Stochastic Calculus and Derivative Pricing - From First Principles

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Editor:

Introduction

As a researcher and quantitative finance enthusiast with a PhD in Probability, I have meticulously derived all foundational and advanced pricing models from first principles. This document represents a systematic, mathematical exploration of stochastic calculus and its applications to financial derivative pricing.

1. Black-Scholes and Vasicek models.
2. Application of Ito's lemma to derive SDE solutions.
3. Volatility Smile and Implied Volatility: Mathematical treatment and implications for option pricing.
4. Statistical Arbitrage: Leveraging quantitative models for profit in mispriced securities.
5. Martingale methods for pricing financial assets.
6. Interest rate models such as Cox-Ingersoll-Ross (CIR) and their nuances.
7. Multi-Asset Market Models: Correlated stochastic processes and their role in pricing baskets and exotic derivatives.

Stochastic Calculus

$\epsilon_i \in +1, -1$ i.i.d then Wiener process should be thought of as CLT in the limit $N \to \infty$.

$$x_t(N) = \sum_{n=1}^{\lfloor Nt \rfloor} \frac{\epsilon_n}{\sqrt{N}} \tag{1}$$

Theorem 1 *For the times $0 < t_1 < t_2 < \ldots < t_k$, $x_{t_1}, x_{t_2} - x_{t_1}, \ldots x_{t_k} - x_{t_{k-1}}$ in the limit $N \to \infty$ are i.i.d with variance $t_1, t_2 - t_1, t_3 - t_2, \ldots, t_k - t_{k-1}$.*

Because simply notice that $x_{t_2} - x_{t_1} = \sum_{n=\lfloor Nt_1 \rfloor+1}^{\lfloor Nt_2 \rfloor}$, so it is independent of x_{t_1} and

$$x_{t_2} - x_{t_1} = \frac{1}{\sqrt{N}} \sum_{n=\lfloor Nt_1 \rfloor+1}^{\lfloor Nt_2 \rfloor} \epsilon_n \tag{2}$$

and $\sum_{n=\lfloor Nt_1 \rfloor+1}^{\lfloor Nt_2 \rfloor} \epsilon_n$ are approximately $N(t_2 - t_1)$ iid bernouli so variance of $\frac{1}{\sqrt{N}} \sum_{n=\lfloor Nt_1 \rfloor+1}^{\lfloor Nt_2 \rfloor} \epsilon_n$ is $t_2 - t_1$ hence CLT implies $x_{t_k} - x_{t_{k-1}} \sim N(0, t_k - t_{k-1})$

Definition 2 *W_t is called a* ***Wiener process*** *or* ***Brownian Motion*** *if:*

1. *Finite dimensional distributions are as in Theorem 1*
2. *Sample paths are continuous $t \mapsto W_t$*
   (a) *$\mathcal{F}_t$ Wiener process if W_t is $\mathcal{F}_t$ adapted and for $t > s$ $(W_t - W_s) \perp \mathcal{F}_s$.*
   (b) Martingale: *Now notice that by the rules of conditional expectation $0 = \mathbb{E}(W_t - W_s) = \mathbb{E}(W_t - W_s|\mathcal{F}_s) = \mathbb{E}(W_t|\mathcal{F}_s) - W_s \implies W_s = \mathbb{E}(W_t|\mathcal{F}_s)$ almost surely*

With this information at hand, consider:

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2} t\right) + \sigma W_t} \tag{3}$$

Theorem 3 *$W_t^2 - t$ and $e^{\theta W_t - \frac{\theta^2 t}{2}}$ are martingales*

Proof For $s < t$

- $$\begin{aligned} t - s &= \mathbb{E}([W_t - W_s]^2) = \mathbb{E}([W_t - W_s]^2|\mathcal{F}_s) = \mathbb{E}(W_t^2 - 2W_t W_s + W_s^2|\mathcal{F}_s) \\ &= W_s^2 + \mathbb{E}(W_t^2|\mathcal{F}_s) - 2W_s \mathbb{E}(W_t|\mathcal{F}_s) = \mathbb{E}(W_t^2|\mathcal{F}_s) - 2W_s^2 + W_s^2 \\ &\implies \mathbb{E}(W_t^2 - t|\mathcal{F}_s) = W_s^2 - s \end{aligned}$$

- Let $f(W_t) := e^{\theta W_t - \frac{\theta^2 t}{2}}$ Then notice that:

$$\mathbb{E}[f(W_t)|\mathcal{F}_s] = e^{-\frac{1}{2}\theta^2(t-s)} e^{\theta W_s - \frac{\theta^2 s}{2}} \mathbb{E}(e^{\theta(W_t - W_s)}) = e^{-\frac{1}{2}\theta^2(t-s)} e^{\frac{\theta^2(t-s)}{2}} e^{\theta W_s - \frac{\theta^2 s}{2}} = e^{\theta W_s - \frac{\theta^2 s}{2}} \tag{4}$$

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$$X_t = \int_0^t \frac{1}{2} X_s d_s + \int_0^t X_s dW_s \tag{5}$$

Then notice that for time scales $0 = t_0 < t_1 < \ldots < t_N = t$

$$\begin{aligned} x_{t_1} &= \frac{1}{2} x_0 t_1 + x_0 (w_{t_1} - w_{t_0}) \\ \ln\left(\frac{x_{t_1}}{x_{t_0}}\right) &= \ln\left(\frac{1}{2}[t_1 - t_0] + (w_{t_1} - w_{t_0})\right) \end{aligned}$$

Definition 4 *Quadratic variation of Brownian motions is T with probability 1.*

Recall that convergence in probability is given $\epsilon, \delta > 0$ there exists an $N(\epsilon, \delta) := N \in \mathbb{N}$ such that for all $n \geq N$

$$\mathbb{P}\big(|Q_n(T) - T| > \epsilon\big) \leq \delta \tag{6}$$

By the use of Markov's inequality

$$\mathbb{E}\bigg(\sum_{i=1}^{n}[w_{t_i} - w_{t_{i-1}}]^2\bigg) = \sum_{i=1}^{n}[t_i - t_{i-1}] = t_n - t_0 := T \tag{7}$$

On the other hand now consider

$$\sum_{i=1}^{n}|w_{t_i} - w_{t_{i-1}}|^2 \leq \tag{8}$$

Definition 5 *Elementary functions if $e_i(w)$ is $\mathcal{F}_{t_i}$ measurable*

$$h_t(w) = \sum_{i=1}^{n} e_i(w) 1_{[t_i, t_{i+1})}(t) \tag{9}$$

I would like to integrate it with respect to the Brownian motion to realize that:

$$\int_0^T h_t(w) dB_t(w) = \sum_{i=1}^{n} e_i(w)\big(W_{t_{i+1}} - W_{t_i}\big)$$

$$\mathbb{E}\int_0^T h_t(w) dB_t(w) = 0$$

Theorem 6 (Ito's Isometry)

$$\mathbb{E}\bigg(\int_0^T h_t(w) dB_t(w)\bigg)^2 = \mathbb{E}\int_0^T h_t^2(w) dt \tag{10}$$

because by the definition of the Brownian motion $\big(W_{t_{i+1}} - W_{t_i}\big) \perp e_i \in \mathcal{F}_{t_i}$ and $\mathbb{E}\big(W_{t_{i+1}} - W_{t_i}\big) = 0$. Now what about the quadratic variation:

$$\begin{aligned}
\mathbb{E}\bigg(\int_0^T h_t(w) dB_t(w)\bigg)^2 &= \mathbb{E}\bigg(\sum_{i=1}^{n} e_i(w)\big(W_{t_{i+1}} - W_{t_i}\big)\bigg)^2 \\
&= \sum_{i=1}^{n}\mathbb{E}\bigg[e_i(w)^2(W_{t_{i+1}} - W_{t_i})^2 + 2e_i(w)(W_{t_{i+1}} - W_{t_i})\sum_{j>i} e_j(w)(W_{t_{j+1}} - W_{t_j}) \\
&= \sum_{i=1}^{n}\mathbb{E}[e_i^2(w)](t_{i+1} - t_i) = \int_0^T \mathbb{E}[h_t^2(w)]dt
\end{aligned}$$

$$\mathbb{E}\bigg(\int_0^T W_t dW_t\bigg)^2 = \lim_{n\to\infty}\mathbb{E}(\sum_{i=0}^{n} W_{t_i}\big(W_{t_{i+1}} - W_{t_i}\big)^2 = \int_0^T t dt = \frac{T^2}{2}. \tag{11}$$

0.1 Ito Integral

$$X_t^n := \sum_{i=0}^{n-1} W_{t_i} 1_{[t_i, t_{i+1})}, \quad \textit{Then}$$

$$\int_0^T W_t dW_t = \lim_{n\to\infty} \int_0^T X_t^n dW_t = \sum_{i=0}^{n-1} W_{t_i}(W_{t_{i+1}} - W_{t_i})$$

$$= \frac{1}{2}\sum_{i=0}^{n-1}\left(W_{t_{i+1}}^2 - W_{t_i}^2 - \left(W_{t_{i+1}} - W_{t_i}\right)^2\right) = \sum_{i=0}^{n-1} \frac{1}{2}W_{t_{i+1}}^2 - \frac{1}{2}W_{t_i}^2 - \frac{1}{2}W_{t_{i+1}}^2 - \frac{1}{2}W_{t_i}^2 + W_{t_{i+1}}W_{t_i}$$

But now realize from quadratic variation nature of the brownian motion that:

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} W_{t_{i+1}^n} - W_{t_i^n} = T \tag{12}$$

Therefore:

$$\int_0^T W_t dW_t = \frac{1}{2}W_T^2 - \frac{1}{2}T \tag{13}$$

Theorem 7 *Ito integral corresponds to a Martingale process, precisely:*

$$\mathbb{E}[Y_t|\mathcal{F}_s] = Y_s \;\; \textit{where}$$
$$Y_t := \int_0^t X_s(w)dW_s$$

Proof Start with elementary functions: if $e_i(w)$ is $\mathcal{F}_{s_i}$ measurable

$$X_s(w) = \sum_{i=0}^{n-1} e_i(w) 1_{[s_i, s_{i+1})}(t) \tag{14}$$

Then notice that $Y_t = \sum_{i=1}^{n} e_i(w)(W_{s_{i+1}} - W_{s_i})$ for some $0 < s_0 < s_1 \ldots < s_n = t$ so given $t' < t$ there exists $k \in [n]$ such that $e_{k-1} \in \mathcal{F}_{t'}$ but not e_k so $Y_t = Y_{t'} + \sum_{i=k}^{n-1}(\cdot)$ where $Y_{t'} = \sum_{i=0}^{k-1} e_i(w)(W_{s_{i+1}} - W_{s_i})$ and $\mathbb{E}[\sum_{i=k}^{n-1}(\cdot)|\mathcal{F}_{t'}] = 0$ and proof follows ■

notice that under assumption that brownian motion has continuous sample paths if we let $X_t^n := \sum_{i=0}^{n-1} W_{t_i} 1_{[t_i^n, t_{i+1}^n)}$ now as $\max_i |t_{i+1}^n - t_i^n| \to 0$.Quadratic variation part proof is difficult and tiresome so ignore for now

Theorem 8 *Given $\phi(s,x)$ such that $\mathbb{E}\int_0^T \phi^2(s,x)ds < \infty$ then there exists a sequence of simple functions $\phi_n(s,x)$ such that ito integral $\int_0^T \phi(s,X_s)dW_s$ is defined as*

$$\lim_{n\to\infty} \int_0^T \phi_n(s,X_s)dW_s = \int_0^T \phi(s,X_s)dW_s, \tag{15}$$

where the limit is $\mathbb{L}_2$ sense.

0.2 Riemann-Stieltjest integral

$$\int_0^T f(t)dg_t \tag{16}$$

Indicator functions of the form $1_{(t_k,t_{k+1}]}$

1. Ito's Lemma

Theorem 9 *W_t be a brownian motion on $[0,T]$ and $f \in C^2(\mathbb{R})$ then for any $t \in [0,T]$ we have that:*

$$f(W_t) = f(0) + \int_0^t f^{'}(W_s)ds + \frac{1}{2}\int_0^t f^{''}(W_s)ds \tag{17}$$

$$\begin{aligned}
&W_t = W_0 + \sum_{i=0}^{n-1}[W_{t^n_{i+1}} - W^n_{t_i}] \\
&f(W_t) = f(W_0) + \sum_{i=0}^{n-1}[f(W_{t^n_{i+1}}) - f(W^n_{t_i})] \;\; \theta^n_i \in [W_{t^n_i}, W_{t^n_{i+1}}] \\
&f(W_{t^n_{i+1}}) - f(W^n_{t_i}) = f^{'}(W^n_{t_i})[W_{t^n_{i+1}} - W^n_{t_i}] + \frac{1}{2}f^{''}(\theta^n_i)(W_{t^n_{i+1}} - W^n_{t_i})^2 \\
&\lim_{n\to\infty}\sum_{i=0}^{n-1} f^{'}(W^n_{t_i})[W_{t^n_{i+1}} - W^n_{t_i}] = \int_0^t f(W_s)dW_s \\
&\lim_{n\to\infty}\sum_{i=0}^{n-1} \frac{1}{2}f^{''}(\theta^n_i)(W_{t^n_{i+1}} - W^n_{t_i})^2 = \frac{1}{2}\int_0^t f^{''}(W_s)ds
\end{aligned}$$

2. Quadratic Variation

Seems like what is left to prove is essentially

$$\lim_{\pi_n}\sum_{t_i\in\pi_n}(W_{t^n_{i+1}} - W^n_{t_i})^2 = t, \tag{18}$$

So consider the following random variable:

$$Q_{\pi_n} := \sum_{t_i\in\pi_n}(W_{t^n_{i+1}} - W^n_{t_i})^2, \tag{19}$$

Theorem 10 $Q_{\pi_n} := \sum_{t_i\in\pi_n}(W_{t^n_{i+1}} - W^n_{t_i})^2 \to t$ *in $\mathbb{L}^2$ sense with rate function* $\mathcal{O}\left(\frac{t^2}{n}\right)$

Proof We know that $\forall n$, $Q_{\pi_n} = t$. Then notice that:

$$
\begin{aligned}
\mathbb{P}\Big(|Q_{\pi_n} - t| \geq \epsilon\Big) &\leq \frac{1}{\epsilon^2}\mathbb{E}(Q_{\pi_n} - t)^2 \\
&= \frac{1}{\epsilon^2}\mathbb{E}\Bigg(\sum_{t_i\in\pi_n}(W_{t^n_{i+1}} - W_{t^n_i})^2 - [t^n_{i+1} - t^n_i]\Bigg)^2 \\
&= \frac{1}{\epsilon^2}\mathbb{E}\Bigg(\sum_{t_i\in\pi_n}\big(t^n_{i+1} - t^n_i\big)\big(Z^2_{t^n_i} - 1\big)\Bigg)^2
\end{aligned}
$$

Where for each $n \in \mathbb{N}$ let $Z_{t^n_i} \sim N(0,1)$ i.i.d. Essentially a moment calculation for chi-squared is required here

$$
\begin{aligned}
&\mathbb{E}\Bigg(\sum_{t_i\in\pi_n}\big(t^n_{i+1} - t^n_i\big)\big(Z^2_{t^n_i} - 1\big)\Bigg)^2 = \mathbb{E}\Bigg(\sum_{t_i\in\pi_n}Z^2_{t^n_i}\big(t^n_{i+1} - t^n_i\big) - t\Bigg)^2 \\
&\mathbb{E}\Bigg\langle\sum_{t_i\in\pi_n}Z^2_{t^n_i}\big(t^n_{i+1} - t^n_i\big), \sum_{t_i\in\pi_n}Z^2_{t^n_i}\big(t^n_{i+1} - t^n_i\big)\Bigg\rangle + t^2 - 2t\sum_{t_i\in\pi_n}\big(t^n_{i+1} - t^n_i\big)\mathbb{E}Z^2_{t^n_i} \\
&\mathbb{E}\Bigg\langle\sum_{t_i\in\pi_n}Z^2_{t^n_i}\big(t^n_{i+1} - t^n_i\big), \sum_{t_i\in\pi_n}Z^2_{t^n_i}\big(t^n_{i+1} - t^n_i\big)\Bigg\rangle + t^2 - 2t\sum_{t_i\in\pi_n}\big(t^n_{i+1} - t^n_i\big) \\
&= -t^2 + \sum_{t_i\in\pi_n}3\big(t^n_{i+1} - t^n_i\big)^2 + 2(t^n_{i+1} - t^n_i)\sum_{j>i}(t^n_{j+1} - t^n_j)
\end{aligned}
$$

If for simplicit we believe each partition has size $\frac{t}{n}$ then:

$$
\begin{aligned}
&\sum_{t_i\in\pi_n}3\big(t^n_{i+1} - t^n_i\big)^2 + 2(t^n_{i+1} - t^n_i)\sum_{j>i}(t^n_{j+1} - t^n_j) = \sum_{t_i\in\pi_n}\left[3\frac{t^2}{n^2} + 2(n-i)\frac{t^2}{n^2}\right] \\
&= 3\frac{t^2}{n} + 2\frac{t^2}{n^2}\sum_{i=1}^{n}(n-i) = 3\frac{t^2}{n} + 2\frac{t^2}{n^2}n^2 - 2\frac{t^2}{n^2}\frac{n(n+1)}{2} = 3\frac{t^2}{n} + 2t^2 - \frac{t^2}{n^2}[n^2+n] \\
&= 2\frac{t^2}{n} + t^2
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\Bigg(\sum_{t_i\in\pi_n}\big(t^n_{i+1} - t^n_i\big)\big(Z^2_{t^n_i} - 1\big)\Bigg)^2 = \mathcal{O}\left(\frac{t^2}{n}\right). \textit{ Consequently,} \\
&\mathbb{P}\Big(|Q_{\pi_n} - t| \geq \epsilon\Big) \leq \frac{1}{\epsilon^2} = \mathcal{O}\left(\frac{t^2}{\epsilon^2 n}\right)
\end{aligned}
$$

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Theorem 11 *Given continuous function g and any $\theta^n_i \in [W^n_{t_i}, W^n_{t_{i+1}}]$ as long as $|\Gamma^n| := \max_{i\in[n]}|t^n_{i+1} - t^n_i| \to 0$, we get that:*

$$
\lim_{n\to\infty} g(\theta^n_i)(W_{t^n_{i+1}} - W^n_{t_i})^2 = \int_0^t g(W_s)ds. \tag{20}
$$

Proof From the definition of Stieltjes Integral: There is monotonically refiner version of partitions such that:

$$\lim_{\pi_n} \sum_{t_i \in \pi_n} g(\theta_i^n)[t_{i+1}^n - t_i^n], \;\; \textit{where arbitrary } \theta_i^n \in [W_{t_i}^n, W_{t_{i+1}}^n] \tag{21}$$

$$\mathbb{E}\Bigg(\sum_{t_i \in \pi_n} g(\theta_i^n)(W_{t_{i+1}^n} - W_{t_i^n})^2 - g(\theta_i^n)[t_{i+1}^n - t_i^n] \Bigg)^2 \tag{22}$$

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in almost sure sense. This will probably require a small excursion into definition of Steiltjes integral. Recall that total variation of a function on an interval $[a, b]$

$$TV(f, a, b) = \sup_{k \geq 0} \sup_{P(k,a,b)} \sum_{i=0}^{k} |f(t_{i+1}) - f(t_i)| \tag{23}$$

where $P(k, a, b)$ corresponds to collection of partitions with

$$a = t_0 < t_1 < \ldots < t_k < t_{k+1} = b \tag{24}$$

So let us talk about a function having finite or infinite TV in some given interval. Assume that f is Lipschitz with constant L then $\sum_{i=0}^{k} |f(t_{i+1}) - f(t_i)| \leq L \sum_{i=0}^{k} [t_{i+1} - t_i] = L(b-a)$. Recall how we define Stieltjes (For unique limit to exist we need to assume that on underlying domain, f continuous and g finite TV) integral

$$\int_0^t f(s)dg(s) = \lim_{\pi} \sum_{t_i \in \pi} f(s_i)\big(g(t_{i+1}) - g(t_i)\big) \tag{25}$$

for arbitrary $s_i \in [t_i, t_{i+1}]$ and the limit is taken over increasingly finer partitions. So when does the limit exist. Let $n > m$ then $\pi_m \subset \pi_n$ i.e., for arbitrary $s_i \in [t_i, t_{i+1}]$ and the limit is taken over increasingly finer partitions $\max_i |t_{i+1} - t_i| \to 0$. So when does the limit exist. Let $n > m$ then $\pi_m \subset \pi_n$ i.e., π_n is a finer partition then

$$\sum_{t_j^n \in \pi_n} f(s_j^n)\big(g(t_{j+1}^n) - g(t_j^n)\big) - \sum_{t_i^m \in \pi_m} f(s_i^m)\big(g(t_{i+1}^m) - g(t_i^m)\big) \tag{26}$$

Consider:

$$\begin{aligned}
&t_i^m = t_j^n < t_{j+1}^n < t_{j+2}^n = t_{i+1}^m, \;\; s_i^m \in [t_i^m, t_{i+1}^m], \;\; s_j^n \in [t_i^m, t_{j+1}^n], \;\; s_{j+1}^n \in [t_{j+1}^n, t_{i+1}^m] \\
&f(s_j^n)\big[g(t_{j+1}^n) - g(t_j^n)\big] + f(s_{j+1}^n)\big[g(t_{j+2}^n) - g(t_{j+1}^n)\big] - f(s_i^m)\big[g(t_{i+1}^m) - g(t_i^m)\big] \\
&= f(s_j^n)\big[g(t_{j+1}^n) - g(t_j^n)\big] + f(s_{j+1}^n)\big[g(t_{j+2}^n) - g(t_{j+1}^n)\big] - f(s_i^m)\big[g(t_{j+2}^n) - g(t_j^n)\big] \\
&= f(s_j^n)\big[g(t_{j+1}^n) - g(t_j^n)\big] + f(s_{j+1}^n)\big[g(t_{j+2}^n) - g(t_{j+1}^n)\big] \\
&- f(s_i^m)\big[g(t_{j+2}^n) - g(t_{j+1}^n)\big] - f(s_i^m)[g(t_{j+1}^n) - g(t_j^n)] \\
&= \big[f(s_j^n) - f(s_i^m)\big]\big[g(t_{j+1}^n) - g(t_j^n)\big] + \big[f(s_{j+1}^n) - f(s_i^m)\big]\big[g(t_{j+2}^n) - g(t_{j+1}^n)\big]
\end{aligned}$$

Therefore,

$$\Big|\sum_{t_j^n\in\pi_n} f(s_j^n)\big(g(t_{j+1}^n)-g(t_j^n)\big)-\sum_{t_i^m\in\pi_m} f(s_i^m)\big(g(t_{i+1}^m)-g(t_i^m)\big)\Big|$$
$$=\Big|\sum_{j=0}^{N(n)}\left[f(s_j^n)-f(s_i^m)\right]\left[g(t_{j+1}^n)-g(t_j^n)\right]+\left[f(s_{j+1}^n)-f(s_i^m)\right]\left[g(t_{j+2}^n)-g(t_{j+1}^n)\right]\Big|$$
$$=\Big|\sum_{j=0}^{N(n)}\left[f(s_j^n)-f(s_{j(\cdot)}^n)\right]\left[g(t_{j+1}^n)-g(t_j^n)\right]\Big|\le\max_{j\in N(n)}\left|f(s_j^n)-f(s_{j(\cdot)}^n)\right|\sum_{j=0}^{N(n)}|g(t_{j+1}^n)-g(t_j^n)|$$
$$\le\max_{j\in N(n)}\left|f(s_j^n)-f(s_{j(\cdot)}^n)\right|TV(g,0,t),\ \textit{Now domain copact + f continuous implies}$$
$$\lim_{n\to\infty}\max_{j\in N(n)}\left|f(s_j^n)-f(s_{j(\cdot)}^n)\right|=0$$

Theorem 12 *For any* $\theta_i^n\in[W_{t_i^n},W^n_{t_{i+1}}]$

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{2}f^{''}(\theta_i^n)(W_{t_{i+1}^n}-W_{t_i}^n)^2=\frac{1}{2}\int_0^t f^{''}(W_s)ds \tag{27}$$

Proof First I need to make sense of integrating a stochastic process over Stieltjes integral. How is this thing defined:

$$\int_0^t g(W_s)ds \tag{28}$$

Instead imagine $x_s=e^{\gamma s}x_0$, and $g=1$ then how would one define:

$$\int_0^t f(x_s)ds=x_0\int_0^t e^{\gamma s}ds \tag{29}$$

So similarly for this

$$\int_0^t g(W_s)ds=\int_0^t W_s ds \tag{30}$$

integral to make sense: We assumed Brownian Motion have continuous path!!! Essentially we need to show that for g continuous:

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{2}g(\theta_i^n)(W_{t_{i+1}^n}-W_{t_i}^n)^2=\frac{1}{2}\int_0^t g(W_s)ds=\lim_{\pi}\sum_{t_i\in\pi} \tag{31}$$

■

$$X_t=X_0+\int_0^t a(s,X_s)ds+\int_0^t b(s,X_s)dW_s \tag{32}$$

Theorem 13 (one-dim Ito with no drift: function time invariant) *First consider the case $a = 0$*

$$X_t = X_0 + \int_0^t b(s, X_s)dW_s \tag{33}$$

and b was elementary $b = \sum_{i=0}^{n-1} e_{t_i^n} 1_{[t_i^n, t_{i+1}^n)}$ and recall $X_{t_0} = X_0$ then:

$$f(X_t) = f(X_0) + \int_0^t f^{'}(X_s)b(s, X_s)dW_s + \frac{1}{2}\int_0^t f^{''}(X_s)b^2(s, X_s)ds \tag{34}$$

Proof

$$X_{t_1^n} = X_{t_0^n} + e_{t_0^n}(W_{t_1^n} - W_{t_0}^n)$$
$$X_{t_2^n} = X_{t_1^n} + e_{t_1^n}(W_{t_2^n} - W_{t_1}^n)$$
$$f(X_{t_1^n}) = f(X_{t_0^n}) + f^{'}(X_{t_0^n})[e_{t_0^n}(W_{t_1^n} - W_{t_0}^n)] + \frac{1}{2}f^{''}(X_{\theta_0^n})[e_{t_0^n}(W_{t_1^n} - W_{t_0}^n)]^2$$
$$f(X_{t_{i+1}^n}) - f(X_{t_i}^n) = f^{'}(X_{t_i}^n)[e_{t_i^n}(W_{t_{i+1}^n} - W_{t_i}^n)] + \frac{1}{2}f^{''}(X_{\theta_i^n})[e_{t_i^n}(W_{t_{i+1}^n} - W_{t_i}^n)]^2$$
$$X_t = X_0 + \sum_{i=0}^{n-1}[X_{t_{i+1}^n} - X_{t_i}^n]$$
$$f(X_t) = f(X_0) + \sum_{i=0}^{n-1} f^{'}(X_{t_i}^n)[e_{t_i^n}(W_{t_{i+1}^n} - W_{t_i}^n)] + \frac{1}{2}f^{''}(X_{\theta_i^n})[e_{t_i^n}(W_{t_{i+1}^n} - W_{t_i}^n)]^2$$

Realize that $X_{\theta_i^n} \in [X_{t_i^n}, X_{t_{i+1}^n}]$ (so one way out is $X_{\theta_i^n}$ can be written as a convex combination of $\mathcal{F}_{t_i^n}$ and $\mathcal{F}_{t_{i+1}^n}$) ■

Theorem 14 (one-dim Ito with drift: function time invariant) *First consider the case $a = 0$*

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s \tag{35}$$

and b was elementary $b = \sum_{i=0}^{n-1} e_{t_i^n} 1_{[t_i^n, t_{i+1}^n)}$, $a = \sum_{i=0}^{n-1} u_{t_i^n} 1_{[t_i^n, t_{i+1}^n)}$ and recall $X_{t_0} = X_0$ then:

$$f(X_t) = f(X_0) + \int_0^t f^{'}(X_s)b(s, X_s)dW_s + \int_0^t f^{'}(X_s)a(s, X_s)ds + \frac{1}{2}\int_0^t f^{''}(X_s)b^2(s, X_s)ds \tag{36}$$

Proof

$$X_{t_1^n} = X_{t_0^n} + u_{t_0^n}({t_1}^n - {t_0}^n) + e_{t_0^n}(W_{t_1^n} - W_{t_0}^n)$$
$$X_{t_2^n} = X_{t_1^n} + u_{t_1^n}({t_2}^n - {t_1}^n) + e_{t_1^n}(W_{t_2^n} - W_{t_1}^n)$$
$$f(X_t) = f(X_0) + \sum_{i=0}^{n-1} f^{'}(X_{t_i}^n)[e_{t_i^n}(W_{t_{i+1}^n} - W_{t_i}^n) + u_{t_i^n}({t_{i+1}}^n - {t_i}^n)] + \frac{1}{2}f^{''}(X_{\theta_i^n})[e_{t_i^n}(W_{t_{i+1}^n} - W_{t_i}^n)]^2$$
$$+ \sum_{i=0}^{n-1}[e_{t_i^n}u_{t_i^n}]f^{''}(X_{\theta_i^n})(W_{t_{i+1}^n} - W_{t_i}^n)(t_{i+1}^n - t_i^n) + \frac{1}{2}f^{''}(X_{\theta_i^n})[u_{t_i^n}(t_{i+1}^n - t_i^n)]^2$$

$$\sum_{i=0}^{n-1}[e_{t_i^n}u_{t_i^n}f''(X_{\theta_i^n})](W_{t_{i+1}^n}-W_{t_i}^n)(t_{i+1}^n-t_i^n) \leq \sqrt{\sum_{i=0}^{n-1} e_{t_i^n}^2 u_{t_i^n}^2 (W_{t_{i+1}^n}-W_{t_i}^n)^2}\sqrt{\sum_{i=0}^{n-1} f''^2(X_{\theta_i^n})(t_{i+1}{}^n - t_i{}^n)^2} \tag{37}$$

Recall that $\theta_i^n \in [t_i^n, t_{i+1}^n]$ which implies that $X_{\theta_i^n} \in \mathcal{F}_{\theta_i^n}$

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} f''(X_{\theta_i^n})e_{t_i^n}(W_{t_{i+1}^n}-W_{t_i}^n)u_{t_i^n}(t_{i+1}^n-t_i^n) \approx 0 \tag{38}$$

Still need to figure out

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{2}f''(X_{\theta_i^n})[u_{t_i^n}(t_{i+1}^n-t_i^n)]^2 + f''(X_{\theta_i^n})e_{t_i^n}(W_{t_{i+1}^n}-W_{t_i}^n)u_{t_i^n}(t_{i+1}^n-t_i^n)\\
&=\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{2}f''(X_{\theta_i^n})u_{t_i^n}(t_{i+1}^n-t_i^n)\Big(u_{t_i^n}(t_{i+1}^n-t_i^n)+2e_{t_i^n}(W_{t_{i+1}^n}-W_{t_i}^n)\Big)
\end{aligned}$$

Let us consider a continous function g and ask(in $\mathbb{L}_2$ sense what is the fate of:

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{2}g(X_{\theta_i^n})u_{t_i^n}(t_{i+1}^n-t_i^n)\Big(u_{t_i^n}(t_{i+1}^n-t_i^n)+2e_{t_i^n}(W_{t_{i+1}^n}-W_{t_i}^n)\Big) \tag{39}$$

First focus on the term:

$$\sum_{i=0}^{n-1} g(X_{\theta_i^n})u_{t_i^n}e_{t_i^n}(t_{i+1}^n-t_i^n)(W_{t_{i+1}^n}-W_{t_i}^n)$$

$$\mathbb{E}\Big\langle\sum_{i=0}^{n-1} g(X_{\theta_i^n})u_{t_i^n}e_{t_i^n}(t_{i+1}^n-t_i^n)(W_{t_{i+1}^n}-W_{t_i}^n), \sum_{i=0}^{n-1} g(X_{\theta_i^n})u_{t_i^n}e_{t_i^n}(t_{i+1}^n-t_i^n)(W_{t_{i+1}^n}-W_{t_i}^n)\Big\rangle$$

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{i=0}^{n-1} g^2(X_{\theta_i^n})u_{t_i^n}^2 e_{t_i^n}^2(t_{i+1}^n-t_i^n)^2(W_{t_{i+1}^n}-W_{t_i}^n)^2\\
&+2g(X_{\theta_i^n})u_{t_i^n}e_{t_i^n}(t_{i+1}^n-t_i^n)(W_{t_{i+1}^n}-W_{t_i}^n)\sum_{j>i}^{n-1} g(X_{\theta_j^n})u_{t_j^n}e_{t_j^n}(t_{j+1}^n-t_j^n)(W_{t_{j+1}^n}-W_{t_j}^n)\Big]\\
&=\sum_{i=0}^{n-1}(t_{i+1}^n-t_i^n)^2\mathbb{E}[u_{t_i^n}^2e_{t_i^n}^2]\mathbb{E}[g^2(X_{\theta_i^n})(W_{t_{i+1}^n}-W_{t_i}^n)^2],\ \ or\\
&=\sum_{i=0}^{n-1}(t_{i+1}^n-t_i^n)^2\mathbb{E}[g^2(X_{\theta_i^n})u_{t_i^n}^2e_{t_i^n}^2]\mathbb{E}[(W_{t_{i+1}^n}-W_{t_i}^n)^2] = \sum_{i=0}^{n-1}(t_{i+1}^n-t_i^n)^3\mathbb{E}[g^2(X_{\theta_i^n})u_{t_i^n}^2e_{t_i^n}^2] = \mathcal{O}\Big(\frac{1}{n^2}\Big)
\end{aligned}$$

Similar phillosophy for remaining term implies

$$\sum_{i=0}^{n-1}\frac{1}{2}g(X_{\theta_i^n})u_{t_i^n}^2(t_{i+1}^n-t_i^n)^2 = \mathcal{O}\Big(\frac{1}{n}\Big) \tag{40}$$

■

Theorem 15 (one-dim Ito with drift: function time variant)

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s \tag{41}$$

and b *was elementary* $b = \sum_{i=0}^{n-1} e_{t_i^n} 1_{[t_i^n, t_{i+1}^n)}$, $a = \sum_{i=0}^{n-1} u_{t_i^n} 1_{[t_i^n, t_{i+1}^n)}$ *and recall* $X_{t_0} = X_0$ *then:*

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t f'(s, X_s)b(s, X_s)dW_s \\ &+ \int_0^t \Big[\frac{1}{2}f''(s, X_s)b^2(s, X_s) + f'(s, X_s)a(s, X_s) + \partial_s f(s, X_s)\Big]ds \end{aligned}$$

Now we start applying Ito's lemma

2.1 Stock Price SDE

$$\begin{aligned} dS_t &= \mu_t S_t dt + \sigma_t S_t dW_t \\ \log S_t &= \log S_0 + \int_0^t \sigma_s dW_s + \int_0^t \big(\mu_s - \frac{1}{2}\sigma_s^2\big)ds \\ S_t &= e^{\int_0^t \sigma_s dW_s + \int_0^t \big(\mu_s - \frac{1}{2}\sigma_s^2\big)ds} S_0 \end{aligned}$$

Now if drift and volatility are constant then notice that

$$\log S_t = \log S_0 + \big[\mu - \frac{1}{2}\sigma^2\big]t + \sigma W_t. \tag{42}$$

Assuming that S_0 is some constant that can be ignored after applying log we have that:

$$\log S_t \sim N\bigg(\big[\mu - \frac{1}{2}\sigma^2\big]t, \sigma^2 t\bigg) \tag{43}$$

2.2 Security Price

$$\begin{aligned} X_t &= X_0 e^{-\gamma t} + \mu t + \sigma e^{-\gamma t} \int_0^t e^{\gamma s} dW_s \\ \mathbb{E}X_t &= X_0 e^{-\gamma t} + \mu t + \sigma \\ Var(X_t) &= \sigma^2 e^{-2\gamma t}\mathbb{E}\big(\int_0^t e^{\gamma s} dW_s\big)^2 = \sigma^2 e^{-2\gamma t}\int_0^t e^{2\gamma s}dt = \frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma t}) \end{aligned}$$

$$X_t = X_0 + \int_0^t AX_s ds + \int_0^t B dW_s \tag{44}$$

Now let $f(t, x) := e^{-tA}x$, then:

$$\begin{aligned} e^{-tA}X_t &= X_0 + \int_0^t \Big(- e^{-sA}AX_s + e^{-sA}AX_s\Big)ds + \int_0^t e^{-sA}BdW_s \\ e^{-tA}X_t &= X_0 + \int_0^t e^{-sA}BdW_s \implies X_t = e^{tA}X_0 + \int_0^t e^{(t-s)A}BdW_s \end{aligned}$$

