Functions occur in almost every application of mathematics. In this section we give a brief introduction from a geometric point of view to certain functions mapping R^n into R^m. Since we wish to picture these functions, called matrix transformations, we limit most of our discussion in this section to the situation where m and n have the values 2 or 3. In the next section we give an application of these functions to computer graphics in the plane, that is, for m and n equal to 2. In Chapter 4 we consider in more detail a more general function, called a linear transformation mapping R^n into R^m. Since every matrix transformation is a linear transformation, we then learn more about the properties of matrix transformations.

Linear transformations play an important role in many areas of mathematics, as well as in numerous applied problems in the physical sciences, the social sciences, and economics.

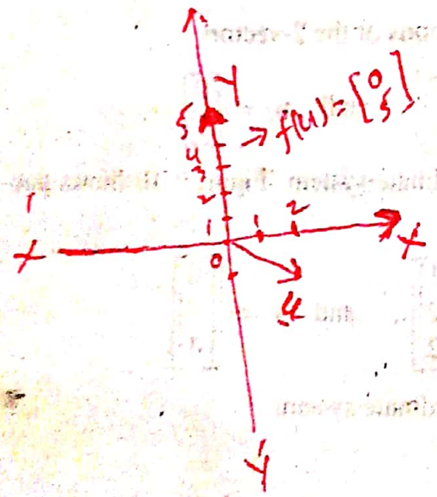
If A is an $m \times n$ matrix and $\mathbf{u}$ is an n-vector, then the matrix product $A\mathbf{u}$ is an m-vector. A function f mapping R^n into R^m is denoted by $f: R^n \to R^m$.[1] A **matrix transformation** is a function $f: R^n \to R^m$ defined by $f(\mathbf{u}) = A\mathbf{u}$. The vector $f(\mathbf{u})$ in R^m is called the **image** of $\mathbf{u}$, and the set of all images of the vectors in R^n is called the **range** of f. Although we are limiting ourselves in this section to matrices and vectors with only real entries, an entirely similar discussion can be developed for matrices and vectors with complex entries. (See Appendix A.2.)

EXAMPLE 2 (a) Let f be the matrix transformation defined by

$$f(\mathbf{u}) = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}\mathbf{u}.$$

The image of $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is

$$f(\mathbf{u}) = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix},$$



and the image of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is $\begin{bmatrix} 10 \\ 5 \end{bmatrix}$ (verify).

(b) Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$, and consider the matrix transformation defined by

$$f(\mathbf{u}) = A\mathbf{u}.$$

Then the image of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, the image of $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ is $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, and the image of $\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$ is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (verify). ∎

Observe that if A is an $m \times n$ matrix and $f: R^n \to R^m$ is a matrix transformation mapping R^n into R^m that is defined by $f(\mathbf{u}) = A\mathbf{u}$, then a vector $\mathbf{w}$ in R^m is in the range of f only if we can find a vector $\mathbf{v}$ in R^n such that $f(\mathbf{v}) = \mathbf{w}$.

[1] Appendix A, dealing with sets and functions, may be consulted as needed.

EXAMPLE 3 Let $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$ and consider the matrix transformation defined by $f(\mathbf{u}) = A\mathbf{u}$. Determine if the vector $\mathbf{w} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ is in the range of f.

Solution The question is equivalent to asking if there is a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that $f(\mathbf{v}) = \mathbf{w}$. We have

$$A\mathbf{v} = \begin{bmatrix} v_1 + 2v_2 \\ -2v_1 + 3v_2 \end{bmatrix} = \mathbf{w} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

or

$$\begin{aligned} v_1 + 2v_2 &= 4 \\ -2v_1 + 3v_2 &= -1. \end{aligned}$$

Solving this linear system of equations by the familiar method of elimination we get $v_1 = 2$ and $v_2 = 1$ (verify). Thus $\mathbf{w}$ is in the range of f. In particular, if $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then $f(\mathbf{v}) = \mathbf{w}$. ■

