Australia Mortality Modelling for Retirement Income Policy

1. Introduction

Australia is amongst the very few countries in the world that boast free, high-quality public healthcare. Owing to this factor, the country has observed a significant fall in mortality rates and an increase in the life expectancy of its citizens over the last century. The current Statistics indicate that the life expectancy of Australians in retirement has almost doubled in the last 150 years[1]. This increase in life expectancy stresses the Australian government's retirement policy and burdens the retirement funds. Amongst the various measures required to tackle this scenario, an important one involves the government addressing the issue of the lack of annuitizations (caused by the lumpsum payments of superannuation savings) which exposes the senior citizens to the risks of longevity, inflation, and investment[2]. A proposed solution is to replace the current mandatory superannuation scheme with a new scheme where adult individuals make contributions to a government-regulated pool of funds during their working lives and at retirement receive a life annuity covering their expenses till death. Therefore, the purpose of this report is to analyse Australia's past mortality data and to prepare models for mortality projection, with an objective to assist in the evaluation of the proposed solution's viability.

Report Structure and variable description:

The statistical modelling structure adopted for this report will consist of a **Preliminary analysis** of Australia's mortality data, **fitting and evaluation** of Natural Cubic and Smoothing spline models for the graduation of mortality data, **mortality projection** using Lee Carter Model and finally a **Model Comparison** to discuss the various shortcomings of different models used throughout the report. The entire modelling is performed using R software and a separate R script is provided to verify the results and plots.

The Australian mortality raw data used in this report is obtained using R's "demography" library, from the website Human Mortality Database[3]. The obtained data set carries various variables of interest used in the report including:

- **Age:** This represents the age of the individuals. The age ranges from 0 to 110 and is based on the principle of age last birthday where the individual's age remains the same throughout the rate interval. It is important to note that the age 110 records observations for people 110 and above.
- **Year:** The variable that indicates the year of the mortality observations. Ranging from 1921 to 2019.
- **Central Rate of Mortality ("rate"):** The mortality rate calculated by dividing the number of deaths of individuals aged x last birthday by the central exposed to risk for the period. Its values in the data set ranges from 0 to 6.
- **Central exposed to risk("pop"):** The population exposed to the risk of death (segregated by age and year). It is represented as a matrix of size [111 x 99] with values ranging from 0 to 387252.3

Note: The lower bound for Central Rate of Mortality and Central Exposed to Risk is set to 0 in the data set. Furthermore, many missing data values can be observed in the earlier years, mostly for ages above 100.

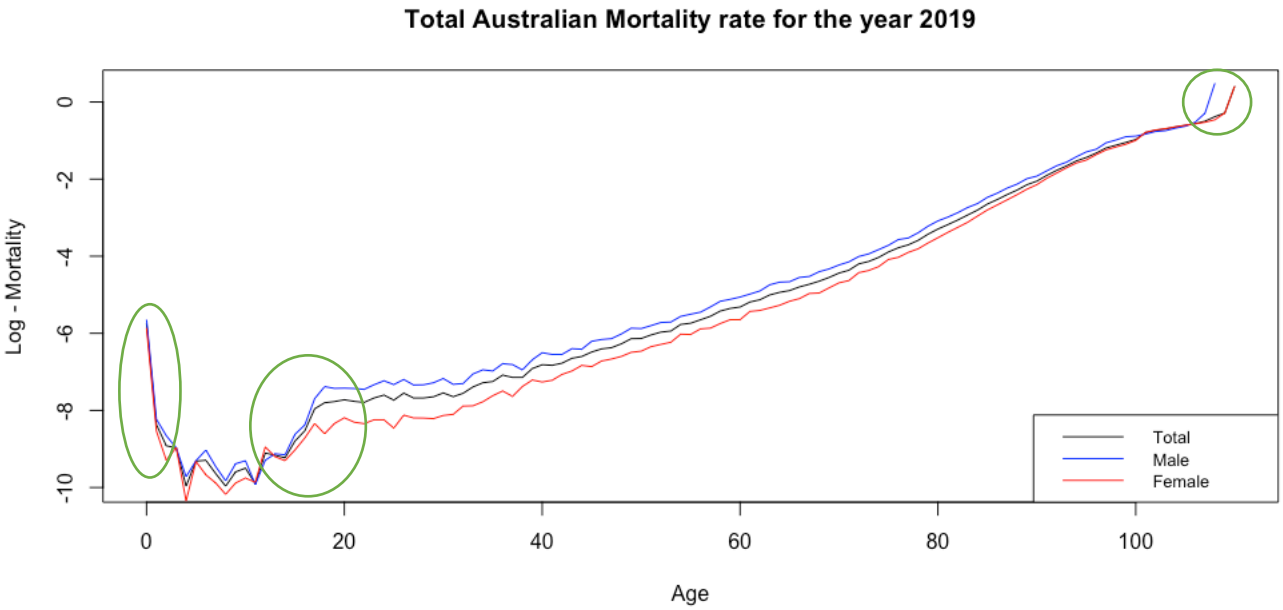
[1] Franks, T (2019), *Australia's retirement challenge*, Household Capital™, viewed 14 September 2022, https://householdcapital.com.au/retirement-incomes/australias-retirement-challenge

[2] Agnew, J 2013, *Australia's Retirement System: Strengths, Weaknesses, And Reforms*, *Centre for Retirement Research at Boston College*, viewed 14 September 2022, https://crr.bc.edu/wp-content/uploads/2013/04/IB_13-5-508.pdf

[3] Human Mortality Database (2016). University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Retrieved from http://www.mortality.org/ accessed on 15 Sep 2022.

