IE 203 PS 3 – Solutions

Q1. Project Selection via Branch-and-Bound

Part a) IP Formulation

Let $x_i \in \{0,1\}$ equal 1 if project P_i is selected, 0 otherwise.

$$\begin{aligned} \max \quad & Z = 11x_1 + 14x_2 + 7x_3 + 12x_4 + 5x_5 \\ \text{s.t.} \quad & 14x_1 + 2x_2 + 9x_3 + 8x_4 + 4x_5 \leq 20 \qquad \text{(Budget)} \\ & x_i \in \{0,1\}, \quad \forall\, i \in \{1,2,3,4,5\} \end{aligned}$$

Part b) Branch-and-Bound Solution

Sort projects by profit-to-cost ratio p_i/c_i in decreasing order:

Rank	**Project**	**Cost** c_i	**Profit** p_i	**Ratio** p_i/c_i
1	P_2	2	14	7.000
2	P_4	8	12	1.500
3	P_5	4	5	1.250
4	P_1	14	11	0.786
5	P_3	9	7	0.778

We search the B&B tree using **Depth-First Search (DFS)** (LIFO queue), branching on the **fractional** variable that follows the ranking above. At each node, we compute the LP relaxation value z_{LP} via the greedy fractional knapsack heuristic (which yields the optimal solution to the LP relaxation of the knapsack problem). Since all objective coefficients are integer, any feasible integer solution has an integer profit. Thus, we use the tighter local upper bound (UB) $\text{UB} = \lfloor z_{\text{LP}} \rfloor$.

Fathoming Rules:
1. **By Bound:** If $\text{UB} \leq \text{LB}$ (where **LB** is the profit of the best integer-feasible solution found so far).
2. **By Integrality:** If the LP solution naturally takes integer values ($x_i \in \{0,1\}$). We update $\text{LB} = z_{\text{LP}}$ if it improves our best known integer-feasible profit.
3. **By Infeasibility:** If the fixed items violate the capacity constraint.

Step 1 — Node 1 (Root)

Fixed assignments: none (root node).

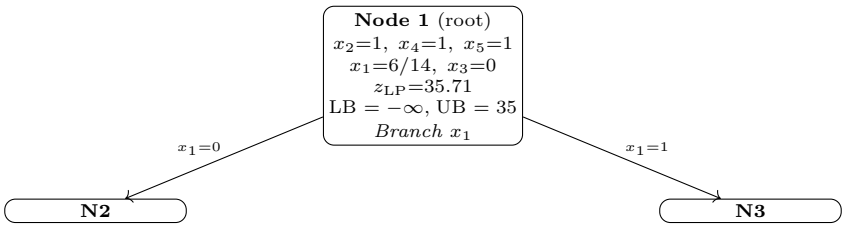
	P_2	P_4	P_5	P_1	P_3
c_i	2	8	4	14	9
x_i	1	1	1	6/14	0
Rem. cap	18	10	6	0	0

$z_{\text{LP}} = 14 + 12 + 5 + \frac{6}{14} \times 11 = 35.71 \quad \Rightarrow \quad \textbf{UB} = \lfloor 35.71 \rfloor = 35, \quad \textbf{LB} = -\infty.$

