1

Topic: 2.3)

COMPUTER GRAPHICS

try. Computer graphics also play a major role in the manufacturing world. *Computer-aided design* (CAD) is used to design a computer model of a product and then, by subjecting the computer model to a variety of tests (carried out on the computer), changes to the current design can be implemented to obtain an improved design. One of the notable successes of this approach has been in the automobile industry, where the computer model can be viewed from different angles to obtain a most pleasing and popular style and can be tested for strength of components, for roadability, for seating comfort, and for safety in a crash.

In this section we give illustrations of matrix transformations $f: R^2 \to R^2$ that are useful in two-dimensional graphics.

EXAMPLE 1 Let $f: R^2 \to R^2$ be the matrix transformation that performs a reflection with respect to the x-axis. (See Example 5 in Section 1.5.) Then f is defined by $f(\mathbf{v}) = A\mathbf{v}$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus we have

$$f(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}.$$

To illustrate a reflection with respect to the x-axis in computer graphics, let the triangle T in Figure 2.15(a) have vertices

$$(-1, 4), \quad (3, 1), \quad \text{and} \quad (2, 6).$$

To reflect T with respect to the x-axis, we let

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

and compute the images $f(\mathbf{v}_1)$, $f(\mathbf{v}_2)$, and $f(\mathbf{v}_3)$ by forming the products

$$A\mathbf{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix},$$

$$A\mathbf{v}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix},$$

$$A\mathbf{v}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}.$$

These three products can be written in terms of partitioned matrices as

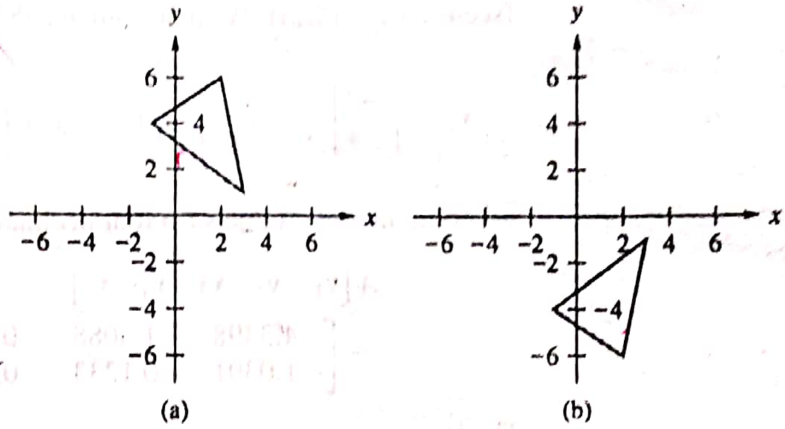
$$A\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 2 \\ -4 & -1 & -6 \end{bmatrix}.$$

Thus the image of T has vertices

$$(-1, -4), \quad (3, -1), \quad \text{and} \quad (2, -6)$$

and is displayed in Figure 2.15(b).

Figure 2.15 ►



EXAMPLE 2 The matrix transformation $f: R^2 \to R^2$ that performs a reflection with respect to the line $y = -x$ is defined by

$$f(\mathbf{v}) = B\mathbf{v},$$

where

$$B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

To illustrate reflection with respect to the line $y = -x$, we use the triangle T as defined in Example 1 and compute the products

$$B\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 3 & 2 \\ 4 & 1 & 6 \end{bmatrix} = \begin{bmatrix} -4 & -1 & -6 \\ 1 & -3 & -2 \end{bmatrix}.$$

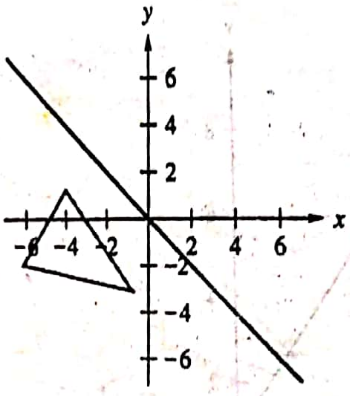


Figure 2.16 ▲

Thus the image of T has vertices

$$(-4, 1), \quad (-1, -3), \quad \text{and} \quad (-6, -2)$$

and is displayed in Figure 2.16. ■

To perform a reflection with respect to the x-axis on the triangle T of Example 1 followed by a reflection with respect to the line $y = -x$, we compute

$$B(A\mathbf{v}_1), \quad B(A\mathbf{v}_2), \quad \text{and} \quad B(A\mathbf{v}_3).$$

It is not difficult to show that reversing the order of these matrix transformations produces a different image (verify). Thus the order in which graphics transformations are performed is important. This is not surprising, since matrix multiplication, unlike multiplication of real numbers, does not satisfy the commutative property.