EXAMPLE 4 (Production) A book publisher publishes a book in three different editions: trade, book club, and deluxe. Each book requires a certain amount of paper and canvas (for the cover). The requirements are given (in grams) by the matrix

$$A = \begin{array}{c} \\ \\ \end{array}\begin{array}{ccc} \text{Trade} & \begin{array}{c}\text{Book}\\\text{Club}\end{array} & \text{Deluxe} \\ \end{array}$$
$$A = \begin{bmatrix} 300 & 500 & 800 \\ 40 & 50 & 60 \end{bmatrix} \begin{array}{l} \text{Paper} \\ \text{Canvas} \end{array}$$

Let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

denote the production vector, where x_1, x_2, and x_3 are the number of trade, book club, and deluxe books, respectively, that are published. The matrix transformation $f: R^3 \to R^2$ defined by $f(\mathbf{x}) = A\mathbf{x}$ gives the vector

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

where y_1 is the total amount of paper required and y_2 is the total amount of canvas required. ■

For matrix transformations where m and n are 2 or 3, we can draw pictures showing the effect of the matrix transformation. This will be illustrated in the examples that follow.

EXAMPLE 5 Let $f: R^2 \to R^2$ be the matrix transformation defined by

$$f(\mathbf{u}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{u}.$$

Thus, if $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$, then

$$f(\mathbf{u}) = f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix}.$$

The effect of the matrix transformation f, called **reflection with respect to the x-axis in R^2**, is shown in Figure 1.11. In Exercise 2 we consider reflection with respect to the y-axis. ■

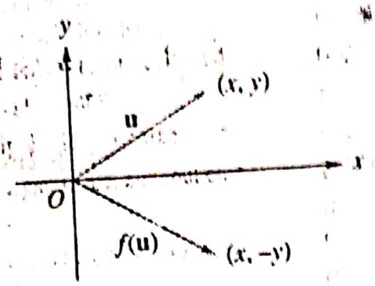


Figure 1.11 ▶ Reflection with respect to the x-axis

EXAMPLE 6 Let $f: R^3 \to R^2$ be the matrix transformation defined by

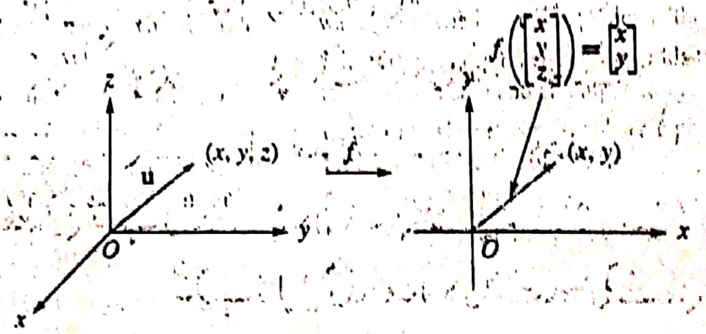
$$f(\mathbf{u}) = f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Then

$$f(\mathbf{u}) = f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Figure 1.12 shows the effect of this matrix transformation. (Warning: Carefully note the axes in Figure 1.12.)

Figure 1.12 ▶



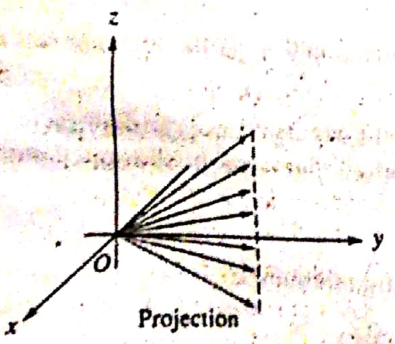


Figure 1.13 ▲

Observe that if

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ s \end{bmatrix},$$

where s is any scalar, then

$$f(\mathbf{v}) = \begin{bmatrix} x \\ y \end{bmatrix} = f(\mathbf{u}).$$

Hence, infinitely many 3-vectors have the same image vector. See Figure 1.13. The matrix transformation f is an example of a type of matrix transformation called **projection**. In this case f is a projection of R^3 into the xy-plane.