2. Preliminary Data Analysis

In this section, we will analyze Australia's logarithmic mortality rates for the year 2019 in order to understand the general mortality trends and to provide a detailed explanation for specific points of interest. We will further examine the differences between male and female mortality rates and their causes.



It is observed that the overall shape of the Australian mortality curve is consistent with the general mortality findings. The existence of high infant mortality followed by low child mortality which is then followed by a high rise in the mortality rates for young adults (The Accident hump) can be observed. In the later periods, a steady increase in mortality rates can be seen which is then followed by a spike towards the end of the curve. This analysis of the log mortality curve allows us to identify the following points of interest in the Australian mortality rates:

- A high infant mortality followed by a steep drop in child mortality rates. According to the Australian Institute of Health and Welfare two third of the deaths among children aged 0-14 are comprised of Infant deaths (death before the first birthday) with a majority of these deaths taking place within 1 month of birth. This high infant mortality is caused due to two major factors including perinatal conditions and congenital anomalies[4] which tend to ease after the perinatal period, therefore we can observe a significant fall in mortality rates for children up to the age of 14.
- A spike in mortality rates for individuals aged 15-24. This spike in the mortality rate is caused due to higher risk-seeking behaviour of young adults. Current statistics show that almost three fourth of the deaths which occur in this age are caused due to injuries, where almost half of these deaths result from intentional injuries caused by factors including homicides, mental illness, crime, and violence while the other half result from unintentional injuries caused by factors including land transport accidents, accidental poisoning etc[5]. In addition to this according to the Australian Institute of Health and Welfare, the death rate for males is more than twice the death rate for females, this is indicated by a higher spike in the male mortality curve than in the female mortality curve.
- A gradual increase in mortality rates for individuals aged 15 to 109 followed by a spike at the age of 110. This gradual increase in mortality is caused due to higher exposure of older people to the leading causes of death (including ischemic heart diseases, dementia etc[5]) while the spike observed at the end of the curve results due to the age 110 being an open interval and accounting for mortality of people aged 110 and above.
- The final point of interest observed from the mortality curve is that the male mortality rates are higher than the female mortality rates for most ages. This difference is caused due to various factors including riskier occupations, higher stress levels, higher addiction to toxic substances (such as cigarettes) etc.

[4]Australian Institute of Health and Welfare 2022, *Australia's children, Infant and child deaths*, Australian Institute of Health and Welfare, viewed 15 September 2022, https://www.aihw.gov.au/reports/children-youth/australias-children/contents/health/infant-child-deaths
[5]Australian Institute of Health and Welfare 2021, *Australia's youth: Deaths*, Australian Institute of Health and Welfare, viewed 15 September 2022, https://www.aihw.gov.au/reports/children-youth/deaths

3. Parametric curve fitting – Spline models

In this section, we will choose the hyperparameters and fit the Natural Cubic splines and the Smoothing splines models to our data set. Furthermore, we will compare the fitted models using the test data set and then apply graduation tests to the superior model. Finally, we will evaluate any identified shortcomings of the discussed models.

Note:

- We assume no mortality improvement with time. This assumption is used to justify the absence of a time-series component affecting mortality rates in both the Natural Cubic spline and the Smooth spline models.
- The modelling objective in our context is accurate future mortality projection, therefore we prefer prediction accuracy over model interpretability, hence we do not add a penalty term to our loss functions for a higher number of parameters in the model.
- The datasets used are (Training data set 2017), (Validation data set 2018) and (Testing dataset 2019)
- We set the maximum age limit to 109 to prevent any inconsistency from the open interval 110.

I. Natural Cubic spline:

The natural cubic spline is a parametric mathematical construct that is used for curve fitting. The NCSs are continuous to the second derivative at all ages and are an efficient way of graduating mortality data because of their ability to capture nonlinearity in the data sets.
One of the expressions for the natural cubic splines is:

$$\mu_x = a_0 + a_1 x + \sum_{j=1}^{n} bj \times \phi j\,(x)$$

$$\text{Where: } \phi j\,(x) = \begin{cases} 0 & , \quad x < xj \\ (x - xj)^3 & , \quad x \geq xj \end{cases}$$

In this equation n is the number of knots selected and the parameters a_0 , a_1 and $b1 \ldots bn$ are the coefficients to be estimated. It is important to note that the natural cubic splines are linear before the first and after the last knot.

Hyperparameter tuning
To fit the Natural cubic spline, we first need to decide the optimal ages to place the knots. The approach adopted involves letting the knots' ages take values from 25,35 … 95 (Adult ages only). We then create a list of knots at all the possible combinations of these ages. This provides us with all the possible combinations of knots for adult ages. For all these possible combinations of knots, we first obtain the basis matrix for Natural cubic spline using the "ns" function of R's "spline" library and then use this basis matrix to fit the linear models on the training data set (2017) using the Central Exposed to risk of the training data set as the weight. The calibrated linear model provides us with estimates of the coefficients for parameters a_0 , a_1 and $b1 \ldots bn$ using the weighted least square method.