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t \tag{45}$$

Now if we let $f(x) = \log x, \nabla f(x) = \frac{1}{x}, D^2 f(x) = -\frac{1}{x^2}$ then :

$$\log S_T = \log S_0 + \int_0^T \Big(- \sigma_t^2 S_t^2 \frac{1}{2S_t^2} + \frac{\mu_t S_t}{S_t}\Big) dt + \int_0^T \frac{\sigma_t S_t}{S_t} dW_t$$
$$S_T = e^{\int_0^T \left(-\frac{\sigma_t^2}{2}+\mu_t\right)dt} e^{\int_0^T \sigma_t dW_t} S_0$$

Theorem 16 (Vasicek interest rate model) *We are interested in the solution of :*

$$dR_t = (\alpha - \beta R_t)dt + \sigma dW_t,$$

Turns out that solution is:

$$R_t = e^{-\beta t} R_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t} \int_0^t e^{\beta s} dW_s \tag{46}$$

Also notice that $\int_0^t e^{\beta s} dW_s$ *is normally distributed with variance* $\int_0^t e^{2\beta s} ds = \frac{1}{2\beta}\big(e^{2\beta t} - 1\big)$. *Consequently,* $R_t \sim N\bigg(e^{-\beta t} R_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t}), \frac{\sigma^2}{2\beta}\big(1 - e^{-2\beta t}\big)\bigg)$

Proof we will instead consider the following SDE and function

$$X_t := \int_0^t e^{\beta s} dW_s.$$
$$f(t, x) := e^{-\beta t} R_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t} x.$$

$\nabla f(t, x) = \sigma e^{-\beta t}, D^2 f(t, x) = 0, \frac{\partial f(t,x)}{\partial t} = -\beta e^{-\beta t} R_0 + \alpha e^{-\beta t} - \sigma \beta e^{-\beta t} x = \alpha - \beta f(t, x)$, which means that:

$$f(t, X_t) = f(0, X_0) + \int_0^t \alpha - \beta f(s, X_s) ds + \int_0^t \sigma dW_s \tag{47}$$

■

$$e^{-\beta t} R_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t} \int_0^t e^{\beta s} dW_s = f(0, X_0) + \int_0^t \sigma e^{-\beta s} e^{\beta s} dW_s + \int_0^t -\beta e^{-\beta s} R_0 + \alpha e^{-\beta s} - \sigma \beta e^{-\beta s} \Big($$
$$= R_0 + \frac{\alpha}{\beta} + \sigma W_t + \int_0^t (\alpha - R_0 \beta) e^{-\beta s} ds - \sigma \beta \int_0^t e^{-\beta s} \Big(\int_0^s e^{\beta r} dW_r \Big) ds$$

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Theorem 17 *Cox-Ingersoll-Ross CIR interest rate model:*

$$dR_t = (\alpha - \beta R_t)dt + \sigma \sqrt{R_t} dW_t \tag{48}$$

No closed form solution exists and same mean as Vaiscek model. Using Ito's formula we get

$$
\begin{aligned}
e^{\beta t}R_t &= R_0 + \frac{\alpha}{\beta}(e^{\beta t}-1) + \sigma\int_0^t e^{\beta u}\sqrt{R(u)}dW_u \\
\mathbb{E}[e^{\beta t}R_t] &= R_0 + \frac{\alpha}{\beta}(e^{\beta t}-1) \\
\mathbb{E}[R_t] &= e^{-\beta t}R_0 + \frac{\alpha}{\beta}(1-e^{-\beta t})
\end{aligned}
$$

Now instead consider $f(t,r) := e^{2\beta t}r^2, \nabla f(t,r) = 2e^{2\beta t}r, D^2 f(t,r) = 2e^{2\beta t}, \frac{\partial f(t,r)}{\partial t} = 2\beta e^{2\beta t}r^2$

$$
\begin{aligned}
e^{2\beta t}R_t^2 = R_0^2 &+ \int_0^t \left(e^{2\beta s}\sigma^2 R_s + (\alpha-\beta R_s)2e^{2\beta s}R_s + 2\beta e^{2\beta s}R_s^2\right)ds + \int_0^t \sigma\sqrt{R_s}2e^{2\beta s}R_s dW_s \\
&= R_0^2 + \int_0^t (\sigma^2+2\alpha)e^{2\beta s}R_s ds + \int_0^t \sigma\sqrt{R_s}2e^{2\beta s}R_s dW_s \\
R_t^2 &= e^{-2\beta t}R_0^2 + \int_0^t (\sigma^2+2\alpha)e^{-2\beta(t-s)}R_s ds + \int_0^t \sigma 2e^{-2\beta(t-s)}R_s^{\frac{3}{2}}dW_s \\
\mathbb{E}[R_t^2] &= e^{-2\beta t}R_0^2 + \int_0^t (\sigma^2+2\alpha)e^{-2\beta(t-s)}\left(e^{-\beta s}R_0 + \frac{\alpha}{\beta}[1-e^{-\beta s}]\right)ds \\
&= e^{-2\beta t}\left[R_0^2 + (\sigma^2+2\alpha)\int_0^t \left(e^{\beta s}R_0 + \frac{e^{2\beta s}}{\beta}[1-e^{-\beta s}]\right)ds\right] \\
&= e^{-2\beta t}\left[R_0^2 + (\sigma^2+2\alpha)\left(R_0 - \frac{1}{\beta}\right)\int_0^t e^{\beta s}ds + \left(\frac{\sigma^2+2\alpha}{\beta}\right)\int_0^t e^{2\beta s}ds\right] \\
&= e^{-2\beta t}\left[R_0^2 + (\sigma^2+2\alpha)\left(R_0 - \frac{1}{\beta}\right)\left(\frac{e^{\beta t}-1}{\beta}\right) + \left(\frac{\sigma^2+2\alpha}{\beta}\right)\left(\frac{e^{2\beta t}-1}{2\beta}\right)\right]
\end{aligned}
$$

which means that:

Let us try computing the variance of this model

Theorem 18 (Multi-dim Ito) *Let $X_t, \mu_{t,X_t} \in \mathbb{R}^d$, m-dimensional brownian motion so we have $\sigma_{t,X_t} \in \mathbb{R}^{d\times m}$*

$$
dX_t = \mu_{t,X_t}dt + \sigma_{t,X_t}dW_t \tag{49}
$$

$f : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$
f(X_t) = f(X_0) + \int_0^T \frac{1}{2}Tr\left(\sigma_{t,X_t}^T D^2 f(X_t)\sigma_{t,X_t}\right) + \frac{\partial f}{\partial t}(t,X_t) + \langle\nabla f(X_t),\mu_{t,X_t}\rangle dt + \int_0^T \langle\nabla f(X_t),\sigma_{t,X_t}dW_t\rangle \tag{50}
$$

Proof

$$f(X_t) = f(X_0) + \sum_{i=0}^{n-1} \big\langle \nabla f(X_{t_{i+1}^n}), X_{t_{i+1}^n} - X_{t_i^n} \big\rangle + \frac{1}{2}\big\langle (X_{t_{i+1}^n} - X_{t_i^n}), D^2 f(\theta_i^n)(X_{t_{i+1}^n} - X_{t_i^n}) \big\rangle \tag{51}$$

$$\begin{aligned}
&\big\langle \nabla f(X_{t_{i+1}^n}), X_{t_{i+1}^n} - X_{t_i^n} \big\rangle + \frac{1}{2}\big\langle (X_{t_{i+1}^n} - X_{t_i^n}), D^2 f(\theta_i^n)(X_{t_{i+1}^n} - X_{t_i^n}) \big\rangle \\
&= \big\langle \nabla f(X_{t_{i+1}^n}), u_{t_i^n}(t_{i+1}^n - t_i^n) + e_{t_i^n}(W_{t_{i+1}^n} - W_{t_i^n}) \big\rangle \\
&+ \frac{1}{2}\big\langle u_{t_i^n}(t_{i+1}^n - t_i^n) + e_{t_i^n}(W_{t_{i+1}^n} - W_{t_i^n}), D^2 f(\theta_i^n)[u_{t_i^n}(t_{i+1}^n - t_i^n) + e_{t_i^n}(W_{t_{i+1}^n} - W_{t_i^n})] \big\rangle
\end{aligned}$$

First term in the limit will go to

$$\int_0^T \big\langle \nabla f(X_t), \mu_{t,X_t} \big\rangle dt + \int_0^T \big\langle \nabla f(X_t), \sigma_{t,X_t} dW_t \big\rangle \tag{52}$$

where the second term turns out to equal:

$$\begin{aligned}
&(t_{i+1}^n - t_i^n)\langle D^2 f(\theta_i^n) u_{t_i^n}, e_{t_i^n}(W_{t_{i+1}^n} - W_{t_i^n})\rangle + \frac{1}{2}(t_{i+1}^n - t_i^n)^2 \langle u_{t_i^n}, D^2 f(\theta_i^n) u_{t_i^n}\rangle \\
&+ \frac{1}{2}\langle e_{t_i^n}(W_{t_{i+1}^n} - W_{t_i^n}), D^2 f(\theta_i^n) e_{t_i^n}(W_{t_{i+1}^n} - W_{t_i^n})\rangle
\end{aligned}$$

where only last term will be consequential: precisely

$$\frac{1}{2}\langle W_{t_{i+1}^n} - W_{t_i^n}, e_{t_i^n}^T D^2 f(\theta_i^n) e_{t_i^n}(W_{t_{i+1}^n} - W_{t_i^n})\rangle \to \int_0^T \frac{1}{2} Tr\big(\sigma_{t,X_t}^T D^2 f(X_t) \sigma_{t,X_t}\big) dt$$

■

3. No drift Brownian motion

$$dX_t = \sigma_t dW_t \tag{53}$$

For $k \in [d]$ and $j \in [m]$

$$X_t^k = X_0^k + \sum_{i=0}^{n-1} \sum_{j=1}^{m} e_{t_i^n}^{k,j} (W_{t_{i+1}^n}^j - W_{t_i^n}^j) \tag{54}$$

Then for some $k_1 \neq_{,} 2 \in [d]$ we let $f(x) = x_1 x_2$, wlog assume $k_1 = 1$ and $k_2 = 2$, then $\nabla f(x) = [\partial_{x_1} f(x), \partial_{x_2} f(x), \ldots, \partial_{x_d} f(x)] = [x_2, x_1, 0, \ldots, 0]$, and

$$D_2 f(x) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \quad \sigma_t = \begin{bmatrix} \sigma_t^{1,1} & \sigma_t^{1,2} & \ldots & \sigma_t^{1,m} \\ \sigma_t^{2,1} & \sigma_t^{2,2} & \ldots & \sigma_t^{2,m} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_t^{d,1} & \sigma_t^{d,2} & \ldots & \sigma_t^{d,m} \end{bmatrix}, \quad \mu_t := \begin{bmatrix} \mu_t^1 \\ \mu_t^2 \\ \vdots \\ \mu_t^d \end{bmatrix}$$

$$\langle \nabla f(x), \sigma_t dWt \rangle = X_t^2 \big(\sum_{j=1}^{m} \sigma_t^{1,j} dW_t^j \big) + X_t^1 \big(\sum_{j=1}^{m} \sigma_t^{2,j} dW_t^j \big) = \sum_{j=1}^{m} \int_0^t \big(X_s^2 \sigma_s^{1,j} + X_s^1 \sigma_s^{2,j} \big) dW_s^j \tag{55}$$

On the other hand

$$\frac{1}{2} Tr\big(\sigma_t^T D_2 f(X_t) \sigma_t \big) = \Big(\sigma_t^{1,1} \sigma_t^{2,1} + \sigma_t^{1,2} \sigma_t^{2,2} + \sigma_t^{1,3} \sigma_t^{2,3} + \ldots + \sigma_t^{1,m} \sigma_t^{2,m} \Big) = \sum_{j=1}^{m} \sigma_t^{1,j} \sigma_t^{2,j} \tag{56}$$

Consequently we get:

$$d\big(X_t^1 X_t^2 \big) = \big(X_t^2 \mu_t^1 + X_t^1 \mu_t^2 \big) dt + \sum_{j=1}^{m} \sigma_t^{1,j} \sigma_t^{2,j} dt + \big(X_t^2 \sigma_t^{1,j} + X_t^1 \sigma_t^{2,j} \big) dW_t^j \tag{57}$$

Can I come up with some composition rule. Individual composition rules are.

$$\begin{aligned} dX_t^1 &= \mu_t^1 dt + \sum_{j=1}^{m} \sigma_t^{1,j} dW_t^j \\ dX_t^2 &= \mu_t^2 dt + \sum_{j=1}^{m} \sigma_t^{2,j} dW_t^j \end{aligned}$$

So it seems like

$$\begin{aligned} &d\big(X_t^1 X_t^2 \big) = dX_t^1 dX_t^2 + X_t^1 dX_t^2 + X_t^2 dX_t^1 \\ &= \Big(\mu_t^1 dt + \sum_{j=1}^{m} \sigma_t^{1,j} dW_t^j \Big) \Big(\mu_t^2 dt + \sum_{j=1}^{m} \sigma_t^{2,j} dW_t^j \Big) + X_t^1 \sum_{j=1}^{m} \sigma_t^{2,j} dW_t^j + X_t^2 \sum_{j=1}^{m} \sigma_t^{1,j} dW_t^j + X_t^1 \mu_t^2 dt + X_t^2 \mu_t^1 dt \\ &= X_t^1 \mu_t^2 dt + X_t^2 \mu_t^1 dt + \sum_{j=1}^{m} \sigma_t^{1,j} \sigma_t^{2,j} dt + \Big(X_t^1 \sigma_t^{2,j} + X_t^2 \sigma_t^{1,j} \Big) dW_t^j \end{aligned}$$

Theorem 19

$$\begin{aligned} &dt dW_t = 0 \\ &dt dt = 0 \\ &dW_t^j dW_t^i = \delta_j(i) dt \end{aligned}$$

4. Exercise related to Multi-dimensional Ito's formula

5. Exercise 4.13 Shreve (Correlated Brownian Motion)

Suppose that $B_1(t)$ and $B_2(t)$ are brownian motions and stochastic process $\rho(t) \in (-1, 1)$ such that:

$$dB_1(t) dB_2(t) = \rho(t) dt \tag{58}$$

$$
\begin{aligned}
&B_1(t) = W_1(t) \\
&B_2(t) = \int_0^t \rho(s) dW_1(s) + \int_0^t \sqrt{1-\rho^2(s)} dW_2(s)
\end{aligned}
$$

Show that $W_1(t)$ and $W_2(t)$ are independent brownian motions.
Proof Notice that

$$
\begin{aligned}
&dB_2(t) = \rho(t)dW_1(t) + \sqrt{1-\rho^2(t)}dW_2(t) \\
&dB_1(t)dB_2(t) = \rho_t dW_1(t)dW_1(t) + \sqrt{1-\rho^2(t)}dW_1(t)dW_2(t) = \rho(t)dt + \sqrt{1-\rho^2(t)}dW_1(t)dW_2 =: \rho(t)dt
\end{aligned}
$$

implies that $W_1(t) \perp W_2(t)$ ■

In more compact form

$$
dB_t = \sigma_t dW_t, \tag{59}
$$

where

$$
\sigma_t = \begin{bmatrix} 1 & 0 \\ \rho(t) & \sqrt{1-\rho^2(t)} \end{bmatrix}, \ \sigma_t^{-1} = \frac{1}{\sqrt{1-\rho^2(t)}} \begin{bmatrix} \sqrt{1-\rho^2(t)} & 0 \\ -\rho(t) & 1. \end{bmatrix}
$$

Which implies that:

$$
\begin{aligned}
&dW_t = \sigma_t^{-1} dB_t \\
&W_1(t) = B_1(t) \\
&W_2(t) = \int_0^t \frac{-\rho(s)}{\sqrt{1-\rho^2(s)}} dB_1(s) + \int_0^t \frac{1}{\sqrt{1-\rho^2(s)}} dB_2(s) \ \textit{for } t \textit{ and } r \textit{ positive} \\
&\mathbb{E}\big[W_2(t+r) - W_2(r)\big]^2 = \int_r^{t+r} \frac{\rho^2(s)}{1-\rho^2(s)} ds + \int_r^{t+r} \frac{1}{1-\rho^2(s)} ds \\
&+ 2\mathbb{E}\bigg(\int_r^{t+r} \frac{-\rho(s)}{\sqrt{1-\rho^2(s)}} dB_1(s) \times \int_0^t \frac{1}{\sqrt{1-\rho^2(s)}} dB_2(s)\bigg)
\end{aligned}
$$

Now recall that $dB_1(t)dB_2(t) = \rho(t)dt$ along with our usual approximation technique

$$
\bigg(\sum_{i=0}^{n-1} \frac{-\rho(t_i^n)}{\sqrt{1-\rho^2(t_i^n)}} \big(B_1(t_{i+1}^n) - B_1(t_{i+1}^n)\big) \times \sum_{i=0}^{n-1} \frac{1}{\sqrt{1-\rho^2(t_i^n)}} \big(B_2(t_{i+1}^n) - B_2(t_{i+1}^n)\big)\bigg) \tag{60}
$$

which implies that

$$
\mathbb{E}\big[W_2(t+r) - W_2(r)\big]^2 = \int_r^{t+r} \frac{\rho^2(s) + 1 - 2\rho^2(s)}{1-\rho^2(s)} ds = t.
$$

We have $m-$ dimensional brownian motion W_t and $n-$ dimensional ito process X_t satisfying:

$$
dX_t = \mu_{t,X_t} dt + \sigma_{t,X_t} dW_t, \tag{61}
$$

where $\mu_{t,X_t} \in \mathbb{R}^{n\times 1}$ and $\sigma_{t,X_t} \in \mathbb{R}^{n\times m}$

$$f(X_t) = f(X_0) + \int_0^T \frac{1}{2} Tr\big(\sigma_{t,X_t}^T D^2 f(X_t)\sigma_{t,X_t}\big) + \big\langle \nabla f(X_t), \mu_{t,X_t}\big\rangle dt + \int_0^T \big\langle \nabla f(X_t), \sigma_{t,X_t} dW_t\big\rangle \tag{62}$$

$$\begin{aligned} dX_t &= \mu_t^1 dt + \sigma_t^{(1,1)} dW_t^1 \\ dY_t &= \mu_t^2 dt + \sigma_t^{(2,1)} dW_t^1 + \sigma_t^{(2,2)} dW_t^2 \end{aligned}$$

Ito formula for $Z_t := X_t Y_t$, If we let $f(x,y) := xy$ then $\nabla f(x,y) = [y,x], \mu_t = [\mu_t^1, \mu_t^2]$

$$D^2 f(x,y) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_t = \begin{bmatrix} \sigma_t^{(1,1)} & 0 \\ \sigma_t^{(2,1)} & \sigma_t^{(2,2)} \end{bmatrix}, \sigma_{t,X_t}^T D^2 f(X_t)\sigma_{t,X_t} = \begin{bmatrix} 2\sigma_t^{(1,1)}\sigma_t^{(2,1)} & \sigma_t^{(1,1)}\sigma_t^{(2,2)} \\ \sigma_t^{(1,1)}\sigma_t^{(2,2)} & 0 \end{bmatrix}$$

$$\sigma_t W_t = \begin{bmatrix} \sigma_t^{(1,1)} W_t^1 \\ \sigma_t^{(2,1)} W_t^1 + \sigma_t^{(2,2)} W_t^2 \end{bmatrix}$$

Hence we get:

$$\begin{aligned} Z_T =& Z_0 + \int_0^T \Big(\sigma_t^{(1,1)}\sigma_t^{(2,1)} + \mu_t^1 Y_t + \mu_t^2 X_t\Big) dt \\ &+ \int_0^T \big(Y_t \sigma_t^{(1,1)} + X_t \sigma_t^{(2,1)}\big) dW_t^1 + \int_0^T X_t \sigma_t^{(2,2)} dW_t^2 \end{aligned}$$

6. The Martingale Rep Theorem

Theorem 20 ***Levy's theorem*** *A continuous martingale is a Brownian motion iff its' quadratic variation over each interval* $[0,t]$ *is* t.

Example 1 $W_1(t), W_2(t)$ *be two independent brownian motions, use Levy's theorem to show that:*

$$W(t) := \rho W_1(t) + \sqrt{1-\rho^2} W_t^2 \tag{63}$$

is also a Brownian motion for given constant ρ

Theorem 21 *Given* $\mathcal{F}_t$*- martingale* M_t*, then there exists adapted stochastic process* ϕ_t *such that:*

$$M_t = M_0 + \int_0^t \phi_s^* dW_s \tag{64}$$

Theorem 22 $F := W_T^3$ *and let* $M_t := \mathbb{E}_t[F] = \mathbb{E}[F|\mathcal{F}_t]$*. First notice that* $M_t := \mathbb{E}[W_T^3|\mathcal{F}_t]$ *is* $\mathcal{F}_t$ *adapted martingale, because for* $s < t$ *using conditional expectation property,* $\mathbb{E}\big[M_t|\mathcal{F}_s\big] = \mathbb{E}\big[\mathbb{E}[W_T^3|\mathcal{F}_t]|\mathcal{F}_s\big] = \mathbb{E}[W_T^3|\mathcal{F}_s] =: M_s$ *almost surely.*

$$M_t = 3\int_0^t (T - s + W_s^2) dW_s \tag{65}$$

Proof

$$\mathbb{E}\Big[\big(W_T - W_t + W_t\big)^3\Big] = \sum_{k=0}^{3} \binom{3}{k} [W_T - W_t]^k W_t^{3-k}$$
$$\mathbb{E}\Big[\big(W_T - W_t + W_t\big)^3\Big] = \sum_{k=0}^{3} \binom{3}{k} \mathbb{E}_t\big[(W_T - W_t)^k W_t^{3-k}\big]$$
$$= \sum_{k=0}^{3} \binom{3}{k} W_t^{3-k} \mathbb{E}\big[(W_T - W_t)^k\big] = W_t^3 + 3W_t(T-t)$$
$$M_t = \mathbb{E}\Big[\big(W_T - W_t + W_t\big)^3 | \mathcal{F}_t\Big] = W_t^3 + 3W_t(T-t)$$
$$M_t := W_t^3 + 3W_t(T-t)$$
$$f(t,w) := w^3 + 3w(T-t), \nabla f(t,w) = 3(w^2 + T - t),\ \ \sigma_t = 1,\ \ \textit{Using Ito implies}$$
$$f(t,W_t) = M_t = 3\int_0^t (W_s^2 + T - s) dW_s$$

■

7. Gaussian Processes

We know that moment generating function of $Y = N(\mu, \sigma^2)$, satisfy for all $m \in \mathbb{R}$

$$\mathbb{E}[e^{mY}] = e^{m\mu + \frac{1}{2}\sigma^2 m^2}. \tag{66}$$

Theorem 23 *If $\delta(s)$ is a deterministic function then:*

$$X_t = \int_0^t \delta(s) dW_s \tag{67}$$

Recall from Ito's isometry *that:*

$$\mathbb{E}[X_t^2] = \int_0^t \delta(s)^2 ds \tag{68}$$

is a Gaussian process in particular:

$$\mathbb{E}[e^{mX_t}] = e^{\frac{1}{2}m^2 \int_0^t \delta(s)^2 ds}. \tag{69}$$

Proof For arbitrary $m \in \mathbb{R}$, let $f(x) := e^{mx}, \nabla f(x) = me^{mx}, D^2 f(x) = m^2 e^{mx}$. Using Ito's formula we have :

$$
\begin{aligned}
e^{mX_t} &= \underbrace{e^{mX_0}}_{=1} + \int_0^t m\delta(s)e^{mX_s}dW_s + \frac{1}{2}m^2 \int_0^t \delta(s)^2 e^{mX_s}ds \\
\mathbb{E}[e^{mX_t}] &= 1 + \frac{1}{2}m^2 \int_0^t \delta(s)^2 \mathbb{E}[e^{mX_s}]ds + \underbrace{\mathbb{E}\int_0^t m\delta(s)e^{mX_s}dW_s}_{=0, Martinagle\ term} \\
\mathbb{E}[e^{mX_t}] &= 1 + \frac{1}{2}m^2 \int_0^t \delta(s)^2 \mathbb{E}[e^{mX_s}]ds \\
y_t &= 1 + \frac{1}{2}m^2 \int_0^t \delta(s)^2 y_s ds. \ Solution\ to\ this
\end{aligned}
$$

Check that $y_t = e^{\frac{1}{2}m^2 \int_0^t \delta(s)^2 ds}$. Consequently,

$$
\mathbb{E}\left[e^{\int_0^t m\delta(s)dW_s - \frac{1}{2}\int_0^t [m\delta(s)]^2 ds}\right] = 1 \tag{70}
$$

recall that for any $m \in \mathbb{R}$:

$$
e^{mW_t - \frac{m^2 t}{2}} \tag{71}
$$

is a martingale. ∎

7.1 Self financing portfolio

X_k be your portfolio at time k, Δ_k be the number of shares of stock that you hold at time k, you invest $X_k - \Delta_k S_k$ in money market so your portfolio at time $k+1$ will be

$$
\begin{aligned}
X_{k+1} &= \Delta_k S_{k+1} + (1+r)\big(X_k - \Delta_k S_k\big) \implies \\
X_{k+1} - X_k &= \Delta_k\big(S_{k+1} - S_k\big) + r(X_k - \Delta_k S_k)
\end{aligned}
$$

So a continuous time version that makes sense is

$$
dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t)dt \tag{72}
$$

Now instead let us say that at time $t = 0$ we have X_0 portfolio and Δ_0 invested in stock and remaining in money market:

$$
\begin{aligned}
X_1 &= \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) \qquad (73) \\
X_2 &= \Delta_1 S_2 + (1+r)\big(X_1 - \Delta_1 S_1\big) \\
X_2 &= \Delta_1 S_2 + (1+r)\Big(\Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) - \Delta_1 S_1\Big) \\
X_2 &= \Delta_1 S_2 + (1+r)^2(X_0 - \Delta_0)S_0 + (1+r)(\Delta_0 - \Delta_1)S_1
\end{aligned}
$$

Now let:

$$M_k = (1+r)^k \tag{74}$$

and find a parameter Γ_k such that

$$X_k = \Delta_k S_k + \Gamma_k M_k \tag{75}$$

so that:

$$X_{k+1} = \Delta_k S_{k+1} + (1+r)\Gamma_k M_k = \Delta_k S_{k+1} + \Gamma_k M_{k+1}$$

So let us begin by equating from (73)

$$X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) = \Delta_1 S_1 + \Gamma_1 M_1 \tag{76}$$

Start with time $t = 0$ and capital X_0

$$\begin{aligned} X_0 &= \Delta_0 S_0 + (1+r)^0 (X_0 - \Delta_0 S_0) \\ X_1 &= \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) \\ X_2 &= \Delta_1 S_2 + (1+r)(X_1 - \Delta_1 S_1) \end{aligned}$$

This way

Theorem 24 *Some portfolio valued at X_t*

$$dS_t = \alpha S_t dt + \sigma S_t dW_t \tag{77}$$

Suppose at each time t the investor holds Δ_t stocks and remaining $X_t - \Delta_t S_t$ are invested in a money market of fixed interest rate r, then:

$$dX_t = \underbrace{rX_t + \Delta_t(\alpha - r)S_t}_{:=\Theta^{\alpha,r}_{t,S_t}} dt + \Delta_t \sigma S_t dW_t \tag{78}$$

Let $X_t := f(t, S_t)$,then we know that:

$$f_x(t, S_t) = \Delta_t \tag{79}$$

and also,

$$\begin{aligned} f_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 f_{xx}(t, S_t) &= rf(t, S_t) - r\Delta_t S_t \\ \implies f_t(t, S_t) &= rf(t, S_t) - r\Delta_t S_t - \frac{1}{2}\sigma^2 S_t^2 f_{xx}(t, S_t) \end{aligned}$$

This implies that $g(t, S_t) := e^{-rt} X_t = e^{-rt} f(t, S_t)$ satisfies the equation

$$g(t, S_t) = \left(g_t(t, S_t) + \langle g_x(t, S_t), \alpha S_t \rangle + \frac{1}{2}\sigma^2 S_t^2 g_{xx}(t, S_t) \right) + \langle g_x(t, S_t), \sigma S_t \rangle dW_t. \tag{80}$$

where now one simply realise that $g_t(t,S_t) := e^{-rt}f(t,S_t) = e^{-rt}[f_t(t,S_t) - rf(t,S_t)]$. *Consequentlt,*

$$\begin{aligned} d\big(e^{-rt}f(t,S_t)\big) =& e^{-rt}\Bigg(- rf(t,S_t) + \underbrace{f_t(t,S_t) + \langle f_x(t,S_t), \alpha S_t\rangle + \frac{1}{2}\sigma^2 S_t^2 f_{xx}(t,S_t)}_{=rf(t,S_t)+\Delta_t(\alpha-r)S_t}\Bigg)dt \\ &+ e^{-rt}\langle f_x(t,S_t), \sigma S_t\rangle dW_t \\ &= e^{-rt}\Delta_t(\alpha - r)S_t dt + e^{-rt}\underbrace{\langle f_x(t,S_t)}_{=\Delta_t}, \sigma S_t\rangle dW_t \end{aligned}$$

Discounted investment portfolio will read:

$$d(e^{-rt}X_t) = e^{-rt}\Delta_t(\alpha - r)S_t dt + e^{-rt}\Delta_t\sigma S_t dW_t \tag{81}$$

Proof First let us consider the discretized version:

$X_{k+1} = \Delta_k S_{k+1} + (1+r)\big(X_k - \Delta_k S_k\big)$
$X_{k+1} - X_k = \Delta_k\big(S_{k+1} - S_k\big) + r\big(X_k - \Delta_k S_k\big)$, *continuous time version makes sense following way*
$dX_t = rX_t dt + \Delta_t(\alpha - r)S_t dt + \Delta_t\sigma S_t dW_t dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t)dt$
$= \Delta_t\alpha S_t dt + \Delta_t\sigma S_t dW_t + r(X_t - \Delta_t S_t)dt = \underbrace{(\alpha - r)\Delta_t S_t + rX_t}_{:=\Theta^{\alpha,r}_{t,S_t}} dt + \Delta_t\sigma S_t dW_t$

Now if we let $X_t := g(t,S_t)$, then we know that

$$e^{-rt}g(t,S_t) = e^{-rt}\big(\Theta^{\alpha,r}_{t,S_t} - rg(t,S_t)\big)dt + e^{-rt}\Delta_t\sigma S_t dW_t \tag{82}$$

■

Now think of $c(t,S_t)$ as the price of the call option, then:

$$dc(t,S_t) = \big(c_t(t,S_t) + \alpha S_t\nabla c(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 D^2 c(t,S_t)\big)dt + \sigma S_t\nabla c(t,S_t)dW_t \tag{83}$$

8. Black-Scholes PDE

If we require for all $t \in (0,T]$

$$d(e^{-rt}X_t) = d(e^{-rt}c(t,S_t)) \tag{84}$$

and for the discounted process we have that:

$$de^{-rt}c(t,S_t) = e^{-rt}\big(c_t(t,S_t) + \alpha S_t\nabla c(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 D^2 c(t,S_t) - rc(t,S_t)\big)dt + \sigma e^{-rt}S_t\nabla c(t,S_t)dW_t \tag{85}$$

equating it to :

$$e^{-rt}\Delta_t(\alpha - r)S_t dt + e^{-rt}\Delta_t\sigma S_t dW_t \tag{86}$$

If $c(0,S_0) = X_0$

1. **Delta Hedge:** $\Delta_t = \nabla_x c(t,x)$ for all $x \in \mathbb{R}$
2. **Black-Scholes Merton Partial Differential equation:**

$$c_t(t,x) + rx\nabla_x c(t,x) + \frac{1}{2}\sigma^2 x^2 \partial_{xx} c(t,x) = rc(t,x), \;\; \forall t \in (0,T], x \in \mathbb{R}^+ \tag{87}$$
$$c(T,x) = \max\big(x - K, 0\big)$$

Partial differential equation from (87) is called backward parabolic, along with terminal condition, we need boundary conditions at $x = 0$ and $x = \infty$(again from (87))

$$\begin{aligned} &c_t(t,0) = rc(t,0), \;\; \textit{an ordinary diff equation with solution} \\ &c(t,0) = e^{rt}c(0,0), \;\; \textit{but terminal cost with stock price 0} = \max(-K,0) = 0 \\ &\implies c(0,0) = 0, \;\; \implies c(t,0) == 0\,\forall t \in (0,T]. \end{aligned}$$

Boundary condition for $x = \infty$ for the European call option

$$\lim_{x\to\infty}\Big[c(t,x) - \big(x - e^{-r(T-t)}K\big)\Big] = 0 \tag{88}$$

$$\begin{aligned} &c(t,x) = xN\big(d_+(T-t,x)\big) - Ke^{-r(T-t)}N\big(d_-(T-t,x)\big), \;\; t \in [0,T), x > 0 \\ &d_\pm\big((T-t)),x\big) = \frac{1}{\sigma\sqrt{T-t}}\left[\log\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)\right] \\ &N(y) := \frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty} e^{-\frac{z^2}{2}}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{z^2}{2}}dx \end{aligned}$$

$$\begin{aligned} &\frac{x}{\sqrt{2\pi}}\int_{-\infty}^{\frac{1}{\sigma\sqrt{T-t}}\left[\log\left(\frac{x}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t)\right]} e^{-\frac{z^2}{2}}dz \\ &-\frac{Ke^{-r(T-t)}}{\sqrt{2\pi}}\int_{-\infty}^{\frac{1}{\sigma\sqrt{(T-t))}}\left[\log\left(\frac{x}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)\right]} e^{-\frac{z^2}{2}}dz \end{aligned}$$

In order to make sense of this thing first we focus on

$$\frac{x}{\sqrt{2\pi}}\int_{-\infty}^{\frac{1}{\sigma\sqrt{\tau}}\log\left(\frac{x}{K}\right)\tau(T-t)x} e^{-\frac{z^2}{2}}dz \tag{89}$$

Missing part is some sort of change of variables, recall Distribution function of a Gaussian with mean μ and deviation σ is:

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{x} e^{-\frac{(z-\mu)^2}{2\sigma^2}}dz =: F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-\frac{t^2}{2}}dt \tag{90}$$

Theorem 25

$$c(t,x) = xN\big(d_+(T-t,x)\big) - Ke^{-r(T-t)}N\big(d_-(T-t,x)\big) \tag{91}$$

Proof

1. $Ke^{-r(T-t)}N^{'}(d_-) = xN^{'}(d_+)$

$$N^{'}(x) := \frac{1}{\sqrt{2\pi}} \lim_{\epsilon\to 0} \frac{\int_x^{x+\epsilon} e^{-\frac{z^2}{2}}dz}{\epsilon} = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

$$xN^{'}(d_+) = \frac{x}{\sqrt{2\pi}} e^{-\frac{\frac{1}{\sigma^2(T-t)}\left[\log\left(\frac{x}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t)\right]^2}{2}}$$

$$Ke^{-r(T-t)}N^{'}(d_-) = \frac{Ke^{-r(T-t)}}{\sqrt{2\pi}} e^{-\frac{\frac{1}{\sigma^2(T-t)}\left[\log\left(\frac{x}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t)-\sigma^2(T-t)\right]^2}{2}}$$

$$\frac{1}{2\sigma^2(T-t)}\left[\log\left(\frac{x}{K}\right)+\left(r+\frac{2\sigma^2}{2}\right)(T-t)-\sigma^2(T-t)\right]^2 = \frac{1}{2\sigma^2(T-t)}\left[\log\left(\frac{x}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t)\right.$$
$$+\frac{1}{2\sigma^2(T-t)}\sigma^4(T-t)^2 - \frac{1}{2\sigma^2(T-t)}2\sigma^2(T-t)\log\left(\frac{x}{K}\right) - \frac{1}{2\sigma^2(T-t)}2\sigma^2(T-t)\left(r+\frac{\sigma^2}{2}\right)(T-t)$$

