IE 203 PS 2

2026-02-19

Q1. Set Packing Problem

A specialized freelance consultant has received offers for 5 different short-term contracts. Each contract requires specific days of the work week (Monday through Friday) to be completed on-site. The consultant cannot work on two different contracts on the same day.

The available contracts, their required days, and the total payment for each are listed below:

Contract (j)	**Required Days**	**Set Representation**	**Payment ($)**
1	Monday, Tuesday	{Mon, Tue}	200
2	Tuesday, Wednesday	{Tue, Wed}	250
3	Monday, Thursday	{Mon, Thu}	180
4	Wednesday, Friday	{Wed, Fri}	300
5	Thursday, Friday	{Thu, Fri}	220

Formulate an Integer Programming (IP) model to determine which subset of contracts the consultant should accept to maximize their total income without creating a scheduling conflict.

Q2. LP Relaxation of the 0–1 Knapsack Problem

A hiker has a knapsack with a weight capacity of **15 kg**. There are 6 items available to pack. Each item has a weight w_i (kg) and a value v_i ($). The hiker wants to maximize the total value of items packed.

Item	**Weight** w_i **(kg)**	**Value** v_i **($)**
A	5	12
B	7	14
C	2	6
D	6	8
E	3	7
F	4	6

a) Formulate the original 0–1 Integer Programming (IP) model for this knapsack problem.

b) Write the **LP relaxation** of this model (i.e., relax the integrality constraints).

c) Solve the LP relaxation. Explain your solution method and why it gives the optimal LP solution.

Q3. Convexity of a Polyhedron

Consider the polyhedron P in $\mathbb{R}^2$ defined by the following system of inequalities:

$$\begin{aligned} -2x_1 + x_2 &\leq 1 \quad \text{(C1)} \\ x_1 - x_2 &\leq 1 \quad \text{(C2)} \\ x_1 + x_2 &\geq 4 \quad \text{(C3)} \\ x_1, x_2 &\geq 0 \end{aligned}$$

Questions:

a) Show that this specific polyhedron P is a convex set.

b) Prove generally that *any* polyhedron defined by $Ax \leq b$ is a convex set.

c) Determine the extreme points and extreme directions of P, and express P using the Resolution Theorem (Representation Theorem).

d) Draw the polyhedron, clearly labeling the axes, constraints, extreme points, and extreme directions.

Q4. Convexity of Sets

Determine whether each of the following sets S is a **convex set**. Provide a formal proof if the set is convex, or a counterexample/disproof if it is not.

a) $S = \{x_1 \in \mathbb{R} \mid 1 \leq x_1 \leq 5\}$

b) $S = \{x_1 \in \mathbb{Z} \mid 1 \leq x_1 \leq 5\}$

c) $S = \{x_1 \in \mathbb{Z} \mid 1 \leq x_1 \leq 1.5\}$

d) $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 - x_2 \geq 0\}$

e) $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 - x_2 \leq 0\}$

f) $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 - x_2 \geq 0\}$

g) $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 - x_2 = 0\}$

h) $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 - x_2 = 0\}$

i) $S = S_1 \cup S_2$, where $S_1 = \{(x_1, x_2) \mid 0 \leq x_1 \leq 1,\ x_2 = 1\}$ and $S_2 = \{(x_1, x_2) \mid x_1 = 1,\ 0 \leq x_2 \leq 1\}$.

j) $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_2| \leq x_1\}$

k) $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_2| \leq |x_1|\}$

Q5. Properties of the Euclidean Norm

Let $x \in \mathbb{R}^n$. The Euclidean norm (or L_2 norm) is defined as:

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{x^T x}$$

Prove that the Euclidean norm satisfies the three fundamental properties of a norm:

1. **Non-negativity:** $\|x\| \geq 0$ for all x, and $\|x\| = 0$ if and only if $x = 0$.
2. **Homogeneity (Scalar Multiplication):** $\|\alpha x\| = |\alpha|\|x\|$ for any scalar $\alpha \in \mathbb{R}$.
3. **Triangle Inequality:** $\|x + y\| \leq \|x\| + \|y\|$ for all vectors $x, y \in \mathbb{R}^n$.