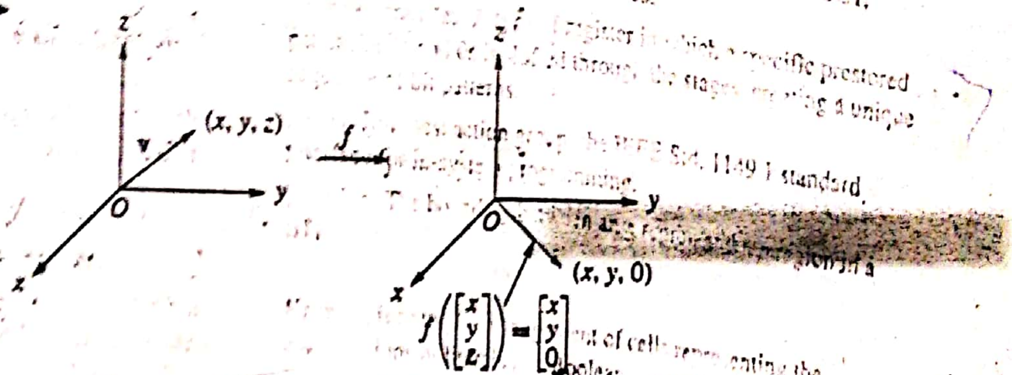
Note that the image of the 3-vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ under the matrix transformation

$f: R^3 \to R^3$ defined by

$$f(\mathbf{v}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{v}$$

is $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$. The effect of this matrix transformation is shown in Figure 1.14. The picture is almost the same as Figure 1.12, where the image is a 2-vector that lies in the xy-plane, whereas in Figure 1.14 the image is a 3-vector that lies in the xy-plane. Observe that $f(\mathbf{v})$ appears to be the shadow cast by $\mathbf{v}$ onto the xy-plane. ■

Figure 1.14 ▶



EXAMPLE 7 Let $f: R^3 \to R^3$ be the matrix transformation defined by

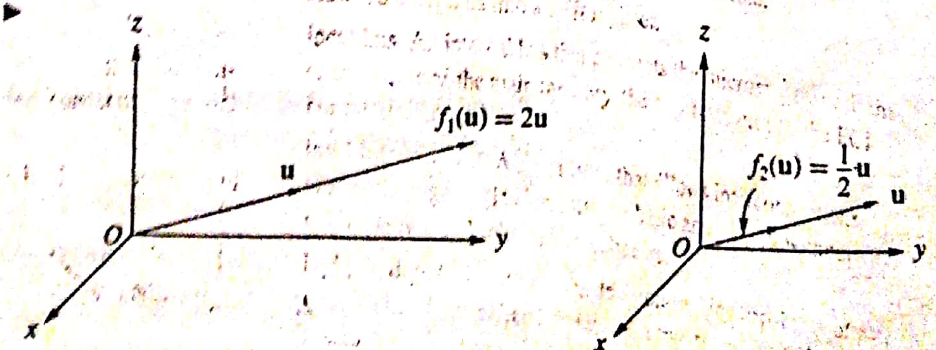
$$f(\mathbf{u}) = \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix} \mathbf{u},$$

where r is a real number. It is easily seen that $f(\mathbf{u}) = r\mathbf{u}$. If $r > 1$, f is called **dilation**; if $0 < r < 1$, f is called **contraction**. In Figure 1.15(a) we show the vector $f_1(\mathbf{u}) = 2\mathbf{u}$ and in Figure 1.15(b) the vector $f_2(\mathbf{u}) = \frac{1}{2}\mathbf{u}$. Thus dilation stretches a vector, and contraction shrinks it. Similarly, we can define the matrix transformation $g: R^2 \to R^2$ by

$$g(\mathbf{u}) = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \mathbf{u}.$$