EXAMPLE 3 Rotations in a plane have been defined in Example 9 of Section 1.5. A plane figure is rotated counterclockwise through an angle ϕ by using the matrix transformation $f: R^2 \to R^2$ defined by $f(\mathbf{v}) = A\mathbf{v}$, where

$$A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

Now suppose that we wish to rotate the parabola $y = x^2$ counterclockwise through 50°. We start by choosing a sample of points from the parabola, say,

$$(-2, 4), \quad (-1, 1), \quad (0, 0), \quad \left(\tfrac{1}{2}, \tfrac{1}{4}\right), \quad \text{and} \quad (3, 9)$$

[see Figure 2.17(a)]. We then compute the images of these points. Thus letting

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 3 \\ 9 \end{bmatrix},$$

we compute the products (to four decimal places) (verify)

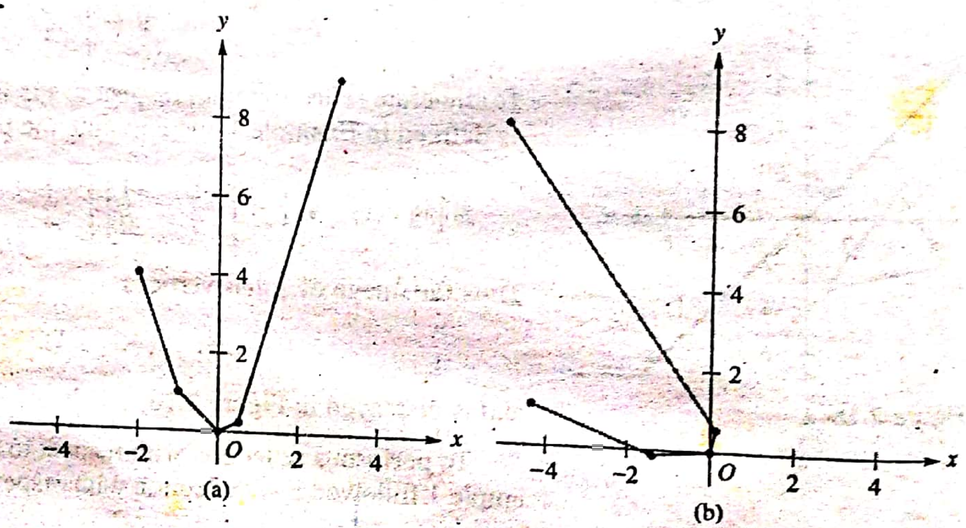
$$A\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5\end{bmatrix} = \begin{bmatrix} -4.3498 & -1.4088 & 0 & 0.1299 & -4.9660 \\ 1.0391 & -0.1233 & 0 & 0.5437 & 8.0832 \end{bmatrix}.$$

The image points

$$(-4.3498, 1.0391), \quad (-1.4088, -0.1233), \quad (0, 0),$$
$$(0.1299, 0.5437), \quad \text{and} \quad (-4.9660, 8.0832)$$

are plotted, as shown in Figure 2.17(b), and successive points are connected showing the approximate image of the parabola. ∎

Figure 2.17 ►



Rotations are particularly useful in achieving the sophisticated effects seen in arcade games and animated computer demonstrations. For example, to show a wheel spinning we can rotate the spokes through an angle θ_1 followed by a second rotation through an angle θ_2 and so on. Let the 2-vector $\mathbf{u} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ represent a spoke of the wheel; let $f: R^2 \to R^2$ be the matrix transformation defined by $f(\mathbf{v}) = A\mathbf{v}$, where

$$A = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix};$$

and let $g: R^2 \to R^2$ be the matrix transformation defined by $g(\mathbf{v}) = B\mathbf{v}$, where

$$B = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}.$$

We represent the succession of rotations of the spoke $\mathbf{u}$ by

$$g(f(\mathbf{u})) = g(A\mathbf{u}) = B(A\mathbf{u}).$$

The product $A\mathbf{u}$ is performed first and generates a rotation of $\mathbf{u}$ through the angle θ_1; then the product $B(A\mathbf{u})$ generates the second rotation. We have

$$B(A\mathbf{u}) = B(a_1\mathrm{col}_1(A) + a_2\mathrm{col}_2(A)) = a_1 B\mathrm{col}_1(A) + a_2 B\mathrm{col}_2(A)$$

and the final expression is a linear combination of column vectors $B\mathrm{col}_1(A)$ and $B\mathrm{col}_2(A)$, which we can write as the product

$$\begin{bmatrix} B\mathrm{col}_1(A) & B\mathrm{col}_2(A) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