These fitted models are then used to make projections for 2018 Validation mortality rates. As noted at the start of this section we do not add a penalty for the higher number of knots therefore we choose the Means Squared error as our out-of-sample model evaluation statistic and select the number of knots that provide the lowest MSE between the (Validation 2018) mortality rates and the predicted mortality rates. This approach provides us with the optimal Knots ages of [**25 35 45 55 75 95**] corresponding to an MSE of **0.0002212799**.

Note: The default boundary Knots selected are at the range of data set i.e., ages 18 and 109. This improves the model performance at extreme ages by not enforcing linearity in the ages of interest.

Final Natural Cubic Spline Model Fitting:
After obtaining the optimal ages for the placement of knots we then use these ages to obtain the basis matrix for our final model, which is then used to fit the final Natural Cubic spline model on our training data set using the "lm" function of R. This provides us with the following coefficient estimates for our parameters:

A0	NC.B1	NC.B2	NC.B2	NC.B4	NC.B5	NC.B6	NC.B7
0.0003683	0.0001814	0.0011076	0.0028580	0.0091581	0.0707313	0.5558420	0.7554649

II. Smoothing spline

Smoothing spline is a nonparametric mathematical model which is widely used for the graduation of mortality data. The reason behind the popularity of smoothing splines is its ability to circumvent the problem of knot selection and its ability to decide the smoothness of graduation rates based on its hyperparameter $\lambda's$ value.
In the fitting of smoothing spline, we seek to find the function f which minimizes the following expression:

$$\sum_{i=1}^{n}[\hat{\mu}_{xi} - f(xi)]^2 + \lambda \int_{x1}^{xn} f''(t)^2 dt$$

Where $\hat{\mu}_{xi}$ are the crude mortality rates and λ is the hyperparameter that decides the smoothness level of the graduated rates.

Hyperparameter tuning:

Note: In our application of the smoothing splines model, instead of finding the optimal values for unbounded λ, we find the optimal value for 'spar' which is a hyperparameter bounded between (0,1] and is a monotone function of λ.

To fit the smoothing splines model, we first need to decide the optimal value for the tuning parameter "spar". The approach adopted involves letting the spar value range from 0.01 to 1 by a difference of 0.001. We then use these 1000 spar values to fit 1000 smoothing spline models on our training dataset (2017) using the function "smooth.spline" from the R library "splines".

These 1000 fitted models are then used to make projections for (2018) Validation mortality rates. As noted at the start of this section we choose the Means Squared error as our out-of-sample model evaluation statistic and select the "spar" value that provides the lowest MSE between the (Validation 2018) mortality rates and the predicted mortality rates. This approach provides us with the optimal "spar" value of **0.346** corresponding to MSE of **0.0002444306**.