We also have $g(\mathbf{u}) = r\mathbf{u}$, so again if $r > 1$, g is called **dilation**; if $0 < r < 1$, g is called **contraction**. ■

Figure 1.15 ▶



(a) Dilation: $r > 1$ (b) Contraction: $0 < r < 1$

EXAMPLE 8 (Production) We return to the book publisher discussed in Example 4. The requirements are given by the production vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

where x_1, x_2, and x_3 are the number of trade, book club, and deluxe books, respectively. The vector

$$\mathbf{y} = A\mathbf{x} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

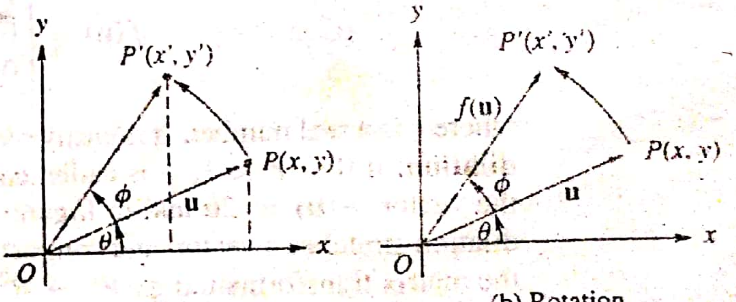
gives y_1, the total amunt of paper required, y_2, the total amount of canvas required. Let c_1 denote the cost per pound of paper and let c_2 denote the cost per pound of canvas. The matrix transformation $g: R^2 \to R^1$ defined by $g(\mathbf{y}) = B\mathbf{y}$, where

$$B = \begin{bmatrix} c_1 & c_2 \end{bmatrix}$$

gives the total cost to manufacture all the books. ■

EXAMPLE 9 Suppose that we rotate every point in R^2 counterclockwise through an angle ϕ about the origin of a rectangular coordinate system. Thus, if the point P has coordinates (x, y), then after rotating, we get the point P' with coordinates (x', y'). To obtain a relationship between the coordinates of P' and those of P, we let $\mathbf{u}$ be the vector $\begin{bmatrix} x \\ y \end{bmatrix}$, which is represented by the directed line segment from the origin to $P(x, y)$. See Figure 1.16(a). Also, let θ be the angle made by $\mathbf{u}$ with the positive x-axis.

Figure 1.16 ▶



Letting r denote the length of the directed line segment from O to P, we see from Figure 1.16(a) that

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{1}$$

and

$$x' = r\cos(\theta + \phi), \qquad y' = r\sin(\theta + \phi). \tag{2}$$

Using the formulas for the sine and cosine of a sum of angles, the equations in (2) become

$$x' = r\cos\theta\cos\phi - r\sin\theta\sin\phi$$
$$y' = r\sin\theta\cos\phi + r\cos\theta\sin\phi.$$

Substituting the expression in (1) into the last pair of equations, we obtain

$$x' = x\cos\phi - y\sin\phi, \qquad y' = x\sin\phi + y\cos\phi. \tag{3}$$

Solving (3) for x and y, we have

$$x = x' \cos\phi + y' \sin\phi \quad \text{and} \quad y = -x' \sin\phi + y' \cos\phi. \tag{4}$$

Equation (3) gives the coordinates of P' in terms of those of P and (4) expresses the coordinates of P in terms of those of P'. This type of rotation is used to simplify the general equation of second degree

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

Substituting for x and y in terms of x' and y', we obtain

$$a'x'^2 + b'x'y' + c'y'^2 + d'x' + e'y' + f' = 0.$$

The key point is to choose ϕ so that $b' = 0$. Once this is done (we might now have to perform a translation of coordinates), we identify the general equation of second degree as a circle, ellipse, hyperbola, parabola, or a degenerate form of these. This topic will be treated from a linear algebra point of view in Section 9.5.