From the definition of matrix multiplication, $\begin{bmatrix} B\mathrm{col}_1(A) & B\mathrm{col}_2(A) \end{bmatrix} = BA$, so we have

$$B(A\mathbf{u}) = (BA)\mathbf{u},$$

which says that instead of applying the transformations in succession, f followed by g, we can achieve the same result by forming the matrix product BA and using it to define a matrix transformation on the spokes of the wheel.

EXAMPLE 4 A **shear in the x-direction** is the matrix transformation defined by

$$f(\mathbf{v}) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\mathbf{v},$$

where k is a scalar. A shear in the x-direction takes the point (x, y) to the point $(x + ky, y)$. That is, the point (x, y) is moved parallel to the x-axis by the amount ky.

Consider now the rectangle R, shown in Figure 2.18(a), with vertices

$$(0, 0), \quad (0, 2), \quad (4, 0), \quad \text{and} \quad (4, 2).$$

If we apply the shear in the x-direction with $k = 2$, then the image of R is the parallelogram with vertices

$$(0, 0), \quad (4, 2), \quad (4, 0), \quad \text{and} \quad (8, 2),$$

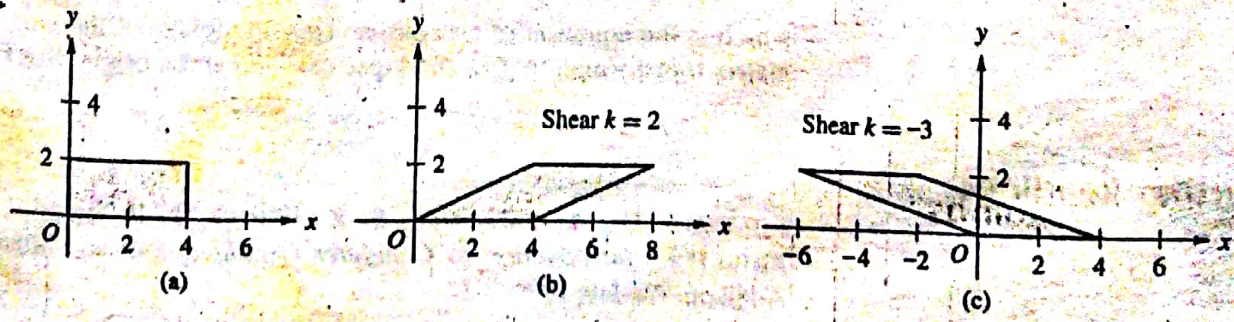
shown in Figure 2.18(b). If we apply the shear in the x-direction with $k = -3$, then the image of R is the parallelogram with vertices

$$(0, 0), \quad (-6, 2), \quad (4, 0), \quad \text{and} \quad (-2, 2),$$

shown in Figure 2.18(c).

In Exercise 3 we consider shears in the y-direction. ■

Figure 2.18 ▶



(5)

Other matrix transformations used in two-dimensional computer graphics are considered in the exercises at the end of this section. For a detailed discussion of computer graphics, the reader is referred to the books listed in the Further Readings at the end of this section.

In Examples 1 and 2, we applied a matrix transformation to a triangle, a figure that can be specified by giving its three vertices. In Example 3, the figure being transformed was a parabola, which cannot be specified by a finite number of points. In this case we chose a number of points on the parabola to approximate its shape and computed the images of these approximating points, which when joined gave an approximate shape of the image figure.

