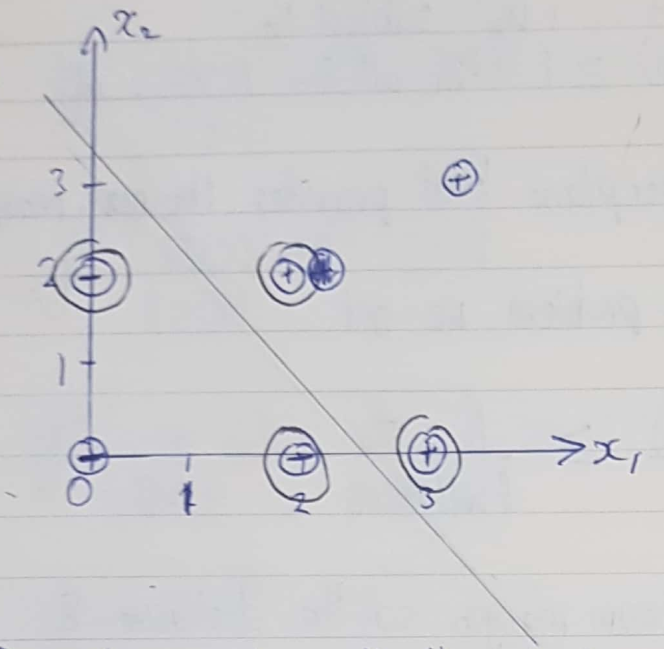
Problem Set 3: SVM & PCA

Problem 1.

a)



Yes, the two classes are linearly separable as clearly observed by inspection.

b) The points with the double circles are the support vectors.

$\left(\frac{5}{2}, 0\right) \left(1, 2\right) \quad m = -\frac{4}{3} \quad \frac{4}{3}x_1 + x_2 - \frac{10}{3} = 0$

$W = \langle \frac{4}{3}, 1 \rangle \quad b = -\frac{10}{3}$

c) Support vector are the ones that determine how much the margin will expand and hold the margin from expanding.
$\therefore$ removing a support vector will cause the margin to either remain the same <u>or</u> increase.

Let w, b be an optimal solution to:

Minimize $||w||^2 = w_1^2 + w_2^2 + \dots + w_n^2$ subject to:

$$y^{(i)}(w \cdot x^{(i)} + b) \geq 1 \quad \text{for all } i = 1, 2, \dots m$$

Then $w \cdot x + b = 0$ is a hyperplane that provides the max. margin

Taking the hard margin optimisation problem, we get

$$\frac{y^{(i)}(w \cdot (x)^{(i)} + b)}{||w||} \geq \frac{1}{||w||}$$

We know that $\frac{1}{||w||}$ is exactly the maximum margin so the distance δ is given by;

$$\delta \geq \frac{1}{||w||}$$

Suppose there exists a margin $> \frac{1}{||w||}$. This would mean the following is true.

$$y^{(i)}(w \cdot (x)^{(i)} + b) \geq 1 + \alpha$$, where α is the greater margin.

This would mean that there are new values of w & b, ~~such~~ w^* & b^* such that

$$w^*(x)^{(i)} + b^* \geq w \cdot x^{(i)} + b$$

But! w & b are optimal solutions so there is a contradiction as shown above.

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b) $z' = \frac{z}{\|z\|M}$ & $d' = \frac{d}{\|z\|M}$

$$\frac{y^{(i)}\left(z.x^{(i)}+d\right)}{\|z\|} \geq M$$

$$\frac{y^{(i)}\left(z.(x)^{i}+d\right)}{\|z\|M} \geq 1$$

$$y^{(i)}\left(\frac{z}{\|z\|M}(x^{i}) + \frac{d}{\|z\|M}\right) \geq 1$$

$y^{(i)}\left(z' x^{(i)} + d'\right) \geq 1 \Rightarrow$ Thus, z' & d' provide a feasible solution for the hard margin.