We may also perform this change of coordinates by considering the matrix transformation $f: R^2 \to R^2$ defined by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{5}$$

Then (5) can be written, using (3), as

$$f(\mathbf{u}) = \begin{bmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

It then follows that the vector $f(\mathbf{u})$ is represented by the directed line segment from O to the point P'. Thus, rotation counterclockwise through an angle ϕ is a matrix transformation. ■

Key Terms

Matrix transformation
Mapping (function)
Image
Range
Reflection
Projection
Dilation
Contraction
Rotation

1.5 Exercises

In Exercises 1 through 8, sketch **u** *and its image under the given matrix transformation f.*

1. $f: R^2 \to R^2$ defined by
$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}; \quad \mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

2. $f: R^2 \to R^2$ (reflection with respect to the y-axis) defined by
$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}; \quad \mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

3. $f: R^2 \to R^2$ is a counterclockwise rotation through $30°$; $\mathbf{u} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

4. $f: R^2 \to R^2$ is a counterclockwise rotation through $\frac{2}{3}\pi$ radians; $\mathbf{u} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

5. $f: R^2 \to R^2$ defined by
$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}; \quad \mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

6. $f: R^2 \to R^2$ defined by
$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}; \quad \mathbf{u} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

7. $f: R^3 \to R^3$ defined by
$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}; \quad \mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

8. $f: R^3 \to R^3$ defined by
$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}; \quad \mathbf{u} = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}$$

In Exercises 9 through 11, let $f: R^2 \to R^2$ be the matrix transformation defined by $f(\mathbf{x}) = A\mathbf{x}$, where
$$A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}.$$

Determine whether the given vector **w** *is in the range of f.*

9. $\mathbf{w} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$ 10. $\mathbf{w} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ 11. $\mathbf{w} = \begin{bmatrix} -1 \\ -9 \end{bmatrix}$

In Exercises 12 through 14, let $f: R^2 \to R^3$ be the matrix transformation defined by $f(\mathbf{x}) = A\mathbf{x}$, where
$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Determine whether the given vector **w** *is in the range of f.*

12. $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ 13. $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ 14. $\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

In Exercises 15 through 17, give a geometric description of the matrix transformation $f: R^2 \to R^2$ defined by $f(\mathbf{u}) = A\mathbf{u}$ for the given matrix A.

15. (a) $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

16. (a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ Ref

17. (a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ P

18. Some matrix transformations f have the property that $f(\mathbf{u}) = f(\mathbf{v})$, when $\mathbf{u} \neq \mathbf{v}$. That is, the images of different vectors can be the same. For each of the following matrix transformations $f: R^2 \to R^2$ defined by $f(\mathbf{u}) = A\mathbf{u}$, find two different vectors **u** and **v** such that $f(\mathbf{u}) = f(\mathbf{v}) = \mathbf{w}$ for the given vector **w**.

(a) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

19. Let $f: R^2 \to R^2$ be the linear transformation defined by $f(\mathbf{u}) = A\mathbf{u}$, where
$$A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$
For $\phi = 30°$, f defines a counterclockwise rotation by an angle of $30°$.

(a) If $T_1(\mathbf{u}) = A^2\mathbf{u}$, describe the action of T_1 on **u**.

(b) If $T_2(\mathbf{u}) = A^{-1}\mathbf{u}$, describe the action of T_2 on **u**.

(c) What is the smallest positive value of k for which $T(\mathbf{u}) = A^k\mathbf{u} = \mathbf{u}$?

Theoretical Exercises

T.1. Let $f: R^n \to R^m$ be a matrix transformation defined by $f(\mathbf{u}) = A\mathbf{u}$, where A is an $m \times n$ matrix.

(a) Show that $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$ for any **u** and **v** in R^n.

(b) Show that $f(c\mathbf{u}) = cf(\mathbf{u})$ for any **u** in R^n and any real number c.