Consequently:

$$\frac{N^{'}(d_-)}{N^{'}(d_+)} = \left(\frac{x}{K}\right)e^{r(T-t)} \tag{92}$$
$$Ke^{-r(T-t)}N^{'}(d_-) = xN^{'}(d_+)$$

2. $c_x = N\big(d_+(T-t,x)\big)$ Eeasy to compute that $\partial_x d_\pm(T-t,x) = \frac{1}{\sigma x\sqrt{T-t}}$Now, recall that by hypothesis

$$c(t,x) = xN\big(d_+(T-t,x)\big) - Ke^{-r(T-t)}N\big(d_-(T-t,x)\big) \implies$$
$$c_x(t,x) = N\big(d_+(T-t,x)\big) + xN^{'}\big(d_+(T-t,x)\big)\partial_x d_+(T-t,x) - Ke^{-r(T-t)}N^{'}\big(d_-(T-t,x)\big)\partial_x d_-(T-t,$$
$$= N\big(d_+(T-t,x)\big) + xN^{'}\big(d_+(T-t,x)\big)\frac{1}{\sigma x\sqrt{T-t}} - Ke^{-r(T-t)}N^{'}\big(d_-(T-t,x)\big)\frac{1}{\sigma x\sqrt{T-t}} \tag{93}$$
$$N\big(d_+(T-t,x)\big) + \frac{1}{\sigma x\sqrt{T-t}}\Big(\underbrace{xN^{'}\big(d_+(T-t,x)\big) - Ke^{-r(T-t)}N^{'}\big(d_-(T-t,x)\big)}_{=0}\Big)$$

where the last equation follows from (92). It is better to also compute c_{xx} at this point

$$c_{xx} = N^{'}\big(d_+(T-t,x)\big)\frac{1}{\sigma x\sqrt{T-t}} \tag{94}$$

3. Show that(Theta option):

$$c_t = -xN^{'}\big(d_+(T-t,x)\big)\frac{\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N\big(d_-(T-t,x)\big) \tag{95}$$

First notice that

$$\begin{aligned}
\frac{d}{dt}N\big(d_\pm(T-t,x)\big) &= N^{'}\big(d_\pm(T-t,x)\big)\frac{d}{dt}d_\pm\big((T-t),x\big) \\
&= N^{'}\big(d_\pm(T-t,x)\big)\frac{d}{dt}\frac{1}{\sigma\sqrt{T-t}}\left[\log\left(\frac{x}{K}\right)+\left(r\pm\frac{\sigma^2}{2}\right)(T-t)\right] \\
&= N^{'}\big(d_\pm(T-t,x)\big)\left[\log\left(\frac{x}{K}\right)\frac{(T-t)^{-\frac{3}{2}}}{2\sigma} - \left(r\pm\frac{\sigma^2}{2}\right)\frac{(T-t)^{-\frac{1}{2}}}{2\sigma}\right]
\end{aligned}$$

Combined with preceding observations, result follows

4. Show that for $x > K$, $\lim_{t\to T} d_\pm = \infty$ and for $x < K$, $\lim_{t\to T} d_\pm = -\infty$

$$c(t,x) = xN\big(d_+(T-t,x)\big) - Ke^{-r(T-t)}N\big(d_-(T-t,x)\big) \tag{96}$$

5. **Terminal Condition** Show that for $x > K, \lim_{t\to T} d_\pm(T-t,x) = \infty$ and for $0 < x < K, \lim_{t\to T} d_\pm(T-t,x) = -\infty$ Notice that for $x > K$ and use it to conclude

$$\begin{aligned}
&c(T,x) = (x-K)^+,\ \ x \geq 0 \\
&c(t,0) = 0,\ \ 0 \leq t \leq T
\end{aligned}$$

Recall that:

$$d_\pm\big(T-t,x\big) = \frac{1}{\sigma\sqrt{T-t}}\left[\log\left(\frac{x}{K}\right)+\left(r\pm\frac{\sigma^2}{2}\right)(T-t)\right] \tag{97}$$

So when $x > K$, implies that $\log\left(\frac{x}{K}\right) > 0$. Therefore

$$\lim_{t\to T}\frac{1}{\sigma\sqrt{T-t}}\log\left(\frac{x}{K}\right) = \infty \tag{98}$$

and when $x < K$, implies that $\log\left(\frac{x}{K}\right) < 0$. Therefore

$$\lim_{t\to T}\frac{1}{\sigma\sqrt{T-t}}\log\left(\frac{x}{K}\right) = -\infty \tag{99}$$

We know that $N(\infty) = 1, N(-\infty) = 0$, which means that if $x > K$:

$$\lim_{t\to T} c(t,x) = \lim_{t\to T} xN\big(d_+(T-t,x)\big) - \lim_{t\to T} Ke^{-r(T-t)}N\big(d_-(T-t,x)\big) = x - K \tag{100}$$

On the other hand when $x < K$

$$\lim_{t\to T} c(t,x) = \lim_{t\to T} xN\big(d_+(T-t,x)\big) - \lim_{t\to T} Ke^{-r(T-t)}N\big(d_-(T-t,x)\big) = x\times 0 - K\times 0 = 0 \tag{101}$$

■

8.1 The Greeks

The derivatives of the function $c(t,x)$ of (4.5.19) with respect to various variables are called the Greeks, e.g., *delta*

$$c_x(t,x) = N\big(d_+(T-t,x)\big) \geq 0. \tag{102}$$

and *theta*, $c_t(t,x)$ which satisfies:

$$c_t = -xN^{'}\big(d_+(T-t,x)\big)\frac{\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N\big(d_-(T-t,x)\big) \leq 0 \tag{103}$$

Because both N and N' are always positive, delta is always positive and theta is always negative. Another of the Greeks is gamma, which is $c_{xx}(t,x)$ satisfying

$$c_{xx} = N^{'}\big(d_+(T-t,x)\big)\frac{1}{\sigma x\sqrt{T-t}} \geq 0 \tag{104}$$

which means that the *price of call option is convex w.r.t initial value of the stock* i.e., for any $\alpha \in (0,1)$

$$c\big(t, \alpha x_1 + (1-\alpha)x_2\big) \leq \alpha c(t,x_1) + (1-\alpha)c(t,x_2). \tag{105}$$

1. **To hedge a short position in a call option one must borrow money:** Given current stock price $S_t = x$, one starts with initial capital of $xc_x - Ke^{-r(T-t)}N(d_-)$. c_x shares of the underlying stock are bought at price of unit equals x. Remaining is to be invested in money market, but remaining is $xc_x - Ke^{-r(T-t)}N(d_-) - xc_x = -Ke^{-r(T-t)}N(d_-)$, as $N(d_-) \geq 0$ so short position needs to borrow.*Had been hedging the short position throughout these notes*

2.

Definition 26 ***Rebalancing the Gamma:*** *corresponds to buying and selling of the underlying security to replicate the options pay-off at time of the expiry.*

Definition 27 ***Short position for the call option****: start with initial capital of your portfolio X_t being equal to $c(t,x)$ and hedge the portfolio between money market and buying the shares of the underlying stock then at time of expiry almost surely $X_T = c(T, S_T)$*

9. Solving the generalized Brownian motion

$$dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t \tag{106}$$

Now if we let $f(x) = \log x$, $f_x(x) = \frac{1}{x}$ and $f_{xx} = -\frac{1}{x^2}$ then:

$$\begin{aligned}
\log S_t &= \log S_0 + \int_0^t \left(\alpha_s + \frac{1}{2}\sigma_s^2 S_s^2\big(-\frac{1}{S_s^2}\big)\right)ds + \int_0^t \sigma_s dW_s \\
&= \log S_0 + \int_0^t \left(\alpha_s - \frac{1}{2}\sigma_s^2\right)ds + \int_0^t \sigma_s dW_s \\
\implies S_t &= e^{\int_0^t \left(\alpha_s - \frac{1}{2}\sigma_s^2\right)ds + \int_0^t \sigma_s dW_s} S_0
\end{aligned}$$

Now for $p > 0$, $f(x) = x^p, f_x = px^{p-1}$ and $f_{xx} = p(p-1)x^{p-2}$

$$
\begin{aligned}
S_t^p &= S_0^p + \int_0^t \big(pS_s^{p-1}\alpha_s S_s + \frac{1}{2}\sigma_s^2 S_s^2 p(p-1)S_s^{p-2}\big)ds + \int_0^t pS_s^{p-1}\sigma_s S_s dW_s \\
&= \int_0^t \left(\alpha_s p S_s^p + \frac{p(p-1)}{2}\sigma_s^2 S_s^p\right)ds + \int_0^t p\sigma_s S_s^p dW_s
\end{aligned}
$$

10. Exercise 4,7 Shreve

$$
\begin{aligned}
&f(t,x) := x^4, \;\; f_x(t,x) = 4x^3 \;\; and f_{xx}(t,x) = 12x^2 \\
&X_t := \int_0^t dW_s,
\end{aligned}
$$

Then one notices that:

$$
\begin{aligned}
W_T^4 &= W_0^4 + \frac{12}{2}\int_0^T \big(\int_0^s dW_r\big)^2 ds + 4\int_0^T \big(\int_0^s dW_r\big)^3 dW_s \\
&= W_0^4 + 6\int_0^T W_s^2 ds + 4\int_0^T W_s^3 dW_s \\
&\implies \mathbb{E}[W_T^4] = 6\int_0^T \mathbb{E}[W_s^s]ds = 6\int_0^T sds = 3T^2.
\end{aligned}
$$

11. Exercise 4.11 Shreve(conservative estimate of stock variance)

The value of portfolio at time $k+1$

$$
X_{k+1} = \Delta_k S_{k+1} + (1+r)(X_k - \Delta_k S_k) \tag{107}
$$

1. Being long a stock means you own it and will profit if the stock rises
2. Being short a stock means that you have a negative position in the stock and will profit of the stock falls
3. Let Δ_t be the number of shares of stock held at time t.
4. Γ_t denote the number of shares of money market account held at time t.

For $\sigma_2 > \sigma_1$

$$
dS_t = \mu S_t dt + \sigma_2 S_t dW_t. \tag{108}
$$

So miscalculated the call price as actual stock variance is σ_2. We set up a portfolio whose value at time t is X_t and $X_0 = 0$. At each time t, the portfolio is long one Eurpoean call, is short $c_x(t, S_t)$ shares of stock and thus has a cash position

$$
X_t - c(t, S_t) + c_x(t, S_t)S_t \tag{109}
$$

which is invested at constant rate r, we also remove cash from this portfolio at a constant rate of $\frac{1}{2}(\sigma_2^2 - \sigma_1^2)S_t^2 c_{xx}(t, S_t)$. Therefore,

$$dX_t = dc(t, S_t) - c_x(t, S_t)dS_t + r\Big(X_t - c(t, S_t) + c_x(t, S_t)S_t\Big)dt - \frac{1}{2}\Big(\sigma_2^2 - \sigma_1^2\Big)S_t^2 c_{xx}(t, S_t)dt$$

which means that now if we let $f(t, S_t) = X_t$, then

$$\begin{aligned}
&f(t, S_t) - f(t, S_0) := c(t, S_t) - c(0, S_0) - c_x(t, S_t)(S_t - S_0) \\
&+ r\Big(X_t - c(t, S_t) + c_x(t, S_t)S_t\Big)dt - \frac{1}{2}\Big(\sigma_2^2 - \sigma_1^2\Big)S_t^2 c_{xx}(t, S_t)dt
\end{aligned}$$

But recall that actual stock evolves according to variance σ_2. Therefore after simple algebra

$$f(t, S_t) - f(t, S_0) = \Big(c_t(t, S_t) + rf(t, S_t) - rc(t, S_t) + rS_t c_x(t, S_t) + \frac{1}{2}\sigma_1^2 c_{xx}(t, S_t)\Big)dt \tag{110}$$

But now using discounted shitt

$$\begin{aligned}
d(e^{-rt}f(t, S_t)) &= e^{-rt}\Big(c_t(t, S_t) + rf(t, S_t) - rc(t, S_t) + rS_t c_x(t, S_t) + \frac{1}{2}\sigma_1^2 c_{xx}(t, S_t) - rf(t, S_t)\Big)dt \\
&= e^{-rt}\Big(\underbrace{c_t(t, S_t) - rc(t, S_t) + rS_t c_x(t, S_t) + \frac{1}{2}\sigma_1^2 c_{xx}(t, S_t)}_{=0, via BSM}\Big)dt
\end{aligned}$$

Where last equalit follows from Black Scholes Merton PDE:

$$c_t(t, x) - rc(t, x) + rxc_x(t, x) + \frac{1}{2}\sigma_1^2 x^2 c_{xx}(t, x) = 0 \tag{111}$$

12. Instantaneous correlations 4.17

$$\begin{aligned}
dX_1(t) &= \Theta(1)dt + \sigma_1(t)dB_1(t) \\
dX_2(t) &= \Theta(2)dt + \sigma_2(t)dB_2(t)
\end{aligned}$$

where $B_1(t)$ and $B_2(t)$ are Brownian motions satisfying $dB_1(t)dB_2(t) = \rho(t)dt$ and all the drifts and variances are adapted processes then

$$dX_1(t)dX_2(t) = \sigma_1(t)\sigma_2(t)\rho(t)dt \tag{112}$$

But by definition we know that:

But recall that:

$$\begin{aligned}
&d\Big(B_1(t)B_2(t)\Big) = dB_1(t)dB_2(t) + B_1(t)dB_2(t) + B_2(t)dB_1(t) \\
&\rho_t dt + B_1(t)dB_2(t) + B_2(t)dB_1(t) \implies \\
&B_1(t_0+\epsilon)B_2(t_0+\epsilon) - B_1(t_0)B_2(t_0) = \big(B_1(t_0+\epsilon) - B_1(t_0)\big)\big(B_2(t_0+\epsilon) - B_2(t_0)\big) \\
&+ B_1(t_0)\big(B_2(t_0+\epsilon) - B_2(t_0)\big) + B_2(t_0)\big(B_1(t_0+\epsilon) - B_1(t_0)\big)
\end{aligned}$$

Similarly,

$$d\big(X_1(t)X_2(t)\big) = dX_1(t)dX_2(t) + X_1(t)dX_2(t) + X_2(t)dX_1(t)$$
$$= \sigma_1(t)\sigma_2(t)\rho(t)dt + X_1(t)\Theta_2(t)dt + X_1(t)\sigma_2(t)dB_2(t) + X_2(t)\Theta_1(t)dt + X_2(t)\sigma_1(t)dB_1(t)$$
$$= \Bigg(\sigma_1(t)\sigma_2(t)\rho(t) + X_1(t)\Theta_2(t) + X_2(t)\Theta_1(t)\Bigg)dt + X_1(t)\sigma_2(t)dB_2(t) + X_2\sigma_1(t)dB_1(t)$$

In the case $\rho, \Theta_1, \Theta_2, \sigma_1, \sigma_2$ are all constants, then:

$$X_1(t) = X_1(0) + \Theta_1 t + \sigma_1 B_1(t)$$
$$X_2(t) = X_2(0) + \Theta_2 t + \sigma_2 B_2(t)$$
$$dX_1(t)dX_2(t) = \big(\Theta_1 t + \sigma_1 B_1(t)\big)\big(\Theta_2 t + \sigma_2 B_2(t)\big) = \big(\Theta_1\Theta_2 t^2 + \big)$$

1. Fix $t_0 > 0$ and let $\epsilon > 0$ be given. Show that (Hint:Ito's product rule)

$$\mathbb{E}\Bigg[\big(B_1(t_0+\epsilon) - B_1(t_0)\big)\big(B_2(t_0+\epsilon) - B_2(t_0)\big)\Bigg|\mathcal{F}_{t_0}\Bigg] = \rho\epsilon \tag{113}$$

   Proof Since $dB_1(t)dB_2(t) = \rho dt$, result follows

2. Mean conditioned on $\mathcal{F}_{t_0}$

$$\mathbb{E}\big[X_i(t_0+\epsilon) - X_i(t_0)\big|\mathcal{F}_{t_0}\big] = \mathbb{E}\Bigg[\Theta_i\epsilon + \sigma_i\big(B_i(t_0+\epsilon) - B_i(t_0)\big)\big|\mathcal{F}_{t_0}\Bigg]$$
$$= \Theta_i\epsilon + \sigma_i \underbrace{\mathbb{E}\big[B_i(t_0+\epsilon) - B_i(t_0)\big|\mathcal{F}_{t_0}\big]}_{=0, def\ of\ Brownian\ motion}$$

3. Covariance conditioned on $\mathcal{F}_{t_0}$

$$\mathbb{E}\Bigg[\big(\Theta_1\epsilon + \sigma_1[B_1(t_0+\epsilon) - B_1(t_0)]\big)\big(\Theta_2\epsilon + \sigma_2[B_2(t_0+\epsilon) - B_1(t_0)]\big)\big|\mathcal{F}_{t_0}\Bigg]$$
$$= \Theta_1\Theta_2\epsilon^2 + \sigma_1\sigma_2\rho\epsilon.$$

4. Make a time varying drift and covariance structure, show that:

$$M_i(\epsilon) := \mathbb{E}\Bigg[X_i(t_0+\epsilon) - X_i(t_0)\big|\mathcal{F}_{t_0}\Bigg] = \Theta_i(t_0)\epsilon + o(\epsilon)$$

   So we necessarily need to prove that

$$\lim_{\epsilon\to 0}\frac{M_i(\epsilon)}{\epsilon} = \Theta_i(t_0) \tag{114}$$

   Essentially under the assumption of $|\Theta_i(u)| \leq M, |\sigma_i(u)| \leq M$ We know that(instead of ϵ, pick $\frac{1}{k}$ for $k \in \mathbb{N}$)

$$X_i(t_0 + \frac{1}{k}) - X_i(t_0) = \int_{t_0}^{t_0+\frac{1}{k}} \Theta_i(s)ds + \int_{t_0}^{t_0+\frac{1}{k}} \sigma_i(s)dB_s \tag{115}$$

Now a mesh with $t_0 = t_0^n < t_1^n < \ldots < t_n^n = t_0 + \frac{1}{k}$, preceding equations simplify as for $l = 1, 2$

$$X_l\big(t_0 + \frac{1}{k}\big) - X_l(t_0) = \sum_{i=0}^{n-1} \Theta_l(t_i^n)(t_{i+1}^n - t_i^n) + \sigma_l(t_i^n)\big(B_l(t_{i+1}^n) - B_l(t_i^n)\big)$$

Now using conditioning properly reveals that:

$$k\mathbb{E}\big[X_i(t_0 + \epsilon) - X_i(t_0)\big|\mathcal{F}_{t_0}\big] = k\Theta_l(t_0)(t_1^n - t_0) + k\sum_{i=1}^{n-1} \mathbb{E}_{t_0}[\Theta_l(t_i^n)](t_{i+1}^n - t_i^n) \tag{116}$$

as $t_1^n - t_0 < \frac{1}{k}$ and $\sum_{i=1}^{n-1} \mathbb{E}_{t_0}[\Theta_l(t_i^n)](t_{i+1}^n - t_i^n)$ where Brownian increments will turn out to be zero by towering property of conditonal expectation and recall that $\Theta_l(t_0)$ is $\mathcal{F}_{t_0}$ measurable.

$$M_i(\epsilon) := \mathbb{E}\Bigg[X_i(t_0 + \epsilon) - X_i(t_0)\Big|\mathcal{F}_{t_0}\Bigg] = \mathbb{E}\Bigg[\Theta_i(t_0)\epsilon + \sigma_i(t_0)\big(B_2(t_{i+1}^n) - B_2(t_i^n)\big)\Big|\mathcal{F}_{t_0}\Bigg]$$
$$= \Theta_i(t_0)\epsilon$$

$\blacksquare$

13. Exercise 4.18

$$dS_t = \alpha S_t dt + \sigma S_t dW_t \tag{117}$$

market price of risk, $\theta := \frac{\alpha - r}{\sigma}$ and *state price density*

$$\zeta_t = e^{-\theta W_t - \Bigg(r + \frac{1}{2}\theta^2\Bigg)t} \tag{118}$$

1. Show that:

$$d\zeta_t = -r\zeta_t dt - \theta\zeta_t dW_t$$

2. Show that $\zeta_t X_t$ is a martingale, where X_t is the investment portfolio:

$$dX_t = \big[rX_t + \Delta_t(\alpha - r)S_t\big]dt + \Delta_t \sigma S_t dW_t \tag{119}$$

Proof Let $f(t,x) = e^{-\theta x - \left(r+\frac{1}{2}\theta^2\right)t}$ where $x := \int_0^t dW_s$. $\partial_t f(t,x) = -\left(r+\frac{1}{2}\theta^2\right) f(t,x),\ \ \partial_x f(t,x) = -\theta f(t,x)\ \ \partial_{xx} f(t,x) = \theta^2 f(t,x)$. Now using Ito formula we get

$$
\begin{aligned}
& f(t,X_t) = \int_0^t -\left(r+\frac{1}{2}\theta^2\right) f(s,x_s) + \frac{1}{2}\theta^2 f(t,x) + \int_0^t -\theta f(s,x) dW_s \\
& = \int_0^t -\left(r+\frac{1}{2}\theta^2 - \frac{1}{2}\theta^2\right) f(s,X_s) - \int_0^t \theta f(s,X_s) dW_s \\
& = -r\int_0^t f(s,X_s)ds - \theta \int_0^t f(s,X_s) dW_s \\
& df(t,X_t) = -rf(t,X_t)dt - \theta f(t,X_t) dW_t \\
& \zeta_t := f(t,X_t)
\end{aligned}
$$

Chain rule

$$
\begin{aligned}
d\big(\zeta_t X_t\big) &= d\zeta_t dX_t + X_t d\zeta_t + \zeta_t dX_t \\
&= \big(-r\zeta_t dt - \theta\zeta_t dW_t\big)\big(rX_t dt + \Delta_t(\alpha-r)S_t dt + \Delta_t \sigma S_t dW_t\big) \\
&+ X_t\big(-r\zeta_t dt - \theta\zeta_t dW_t\big) + \zeta_t\big(rX_t dt + \Delta_t(\alpha-r)S_t dt + \Delta_t\sigma S_t dW_t\big) \\
&= -r\theta\zeta_t X_t dW_t - \sigma\theta\zeta_t\Delta_t S_t dt - rX_t\zeta_t dt \\
&- \theta X_t \zeta_t dW_t + rX_t\zeta_t dt + \zeta_t\Delta_t(\alpha-r)S_t dt + \sigma\zeta_t\Delta_t S_t dW_t \\
&= \zeta_t\Big\{-\big(1+r\big)\theta X_t + \sigma\Delta_t S_t\Big\} dW_t + \zeta_t\Delta_t S_t \underbrace{(\alpha - r - \sigma\theta)}_{=0} dt
\end{aligned}
$$

Drift terms are 0, hence a martingale.
If for some fixed time $T > 0$, investor wants his portfolio to equal $V(T)$- $\mathcal{F}_T$ adapted, then he must begin with $X_0 = \mathbb{E}[\zeta_T V(T)]$

$$
\begin{aligned}
& \zeta_T X_T = \zeta_0 X_0 + \int_0^T \zeta_t\Big\{-\big(1+r\big)\theta X_t + \sigma\Delta_t S_t\Big\} dW_t \\
& \mathbb{E}[\zeta_T X_T | \mathcal{F}_t] = \zeta_t X_t \ \ a.s\ for\ all\ t < T \\
& \zeta_t = e^{-\theta W_t - \left(r + \frac{1}{2}\theta^2\right)t} \implies \zeta_0 = 1\ \ a.s\ and\ consequently \\
& \mathbb{E}[\zeta_T X_T] = \mathbb{E}\mathbb{E}\big[\zeta_T X_T | \mathcal{F}_{t_0}\big] = \zeta_0 X_0 = X_0,\ a.s
\end{aligned}
$$

14. Stop-Loss start-gain paradox, Ex 4.21

GBM with mean rate of return 0

$$dS_t = \sigma S_t dW_t \tag{120}$$

Assume interest rate of 0

■

14.1 Random Walks and Martingales(Green Quant Book)

1. X_n is $\mathcal{F}_n$ **adapted** if X_n is $\mathcal{F}_n$ measurable for all $n \in \mathbb{N}$ which implies $\mathbb{E}[X_n|\mathcal{F}_n] = X_n$ almost surely

2. X_n is $\mathcal{F}_n$ **predictable** if X_n is $\mathcal{F}_{n-1}$ measurable for all $n \in \mathbb{N}$, mathematically $\mathbb{E}[X_n|\mathcal{F}_{n-1}] = X_n$ almost surely.

Gamblers' ruin Let X_i be the capital of the gambler at the end of round $1(X_0 = 0)$. $\mathbb{E}[X_{i+1}|\mathcal{F}_i] = \frac{1}{2}(X_i + 1) + \frac{1}{2}(X_i - 1) = X_i$ so it is a martingale. **What is the probability that gambler is ruined, assuming ruin value is -a** Now we define X

Definition 28 (Martingale transform being a martingale:) *Let M_n be a martingale and A_n be a predictable process then: $(A.M)_n := \sum_{k=1}^{n} A_k(M_k - M_{k-1})$ is again a martingale, provided that A_n and $(A.M)_n$ are in $\mathcal{L}_1$ for all n*

Proof For $m \leq n$,

$$\mathbb{E}[(A.M)_n|\mathcal{F}_m] = \sum_{k=1}^{m} \mathbb{E}[A_k M_k - A_k M_{k-1}|\mathcal{F}_m] + \sum_{k=m+1}^{n} \mathbb{E}[A_k M_k - A_k M_{k-1}|\mathcal{F}_m]$$

$$\sum_{k=1}^{m} \mathbb{E}[A_k M_k - A_k M_{k-1}|\mathcal{F}_m] = \sum_{k=1}^{m} A_k(M_k - M_{k-1})$$

$$\mathbb{E}[A_{m+1}(M_{m+1} - M_m)|\mathcal{F}_m] = A_{m+1}\underbrace{\mathbb{E}[(M_{m+1} - M_m)|\mathcal{F}_m]}_{=0}, \quad \textit{almost surely}$$

$$\mathbb{E}[A_{m+2}(M_{m+2} - M_{m+1})|\mathcal{F}_m] = \mathbb{E}\left[\mathbb{E}[A_{m+2}(M_{m+2} - M_{m+1})|\mathcal{F}_{m+1}]\middle|\mathcal{F}_m\right]$$

$$= \mathbb{E}\left[A_{m+2}\underbrace{\mathbb{E}[(M_{m+2} - M_{m+1})|\mathcal{F}_{m+1}]}_{=0}\middle|\mathcal{F}_m\right]$$

Conditioning property is validated, to show $\mathbb{E}[|(A.M)_n|] < \infty$ for all n follows by hypothesis ■

1. Stopping rule

2. Walds' inequality

3. Martingale and symmetric random walks, example including:

   - Gamblers ruin
   - Drunk man
   - Dice Game
   - Ticket line

$$3\int_0^t (T-s+W_s^2)dW_s = 3TW_t + 3\int_0^t \underbrace{(W_s^2-s)}_{Martingale}dW_s$$

$$\int_0^t (W_s^2-s)dW_s = \lim_{n\to\infty}\sum_{i=0}^{n-1}\left(W_{t_i^n}^2 - t_i^n\right)\left(W_{t_{i+1}^n} - W_{t_i^n}\right)$$

$$= -\lim_{n\to\infty}\sum_{i=0}^{n-1}\left([t_i^n - W_{t_{i+1}^n}^2] + [W_{t_{i+1}^n}^2 - W_{t_i^n}^2]\right)\left(W_{t_{i+1}^n} - W_{t_i^n}\right) \text{ where term after summation operator is}$$

$$= \left(t_i^n - W_{t_{i+1}^n}^2\right)\left(W_{t_{i+1}^n} - W_{t_i^n}\right) + \underbrace{\left(W_{t_{i+1}^n}^3 - W_{t_i^n}^3\right)}_{\to W_t^3} - W_{t_i^n}^2 W_{t_{i+1}^n} - W_{t_{i+1}^n}^2 W_{t_i^n}$$

$$\left(t_i^n - W_{t_{i+1}^n}^2\right)\left(W_{t_{i+1}^n} - W_{t_i^n}\right) - W_{t_i^n}^2 W_{t_{i+1}^n} - W_{t_{i+1}^n}^2 W_{t_i^n} = t_i^n\left(W_{t_{i+1}^n} - W_{t_i^n}\right) + W_{t_{i+1}^n}^3 - W_{t_i^n}^2 W_{t_{i+1}^n}$$

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} t_i^n\left(W_{t_{i+1}^n} - W_{t_i^n}\right) = t_0^n\left(W_{t_1^n} - W_{t_0^n}\right) + t_1^n\left(W_{t_2^n} - W_{t_1^n}\right) + t_2^n\left(W_{t_3^n} - W_{t_2^n}\right) + t_3^n\left(W_{t_4^n} - W_{t_3^n}\right)$$

$$t_4^n\left(W_{t_5^n} - W_{t_4^n}\right) + \ldots + t_-^n(W_t - W_{t_-^n})$$

$$\mathbb{E}\left(\sum_{i=0}^{n-2} t_i^n\left(W_{t_{i+1}^n} - W_{t_i^n}\right) + t_-^n(W_t - W_{t_-^n}) - tW_t\right)^2$$

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} W_{t_{i+1}^n}\left(W_{t_{i+1}^n}^2 - W_{t_i^n}^2\right) + \left(W_{t_{i+1}^n}^3 - W_{t_i^n}^3\right)$$

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} W_{t_{i+1}^n}^3 - W_{t_i^n}^3 = W_t^3 \ \textit{need to show}$$

$$\mathbb{E}\left\langle \sum_{i=0}^{n-1} W_{t_{i+1}^n}\left(W_{t_{i+1}^n}^2 - W_{t_i^n}^2\right), \sum_{i=0}^{n-1} W_{t_{i+1}^n}\left(W_{t_{i+1}^n}^2 - W_{t_i^n}^2\right)\right\rangle$$

$$= \mathbb{E}\sum_{i=0}^{n-1}\left(W_{t_{i+1}^n}^2\left(W_{t_{i+1}^n}^2 - W_{t_i^n}^2\right)^2 + W_{t_{i+1}^n}\left(W_{t_{i+1}^n}^2 - W_{t_i^n}^2\right)\sum_{j>i}^{n-1} W_{t_{j+1}^n}\left(W_{t_{j+1}^n}^2 - W_{t_j^n}^2\right)\right)$$

Let $s \sim U[0,1]$ and $X(s) = s$, we consider the following sequence of random variables $X_2(s) = s + 1_{[0,\frac{1}{2}]}$, $X_3(s) = s + 1_{[\frac{1}{2},1]}$, $X_4(s) = s + 1_{[0,\frac{1}{3}]}$, $X_5(s) = s + 1_{[\frac{1}{3},\frac{2}{3}]}$, $X_6(s) = s + 1_{[\frac{2}{3},1]}$ and so on.

$$\mathbb{P}\left(|X_n(s) - X(s)| > \epsilon\right) \leq \frac{1}{n} \tag{121}$$

Therefore converges in probability but not almost surely(will comeback to this later) stochastic convergence requires X_n and X to be defined on the same probability space

Quadratic mean convergence

$$\|X_n - X\|_{L_2} \to 0 \tag{122}$$

An example of convergence in distribution: Consider $X = 0$, then $F_X(x) = 0$ for $x < 0$ and 1 for $x > 0$ so we have a discontinuity at 0.

Definition 29 *X_n converges in disrtibution to X if for all points x where F_X is continuous:*

$$F_{X_n}(x) \to F_X(x) \tag{123}$$

$X_n = N(0, \frac{1}{n})$, then $\mathbb{P}\big(X_n \leq x\big) = \mathbb{P}\big(\frac{Z}{\sqrt{n}} \leq x\big) = \mathbb{P}\big(Z \leq \sqrt{n}x\big) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sqrt{n}x} e^{-\frac{y^2}{2}} dy$ Now if $x > 0$ then this converges to 1 and when $x < 0$ convergest to 0. Therefore

$$X_n(x) \implies \delta_0(x) \tag{124}$$

Also notice that $\mathbb{E}[X_n^2] = \frac{1}{n}$ so X_n also converges in probability to 0.