Since $\vec{w}$ & b are the optimal solutions, it follows that

$|\vec{w}|^2 \leq |\vec{z}'|^2$, which turn follows that $|\vec{w}| \leq |\vec{z}'|$

c) Consider $|\vec{z}'|$

$\cancel{|\vec{z}'| = \sqrt{\vec{z}'.\vec{z}'}}$

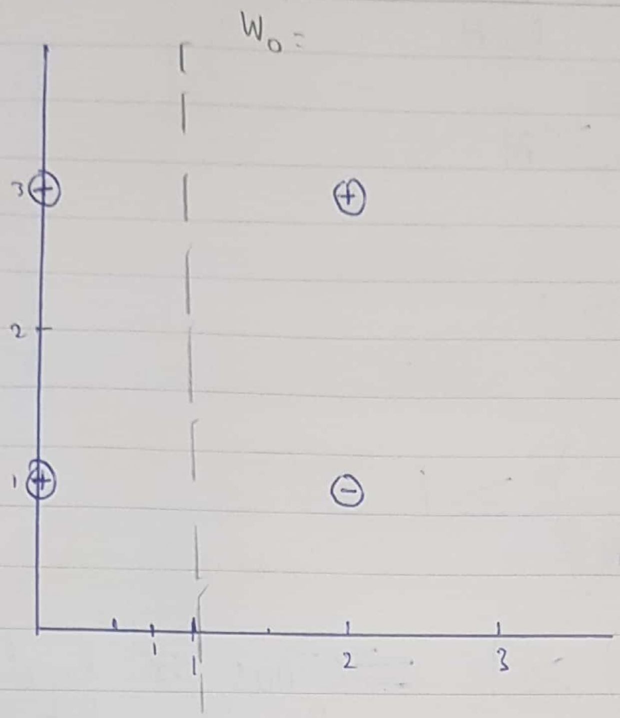
$$|\vec{z}'| = \sqrt{z'.z'} = \sqrt{\frac{\vec{z}}{|\vec{z}|M} \cdot \frac{\vec{z}}{|\vec{z}|M}} = \frac{|\vec{z}|}{|\vec{z}|M} = \frac{1}{M}$$

we know that $|w| \leq |z'|$ from b)

$\therefore \frac{1}{|w|} \geq \frac{1}{|z|} \Rightarrow \frac{1}{w} \geq M$ where $\frac{1}{w}$ is the exact max. margin and M is the margin of any general case.

$\therefore \frac{1}{w} \geq M$ for any z

Problem 3



1) For a hard margin to exist, the points have to be linearly separable. It is very clear that the convex hull of the $\oplus$ set intersects with the convex hull of the ~~negative~~ $\ominus$ set. This means that the two convex hulls are not disjoint so the points are not linearly separable. *

2) Taking $\langle w_0, w_1, b\rangle$ to be $\langle 0,1,0\rangle$ as shown above, we get

$$\varepsilon^{(i)} = \max(0, 1 - y^{(i)}(w.x^{(i)} + b))$$

$$\varepsilon^1 = \max(0, 1-1(\langle 0,1\rangle.\langle 0,1\rangle + 0)) \Rightarrow \max(0,0) \Rightarrow \underline{\underline{0}}$$

$$\varepsilon^2 = \max(0, 1-1(\langle 0,1\rangle.\langle 2,3\rangle + 0) \Rightarrow \max(0,-2) = \underline{\underline{0}}$$

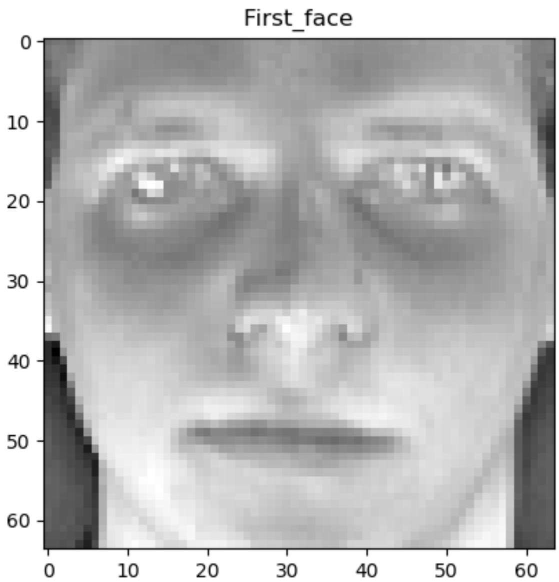
$$\varepsilon^3 = \max(0, 1+1(\langle 0,1\rangle.(0,3\rangle + 0) = \max(0,4) = \underline{\underline{4}}$$