(c) Show that $f(c\mathbf{u} + d\mathbf{v}) = cf(\mathbf{u}) + df(\mathbf{v})$ for any **u** and **v** in R^n and any real numbers c and d.

Exercise 1.5

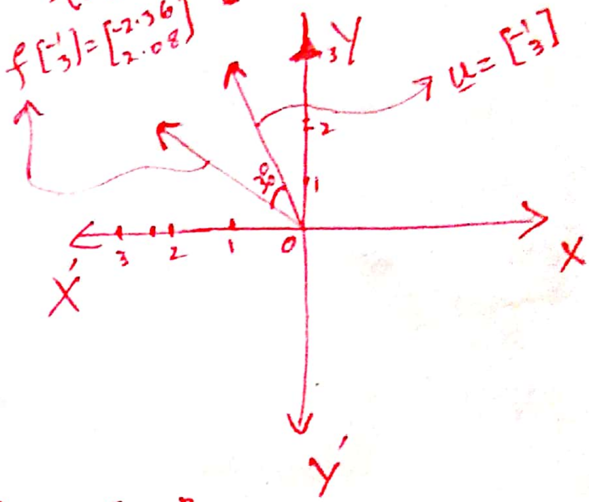
Q1 — Q8: sketch $\underline{u}$ and its image under the given matrix transformation f.

Q3: $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a counterclockwise rotation through 30°; $\underline{u} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Sol: Here f is defined by $f(\underline{u}) = A\underline{u}$ —(1), where $A = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 0.86 & -0.5 \\ 0.5 & 0.86 \end{bmatrix}$. Thus (1) $\Rightarrow f\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 0.86 & -0.5 \\ 0.5 & 0.86 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -0.86-1.5 \\ -0.5+2.58 \end{bmatrix}$

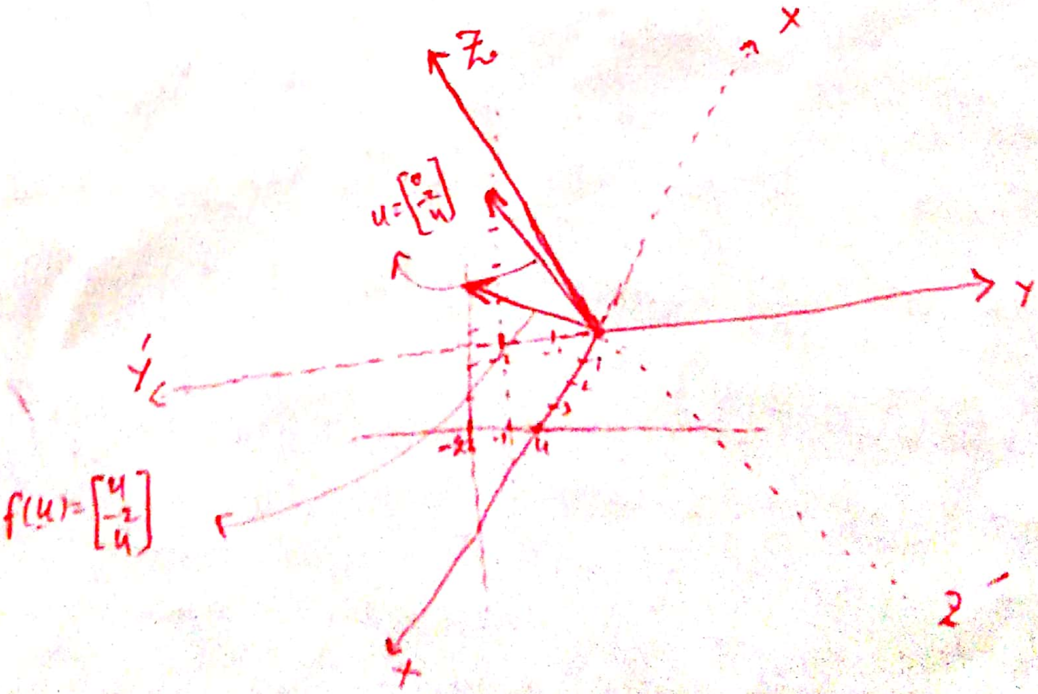
$\Rightarrow f\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -2.36 \\ 2.08 \end{bmatrix}$



Q8: $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ —(1)}; \quad \underline{u} = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}$$

(1) $\Rightarrow f\left(\begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 4 \end{bmatrix}$ is the desired image of $\underline{u}$ under f.

