 = -1.$	Fails (N1): $x_1 = -2 < 0$. **Rejected.**
C1 & C3	$-2x_1 + x_2 = 1,\ x_1 + x_2 = 4.$ Subtracting: $-3x_1 = -3 \Rightarrow x_1 = 1,\ x_2 = 3.$	(C2): $1 - 3 = -2 \leq 1$ ✓ (N1): $1 \geq 0$ ✓ (N2): $3 \geq 0$ ✓. **Valid:** $\bar{v}_1 = (1,3)$**.**
C1 & N1	$x_1 = 0 \Rightarrow x_2 = 1.$	(C3): $0 + 1 = 1 \ngeq 4$. **Rejected.**
C1 & N2	$x_2 = 0 \Rightarrow -2x_1 = 1 \Rightarrow x_1 = -\frac{1}{2}.$	Fails (N1): $x_1 < 0$. **Rejected.**
C2 & C3	$x_1 - x_2 = 1,\ x_1 + x_2 = 4.$ Adding: $2x_1 = 5 \Rightarrow x_1 = 2.5,\ x_2 = 1.5.$	(C1): $-5 + 1.5 = -3.5 \leq 1$ ✓ (N1): $2.5 \geq 0$ ✓ (N2): $1.5 \geq 0$ ✓. **Valid:** $\bar{v}_2 = (2.5, 1.5)$**.**
C2 & N1	$x_1 = 0 \Rightarrow -x_2 = 1 \Rightarrow x_2 = -1.$	Fails (N2): $x_2 = -1 < 0$. **Rejected.**
C2 & N2	$x_2 = 0 \Rightarrow x_1 = 1.$	(C3): $1 + 0 = 1 \ngeq 4$. **Rejected.**
C3 & N1	$x_1 = 0 \Rightarrow x_2 = 4.$	(C1): $0 + 4 = 4 \nleq 1$. **Rejected.**
C3 & N2	$x_2 = 0 \Rightarrow x_1 = 4.$	(C2): $4 - 0 = 4 \nleq 1$. **Rejected.**
N1 & N2	$x_1 = 0,\ x_2 = 0.$	(C3): $0 + 0 = 0 \ngeq 4$. **Rejected.**

P has exactly two extreme points: $\bar{v}_1 = (1,\, 3)$ and $\bar{v}_2 = (2.5,\, 1.5)$.

2. Extreme Directions

A direction $d = (d_1, d_2)$ belongs to the recession cone of P if moving in that direction from any feasible point keeps us in P for all $t \geq 0$. Each constraint must hold with its right-hand side set to 0 (preserving sign):

1. (C1): $-2d_1 + d_2 \leq 0 \;\Rightarrow\; d_2 \leq 2d_1$
2. (C2): $d_1 - d_2 \leq 0 \;\Rightarrow\; d_1 \leq d_2$
3. (C3): $d_1 + d_2 \geq 0 \quad$ (automatically satisfied when $d_1, d_2 \geq 0$)
4. (N1): $d_1 \geq 0$
5. (N2): $d_2 \geq 0$

Combining: we need $d_1 \geq 0$, $d_2 \geq 0$, and $d_1 \leq d_2 \leq 2d_1$. The extreme rays are at the boundary of this cone:

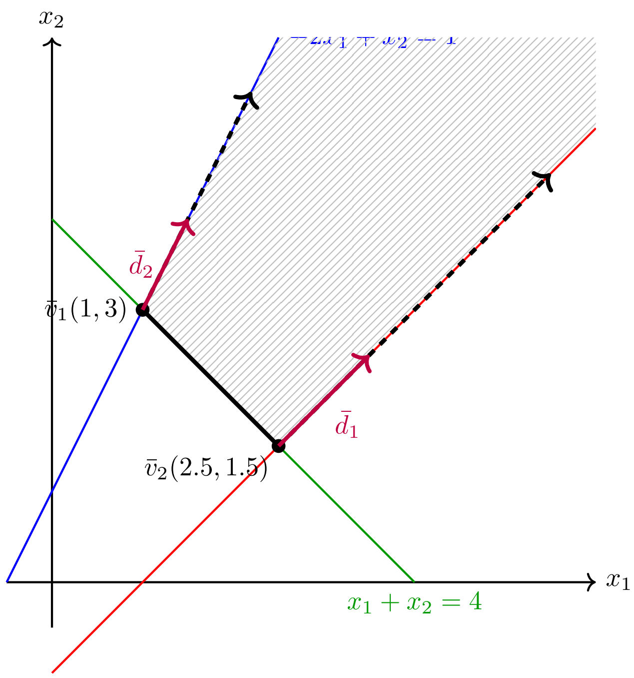
- **(C2) active** ($d_2 = d_1$, $d_1 = 1$): $\bar{d}_1 = (1,1)$. Check: (C1) $-2+1 = -1 \leq 0\,\checkmark$, (C2) $0 \leq 0\,\checkmark$, (C3) $2 \geq 0\,\checkmark$, (N1) $\checkmark$, (N2) $\checkmark$. **Extreme direction $\bar{d}_1 = (1,1)$.**
- **(C1) active** ($d_2 = 2d_1$, $d_1 = 1$): $\bar{d}_2 = (1,2)$. Check: (C1) $-2+2 = 0 \leq 0\,\checkmark$, (C2) $1-2 = -1 \leq 0\,\checkmark$, (C3) $3 \geq 0\,\checkmark$, (N1) $\checkmark$, (N2) $\checkmark$. **Extreme direction $\bar{d}_2 = (1,2)$.**

3. Representation (Resolution Theorem)

Every point in P can be written as a convex combination of the extreme points plus a non-negative combination of the extreme directions:

$$P = \left\{ \lambda_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2.5 \\ 1.5 \end{pmatrix} + \mu_1 \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\bar{d}_1} + \mu_2 \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\bar{d}_2} \;\middle|\; \lambda_1 + \lambda_2 = 1,\ \lambda_1, \lambda_2 \geq 0,\ \mu_1, \mu_2 \geq 0 \right\}$$

Part d) Visualization



Q4. Convexity of Sets

Solution

Definition: A set S is convex if for any pair of points $x, y \in S$ and any $\lambda \in [0,1]$, the point $z = \lambda x + (1-\lambda)y$ is also in S.

a) Convex.
S is a closed interval $[1,5]$ in $\mathbb{R}$.

- Let $x, y \in S$, so $1 \leq x \leq 5$ and $1 \leq y \leq 5$. For any $\lambda \in [0,1]$:

$$z = \lambda x + (1-\lambda)y \geq \lambda(1) + (1-\lambda)(1) = 1$$

$$z = \lambda x + (1-\lambda)y \leq \lambda(5) + (1-\lambda)(5) = 5$$

- Thus $1 \leq z \leq 5$, so $z \in S$.

b) Not Convex.
$S = \{1,2,3,4,5\}$ is a discrete set of integers.

- **Counterexample:** Let $x = 1, y = 2$, $\lambda = 0.5$.
- $z = 0.5(1) + 0.5(2) = 1.5$.
- Since $1.5 \notin \mathbb{Z}$, $z \notin S$.

c) Convex.
The only integer satisfying $1 \leq x_1 \leq 1.5$ is $x_1 = 1$. Thus $S = \{1\}$ (a singleton).

- Let $x, y \in S$. Then $x = 1, y = 1$.
- $z = \lambda(1) + (1-\lambda)(1) = 1 \in S$.

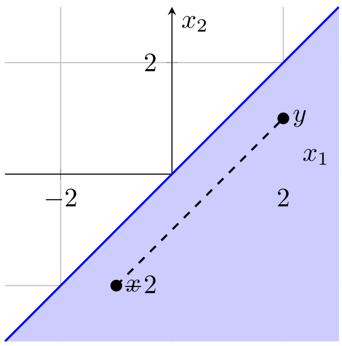
d) Convex.
S is a closed half-space defined by $x_1 - x_2 \geq 0$, i.e., $x_1 \geq x_2$.

- Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be in S. Then $x_1 \geq x_2$ and $y_1 \geq y_2$.
- Let $z = \lambda x + (1-\lambda)y$. Then:

$$z_1 - z_2 = \lambda(x_1 - x_2) + (1-\lambda)(y_1 - y_2) \geq 0$$

  since both terms in parentheses are ≥ 0 and $\lambda \geq 0$, $(1-\lambda) \geq 0$.
- Thus $z_1 \geq z_2$, so $z \in S$.

$x_1 \geq x_2$ (Convex)



e) Convex.
S is the region above the parabola: $S = \{(x_1, x_2) \mid x_2 \geq x_1^2\}$.

- Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be in S, so $x_2 \geq x_1^2$ and $y_2 \geq y_1^2$.
- Let $z = \lambda x + (1-\lambda)y$ for any $\lambda \in [0,1]$. We need to show $z_2 \geq z_1^2$, i.e. $z_2 - z_1^2 \geq 0$.

- We compute $z_2 - z_1^2$ directly:

$$z_2 - z_1^2 = \big[\lambda x_2 + (1-\lambda)y_2\big] - \big[\lambda x_1 + (1-\lambda)y_1\big]^2$$

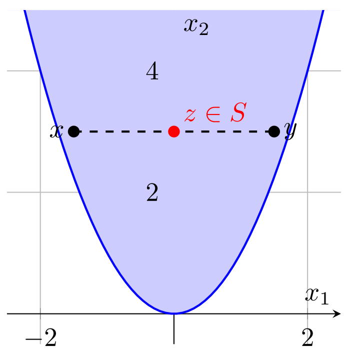
Since $x_2 \ge x_1^2$ and $y_2 \ge y_1^2$, replacing x_2 with x_1^2 and y_2 with y_1^2 only makes the first bracket smaller. Therefore:

$$\begin{aligned} z_2 - z_1^2 &\ge \big[\lambda x_1^2 + (1-\lambda)y_1^2\big] - \big[\lambda x_1 + (1-\lambda)y_1\big]^2 \\ &= \lambda x_1^2 + (1-\lambda)y_1^2 - \lambda^2 x_1^2 - 2\lambda(1-\lambda)x_1y_1 - (1-\lambda)^2 y_1^2 \\ &= \lambda(1-\lambda)\,x_1^2 - 2\lambda(1-\lambda)\,x_1y_1 + \lambda(1-\lambda)\,y_1^2 \\ &= \lambda(1-\lambda)\,(x_1-y_1)^2 \;\ge\; 0, \end{aligned}$$

since $\lambda \ge 0$, $(1-\lambda) \ge 0$, and any square is ≥ 0.

- Therefore $z_2 - z_1^2 \ge 0$, i.e. $z_2 \ge z_1^2$, so $z \in S$.

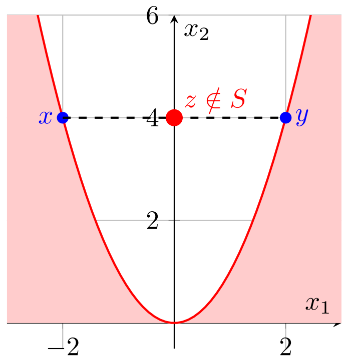
$x_2 \ge x_1^2$ (Convex)



f) Not Convex.
S is the region $x_2 \le x_1^2$.

- **Counterexample:** Let $x = (-2, 4)$ and $y = (2, 4)$. Check: $4 \le (-2)^2 = 4$ ✓and $4 \le 4$ ✓.
- Let $\lambda = 0.5$. Midpoint $z = (0, 4)$.
- Check: Is $4 \le 0^2 = 0$? No. So $z \notin S$.

$x_2 \le x_1^2$ (Not Convex)



g) Convex.
S is the line $x_1 = x_2$.

- Let $x = (x_1, x_1)$ and $y = (y_1, y_1)$ be in S.
- $z = \lambda x + (1-\lambda)y = (\lambda x_1 + (1-\lambda)y_1,\ \lambda x_1 + (1-\lambda)y_1)$.
- Both components are equal, so $z \in S$.

h) Not Convex.
S is the curve $x_2 = x_1^2$ (a parabola).