Final Smooth Spline Model Fitting:
After obtaining the optimal value for the smoothing hyperparameter we use this value to calibrate our final smoothing spline model on our training dataset (2017). This provides us with the following statistics

```
Smoothing Parameter  spar= 0.346  lambda= 1.607881e-06
Equivalent Degrees of Freedom (Df): 27.27836
Penalized Criterion (weighted RSS): 4.131478e-05
GCV: 9.073905e-07
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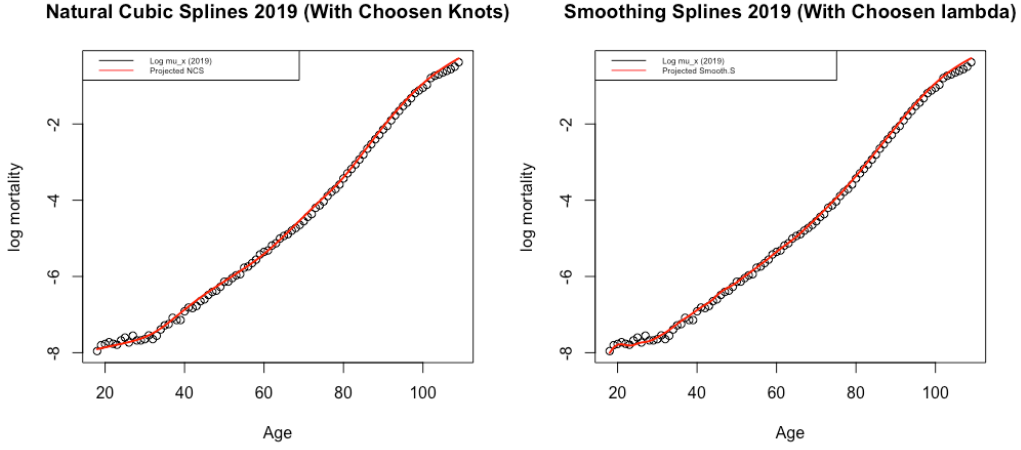
III. Comparing model performance on 2019 test data

Note: The mortality rates are graphed on a logarithmic scale to assist understanding. The test statistic used for the comparison of the models is Weighted Test MSE with Central Exposed to Risk of the test data used as weight. Weighted MSE is used to provide higher weight to mortality rates at ages with higher population.



From the graph above it can be observed that the Smoothing spline model fits the data better than the Natural cubic spline model. This result can be verified from the respective WMSE obtained using the following formulae:

$$WMSE = \frac{\sum_{x=18}^{109} E_x^c (\widehat{mx} - \dot{mix})^2}{92 \times \sum_{x=18}^{109} E_x^c}$$

Test Weighted MSE Natural Cubic Spline Model	Test Weighted MSE Smoothing spline Model
$\mathbf{4.874023 \times 10^{-8}}$	$\mathbf{4.659097 \times 10^{-8}}$

This difference exists due to the nature of the NCS model being parametric and the knots' ages being decided by the user whereas, in the smoothing spline model, the user only decides the optimal value of the smoothing parameter and the "smooth.spline" function automatically selects a large subset of ages to place the knots. Both, the NCS model and the Smoothing spline model use cubic curves, but the performance difference is caused due to the number of curves combined which depends on the number of knots selected.

IV. Graduation Testing for Smoothing spline

In this section, we will carry out various Graduation tests to check whether the smoothing spline model fits the data well. The following information will be used throughout this section unless stated otherwise:

- All the mortality observations are independent of each other
- The null hypothesis **H0:** Model fits the data well, or in other words, the graduated mortality rates reflect the actual data features therefore the actual mortality experience aligns with the graduated rates.
- Under the null hypothesis, the standardized deviations are standard normal $\mathbf{Zx} = \frac{D_x - E_x^c \dot{\mu}_x}{\sqrt{E_x^c \dot{\mu}_x}} \sim \boldsymbol{N(0,1)}$, where D_x is the actual number of deaths, $\dot{\mu}_x$ refer to the graduated mortality rates from smoothing spline model and $E_x^c \dot{\mu}_x$ refer to the expected number of deaths.
- The alternate hypothesis **H1:** The model doesn't fit the data well, in other words, the actual mortality experience doesn't align with the graduated rates.
- All the tests are carried out at 5% significance level.

Chi-squared test of fit: The Chi-squared test of fit examines the overall goodness of fit for the smoothing spline model by testing whether the observed number of deaths is consistent with the expected number of deaths. The test statistic for the Chi-squared test can be obtained by summing the square of standardised deviations over the entire age range (under consideration) as follows:

$$\boldsymbol{X = \sum_{x=18}^{109} \frac{(D_x - E_x^c \dot{\mu}_x)^2}{E_x^c \dot{\mu}_x} = 88.50114 \sim X_{91}^2}$$

The smoothing spline is a non-parametric model, but we still optimised the value of the smoothing parameter λ therefore we will deduct 1 degree of freedom from the number of observations, hence we obtain the value of Chi-squared tests degree of freedom as 92-1= 91.

Since $\mathbf{X = 88.50114 < X_{91,0.95}^2 = 114.2679}$, we retain the null hypotheses at 5% confidence level. Therefore, we can conclude that the model fits the data well and the mortality experience is in line with the graduated rates. On the other hand, from the Chi-squared test, we are unable to ensure that a few large biases are not masked by many small biases, neither are we able to detect the direction of the biases because the test statistic only informs us about the magnitude of squared deviations.

The Standardized deviations test: This is also a goodness of fit test which checks the normality of the standardized deviations in order to ensure that they are consistent with standard normal distribution. The distribution of the actual and expected standardized deviations is summarized in the table below:

Interval	(-∞,-1)	(-1,0)	(0,1)	(1, ∞)
Actual Observed	13	31	36	12
Expected	14.59628	31.40372	31.40372	14.59628
Contribution	0.174573246	0.005190058	0.672717213	0.461808470

The degree of freedom for our test is equal to the number of intervals used – 1. Therefore, we obtain the degree of freedom of **4-1 = 3**. The Chi-squared test statistic for our Standardized deviations test is:

$$X = \sum_{i=1}^{4} \frac{(O_i - E_i)^2}{E_i} = 1.314289 \sim X_3^2$$

Since $X = 1.314289 < X_{3,0.95}^2 = 7.814728$, we retain the null hypothesis at 5% significance level. Therefore, it can be concluded that the deviations do confirm to a standard normal distribution. On the other hand, the actual distribution of deviations is slightly more concentrated in the middle than the standard normal distribution indicating slight under-graduation.

Signs test: The signs test concentrates on checking the direction of the deviations in order to identify overall bias, it does so by examining the number of positive and negative deviations for an imbalance. The null and alternate hypotheses for the sings test are:

- H0: The number of positive deviations is equal to the number of negative deviations.
- H1: The number of positive deviations is not equal to the number of negative deviations.

Under the null hypotheses the positive deviations **P ~ B (92,0.5),** therefore we can obtain the acceptance region for our test null hypothesis to be [**37,55**]. The actual number of positive deviations observed is 48, this falls in the acceptance region hence we retain the null hypothesis at 5% level of significance. Therefore, we can conclude that the data is properly graduated using the smoothing spline model. On the other hand, from the results of the signs test we are unable to comment on the magnitude of these positive and negative deviations.

Cumulative deviations test: The cumulative deviations test looks at the sum of deviations in order to check whether the overall number of deaths confirm to the model. The test identifies large cumulative deviations and therefore is able to find overall bias. Under the null hypothesis, the test statistic for our cumulative deviations test is:

$$X = \frac{\sum_{x=18}^{109}(D_x - E_x^c \dot{\mu}_x)}{\sqrt{\sum_{x=18}^{109}(E_x^c \dot{\mu}_x)}} = 5.60923 \times 10^{-16} \sim N(0,1)$$

Since the test statistic 5.60923×10^{-16} falls in the acceptance region of (-1.96, 1.96) we retain the null hypothesis under 5% significance level and conclude that the overall number of deaths is in line with the graduated mortality rates. On the other hand, from the analysis of the individual deviations, it can be observed that large positive deviations were cancelled out by large negative deviations.

Grouping of signs test: The test focuses on identifying over-graduation by looking for long series or clumps of deviations with the same sign. Under the correct graduation, we expect the graduated mortality curve to oscillate around the crude mortality rate, therefore we expect the test statistic **G** (The groups of positive deviations) to be high. From our data set, we identify (m=92) total deviations out of which (n1=48) are positive deviations whereas (n2=44) are negative deviations. In addition to this, we identify (G=26) groups of positive deviations. In order to calculate the critical value K such that P ($G \leq k$) ≥ 0.05, we use the following normal approximation:

$$G \sim N\left(\frac{48(44+1)}{92}, \frac{(48 \times 44)^2}{(48+44)^3}\right)$$

At 5% significance level the Critical Value K obtained is 20 which is smaller than our test statistic of 26, hence we retain the null hypothesis and conclude that appropriate graduation has taken place. On the other hand, the conclusion would have been different if we would have selected the test statistic as the number of groups with negative deviations.

Note: In our context, we will use the serial correlation test as a one-sided test for over-graduation.

Serial correlation test: The objective of the test is to identify over-graduation by finding clumps of deviations with the same sign. The process involves the analysis of the correlation between standardised deviations for consecutive ages which are assumed to be independent under the null hypothesis. Therefore, we can find the correlation coefficient for two sequences of standardised deviations with a lag of 1 using the following equation:

$$r_1 = \frac{\sum_{i=1}^{91}(z_i - \bar{z}^{(1)})(z_{i+1} - \bar{z}^{(2)})}{\sqrt{\sum_{i=1}^{91}(z_i - \bar{z}^{(1)})^2 \sum_{i=1}^{91}(z_{i+1} - \bar{z}^{(2)})^2}} = -0.47849$$

Where $\bar{z}^{(1)}, \bar{z}^{(2)}$ *are the respective mean for the sequences with* $\bar{z}^{(1)} = 0.04443851$ *and* $\bar{z}^{(2)} = 0.05441476$

Under the null hypothesis, we can obtain the test statistic $r_1\sqrt{m} = r_1\sqrt{92} = -4.589515 \sim N(0,1)$, and the critical value of 1.64. Since the test statistic is less than the critical value, we retain the null hypothesis under 5% significance level and conclude that no over graduation of data has taken place. On the other hand, it should be noted that there is a possibility of a positive correlation in a part of the age range getting cancelled by a negative correlation in another part of the age range.

Overall Evaluation: All 6 graduation tests conclude that the graduated data adheres to the mortality features, therefore it is safe to conclude that the smooth spline model with the optimised tuning parameter is a good fit for the data set and makes reasonable mortality projections.