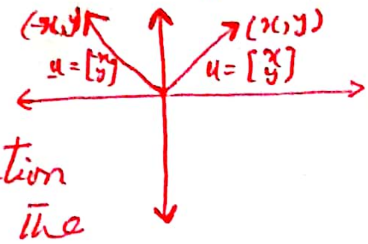


Q9 — Q14 : similar to Example-3 on Page-55.

Q15 — Q17: Give a geometric description of the matrix transformation $f:\mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(\underline{u}) = A\underline{u}$ ① for the given matrix A.

Q15(a) $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$; for $u = \begin{bmatrix} x \\ y \end{bmatrix}$, (1) $\Rightarrow f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y \end{bmatrix}$

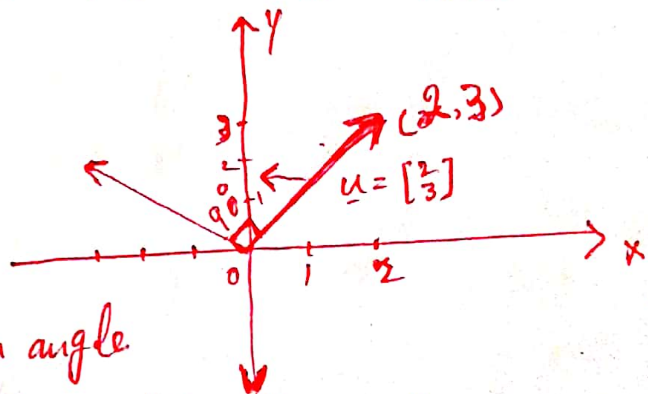


Here f/A represents reflection transformation with respect to the y-axis.

b) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$; for $\underline{u} = \begin{bmatrix} x \\ y \end{bmatrix}$, ① $\Rightarrow f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x \end{bmatrix}$

$\Rightarrow f\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

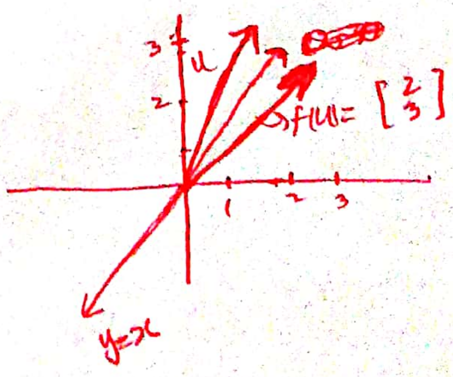


Here f/A represents the counterclockwise rotation by an angle of 90° as $A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Q16(a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; for $\underline{u} = \begin{bmatrix} x \\ y \end{bmatrix}$, ① $\Rightarrow f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$

$\Rightarrow f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}$ — ②

Ex: For $u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, ② $\Rightarrow f\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$



Here f/A represents the reflection transformation with respect to the line $y = x$. Similarly you can do (b) and Q17

Q18: Given that $f(\underline{u}) = f(\underline{v})$ when $\underline{u} \neq \underline{v}$
where $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is defined by
$f(\underline{u}) = A\underline{u}$ — ① we need to find two different vectors
$\underline{u}$ and $\underline{v}$ such that $f(\underline{u}) = f(\underline{v}) = \underline{w}$ — ② for the given vector $\underline{w}$

a) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}_{2\times3}$, $\underline{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{2\times1}$