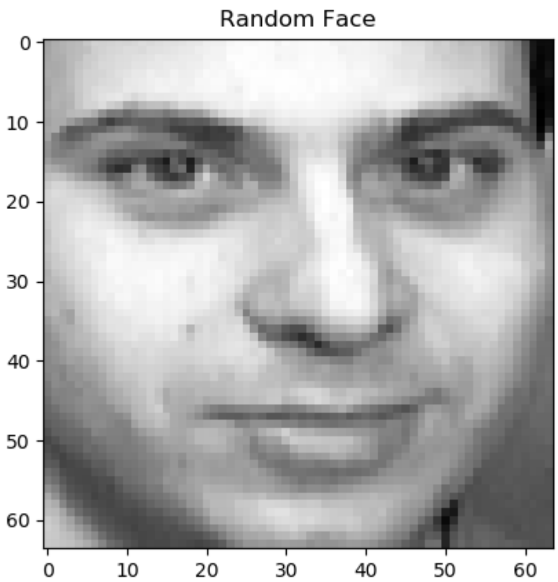
$$\varepsilon^4 = \max(0, 1+1(\langle 0,1\rangle.\langle 2,1\rangle + 0) = \max(0,2) = \underline{\underline{2}}$$

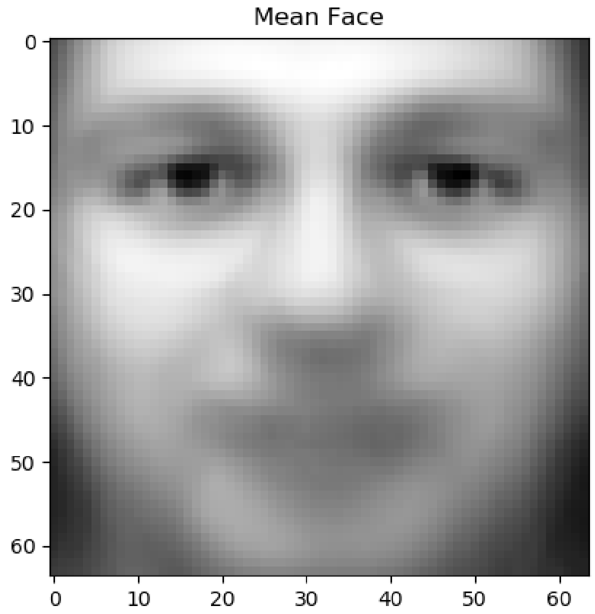
$$(w_0, w_1, b, \varepsilon^1, \varepsilon^2, \varepsilon^3, \varepsilon^4) = (0,1,0,0,0,4,2)$$