EXAMPLE 5 Let $f: R^2 \to R^2$ be the matrix transformation defined by $f(\mathbf{v}) = A\mathbf{v}$, where

$$A = \begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix}$$

with h and k both nonzero. Suppose that we now wish to apply this matrix transformation to a circle of radius 1 that is centered at the origin (the unit circle). Unfortunately, a circle cannot be specified by a finite number of points. However, each point on the unit circle is described by an ordered pair $(\cos\theta, \sin\theta)$, where the angle θ takes on all values from 0 to 2π radians. Thus, we now represent an arbitrary point on the unit circle by the vector $\mathbf{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$. Hence, the images of the unit circle that are obtained by applying the matrix transformation f are given by

$$f(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} h\cos\theta \\ k\sin\theta \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

We recall that a circle of radius 1 centered at the origin is described by the equation

$$x^2 + y^2 = 1.$$

By Pythagoras' identity, $\sin^2\theta + \cos^2\theta = 1$. Thus, the points $(\cos\theta, \sin\theta)$ lie on the circumference of the unit circle. We now want to obtain an equation describing the image of the unit circle. We have

$$x' = h\cos\theta \quad \text{and} \quad y' = k\sin\theta$$

so

$$\frac{x'}{h} = \cos\theta, \quad \frac{y'}{k} = \sin\theta.$$

It then follows that

$$\left(\frac{x'}{h}\right)^2 + \left(\frac{y'}{k}\right)^2 = 1,$$

which is the equation of an ellipse. Thus the image of the unit circle by the matrix transformation f is an ellipse centered at the origin. See Figure 2.19. ■

Further Readings

FOLEY, J. D., A. VAN DAM, S. K. FEINER, J. F. HUGHES, and R. L. PHILLIPS. *Introduction to Computer Graphics*, 2nd ed. Reading, Mass.: Addison-Wesley, 1996.

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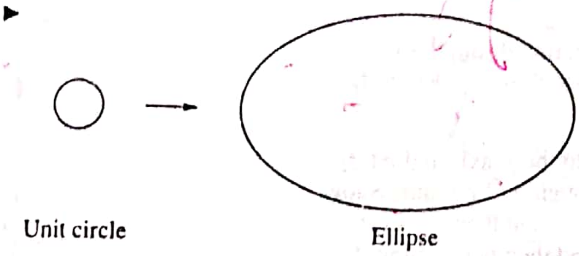
Figure 2.19 ▸



MORTENSON, M. E. *Mathematics for Computer Graphics Applications*, 2nd ed. New York: Industrial Press, Inc., 1999.

ROGERS, D. F., and J. A. ADAMS. *Mathematical Elements for Computer Graphics*, 2nd ed. New York: McGraw-Hill, 1989.

Key Terms

Computer graphics	Reflection	Dilation
Computer-aided design	Rotation	Contraction
Image	Shear	

11 – 13

2.3 Exercises

1. Let $f: R^2 \to R^2$ be the matrix transformation defined by $f(\mathbf{v}) = A\mathbf{v}$, where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix};$$

that is, f is a reflection with respect to the y-axis. Find and sketch the image of the rectangle R with vertices $(1, 1)$, $(2, 1)$, $(1, 3)$, and $(2, 3)$.

2. Let R be the rectangle with vertices $(1, 1)$, $(1, 4)$, $(3, 1)$, and $(3, 4)$. Let f be the shear in the x-direction with $k = 3$. Find and sketch the image of R.

3. A **shear in the y-direction** is the matrix transformation $f: R^2 \to R^2$ defined by $f(\mathbf{v}) = A\mathbf{v}$, and

$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix},$$

and k is a scalar. Let R be the rectangle defined in Exercise 2 and let f be the shear in the y-direction with $k = -2$. Find and sketch the image of R.

4. The matrix transformation $f: R^2 \to R^2$ defined by $f(\mathbf{v}) = A\mathbf{v}$, where

$$A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix},$$

and k is a real number, is called **dilation** if $k > 1$ and a **contraction** if $0 < k < 1$. Thus, dilation stretches a vector, whereas contraction shrinks it. If R is the rectangle defined in Exercise 2, find and sketch the image of R for

(a) $k = 4$ (b) $k = \frac{1}{4}$

5. The matrix transformation $f: R^2 \to R^2$ defined by $f(\mathbf{v}) = A\mathbf{v}$, where

$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}, \quad R_2\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

and k is a real number, is called **dilation in the x-direction** if $k > 1$ and a **contraction in the x-direction** if $0 < k < 1$. If R is the unit square and f is dilation in the x-direction with $k = 2$, find and sketch the image of R.