V. Model's shortcomings Evaluation:

The analysis conducted in this section has indicated the following shortcomings of the Smoothing Spline model:

- The smoothing parameter selection is subjective to a certain extent, and this subjectivity can greatly impact the model's performance. A high smoothing parameter can lead to over graduation, compromising the model's ability to pick up mortality trends, whereas a low smoothing parameter can cause under graduation causing the model to pick up random noise. Therefore, caution is required when selecting the optimal value for tuning parameter in order to have accurate mortality forecasts.
- The smoothing spline model is highly data-driven and therefore is unsuitable for scenarios where a small sample size is available. This shortcoming is verified by a high magnitude of deviations at later ages due to the availability of a small sample population. Therefore, it is advisable not to completely rely on smoothing spline mortality projections for later ages.
- The smoothing spline mortality model lacks interpretability. We can obtain accurate mortality projections using the smoothing spline model, but the model doesn't consist of covariates directly explaining the mortality changes. Therefore, it is hard to interpret the results obtained using the smoothing spline model. It is possible for us to obtain accurate future lifetables, but it is hard to interpret the causes behind the mortality trends.
- The smoothing spline model doesn't consist of any components that consider the time factor affecting mortality. This assumption of mortality rates being independent of the time effect is inaccurate and therefore will lead to inaccurate mortality forecasts for the future. This shortcoming is verified by a consistent fall in mortality rates over the last century due to factors including higher health awareness and medical advancements which make the mortality models of the past unsuitable for the present day.

4. Mortality projection fitting – Lee-Carter Model

In this section, we will use the Lee-Carter (LC) model for mortality projections. This is a stochastic model which takes into account both the age and period effects in order to obtain accurate mortality forecasts. The mathematical expression for the lee carter model is:

$$\ln(m_{x,t}) = \alpha_x + \beta_x \kappa_t + \epsilon_{x,t}$$

Where:

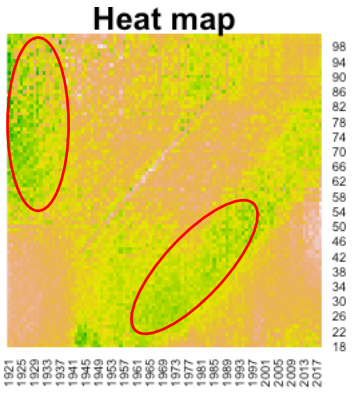
- $m_{x,t}$ is the central mortality rate for the age x in the year t, represented by the variable "rate" in our dataset.
- α_x is the average for $\ln\ (m_{x,t})$ for a specific age x over the entire range of years t in the dataset. It represents the general shape of mortality for a particular age x.
- κ_t represents the time trend of mortality at a particular time t for all ages. It depicts the difference between the average mortality and the mortality rates at time t to indicate the change caused by the time effect.
- β_x represents the change in mortality rates at a particular age x caused in response to the time trend κ_t. It depicts the effect of mortality improvement for a particular age x.
- $\epsilon_{x,t}$ are independently distributed normal random errors with mean 0 and variance σ^2

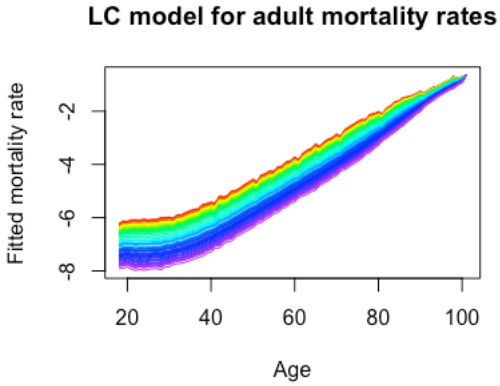
Lee-Carter model fitting:

The lee carter model fitting involves calibration of parameters including $\alpha_x, \beta_x\ and\ \kappa_t$. α_x estimate is calculated by obtaining time-averaged log mortality rates $\ln\ (m_{x,t})$, whereas Singular Value Decomposition is used to obtain estimates for $\beta_x\ and\ \kappa_t$ while keeping the following restrains in consideration:

$$\sum_x \beta_x = 1, \qquad \sum_t \kappa_t = 0$$

We fit a Lee-Carter model on the Australian adult mortality data from the years 1921 to 2018 in the age range 18 to 101. The age range is capped at 101 in order to prevent inconsistencies caused by the missing mortality data values for the higher ages. The following plots indicate the residual heatmap and fitted adult mortality rates from the LC model:





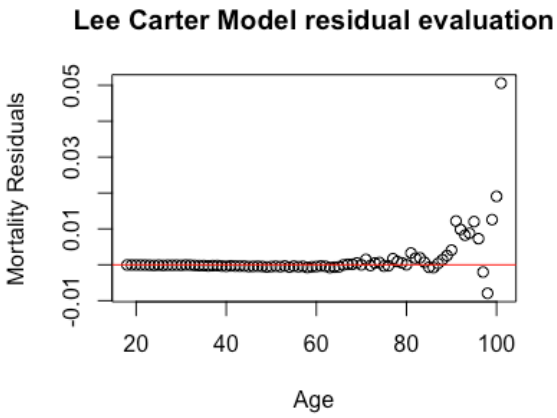
The heat map depicts two strong patterns. Firstly, a strong pattern of negative residuals can be observed for mortality rates in the early years for the older ages. A possible explanation for this is poor record keeping in that period. Secondly, it can be observed that the lee carter model was unable to take into account the cohort effect on mortality rates. This is indicated by the pattern of negative residuals in the diagonal form for the cohort that represents the young adults alive at the time of World War two (1939:1945).

From the plot for the fitted mortality rates for the Lee carter model, it is observed that the mortality estimates adhere to the general mortality trends. In addition to this, a consistent decline in the overall mortality rates can be observed. This indicates mortality improvement over time and is in line with the empirical evidence.