Theorem 30 *$X_1, X_2, \ldots$ be iid $U[0,1]$ and $M_n := \max_{i \in [n]} X_i$ then M_n converges in prob to 1 and $n(1 - M_n)$ to Exp(1).*

14.2 Integration in polar coordinates

14.3 Modes of convergence

X_n converges to X in probability, if:

$$\mathbb{P}\left(\bigcap_{\epsilon, \delta > 0} \bigcup_{N(\epsilon,\delta)} \bigcap_{n \geq N(\epsilon,\delta)} |X_n(w) - X(w)| \geq \epsilon \right) \leq \delta \tag{125}$$

$$\begin{aligned}
X_{t_1^n} &= \mu^1_{t_0^n}(X_{t_0^n})(t_1^n - t_0^n) + \sigma^{(1,1)}_{t_0^n}(X_{t_0^n})(W^1_{t_1^n} - W^1_{t_0^n}) \\
Y_{t_1^n} &= \mu^2_{t_0^n}(Y_{t_0^n})(t_1^n - t_0^n) + \sigma^{(2,1)}_{t_0^n}(Y_{t_0^n}, X_{t_0^n})(W^1_{t_1^n} - W^1_{t_0^n}) + \sigma^{(2,2)}_{t_0^n}(Y_{t_0^n})(W^2_{t_1^n} - W^2_{t_0^n}) \\
X_{t_2^n} &= \mu^1_{t_0^n}(X_{t_1^n})(t_2^n - t_1^n) + \sigma^{(1,1)}_{t_0^n}(X_{t_1^n})(W^1_{t_2^n} - W^1_{t_1^n}) \\
Y_{t_2^n} &= \mu^2_{t_0^n}(Y_{t_1^n})(t_2^n - t_1^n) + \sigma^{(2,1)}_{t_0^n}(Y_{t_1^n}, X_{t_1^n})(W^1_{t_2^n} - W^1_{t_1^n}) + \sigma^{(2,2)}_{t_0^n}(Y_{t_1^n})(W^2_{t_1^n} - W^2_{t_0^n})
\end{aligned}$$

This means that:

$$\begin{aligned}
Z_{t_1^n} &= \big(\mu^1_{t_0^n}(t_1^n - t_0^n) + \sigma^{(1,1)}_{t_0^n}(W^1_{t_1^n} - W^1_{t_0^n})\big)\big(\mu^2_{t_0^n}(t_1^n - t_0^n) + \sigma^{(2,1)}_{t_0^n}(W^1_{t_1^n} - W^1_{t_0^n}) + \sigma^{(2,2)}_{t_0^n}(W^2_{t_1^n} - W^2_{t_0^n})\big) \\
&= \mu^1_{t_0^n}\mu^2_{t_0^n}(t_1^n - t_0^n)^2 + \mu^1_{t_0^n}(t_1^n - t_0^n)\sigma^{(2,1)}_{t_0^n}(W^1_{t_1^n} - W^1_{t_0^n})) + \mu^1_{t_0^n}(t_1^n - t_0^n)\sigma^{(2,2)}_{t_0^n}(W^2_{t_1^n} - W^2_{t_0^n}) \\
&+ \mu^2_{t_0^n}(t_1^n - t_0^n)\sigma^{(1,1)}_{t_0^n}(W^1_{t_1^n} - W^1_{t_0^n}) + \sigma^{(1,1)}_{t_0^n}(W^1_{t_1^n} - W^1_{t_0^n})\sigma^{(2,1)}_{t_0^n}(W^1_{t_1^n} - W^1_{t_0^n})) + \sigma^{(1,1)}_{t_0^n}(W^1_{t_1^n} - W^1_{t_0^n})\sigma^{(2,2)}_{t_0^n}(W^2_{t_1^n} - W^2_{t_0^n})
\end{aligned}$$

$$Z_t = Z_0 + \sum_{i=0}^{n-1} \left(Z_{t_{i+1}^n} - Z_{t_i^n} \right) \tag{126}$$

$$\begin{aligned}
dX_t &= dc(t, S_t) - c_x(t, S_t)dS_t + r\left(X_t - c(t, S_t) + c_x(t, S_t)S_t\right)dt - \frac{1}{2}\left(\sigma_2^2 - \sigma_1^2\right)S_t^2 c_{xx}(t, S_t)dt \\
&= \big(c_t(t, S_t) + \alpha S_t \nabla c(t, S_t) + \frac{1}{2}\sigma_1^2 S_t^2 D^2 c(t, S_t)\big)dt + \sigma_1 S_t \nabla c(t, S_t)dW_t - c_x(t, S_t)\big(\mu S_t d_t + \sigma_2 S_t dW_t\big) \\
&+ r\left(X_t - c(t, S_t) + c_x(t, S_t)S_t\right)dt - \frac{1}{2}\left(\sigma_2^2 - \sigma_1^2\right)S_t^2 c_{xx}(t, S_t)dt \\
&= \left[c_t(t, S_t) + (\alpha + r - \alpha)S_t c_x(t, S_t) + rX_t - rc(t, S_t) - \frac{1}{2}(\sigma_2^2 - 2\sigma_1^2)S_t^2 c_{xx}(t, S_t)\right]dt \\
&+ c_x(t, S_t)S_t(\sigma_1 - \sigma_2)dW_t
\end{aligned}$$

Which means that

$$d(e^{-rt}X_t) = e^{-rt}\left(c_t(t,S_t) + rc_x(t,S_t) + \frac{1}{2}\sigma_1^2 S_t^2 c_{xx}(t,S_t) - rc(t,S_t) - \frac{1}{2}(\sigma_2^2 - \sigma_1^2)S_t^2 c_{xx}(t,S_t)\right)dt$$
$$+ e^{-rt}c_x(t,S_t)S_t(\sigma_1 - \sigma_2)dW_t,$$

but recall:

$$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma_1^2 x^2 c_{xx}(t,x) = rc(t,x), \;\; \forall t \in (0,T], x \in \mathbb{R}^+ \tag{127}$$

implying that:

$$d(e^{-rt}X_t) = -e^{-rt}\frac{1}{2}(\sigma_2^2 - \sigma_1^2)S_t^2 c_{xx}(t,S_t)dt + +e^{-rt}c_x(t,S_t)S_t(\sigma_1 - \sigma_2)dW_t \tag{128}$$

So total value of portfolio at time t

$$X_t = \Delta_t S_t + \Gamma_t M_t \tag{129}$$

Let $c(t,x)$ be the price of a call option at time t, if the stock price at time t is $S_t = x$

15. Try again

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t \\ dX_t := df(t,S_t) &= \Big(rf(t,S_t) + \Delta_t(\alpha - r)S_t\Big)dt + \Delta_t \sigma S_t dW_t \end{aligned}$$

This means that $f_x(t,x)_{x=S_t} = \Delta_t, \;\; \frac{1}{2}\sigma^2 x^2 f_{xx}(t,x)$

15.1 Call price

$C(T,x) = (x-K)^+$ on the other hand if

CHAPTER: RISK NEUTRAL PRICING

15.2 Change of measure

recall that if $\mathbb{Q}_1$ is some probability measure then

$$\mathbb{E}_{\mathbb{Q}_1} f = \int f(w)\mathbb{Q}_1(dw) \tag{130}$$

Let $\mathbb{Q}_2$ be another probability such that $\mathbb{Q}_1 << \mathbb{Q}_2$ i.e., $\frac{d\mathbb{Q}_1}{d\mathbb{Q}_2}$ exists. Mathematically speaking for any borel measurable set A

$$\mathbb{Q}_1(A) = \int_A \frac{d\mathbb{Q}_1}{d\mathbb{Q}_2}(w)\mathbb{Q}_2(dw) \tag{131}$$

Often in practice we can not compute $\mathbb{E}_{\mathbb{Q}_1} f$, either because density function of $\mathbb{Q}_1$ is intractable. So instead assume we have Y_i be i.i.d random variable such that $Y_i \sim \mathbb{Q}_2$ and we have knowledge of Radon-Nikodym derivative $\frac{d\mathbb{Q}_1}{d\mathbb{Q}_2}$, then

$$\frac{1}{n}\sum_{i=1}^{n} f(y_i)\frac{d\mathbb{Q}_1}{d\mathbb{Q}_2}(y_i) \approx \mathbb{E}_{\mathbb{Q}_1} f \tag{132}$$

Thanks to Law of Large Numbers Let X be a standard normal, θ is some constant and

$$Z = e^{-\theta X - \frac{1}{2}\theta^2}, \tag{133}$$

being the radon-nikodym derviative. Then $Y := X + \theta$ is standard normal under the change of measure

$$\mathbb{E}_{X\sim N(0,1)}\left[e^{-\theta X - \frac{1}{2}\theta^2}\right] = e^{-\frac{1}{2}\theta^2}\mathbb{E}_{X\sim N(0,1)}\underbrace{\left[e^{-\theta X}\right]}_{e^{\frac{1}{2}\theta^2}} = 1 \tag{134}$$

So it is a valid radon-nikodym derivative. Furthermore, for every $m \in \mathbb{R}$

$$\mathbb{E}_{X\sim N(0,1)}\left[e^{mY}e^{-\theta X - \frac{1}{2}\theta^2}\right] = e^{m\theta - \frac{1}{2}\theta^2}\mathbb{E}_{X\sim N(0,1)}\left[e^{(m-\theta)X}\right] = e^{m\theta-\frac{1}{2}\theta^2}e^{\frac{1}{2}(m-\theta)^2} = e^{\frac{1}{2}m^2} \tag{135}$$

Theorem 31 (Girsanov) *Let W_t be $\mathcal{F}_t$ adapted on the probability space $\left(\Omega, \mathcal{F}_{t\in[0,T]}, \mathcal{F}, \mathbb{P}\right)$ $X_t, W_t \in \mathbb{R}^d$ and $F_s \in \mathbb{R}^{d\times 1}$*

$$X_t - X_s = \int_s^r F_r dr + (W_t - W_s), \;\; t \in [0,T]$$

$$\Lambda := e^{-\int_0^T \langle F_s, dW_s\rangle - \frac{1}{2}\int_0^T \langle F_s, F_s\rangle ds}$$

Novikov's condition valid, i.e., $\mathbb{E}_{\mathbb{P}}\left[e^{\frac{1}{2}\int_0^T \langle F_s, F_s\rangle ds}\right] < \infty$

Then X_t is an $\mathcal{F}_t-$ Wiener process under $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}\left[\Lambda 1_A\right]$

Proof We first need the following prop

Proposition 32 *Let $Y \in \mathcal{F}_t$, then $\mathbb{E}_{\mathbb{Q}}[Y] = \mathbb{E}_{\mathbb{P}}[YM_t]$ and $M_s\mathbb{E}_{\mathbb{Q}}\left[Y|\mathcal{F}_s\right] = \mathbb{E}_{\mathbb{P}}\left[YM_t|\mathcal{F}_s\right]$, where $\mathbb{Q}$ represents radon-nikodym density under $\Lambda := M_T$.*

W.l.o.g whenever expectation is without subscript it is wrt original measure ı. Now notice $\mathbb{E}_{\mathbb{Q}}[Y] = \mathbb{E}[YM_T]$, but by the definition of conditional expectation $\mathbb{E}[YM_T] = \mathbb{E}[Y\mathbb{E}[M_T|\mathcal{F}_t]] := \mathbb{E}[YM_t]$.

By definition of conditional expectation in Kolmogorovs' sense: for $\eta \in \mathcal{F}_s \subset \mathcal{F}_t$:

$$\mathbb{E}_{\mathbb{Q}}\left[\eta\mathbb{E}_{\mathbb{Q}}\left[Y|\mathcal{F}_s\right]\right] = \mathbb{E}_{\mathbb{Q}}\left[\eta Y\right] = \mathbb{E}_{\mathbb{P}}\left[\eta Y M_T\right] = \mathbb{E}_{\mathbb{Q}}\left[\eta Y\right] = \mathbb{E}_{\mathbb{P}}\left[\eta Y M_T\right] =$$
$$\mathbb{E}_{\mathbb{P}}\left[\eta\mathbb{E}_{\mathbb{P}}\left[YM_T|\mathcal{F}_s\right]\right]$$

But $\mathbb{E}[\eta Y M_T] = \mathbb{E}[\eta Y M_t]$. Therefore,

$$\int \eta \mathbb{E}_{\mathbb{Q}}\big[Y|\mathcal{F}_s\big] M_T d\mathbb{P} = \int \eta \mathbb{E}_{\mathbb{Q}}\big[Y|\mathcal{F}_s\big] M_s d\mathbb{P} = \int \eta Y M_t d\mathbb{P} = \int \eta \mathbb{E}_{\mathbb{P}}[Y M_t|\mathcal{F}_s] d\mathbb{P} \tag{136}$$

which means that:

$$\mathbb{E}_{\mathbb{Q}}\big[Y|\mathcal{F}_s\big] = \frac{\mathbb{E}_{\mathbb{P}}[Y M_t|\mathcal{F}_s]}{M_s} \;\; \textit{almost surely w.r.t} - \mathbb{P} \tag{137}$$

Let ζ_t be defined as follows with differential and stuff

$$\begin{aligned}
&\zeta_t := e^{-\int_0^t F_s dW_s - \frac{1}{2}\int_0^t \langle F_s, F_s\rangle ds} \\
&d\zeta_t = -\zeta_t F_t dW_t \\
&\zeta_t = 1 - \int_0^t \zeta_s F_s dW_s
\end{aligned}$$

.So ζ_t is a martingale itself and $\zeta_T = \Lambda$. Furthermore notice that:

$$\begin{aligned}
&d(\zeta_t X_t) = -\zeta_t F_t dW_t(F_t dt + dW_t) + \zeta_t(F_t dt + dW_t) - X_t \zeta_t F_t dW_t \\
&= \zeta_t(1 - X_t F_t) dW_t.
\end{aligned}$$

Therefore, $\zeta_t X_t$ is again a Martinagle and recall.

$$\mathbb{E}_{\mathbb{Q}}\big[X_t|\mathcal{F}_s\big] = \frac{\mathbb{E}_{\mathbb{P}}[X_t \zeta_t|\mathcal{F}_s]}{\zeta_s} = \frac{X_s \zeta_s}{\zeta_s} = X_s \tag{138}$$

. Hence a brownian motion ■

Recall that $M_s \mathbb{E}_{\mathbb{Q}}\big[Y|\mathcal{F}_s\big] = \mathbb{E}_{\mathbb{P}}\big[Y M_t|\mathcal{F}_s\big]$ Now I will attempt via moment generating functions

$$\mathbb{E}_{\mathbb{Q}}[e^{m(X_t - X_s)}|\mathcal{F}_s] = \frac{\mathbb{E}[e^{m(X_t - X_s)}\zeta_t|\mathcal{F}_s]}{\zeta_s} \tag{139}$$

Proof Let $0 < t_1 < t_2 < T$, and start with scalar case $d = 1$ then suffices to show that:

1. For any $m \in \mathbb{R}$ and $t \in [0, T]$

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\big[e^{m X_t}\Lambda\big] = e^{\frac{m^2 t}{2}} \\
&= \mathbb{E}_{\mathbb{P}}\big[e^{m X_t}\Lambda\big] = \mathbb{E}_{\mathbb{P}}[e^{m\int_0^t F_s ds + W_t} e^{-\int_0^T \langle F_s, dW_s\rangle - \frac{1}{2}\int_0^T \langle F_s, F_s\rangle ds}] \\
&= \mathbb{E}_{\mathbb{P}}[e^{\int_0^t \left(m F_s - \frac{1}{2}\langle F_s, F_s\rangle\right) ds + \int_0^t (m - F_s) dW_s} e^{-\int_t^T F_s dW_s - \frac{1}{2}\int_t^T \langle F_s, F_s\rangle ds}] \\
&= \mathbb{E}_{\mathbb{P}}\Big[\mathbb{E}_{\mathbb{P}}\big[e^{\int_0^t \left(m F_s - \frac{1}{2}\langle F_s, F_s\rangle\right) ds + \int_0^t (m - F_s) dW_s} e^{-\int_t^T F_s dW_s - \frac{1}{2}\int_t^T \langle F_s, F_s\rangle ds}\big|\mathcal{F}_t\big]\Big] \\
&= \mathbb{E}_{\mathbb{P}}\Big[e^{\int_0^t \left(m F_s - \frac{1}{2}\langle F_s, F_s\rangle\right) ds + \int_0^t (m - F_s) dW_s} \underbrace{\mathbb{E}_{\mathbb{P}}\big[e^{-\int_t^T F_s dW_s - \frac{1}{2}\int_t^T \langle F_s, F_s\rangle ds}\big|\mathcal{F}_t\big]}_{:= g(t, W_t)}\Big]
\end{aligned}$$

For $t < T$, let $M_t := \mathbb{E}[\Lambda|\mathcal{F}_t]$ be a *martingale*, because:

$$M_t = \mathbb{E}_{\mathbb{P}}[e^{-\int_0^T \langle F_s, dW_s\rangle - \frac{1}{2}\int_0^T \langle F_s, F_s\rangle ds}|\mathcal{F}_t] \ \textit{for } r<t$$
$$\mathbb{E}_{\mathbb{P}}\big[M_t\big|\mathcal{F}_r\big] = \mathbb{E}_{\mathbb{P}}\big[\mathbb{E}_{\mathbb{P}}[e^{-\int_0^T \langle F_s, dW_s\rangle - \frac{1}{2}\int_0^T \langle F_s, F_s\rangle ds}|\mathcal{F}_t]\big|\mathcal{F}_r\big]$$
$$= \mathbb{E}_{\mathbb{P}}[e^{-\int_0^T \langle F_s, dW_s\rangle - \frac{1}{2}\int_0^T \langle F_s, F_s\rangle ds}|\mathcal{F}_r] =: M_r$$

This means by Martingale representation theorem exist an adapted process ϕ_t such that:

$$M_t = 1 + \int_0^t \phi_s dW_s \tag{140}$$

Therefore simply we get:

$$\mathbb{E}_{\mathbb{Q}}\big[e^{m(X_t - X_s)}|\mathcal{F}_s\big] = \frac{\mathbb{E}_{\mathbb{P}}\big[e^{m\int_s^t F_r dr} e^{m(W_t - W_s)} M_t|\mathcal{F}_s\big]}{M_s} = e^{\frac{1}{2}m^2(t-s)} \frac{\mathbb{E}_{\mathbb{P}}\big[e^{m\int_s^t F_r dr} M_t|\mathcal{F}_s\big]}{M_s}$$

Let us focus on term:

$$g(t, X_t) := \mathbb{E}_{\mathbb{P}}\big[e^{-\int_t^T F_s dW_s - \frac{1}{2}\int_t^T \langle F_s, F_s\rangle ds}\big|\mathcal{F}_t\big] \ \ as$$
$$M_t = \underbrace{e^{-\int_0^t F_s dW_s - \frac{1}{2}\int_0^t \langle F_s, F_s\rangle ds}}_{\zeta_t} g(t, X_t)$$

Now we should apply ito's formula $f(x) = e^{-x}$ on $x := \int_0^t F_s dW_s + \frac{1}{2}\int_0^t \langle F_s, F_s\rangle ds$ to get:

$$df(X_t) = [f(X_t)\frac{1}{2}F_t^2 - \frac{1}{2}f(X_t)F_t^2]dt - F_t f(X_t)dW_t = -f(X_t)F_t dW_t \tag{141}$$

Hypothetically assume:

$$dg(t, X_t) = \Theta_t dt + \sigma_t dW_t \tag{142}$$

and now let us apply product rule on dM_t

$$d(M_t) = d\big(\zeta_t g_t\big) = d\zeta_t dg_t + \zeta_t dg_t + g_t d\zeta_t = (-\zeta_t F_t dW_t + \zeta_t)dg_t - g_t\zeta_t F_t dW_t$$
$$= (-\zeta_t F_t dW_t + \zeta_t)(\Theta_t dt + \sigma_t dW_t) - g_t\zeta_t F_t dW_t = -\zeta_t\sigma_t F_t dt + \zeta_t\Theta_t dt + \zeta_t\sigma_t dW_t - g_t\zeta_t F_t dW_t$$
$$= \zeta_t\big(\Theta_t - \sigma_t F_t\big)dt + \zeta_t\big(\sigma_t - g_t F_t\big)dW_t.$$

Recall that:

$$dX_t = F_t dt + dW_t \tag{143}$$

Therefore,

$$dg(t, X_t) = \big\{g_t(t, X_t) + g_x(t, X_t)F_t + \frac{1}{2}g_{xx}(t, X_t)\big\}dt + g_x(t, X_t)dW_t \tag{144}$$

$$\Big(-\zeta_t F_t dW_t + \zeta_t\Big)\left(\Big\{g_t(t,X_t) + g_x(t,X_t)F_t + \frac{1}{2}g_{xx}(t,X_t)\Big\}dt + g_x(t,X_t)dW_t\right) - g(t,X_t)\zeta_t F_t dW_t$$

$$= \zeta_t\left(-F_t g_x(t,X_t) + g_t(t,X_t) + g_x(t,X_t)F_t + \frac{1}{2}g_{xx}(t,X_t)\right)dt + \zeta_t\Big(g_x(t,X_t) - g(t,X_t)F_t\Big)dW_t$$

$$\implies d(M_t) = \zeta_t\left(g_t(t,X_t) + \frac{1}{2}g_{xx}(t,X_t)\right)dt + \zeta_t\left(g_x(t,X_t) - g(t,X_t)F_t\right)dW_t$$

Recall that $d\zeta_t = -\zeta_t F_t dW_t \implies dM_t = \zeta_t g_x(t,X_t)dW_t + g(t,X_t)d\zeta_t$

$$M_t = M_0 + \zeta_t\left\{g_x(t,X_t) - g(t,X_t)F_t\right\}dW_t \tag{145}$$

Consequently:

$$g_t(t,X_t) + \frac{1}{2}g_{xx}(t,X_t) = 0 \tag{146}$$

where, recall that

$$g(t,X_t) = \mathbb{E}_{\mathbb{P}}\big[e^{-\int_t^T F_s dW_s - \frac{1}{2}\int_t^T \langle F_s,F_s\rangle ds}\big|\mathcal{F}_t\big] \tag{147}$$

implying that

$$g(t,X_t) = e^{\sigma X_t}e^{-\frac{1}{2}\sigma^2 t}, \tag{148}$$

because $g_t(t,X_t) = -\frac{1}{2}\sigma^2 e^{\sigma X_t}e^{-\frac{1}{2}\sigma^2 t}$ and $g_x(t,X_t) = \sigma e^{\sigma X_t}e^{-\frac{1}{2}\sigma^2 t}$ and $g_{xx}(t,X_t) = \sigma^2 e^{\sigma X_t}e^{-\frac{1}{2}\sigma^2 t}$. Therefore $\frac{1}{2}g_{xx}(t,X_t) + g_t(t,X_t) = 0$. Recall

$$dM_t = \zeta_t g_x(t,X_t)dW_t + g(t,X_t)d\zeta_t \tag{149}$$

and now

$$\mathbb{E}_{\mathbb{P}}[e^{mX_t}\Lambda] = \mathbb{E}_{\mathbb{P}}\left[e^{\int_0^t\left(mF_s - \frac{1}{2}\langle F_s,F_s\rangle\right)ds + \int_0^t(m-F_s)dW_s}\mathbb{E}_{\mathbb{P}}\big[e^{-\int_t^T F_s dW_s - \frac{1}{2}\int_t^T\langle F_s,F_s\rangle ds}\big|\mathcal{F}_t\big]\right]$$

$$= e^{-\frac{1}{2}m^2 t}\mathbb{E}_{\mathbb{P}}\left[e^{\int_0^t\left(mF_s - \frac{1}{2}\langle F_s,F_s\rangle\right)ds + \int_0^t(m-F_s)dW_s}e^{mX_t}\right]$$

$$= e^{-\frac{1}{2}m^2 t}\mathbb{E}_{\mathbb{P}}\left[e^{2mX_t}\underbrace{e^{-\int_0^t\frac{1}{2}\langle F_s,F_s\rangle ds - \int_0^t F_s dW_s}}_{\zeta_t}\right] = e^{-\frac{1}{2}m^2 t}\mathbb{E}_{\mathbb{P}}\left[e^{2mX_t}\Big(1 - \int_0^t \zeta_s F_s dW_s\Big)\right]$$

$$= e^{-\frac{1}{2}m^2 t}\mathbb{E}_{\mathbb{P}}\big[e^{2mX_t}\big] - e^{-\frac{1}{2}m^2 t}\mathbb{E}_{\mathbb{P}}\left[e^{2mX_t}\int_0^t \zeta_s F_s dW_s\Big)\right]$$

Let $f(x) = e^{2mx}$, then

$$e^{2mX_t} = 1 + \int_0^t 2me^{2mX_s}\left(F_s + 4m^2 e^{2mX_s}\right)dt + \int_0^t 2me^{2mX_s}dW_s \tag{150}$$

$\blacksquare$

15.3 Feynman-Kac

$$
\begin{aligned}
&dX_t = \mu_{t,X_t} dt + \sigma_{t,X_t} dW_t \\
&f(t,x) := \mathbb{E}_t^x\Bigg[\int_t^T \phi_s^{(t)} h(X_s,s)ds + \phi_T^t g(X_T)\Bigg] \\
&\phi_s^{(t)} := e^{-\int_t^s r(X_u,u)du}
\end{aligned}
$$

where E_t^x should be understood as conditioned on time t, $X_t = x$.

$$
f(t,x) = \mathbb{E}_t^x\Bigg[\int_t^T e^{-\int_t^s r(X_u,u)du} h(X_s,s)ds + e^{-\int_t^T r(X_u,u)du} g(X_T)\Bigg]
$$

Notice that:

$$
f(t,X_t) := \mathbb{E}\Bigg[\int_t^T e^{-\int_t^s r(X_u,u)du} h(X_s,s)ds + e^{-\int_t^T r(X_u,u)du} g(X_T)\Bigg|\mathcal{F}_t\Bigg]
$$

is not a martingale. Consider $M_t := \mathbb{E}\big[\int_t^T f(W_r)dr|\mathcal{F}_t\big]$, then for $s < t$,

$$
\begin{aligned}
\mathbb{E}\Big[M_t\Big|\mathcal{F}_s\Big] &= \mathbb{E}\Big[\mathbb{E}\big[\int_t^T f(W_r)dr|\mathcal{F}_t\big]\Big|\mathcal{F}_s\Big] = \mathbb{E}\Big[\int_t^T f(W_r)dr\Big|\mathcal{F}_s\Big] \\
&= M_s - \mathbb{E}\Big[\int_s^t f(W_r)dr\Big|\mathcal{F}_s\Big]
\end{aligned}
$$

This suggests making $f(t,X_t)$ a martingle by addition and multiplication of some terms.

$$
\begin{aligned}
&e^{-\int_0^t r(X_u,u)du}\Bigg(f(t,X_t) + \int_0^t e^{-\int_t^s r(X_u,u)du} h(X_s,s)ds\Bigg) \\
&= \mathbb{E}\Bigg[\int_t^T e^{-\int_0^t r(X_u,u)du} e^{-\int_t^s r(X_u,u)du} h(X_s,s)ds + \int_0^t e^{-\int_0^t r(X_u,u)du} e^{-\int_t^s r(X_u,u)du} h(X_s,s)ds + e^{-\int_0^t r(X_u,u)du} e^{-\int} \\
&= \mathbb{E}\Bigg[\int_t^T e^{-\int_0^s r(X_u,u)du} h(X_s,s)ds + \int_0^t e^{-\int_0^s r(X_u,u)du} h(X_s,s)ds + e^{-\int_0^T r(X_u,u)du} g(X_T)\Bigg|\mathcal{F}_t\Bigg] \\
&= \mathbb{E}\Bigg[\int_0^T e^{-\int_0^s r(X_u,u)du} h(X_s,s)ds + e^{-\int_0^T r(X_u,u)du} g(X_T)\Bigg|\mathcal{F}_t\Bigg]
\end{aligned}
$$

$$
\mathbb{E}\Bigg[\int_0^T e^{-\int_0^s r(X_u,u)du} h(X_s,s)ds + e^{-\int_0^T r(X_u,u)du} g(X_T)\Bigg|\mathcal{F}_t\Bigg] \tag{151}
$$

Now if we let:

$$
M_t := \underbrace{e^{-\int_0^t r(X_u,u)du}}_{\phi_0^{(t)}}\Bigg(f(t,X_t) + \int_0^t e^{-\int_t^s r(X_u,u)du} h(X_s,s)ds\Bigg) \tag{152}
$$

$$
dX_t = \mu_{t,X_t} dt + \sigma_{t,X_t} dW_t
$$

then by Martingale representation theorem after applying Ito's formula on M_t terms corresponding to dt must be zero

$$
f_t(t,X_t) + f_x(t,X_t)\mu_{t,X_t} + \frac{1}{2} f_{xx}(t,X_t)\sigma_{t,X_t}^2 - r(x,t)f(x,t) + h(x,t) = 0 \tag{153}
$$

16. Pricing under the risk neutral measure

$$dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t$$
$$S_t = S_0 e^{\int_0^t \sigma_s dW_s + \int_0^t \left(\alpha_s - \frac{1}{2}\sigma_s^2\right) ds}$$

Now we also have *an adapted interest rate process* R_t. We define the *discount process*

$$D_t = e^{-\int_0^t R_s ds}$$

Let $f(x) = e^{-x}, f'(x) = -f(x), f''(x) = f(x)$, for $dX_t = R_t dt$. Therefore:

$$df(X_t) = -f(X_t)R_t dt, \implies dD_t = -D_t R_t dt$$

and discounted stock price is

$$d(D_t S_t) = D_t dS_t = (\alpha_t - R_t)D_t S_t dt + \sigma_t D_t S_t dW_t = D_t S_t \sigma_t \big(\Theta_t dt + dW_t\big)$$
$$D_t S_t = e^{\int_0^t \sigma_s dW_s + \int_0^t \left(\alpha_s - R_s - \frac{1}{2}\sigma_s^2\right)} D_0 S_0,$$

where $\Theta_t := \frac{\alpha_t - R_t}{\sigma_t}$, is *market price of risk.* Now define a new measure based on the market price of risk $\mathbb{Q}$ via Radon-Nikodym derivative

$$Z_t = e^{-\frac{1}{2}\int_0^t \Theta_s^2 ds - \int_0^t \Theta_s dW_s} \tag{154}$$

and recall that $B_t := \int_0^t \Theta_s dt + W_t$ is brownian motion under $\mathbb{Q}$. Hence:

$$d(D_t S_t) = D_t S_t \sigma_t dB_t$$
$$D_t S_t = S_0 + \int_0^t \sigma_s D_s S_s dB_s$$

Hence under $\mathbb{Q}$ discounted stock price is a martingale with mean rate of return S_0. On the other hand actual stock price under $\mathbb{Q}$ is essentially

$$dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t = \alpha_t S_t dt + \sigma_t S_t (dB_t - \Theta_t dt)$$
$$= \big(\alpha_t - \sigma_t \Theta_t\big) S_t dt + \sigma_t S_t dB_t = R_t S_t dt + \sigma_t S_t dB_t$$

Hence mean rate of return of the stock under $\mathbb{Q}$, is interest rate hence the name *risk neutral measure or equivalent martingale measure(EMM).* For discounted portfolio recall that:

$$dX_t = R_t X_t dt + \Delta_t \sigma_t S_t dB_t$$
$$d(D_t X_t) = \Delta_t \sigma_t D_t S_t dB_t,$$
$$D_t X_t = \mathbb{E}_{\mathbb{Q}}[D(T)X_T|\mathcal{F}_t]$$

So $D_t X_t$ is a martingale in risk neutral measure We wish to know what initial capital X_0 and hedging strategy Δ_t to use as to make sure $X(T) = V(T)$ a.s then

$$D_t X_t = \mathbb{E}_{\mathbb{Q}}[D_T X_T|\mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}[D_T V_T|\mathcal{F}_t] \tag{155}$$

therefore X_t is essentially the value of the derivative security at time t before pay-off and consequently,

$$D_t V_t = \mathbb{E}_{\mathbb{Q}}[D_T V_T|\mathcal{F}_t] \implies V_t = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} V_T|\mathcal{F}_t\right] \tag{156}$$

and *correct initial capital*, $V_0 = \mathbb{E}_{\mathbb{Q}}[D_T V_T]$

17. Digressions

So how do I price a european call option. I know that if strike price is K and time of maturity is T then worth of call option at time $t < T$ is $e^{-r(T-t)}(S_T - K)^+$. Now if I start with initial portfolio of X_0 and can hedge i.e.

$$\begin{aligned}
&X_{k+1} = \Delta_k S_{k+1} + (1+r)\{X_k - \Delta_k S_k\} \\
&X_{k+1} - X_k = \Delta_k(S_{k+1} - S_k) + r(X_k - S_k) \\
&dX_t = \Delta_t dS_t + r(X_t - S_t)dt \\
&d(e^{-rt}X_t) = e^{-rt}\Delta_t(\alpha - r)S_t dt + e^{-rt}\Delta_t \sigma S_t dW_t
\end{aligned}$$

On the other hand

$$dS_t = \alpha S_t dt + \sigma S_t dW_t$$

$$dc(t, S_t) = \left\{ c_t(t, S_t) + c_x(t, S_t) + \frac{1}{2}c_{xx}(t, S_t)\sigma^2 S_t^2 \right\} dt + c_x(t, S_t)\sigma S_t dW_t$$

$$d\big(e^{-rt}c(t, S_t)\big) = e^{-rt}\left\{ -rc(t, S_t) + c_t(t, S_t) + c_x(t, S_t)\alpha S_t + \frac{1}{2}c_{xx}(t, S_t)\sigma^2 S_t^2 \right\} dt + e^{-rt}c_x(t, S_t)\sigma S_t dW_t$$

Hence $\Delta_t = c_x(t, S_t)$ and

$$\begin{aligned}
&rc_x(t, S_t)S_t - rc(t, S_t) + c_t(t, S_t) + \frac{1}{2}c_{xx}(t, S_t)\sigma^2 S_t^2 = 0 \\
&c_t(t, x) - rc(t, x) + rxc_x(t, x) + \frac{1}{2}\sigma_1^2 x^2 c_{xx}(t, x) = 0
\end{aligned}$$

implying that if I start with $X_0 = C(0, S_0)$ then time value of my portfolio at all the times $t \in (0, T]$, $e^{-rt}X_t$ should be equal to price of the call option.

17.1 Martingale and Brownian motion

Theorem 33 (Levy's theorem) *Let M_t be a martingale w.r.t filteration $\mathcal{F}_t$, with* continuous path *and* Quadratic variation *t, then M_t is a Brownian motion*

$M_0 = 0$ almost surely(as M_0 has quadratic variation 0)

$$M_t = M_0 + \sum_{i=0}^{n-1}\left(M_{t_{i+1}^n} - M_{t_i^n}\right)$$

$$f(M_t) = f(M_0) + \sum_{i=0}^{n-1} f'(M_{t_i^n})\big(M_{t_{i+1}^n} - M_{t_i^n}\big) + \frac{1}{2}f''(\theta_{t_i^n})\big(M_{t_{i+1}^n} - M_{t_i^n}\big)^2, \ \ \theta_{t_i^n} \in [M_{t_i^n}, M_{t_{i+1}^n}]$$

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} f'(M_{t_i^n})\big(M_{t_{i+1}^n} - M_{t_i^n}\big) = \int_0^t f'(M_s)dM_s$$

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} \frac{1}{2}f''(\theta_{t_i^n})\big(M_{t_{i+1}^n} - M_{t_i^n}\big)^2 = \frac{1}{2}\int_0^t f''(M_s)ds \ \ \textit{This is where 2-var and cont of sample path are used}$$

$$f(t, M_t) = f(0, M_0) + \int_0^t \left\{ \frac{1}{2}f''(s, M_s) + f_s(s, M_s) \right\} ds + \int_0^t f'(s, M_s)dM_s$$

Proposition 34 *$\int_0^t f^{'}(s, M_s)dM_s$ is a Martingale*

Proof

Notice that for $s < t$ and martingale property, $\mathbb{E}[f(M_s)(M_t - M_s)|\mathcal{F}_s] = f(M_s)\mathbb{E}[M_t - M_s|\mathcal{F}_s] = f(M_s)(M_s - M_s) = 0$ almost surely. Now for $m < n$

$$
\begin{aligned}
&Z_m := \sum_{i=0}^{m-1} f^{'}(M_{t_i^n})\big(M_{t_{i+1}^n} - M_{t_i^n}\big) \\
&\mathbb{E}\Big[\sum_{i=0}^{n-1} f^{'}(M_{t_i^n})\big(M_{t_{i+1}^n} - M_{t_i^n}\big)\big|\mathcal{F}_{t_m^n}\Big] = \sum_{i=0}^{m-1} f^{'}(M_{t_i^n})\big(M_{t_{i+1}^n} - M_{t_i^n}\big) + \mathbb{E}\Big[f^{'}(M_{t_m^n})\big(M_{t_{m+1}^n} - M_{t_m^n}\big)\big|\mathcal{F}_{t_m^n}\Big] \\
&= \sum_{i=0}^{m-1} f^{'}(M_{t_i^n})\big(M_{t_{i+1}^n} - M_{t_i^n}\big) + f^{'}(M_{t_m^n})\mathbb{E}\Big[\big(M_{t_{m+1}^n} - M_{t_m^n}\big)\big|\mathcal{F}_{t_m^n}\Big] = \sum_{i=0}^{m-1} f^{'}(M_{t_i^n})\big(M_{t_{i+1}^n} - M_{t_i^n}\big) + f^{'}(M_{t_m^n})\big(M_t
\end{aligned}
$$

■

Now for any $u \in \mathbb{R}$, let $f(t, x) = e^{ux - \frac{1}{2}u^2 t}$, then $f_t(t, x) = -\frac{1}{2}u^2 f(t, x), f^{'}(t, x) = uf(t, x)$ and $f^{''}(t, x) = u^2 f(t, x)$. Consequently

$$
\begin{aligned}
&f(t, M_t) = 1 + \int_0^t \Big\{ \underbrace{\frac{1}{2}f^{''}(s, M_s) + f_s(s, M_s)}_{=0} \Big\}ds + \int_0^t f^{'}(s, M_s)dM_s \\
&\mathbb{E}[f(t, M_t)] = 1 + \underbrace{\mathbb{E}\big[\int_0^t f^{'}(s, M_s)dM_s\big]}_{=0, Martingale}, \ \textit{Hence} \\
&\mathbb{E}\big[e^{uM_t - \frac{1}{2}u^2 t}\big] = 1 \implies \mathbb{E}\big[e^{uM_t}\big] = e^{\frac{1}{2}u^2 t}
\end{aligned}
$$

Hence M_t is a Brownian motion

Theorem 35 (Martingale rep theorem) *Let W_t be a Brownian motion $\mathcal{F}_t^{\mathcal{W}} := \sigma\{W_s : s \leq t\}$ be the filteration it generates and M_t is a $\mathcal{F}_t^{\mathcal{W}}$ martingale, then there is a unique $\mathcal{F}_t^{\mathcal{W}}$ adapted process Γ_t such that:*

$$
M_t = M_0 + \int_0^t \Gamma_s dW_s \tag{157}
$$

On the other hand for *Girsanov* remember that W_t is $\mathcal{F}_t$ Wiener process.Recall that:

$$
\Lambda_t := e^{-\int_0^t F_s dW_s - \frac{1}{2}\int_0^t F_s^2 ds} \tag{158}
$$

is a martingale, which we now show Notice that if $dX_t := F_t dW_t + \frac{1}{2}F_t^2 dt$, if we let $f(X_t) = e^{-X_t}$, then $f_x(X_t) = -f(X_t)$ and $f_{xx}(X_t) = f(X_t)$. Hence via Ito's formula :

$$
\begin{aligned}
&f(X_t) = f(X_0) + \int_0^t \Big\{ -\frac{1}{2}F_s^2 f(X_s) + \frac{1}{2}F_s^2 f(X_s)\Big\}dt - \int_0^t F_s f(X_s)dW_s \\
&\Lambda_t = \Lambda_0 - \int_0^t F_s \Lambda_s dW_s
\end{aligned}
$$