- **Counterexample:** Let $x = (-1, 1)$ and $y = (1, 1)$. Both satisfy $x_2 = x_1^2$.
- Midpoint $z = (0, 1)$. Check: $1 \neq 0^2 = 0$. So $z \notin S$.

i) Not Convex.
S is the "L"-shaped union of two perpendicular segments.

- **Counterexample:** Take $x = (0, 1) \in S_1$ and $y = (1, 0) \in S_2$.
- Midpoint $z = (0.5, 0.5)$. For $z \in S_1$ we need $x_2 = 1$; for $z \in S_2$ we need $x_1 = 1$. Neither holds.
- $z \notin S$.

j) Convex.
$S = \{(x_1, x_2) : |x_2| \leq x_1\}$ is the region bounded by $x_2 = x_1$ and $x_2 = -x_1$ (right-facing V-shape).

- Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be in S, so $|x_2| \leq x_1$ and $|y_2| \leq y_1$.
- Let $z = \lambda x + (1 - \lambda)y$ for any $\lambda \in [0, 1]$. We need to show $|z_2| \leq z_1$.
- For any real number a, both $a \leq |a|$ and $-a \leq |a|$ hold. Multiplying the inequalities $x_2 \leq |x_2|$ and $y_2 \leq |y_2|$ by $\lambda \geq 0$ and $(1 - \lambda) \geq 0$ respectively and adding:

$$z_2 = \lambda x_2 + (1 - \lambda)y_2 \ \leq \ \lambda|x_2| + (1 - \lambda)|y_2|.$$

  Doing the same with $-x_2 \leq |x_2|$ and $-y_2 \leq |y_2|$:

$$-z_2 = \lambda(-x_2) + (1 - \lambda)(-y_2) \ \leq \ \lambda|x_2| + (1 - \lambda)|y_2|.$$

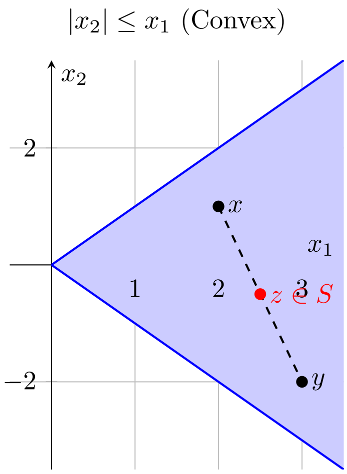
  Since $|z_2|$ equals either z_2 or $-z_2$ (whichever is ≥ 0), both cases give:

$$|z_2| \ \leq \ \lambda|x_2| + (1 - \lambda)|y_2|.$$

- Finally, using $|x_2| \leq x_1$ and $|y_2| \leq y_1$:

$$|z_2| \ \leq \ \lambda|x_2| + (1 - \lambda)|y_2| \ \leq \ \lambda x_1 + (1 - \lambda)y_1 = z_1.$$

- Therefore $|z_2| \leq z_1$, so $z \in S$.

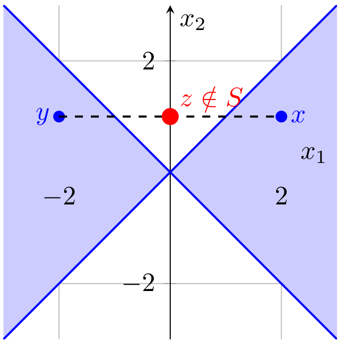


k) Not Convex.
$S = \{(x_1, x_2) : |x_2| \leq |x_1|\}$ contains points of the form where both $x_1 \geq 0$ and $x_1 \leq 0$ regions are included, giving an "X"-shaped (bowtie/hourglass) region of four quadrant sectors.

- **Counterexample:** Let $x = (2, 1) \in S$ (check: $|1| \leq |2|$ ✓) and $y = (-2, 1) \in S$ (check: $|1| \leq |-2|$ ✓).
- Let $\lambda = 0.5$. Midpoint $z = (0, 1)$.