* As given by the definition of Linear Separability

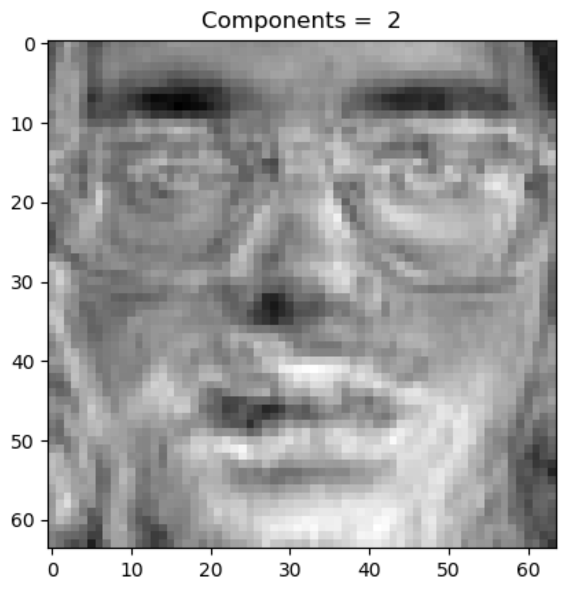
ProMate


First_face

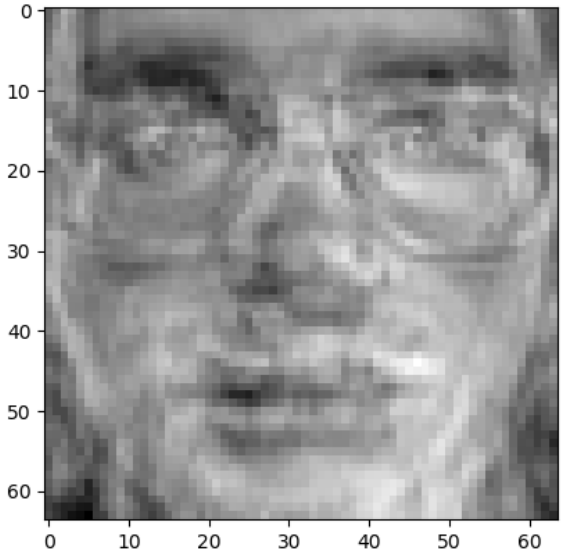

Random Face



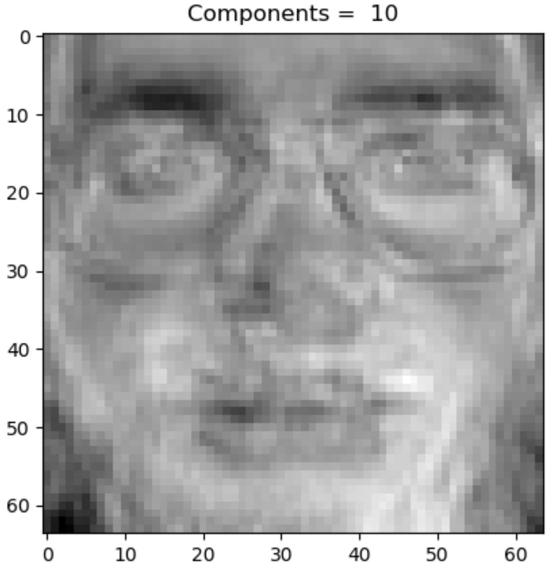
As instructed, the 99th index was used to calculate the components.