6. The matrix transformation $f: R^2 \to R^2$ defined by $f(\mathbf{v}) = A\mathbf{v}$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix},$$

where k is a real number, is called **dilation in the y-direction** if $k > 1$ and **contraction in the y-direction** if $0 < k < 1$. If R is the unit square and f is the contraction in the y-direction with $k = \frac{1}{2}$, find and sketch the image of R.

7. Let T be the triangle with vertices $(5, 0)$, $(0, 3)$, and $(2, -1)$. Find the coordinates of the vertices of the image of T under the matrix transformation f defined by

$$f(\mathbf{v}) = \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix}\mathbf{v}.$$

8. Let T be the triangle with vertices $(1, 1)$, $(-3, -3)$, and $(2, -1)$. Find the coordinates of the vertices of the image of T under the matrix transformation defined by

$$f(\mathbf{v}) = \begin{bmatrix} 4 & -3 \\ -4 & 2 \end{bmatrix}\mathbf{v}.$$

(7)

9. Let f be the counterclockwise rotation through 60°. If T is the triangle defined in Exercise 8, find and sketch the image of T under f.

10. Let f_1 be reflection with respect to the y-axis and let f_2 be counterclockwise rotation through $\pi/2$ radians. Show that the result of first performing f_2 and then f_1 is not the same as first performing f_1 and then performing f_2.

11. Let A be the singular matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and let T be the triangle defined in Exercise 8. Describe the image of T under the matrix transformation $f: R^2 \to R^2$ defined by $f(\mathbf{v}) = A\mathbf{v}$.

12. Let f be the matrix transformation defined in Example 5. Find and sketch the image of the rectangle with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$ for $h = 2$ and $k = 3$.

13. Let $f: R^2 \to R^2$ be the matrix transformation defined by $f(\mathbf{v}) = A\mathbf{v}$, where

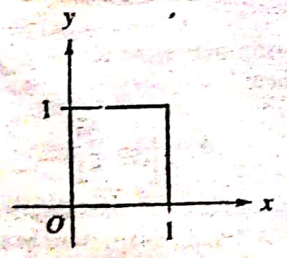
$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}.$$

Find and sketch the image of the rectangle defined in Exercise 12.

In Exercises 14 and 15, let f_1, f_2, f_3, and f_4 be the following matrix transformations:

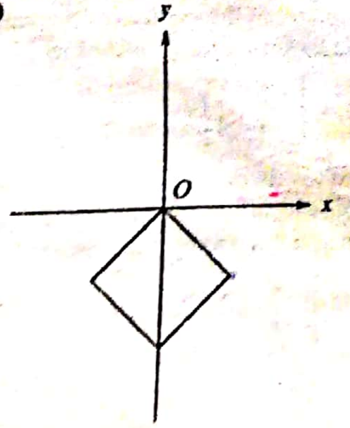
f_1: counterclockwise rotation through the angle ϕ
f_2: reflection with respect to the x-axis
f_3: reflection with respect to the y-axis
f_4: reflection with respect to the line $y = x$

14. Let S denote the unit square.

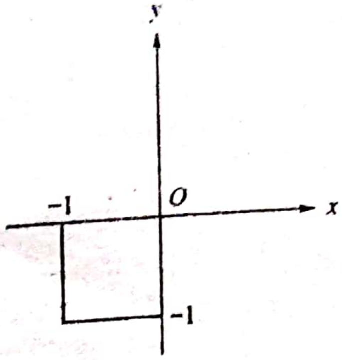


Determine two distinct ways to use the matrix transformations defined on S to obtain the given image. You may apply more than one matrix transformation in succession.

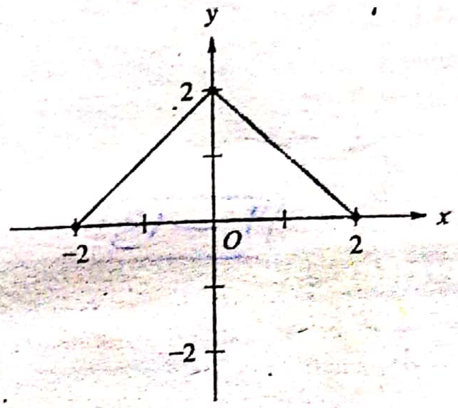
(a)



(b)