P_1 **is fractional** $\Rightarrow$ **branch on** x_1**.**

Subproblem list:
- **[Node 3]**

- **[Node 2]**



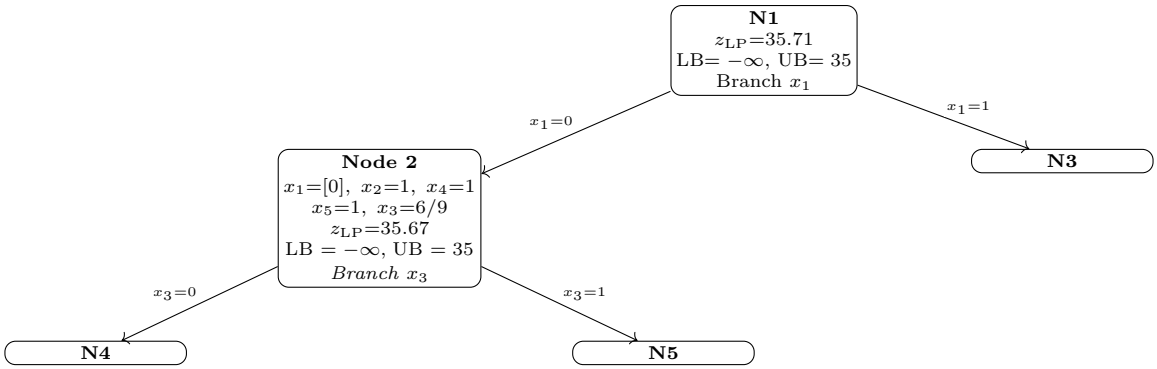
Fixed: $x_1 = 0$.

	P_1	P_2	P_4	P_5	P_3
c_i	14	2	8	4	9
x_i	[0]	1	1	1	6/9
Rem. cap	20	18	10	6	0

$z_{\text{LP}} = 14 + 12 + 5 + \frac{6}{9}\times 7 = 35.67 \quad \Rightarrow \quad \textbf{UB} = \lfloor 35.67 \rfloor = 35, \quad \textbf{LB} = -\infty.$

P_3 **is fractional** $\Rightarrow$ **branch on** x_3**.**

Subproblem list:
- [Node 3]
- ~~[Node 2]~~
- **[Node 5]**
- **[Node 4]**



Fixed: $x_1 = 0$, $x_3 = 0$.

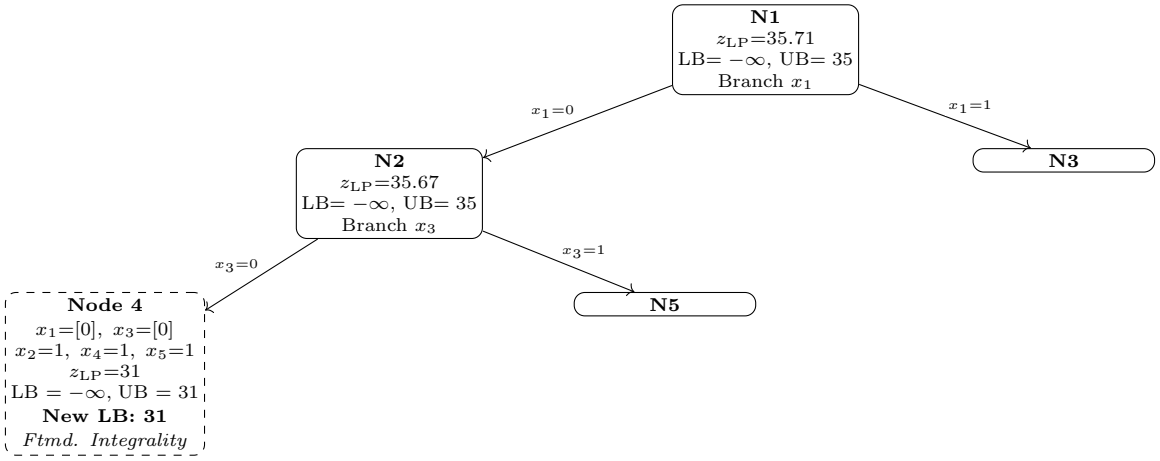
	P_1	P_3	P_2	P_4	P_5
c_i	14	9	2	8	4
x_i	[0]	[0]	1	1	1
Rem. cap	20	20	18	10	6

$z_{\text{LP}} = 14 + 12 + 5 = 31 \quad \Rightarrow \quad \textbf{UB} = 31, \quad \textbf{LB} = -\infty.$

LP solution is integer. $z = 31 \Rightarrow$ **LB updated to 31**.

Status: Fathomed by Integrality.

Subproblem list:
- [Node 3]
- ~~[Node 2]~~
- [Node 5]
- ~~[Node 4]~~



Fixed: $x_1 = 0$, $x_3 = 1$.