18. Fundamental theorems of assett pricing

18.1 Hedging Strategy

Assumption: $\mathcal{F}_t^{\mathcal{W}} = \sigma\{W_s : s \leq t\}$ is generated by Brownian motion W_t(So from now on assume that volatility σ_t and drift of the underlying financial assett α_t is $\mathcal{F}_t^{\mathcal{W}}$ adapted). Given $V(T) \in \mathcal{F}_T$ (pay of a security at time of maturity T) we will show that for some initial capital X_0 and hedging strategy Δ_t, we can have $X(T) = V(T)$ almost surely

Proof Since $V(T)$ is $\mathcal{F}_T$, this means that for $t < T$

$$\begin{aligned}
&G_t := \mathbb{E}_{\mathbb{Q}}\big[D_T V_T | \mathcal{F}_t\big],\ \ \textit{is a Martingale, because for } s < t \\
&\mathbb{E}_{\mathbb{Q}}[G_t|\mathcal{F}_s] = G_s\ \ \textit{almost surely and by MRT exists}\, \Gamma_s : \\
&G_t = G_0 + \int_0^t \Gamma_s dB_s
\end{aligned}$$

but also recall that

$$D_t X_t = X_0 + \int_0^t \Delta_s \sigma_s D_s S_s dB_s \tag{159}$$

This means that with initial capital for portfolio

$$X_0 = G_0 := \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T R_s ds} V(T) \big| \mathcal{F}_0\right] \tag{160}$$

and hedging strategy

$$\Delta_t = \frac{\Gamma_t}{\sigma_s D_s S_s} \tag{161}$$

implies $X(T) = V(T)$ almost surely. ■

Reiterating as $X(T) = V(T)$ almost surely

$$\begin{aligned}
&D_t X_t = \mathbb{E}_{\mathbb{Q}}[D_T X_T|\mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}[D_T V_T|\mathcal{F}_t],\ \ \textit{almost surely} \\
&\implies X_t \overset{a.s}{=} \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T R_s ds} V_T|\mathcal{F}_t\right].
\end{aligned}$$

That is portfolio thats replicate pay off of the security at time T equalling $V(T)$, for $t < T$ must be valued at $X_t \overset{a.s}{=} \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T R_s ds} V_T|\mathcal{F}_t\right]$ Therefore, fair price of the derivative security should satisfy:

$$V_t = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T R_s ds} V_T|\mathcal{F}_t\right] \tag{162}$$

Martingale price of a call at time t, V_t

$$V_t = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} V_T\big|\mathcal{F}_t\right] \tag{163}$$

Now let us try to verify it for the Call option(Thanks to Markovian nature of GBM)

$$
\begin{aligned}
&S_T = S_t e^{\sigma\left(B_T - B_t\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)} \\
&C_{t,S_t} = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\big(S_T - K\big)^+ \big| \mathcal{F}_t\right] = \frac{1}{\sqrt{2\pi}}\int e^{-r(T-t)}\big(S_t e^{\sigma\sqrt{T-t}y} e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)} - K\big)^+ e^{-\frac{y^2}{2}}dy \\
&= \frac{1}{\sqrt{2\pi}}\int \big(S_t e^{\sigma\sqrt{T-t}y} e^{-\frac{1}{2}\sigma^2(T-t)} - Ke^{-r(T-t)}\big)^+ e^{-\frac{y^2}{2}}dy
\end{aligned}
$$

It will require

$$
\begin{aligned}
&e^{\sigma\sqrt{T-t}y} \geq \frac{K}{S_t} e^{\left(\frac{1}{2}\sigma^2 - r\right)(T-t)} \\
&\sigma\sqrt{T-t}y \geq \ln\left(\frac{K}{S_t}\right) + (T-t)\left(\frac{1}{2}\sigma^2 - r\right) \\
&y \geq \frac{\ln\left(\frac{K}{S_t}\right)}{\sigma\sqrt{T-t}} + \frac{\sqrt{T-t}}{\sigma}\left(\frac{1}{2}\sigma^2 - r\right) =: N(d)
\end{aligned}
$$

and arithmetics will reveal that

$$
C_{t,S_t} = S_t N\big(d_+(T-t, S_t)\big) - Ke^{-r(T-t)} N\big(d_-(T-t, S_t)\big) \tag{164}
$$

Theorem 36 (Girsanov Multi dim) *Let W_t be $\mathcal{F}_t$ adapted on the probability space $\Big(\Omega, \mathcal{F}_{t\in[0,T]}, \mathcal{F}, \mathbb{P}\Big)$ $X_t, W_t \in \mathbb{R}^d$ and $F_s \in \mathbb{R}^{d\times 1}$*

$$
\begin{aligned}
&X_t - X_s = \int_s^r F_r dr + (W_t - W_s),\ \ t \in [0,T] \\
&\Lambda := e^{-\int_0^T \langle F_s, dW_s\rangle - \frac{1}{2}\int_0^T \langle F_s, F_s\rangle ds} \\
&\textit{Novikov's condition valid, i.e., } \mathbb{E}_{\mathbb{P}}\left[e^{\frac{1}{2}\int_0^T \langle F_s, F_s\rangle ds}\right] < \infty
\end{aligned}
$$

Then X_t is an $\mathcal{F}_t-$ Wiener process under $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}\big[\Lambda 1_A\big]$

Theorem 37 (Multi-dim Martingale rep theorem) *Let W_t be a $d-$ dimensional Brownian motion and let $\mathcal{F}_t$ be the filteration it generates. If M_t is a martingale w.r.t $\mathcal{F}_t$ then there is an adapter process $\Gamma(s) \in \mathbb{R}^d$ such that:*

$$
M_t = M_0 + \int_0^t \langle \Gamma(s), dW_s\rangle \tag{165}
$$

Definition 38 (Multi dimensional market models) *Let $S_t \in \mathbb{R}^m$ driven by $d-$ dimensioanl Brownian motion. For each $i \in [m]$*

$$dS_i(t) = \alpha_i(t)S_i(t)dt + S_i(t)\sum_{j=1}^{d}\sigma_{i,j}(t)dW_j(t),\ \ \sigma_i(t) := \sum_{j=1}^{d}\sqrt{\sigma_{i,j}^2(t)}$$

$$B_i(t) := \sum_{j=1}^{d}\int_0^t \frac{\sigma_{i,j}(s)}{\sigma_i(s)}dW_j(s).\ dB_i(t)dB_i(t) = dt \textit{(Hence brownian motion)}$$

$$dS_i(t) = \alpha_i(t)S_i(t)dt + \sigma_i(t)S_i(t)dB_i(t),\ \ dB_i(t)dB_k(t) = \underbrace{\sum_{j=1}^{d}\frac{\sigma_{ij}(t)\sigma_{kj}(t)}{\sigma_i(t)\sigma_k(t)}}_{=:\rho_{ik}(t)}dt$$

$$Cov[B_i(t), B_k(t)] = \mathbb{E}\int_0^t \rho_{ik}(s)ds$$

$$dS_i(t)dS_k(t) = \rho_{ik}(t)\sigma_i(t)\sigma_k(t)S_i(t)S_k(t)dt$$

$$\frac{dS_i(t)}{S_i(t)}\frac{dS_k(t)}{S_k(t)} = \rho_{ik}(t)\sigma_k(t)\sigma_i(t)dt\ \ \textit{(Relative differentials)}$$

$$d\big(D_tS_i(t)\big) = D_tS_i(t)\Big\{(\alpha_i(t) - R_t)dt + \sigma_i(t)dB_i(t)\Big\},\ \ i \in [m]$$

Recall the risk neutral part in one dimension

$$d(D_tS_t) = D_tdS_t = (\alpha_t - R_t)D_tS_tdt + \sigma_tD_tS_tdW_t = D_tS_t\sigma_t\big(\Theta_tdt + dW_t\big) \tag{166}$$

Now recall that:

$$dS_i(t) = \alpha_i(t)S_i(t)dt + S_i(t)\sum_{j=1}^{d}\sigma_{i,j}(t)dW_j(t)\ \ dD_t = -R_tD_tdt$$

$$d\big(D_tS_i(t)\big) = D_t\bigg(\alpha_i(t)S_i(t)dt + S_i(t)\sum_{j=1}^{d}\sigma_{i,j}(t)dW_j(t)\bigg) - S_i(t)R_tD_tdt$$

$$d\big(D_tS_i(t)\big) = D_tS_i(t)\Big\{(\alpha_i(t) - R_t)dt + \sum_{j=1}^{d}\sigma_{i,j}(t)dW_j(t)\Big\}$$

If we can find a market price of risk Θ_j such that:

$$d\big(D_tS_i(t)\big) = D_tS_i(t)\Big\{(\alpha_i(t) - R_t)dt + \sum_{j=1}^{d}\sigma_{i,j}(t)dW_j(t)\Big\} = D_tS_i(t)\sum_{j=1}^{d}\sigma_{i,j}(t)\big\{\Theta_j(t)dt + dW_j(t)\big\}, \tag{167}$$

Essentialy:

$$\alpha_t^R = \begin{bmatrix}\alpha_1(t) - R_t\\ \alpha_2(t) - R_t\\ \vdots\\ \alpha_m(t) - R_t\end{bmatrix}\ \ \sigma_t = \begin{bmatrix}\sigma_t^{1,1} & \sigma_t^{1,2} & \dots & \sigma_t^{1,d}\\ \sigma_t^{2,1} & \sigma_t^{2,2} & \dots & \sigma_t^{2,m}\\ \vdots & \vdots & \vdots & \vdots\\ \sigma_t^{m,1} & \sigma_t^{m,2} & \dots & \sigma_t^{m,d}\end{bmatrix},\ \ \Theta_t := \begin{bmatrix}\Theta_1(t)\\ \Theta_2(t)\\ \vdots\\ \Theta_d(t)\end{bmatrix}$$

If exists Θ_t such that $\sigma_t \Theta_t = \alpha_t^R$ then we good, else the other hand if one can not solve market price of risk then their is arbitrage lurking

Example 2

$$
\begin{aligned}
& dS_1(t) = \alpha_1 S_1(t)dt + S_1\sigma_1 dW_t \\
& dS_2(t) = \alpha_2 S_2(t)dt + S_2\sigma_2 dW_t \\
& \frac{\alpha_1 - r}{\sigma_1} > \frac{\alpha_2 - r}{\sigma_2}, \quad \mu := \frac{\alpha_1 - r}{\sigma_1} - \frac{\alpha_2 - r}{\sigma_2} > 0 \\
& dX_t = \Delta_1(t)dS_1(t) + \Delta_2 dS_2(t) + r\big(X_t - \Delta_1(t)S_1(t) - \Delta_2(t)S_2(t)\big)dt \\
& \Delta_1(t) = \frac{1}{S_1(t)\sigma_1}, \quad \Delta_2(t) = -\frac{1}{S_2(t)\sigma_2} \\
& dX_t = \left(\frac{1}{S_1(t)\sigma_1}\right)dS_1(t) - \left(\frac{1}{S_2(t)\sigma_2}\right)dS_2(t) + r\left(X_t - \frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right) \implies \\
& dX_t = \mu dt + rX_t dt \\
& d(D_t X_t) = \mu D_t dt
\end{aligned}
$$

Therefore portfolio will make money for sure and do faster than interest rate

Now we have an important take away

Theorem 39 *No solution for market price of risk, then deterministic arbitrage exists*

Solution for market price of risk exists then

$$
dX_t = R_t X_t d_t + \sum_{i=1}^m \Delta_i(t) \frac{1}{D_t} D_t\big(dS_i(t) - R_t S_i(t)dt\big) = \sum_{i=1}^m \Delta_i(t) \frac{d\big(D_t S_i(t)\big)}{D_t}
$$

$$
\sum_{i=1}^m \Delta_i(t) \frac{D_t S_i(t) \sum_{j=1}^d \sigma_{i,j}(t)\big\{\Theta_j(t)dt + dW_j(t)\big\}}{D_t} = \sum_{i=1}^m \Delta_i(t) S_i(t) \sum_{j=1}^d \sigma_{i,j}(t)\big\{\Theta_j(t)dt + dW_j(t)\big\}
$$

Then under new measure $\mathbb{Q}$ with radon-nikodym derivative

$$
\Lambda_T := e^{-\frac{1}{2}\int_0^T \|\Theta_s\|^2 ds - \int_0^T \langle \Theta_s, dW_s \rangle}, \tag{168}
$$

$B_t := \int_0^t \Theta_s ds + W_t$ will be $\mathbb{R}^d$ dimensional Brownian motion and under $\mathbb{Q}$, $D_t S_i(t)$ is a Martingale as

$$
d\big(D_t S_i(t)\big) = D_t S_i(t) \sum_{j=1}^d \sigma_{i,j}(t) dB_j(t) \tag{169}
$$

Definition 40 $\mathbb{Q}$ *is* risk neutral*:*

1. $\mathbb{Q} << \mathbb{P}$ *and* $\mathbb{P} << \mathbb{Q}$

2. Under $\mathbb{Q}$*,* $D_t S_i(t)$ *is a martingale for every* $i \in [m]$

Definition 41 *Portfolio process $X_1(t)$ withn initial capital $X_1(0) = 0$*

$$\mathbb{P}\big\{X_1(T) \geq 0\big\} = 1, \quad \mathbb{P}\big\{X_1(T) > 0\big\} > 0 \tag{170}$$

Proof *Y_t is money market portfolio with initial capital $X_2(0)$. Define $X_2(t)$ as a linear combination of portfolio Y_t and X_t. Portfolio process $X_2(t) := \underbrace{X_2(0)e^{\int_0^t r_s ds}}_{Y_t} + X_1(t)$. Then*

$$\mathbb{P}\bigg\{X_2(T) \geq \frac{X_2(0)}{D_T}\bigg\} = \mathbb{P}\bigg\{Y_T + X_1(T) \geq \frac{X_2(0)}{D_T}\bigg\} = \mathbb{P}\bigg\{X_1(T) \geq \underbrace{\frac{X_2(0)}{D_T} - Y_T}_{=0}\bigg\} \tag{171}$$

and the result follows similarly for the other part. ∎

Theorem 42 (First fundamental theorem of assett pricing) *If a risk neutral measure exists then no arbitrage*

Proof *Risk neutral measure implies $D_t X_t$ is a martingale under $\mathbb{Q}$ so if $X_0 = 0$, then $\mathbb{E}_{\mathbb{Q}}[D_T X_T] = X_0 = 0$ for all times T. Now assume that arbitrage exist even though we believe a risk neutral measure exist $\mathbb{P}\big(X_1(T) < 0\big) = 0 \implies \mathbb{P}_{\mathbb{Q}}\big(X_1(T) < 0\big) = 0$ as $\mathbb{Q} << \mathbb{P}$. So $\mathbb{P}_{\mathbb{Q}}\big(X_1(T) \geq 0\big) = 1$. In fact risk neutral by definition $\mathbb{Q} << \mathbb{P}$ and $\mathbb{P} << \mathbb{Q}$. So $\mathbb{P}_{\mathbb{Q}}\big(X_1(T) > 0\big) > 0 \implies \mathbb{P}_{\mathbb{Q}}\big(D_T X_1(T) > 0\big) > 0$ which by Markov's inequality mean that*

$$0 < \mathbb{P}_{\mathbb{Q}}\big(D_T X_1(T) > 0\big) \leq \mathbb{E}_{\mathbb{Q}}[D_T X_t] = 0 \tag{172}$$

Hence a contradiction. ∎

Back to fair price of a derivative security: Martingale approach *Let $V(T)$ be $\mathcal{F}_T$ adapted process (pay off derivative security at time of expiry T).*

Proposition 43 *If underlying assets have risk neutral measure $\mathbb{Q}$, then there exists a portfolio +hedge(initial capital X_0 and Δ_t) such that $X_T = V(T)$ almost surely.*

$dD_t X_t = \Delta_s \sigma_s D_s S_s dB_s$ is a martingale, combined with prop

$$\begin{aligned} D_t X_t &= \mathbb{E}_{\mathbb{Q}}[D_T X_T | \mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}[D_T V_T | \mathcal{F}_t] \\ X_t &= e^{\int_0^t r_s ds} \mathbb{E}_{\mathbb{Q}}[e^{-\int_0^T r_s ds} V_T | \mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^T r_s ds} V_T | \mathcal{F}_t] \end{aligned}$$

essentially saying at time t- with capital X_t I can hedge between times $[t, T)$ to produce $X_T = V_T$, the pay-off for derivative security. Hence it makes sense that fair of derivative security *at time t,*

$$V_t := \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^T r_s ds} V_T | \mathcal{F}_t] \tag{173}$$

Derivative security of a multi-dimensional correlated asset

Definition 44 (Complete Market Model) *if every derivative security can be hedged*

Theorem 45 *Similar to one dimension,in this $d-$dimensional space D_tV_t is a martingale and by MRT*

$$D_tV_t = V_0 + \int_0^t \langle \Gamma(s), dB_s \rangle \tag{174}$$

For, $m-$ dimensional stocks and $d-$ dimensional Brownian motion. Hedgings $\Delta_1(t), \Delta_2(t), \ldots, \Delta_m(t)$ satisfy for $j \in [d]$

$$\frac{\Gamma_j(t)}{D_t} = \sum_{i=1}^{m} \Delta_i(t) S_i(t) \sigma_{ij}(t) \tag{175}$$

These are $d-$ equations in $m-$ unknown processes $\Delta_1(t), \Delta_2(t), \ldots, \Delta_m(t)$ (Irony: we do not even know $\Gamma(t)$)

Theorem 46 (Second fundamental theorem of asset pricing) *Consider a market model with risk free prob measure, it is complete iff the risk neutral probability measure is unique.*

Recall that:

$$\begin{aligned}
Z_T &:= e^{-\frac{1}{2}\int_0^T \Theta_s^2 ds - \int_0^T \Theta_s dW_s}, \\
dZ_t &= -Z_t \Theta_t dW_t, \quad \Theta_t = \frac{\alpha_t - r_t}{\sigma_t}
\end{aligned}$$

Hence Z_t is $\mathcal{F}_t$ martingale. Recall that Girsanov implies:

$$B_t := \int_0^t \Theta_s ds + W_t \tag{176}$$

is $\mathcal{F}_t-$ Wiener process under measure $\mathbb{Q}$, which has Radon-Nikodym derivative defined

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T \tag{177}$$

Now recall via ito's product rule and $D_t := e^{-\int_0^t r_s ds}$

$$\begin{aligned}
d(D_tX_t) = dD_t dX_t + dD_t X_t + D_t dX_t &= \underbrace{-D_t R_t dt\left(\left\{\Delta_t(\alpha - r)S_t + rX_t\right\}dt + \Delta_t \sigma S_t dW_t\right)}_{=0} \\
&- X_t D_t R_t dt + D_t\left(\left\{\Delta_t(\alpha - r)S_t + r_t X_t\right\}dt + \Delta_t \sigma S_t dW_t\right) \\
&= D_t\left\{\Delta_t(\alpha_t - r_t)S_t + r_t X_t - r_t X_t\right\}dt + D_t \Delta_t \sigma_t S_t dW_t = D_t \Delta_t \sigma_t \Theta_t S_t dt + D_t \Delta_t S_t \sigma_t dW_t \\
&\implies d\big(D_t X_t\big) = D_t \Delta_t \sigma_t S_t \big(\Theta_t dt + dW_t\big) \text{ under change of measure}
\end{aligned}$$

$$d\big(D_tX_t\big) \underbrace{=}_{\mathbb{Q}} D_t\Delta_t\sigma_tS_tdB_t$$

which implies that D_tX_t is an $\mathcal{F}_t$ martingale under $\mathbb{Q}$ So essentially if risk neutral measure $\mathbb{Q}$ exists then D_tX_t is a martingale. Hence:

$$D_tX_t \quad \underbrace{=}_{\textit{almost surely}} \quad \mathbb{E}_{\mathbb{Q}}[D_TX_T|\mathcal{F}_t]$$

Exercise 5.2 Shreve: Show

$$D_tX_t = \mathbb{E}_{\mathbb{Q}}\big[D_TX_T|\mathcal{F}_t\big] \tag{178}$$

may be re-written as :

$$D_tX_tZ_t = \mathbb{E}_{\mathbb{P}}\big[D_TX_tZ_T|\mathcal{F}_t\big] \tag{179}$$

Now recall that by definition:

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\big[D_TX_T\big] := \mathbb{E}_{\mathbb{P}}\big[D_TX_TZ_T\big], \ \ \textit{where}\\ &Z_T := e^{-\frac{1}{2}\int_0^T \Theta_s^2 ds - \int_0^T \Theta_s dW_s}\end{aligned}$$

and from Proposition 32 regarding conditional expectations and radon-nikodym derivatives we know that:

$$\mathbb{E}_{\mathbb{Q}}\big[D_TX_T|\mathcal{F}_t\big] = \frac{\mathbb{E}_{\mathbb{P}}\big[D_TX_TZ_T|\mathcal{F}_t\big]}{Z_t} \tag{180}$$

from hypothesis we know

$$D_tX_t = \mathbb{E}_{\mathbb{Q}}\big[D_TX_T|\mathcal{F}_t\big]. \tag{181}$$

Hence

$$\begin{aligned}&\frac{\mathbb{E}_{\mathbb{P}}\big[D_TX_TZ_T|\mathcal{F}_t\big]}{Z_t} \quad \underbrace{=}_{\textit{almost surely}} \quad D_tX_t\\ &\implies \mathbb{E}_{\mathbb{P}}\big[D_TX_TZ_T|\mathcal{F}_t\big] = D_tX_tZ_t\end{aligned}$$

18.2 Security pricing via conditional expectations

Theorem 47 *Assuming almost surely we can hedge portfolio(X.) such that at time of maturity T it equals the pay of security (V.), i.e., almost surely $X_T = V_T$. Furthermore, underlying assett has a risk neutral measure $\mathbb{Q}$ then price of security at time $t < T$ should be*

$$V_t = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}V_T\big|\mathcal{F}_t\right] \tag{182}$$

or simply written in original measure $\mathbb{P}$ as:

$$V_t = \frac{\mathbb{E}_{\mathbb{P}}\left[e^{-\int_t^T r_s ds}V_TZ_T\big|\mathcal{F}_t\right]}{Z_t} \tag{183}$$

For risk neutral measure $\mathbb{Q}$:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$
$$dB_t = \left(\frac{\mu - r}{\sigma}\right) dt + dW_t$$
$$dS_t = \mu S_t dt + \sigma S_t \big(dB_t - (\frac{\mu - r}{\sigma}) dt\big)$$
$$dS_t = \mu S_t dt + \sigma S_t dB_t - (\mu - r) S_t dt \implies$$
$$dS_t \underbrace{=}_{\mathbb{Q}} r S_t dt + \sigma S_t dB_t$$
$$S_T \underbrace{=}_{\mathbb{Q}} S_t e^{\left(r - \frac{\sigma^2}{2}\right)(T-t)} e^{\sigma\left(B_T - B_t\right)} \quad \textit{implies}$$
$$S_T \underbrace{=}_{\mathbb{Q}} S_t e^{\left(r - \frac{\sigma^2}{2}\right)(T-t)} e^{-\sigma\sqrt{T-t}\left[\frac{-(B_T - B_t)}{\sqrt{T-t}}\right]}, \quad z := \left[\frac{-(B_T - B_t)}{\sqrt{T-t}}\right] \sim N(0,1)$$

which now implies that

$$E_{\mathbb{Q}}\left[\max(S_T - K, 0)\middle|\mathcal{F}_t\right] = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int \max(S_t e^{\left(r - \frac{\sigma^2}{2}\right)(T-t)} e^{-\sigma\sqrt{T-t}z} - K, 0) e^{-\frac{z^2}{2}} dz$$

$$= S_t \frac{e^{-\frac{\sigma^2}{2}(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\log\left(\frac{S_t}{K}\right) + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}} e^{-\sigma\sqrt{T-t}z} e^{-\frac{z^2}{2}} dz - K e^{-r(T-t)} N\Big(d_-(T-t, S_t)\Big)$$

$$V_t = \frac{e^{-r(T-t)}}{}$$

$$E_{\mathbb{Q}}\left[\max(S_T - K, 0)\middle|\mathcal{F}_t\right] = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int \max(S_t e^{\left(r - \frac{\sigma^2}{2}\right)(T-t)} e^{-\sigma\sqrt{T-t}z} - K, 0) e^{-\frac{z^2}{2}} dz$$

$$= S_t \frac{e^{-\frac{\sigma^2}{2}(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\log\left(\frac{S_t}{K}\right) + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}} e^{-\sigma\sqrt{T-t}z} e^{-\frac{z^2}{2}} dz - K e^{-r(T-t)} N\Big(d_-(T-t, S_t)\Big)$$

Now only remaining term to unravel is:

$$S_t \frac{e^{-\frac{\sigma^2}{2}(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\log\left(\frac{S_t}{K}\right) + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}} e^{-\sigma\sqrt{T-t}z} e^{-\frac{z^2}{2}} dz$$

$$= \frac{S_t}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\log\left(\frac{S_t}{K}\right) + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{1}{2}\left(z + \sigma\sqrt{T-t}\right)^2} dz = \frac{S_t}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\log\left(\frac{S_t}{K}\right) + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} + \sigma\sqrt{T-t}} e^{-\frac{\hat{z}^2}{2}} d\hat{z}$$

$$S_t N\Big(d_+(T-t, S_t)\Big)$$

where we used change of variable, $\hat{z} := z + \sigma\sqrt{T-t}$ and simple algebra that:

$$\frac{\log\left(\frac{S_t}{K}\right) + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} + \sigma\sqrt{T-t} = \frac{\log\left(\frac{S_t}{K}\right) + \left(\mu + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} =: d_+(T-t, S_t) \tag{184}$$

Concluding that:

$$c(t, S_t) = S_t N\Big(d_+(T-t, S_t)\Big) - Ke^{-r(T-t)} N\Big(d_-(T-t, S_t)\Big) \tag{185}$$

Where simply recall that:

$$\begin{aligned} d_-(T-t, S_t) &= \frac{\log\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\ &= \underbrace{\frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}_{:=d_+(T-t,S_t)} - \sigma\sqrt{T-t} \end{aligned}$$

CHAPTER 4: COMPUTING PRICES OF DERIVATIVE SECURITIES

Revision

Replicating the price of a call option via portfolio hedging. Recall that stock prices, price of call option and value of portfolio

$$dS_t = \alpha S_t dt + \sigma S_t dW_t$$

$$c(T, S_T) = c(t, S_t) + \int_t^T \left\{ c_r(r, S_r) + c_x(r, S_r)\alpha S_r + \frac{1}{2}c_{xx}(r, S_r)\sigma^2 S_r^2 \right\} dr + \int_t^T c_x(r, S_r)\sigma S_r dW_r$$

$$dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t)dt = \Big\{ \Delta_t(\alpha - r)S_t + rX_t \Big\} dt + \Delta_t \sigma S_t dW_t$$

This means that hedging strategy is:

$$\Delta_t = c_x(t, S_t) \tag{186}$$

and

$$c_r(r, S_r) + c_x(r, S_r)\alpha S_r + \frac{1}{2}c_{xx}(r, S_r)\sigma^2 S_r^2 = \Delta_t(\alpha - r)S_t + rX_t \tag{187}$$

Since there appears an rX_t term on the RHS, we need discounted pay of where after applying ito's rule we will require:

$$\begin{aligned} &e^{-rt}\Big\{ c_x(t, S_t)(\alpha - r)S_t \Big\} = e^{-rt}\Big\{ -rc(t, S_t) + c_t(t, S_t) + c_x(t, S_t)\alpha S_t + \frac{1}{2}c_{xx}(t, S_t)\sigma^2 S_t^2 \Big\} \\ &\implies c_t(t, S_t) + rS_t c_x(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{xx}(t, S_t) = rc(t, S_t) \end{aligned}$$

Now recall that if we let:

$$
\begin{aligned}
&d_{\pm}\big(T-t, S_t\big) = \frac{1}{\sigma\sqrt{T-t}}\left[\log\left(\frac{S_t}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)\right] \\
&c(t, S_t) = S_t N\big(d_+(T-t, S_t)\big) - Ke^{-r(T-t)}N\big(d_-(T-t, S_t)\big) \\
&c(T, S_T) = (S_T - K)^+
\end{aligned}
$$

Now the question is what is the probability of me profiting from buying a call option:

$$
\begin{aligned}
\mathbb{P}\bigg(S_T - e^{r(T-t)}c(t, S_t) > K\bigg) &\leq \frac{\mathbb{E}\big[S_T - e^{r(T-t)}c(t, S_t)\big]^2}{K^2} \\
&= \frac{\mathbb{E}_{S_t}[S_T^2] + e^{2r(T-t)}c(t, S_t)^2 - 2e^{r(T-t)}c(t, S_t)\mathbb{E}_{S_t}[S_T]}{K^2}
\end{aligned}
$$

Now we will use the fact that the model follows geometric Brownian motion

$$
S_T = S_t e \tag{188}
$$

Simulation of security prices

$$
\begin{aligned}
V_t &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} V_T \big| \mathcal{F}_t\right] = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\big[\max(S_T - K, 0)|\mathcal{F}_t\big] \\
&= e^{-\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2(T-t)}e^{-r(T-t)}\mathbb{E}_{\mathbb{P}}\left[\max\big(S_T - K, 0\big)e^{-\left(\frac{\mu-r}{\sigma}\right)[W_T - W_t]}\big|\mathcal{F}_t\right]
\end{aligned}
$$

We know that:

$$
V_t = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\big[\max(S_T - K, 0)|\mathcal{F}_t\big] = e^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2(T-t)}e^{-r(T-t)}\mathbb{E}_{\mathbb{P}}\left[\max\big(S_T - K, 0\big)e^{-\left(\frac{\mu-r}{\sigma}\right)[W_T - W_t]}\big|\mathcal{F}_t\right]
$$

Now for constant variance, interest rate and drift:

$$
V_t = e^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2(T-t)}e^{-r(T-t)}\mathbb{E}_{\mathbb{P}}\left[\max\big(S_T - K, 0\big)e^{-\left(\frac{\mu-r}{\sigma}\right)(W_T - W_t)}\big|\mathcal{F}_t\right] \tag{189}
$$

Now GBM recall that:

$$
S_T = S_t e^{\left(\alpha - \frac{\sigma^2}{2}\right)(T-t)}e^{\sigma\left(W_T - W_t\right)} \tag{190}
$$

So essentially $S_T|S_t = S_t e^{\left(\alpha - \frac{\sigma^2}{2}\right)(T-t)}e^{\sigma\sqrt{T-t}Z}$ where Z is standard normal. Hence:

$$
\begin{aligned}
&c(t, S_t) = V_t = e^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2(T-t)}e^{-r(T-t)}\mathbb{E}_{\mathbb{P}}\left[\max\big(S_T - K, 0\big)e^{-\left(\frac{\mu-r}{\sigma}\right)(W_T - W_t)}\big|\mathcal{F}_t\right] \\
&= e^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2(T-t)}e^{-r(T-t)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\max\left(S_t e^{\left(\alpha - \frac{\sigma^2}{2}\right)(T-t)}e^{\sigma\sqrt{T-t}z} - K, 0\right)e^{-\left(\frac{\mu-r}{\sigma}\right)\sqrt{T-t}z}e^{-\frac{z^2}{2}}dz
\end{aligned}
$$

$$
S_t e^{\left(\alpha - \frac{\sigma^2}{2}\right)(T-t)}e^{\sigma\sqrt{T-t}z} \geq K \implies z \geq \frac{\log\left(\frac{K}{S_t}\right) - \left(\mu - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\max\left(S_t e^{\left(\alpha-\frac{\sigma^2}{2}\right)(T-t)}e^{\sigma\sqrt{T-t}z}-K,0\right)e^{-\left(\frac{\mu-r}{\sigma}\right)\sqrt{T-t}z}e^{-\frac{z^2}{2}}dz\\
&=\frac{1}{\sqrt{2\pi}}\int_{\frac{\log\left(\frac{K}{S_t}\right)-\left(\mu-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}^{\infty}\left(S_t e^{\left(\alpha-\frac{\sigma^2}{2}\right)(T-t)+\sigma\sqrt{T-t}z}-K\right)e^{-\left(\frac{\mu-r}{\sigma}\right)\sqrt{T-t}z}e^{-\frac{z^2}{2}}dz
\end{aligned}
$$

$$
c(t,S_t)=S_t e^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2(T-t)}e^{-r(T-t)}e^{\left(\mu-\frac{\sigma^2}{2}\right)(T-t)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\log\left(\frac{S_t}{K}\right)+\left(\mu-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}e^{\left(\sigma-\left(\frac{\mu-r}{\sigma}\right)\right)\sqrt{T-t}z}e^{-\frac{z^2}{2}}dz \tag{191}
$$

$$
-Ke^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2(T-t)}e^{-r(T-t)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\log\left(\frac{S_t}{K}\right)+\left(\mu-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}e^{-\left(\frac{\mu-r}{\sigma}\right)\sqrt{T-t}z}e^{-\frac{z^2}{2}}dz
$$

Markovian nature of stock prices:

$$
\begin{aligned}
&dS_t=\mu S_t dt+\sigma S_t dW_t \ \textit{ and its' solution}\\
&S_T=S_t e^{\left(\mu-\frac{\sigma^2}{2}\right)(T-t)}e^{\sigma\left(W_T-W_t\right)}
\end{aligned}
$$

Now we know that S_t is $\mathcal{F}_t$ adapted and by the definition of Brownian motion $\left(W_T-W_t\right)\perp\mathcal{F}_t$ and consequently:

$$
\begin{aligned}
\mathbb{E}\left[S_T|\mathcal{F}_t\right]=\mathbb{E}\left[S_t e^{\left(\mu-\frac{\sigma^2}{2}\right)(T-t)}e^{\sigma\left(W_T-W_t\right)}|\mathcal{F}_t\right]&=S_t e^{\left(\mu-\frac{\sigma^2}{2}\right)(T-t)}\mathbb{E}\left[e^{\sigma(W_T-W_t)}|\mathcal{F}_t\right]\\
&=\mathbb{E}_{Z\sim N(0,1)}\left[e^{\sigma\sqrt{T-t}Z}\right]
\end{aligned}
$$

5.3 Shreve: So from here we know that:

$$
c(t,x)=e^{-r(T-t)}\mathbb{E}_{z\sim N(0,1)}\left[\max\left(xe^{\left(r-\frac{\sigma^2}{2}\right)(T-t)}e^{-\sigma\sqrt{T-t}Z_i}-K,0\right)\right] \tag{192}
$$

and simply:

$$
c(0,x)=e^{-rT}\mathbb{E}_{z\sim N(0,1)}\left[\max\left(xe^{\left(r-\frac{\sigma^2}{2}\right)T}e^{-\sigma\sqrt{T}Z_i}-K,0\right)\right] \tag{193}
$$

Interested in figuring out $c_x(0,x)$: simply notice that:

$$c(0,x) = \int_{-\infty}^{\infty} \max\left(xe^{\left(r-\frac{\sigma^2}{2}\right)T}e^{-\sigma\sqrt{T}z} - K, 0\right)e^{-\frac{z^2}{2}}dz$$

$$c_x(0,x) = \lim_{\delta_x \to 0} \frac{\int_{-\infty}^{\infty}\left\{\max\left([x+\delta_x]e^{\left(r-\frac{\sigma^2}{2}\right)T}e^{-\sigma\sqrt{T}z} - K, 0\right) - \max\left(xe^{\left(r-\frac{\sigma^2}{2}\right)T}e^{-\sigma\sqrt{T}z} - K, 0\right)\right\}e^{-\frac{z^2}{2}}dz}{\delta_x}$$

$$= \int_{-\infty}^{\frac{\log\left(\frac{S_t}{K}\right)+\left(\mu-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}} e^{\left(r-\frac{\sigma^2}{2}\right)T}e^{-\sigma\sqrt{T}z}e^{-\frac{z^2}{2}}dz := N(d_+(T,x))$$

We can equivalently write it as, since

$$S_t = xe^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma B_t}$$

$$c_x(0,x) = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[1_{[S_T>K]}e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma B_T}\right] = \mathbb{E}_{\mathbb{Q}}\left[1_{[S_T>K]}e^{-\frac{\sigma^2}{2}T+\sigma B_T}\right]$$