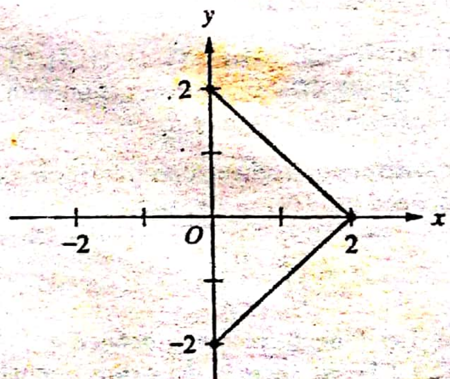


15. Let S denote the triangle shown in the figure.

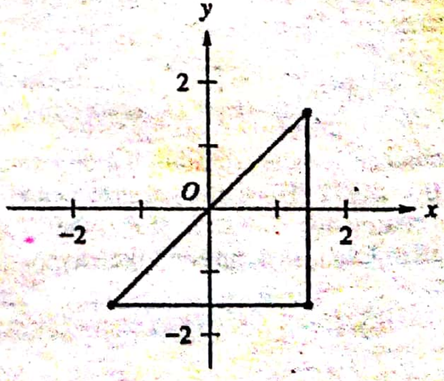


Determine two distinct ways to use the matrix transformations defined on S to obtain the given image. You may apply more than one matrix transformation in succession.

(a)



(b)



DATE :-

Excercise 2.3 (8)

Q1. $f(V) = AV$ — ①, where $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Here f is the reflection transformation with the respect to the y-axis as for an arbitrary vector $V = \begin{bmatrix} x \\ y \end{bmatrix}$,

$$① \Rightarrow f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y \end{bmatrix} — ②$$

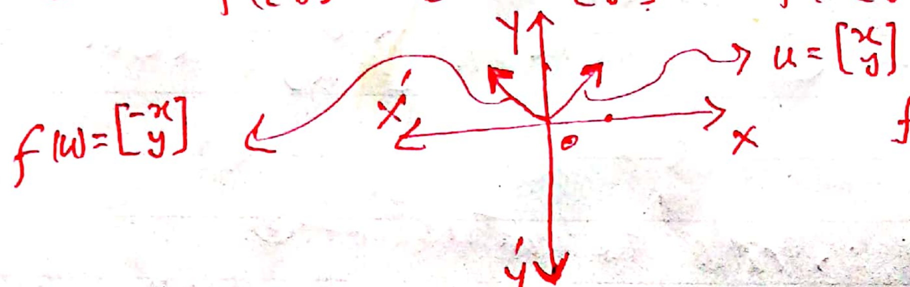
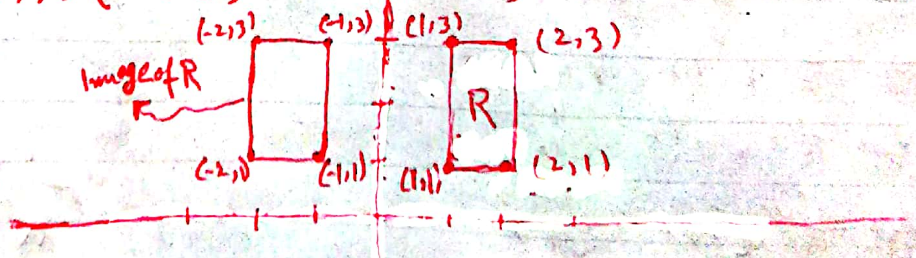


fig. for equation ②

The question is to find and sketch the image of of the rectangle R with vertices (1, 1), (2, 1), (1, 3) and (2, 3). Let $V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $V_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $V_3 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $V_4 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ be the position vectors of the vertices of R respectively. Then by (2) $f(V_1) = f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $f(V_2) = f\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$f(V_3) = f\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $f(V_4) = f\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

Thus the image of the rectangle R has the following vertices (−1, 1), (−2, 1), (−1, 3) and (−2, 3). Now we draw R and its image as.



Exercise 2-3

Q2: similar to Ex-4 (P-139)

Q3: similar to ex-4 (P-139) but here the matrix is $A=\begin{bmatrix}1&0\\K&1\end{bmatrix}$ instead of $A=\begin{bmatrix}1&K\\0&1\end{bmatrix}$ with $K=-2$.