Let $\underline{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be an arbitrary vector in $\mathbb{R}^3$, Then from ① and ②
we have $A\underline{u} = \underline{w} \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} x+2y \\ y-z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$$\Rightarrow \left.\begin{aligned} x + 2y &= 0 \quad -(3) \\ y - z &= -1 \quad -(4) \end{aligned}\right\} \quad \begin{aligned} &(4) \Rightarrow y = z-1 \Rightarrow y = r-1, \text{ where } z = r \in \mathbb{R} \\ &(3) \Rightarrow x = -2y \Rightarrow x = -2r+2 \end{aligned}$$

Thus $\underline{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2r+2 \\ r-1 \\ r \end{bmatrix}$, $r \in \mathbb{R}$

For $r = 0$ say, $\underline{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and for $r = 1$ say $\underline{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Thus $\underline{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are two different vectors
for which $f(\underline{u}) = f(\underline{v}) = \underline{w}$

$$\left.\begin{aligned} &\text{i.e } f(\underline{u}) = A\underline{u} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2-2+0 \\ 0-1-0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \underline{w} \\ &\text{and } f(\underline{v}) = A\underline{v} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+0+0 \\ 0+0-1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \underline{w} \end{aligned}\right\} \Rightarrow f(\underline{u}) = f(\underline{v}) = \underline{w}$$

Exercise 1.5 (4)

Q19: $f: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(\underline{u}) = A\underline{u}$, where

$A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$. For $\phi = 30^\circ$, f represents counterclockwise rotation by 30°.

a) For $T_1(\underline{u}) = A^2\underline{u}$, the action of T_1 on $\underline{u}$?

As $A^2 = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos^2\phi - \sin^2\phi & -\sin\phi\cos\phi - \sin\phi\cos\phi \\ \sin\phi\cos\phi + \sin\phi\cos\phi & -\sin^2\phi + \cos^2\phi \end{bmatrix}$

$\Rightarrow A^2 = \begin{bmatrix} \cos 2\phi & -\sin 2\phi \\ \sin 2\phi & \cos 2\phi \end{bmatrix}$

Thus the action of T_1 on $\underline{u}$ is the counterclockwise rotation by the angle of $2\phi = 2 \times 30^\circ = 60^\circ$

b) For $T_2(\underline{u}) = A^{-1}\underline{u}$, the action of T_2 on $\underline{u}$?

As $A^{-1} = \begin{bmatrix} \cos(-\phi) & -\sin(-\phi) \\ \sin(-\phi) & \cos(-\phi) \end{bmatrix}$ (the speciality of this matrix that power is multiplied to the angle)

$\Rightarrow A^{-1} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$. It indicates that the action of T_2 on $\underline{u}$ is the clockwise rotation by an angle of $\phi = 30^\circ$

c) $T(\underline{u}) = A^k\underline{u} = \underline{u} \Rightarrow A^k\underline{u} = \underline{u} \Rightarrow \underline{u} - A^k\underline{u} = 0$

$\Rightarrow (I_2 - A^k)\underline{u} = 0 \Rightarrow I_2 - A^k = 0$, as $\underline{u} \neq 0$

$\Rightarrow A^k = I_2 \Rightarrow \begin{bmatrix} \cos k\phi & -\sin k\phi \\ \sin k\phi & \cos k\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \left.\begin{array}{l} \cos k\phi = 1 \\ \sin k\phi = 0 \end{array}\right\} \Rightarrow k\phi = 2\pi$ as $k\phi \neq 0$ [if $k\phi = 0 \Rightarrow k = 0$ as $\phi = 30^\circ \neq 0$ but k is +ve, so contradicting]

$k\phi = 2\pi \Rightarrow k = \frac{2\pi}{\phi} \Rightarrow k = \frac{2\pi}{\pi/6} \Rightarrow k = 12$ which is the smallest +ve value of k for which $T(\underline{u}) = A^k\underline{u} = \underline{u}$ holds.