D.E

System of Differential Equation:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

System of linear equations.

$\downarrow$

$$Ax = b$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Solve by elimination method

Q#1:
$$\frac{dx}{dt} = 3y \quad \text{--- (1)}$$
$$\frac{dy}{dt} = 2x \quad \text{--- (2)}$$

Sol: let $D = \frac{d}{dt}$

$$Dx - 3y = 0 \quad \text{--- (3)}$$
$$Dy - 2x = 0 \quad \text{--- (4)}$$

$$Dx - 3y = 0 \quad \text{--- (5)}$$
$$-2x + Dy = 0 \quad \text{--- (6)}$$

Multiply D with eq (5) & 3 with eq (6)

$$D^2x - 3Dy = 0$$
$$-6x + 3Dy = 0 \quad \text{Add}$$

$$D^2x - 6x = 0 \quad \text{--- (7)}$$

$$(D^2 - 6)x = 0$$

$$\frac{d^2x}{dt} - 6x = 0$$

let $y = e^{\lambda t}$

$y' = \lambda e^{\lambda t}$

$y'' = \lambda^2 e^{\lambda t}$

$\vdots$

$D^2 - 6 = 0$

$\because \lambda^2 - 6 = 0$

$\lambda^2 = 6$

$\lambda = \pm\sqrt{6}$

So, $\boxed{x(t) = c_1 e^{\sqrt{6}t} + c_2 e^{-\sqrt{6}t}} \rightarrow (8)$

Multiply 2 with eq (5) & D with eq (6)

$$\begin{array}{r} 2Dx - 6y = 0 \\ -2Dx + D^2y = 0 \\ \hline -6y + D^2y = 0 \end{array} \quad — (9)$$

$(D^2 - 6)y = 0$

let $y = e^{\lambda t}$

$\vdots$

$D^2 - 6 = 0$

$\because \lambda^2 - 6 = 0$

$\lambda = \pm\sqrt{6}$

So, $\boxed{y(t) = c_3 e^{\sqrt{6}t} + c_4 e^{-\sqrt{6}t}} \rightarrow (10)$

Equalizing the coefficients:-

$$\frac{dx}{dt} = 3y$$

$$\sqrt{6}C_1 e^{\sqrt{6}t} - \sqrt{6}C_2 e^{-\sqrt{6}t} = 3\left(C_3 e^{\sqrt{6}t} + C_4 e^{-\sqrt{6}t}\right)$$

$$\sqrt{6}C_1 e^{\sqrt{6}t} - \sqrt{6}C_2 e^{-\sqrt{6}t} = 3C_3 e^{\sqrt{6}t} + 3C_4 e^{-\sqrt{6}t}$$

$\rightarrow$ $\sqrt{6}C_1 = 3C_3$

$$C_3 = \frac{\sqrt{6}}{3}C_1$$

$\rightarrow$ $-\sqrt{6}C_2 = 3C_4$

$$C_4 = -\frac{\sqrt{6}}{3}C_2$$

$$x(t) = C_1 e^{\sqrt{6}t} + C_2 e^{-\sqrt{6}t}$$

$$y(t) = \frac{\sqrt{6}}{3}C_1 e^{\sqrt{6}t} - \frac{\sqrt{6}}{3}C_2 e^{-\sqrt{6}t}$$ Ans.

Q#2:- $\frac{dx}{dt} = x + 2y$ — (1)

$\frac{dy}{dt} = 4x + 3y$ — (2)

Sol:- let $D = \frac{d}{dt}$

$Dx - x - 2y = 0$ — (3)

$Dy - 4x - 3y = 0$ — (4)

$(D-1)x - 2y = 0$ — (5)

$-4x + (D-3)y = 0$ — (6)

Multiply 4 by eq (5) & (D-1) by eq (6)

$$4(D-1)x - 8y = 0$$

$$-4(D-1)x + (D-3)(D-1)y = 0 \quad \text{Add}$$

$$(D-3)(D-1)y - 8y = 0$$

$$D^2y - 4Dy + 3y - 8y = 0$$

$$D^2y - 4Dy - 5y = 0$$

let $y = e^{\lambda t}$

⋮

$$(D^2 - 4D - 5)y = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$\lambda^2 - 5\lambda + \lambda - 5 = 0$$

$$\lambda(\lambda - 5) + 1(\lambda - 5) = 0$$

$$\lambda = -1, \quad \lambda = 5$$

⋮

$$y(t) = c_1 e^{5t} + c_2 e^{-t}$$

$$x(t) = c_3 e^{5t} + c_4 e^{-t}$$

Equalizing coefficients

$$\frac{dx}{dt} = x + 2y$$

$$5c_3e^{5t} - c_4e^{-t} = c_3e^{5t} + c_4e^{-t} + 2c_1e^{5t} + 2c_2e^{-t}$$
$$5c_3e^{5t} - c_4e^{-t} = (c_3 + 2c_1)e^{5t} + (c_4 + 2c_2)e^{-t}$$

$\longrightarrow$ $5c_3 = c_3 + 2c_1$

$4c_3 = 2c_1$

$c_3 = \frac{1}{2}c_1$

$\longrightarrow$ $-c_4 = c_4 + 2c_2$

$-2c_4 = 2c_2$

$c_4 = -c_2$

$y(t) = c_1e^{5t} + c_2e^{-t}$

$x(t) = \frac{1}{2}c_1e^{5t} - c_2e^{-t}$ Ans.