Q4: Here the matrix transformation $f:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined by $f(V)=AV$ — ①, where $A=\begin{bmatrix}K&0\\0&K\end{bmatrix}$, $K\in\mathbb{R}$

For $K>1$, f/A, is called the dilation — ②

For $0<K<1$, f/A, is called the contraction — ③

The question is to find and sketch the image of the rectangle R with vertices (1,1), (1,4), (3,1) and (3,4), for

a) $K=4$, b) $K=1/4$

For an arbitrary vector $V=\begin{bmatrix}x\\y\end{bmatrix}$, ① $\Rightarrow f\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}K&0\\0&K\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\Rightarrow$

$f\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}Kx\\Ky\end{bmatrix}$ — ④

Let $V_1=\begin{bmatrix}1\\1\end{bmatrix}$, $V_2=\begin{bmatrix}1\\4\end{bmatrix}$, $V_3=\begin{bmatrix}3\\1\end{bmatrix}$ and $V_4=\begin{bmatrix}3\\4\end{bmatrix}$ be the position vectors of the vertices of the rectangle R respectively. Then by (4), we have

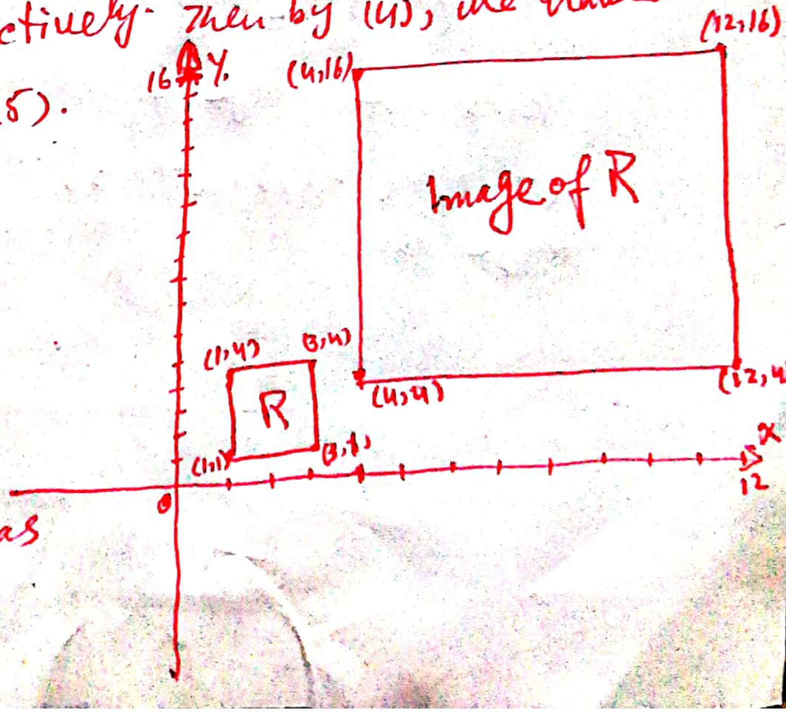
(a) for $K=4$, $f\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}4x\\4y\end{bmatrix}$ — (5).

(5) $\Rightarrow f(V_1)=f\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}4\\4\end{bmatrix}$

(5) $\Rightarrow f(V_2)=f\left(\begin{bmatrix}1\\4\end{bmatrix}\right)=\begin{bmatrix}4\\16\end{bmatrix}$

(5) $\Rightarrow f(V_3)=f\left(\begin{bmatrix}3\\1\end{bmatrix}\right)=\begin{bmatrix}12\\4\end{bmatrix}$

(5) $\Rightarrow f(V_4)=f\left(\begin{bmatrix}3\\4\end{bmatrix}\right)=\begin{bmatrix}12\\16\end{bmatrix}$



Thus the vertices of the image of R are as (4,4), (4,16), (12,4), (12,16)

similarly we can do for b) $K=1/4$.

Exercise 2.3 (10)

Q5 — Q8: similar to Q4 with matrices different from Q4.

Q9: Here $f: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$f(V) = AV \quad \text{——} ① \quad , \text{ where } A = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 0.5 & -0.86 \\ 0.86 & 0.5 \end{bmatrix}$$ counter clockwise rotation by $\phi = 60^\circ$

The question is to find and sketch the image of the triangle T with vertices (1,1), (-3,-3) and (2,-1).