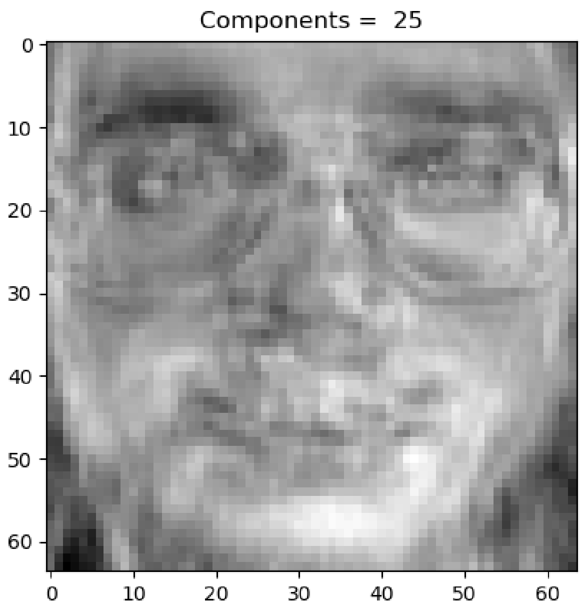


Components = 5

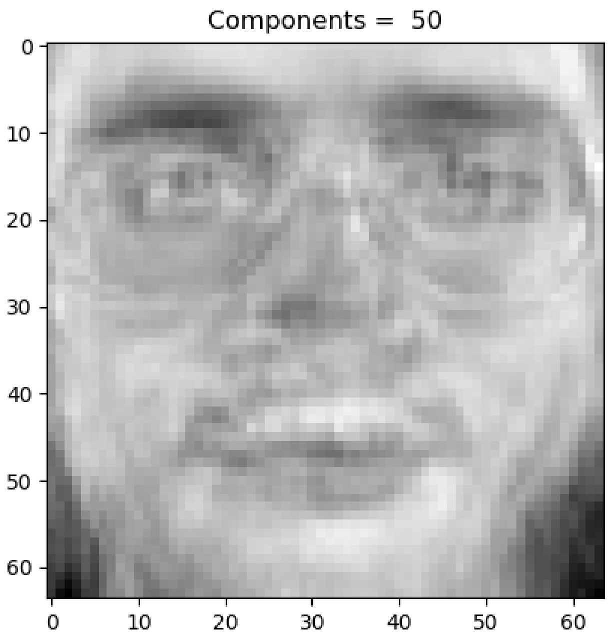


Components = 10

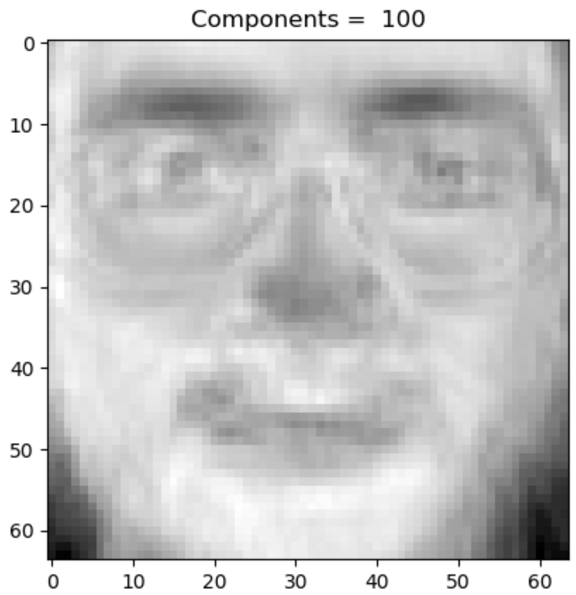


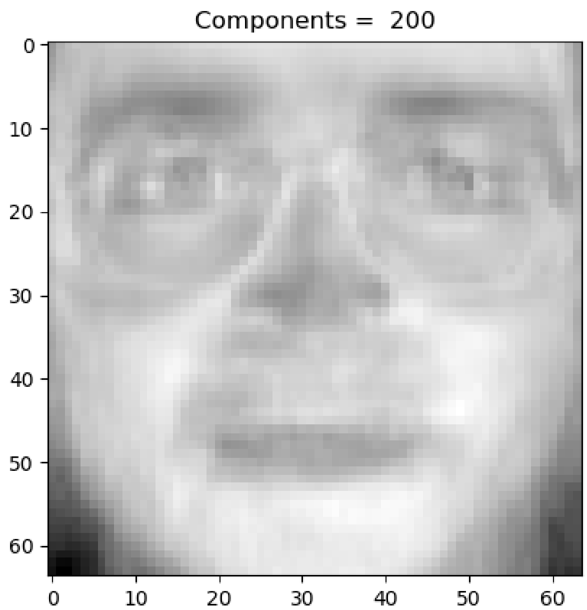


Components = 25
0
10
20
30
40
50
60
0
10
20
30
40
50
60



Components = 50
0
10
20
30
40
50
60
0
10
20
30
40
50
60


Components = 100
0
10
20
30
40
50
60
0
10
20
30
40
50
60


Components = 200
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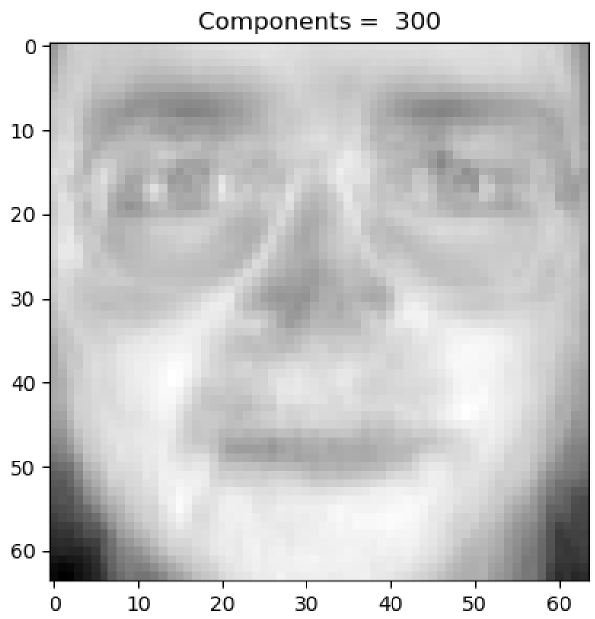
30
40
50
60
0
10
20
30
40
50
60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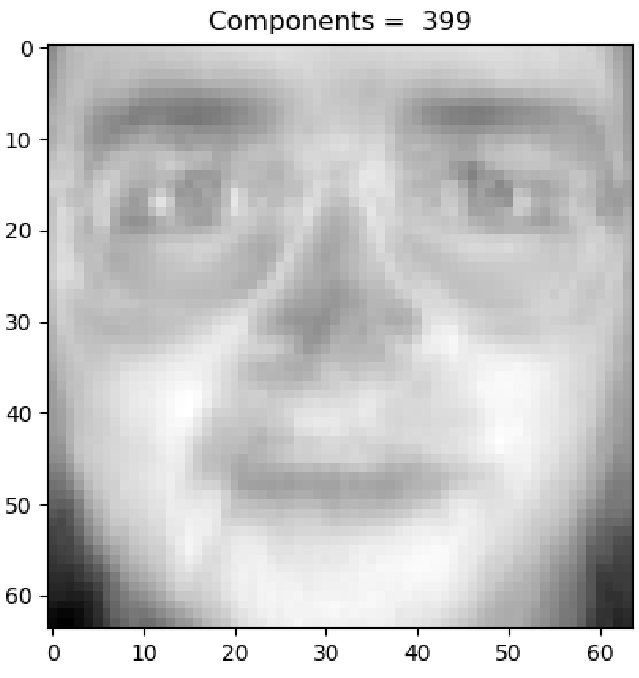
40
50
60
0
10
20
30
40
50
60