Girsanov means $\frac{d\mathbb{R}}{d\mathbb{Q}} := e^{-\frac{\sigma^2}{2}T+\sigma B_T}$ such that under $\mathbb{R}$,

$$\hat{B}_t := B_t - \sigma t, \;\; \textit{is Brownian motion}$$

$$S_t = xe^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma\hat{B}_t+\sigma^2 t} = xe^{\left(r+\frac{\sigma^2}{2}\right)t+\sigma\hat{B}_t}$$

$$c_x(0,x) = \mathbb{P}_{\mathbb{R}}\Big[S_T > K\Big] = \mathbb{P}_{\mathbb{R}}\left[-\frac{\hat{B}_T}{\sqrt{T}} < d_+(T,x)\right]$$

Now simply notice that:

We know that: $c(0,x) = N\big(d_-(T,x)\big)$ and $c_x(0,x) = N\big(d_+(T,x)\big)$

Theorem 48 *Show that*

$$\mathbb{P}_{\mathbb{Q}}\big\{S_T > K\big\} = N\big(d_+(T,x)\big) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\log\left(\frac{S_t}{K}\right)+\left(\mu-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}+\sigma\sqrt{T-t}} e^{-\frac{\hat{z}^2}{2}}d\hat{z} \qquad (194)$$

$$\mathbb{P}_{\mathbb{Q}}\big(S_T > K\big)$$

$$S_T \underbrace{=}_{\mathbb{Q}} S_t e^{\left(r-\frac{\sigma^2}{2}\right)(T-t)} e^{-\sigma\sqrt{T-t}Z}, \quad Z \sim N(0,1) \ \textit{Therefore,}$$

$$\mathbb{P}_{\mathbb{Q}}\big(S_T > K\big) = \mathbb{P}\left(e^{-\sigma\sqrt{T-t}Z} > \frac{K}{S_t} e^{\left(\frac{\sigma^2}{2}-r\right)(T-t)}\right) = \mathbb{P}\left\{e^{-\sigma\sqrt{T-t}Z} > e^{\ln\left(\frac{K}{S_t}\right)+\left(\frac{\sigma^2}{2}-r\right)(T-t)}\right\}$$

$$\leq \inf_{m>0} \mathbb{E}\big[e^{-\sigma m\sqrt{T-t}Z}\big] e^{-m\left[\ln\left(\frac{K}{S_t}\right)+\left(\frac{\sigma^2}{2}-r\right)(T-t)\right]} = \inf_{m>0} e^{\frac{\sigma^2 m^2 (T-t)}{2}} e^{-m\left[\ln\left(\frac{K}{S_t}\right)+\left(\frac{\sigma^2}{2}-r\right)(T-t)\right]} \ \textit{suffices to mini}$$

$$\frac{d}{dm}\left\{\frac{\sigma^2 m^2 (T-t)}{2} - m\ln\left(\frac{K}{S_t}\right) - m\big(\frac{\sigma^2}{2}-r\big)(T-t)\right\} = 0, \implies$$

$$m\sigma^2(T-t) - \ln\left(\frac{K}{S_t}\right) - \big(\frac{\sigma^2}{2}-r\big)(T-t) = 0, \implies$$

$$m^* = \frac{\ln\left(\frac{K}{S_t}\right)}{\sigma^2(T-t)} + \frac{\left(\frac{\sigma^2}{2}-r\right)}{\sigma^2}$$

$$e^{\frac{\sigma^2\left\{\frac{\ln\left(\frac{K}{S_t}\right)}{\sigma^2(T-t)}+\frac{\left(\frac{\sigma^2}{2}-r\right)}{\sigma^2}\right\}^2(T-t)}{2}} e^{-\left\{\frac{\ln^2\left(\frac{K}{S_t}\right)}{\sigma^2(T-t)}+2\frac{\left(\frac{\sigma^2}{2}-r\right)\ln\left(\frac{K}{S_t}\right)}{\sigma^2}+\frac{\left(\frac{\sigma^2}{2}-r\right)^2(T-t)}{\sigma^2}\right\}}$$

$$= e^{-\frac{\sigma^2\left\{\frac{\ln\left(\frac{K}{S_t}\right)}{\sigma^2(T-t)}+\frac{\left(\frac{\sigma^2}{2}-r\right)}{\sigma^2}\right\}^2(T-t)}{2}}$$

$$dS_t = \alpha S_t + \sigma S_t dW_t \tag{195}$$

Exercise 5.4 Shreve, time-varying Black Scholes Given that r_t and σ_t are non-random.

$$dS_t = r_t S_t dt + \sigma_t dB_t$$

$$c(0, S_0) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r_t dt} \max\left(S_T - K, 0\right)\right]$$

$$S_t = S_0 e^{\int_0^t \left(r_s - \frac{\sigma_s^2}{2}\right)dt + \int_0^s \sigma_s dW_s}$$

$$X := \int_0^t \left(r_s - \frac{\sigma_s^2}{2}\right)dt + \int_0^s \sigma_s dW_s \sim N\left(\int_0^t \left(r_s - \frac{\sigma_s^2}{2}\right)dt, \int_0^t \sigma_s^2 ds\right)$$

$$c(0, S_0) = e^{-\int_0^T r_t dt} \int_{-\infty}^{\infty} \max\left(S_0 e^{\int_0^T \left(r_t - \frac{\sigma_t^2}{2}\right)dt - \sqrt{\int_0^T \sigma_t^2 dt} z} - K\right) e^{-\frac{z^2}{2}} dz$$

$$S_0 e^{\int_0^T \left(r_t - \frac{\sigma_t^2}{2}\right)dt - \sqrt{\int_0^T \sigma_t^2 dt} z} > K \implies z < \frac{\ln\left(\frac{S_0}{K}\right) + \int_0^T \left(r_t - \frac{\sigma_t^2}{2}\right)dt}{\sqrt{\int_0^T \sigma_t^2 dt}}$$

$$c(0, S_0) = S_0 \Phi\left(\frac{\ln\left(\frac{S_0}{K}\right) + \int_0^T \left(r_t + \frac{\sigma_t^2}{2}\right)dt}{\sqrt{\int_0^T \sigma_t^2 dt}}\right) - K e^{-\int_0^T r_t dt} \Phi\left(\frac{\ln\left(\frac{S_0}{K}\right) + \int_0^T \left(r_t - \frac{\sigma_t^2}{2}\right)dt}{\sqrt{\int_0^T \sigma_t^2 dt}}\right)$$

From here we see that model with fixed volatility of $\sigma := \frac{\int_0^T \sigma_t^2 dt}{T}$ would have given the same call price On the Gaussian nature of $dX_t := \sigma_t dW_t$ by Ito's formula: $\mathbb{E}[e^{mX_t}] = 1 + \frac{1}{2} m^2 \int_0^t \sigma_s^2 \mathbb{E}[e^{mX_s}] ds$. Now if we let $u_t := \mathbb{E}[e^{mX_t}]$

$$u_t = 1 + \frac{1}{2} m^2 \int_0^t \sigma_s^2 u_s ds$$

$$\frac{d}{dt} u_t = \frac{1}{2} m^2 \lim_{\delta_t \to 0} \frac{\int_t^{t+\delta t} \sigma_s^2 u_s ds}{\delta_t} = \frac{1}{2} m^2 \sigma_t^2 u_t$$

$$u_t := e^{\frac{1}{2} m^2 \int_0^t \sigma_s^2 ds}$$

For all MGF($e^{t\mu + \frac{1}{2}\sigma^2 t^2}$) vs CHAR FUN($e^{it\mu - \frac{1}{2}\sigma^2 t^2}$) of Gaussian with mean μ and variance σ^2

Exercise 5.8 think of interest rates as time varying Sigma algebra generated by underlying Brownian motion $\mathcal{F}_t^W$, $\mathbb{Q}$ is risk neural measure and V_T is $\mathcal{F}_T^W$ and almost surely positive measurable price or value at time t is:

$$V_t := \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} V_T \middle| \mathcal{F}_t^W\right] \tag{196}$$

Let $G_t := D_t V_t = e^{-\int_0^t r_s ds} \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} V_T \middle| \mathcal{F}_t^W\right] = \mathbb{E}_{\mathbb{Q}}[D_T V_T | \mathcal{F}_t^W]$ which is a martingale and by MRT exists adapted ˆ

$$d(D_t V_t) = \Gamma_t dB_t$$

But recall that $dD_t = -R_tD_tdt$ along with ito product rule:

$$\begin{aligned}
&d(D_tV_t) = (dD_t + D_t)dV_t + dD_tV_t = -R_tD_tV_tdt + D_tdV_t, \ \textit{implying that} \\
&- R_tD_tV_tdt + D_tdV_t = \Gamma_t dB_t \ \textit{implying that} \\
&dV_t = R_tV_tdt + \frac{\Gamma_t}{D_t}dB_t
\end{aligned}$$

recall that under risk neutral measure

$$d(D_tS_t) = D_tS_t\sigma_t dB_t \tag{197}$$

19. Fixed Income Securities

Traditionally, the term fixed income securities refers to securities whose cash flows are fixed in advance and whose values therefore depend largely on the current level of interest rates. The classic example of a fixed income security is a bond which pays a fixed coupon every period until expiration when the final coupon and original principal is paid

20. Stocks with Dividends

20.1 Continuous Dividends

$$dS_t = \alpha_tS_tdt + \sigma_tS_tdW_t - A_tS_tdt \tag{198}$$

What happens to the portfolio $X_{t+1} = \Delta_tS_{t+1} + (1+r)[X_t - \Delta_tS_t]$ implying: $dX_t = \Delta_t dS_t + r[X_t - \Delta_tS_t]dt = \Delta_t\big(\alpha_tS_tdt + \sigma_tS_tdW_t - A_tS_tdt\big) + r[X_t - \Delta_tS_t]dt$

$$dX_t = \Bigg\{\Delta_t\big(\alpha_t - A_t - r\big)S_t + rX_t\Bigg\}dt + \Delta_t\sigma_tS_tdW_t \tag{199}$$

$$S_{t_j} = S_{t_{j-}} - a_jS_{t_{j-}} = (1-a_j)S_{t_{j-}} \tag{200}$$

with $a_j \in [0,1]$ and $a_j \in \mathcal{F}_{t_j}$. When dividends are not paid stock follows usual GBM

$$dS_t = \alpha_tS_tdt + \sigma_tS_tdW_t \tag{201}$$

Definition 49 *A forward contract is an agreement to pay a specified delivery price K at a delivery date T, where $0 \le T \le \overline{T}$, for the asset whose price at time t is S_t . The $T-$forward price $For_s(t,T)$ of this asset at time t, where $0 \le t \le T \le \overline{T}$ is the value of K that makes the forward contract have no-arbitrage price zero at time t.*

Exercise 5.10 Choosers Option:

Stochastic Differential Equations

Conditioned on $X_0 = x$, let:

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t u(s, X_s)dW_s \tag{202}$$

If their exists a stochastic process X_t that satisfies the preceding equation we say that X_t solves the stochastic differential equation.

Linear SDE Let $X_t \in \mathbb{R}^n$, $W_t \in \mathbb{R}^m$ and $B \in \mathbb{R}^{n\times m}$

$$
\begin{aligned}
&X_t = x + \int_0^t AX_s ds + \int_0^t \langle B, dW_s\rangle && (203)\\
&Z_t := \int_0^t e^{-As}\langle B, dW_s\rangle\\
&f(t,z) := e^{At}(x+z).\ f_t(t,z) = Af(t,z),\ f_z(t,z) = e^{At},\ f_{zz}(t,z) = 0.\\
&\textit{Ito implies } f \textit{ satisfies } f(t,Z_t) = \underbrace{f(0,Z_0)}_{=x} + \int_0^t Af(s,Z_s)ds + \int_0^t \langle e^{As}, BdW_s\rangle
\end{aligned}
$$

which means that:

$$
Y_t := e^{At}\left(x + \int_0^t e^{-As}BdW_s\right) \tag{204}
$$

is a solution to Linear SDE (203).

Example 3 ***Stock price as Markov Chain***

$$
\begin{aligned}
&dS_t = \alpha S_t dt + \sigma S_t dW_t\\
&S_T = S_t e^{\left(\alpha - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)}\\
&\mathbb{E}\big[f(S_T)|\mathcal{F}_t\big] = \mathbb{E}\big[f(S_T)|S_t\big] = g_{T-t}(X_t)
\end{aligned}
$$

By the definition of conditional expectation, for all measurable $g(W_{[0,t)})$

$$
\mathbb{E}\big[f(S_T)g(W_{[0,t)})\big] := \mathbb{E}\Big[\mathbb{E}\big[f(S_T)|\mathcal{F}_t\big]g(W_{[0,t)})\Big] \tag{205}
$$

First need to figure out whether time homogeneous or not, given $X_t := x$ we want to establish X_T. Step size is δ such that for $N \in \mathbb{N}$, $t + \delta N = T$.

$$
\begin{aligned}
&dX_u = \beta(u, X_u)du + \gamma(u, X_u)dW_u\\
&X_{t+\delta} = x + \beta(t,x)\delta + \gamma(t,x)\big[W_{t+\delta} - W_t\big]\\
&X_{t+2\delta} = X_{t+\delta} + \beta(t+\delta, X_{t+\delta})\delta + \gamma(t+\delta, X_{t+\delta})\big[W_{t+2\delta} - W_{t+\delta}\big]
\end{aligned}
$$

- Markovian nature is now obvious
- $\epsilon_i \sim N(0,1)$ for each $i \in [k]$ and independent of each other

  $$
  \begin{aligned}
  &X_{t+\delta} = x + \beta(t,x)\delta + \gamma(t,x)\sqrt{\delta}\epsilon_1\\
  &X_{t+2\delta} = X_{t+\delta} + \beta(t+\delta, X_{t+\delta})\delta + \gamma(t+\delta, X_{t+\delta})\sqrt{\delta}\epsilon_2
  \end{aligned}
  $$

  This allows us to sequentially simulate an entire path $X^1_{[t,T]} := \left(X^1_t, X^1_{t+\delta}, \ldots, X^1_T\right)$ and if we want to approximate $\mathbb{E}[f(X_T)|X_t]$, we can simulate multiple independent copies of the sample path $[X^j_{[t,T]}]^N_{j=1}$ such that:

  $$
  \frac{1}{N}\sum_{j=1}^N f(X^j_T) \approx \mathbb{E}[f(X_T)|X_t] \tag{206}
  $$

- **Fix the final time** T, now Markovian property implies that $\mathbb{E}[h(X_T)|\mathcal{F}_t] = g(t, X_t)$. Recall that underlying SDE is

$$dX_u = \beta(u, X_u)du + \gamma(u, X_u)dW_u. \tag{207}$$

Ito and MRT now implies that:

$$\partial_t g(t, X_t) + g_x(t, X_t)\beta(t, X_t) + \frac{1}{2}\sigma^2(t, X_t)g_{xx}(t, X_t) = 0 \tag{208}$$

Simply meaning that

$$\partial_t g(t, x) + g_x(t, x)\beta(t, x) + \frac{1}{2}\sigma^2(t, x)g_{xx}(t, x) = 0 \tag{209}$$

- Consequently, we can compute derivative security prices via *numerically solving the partial differential equation or Monte Carlo method*

Recall that when T is fixed.

$$\begin{aligned}
&g(t, X_t) := \mathbb{E}[h(X_T)|X_t] \\
&g(t, x) := \mathbb{E}^{t,x}[h(X_T)] = \mathbb{E}[h(X_T)|X_t = x] \\
&g(T, x) = h(x)
\end{aligned}$$

Discounted Feynman-Kac:

$$\begin{aligned}
&f(t, x) := \mathbb{E}_{t,x}\left[e^{-r(T-t)}h(X_T)|X_t\right] \ \textit{satisfies} \\
&\partial_t f(t, x) + f_x(t, x)\beta(t, x) + \frac{1}{2}\sigma^2(t, x)f_{xx}(t, x) = rf(t, x), \\
&\textit{with terminal condition } f(T, x) = h(x)
\end{aligned}$$

21. Hamilton Jacobi Bellman Equation

$$\begin{aligned}
&V(x_0, 0) := \min_u \Big\{ \int_0^T C(x_t, u_t)dt + D(x_T) \Big\} \\
&\dot{x}_t = F(x_t, u_t)
\end{aligned}$$

This implies that:

$$V(x, t+\delta_t) - V(x, t) = \min_u \Big\{ \int_t^T C(x_s, u_s)ds + D(x_T) \Big\} - \min_u \Big\{ \int_{t+\delta_t}^T C(x_s, u_s)ds + D(x_T) \Big\} \tag{210}$$

22. Come back in the end

If β and γ are time invariant then:

$$g(t,x) = \mathbb{E}[h(X_t)|X_0 = x] = \int h(y)p(t,x,dy) = P^t h(x)$$

$$\lim_{\delta t \to 0} \frac{P^{t+\delta_t} f(x) - P^t f(x)}{\delta_t} = \int f(y)\partial_t p(t,x,dy)$$

$$dX_u = \beta(u, X_u)du + \gamma(u, X_u)dW_u$$
$$\mathbb{E}\big[h(X_T)|X_t = x\big] = P^{T-t}h(x) \ \textit{After Markovian nature is established}$$
$$\mathbb{E}[h(X_t)|X_0 = x] =: g(t,x).$$

What about the Markovian nature for $T > t > s$:

$$M_t := \mathbb{E}\big[f(Y_T)|\mathcal{F}_t^W\big]$$
$$\mathbb{E}\big[M_t|\mathcal{F}_s^W\big] = M_s \ \textit{implies via MRT}$$

Example 4 ***Hull-White interest rate models*** *$a(u), b(u), \sigma(u)$ positive and non-random*

$$dR(u) = \big(a(u) - b(u)R(u)\big)du + \sigma(u)dB(u)$$

Given initial condition $R(t) = r$ solution of this SDE

23. Implied volatility, volatility smile and simulations

Something to do with inter-arrival time(exponential random variable) and Poisson random variable.

Example 5 *(Simulating Delta-Hedging in a Black-Scholes Economy) In this extended example we consider the use of the Black-Scholes model to hedge a vanilla European call option. Moreover, we will assume that the assumptions of Black-Scholes are correct so that the security price has GBM dynamics, it is possible to trade continuously at no cost and borrowing and lending at the risk-free rate are also possible. It is then possible to dynamically replicate the payoff of the call option using a self-financing (s.f.) trading strategy. The initial value of this s.f. strategy is the famous Black-Scholes arbitrage-free price of the option. The s.f. replication strategy requires the continuous delta-hedging of the option but of course it is not practical to do this and so instead we hedge periodically. (Periodic or discrete hedging then results in some replication error but this error goes to 0 as the time interval between re-balancing goes to 0). Towards this end, let P_t denote the time t value of the discrete-time s.f. strategy that attempts to replicate the option payoff and let C_0 denote the initial value of the option. The replicating strategy is then given by:*

$$P_0 := C_0.$$
$$P_{t_{i+1}} - P_{t_i} = r\big(P_{t_i} - \Delta_{t_i} S_{t_i}\big)dt + \Delta_{t_i}\big(S_{t_{i+1}} - S_{t_i}\big)$$
$$dt = t_{i+1} - t_i.$$
$$S_{t_{i+1}} = S_{t_i} e^{(\mu - \frac{\sigma^2}{2})dt + \sigma\sqrt{dt}Z}, \ Z \sim N(0,1).$$

In the case of a short position in a call option with strike K and maturity T, the final trading $P\&L$ is then defined as:

$$P\&L := P_T - (S_T - K)^+ \tag{211}$$

where P_T is the terminal value of the replicating strategy in (6). In the Black-Scholes world we have $\sigma = \sigma_{imp}$ and the $P\&L$ will be 0 along every price path in the limit as $dt \to 0$ Recall that Delta

$$\Delta_{t_i} = \nabla_x c(t_i, x) = N\big(d_+(T - t_i, x)\big) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{1}{\sigma\sqrt{T-t_i}}\left[\log\left(\frac{x}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t_i)\right]} e^{-\frac{y^2}{2}} dy \tag{212}$$

In practice, however, we do not know σ and so the market (and hence the option hedger) has no way to ensure a value of σ_{imp} such that $\sigma = \sigma_{imp}$. This has interesting implications for the trading PL and it means in particular that we cannot exactly replicate the option even if all of the assumptions of Black-Scholes are correct. In Figure 2 we display histograms of the $P\&L$ in (7) that results from simulating 100,000 sample paths of the underlying price process with $S_0 = K = 100$. (Other parameters and details are given below the figure.) In the case of the first histogram the true volatility was $\sigma = 0.3$ with $\sigma_{imp} = 0.2$ and the option hedger makes (why?) substantial loses. In the case of the second histogram the true volatility was $\sigma = 0.3$ with $\sigma_{imp} = 0.4$ and the option hedger makes (why?) substantial gains. Clearly then this is a situation where substantial errors in the form of non-zero hedging $P\&L$'s are made and this can only be due to the use of incorrect model parameters. This example is intended to highlight the importance of not just having a good model but also having the correct model parameters. Note that the payoff from delta-hedging an option is in general path-dependent, i.e. it depends on the price path taken by the stock over the entire time interval. In fact, it can be shown that the payoff from continuously delta-hedging an option satisfies

It has been observed in markets that if one assumes a constant volatility, the parameter σ that makes the theoretical option price given by MRT or Black Scholes agree with the market price, the so-called implied volatility, is different for options having different strikes. In fact, this implied volatility is generally a convex function of the strike price. One refers to this phenomenon as the volatility smile.

24. Hedging short and long positions

Definition 50 *To hedge a short position in the call option: you take a short position on a security or asset when you think its value will fall in the future, and to hedge a short position means that you mitigate the risk of loss if the value of underlying asset or security increase by some alternate transaction. You take this position by shorting call option and purchasing stock*

Definition 51 *To hedge a long position in the call option: you take a long position on a security or asset when you think its value will fall in the future, and to hedge a long position* *means that you mitigate the risk of loss if the value of underlying asset or security decrease by some alternate transaction. You take this position by purchasing call option and shorting the stock*

Suppose at time t the stock price is x_1 and we wish to take a long position in the option and hedge it. We do this by purchasing the option for $c(t, x_1)$, *shorting* $c_x(t, x_1)$ shares of stock, which generates income $x_1 c_x(t, x_1)$, and investing the difference:

$$M := x_1 c_x(t, x_1) - c(t, x_1), \tag{213}$$

in the money market account. Overall value of our portfolio equals zeros as:

$$c(t, x_1) - x_1 c_x(t, x_1) + M = 0. \tag{214}$$

1. Now if the value of stock goes down to $x_0 < x_1$: then new portfolio value is:

   $$c(t, x_0) - x_0 c_x(t, x_1) + x_1 c_x(t, x_1) - c(t, x_1) = c(t, x_0) + (x_1 - x_0) c_x(t, x_1) - c(t, x_1). \tag{215}$$

   Taylors' expansion theorem implies that for some $z \in (x_0, x_1)$:

   $$c(t, x_0) - c(t, x_1) = c_x(t, x_1)(x_0 - x_1) + \frac{1}{2}(x_0 - x_1)^2 c_{xx}(t, z), \tag{216}$$

   which implies that our current value of the portfolio is

   $$c_x(t, x_1)(x_0 - x_1) + \frac{1}{2}(x_0 - x_1)^2 c_{xx}(t, z) + (x_1 - x_0) c_x(t, x_1) = \frac{1}{2}(x_0 - x_1)^2 \underbrace{c_{xx}(t, z)}_{\geq 0,\ \textit{convexity}}. \tag{217}$$

   Thanks to convexity, even if the stock price drops your instantaneous portfolio value increases.

2. Now if the stock price increases from x_1 to x_2, then overall value of the portfolio is:

   $$c(t, x_2) - x_2 c_x(t, x_1) + M = c(t, x_2) + (x_1 - x_2) c_x(t, x_1) - c(t, x_1) \tag{218}$$

   Definition 52 *This position is delta neutral as increase in the value of portfolio due to increase in the value of stock from x_1 to x_2 is compensated by the increase in liability of shorted stocks.*

   again using the Taylors' expansion, for some $z \in (x_1, x_2)$

   $$c(t, x_2) - c(t, x_1) = c_x(t, x_1)(x_2 - x_1) + \frac{1}{2} c_{xx}(t, z)(x_2 - x_1)^2. \tag{219}$$

   Consequently the value of instantaneous portfolio becomes:

   $$c_x(t, x_1)(x_2 - x_1) + \frac{1}{2} c_{xx}(t, z)(x_2 - x_1)^2 + (x_1 - x_2) c_x(t, x_1) = \frac{1}{2} c_{xx}(t, z)(x_2 - x_1)^2 \geq 0, \tag{220}$$

   again thanks to convexity.

The portfolio we have set up is called *delta-neutral* and *long gamma*. Long gamma because it takes the advantage of convexity of $c(t, x)$ w.r.t the x argument. If there is an instantaneous rise or an instantaneous fall in the stock price, the value of the portfolio increases. A long gamma portfolio is profitable in times of high stock volatility. *Delta-neutral* refers to the fact that the line in Figure 4.5.1 is tangent to the curve $y = c(t, x)$. Therefore, when the stock price makes a small move, the change of portfolio value due to the corresponding change in option price is nearly offset by the change in the value of our short position in the stock. The straight line is a good approximation to the option price for small stock price moves. If the straight line were steeper than the option price curve at the starting point x_1 , then we would be *short delta*; an upward move in the stock price would hurt the portfolio because the liability from the short position in stock would rise faster than the value of the option. On the other hand, a downward move would increase the portfolio value because the option price would fall more slowly than the rate of decrease in the liability from the short stock position. Unless a trader has a view on the market, he tries to set up portfolios that are *delta-neutral*. If he expects high volatility, he would at the same time try to choose the portfolio to be *long gamma*.

We wish to consider the *sensitivity to stock price* changes of the portfolio that has these three components: *long option*, *short stock*, and *long money market account*. The initial portfolio value:

Put-Call Parity

Definition 53 *A* forward contract *with delivery price K obligates its holder to buy one share of the stock at expiration time T in exchange for payment K.*

So at the time of expiry, value of the forward contract is $S_T - K$. Obviously value is positive if $K < S_T$. Now what about $f(t, x)$? *the current value of a forward contract (current time is t), expiring at future time T with delivery price K*. We argue that

$$f(t, x) = x - e^{-r(T-t)}K. \tag{221}$$

without loss of generality, let $t = 0$ and initial stock price $S_0 = x$ and we start with the initial capital of:

$$f(0, S_0) = S_0 - e^{-rT}K, \tag{222}$$

Then at time T our portfolio stands at

$$S_T - e^{rT}e^{-rT}K = S_T - K. \tag{223}$$

Because the agent has been able to replicate the payoff of the forward contract with a portfolio whose value at each time t is:

$$f(t, S_t) = S_t - e^{-r(T-t)}K. \tag{224}$$

Definition 54 *The forward price of a stock at time t is defined to be the value of K that causes the forward contract at time t to have value zero* i.e., K such that

$$S_t - e^{-r(T-t)}K = 0 \implies K = S_t e^{r(T-t)}$$
$$For(t) = S_t e^{r(T-t)}.$$

We observe that for any x,

$$x - K = (x-K)^+ - (K-x)^+. \tag{225}$$

If $x < K$, then $(x-K)^+ = 0$ and $(K-x)^+ = (K-x)$ and $-(K-x) = x-K$. Now pay off for a call option is $(x-K)^+$ and for put option is $(K-x)^+$. Let $c(t,x)$ be the value of call option at time $t < T$, where T is expiration date and strike price K and $S_t = x$. Similarly $p(t,x)$ the put option with same maturity date and strike price, then:

$$S_T - K = f(T, S_T) = c(T, S_T) - p(T, S_T), \tag{226}$$

i.e., the payoff of the forward contract agrees with the payoff of a portfolio that is long one call and short one put. Since the value at time T of the forward contract agrees with the value of the portfolio that is long one call and short one put, these values must agree at all previous time.

$$f(t,x) = c(t,x) - p(t,x) \ \forall x \geq 0, t \geq 0. \tag{227}$$

TAKE AWAY: from call price we can compute put price as

$$p(t, S_t) = c(t, S_t) - S_t + e^{-r(T-t)}K. \tag{228}$$

Assume that $f(t,x) - c(t,x) + p(t,x) > 0$ then $c(t,x) - p(t,x) - f(t,x) < 0$ then one can get paid to have a portfolio long in call and short in put and forward and hold this position up till time T when its value increases to 0.

If this were not the case, one could at some time t either sell or buy the portfolio that is long the forward, short the call, and long the put, realizing an instant profit, and have no liability upon expiration of the contracts. The relationship in preceding equation 227 is called put-call parity.

24.1 Minimizing the residual error at the time of maturity:

Side note on call pde: Call price would satisfy:

$$rc_x(t, S_t)S_t + c_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{xx}(t, S_t) = rc(t, S_t), \tag{229}$$

Recall that portfolio satisfies the following SDE:

$$dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t)dt \tag{230}$$

where Δ_t is the hedging strategy, in practice we can not continuously hedge so discretize things. Assume that call option is expiring at time T with a payoff $C(T, S_T) = (S_T - K)^+$. So introduce sequence of finer meshes each of intensity $n \in \mathbb{N}$ such that: $0 =: t_0^n < t_1^n <$

$\ldots < t_n^n = T$. Now given that the value of underlying security at time 0 is x, i.e., $S_0 = x$. Now let $c(0,x)$ be the price we get from BSM for a call option expiring at time T and strike price K. Initial portfolio value $P_0 = c(0,x)$, then we hedge according to the discretized SDE for our portfolio based SDE,

$$P_{t_{i+1}^n} = P_{t_i^n} + \Delta_{t_i^n}\big(S_{t_{i+1}^n} - S_{t_i^n}\big) + r(P_{t_i^n} - \Delta_{t_i^n} S_{t_i^n})(t_{i+1}^n - t_i^n), \tag{231}$$

for $i \in [n-1]$. Define the residual error in value of call option at time t_{i+1}^n and stock price $S_{t_{i+1}^n}$ as:

$e(t_{i+1}^n, S_{t_{i+1}^n}) := c(t_{i+1}^n, S_{t_{i+1}^n}) - P_{t_{i+1}^n}$. *implies that*
$e^2(t_{i+1}^n, S_{t_{i+1}^n}) = c^2(t_{i+1}^n, S_{t_{i+1}^n}) + P_{t_{i+1}^n}^2 - 2\big\langle c(t_{i+1}^n, S_{t_{i+1}^n}), P_{t_{i+1}^n}\big\rangle$
$e^2(t_{i+1}^n, S_{t_{i+1}^n}) = c^2(t_{i+1}^n, S_{t_{i+1}^n}) + P_{t_{i+1}^n}^2 - 2\big\langle c(t_{i+1}^n, S_{t_{i+1}^n}), P_{t_i^n} + \Delta_{t_i^n}\big(S_{t_{i+1}^n} - S_{t_i^n}\big) + r(P_{t_i^n} - \Delta_{t_i^n} S_{t_i^n})(t_{i+1}^n - t_i^n)\big\rangle$
$e^2(t_1^n, S_{t_1^n}) = c^2(t_1^n, S_{t_1^n}) + P_{t_1^n}^2 - 2\big\langle c(t_1^n, S_{t_1^n}), P_{t_0^n} + \Delta_{t_0^n}\big(S_{t_1^n} - S_{t_0^n}\big) + r(P_{t_0^n} - \Delta_{t_0^n} S_{t_0^n})(t_1^n - t_0^n)\big\rangle$

$$\begin{aligned} e^2(t_1^n, S_{t_1^n}) = {} & c^2(t_1^n, S_{t_1^n}) + P_{t_1^n}^2 - 2[1 + r(t_1^n - t_0)]\langle c(t_1^n, S_{t_1^n}), c(0, S_0)\rangle \\ & - 2\Delta_{t_0^n}\langle c(t_1^n, S_{t_1^n}), (S_{t_1^n} - S_0) - rS_0(t_1^n - t_0)\rangle \end{aligned}$$

So essentially minimizing least square error in atleast for the first iteration would require maximizing in some sense of then parameter

$$\mathbb{E}\big[\Delta_{t_0^n}\langle c(t_1^n, S_{t_1^n}), (S_{t_1^n} - S_0) - rS_0(t_1^n - t_0)\rangle | S_0\big], \tag{232}$$

where $\Delta_{t_0} \in \mathcal{F}_0$ To be hands on:

$e(t_1^n, S_{t_1^n}) = c(t_1^n, S_{t_1^n}) - P_{t_1^n} = c(t_1^n, S_{t_1^n}) - c(0, S_0) + \Delta_{t_0^n}\big(S_{t_1}^n - S_0\big) + r\big(c(0, S_0) - \Delta_{t_0^n} S_0\big)t_1^n.$

as $t_0^n = 0$. Let:

$$\hat{\Delta}_{t_0^n} = arg \min_{\Delta_{t_0^n} \in \mathcal{F}_{t_0^n}} \mathbb{E}\big[e^2(t_1^n, S_{t_1^n}) | S_{t_0^n}\big] \tag{233}$$

Recall that for some $t_{[0,1]^n} \in (0, t_1^n)$

$$c(t_1^n, S_{t_1^n}) = c(0, S_{t_1^n}) + c_t(t_{[0,1]^n}, S_{t_1^n})t_1^n. \tag{234}$$

maximizing in some sense of then parameter

$$\mathbb{E}\big[\Delta_{t_0^n}\langle c(0, S_{t_1^n}) + c_t(t_{[0,1]^n}, S_{t_1^n})t_1^n, (S_{t_1^n} - S_0) - rS_0(t_1^n - t_0)\rangle | S_0\big], \tag{235}$$

Therefore,

$e(t_1^n, S_{t_1^n}) = c(0, S_{t_1^n}) - c(0, S_0) + c_t(t_{[0,1]^n}, S_{t_1^n})t_1^n + \Delta_{t_0^n}\big(S_{t_1}^n - S_0\big) + r\big(c(0, S_0) - \Delta_{t_0^n} S_0\big)t_1^n.$
$e^2(t_1^n, S_{t_1^n}) =$

Now from GBM definition:

$$S_{t_{i+1}^n} = S_{t_i^n} e^{(\alpha - \frac{\sigma^2}{2})[t_{i+1}^n - t_i^n] + \sigma\sqrt{t_{i+1}^n - t_i^n} Z_i}, \tag{236}$$

for $z_i \sim N(0,1)$. Now define $PL(n)$ as the profit loss:

$$PL(n) := P_{t_n^n} - (S_T - K)^+. \tag{237}$$

Pretty obvious that $PL(n)$ is sensitive to the hedging strategy. Replicating the pay-off for a call option requires hedging

$$\Delta_{t_i^n} = c_x(t_i^n, S_{t_i^n}) = N\big(d_+(T - t_i^n, S_{t_i^n})\big) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{1}{\sigma\sqrt{T-t_i^n}}\left[\log\left(\frac{S_{t_i^n}}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T - t_i^n)\right]} e^{-\frac{y^2}{2}}\, dy.$$

Notice that:

$$\begin{aligned}
P_{t_n^n} &= P_{t_0^n} + \sum_{i=0}^{n-1} \left(P_{t_{i+1}^n} - P_{t_i^n}\right) \\
&= P_{t_0^n} + \sum_{i=0}^{n-1} \Delta_{t_i^n}\left(S_{t_{i+1}^n} - S_{t_i^n}\right) + r(P_{t_i^n} - \Delta_{t_i^n} S_{t_i^n})(t_{i+1}^n - t_i^n)
\end{aligned}$$

On the other hand the Geometric brownian motion nature implies

$$S_{t_{i+1}^n} = S_{t_i^n} e^{\left(r - \frac{\sigma^2}{2}\right)(t_{i+1}^n - t_i^n)} e^{\sigma\sqrt{t_{i+1}^n - t_i^n} Z_i}, \tag{238}$$

where for each $i \in [0, \ldots, n-1]$, $Z_i \sim N(0,1)$ iid and also independent of $S_{t_i^n}$. This means that:

$$P_{t_n^n} = P_{t_0^n} + \sum_{i=0}^{n-1} \Delta_{t_i^n} S_{t_i^n}\left[e^{\left(r - \frac{\sigma^2}{2}\right)(t_{i+1}^n - t_i^n)} e^{\sigma\sqrt{t_{i+1}^n - t_i^n} Z_i} - 1\right] + r(P_{t_i^n} - \Delta_{t_i^n} S_{t_i^n})(t_{i+1}^n - t_i^n) \tag{239}$$

25. Logarithmic returns

Recall again that if we take time increment equal to one, then GBM implies:

$$S_{i+1} = S_i e^{(\alpha - \frac{\sigma^2}{2}) + \sigma Z_i}, \tag{240}$$

for $z_i \sim N(0,1)$ independent of S_i. Now let us define *log returns*

$$lR_i := \log\left(\frac{S_{i+1}}{S_i}\right) \sim N\Big(\alpha - \frac{\sigma^2}{2}, \sigma^2\Big). \tag{241}$$

Therefore, log returns at each time instant i are independent of each other and follows preceding normal distribution.