– Check: $|1| \leq |0|$? That is $1 \leq 0$, which is **false**.
– So $z \notin S$, and S is **not convex**.
– *Alternatively, note analytically that the boundary curves $x_2 = x_1$ and $x_2 = -x_1$ divide the plane into four sectors and the set includes the left and right sectors but not the top and bottom. Any path from the right to the left sector crosses through the top or bottom sector.*

$|x_2| \leq |x_1|$ (Not Convex)



Q5. Properties of the Euclidean Norm

Solution

1. Proof of Non-negativity

Statement: $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$.

Proof. By definition, $x_i^2 \geq 0$ for any real number x_i. Consequently, the sum $\sum_{i=1}^n x_i^2 \geq 0$. Since the square root function is monotonically increasing and non-negative for non-negative arguments:

$$\|x\| = \sqrt{\sum_{i=1}^n x_i^2} \geq 0$$

Now, assume $\|x\| = 0$:

$$\sqrt{\sum_{i=1}^n x_i^2} = 0 \implies \sum_{i=1}^n x_i^2 = 0$$

Since each term $x_i^2 \geq 0$, the sum can only be zero if *every* individual term is zero.

$$x_i^2 = 0 \implies x_i = 0 \quad \forall i$$

Therefore, x must be the zero vector $\mathbf{0}$. □

2. Proof of Homogeneity

Statement: $\|\alpha x\| = |\alpha|\|x\|$.

Proof. Let $\alpha \in \mathbb{R}$. The components of the vector αx are $(\alpha x_1, \ldots, \alpha x_n)$.

$$\begin{aligned} \|\alpha x\| &= \sqrt{\sum_{i=1}^n (\alpha x_i)^2} \\ &= \sqrt{\sum_{i=1}^n \alpha^2 x_i^2} \\ &= \sqrt{\alpha^2 \sum_{i=1}^n x_i^2} \end{aligned}$$

Using the property $\sqrt{ab} = \sqrt{a}\sqrt{b}$ (for non-negative reals) and $\sqrt{\alpha^2} = |\alpha|$:

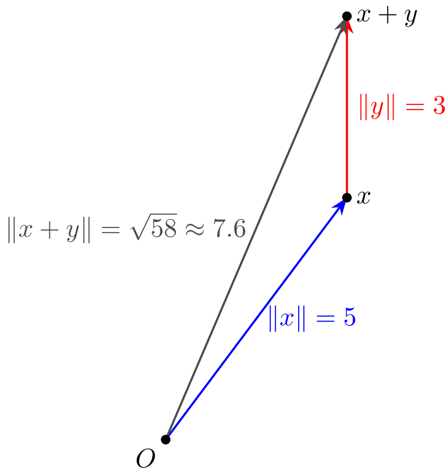
$$= \sqrt{\alpha^2}\sqrt{\sum_{i=1}^n x_i^2} = |\alpha|\|x\|$$

□

3. Proof of Triangle Inequality

Statement: $\|x + y\| \leq \|x\| + \|y\|$.

Geometric meaning: The vectors x and y form two sides of a triangle in $\mathbb{R}^n$; their sum $x + y$ is the third side. The inequality says the length of the third side cannot exceed the sum of the other two — the classical triangle inequality, generalized to any finite dimension.



$$\|x+y\| \le \|x\| + \|y\| \Leftrightarrow 7.6 \le 5+3 = 8 \checkmark$$

Numerical example. Let $x = (3,4)$ and $y = (0,3)$.

$$\|x\| = \sqrt{3^2+4^2} = \sqrt{25} = 5, \quad \|y\| = \sqrt{0^2+3^2} = 3, \quad \|x+y\| = \sqrt{3^2+7^2} = \sqrt{58} \approx 7.62$$

$$\|x+y\| \approx 7.62 \le 5+3 = 8 = \|x\| + \|y\| \quad \checkmark$$

Proof. By the law of cosines, for two vectors x and y with angle θ between them:

$$\|x+y\|^2 = \|x\|^2 + 2\|x\|\|y\|\cos\theta + \|y\|^2.$$

Since $\cos\theta \le 1$ for any angle θ:

$$\|x+y\|^2 \le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2.$$

Taking the square root of both sides (both sides non-negative, $\sqrt{\cdot}$ is increasing):

$$\|x+y\| \le \|x\| + \|y\|. \qquad \square$$