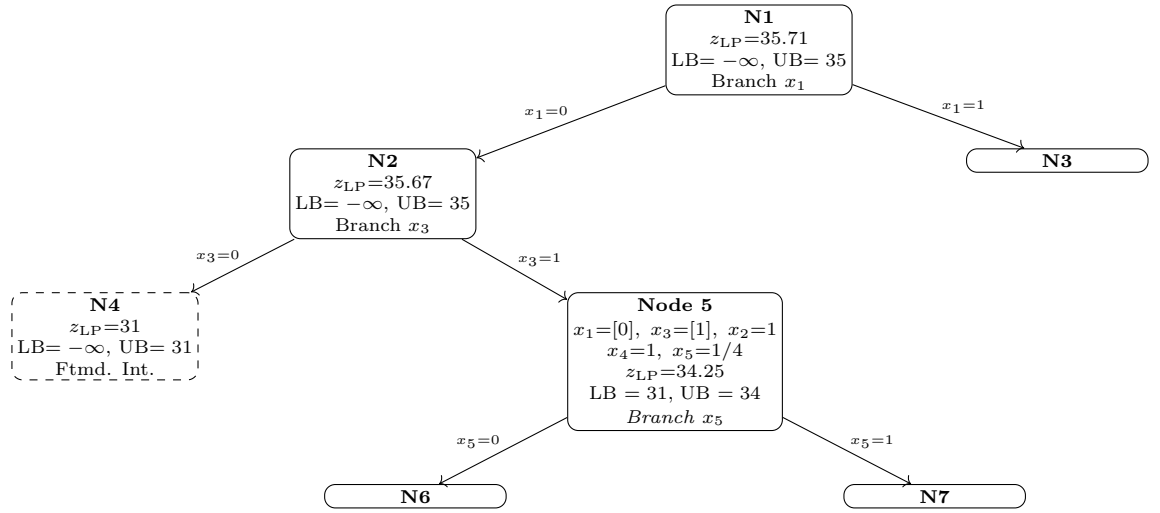
	P_1	P_3	P_2	P_4	P_5
c_i	14	9	2	8	4
x_i	[0]	[1]	1	1	1/4
Rem. cap	20	11	9	1	0

$z_{\text{LP}} = 7 + 14 + 12 + \frac{1}{4}\times 5 = 34.25 \quad \Rightarrow \quad \mathbf{UB} = \lfloor 34.25 \rfloor = 34, \quad \mathbf{LB} = 31.$

P_5 **is fractional** $\Rightarrow$ **branch on** x_5**.**

Subproblem list:

- [Node 3]
- ~~[Node 2]~~
- ~~[Node 5]~~
- ~~[Node 4]~~
- **[Node 7]**
- **[Node 6]**



Fixed: $x_5 = 0$, $x_1 = 0$, $x_3 = 1$.