Example 6 *(Heston stochastic volatility model): Suppose that under the risk neutral measure*

$$\begin{aligned}
&dS_t = rS_t dt + \sqrt{V_t} S_t dB_t^1 \\
&dV_t = (a - bV_t)dt + \sigma\sqrt{V_t} dB_t^2 \\
&dB_t^2 dB_t^1 = \rho dt, \;\; \rho \in (-1, 1). \\
&c(t, S_t, V_t) := \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}(S_T - K)^+ | \mathcal{F}_t]
\end{aligned}$$

$$D_2 f(x) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \quad \sigma_t = \begin{bmatrix} \sigma_t^{1,1} & \sigma_t^{1,2} & \dots & \sigma_t^{1,m} \\ \sigma_t^{2,1} & \sigma_t^{2,2} & \dots & \sigma_t^{2,m} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_t^{d,1} & \sigma_t^{d,2} & \dots & \sigma_t^{d,m} \end{bmatrix}, \ \mu_{t,X_t} := \begin{bmatrix} rS_t \\ (a - bV_t) \end{bmatrix}$$

1. *$e^{-rt}c(t, S_t, V_t)$ is a martingale under the risk neutral measure $\mathbb{Q}$. First i need to ascertain the Markovian nature of things.*

$$\begin{aligned} &S_{t+\delta t} - S_t = rS_t\delta t + \sqrt{V_t}S_t(B^1_{t+\delta t} - B^1_t) \\ &V_{t+\delta t} - V_t = (a - bV_t)\delta t + \sigma\sqrt{V_t}(B^2_{t+\delta t} - B^2_t). \implies \\ &S_{t+\delta t} - S_t = rS_t\delta t + \sqrt{V_{t-\delta t} + (a - bV_{t-\delta t})\delta t + \sigma\sqrt{V_{t-\delta t}}(B^2_{t+\delta t} - B^2_{t-\delta t})}S_t(B^1_{t+\delta t} - B^1_t) \end{aligned}$$

First start with Markovian nature of GBM based stock prices,

$$dS_t = rS_t dt + \sigma S_t dB_t$$

Let $f(t, S_t) := e^{-rt}S_t, \partial_t f(t, S_t) = -rf(t, S_t), \nabla f(t, S_t) = e^{-rt}$. Therefore:

$$\begin{aligned} e^{-rt}S_t &= S_0 + \int_0^t \{e^{-rs}rS_s - re^{-rs}S_s\}ds + \int_0^t e^{-rs}\sigma \\ \implies S_t &= e^{rt}S_0 + \sigma\int_0^t e^{r(t-s)}S_s dW_s. \end{aligned}$$

well that was meaningless instead try with $g(s) = \log(s), \nabla g(s) = \frac{1}{s}, D^2 g(s) = -\frac{1}{s^2}$ and consequently:

$$\begin{aligned} \log S_t &= \log S_0 + \int_0^t \{r - \frac{1}{2S^2}\sigma^2 S_s^2\}ds + \int_0^t \sigma dB_s \\ \implies \log S_t &= \log S_0 + (r - \frac{\sigma^2}{2})t + \sigma(B_t - B_0). \\ S_t &= S_0 e^{(r - \frac{\sigma^2}{2})t + \sigma(B_t - B_0)} \end{aligned}$$

More specifically for any $T > t$:

$$S_T = S_t e^{(r - \frac{\sigma^2}{2}[T-t]) + \sigma(B_T - B_t)}, \tag{242}$$

now the Markovian nature is obvious.

$$\begin{aligned} &dB^1_t dB^2_t = \rho_t dt \\ &dW^1_t = dB^1_t \\ &dW^2_t = -\frac{\rho_t}{\sqrt{1-\rho_t^2}}dB^1_t + \frac{1}{\sqrt{1-\rho_t^2}}dB^2_t \\ \implies &dW^1_t dW^2_t = -\frac{\rho_t}{\sqrt{1-\rho_t^2}}dt + \frac{\rho_t}{\sqrt{1-\rho_t^2}}dt = 0. \ \textit{independence verified!} \end{aligned}$$

A general principle Ito's formula for correlated Brownian motions, using the usual mesh based arguments. For all $n \in \mathbb{N}$, $t_0^n := t < t_1^n < \ldots < t_n^n := T$ and $f : \mathbb{R}^2 \to \mathbb{R}$

$$W_T - W_t = \sum_{i=0}^{n-1} (W_{t_{i+1}^n} - W_{t_i^n}). \;\; X_t := (S_t, V_t)$$
$$S_{t_{i+1}^n} - S_{t_i^n} = rS_{t_i^n}(t_{i+1}^n - t_i^n) + \sqrt{V_{t_i^n}} S_{t_i^n}(B^1_{t_{i+1}^n} - B^1_{t_i^n}).$$
$$V_{t_{i+1}^n} - V_{t_i^n} = [a - bV_{t_i^n}](t_{i+1}^n - t_i^n) + \sigma\sqrt{V_{t_i^n}} S_{t_i^n}(B^2_{t_{i+1}^n} - B^2_{t_i^n}).$$
$$f(S_T, V_T) = f(S_t, V_t) + \sum_{i=0}^{n-1} [f(S_{t_{i+1}^n}, V_{t_{i+1}^n}) - f(S_{t_i^n}, V_{t_i^n})]$$
$$f(S_{t_{i+1}^n}, V_{t_{i+1}^n}) - f(S_{t_i^n}, V_{t_i^n}) = \langle \nabla f(X_{t_i^n}), (X_{t_{i+1}^n} - X_{t_i^n}) \rangle$$
$$+ \frac{1}{2} \langle (X_{t_{i+1}^n} - X_{t_i^n}), D^2 f(\theta_i^n)(X_{t_{i+1}^n} - X_{t_i^n}) \rangle. \; \textit{for some } \theta_i^n \in [X_{t_i^n}, X_{t_{i+1}^n}].$$
$$f(S_T, V_T) = f(S_t, V_t) + \sum_{i=0}^{n-1}$$
$$\left[\partial_s f(S_{t_i^n}) \big(S_{t_{i+1}^n} - S_{t_i^n} \big) + \partial_v f(V_{t_i^n}) \big(V_{t_{i+1}^n} - V_{t_i^n} \big) + \frac{1}{2} \langle (X_{t_{i+1}^n} - X_{t_i^n}), D^2 f(\theta_i^n)(X_{t_{i+1}^n} - X_{t_i^n}) \rangle \right].$$
$$\partial_s f(X_{t_i^n}) \big(S_{t_{i+1}^n} - S_{t_i^n} \big) + \partial_v f(X_{t_i^n}) \big(V_{t_{i+1}^n} - V_{t_i^n} \big) + \frac{1}{2} \langle (X_{t_{i+1}^n} - X_{t_i^n}), D^2 f(\theta_i^n)(X_{t_{i+1}^n} - X_{t_i^n}) \rangle =$$
$$\partial_s f(X_{t_i^n}) \Big[rS_{t_i^n}(t_{i+1}^n - t_i^n) + \sqrt{V_{t_i^n}} S_{t_i^n}(B^1_{t_{i+1}^n} - B^1_{t_i^n}) \Big] + \partial_v f(X_{t_i^n}) \Big[[a - bV_{t_i^n}](t_{i+1}^n - t_i^n) + \sigma\sqrt{V_{t_i^n}} S_{t_i^n}(B^2_{t_{i+1}^n} - B^2_{t_i^n})$$
$$+ \frac{1}{2} \langle (X_{t_{i+1}^n} - X_{t_i^n}), D^2 f(\theta_i^n)(X_{t_{i+1}^n} - X_{t_i^n}) \rangle.$$

If $X_t = (S_t, V_t)$ satsifies

$$dX_t = \mu_{t,X_t} dt + \sigma_{t,X_t} dB_t, \tag{243}$$

where recall that $B_t = (B_t^1, B_t^2)$ is a Brownian motion with $dB_t^1 dB_t^2 = \rho dt$. Then it seems like at least the martingale term satisfy

$$df(X_t) = \ldots dt + \big\langle \nabla f(X_t), \sigma_{t,X_t} dB_t \big\rangle. \tag{244}$$

So the most interesting part is $\frac{1}{2} \langle (X_{t_{i+1}^n} - X_{t_i^n}), D^2 f(\theta_i^n)(X_{t_{i+1}^n} - X_{t_i^n}) \rangle$

$$D_2 f(\theta_i^n) = \begin{bmatrix} \partial_{ss} f(\theta_i^n) & \partial_{sv} f(\theta_i^n) \\ \partial_{sv} f(\theta_i^n) & \partial_{vv} f(\theta_i^n) \end{bmatrix} \;\; D_2 f(\theta_i^n)(X_{t_{i+1}^n} - X_{t_i^n}) = \begin{bmatrix} \partial_{ss} f(\theta_i^n)(S_{t_{i+1}^n} - S_{t_i^n}) + \partial_{sv} f(\theta_i^n)(V_{t_{i+1}^n} - V_{t_i^n}) \\ \partial_{sv} f(\theta_i^n)(S_{t_{i+1}^n} - S_{t_i^n}) + \partial_{vv} f(\theta_i^n)(V_{t_{i+1}^n} - V_{t_i^n}) \end{bmatrix}$$

$$\langle (X_{t_{i+1}^n} - X_{t_i^n}), D^2 f(\theta_i^n)(X_{t_{i+1}^n} - X_{t_i^n})\rangle = (S_{t_{i+1}^n} - S_{t_i^n})\Big[\partial_{ss} f(\theta_i^n)(S_{t_{i+1}^n} - S_{t_i^n}) + \partial_{sv} f(\theta_i^n)(V_{t_{i+1}^n} - V_{t_i^n})\Big]$$
$$+ (V_{t_{i+1}^n} - V_{t_i^n})\Big[\partial_{sv} f(\theta_i^n)(S_{t_{i+1}^n} - S_{t_i^n}) + \partial_{vv} f(\theta_i^n)(V_{t_{i+1}^n} - V_{t_i^n})\Big].$$
$$\Big[S_{t_{i+1}^n} - S_{t_i^n}\Big]\partial_{sv} f(\theta_i^n)\Big[V_{t_{i+1}^n} - V_{t_i^n}\Big] =$$
$$\Big[rS_{t_i^n}(t_{i+1}^n - t_i^n) + \sqrt{V_{t_i^n}}S_{t_i^n}(B^1_{t_{i+1}^n} - B^1_{t_i^n})\Big]\partial_{sv} f(\theta_i^n)\Big[[a - bV_{t_i^n}](t_{i+1}^n - t_i^n) + \sigma\sqrt{V_{t_i^n}}(B^2_{t_{i+1}^n} - B^2_{t_i^n})\Big]$$
$$\overset{L^2}{=}\sqrt{V_{t_i^n}}S_{t_i^n}(B^1_{t_{i+1}^n} - B^1_{t_i^n})\partial_{sv} f(\theta_i^n)\sigma\sqrt{V_{t_i^n}}(B^2_{t_{i+1}^n} - B^2_{t_i^n}) \overset{L^2}{=}\sigma\rho V_{t_i^n}S_{t_i^n}\partial_{sv} f(\theta_i^n)(t_{i+1}^n - t_i^n).$$
$$\Big[rS_{t_i^n}(t_{i+1}^n - t_i^n) + \sqrt{V_{t_i^n}}S_{t_i^n}(B^1_{t_{i+1}^n} - B^1_{t_i^n})\Big]\partial_{ss} f(\theta_i^n)\Big[rS_{t_i^n}(t_{i+1}^n - t_i^n) + \sqrt{V_{t_i^n}}S_{t_i^n}(B^1_{t_{i+1}^n} - B^1_{t_i^n})\Big]$$
$$\overset{L^2}{=}V_{t_i^n}S^2_{t_i^n}\partial_{ss} f(\theta_i^n)(t_{i+1}^n - t_i^n)$$
$$\Big[V_{t_{i+1}^n} - V_{t_i^n}\Big]\partial_{vv} f(\theta_i^n)\Big[V_{t_{i+1}^n} - V_{t_i^n}\Big]$$
$$= \Big[[a - bV_{t_i^n}](t_{i+1}^n - t_i^n) + \sigma\sqrt{V_{t_i^n}}S_{t_i^n}(B^2_{t_{i+1}^n} - B^2_{t_i^n})\Big]\partial_{vv} f(\theta_i^n)\Big[[a - bV_{t_i^n}](t_{i+1}^n - t_i^n) + \sigma\sqrt{V_{t_i^n}}(B^2_{t_{i+1}^n} - B^2_{t_i^n})\Big]$$
$\overset{L^2}{=} \sigma^2 V_{t_i^n}\partial_{vv} f(\theta_i^n)(t_{i+1}^n - t_i^n)$. *Consequently* $\frac{1}{2}\langle (X_{t_{i+1}^n} - X_{t_i^n}), D^2 f(\theta_i^n)(X_{t_{i+1}^n} - X_{t_i^n})\rangle \overset{L^2}{=}$
$\Big[\sigma\rho V_{t_i^n}S^2_{t_i^n}\partial_{sv} f(\theta_i^n) + \frac{1}{2}V_{t_i^n}S^2_{t_i^n}\partial_{ss} f(\theta_i^n) + \frac{1}{2}\sigma^2 V_{t_i^n}S^2_{t_i^n}\partial_{vv} f(\theta_i^n)\Big](t_{i+1}^n - t_i^n)$. *Hence* $df(S_t, V_t) =$
$$\Bigg\{\sigma\rho V_t S_t\partial_{sv} f(X_t) + \frac{1}{2}\underbrace{\big[V_t S_t^2\partial_{ss} f(X_t) + \sigma^2 V_t\partial_{vv} f(X_t)\big]}_{=Tr(\sigma^T_{t,X_t}D^2 f(X_t)\sigma_{t,X_t})} + \big\langle\nabla f(X_t), \mu_{t,X_t}\big\rangle\Bigg\}dt + \big\langle\nabla f(X_t), \sigma_{t,X_t}dB_t\big\rangle.$$

single concise ito formula for correlated Brownian motions? Now we will show that $e^{-rt}c(t, S_t, V_t)$ is a Martingale, where $c(t, S_t, V_t) := \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}(S_T - K)^+|\mathcal{F}_t]$. By Ito's lemma:

$$d\big(e^{-rt}c(t, S_t, V_t)\big) =$$
$$\Bigg\{e^{-rt}\Big(-rc_t(t, S_t, V_t) + \sigma\rho V_t S_t^2\partial_{sv}c(t, S_t, V_t) + \frac{1}{2}\big[V_t S_t^2\partial_{ss}c(t, S_t, V_t) + \sigma^2 V_t S_t^2\partial_{vv}c(t, S_t, V_t)\big] +$$
$$\big\langle\nabla c(t, S_t, V_t), \mu_{t,X_t}\big\rangle\Big)\Bigg\}dt + e^{-rt}\big\langle\nabla c(t, S_t, V_t), \sigma_{t,X_t}dB_t\big\rangle.$$

But notice that $e^{-rt}c(t, S_t, V_t) := \mathbb{E}_{\mathbb{Q}}[e^{-rT}(S_T - K)^+|\mathcal{F}_t]$ is a matringale by iterative conditioning. Therefore,

$$c_t(t, S_t, V_t) - rc(t, S_t, V_t) + \sigma\rho V_t S_t^2\partial_{sv}c(t, S_t, V_t) + \frac{1}{2}\big[V_t S_t^2\partial_{ss}c(t, S_t, V_t) + \sigma^2 V_t S_t^2\partial_{vv}c(t, S_t, V_t)\big]$$
$$+ rS_t\partial_s c(t, S_t, V_t) + (a - bV_t)\partial_v c(t, S_t, V_t) = 0. \quad \textit{Concise notation} \tag{245}$$

Theorem 55 *(Call price based on Heston stochastic volatility model). Given model:*

$$
\begin{aligned}
&dS_t = rS_t dt + \sqrt{V_t} S_t dB_t^1 \\
&dV_t = (a - bV_t)dt + \sigma\sqrt{V_t} dB_t^2 \\
&dB_t^2 dB_t^1 = \rho dt, \ \ \rho \in (-1, 1). \\
&c(t, S_t, V_t) := \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}(S_T - K)^+ | \mathcal{F}_t],
\end{aligned}
$$

Price of the call option satisfies:

$$c_t + rsc_s + (a - bv)c_v + \sigma\rho vsc_{sv} + \frac{1}{2}(vs^2 c_{ss} + \sigma^2 vc_{vv}) = rc. \tag{246}$$

Proposition 56 *Suppose that $f(t, x, v)$ and $g(t, x, v)$ satisfy the following conditions:*

$$
\begin{aligned}
&f_t + (r + \frac{1}{2}v)f_x + (a - bv + \rho\sigma v)f_v + \frac{1}{2}vf_{xx} + \rho\sigma vf_{xv} + \frac{1}{2}\sigma^2 vf_{vv} = 0, \\
&g_t + (r - \frac{1}{2}v)g_x + (a - bv)g_v + \frac{1}{2}vg_{xx} + \rho\sigma vg_{xv} + \frac{1}{2}\sigma^2 vg_{vv} = 0,
\end{aligned}
$$

in the region $t \in [0, T], x \in (-\infty, \infty)$ and $v \geq 0$. Now define:

$$c(t, s, v) := sf(t, \log s, v) - e^{-r(T-t)}Kg(t, \log s, v). \tag{247}$$

Then notice that:

$$
\begin{aligned}
&c_t = sf_t(t, \log s, v) - re^{-r(T-t)}Kg(t, \log s, v) - e^{-r(T-t)}Kg_t(t, \log s, v) \\
&c_s = f(t, \log s, v) + sf_s(t, \log s, v)\frac{1}{s} - e^{-r(T-t)}Kg_s(t, \log s, v)\frac{1}{s}
\end{aligned}
$$

and so on now one can verify remaining stuff easily.

Suppose the pair (X_t, V_t) is governed by the following stochastic process

$$
\begin{aligned}
&dX_t = (r + \frac{1}{2}V_t)dt + \sqrt{V_t}dW_t^1, \\
&dV_t = (a - bV_t = \rho\sigma V_t)dt + \sigma\sqrt{V_t}dW_t^2 \\
&dW_t^1 dW_t^2 = \rho dt. \\
&f(t, X_t, V_t) := \mathbb{E}\big[1_{X_T \geq \log K} | \mathcal{F}_t\big] = \mathbb{P}\big[X_T > \log K | (X_t, V_t)\big]
\end{aligned}
$$

As $f(t, X_t, V_t)$ is a martingale, Ito type argument:

$$f_t + \frac{1}{2}vf_{xx} + \frac{1}{2}\sigma^2 vf_{vv} + \sigma v\rho f_{xv} + (r + \frac{1}{2}v)f_x + (a - bv + \rho\sigma v)f_v = 0 \tag{248}$$

Theorem 57 ***Kolmogorovs backward equation****: consider the SDE*

$$dX_u = \beta(u, X_u)du + \gamma(u, X_u)dW_u \tag{249}$$

assuming that underlying drift and volatility profiles are chosen as to ensure, $X_t \geq 0$. Assuming Markovian SDE and density absolutely continuous wrt Lebesgue measure. The probability density function $P(X_T | X_t = x) = p(t, T, x, \cdot)$ satisfies(Kolmos back eqn):

$$\beta(t, x)p_x(t, T, x, y) + \frac{\sigma^2(t, x)p_{xx}(t, T, x, y)}{2} = -p_t(t, T, x, y) \tag{250}$$

Proof For an arbitrary smooth function $f = f(X_t)$, if we let $g(t, X_t) := \mathbb{E}[f(X_T)|\mathcal{F}_t] = \mathbb{E}[f(X_T)|X_t] = \int_0^\infty f(y)p(t,T,x,y)dy$ This implies that $g(t, X_t)$ is a martingale so ito implies:

$$g_t(t,x) + g_x(t,x)\beta(t,x) + \frac{1}{2}g_{xx}(t,x)\sigma^2(t,x) = 0. \tag{251}$$

and consequently:

$$\int_0^\infty f(y)p_t(t,T,x,y)dy + \int_0^\infty f(y)\beta(t,x)p_x(t,T,x,y)dy + \int_0^\infty f(y)\frac{\sigma^2(t,x)p_{xx}(t,T,x,y)}{2}dy = 0,$$

result follows. ■

Theorem 58 ***Kolmogorovs forward / Fokker-Planck equation***

Given SDE, $dX_u = \beta(u, X_u)du + \gamma(u, X_u)dW_u$. *Transition probability satisfies:*
$\frac{\partial}{\partial T}p(t,T,x,y) = -\frac{\partial}{\partial y}\big(\beta(t,y)p(t,T,x,y)\big) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\big(\gamma^2(T,y)p(t,T,x,y)\big).$

In contrast to the Kolmogorov backward equation, in which T and y were held constant and the variables were t and x, here t and x are held constant and the variables are y and T. The variables t and x are sometimes called the backward variables, and T and y are called the forward variables.

Proof Let b be a positive constant and let $h_b(y) \in C^2$ and $h_b(x) = 0$ for all $x \leq 0$, $h_b^{'}(x) = 0$ for all $x \geq b$, and $h_b(b) = h_b^{'}(b) = 0$. Let X_u be the solution of SDE with initial condition $X_t = x \in (0, b)$, using the Ito's formula we get for any $u > t$,

$$h_b(X_T) - h_b(X_t) = \int_t^T \Big[h_b^{'}(X_s)\beta(s,X_s) + \frac{1}{2}\gamma^2(s,X_s)h_b^{''}(X_s)\Big]ds + \int_t^T h_b^{'}(X_s)\gamma(s,X_s)dW_s.$$

$$\mathbb{E}^{t,x}\Big[h_b(X_T) - h_b(X_t)\Big] = \int_0^b h_b(y)p(t,T,x,y)dy - h_b(x) = \mathbb{E}\Bigg[\int_t^T \Big[h_b^{'}(X_s)\beta(s,X_s) + \frac{1}{2}\gamma^2(s,X_s)h_b^{''}(X_s)\Big]ds|X_t$$

$$\int_0^b h_b(y)p(t,T,x,y)dy - h_b(x) = \int_t^T \mathbb{E}\Big[h_b^{'}(X_s)\beta(s,X_s) + \frac{1}{2}\gamma^2(s,X_s)h_b^{''}(X_s)\Big|X_t = x\Big]ds$$

$$\int_t^T \int_0^b \Big[h_b^{'}(y)\beta(s,y) + \frac{1}{2}\gamma^2(s,y)h_b^{''}(y)\Big]p(t,s,x,y)dyds \text{ implies that}$$

$$\int_0^b h_b(y)p(t,T,x,y)dy - h_b(x) = \int_0^b \Bigg(\int_t^T \Big[h_b^{'}(y)\beta(s,y) + \frac{1}{2}\gamma^2(s,y)h_b^{''}(y)\Big]p(t,s,x,y)ds\Bigg)dy.$$

RHS suggest some sort of integration by parts(boundary conditions 0)

$$\int_0^b \Big[h_b^{'}(y)\beta(s,y) + \frac{1}{2}\gamma^2(s,y)h_b^{''}(y)\Big]p(t,s,x,y)dy \tag{252}$$

First notice that:

$$\int_0^b \frac{1}{2}\gamma^2(s,y)h_b^{''}(y)p(t,s,x,y)dy = -\frac{1}{2}\int_0^b h_b^{'}(y)\frac{\partial}{\partial y}\big(\gamma^2(s,y)p(t,s,x,y)\big)dy$$
$$= \frac{1}{2}\int_0^b h_b(y)\frac{\partial^2}{\partial y^2}\big(\gamma^2(s,y)p(t,s,x,y)\big)dy,$$

and similarly:

$$\int_0^b h_b^{'}(y)\beta(s,y)p(t,s,x,y)dy = -\int_0^b h_b(y)\frac{\partial}{\partial y}\big(\beta(s,y)p(t,s,x,y)\big)dy. \tag{253}$$

implying that :

$$\int_0^b \Big[h_b^{'}(y)\beta(s,y) + \frac{1}{2}\gamma^2(s,y)h_b^{''}(y)\Big]p(t,s,x,y)dy$$
$$= \int_0^b h_b(y)\left[\frac{1}{2}\frac{\partial^2}{\partial y^2}\big(\gamma^2(s,y)p(t,s,x,y)\big) - \frac{\partial}{\partial y}\big(\beta(s,y)p(t,s,x,y)\big)\right]dy.$$

Combining all the equations we have now:

$$\int_t^T \left(\int_0^b h_b(y)\Big[\frac{1}{2}\frac{\partial^2}{\partial y^2}\big(\gamma^2(s,y)p(t,s,x,y)\big) - \frac{\partial}{\partial y}\big(\beta(s,y)p(t,s,x,y)\big)\Big]dy\right)ds$$
$$= \int_0^b h_b(y)p(t,T,x,y)dy - h_b(x).\ \textit{taking partial derivative wrt } T$$
$$\int_0^b h_b(y)\frac{\partial}{\partial T}p(t,T,x,y)dy = \int_0^b h_b(y)\Big[\frac{1}{2}\frac{\partial^2}{\partial y^2}\big(\gamma^2(T,y)p(t,T,x,y)\big) - \frac{\partial}{\partial y}\big(\beta(T,y)p(t,T,x,y)\big)\Big]dy$$
$$\frac{\partial}{\partial T}p(t,T,x,y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}\big(\gamma^2(T,y)p(t,T,x,y)\big) - \frac{\partial}{\partial y}\big(\beta(T,y)p(t,T,x,y)\big).$$

■

26. The Volatility Surface

Example 7 ***Implying the volatility surface*** *Assume that a stock price evolves according to the following stochastic differential equation*

$$dS_u = rS_u du + \sigma(u,S_u)S_u dB_u, \tag{254}$$

where $\sigma(u,x)$ is a function of time and the underlying stock price. B_u is the Brownian motion under the risk neutral measure $\mathbb{Q}$. This is a special case of the stochastic differential equation from the previous two preceding topics, as here $\beta(u,x) = rx$ and $\gamma(u,x) = \sigma(u,x)x$. $\hat{p}(t,T,x,y)$ is used to denote the transition density. As,

$$\frac{\partial}{\partial T}\hat{p}(t,T,x,y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}\big(\gamma^2(T,y)\hat{p}(t,T,x,y)\big) - \frac{\partial}{\partial y}\big(\beta(T,y)\hat{p}(t,T,x,y)\big).$$

Substituting the particulars for this problem give us

$$\frac{\partial}{\partial T}\hat{p}(t,T,x,y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}\big(\sigma^2(T,y)y^2\hat{p}(t,T,x,y)\big) - r\frac{\partial}{\partial y}\big(y\hat{p}(t,T,x,y)\big).$$

Final form:

$$c_T(0,T,x,K) = -rKc_K(0,T,x,K) + \frac{1}{2}\sigma^2(T,K)K^2c_{KK}(0,T,x,K). \tag{255}$$

The idea is $c_T(0,T,x,K), c_K(0,T,x,K)$ and $c_{KK}(0,T,x,K)$ can be inferred from the market price and as long as $c_{KK}(0,T,x,K) \neq 0$ we can solve the preceding equation for $\sigma(T,K)$: the implied volatility.

Let,

$$c(0,T,x,K) := e^{-rT}\int_K^\infty (y-K)\hat{p}(0,T,x,y)dy \tag{256}$$

be the price of a call option at time 0 expiring at time T with strike price K when initial stock price $S_0 = x$.

So how does the price of call option vary with change in expiration time, to see that:

$$c_T(0,T,x,K) = -rc(0,T,x,K) + e^{-rT}\int_K^\infty (y-K)\frac{\partial}{\partial T}\hat{p}(0,T,x,y)dy$$

$$e^{-rT}\int_K^\infty (y-K)\frac{\partial}{\partial T}\hat{p}(0,T,x,y)dy = e^{-rT}\int_K^\infty (y-K)\Big[\frac{1}{2}\frac{\partial^2}{\partial y^2}\big(\sigma^2(T,y)y^2\hat{p}(0,T,x,y)\big) - r\frac{\partial}{\partial y}\big(y\hat{p}(0,T,x,y)\big)\Big]dy$$

Try your luck with integration by parts:

$$-re^{-rT}\int_K^\infty (y-K)\frac{\partial}{\partial y}\big(y\hat{p}(0,T,x,y)\big)dy = re^{-rT}\int_K^\infty y\hat{p}(0,T,x,y)dy \tag{257}$$

and the second term is essentially:

$$\begin{aligned}\frac{1}{2}e^{-rT}\int_K^\infty (y-K)\frac{\partial^2}{\partial y^2}\big(\sigma^2(T,y)y^2\hat{p}(0,T,x,y)\big)dy &= -\frac{1}{2}e^{-rT}\int_K^\infty \frac{\partial}{\partial y}\big(\sigma^2(T,y)y^2\hat{p}(0,T,x,y)\big)dy\\ &= -\frac{1}{2}e^{-rT}\sigma^2(T.K)K^2\hat{p}(0,T,x,K).\end{aligned}$$

and eventually we get

$$\begin{aligned}c_T(0,T,x,K) &= -rc(0,T,x,K) + re^{-rT}\int_K^\infty y\hat{p}(0,T,x,y)dy - \frac{1}{2}e^{-rT}\sigma^2(T,K)K^2\hat{p}(0,T,x,K)\\ &= rKe^{-rT}\int_K^\infty \hat{p}(0,T,x,y)dy - \frac{1}{2}e^{-rT}\sigma^2(T,K)K^2\hat{p}(0,T,x,K)\end{aligned}$$

Not difficult to notice that:

$$\begin{aligned}c_K(0,T,x,K) &= -e^{-rT}K\hat{p}(0,T,x,K) - e^{-rT}\partial_K\Big[K\int_K^\infty \hat{p}(0,T,x,y)dy\Big]\\ &= -e^{-rT}K\hat{p}(0,T,x,K) - e^{-rT}\int_K^\infty \hat{p}(0,T,x,y)dy + Ke^{-rT}\hat{p}(0,T,x,K).\end{aligned}$$

Hence

$$c_K(0,T,x,K) = -e^{-rT}\int_K^\infty \hat{p}(0,T,x,y)dy \tag{258}$$

and second partial

$$c_{KK}(0,T,x,K) = e^{-rT}\hat{p}(0,T,x,K). \tag{259}$$

Everything combined now

Theorem 59 (Implied volatility) *Assume that a stock price evolves according to the following stochastic differential equation*

$$dS_u = rS_u du + \sigma(u,S_u)S_u dB_u, \tag{260}$$

where $\sigma(u,x)$ is a function of time and the underlying stock price. B_u is the Brownian motion under the risk neutral measure $\mathbb{Q}$.

$$c_T(0,T,x,K) = -rKc_K(0,T,x,K) + \frac{1}{2}\sigma^2(T,K)K^2 c_{KK}(0,T,x,K)$$

$$\sigma(T,K) = \sqrt{\frac{2\Big(c_T(0,T,x,K) + rKc_K(0,T,x,K)\Big)}{K^2 c_{KK}(0,T,x,K)}}.$$

So if BSM equation is correct then $\sigma(T,K) = \sigma$ regardless of the stock price and expiration date which from preceding equation will mean:

$$\sqrt{\frac{2\Big(c_T(0,T,x,K) + rKc_K(0,T,x,K)\Big)}{K^2 c_{KK}(0,T,x,K)}} \tag{261}$$

is a constant which does not really make sense as c_T should not be a constant from any sense.

Remark 60 *For stocks and stock indices the shape of the volatility surface is always changing. There is generally a skew, however, so that for any fixed maturity, T, the implied volatility decreases with the strike, K. It is most pronounced at shorter expirations*

Volatility of the basket:

$$\langle \sum_{i=1}^N w_i x_i, \sum_{j=1}^N w_j x_j \rangle = \sum_{i=1}^N w_i^2\sigma_i^2 + 2w_i\sigma_i \sum_{j>i}^N w_j\rho_{i,j}\sigma_j \tag{262}$$

σ_i is the standard deviation of the $i-th$ asset, $\rho_{i,j}$ is the correlation between the $i-th$ and $j-th$ asset.

$$dS_t^i = \sigma_i dW_t^i$$
$$dW_t^i dW_t^j = \rho_{i,j}dt.$$

Fair value at time t_1, of the call option of the basket expiring at t_2 is w.l.o.g assume(strike price is 0)

$$C(t_1, S_{t_1}) := \mathbb{E}\Bigg[\sum_{i=1}^N w_i S_{t_2}^i | S_{t_1}\Bigg]. \tag{263}$$

but how much does the call price change with expected

27. Financial terminologies

A *forward contract* with *delivery price* K obligates its holder to buy one share of stock at expiration time T in exchange for payment K.

$$f(T,S_T)=S_T-K,\implies f(t,S_t)=S_t-e^{-r(T-t)}K.$$
$$f(T,S_T)-c(T,S_T)+p(T,S_T)=0.$$

if for some $t<T$, portfolio which is long one forward, long one put and short one call is positive $f(t,S_t)-c(t,S_t)+p(t,S_t)>0$.

Hedging a long position in the call option: Hedging a long position for the call option, current stock price x_1 at time t_1, then we short $c_x(t_1,x_1)$ shares of the stock and buy one call option for $c(t_1,x_1)$, the remaining amount:

$$M:=x_1c_x(t,x_1)-c(t_1,x_1) \tag{264}$$

is invested in the money market. So overall portfolio is:

$$c(t_1,x_1)-x_1c_x(t_1,x_1)+M \tag{265}$$

now for infinitesimally small increase in time $t_2>t_1$

1. If the value of stock does not change remain at x_1 then our portfolio at t_2 with $S_{t_2}=x_1$ is:

$$c(t_2,x_1)-x_1c_x(t_1,x_1)+e^{r(t_2-t_1)}[x_1c_x(t_1,x_1)-c(t_1,x_1)]. \tag{266}$$

Now taylors expansion imply that for some time $\hat{t}\in(t_1,t_2)$

$$c(t_2,x_1)=c(t_1,x_1)+c_t(\hat{t},x_1)(t_2-t_1). \tag{267}$$

implying the value of our portfolio, is now

$$\begin{aligned}
&c(t_1,x_1)+c_t(\hat{t},x_1)(t_2-t_1)-x_1c_x(t_1,x_1)+e^{r(t_2-t_1)}[x_1c_x(t_1,x_1)-c(t_1,x_1)]\\
&=-c(t_1,x_1)[e^{r(t_2-t_1)}-1]+c_t(\hat{t},x_1)[t_2-t_1]+x_1c_x(t_1,x_1)[e^{r(t_2-t_1)}-1]\\
&\approx=-r(t_2-t_1)c(t_1,x_1)+c_t(\hat{t},x_1)[t_2-t_1]+r(t_2-t_1)x_1c_x(t_1,x_1)\\
&=(t_2-t_1)\Big(r\big[x_1c_x(t_1,x_1)-c(t_1,x_1)\big]+c_t(\hat{t},x_1)\Big)\ \textit{BSM PDE implies}\\
&=(t_2-t_1)\Big(c_t(\hat{t},x_1)-c_t(t_1,x_1)-\frac{1}{2}\sigma^2x_1^2c_{xx}(t_1,x_1)\Big)\\
&=(t_2-t_1)\Big(rKe^{-r(T-t_1)}N\big(d_-(T-t_1,x_1)\big)+c_t(\hat{t},x_1)\Big)\\
&=(t_2-t_1)\Big(rKe^{-r(T-t_1)}N\big(d_-(T-t_1,x_1)\big)+c_t(\hat{t},x_1)\Big)\\
&=e^{-rT}rK(t_2-t_1)\Big(e^{rt_1}N\big(d_-(T-t_1,x_1)\big)-e^{r\hat{t}}N\big(d_-(T-\hat{t},x_1)\big)-\frac{\sigma xe^{rT}}{2rK\sqrt{T-\hat{t}}}N'\left(d_+(T-\hat{t},x_1)\right)\Big)
\end{aligned}$$

because:

$$c_t(\hat{t},x_1)=-rKe^{-r(T-\hat{t})}N\big(d_-(T-\hat{t},x_1)\big)-\frac{\sigma x}{2\sqrt{T-\hat{t}}}N'\left(d_+(T-\hat{t},x_1)\right) \tag{268}$$

- Loss, when:

$$e^{rt_1}N\big(d_-(T-t_1,x_1)\big)-e^{r\hat{t}}N\big(d_-(T-\hat{t},x_1)\big)-\frac{\sigma x e^{rT}}{2rK\sqrt{T-\hat{t}}}N'\left(d_+(T-\hat{t},x_1)\right)\le 0 \iff$$
$$c_t(\hat{t},x_1)-c_t(t_1,x_1)-\frac{1}{2}\sigma^2x_1^2c_{xx}(t_1,x_1)<0.$$

- Profit, when:

$$e^{rt_1}N\big(d_-(T-t_1,x_1)\big)-e^{r\hat{t}}N\big(d_-(T-\hat{t},x_1)\big)-\frac{\sigma x e^{rT}}{2rK\sqrt{T-\hat{t}}}N'\left(d_+(T-\hat{t},x_1)\right)> 0 \iff$$
$$c_t(\hat{t},x_1)-c_t(t_1,x_1)-\frac{1}{2}\sigma^2x_1^2c_{xx}(t_1,x_1)>0.$$

From the point of view of no-arbitrage pricing, it is irrelevant how likely the stock is to go up or down because a delta-neutral position is a hedge against both possibilities. What matters is how much volatility the stock has, for we need to know the amount of profit that can be made from the long gamma position. The more volatile stocks offer more opportunity for profit from the portfolio that hedges a long call position with a short stock position, and hence the call is more expensive. The derivative of the option price with respect to the volatility σ is called vega, and it is positive. As volatility increases, so do option prices in the Black-Scholes-Merton model.