Non-Homogeneous DE

Q :- $Dx + y = t \longrightarrow (1)$

$-x + Dy = t \longrightarrow (2)$

Eliminate x

$\cancel{Dx} + y = t$

Add $-\cancel{Dx} + D^2y = -Dt = -1$ $\left(\because \frac{dt}{dt} = 1\right)$

$D^2y + y = t - 1$

$y(D^2+1) = t - 1 \longrightarrow (3)$

let :

$r^2 + 1 = 0$

$r^2 = -1$

$r = \pm i$

$$\boxed{y_c = c_1\cos t + c_2\sin t}$$

Now

y_p :-

let $y_p = At + B$

$y'_p = A$

$y''_p = 0$

Put in (3)

$D^2 y_p + y_p = t - 1$

$0 + At + B = t - 1$

Compare coefficients

$A = 1$, $B = -1$ so $y_p = t - 1$

$y(t) = y_c + y_p$

$$\boxed{y(t) = c_3 \cos t + c_4 \sin t + t - 1} \quad -(5)$$

Now same working for $x(t)$

⋮

$$\boxed{x(t) = c_1 \cos t + c_2 \sin t + t + 1} \rightarrow (4)$$

Now equalizing contant

Put value in eq (1)

$-c_1 \sin t + c_2 \cos t + \cancel{1} + c_3 \cos t + c_4 \sin t + \cancel{t} - \cancel{1} - \cancel{t} = 0$

$(-c_1 + c_4) \sin t + (c_2 + c_3) \cos t = 0$

$-c_1 + c_4 = 0$ $\qquad$ $c_2 + c_3 = 0$

$c_4 = c_1$ $\qquad$ $c_3 = -c_2$

So

$$\boxed{\begin{aligned} y(t) &= -c_2 \cos t + c_1 \sin t + t - 1 \\ x(t) &= c_1 \cos t + c_2 \sin t + t + 1 \end{aligned}} \rightarrow \underline{\text{Ans}}$$

CamScanner

Q:- $\frac{dx}{dt} - 4x + \frac{d^2y}{dt^2} = t^2$

$\frac{dx}{dt} + x + \frac{dy}{dt} = 0$

let $D = \frac{d}{dt}$

$Dx - 4x + D^2y = t^2$ — (1)

$Dx + x + Dy = 0$ — (2)

$(D-4)x + D^2y = t^2$ — (3)

$(D+1)x + Dy = 0$ — (4)

to eliminate y :-

Multiply $-D$ with (4) and add

$(D-4)x + \cancel{D^2y} = t^2$

$-(D+1)Dx - \cancel{D^2y} = 0$

$(D-4)x - (D+1)Dx = t^2$

$\cancel{Dx} - 4x - D^2x - \cancel{Dx} = t^2$

$-D^2x - 4x = t^2$

$(-D^2 - 4)x = t^2$

$(D^2+4)x = -t^2$ — (5)

let $(D^2+4)x = 0$

$\vdots$

$r^2 + 4 = 0$

$r^2 = -4$

$r = \pm 2i$

$x_c = c_1 \cos 2t + c_2 \sin 2t$

for x_p consider At^2+Bt+C

$$x_p = At^2+Bt+C$$
$$x'_p = 2At+B$$
$$x''_p = 2A$$

Put in eq (5)

$$D^2x_p + 4x_p = -t^2$$
$$2A + 4At^2+4Bt+4C = -t^2$$
$$4At^2+4Bt+2A+4C = -t^2$$

Compare

$4A=-1$, $4B=0$, $2A+4C=0$

$A = -\frac{1}{4}$ $B=0$ $2(-\frac{1}{4})+4C=0$

$$-\frac{1}{2}+4C=0$$
$$C = \frac{1}{2}\times\frac{1}{4}$$
$$C=\frac{1}{8}$$

So, $x_p = -\frac{1}{4}t^2+\frac{1}{8}$

So, $x(t) = x_c + x_p$

$$x(t) = c_1\cos 2t + c_2\sin 2t - \frac{1}{4}t^2+\frac{1}{8}$$

Now,

to eliminate x :-

Multiply $(D+1)$ by eq (3) and $(D-4)$ with eq (4) and subtract

$$\cancel{(D-4)(D+1)x} + D^2y(D+1) = t^2(D+1)$$
$$-\ \cancel{(D+1)(D-4)x} \pm Dy(D-4) = 0$$
$$D^2y(D+1) - Dy(D-4) = t^2(D+1)$$

$$\cancel{D^3y + D^2y} - \cancel{D^2y} + 4Dy = Dt^2 + t^2$$

$$D^3y + 4Dy = 2t + t^2 \quad \left(\because \frac{d}{dt}t^2 = 2t\right) \rightarrow (6)$$

let $D^3y + 4Dy = 0$

$$(D^3 + 4D)\,y = 0$$

$$r^3 + 4r = 0$$

$$r(r^2 + 4) = 0$$

$r = 0$, $r^2 = -4$

$$r = 0 \pm 2i$$

$$y_c(t) = C_3 e^{0t} + C_4 \cos 2t + C_5 \sin 2t.$$

$$= C_3 + C_4 \cos 2t + C_5 \sin 2t.$$

$$y_p = At^3 + Bt^2 + Ct \quad \text{---(7)}$$

It is cube because C_1 and C is same in $At^2 + Bt + C$

$$y'_p = 3At^2 + 2Bt + C$$

$$y''_p = 6At + 2B$$

$$y'''_p = 6A$$

Put in (6)

$$6A + 4At^3 + 4Bt^2 + 4Ct = t^2 + 2t$$

$4A = 0$, $4B = 1$, $4C = 2$, $6A = 0$

$A = 0$ $\quad B = 1/4$ $\quad C = 1/2$

Put in (7)

$$y_p = \frac{1}{4}t^2 + \frac{1}{2}t$$

$$y(t) = y_c + y_p$$

$$y(t) = c_3 + c_4 \cos 2t + c_5 \sin 2t + \frac{1}{4}t^2 + \frac{1}{2}t$$

$\vdots$

$\longleftrightarrow$