	P_5	P_1	P_3	P_2	P_4
c_i	4	14	9	2	8
x_i	[0]	[0]	[1]	1	1
Rem. cap	20	20	11	9	1

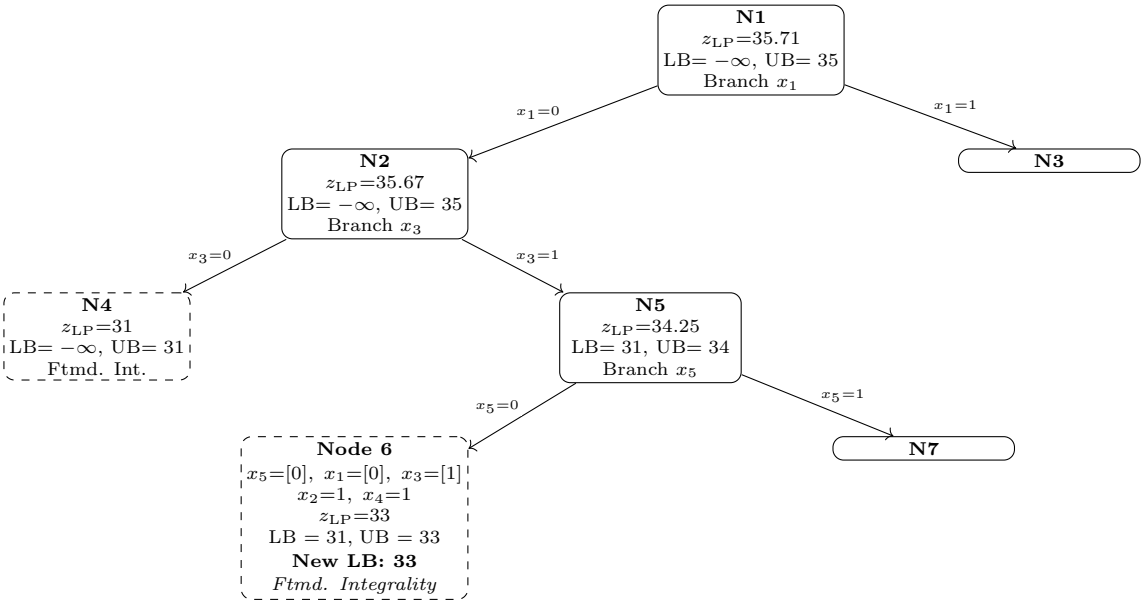
$z_{\text{LP}} = 7 + 14 + 12 = 33 \quad \Rightarrow \quad \mathbf{UB} = 33, \quad \mathbf{LB} = 31.$

LP solution is integer. $z = 33 > 31 \Rightarrow$ **LB updated to 33**.

Status: Fathomed by Integrality.

Subproblem list:

- [Node 3]
- ~~[Node 2]~~
- ~~[Node 5]~~
- ~~[Node 4]~~
- [Node 7]
- ~~[Node 6]~~



Step 6 — Node 7 ($x_5 = 1$, $x_1 = 0$, $x_3 = 1$)

Fixed: $x_5 = 1$, $x_1 = 0$, $x_3 = 1$.

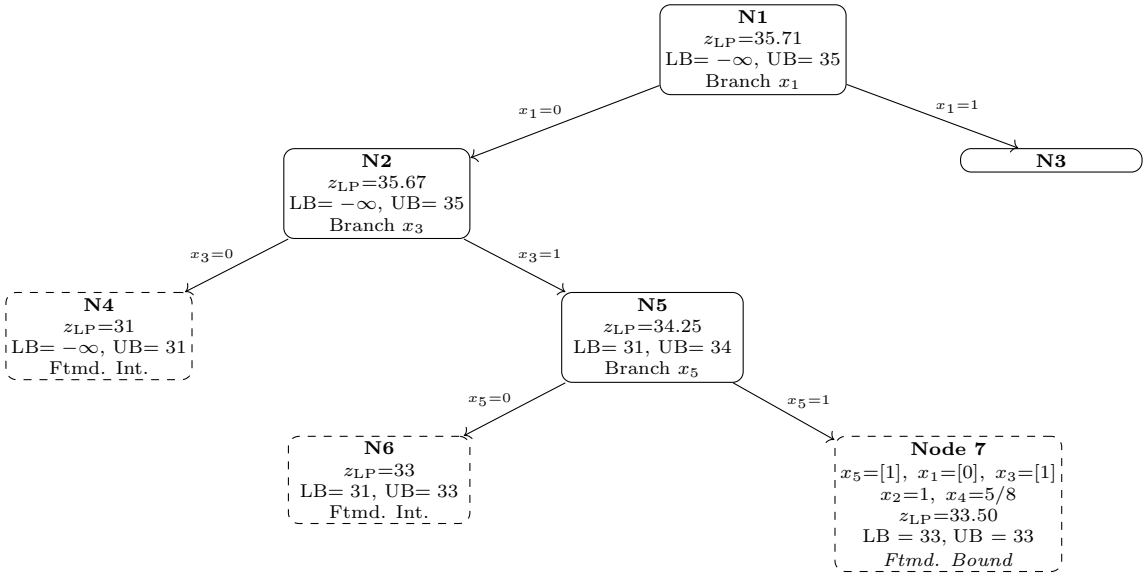
	P_5	P_1	P_3	P_2	P_4
c_i	4	14	9	2	8
x_i	[1]	[0]	[1]	1	5/8
Rem. cap	16	16	7	5	0

$z_{\text{LP}} = 5 + 7 + 14 + \frac{5}{8} \times 12 = 33.50 \quad \Rightarrow \quad \mathbf{UB} = \lfloor 33.50 \rfloor = 33, \quad \mathbf{LB} = 33.$

Status: Fathomed by Bound (UB $= 33 \leq 33 =$ LB).