Exercise 4.10 Shreve: The fundamental idea behind no arbitrage pricing is to reproduce the payoff of a derivative security by trading in the underlying asset (which we call a stock) and the money market account. In discrete time, we let X_k be the value of hedging portfolio at time k and Δ_k be the number of shares of the stock held between times k and $k+1$. Then at time k, after re-balancing i.e., moving from a position Δ_{k-1} to Δ_k in stock), the amount in money market accounts is $\big(X_k-\Delta_kS_k\big)$ and the value of portfolio at time $k+1$ is:

$$\begin{aligned}X_{k+1}&=\Delta_kS_{k+1}+(1+r)\big(X_k-\Delta_kS_k\big)\\X_{k+1}-X_k&=\Delta_k\big(S_{k+1}-S_k\big)+r\big(X_k-\Delta_kS_k\big),\end{aligned}\tag{269}$$

translates into: the gain in portfolio between times k and $k+1$ is the sum of capital gain on the stock holdings, $\Delta_k\big(S_{k+1}-S_k\big)$, and interest earnings on the money market account, $r\big(X_k-\Delta_kS_k\big)$. The obvious continuous time analog is:

$$dX_t=\Delta_tdS_t+r\big(X_t-\Delta_tS_t\big)dt\tag{270}$$

Alternatively, one could define the value of a share of the money market account at time k to be

$$M_k=(1+r)^k\tag{271}$$

and formulate this discrete time model with two processes, one Δ_k as before and Γ_k denoting the number of shares of the money market account held at time k after rebalancing. Then

$$\Delta_{k-1}S_k+(1+r)\big(X_{k-1}-\Delta_{k-1}S_{k-1}\big)=X_k=\Delta_kS_k+\Gamma_kM_k,\tag{272}$$

so that (269) becomes

$$X_{k+1} = \Delta_k S_{k+1} + (1+r)\big(\Delta_k S_k + \Gamma_k M_k - \Delta_k S_k\big) = \Delta_k S_{k+1} + \Gamma_k M_{k+1}. \tag{273}$$

First of all I want to see if I can even re-balance things, given $\Delta_0, \Delta_1, \ldots, \Delta_k$ and X_0, then:

$$\begin{aligned} X_1 &= \Delta_0 S_1 + (1+r)\big(X_0 - \Delta_0 S_0\big) \\ X_2 &= \Delta_1 S_2 + (1+r)\big(X_1 - \Delta_1 S_1\big) \end{aligned}$$

Now we will try to find number of shares of the money market account at time $t = 1$, Γ_1 such that:

$$\begin{aligned} &\Delta_0 S_1 + (1+r)\big(X_0 - \Delta_0 S_0\big) = X_1 = \Delta_1 S_1 + \Gamma_1(1+r), \ \ \textit{implies} \\ &\Delta_0 S_1 + (1+r)\big(X_0 - \Delta_0 S_0\big) = \Delta_1 S_1 + \Gamma_1(1+r), \ \ \textit{implies} \\ &\Gamma_1 = \frac{(\Delta_0 - \Delta_1)S_1 + (1+r)\big(X_0 - \Delta_0 S_0\big)}{(1+r)}, \end{aligned}$$

with this Γ_1:

$$\begin{aligned} X_2 &= \Delta_1 S_2 + (1-r)\big(X_1 - \Delta_1 S_1\big) = \Delta_1 S_2 + (1-r)\big(\Delta_1 S_1 + \Gamma_1 M_1 - \Delta_1 S_1\big) \\ &= \Delta_1 S_2 + \Gamma_1 M_2. \end{aligned}$$

First found Γ_1 such that $X_1 = \Delta_1 S_1 + \Gamma_1 M_1$, then by natural dynamice of self financing portfolio, $X_2 = \Delta_1 S_2 + \Gamma_1 M_2$. Now again I would like to use given information to get Γ_2 such that:

$$\begin{aligned} &X_2 = \Delta_2 S_2 + \Gamma_2 M_2 \\ &\textit{equating}, \ \ \Delta_1 S_2 + \Gamma_1 M_2 = \Delta_2 S_2 + \Gamma_2 M_2 \\ &\textit{implies}, \ \ \Gamma_2 = \frac{(\Delta_1 - \Delta_2)S_2 + \Gamma_1 M_2}{M_2} \end{aligned}$$

Seems like:

$$(\Delta_0, \Delta_1, S_0, S_1) \to \Gamma_1, \ (\Delta_1, \Delta_2, S_2, \Gamma_1) \to \Gamma_2, \ (\Delta_2, \Delta_3, S_3, \Gamma_2) \to \Gamma_3, \ldots \tag{274}$$

Now we have a general purpose formula, for $k \geq 2$

$$\Gamma_k = \frac{(\Delta_{k-1} - \Delta_k)S_k + \Gamma_{k-1} M_k}{M_k}. \tag{275}$$

Anywyas after re-balancing self-financing portfolio can be written as:

$$X_{k+1} = \Delta_k S_{k+1} + \Gamma_k M_{k+1}. \tag{276}$$

again after rebalancing, figuring out $\Gamma_{k+1} = \frac{(\Delta_k - \Delta_{k_1})S_{k+1} + \Gamma_k M_{k+1}}{M_{k+1}}$, we have:

$$X_{k+1} = \Delta_{k+1} S_{k+1} + \Gamma_{k+1} M_{k+1} \tag{277}$$

essentially

$$\begin{aligned}
&\Delta_k S_{k+1} + \Gamma_k M_{k+1} = \Delta_{k+1} S_{k+1} + \Gamma_{k+1} M_{k+1} \\
&\big(\Delta_{k+1} - \Delta_k\big) S_{k+1} + \big(\Gamma_{k+1} - \Gamma_k\big) M_{k+1} = 0,
\end{aligned} \tag{278}$$

which is *discrete-time self-financing condition*. Alternatively written as:

$$\begin{aligned}
\big(\Delta_{k+1} - \Delta_k\big)\big(S_{k+1} - S_k\big) + S_k\big(\Delta_{k+1} - \Delta_k\big) \\
+ \big(\Gamma_{k+1} - \Gamma_k\big)\big(M_{k+1} - M_k\big) + M_k\big(\Gamma_{k+1} - \Gamma_k\big) = 0.
\end{aligned}$$

This is suggestive of continuous time self-financing equation:

$$d\Delta_t dS_t + S_t d\Delta_t + d\Gamma_t dM_t + M_t d\Gamma_t = 0, \tag{279}$$

which we drive below,
Proof

1. In continuous time, let $M_t = e^{rt}$ be the price of a share in money market at time t, let Δ_t be the shares of stock held at time t and let Γ_t denote the number of shares of the money market account held at time t, so that the portfolio value at time t is:

$$\begin{aligned}
&X_t = \Delta_t S_t + \Gamma_t M_t. \\
&\Gamma_t = e^{-rt} X_t - e^{-rt}\Delta_t S_t \implies d\Gamma_t = e^{-rt}\big(- r[X_t - \Delta_t S_t]dt + dX_t - d[\Delta_t S_t]\big) \\
&e^{rt}(d\Gamma_t) = \\
&M_t(d\Gamma_t) = -r[X_t - \Delta_t S_t]dt + dX_t - d[\Delta_t S_t]. \ \textit{but recall that} \\
&dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t)dt, \ \textit{implies} \\
&M_t(d\Gamma_t) = -r[X_t - \Delta_t S_t]dt + \big(\Delta_t dS_t + r(X_t - \Delta_t S_t)dt\big) - d[\Delta_t S_t] \\
&M_t(d\Gamma_t) = \Delta_t dS_t - d[\Delta_t S_t] \\
&\Delta_t dS_t + r(X_t - \Delta_t S_t)dt = d(\Delta_t S_t) + d(\Gamma_t M_t). \, dM_t = rM_t dt.
\end{aligned}$$

Furthermore, for $k \geq 2$

$$\Gamma_k = \frac{(\Delta_{k-1} - \Delta_k)S_k + \Gamma_{k-1}M_k}{M_k}. \implies \Gamma_k M_k - \Gamma_{k-1}M_k = (\Delta_{k-1} - \Delta_k)S_k. \implies (d\Gamma_t)M_t = -(d\Delta_t)S_t.$$

BTW $\Gamma_k M_k - \Gamma_{k-1}M_k = (\Delta_{k-1} - \Delta_k)S_k$ can be alternatively written as:

$$\begin{aligned}
&(\Gamma_k - \Gamma_{k-1})(M_k - M_{k-1}) + M_{k-1}(\Gamma_k - \Gamma_{k-1}) = -(\Delta_k - \Delta_{k-1})(S_k - S_{k-1}) - S_{k-1}(\Delta_k - \Delta_{k-1}), \ \textit{implie} \\
&(d\Gamma_t)(dM_t) + M_t(d\Gamma_t) = -(d\Delta_t)(dS_t) - S_t(d\Delta_t).
\end{aligned}$$

Essentially,

$$(d\Gamma_t)(dM_t) + M_t(d\Gamma_t) + S_t(d\Delta_t) + (d\Delta_t)(dS_t) = 0. \tag{280}$$

which is exactly (279). But we know something more

$$\begin{aligned}
&(d\Gamma_t)(dM_t) + M_t(d\Gamma_t) + S_t(d\Delta_t) + (d\Delta_t)(dS_t) = 0. \ \textit{implies} \\
&e^{rt}(d\Gamma_t) + S_t(d\Delta_t) + (d\Delta_t)(dS_t) = 0.
\end{aligned}$$

because $(d\Gamma_t)(dM_t) = (d\Gamma_t)(rM_t dt)$ and ito $dtdt = dtdw_t = 0$. Now if we try to discretize the equation:

$$e^{rt}(d\Gamma_t) + S_t(d\Delta_t) + (d\Delta_t)(dS_t) = 0. \textit{ discretized}$$
$$\underbrace{M_k}_{:=(1+r)^k}(\Gamma_{k+1} - \Gamma_k) + S_k(\Delta_{k+1} - \Delta_k) + (\Delta_{k+1} - \Delta_k)(S_{k+1} - S_k) = 0.$$

■

From a portfolio that is long the call and short Δ_t shares of the stock with current stock price $S_t = x$ the value of the portfolio N_t, is

$$N_t = c(t, S_t) - \Delta_t S_t. \tag{281}$$

We want to chose Δ_t so that it is 'instantaneously riskless', i.e., $dN_t = rN_t dt$.

$$\begin{aligned} dN_t = dc(t, S_t) - d(\Delta_t S_t) = \left[c_t(t, S_t) + c_x(t, S_t)\alpha S_t + \frac{1}{2}\sigma^2 S_t^2 c_{xx}(t, S_t)\right] dt \\ + \sigma c_x(t, S_t) S_t dW_t - \Delta_t\big(\alpha S_t dt - \sigma S_t dW_t\big) - S_t d\Delta_t - dS_t d\Delta_t = rN_t dt. \end{aligned}$$

Now if we let $\Delta_t = c_x(t, S_t)$, then:

$$dN_t = dc(t, S_t) - d(\Delta_t S_t) = \big[c_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{xx}(t, S_t)\big] dt - S_t d\Delta_t - dS_t d\Delta_t.$$

but we know from continuous time self-financing portfolio that:

$$\begin{aligned} dN_t &= \big[c_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{xx}(t, S_t)\big] dt + (d\Gamma_t)(dM_t) + M_t(d\Gamma_t) \\ &= \big[c_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{xx}(t, S_t)\big] dt + M_t(d\Gamma_t) + \underbrace{(d\Gamma_t)(rM_t dt)}_{=0} \\ &= \big[c_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{xx}(t, S_t)\big] dt + e^{rt}(d\Gamma_t) \end{aligned} \tag{282}$$

$N_t := c(t, S_t) - c_x(t, S_t)S_t$, So $dN_t = rN_t dt$ would require that:

$$rc_x(t, S_t)S_t + c_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{xx}(t, S_t) = rc(t, S_t), \tag{283}$$

which would mean that no arbitrage call price would satisfy:

$$rc_x(t, S_t)S_t + c_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{xx}(t, S_t) = rc(t, S_t), \tag{284}$$

Forgot that Γ_t is derived from Δ_t which was defined to be c_x thing. (come back to it).

27.1 4.10 Shreve

Short Δ_t shares of the stock with current value S_t, buy one call and invest the remaining in the money market. $N_t := c(t, S_t) - \Delta_t S_t$, for self-financing portfolio:

$$dX_t = \Delta_t dS_t + (1+r)(X_t - \Delta_t S_t)dt \tag{285}$$

If we let $\Delta_t == c_x(t, S_t)$ then from self-replicating portfolio propert: $X_t = c(t, S_t)$ which now implies that: $\Gamma_t = \frac{X_t - c_x(t,S_t)S_t}{e^{rt}} = \frac{c(t,S_t) - c_x(t,S_t)S_t}{e^{rt}} = \frac{N_t}{M_t}$. But we have already shown that:

$$\begin{aligned} dN_t &= \left[c_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{xx}(t, S_t)\right]dt + e^{rt}(d\Gamma_t) \\ &= \left[c_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{xx}(t, S_t)\right]dt \end{aligned}$$

28. Additional Stuff

Consequently,

$$\begin{aligned} \Delta_t dS_t + r(X_t - \Delta_t S_t)dt &= (\Delta_t dS_t) + (d\Delta_t)S_t + (d\Delta_t)(dS_t) + (d\Gamma_t)M_t + (dM_t)(\Gamma_t) + (d\Gamma_t)(dM_t) \\ &= (\Delta_t dS_t) + (d\Delta_t)(dS_t) + r\Gamma_t M_t dt + \underbrace{(d\Gamma_t) r M_t dt}_{=0} \end{aligned}$$

which in turn implies that:

$$\begin{aligned} &r(X_t - S_t)dt = (d\Delta_t)(dS_t) + r\Gamma_t M_t dt, \ \ \textit{implies} \\ &r(\Delta_t S_t + \Gamma_t M_t - S_t)dt = (d\Delta_t)(dS_t) + r\Gamma_t M_t dt, \ \ \textit{implies} \\ &r(\Delta_t - 1)S_t dt = (d\Delta_t)(dS_t). \end{aligned}$$

Now weirdly enough Ito's product rule implies $d\Delta_t = drift^{\Delta_t} dt + diff^{\Delta_t} dW_t$

$$\begin{aligned} &diff^{\Delta_t}\sigma S_t = r(\Delta_t - 1)S_t, \ \ \textit{implies} \\ &diff^{\Delta_t} = \frac{r(\Delta_t - 1)}{\sigma}. \end{aligned}$$

Also,

$$dc_x(t, S_t) = \big(\partial_t c_x(t, S_t) + \alpha S_t c_{xx}(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{xxx}(t, S_t)\big)dt + \sigma S_t c_{xx}(t, S_t)dW_t, \tag{286}$$

implies that:

$$\begin{aligned} dN_t = &\left[c_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{xx}(t, S_t) - \sigma^2 S_t^2 c_{xx}(t, S_t)\right]dt \\ &- S_t\big(\partial_t c_x(t, S_t) + \alpha S_t c_{xx}(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{xxx}(t, S_t)\big)dt + \sigma S_t^2 c_{xx}(t, S_t)dW_t. \end{aligned}$$

To cancel out the Brownian motion part $\Delta_t = c_x(t, S_t)$ which means that

$$dN_t = \left[c_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{xx}(t, S_t)\right]dt - S_t d\Delta_t = rN_t dt$$

Recall from the discounted return stuff that: if $D_t = e^{-rt}$, then lets find out what is dD_t.

$$dD_t = D_{t+\delta t} - D_t = e^{-r(t+\delta t)} - e^{-rt} = e^{-rt}\big(e^{-r\delta t} - 1\big)e^{-rt}\big(1 - r\delta t - 1\big) = -rD_t dt. \quad (287)$$

Seems like(if time varying but non-ranodm interest rate)

$$dD_t = -R_t D_t dt. \quad (288)$$

So:

$$d\Delta_t = \big(\partial_t \Delta_t\big)dt. \quad (289)$$

which implies

$$dN_t = dc(t, S_t) - d(\Delta_t S_t) = \big[c_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{xx}(t, S_t)\big]dt - S_t d\Delta_t - \underbrace{dS_t d\Delta_t.}_{=0}$$
$$dN_t = \Big[c_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{xx}(t, S_t) - S_t\big(\partial_t c_x(t, S_t)\big)\Big]dt = rN_t dt.$$

Recall that $N_t = c(t, S_t - c_x(t, S_t)S_t$, $dN_t = rN_t dt$ implies that:

$$c_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{xx}(t, S_t) - S_t\big(\partial_t c_x(t, S_t)\big) = rc(t, S_t) - rc_x(t, S_t)S_t$$
$$rc_x(t, S_t)S_t + c_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{xx}(t, S_t) - S_t\big(\partial_t c_x(t, S_t)\big) = rc(t, S_t).$$

29. Exotic Options

Let W_t be a Brownian motionM on the is given by: space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$. Now let $\alpha \in \mathbb{R}$ be a given number and define:

$$\hat{W}_t = \alpha t + W_t, \ \ 0 \leq t \leq T. \quad (290)$$

We further define:

$$\hat{M}_T := \max_{0 \leq t \leq T} \hat{W}_t. \quad (291)$$

Theorem 61 *The joint density of the pair $(\hat{M}_T, \hat{W}_T)$ is given by:*

$$f_{(\hat{M}_T, \hat{W}_T)} = \frac{2(2m - w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2}, \ \ w \leq m, m \geq 0. \quad (292)$$

Firs consider the exponential martingale:

$$\hat{Z}_t := e^{-\alpha \hat{W}_t + \frac{1}{2}\alpha^2 t}, \ \ \textit{giving us change of measure } \frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = \hat{Z}_T.$$

29.1 Revision:

Theorem 62 (Multidimensional Girsanov) *Let $W_t \in \mathbb{R}^d$ be $\mathcal{F}_t$ adapted on the probability space $\left(\Omega, \mathcal{F}_{t\in[0,T]}, \mathcal{F}, \mathbb{P}\right)$. $X_t \in \mathbb{R}^d$ and so is $F_t \in \mathbb{R}^d$ satisfying:*

$$dX_t = F_t dt + dW_t, \;\; t \in [0, T]$$
$$\Lambda := e^{-\int_0^T \langle F_s, dW_s\rangle - \frac{1}{2}\int_0^T \|F_s\|^2 ds}$$

Then X_t is a $\mathcal{F}_t$ Wiener process under the measure $\mathbb{Q}(A) := \mathbb{E}_{\mathbb{P}}\left[1_A \Lambda\right]$.

Theorem 63 (Multidimensional MRT) *Let W_t be a $d-$ dimensional Brownian motion and the filteration it generates be deoted by $\mathcal{F}_t = \mathcal{F}_t^W$. If M_t is a martingale w.r.t $\mathcal{F}_t$, then there exists an adapted process $\Gamma_s \in \mathbb{R}^d$ such that:*

$$M_t = M_0 + \int_0^t \langle \Gamma_s, dW_s\rangle. \tag{293}$$

29.2 Reviewing Risk Neutral

$$dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t. \tag{294}$$

Now if one defines the discount process by $D_t := e^{-\int_0^t r_s ds}$.

$$\begin{aligned} D_{t+\delta t} - D_t &= e^{-\int_0^t r_s ds - \int_t^{t+\delta t} r_s ds} - e^{-\int_0^t r_s ds} \\ &= e^{-\int_0^t r_s ds}\left(e^{-\int_t^{t+\delta t} r_s ds} - 1\right) \\ &\approx e^{-\int_0^t r_s ds}\left(1 - \int_t^{t+\delta t} r_s ds - 1\right), \end{aligned}$$

which implies that $dD_t = -r_t D_t dt$, so Ito's product rule now implies that:

$$\begin{aligned} d(D_t S_t) = dS_t D_t - S_t r_t D_t dt &= \frac{(\alpha_t - r_t)}{\sigma_t}\sigma_t S_t D_t dt + \sigma_t S_t D_t dW_t \\ &= \sigma_t S_t D_t \underbrace{\left[\frac{(\alpha_t - r_t)}{\sigma_t} dt + dW_t\right]}_{:= dB_t} \end{aligned}$$

Now remind yourself from Girsanov, if we define:

$$\mathbb{Q}(A) := \mathbb{E}_{\mathbb{P}}\left[1_A e^{-\int_0^T \frac{\left(\alpha_s - r_s\right)}{\sigma_s} dW_s - \frac{1}{2}\frac{\left(\alpha_s - r_s\right)^2}{\sigma_s^2} ds}\right], \tag{295}$$

then dB_t is a Brownian motion under $\mathbb{Q}$and therefore:

$$d(D_t S_t) \overset{\mathbb{Q}}{=} \sigma_t S_t D_t dB_t, \tag{296}$$

is a Martingale.

Self-financing portfolio: Δ_t be the number of shares of stock you hold at time t, then:

$$dX_t = \Delta_t dS_t + r_t(X_t - \Delta_t S_t)dt$$

$$\begin{aligned} d(D_tX_t) &= -r_tD_tX_tdt + D_t\Delta_t dS_t + D_tr_t(X_t - \Delta_t S_t)dt \\ &= D_t\Delta_t\big(dS_t - r_tS_tdt\big). \\ &= D_t\Delta_t\big[(\alpha_t - r_t)S_tdt + \sigma_tS_tdW_t\big] \end{aligned}$$

Again via Girsanov D_tX_t is a martingale under $\mathbb{Q}$ and infact:

$$d(D_tX_t) \overset{\mathbb{Q}}{=} \sigma_tD_t\Delta_tS_tdB_t.] \ \textit{Consequently,}$$
$$\mathbb{E}[D_TX_T] = X_0, \textit{almost surely.}$$

Assuming that we have m underlying assets and $d-$ dimensional Brownian motion:

$$dS_t^i = \alpha_t^iS_t^idt + S_t^i\sum_{j=1}^{d}\sigma_t^{i,j}dW_t^j. \tag{297}$$

If we can find a market price of risk Θ_j such that:

$$d\big(D_tS_i(t)\big) = D_tS_i(t)\bigg\{(\alpha_i(t) - R_t)dt + \sum_{j=1}^{d}\sigma_{i,j}(t)dW_j(t)\bigg\} = D_tS_i(t)\sum_{j=1}^{d}\sigma_{i,j}(t)\big\{\Theta_j(t)dt + dW_j(t)\big\}, \tag{298}$$

Essentialy:

$$\alpha_t^R = \begin{bmatrix} \alpha_1(t) - R_t \\ \alpha_2(t) - R_t \\ \vdots \\ \alpha_m(t) - R_t \end{bmatrix} \quad \sigma_t = \begin{bmatrix} \sigma_t^{1,1} & \sigma_t^{1,2} & \dots & \sigma_t^{1,d} \\ \sigma_t^{2,1} & \sigma_t^{2,2} & \dots & \sigma_t^{2,m} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_t^{m,1} & \sigma_t^{m,2} & \dots & \sigma_t^{m,d} \end{bmatrix}, \ \Theta_t := \begin{bmatrix} \Theta_1(t) \\ \Theta_2(t) \\ \vdots \\ \Theta_d(t) \end{bmatrix}$$

If exists Θ_t such that $\sigma_t\Theta_t = \alpha_t^R$, then

$$dB_t := \Theta_tdt + dW_t \tag{299}$$

is a $d-$ dimensional Brownian motion under $\mathbb{Q}$ defined using usual Girsanovs' argument. Unfold one dimensional stuff to find out:

$$dB_t^j = \Theta_t^jdt + dW_t^j. \tag{300}$$

Theorem 64 *(Every strictly positive asset is a generalized geometric Brownian motion). Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and W_t be the Brownian motion that generates the filteration $\mathcal{F}_t := \mathcal{F}_t^W$. Now assume that a unique risk neutral measure $\mathbb{Q}$, with Brownian* *motion dB_t exists under which D_tV_t is a martingale. V_T be an* almost surely positive $\mathcal{F}_T$-*measurable random variable. Also recall that r_t is $\mathcal{F}_t$ adapted. By risk neutral pricing theory, value at time t*

$V_t = \mathbb{E}_{\mathbb{Q}}\big[e^{-\int_t^T r_s ds} V_T|\mathcal{F}_t\big]$. *Then notice*
$D_tV_t = \mathbb{E}_{\mathbb{Q}}[D_TV_T|\mathcal{F}_t]$ *is a martingale, implies by MRT exist* $\eta_t \in \mathcal{F}_t s.t$
$D_tV_t = D_0V_0 + \displaystyle\int_0^t \langle \eta_s, dB_s\rangle = \mathbb{E}_{\mathbb{Q}}[V_T|\mathcal{F}_0] + \int_0^t \langle \eta_s, dB_s\rangle$. *But also via ito product rule*
$d(D_tV_t) = -r_tD_tV_tdt + D_tdV_t$. *Combining preceding two eq*
$dV_t = r_tV_tdt + \langle \dfrac{\eta_t}{D_t}, dB_t\rangle$.

- *Show that for all $t \in [0,T]$, the price of derivative security V_t is almost surely positive*
- *Conclude there exists an adapted process such that:*

$$dV_t = r_tV_tdt + V_t\langle \sigma_t, dB_t\rangle. \tag{301}$$

This simply follows from the preceding fact that $V_t > 0$ almost surely, hence we can let $\sigma_t := \frac{\eta_t}{D_tV_t}$.

30. Change of Numeraire

A numeraire is the unit of account in which other assets are denominated. One usually takes the numeraire to be the currency of a country. One might change the numeraire by changing to the currency of another country. As this example suggests, ´ in some applications one must change the numeraire in which one works because of ´ finance considerations. We shall see that sometimes it is convenient to change the numeraire because of modeling considerations as well. A model can be complicated ´ or simple, depending on the choice of the numeraire for the model.

Assumptions: We have $d-$ dimensional Brownian motion $(W_1(t), \ldots, W_d(t))$ and associated multidimensional market model

$$dS_t^i = \alpha_t^i S_t^i dt + S_t^i \sum_{j=1}^d \sigma_t^{i,j} dW_t^j. \tag{302}$$

Underlying filteration is $F_t = F_t^W$. There is an adapted interest rate process R_t. This can be used to make a money market account whose price per share at time t is:

$$M_t := e^{\int_0^t R_u du}. \tag{303}$$

This is the capital an agent would have if the agent invested one unit of currency in the money market account at time zero and continuously rolled over the capital at the short-term interest rate. *Discounted process* $D_t := \frac{1}{M_t}$. We also assume that *Market price of risk exists* i.e., under $\mathbb{Q}, [D_tS_t^i]_{i\in[m]}$ is a martingale(discounted asset prices) and so is *self-financing* *portfolio* and consequently no deterministic arbitrage. We assume risk neutral measure exists... Θ_t eg such that $\sigma_t \Theta_t = \alpha_t^R$, then

$$dB_t := \Theta_t dt + dW_t \tag{304}$$

is a $d-$ dimensional Brownian motion under $\mathbb{Q}$ defined using usual Girsanovs' argument. Unfold one dimensional stuff to find out:

$$dB_t^j = \Theta_t^j dt + dW_t^j. \tag{305}$$

The risk neutral measure $\mathbb{Q}$, is thus associated with money market account price M_t in the following way: If we were to denominate the ith asset in terms of the money market account, its price would be $\frac{S_t^i}{M_t} = D_t S_t^i$. *In other words, at time t the i-th asset is worth $D_t S_t^i$ shares of the money market.* We therefore call $\mathbb{Q}$ as *risk neutral for money market account numeraire.* When we change the numeraire, denominating the i-th asset in some other unit of account, it is no longer a martingale under Q. When we change the numeraire, we ´ need to also change the risk-neutral measure in order to maintain risk neutrality. The details and some applications of this idea are developed in this chapter.

In principle, we can take any positively priced asset as a numeraire and denominate all other assets in terms of the chosen numeraire. Associated with each numeraire, we shall have a risk-neutral measure. When making this association, we shall take only non-dividend-paying assets as numeraires. In particular, we regard $\mathbb{Q}$ as the risk-neutral measure associated with the domestic money market account, not the domestic currency. Currency pays a dividend because it can be invested in the money market. In contrast, in our model, a share of the money market account increases in value without paying a dividend.

1. $\mathbb{Q}$ will be used for numeraire wrt domestic money market account
2. $\mathbb{Q}^f$ will be used for numeraire wrt foreign money market account
3. $\mathbb{Q}^T$ is the risk neutral measure corresponding to a zero coupon bond maturing at time T.

The asset we take as numeraire could be one of the primary assets given by (302) or it could be a derivative asset. Regardless of which asset we take, it has the stochastic representation provided by the following theorem.

Theorem 65 (Stochastic representation of assets) *Let N be a strictly positive price process for a non-dividend-paying asset, either primary or derivative, in the multidimensional market model in* (302) *Then there exists a vector volatility process $\nu_t \in \mathbb{R}^d$ such that:*

$$dN_t = R_t N_t dt + N_t \langle \nu_t, dB_t \rangle. \tag{306}$$

Proof Proof simply follows from preceding theorem (64) about positive assets. ■

In other words, under the risk-neutral measure, every asset has a mean return equal to the interest rate. The realized risk-neutral return for assets is characterized solely by their volatility vector processes (because initial conditions have no effect on return).

Theorem 66 (Change of risk-neutral measure) *Let S_t and N_t be the prices of two assets denominated in a common currency, and let $\sigma_t := (\sigma_t^1, \ldots, \sigma_t^d)$ and $\nu_t := (\nu_t^1, \ldots, \nu_t^d)$ denote their respective volatility vector processes:*

$$d(D_tS_t) = D_tS_t\langle\sigma_t, dB_t\rangle, \;\; d(D_tN_t) = D_tN_t\langle\nu_t, dB_t\rangle. \tag{307}$$

31. Equivalent measures-not necessarily probability measures

On the probability space $(\Omega, \mathcal{F}, P)$

- Q is absolutely continuous wrt P iff for any $A \in \mathcal{F}, P(A) = 0 \implies Q(A) = 0$.
- P and Q are equivalent iff $P(A) = 0 \iff Q(A) = 0$.

Now given $S_0 = s$ and call option expiring at $T > t$ then for $z \sim N(0,1)$

$$S_T = e^{(\alpha - \frac{\sigma^2}{2})T + \sigma\sqrt{T}z}s$$

$$C(T,s) = E[\max(S_T - K, 0)|S_0 = s] = \frac{1}{\sqrt{2\pi}}\int \max(e^{(\alpha-\frac{\sigma^2}{2})T+\sigma\sqrt{T}z}s - K, 0)e^{-\frac{z^2}{2}}dz.$$

31.1 Simulating the stock prices

$$S_{t_{i+1}} = S_{t_i}e^{\left(\mu - 0.5\sigma^2\right)[t_{i+1}-t_i] + \sigma\sqrt{t_{i+1}-t_i}z_i}$$

$$S_{t_1} = S_{t_0}e^{\left(\mu - 0.5\sigma^2\right)[t_1-t_0] + \sigma\sqrt{t_1-t_0}z_0}$$

$$S_{t_2} = S_{t_1}e^{\left(\mu - 0.5\sigma^2\right)[t_2-t_1] + \sigma\sqrt{t_2-t_1}z_1}$$

$$S_{t_2} = S_{t_0}e^{\left(\mu - 0.5\sigma^2\right)[t_1-t_0] + \sigma\sqrt{t_1-t_0}z_0}e^{\left(\mu - 0.5\sigma^2\right)[t_2-t_1] + \sigma\sqrt{t_2-t_1}z_1}$$

$$P(x) := \frac{1}{N\sqrt{2\pi w^2}}\sum_{k=1}^{N} e^{-\frac{(x_k - x)^2}{2w^2}}$$

$dS_t = \mu S_t dt + \sigma S_t dW_t, \; \approx \; S_{t+\Delta t} - S_t = \mu S_t\Delta t + \sigma S_t(W_t - W_{t-\Delta t}).$ *if time is in years then daily increment* $\Delta t =$

$$\frac{S_{3\Delta t} - S_{2\Delta t}}{S_{2\Delta t}} = \mu\Delta t + \sigma(W_{2\Delta t} - W_{\Delta t})$$

This means that when lag is l

$$S_{k+l} - S_k = \frac{S_{k+l} - S_{k+(l-1)}}{S_{k+(l-1)}}S_{k+(l-1)} + \frac{S_{k+(l-1)} - S_{k+(l-2)}}{S_{k+(l-2)}}S_{k+(l-2)} + \ldots + \frac{S_{k+1} - S_k}{S_k}S_k$$

$$\frac{S_{k+l} - S_{k+(l-1)}}{S_{k+(l-1)}} = \mu\Delta t + \sigma\sqrt{\Delta t}(W_{k+(l-1)} - W_{k+(l-2)})$$

$$S_{k+l}-S_k=\Big[\mu\Delta t+\sigma\sqrt{\Delta t}(W_{k+(l-1)}-W_{k+(l-2)})\Big]S_{k+(l-1)}+\Big[\mu\Delta t+\sigma\sqrt{\Delta t}(W_{k+(l-2)}-W_{k+(l-3)})\Big]S_{k+(l-2)}+\ldots+\Big[\mu\Delta$$

Recall the rules of the game: as W_t is a *Wiener* process implying $E[S_{k+l}-S_k]=0$

$$\left\langle\left(S_{k+l}-S_k\right)^2\right\rangle=(\mu^2\Delta t^2+\sigma^2\Delta t)\sum_{i=1}^{l}ES^2_{k+(l-i)}$$

cross term: $\mathbb{E}\Big[\big(\mu\Delta t+\sigma\sqrt{\Delta t}(W_{k+(l-1)}-W_{k+(l-2)})\big)S_{k+(l-1)}\big(\mu\Delta t+\sigma\sqrt{\Delta t}(W_{k+(l-2)}-W_{k+(l-3)})\big)S_{k+(l-2)}\Big]$

$$\mathbb{E}\Big[\big(\mu\Delta t+\sigma\sqrt{\Delta t}(W_{k+(l-1)}-W_{k+(l-2)})\big)^2S^2_{k+(l-1)}\Big]=\mathbb{E}\Big[[(\mu\Delta t)^2+2\sigma\mu(\Delta t)^{\frac{3}{2}}(W_{k+(l-1)}-W_{k+(l-2)})+\sigma^2\Delta t(W_{k+(l-$$