Test error of projections for the year 2019:

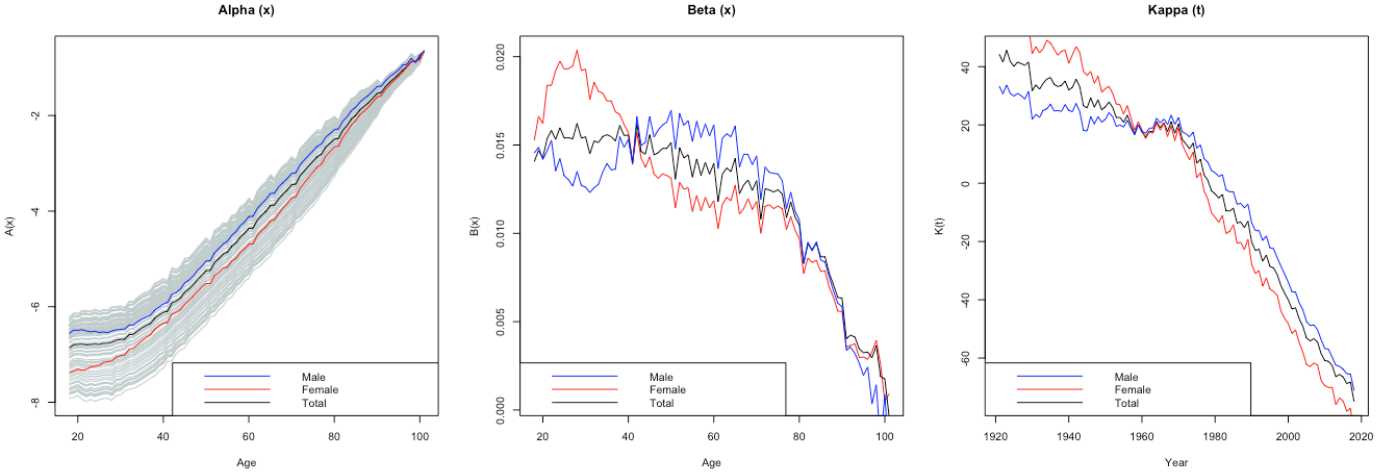
The calibrated Lee-Carter model is used to obtain the mortality projections for the year 2019. These projections are then compared with the crude mortality rates in order to obtain value for our out-of-sample test MSE as follows:

$$MSE = \frac{\sum_{x=18}^{101}(\widehat{mx}-\dot{mx})^2}{84} = 4.517354 \times 10^{-5}$$



The low value of test MSE shows that the Lee-Carter model fits the data well. The projected mortality rates for the year 2019 were close to the observed mortality rates, therefore it can be concluded that the Lee-Carter model makes accurate mortality projections. In addition to this, the residual plot on the right shows that the Lee-carter mortality projections were inaccurate for the ages above 90. This is caused due to the nature of the lee-Carter model being data-driven; hence the model is unable to perform well in situations where the available sample size is small.

Lee-Carter Model Parameters explanation:



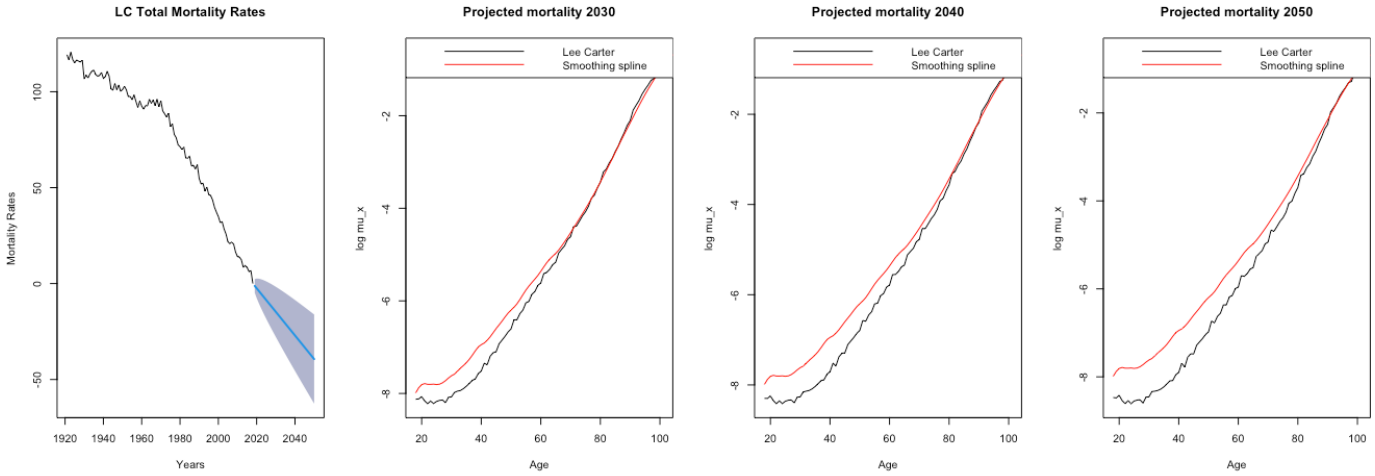
The curves for the alpha x represent the total average $\ln\left(m_{x,t}\right)$ and average $\ln\left(m_{x,t}\right)$ segregated by gender. From the plot, it can be observed that the log mortality rates increase with the age and are almost linear for higher ages. In addition to this, we can observe that the α_x values for males are higher than the α_x values for females across all ages, this indicates that male mortality has been higher than female mortality in the past (1921:2018).

The β_x curve represents the impact of mortality change κ_t on a particular age x. It can be observed that the impact of mortality change is higher for the younger ages than for the older ages. A very rapid fall can be observed in the value of β_x after the age of 80 indicating that the mortality change due to time affect has little to no affect for people above 80. In addition to this we can observe that the β_x values are higher for females before the age of 40 whereas they are higher for males after the age of 40. This indicates that the mortality changes have a higher impact on younger females and older males.

The curve for κ_t represent the improvement in Australia's mortality with time. This improvement is indicated by a continuous fall in the value of κ_t over time. It can be observed that the overall mortality improvement was slow before the year 1970 and benefited males more than females. On the other hand, the mortality improvement became more rapid after 1970 and started benefiting females more than males. The reasons for this mortality improvement can include medical advancements, higher safety measures, reduction in the number of wars etc.

5. Model Comparison:

Mortality Projection: In this section, we will project the future mortality rates for the Australian population using both the smoothing spline model and the Lee-Carter model. For the smoothing spline model, we assume that the mortality rates are independent of the time effect whereas for the lee carter model we project mortality by forecasting κ_t using a simple random walk with drift, $\kappa_t = \mu + \kappa_{t-1} + \xi_t$. The process is executed in R and the following plots are obtained for the years (2030,2040 and 2050):



It can be observed from the Projected mortality curves that the Smoothing spline model ignores the mortality improvement therefore the log(mortality) curve for the smoothing spline model is constant for all the future years. On the other hand, the decreasing curves for the lee-Carter model indicate the impact of the time factor on mortality, this result is also depicted by the declining projected mortality rates in the "LC Total Mortality Rates" plot. This plot also indicates that the variance of the projected mortality rate increases with an increase in the number of years of projection. After examining these plots, we can conclude that the Lee carter model is likely to provide more accurate mortality projections whereas the smoothing spline model's projections will become useless with the increase in the number of years of projection.

Implications of ignoring mortality improvement on lifetime annuities:
From the past empirical data, it is known that mortality is likely to improve with time. This mortality improvement results due to various factors including wider public health initiatives, medical advancements, stricter implementations of regulations etc[7]. Ignoring this mortality improvement in future mortality projections is likely to cause an overestimation of mortality rates and an underestimation of life expectancy.
In the proposed policy, the provision of life annuities to the retirees transfers the longevity risk from the individuals to the government, therefore if the actual mortality experience is better than the projected mortality experience then the government might have to face the following problems:

Incorrect calculations for contributions required: The underestimation of the future life expectancy can lead to underpricing of the life annuity [6] . This underpricing of the annuity will lead to incorrect calculations for the contribution that the individuals are required to make during their working lives.
Incorrect budget calculations: In the long run if the net present value of the annuity payments is higher than expected, the government will become unable to meet its liabilities[6]. This can have an adverse impact on both the government's budget and the people's trust in the government's policies.

On the other hand, from our analysis in the previous sections and the projected mortality curves above, we have identified that the mortality improvements have a much more significant impact on the younger population than on the senior citizens. Since the proposed product is a deferred lifetime annuity with payment made at older ages, therefore the mortality improvement is likely to have a smaller impact on our target population making our assumption slightly justifiable.

[6]Antolin, P. (2007), "Longevity Risk and Private Pensions", OECD Working Papers on Insurance and Private Pensions, No. 3, OECD Publishing. doi:10.1787/261260613084
[7]Marianne C. (2011), MORTALITY IMPROVEMENTS ANALYSIS OF THE PAST AND PROJECTION OF THE FUTURE, Society of Actuaries, Viewed 16 September 2022, Mortality Improvement Analysis

Potential Improvements:

Smoothing spline model:
Through the detailed analysis carried out in this report, we have identified that the smoothing spline model fits the Australian mortality data well but is unable to provide accurate mortality projections due to the lack of a time-series related coefficient. The proposed solution to this problem is the use of the Reduction Factor Method, where the future mortality rates are calculated as a proportion of the baseline mortality rates. This enables us to reflect the fall in mortality while keeping the baseline data features intact. The final mortality rates can be calculated using the following equation:

$$m_{x,t} = m_{x,0} \times R_{x,t}$$

Where $m_{x,t}$ is the future projected mortality rate at time t, $m_{x,0}$ refers to the baseline fitted mortality rates from the smoothing spline model and $R_{x,t}$ refers to the reduction factor at time t. From our analysis, we have identified that the mortality rates are falling at an increasing rate. Therefore, $R_{x,t}$ should be selected as an increasing function of time with the help of appropriate expert opinion.

A further recommendation for the improvement of the smoothing spline model is to update the baseline mortality rates and the future projections every year when the mortality data for the year becomes available.

Lee-Carter Model:
In section 4, through the analysis of the heatmap for the residuals of the Lee-Carter projections, we have identified that the Lee-Carter model is unable to pick up the cohort effects on the mortality rate. This occurs due to the absence of a coefficient that is linearly dependent on both the age and period terms in the Lee-Carter model.
A proposed solution is the use of the age-period-cohort extension to the Lee-Carter model. The extended Lee-Carter model is expressed through the following mathematical expression:

$$\ln(m_{x,t}) = \alpha_x + \beta_x^{(1)}\kappa_t + \beta_x^{(2)}\gamma_{x-t} + \epsilon_{x,t}$$

Where the additional term γ_{x-t} represents the cohort impact on the mortality rates whereas $\beta_x^{(2)}$ represents the proportion of the cohort impact that is applicable to a specific age x. This extension enables the Lee carter model to identify the linear relationship between the age and the period terms.

In addition to this, while forecasting κ_t using a simple random walk with drift, $\kappa_t = \mu + \kappa_{t-1} + \xi_t$, we can identify that the drift is a constant. On the other hand, through our analysis we have established that the mortality rates fall at an increasing rate, this result is depicted by the increased steepness of the kappa(t) curve after the year 1970 (in section 4). Therefore, it is recommended to use a drift term which is an increasing function of time in order to reflect the exponential fall of mortality rates in the mortality forecasts.

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[3]Human Mortality Database (2016). University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Retrieved from http://www.mortality.org accessed on 15 Sep 2022.

[4]Australian Institute of Health and Welfare 2022, *Australia's children, Infant and child deaths*, Australian Institute of Health and Welfare, viewed 15 September 2022, https://www.aihw.gov.au/reports/children-youth/australias-children/contents/health/infant-child-deaths

[5]Australian Institute of Health and Welfare 2021, *Australia's youth: Deaths*, Australian Institute of Health and Welfare, viewed 15 September 2022, https://www.aihw.gov.au/reports/children-youth/deaths

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