Subproblem list:

- [Node 3]
- ~~[Node 2]~~
- ~~[Node 5]~~
- ~~[Node 4]~~
- ~~[Node 7]~~
- ~~[Node 6]~~



Step 7 — Node 3 ($x_1 = 1$)

Fixed: $x_1 = 1$.

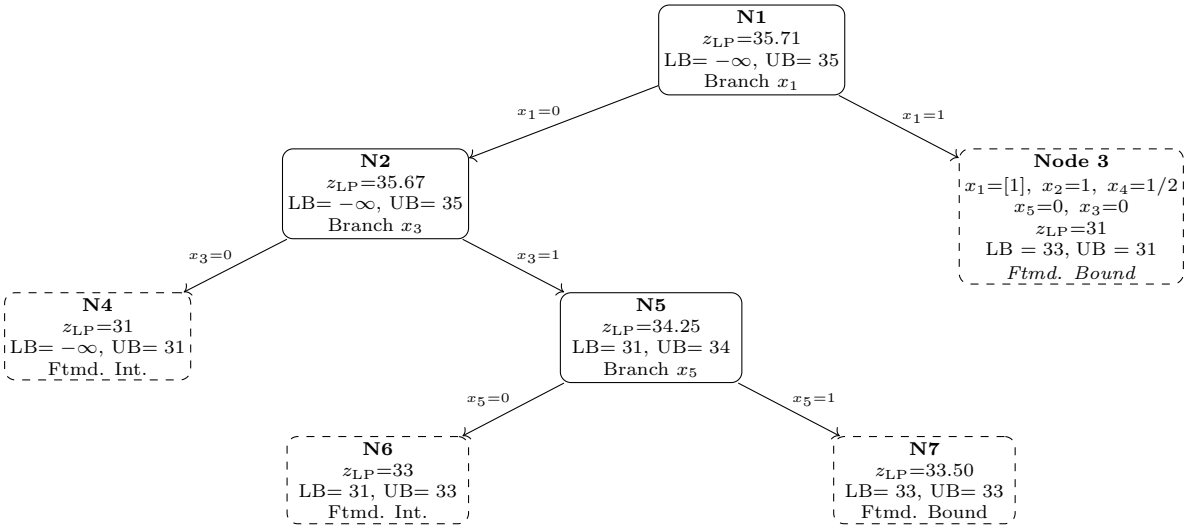
	P_1	P_2	P_4	P_5	P_3
c_i	14	2	8	4	9
x_i	[1]	1	1/2	0	0
Rem. cap	6	4	0	0	0

$z_{\text{LP}} = 11 + 14 + \frac{1}{2}\times 12 = 31 \quad \Rightarrow \quad \mathbf{UB} = 31, \quad \mathbf{LB} = 33.$

Status: Fathomed by Bound (UB = 31 ≤ 33 = LB).

Subproblem list:

- ~~[Node 3]~~
- ~~[Node 2]~~
- ~~[Node 5]~~
- ~~[Node 4]~~
- ~~[Node 7]~~
- ~~[Node 6]~~



Optimal Solution

All subproblems have been fathomed. The optimal solution is:

$$x_2^* = 1, \quad x_3^* = 1, \quad x_4^* = 1, \quad x_1^* = x_5^* = 0$$

$$Z^* = 14 + 7 + 12 = \mathbf{33} \text{ (\$M profit)} \qquad \text{Total cost} = 2 + 9 + 8 = 19 \leq 20 \checkmark$$

Select projects **Software Platform** (P_2), **Logistics Hub** (P_3), and **Regional Office** (P_4).

Branch-and-Bound Tree