Components = 399
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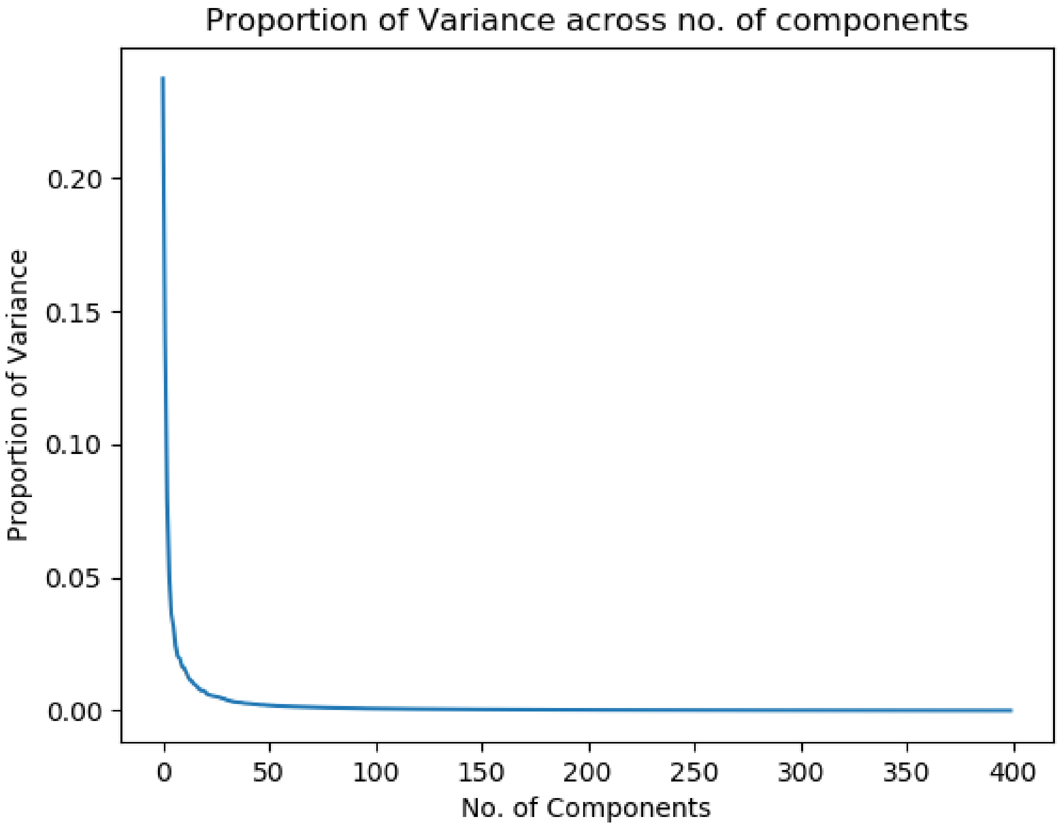
20
30
40
50
60
0
10
20
30
40
50
60



Proportion of Variance across no. of components
Proportion of Variance
0.20
0.15
0.10
0.05
0.00
0
50
100
150
200
250
300
350
400
No. of Components