Let $V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $V_2 = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$ and $V_3 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ be the position vectors of the vertices of triangle T respectively, then by ①, we have

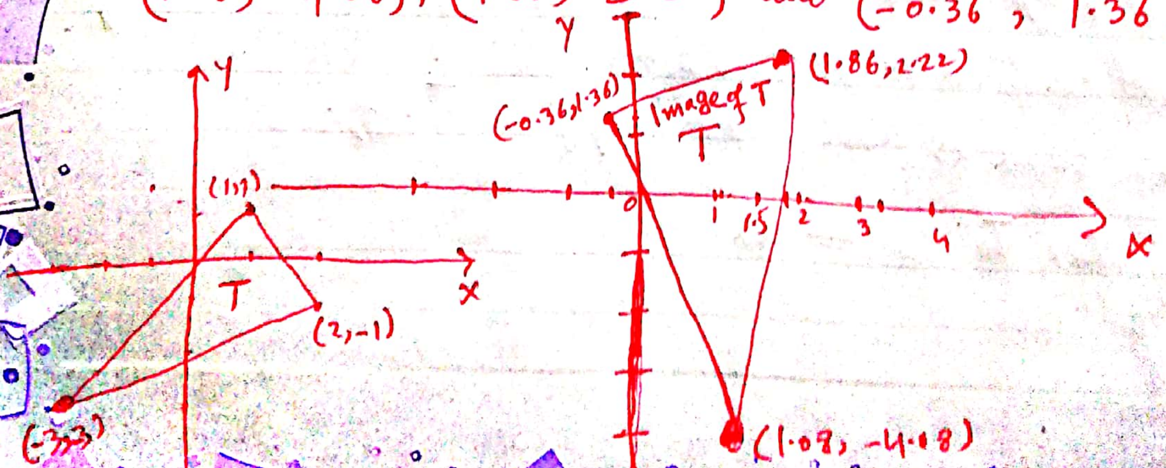
$$f(V_1) = f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0.5 & -0.86 \\ 0.86 & 0.5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5-0.86 \\ 0.86+0.5 \end{bmatrix} = \begin{bmatrix} -0.36 \\ 1.36 \end{bmatrix}$$

$$f(V_2) = f\left(\begin{bmatrix} -3 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} 0.5 & -0.86 \\ 0.86 & 0.5 \end{bmatrix}\begin{bmatrix} -3 \\ -3 \end{bmatrix} = \begin{bmatrix} -1.5+2.58 \\ -2.58-1.5 \end{bmatrix} = \begin{bmatrix} 1.08 \\ -4.08 \end{bmatrix}$$

$$f(V_3) = f\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 0.5 & -0.86 \\ 0.86 & -0.5 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1.00+0.86 \\ 1.72+0.5 \end{bmatrix} = \begin{bmatrix} 1.86 \\ 2.22 \end{bmatrix}$$

Thus the vertices of the image of T are as

(1.08, -4.08), (1.86, 2.22) and (-0.36, 1.36)



Q_{10}: Here $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$f_1(V) = A_1 V$ — ①, where $A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. For $V = \begin{bmatrix} x \\ y \end{bmatrix}$

① $\Rightarrow f_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow f_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y \end{bmatrix}$ — ②

Also $f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$f_2(V) = A_2 V$ — ③, where $A_2 = \begin{bmatrix} \cos\pi/2 & -\sin\pi/2 \\ \sin\pi/2 & \cos\pi/2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

For $V = \begin{bmatrix} x \\ y \end{bmatrix}$, (3) $\Rightarrow f_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow f_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x \end{bmatrix}$ — ④

Now $f_1(f_2(V)) = f_1\left(f_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) = f_1\left(\begin{bmatrix} -y \\ x \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}$ by (4) and then by ②

Also $f_2(f_1(V)) = f_2\left(f_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) = f_2\left(\begin{bmatrix} -x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ -x \end{bmatrix}$ by (2) and then by ④

i.e $f_1(f_2(V)) = \begin{bmatrix} y \\ x \end{bmatrix}$ — (5)

$f_2(f_1(V)) = \begin{bmatrix} -y \\ x \end{bmatrix}$ — (6)

Hence from (5) and (6), we have

$$f_1(f_2(V)) \neq f_2(f_1(V))$$

i.e the result of first performing f_2 and then f_1 is not the same as first performing f_1 and then f_2.

Q_{11} — Q_{13}: Similar to Q_4 — Q_8.