Classical Probability Theory

QWORLD STUDYGROUP #4
Session 3
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Necessary Set Theory Concepts:

Why do we need set theory ?

What is the link with OQS ?

Why is your voice Weird today ?

What is probability?

A way to measure uncertainty.

Whats a set?

- A collection of objects
- E.g, A = {a,b,c,d….z}
- s ε A means s is an element of A
- Sets can be countable as A or uncountable

Countable sets

- Sets such as A = {a,b,c...z} are finitely countable.
- While D = {1,2,3…..} are infinitely countable.

Null set

- Null or empty set contains no elements.
- Notations: {} , Ø
- Is a subset of all sets

Subset

If B={a,b,c,d,e}

and C={b,c,d}

then, $C \subseteq B$

Subsets of C = {b} {c} {d} {bc} {bd} {cd} {bcd} {}

Equal sets

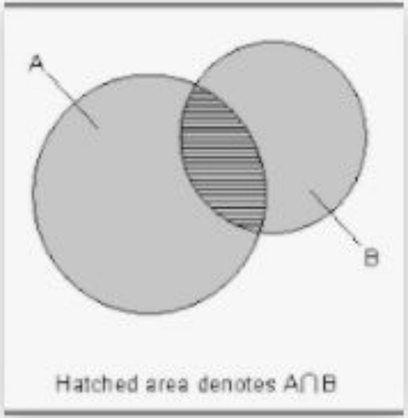
Equal sets: A subset of B and B subset of A, A={1,2,3} B={1,2,3}

Intersection of sets

A = {1, 2, 3, 4, 5}

B = {1, 5, 6, 7, 8}

A intersection B = $A \cap B$ = {1, 5}



What is intersection symbol ...
whatis.techtarget.com

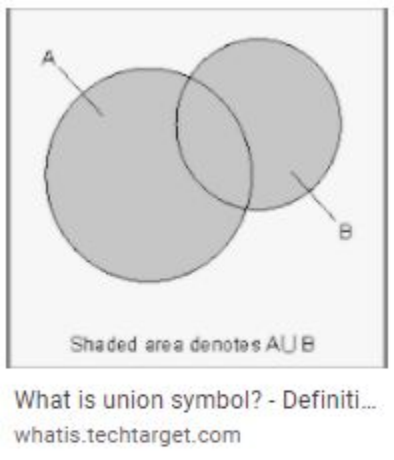
Two sets such as C = {1, 2}, D= {3, 4} disjoint sets if,

$C \cap D = \{\varnothing\}$ = empty/null set.

Union of sets

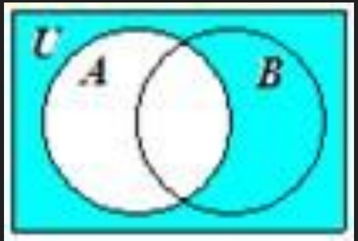
$A=\{1,2,3\}$ $B=\{4,5,6\}$

A Union B = $A \cup B = \{1,2,3,4,5,6\}$



Complement Set

F’,Complement set of F, is the set of all elements in universal set U that are not in F.



The shaded region represents F'.

$F' = \{x \in U : x \notin F\}$.

$U = \{a,b,c,d,e,f\}$ and $F = \{a,b,d\}$. Then $F' = \{c,e,f\}$.

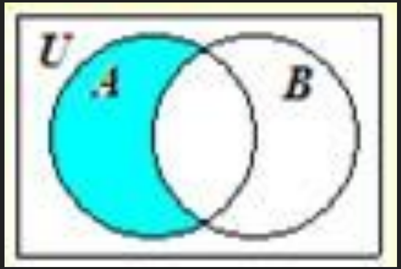
$U' = \varnothing$ Complement of the universal set is the empty set.

$\varnothing' = U$ and vice versa

Set Difference

A – B, the relative complement or set difference, is the set of all elements in A that are not in B.

$A - B = \{x \in U : x \in A \text{ and } x \notin B\} = A \cap B'$.



The shaded region represents A – B

Let A = {a, b, c, d} and B = {b, c, d, e}.

Then A – B = {a} and B – A = {e}.

Let C = {m,o,i,z} and D = {z,i,o,m}. Then $C - D = \varnothing$.

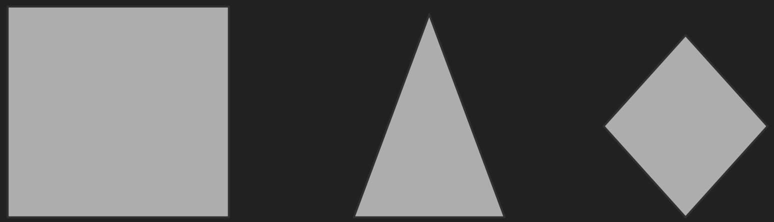
Probability Spaces

The constituents:

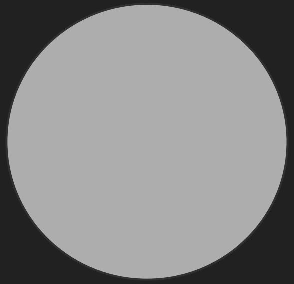
1. Sample space Ω - a (set) place where (all possible) things can happen
2. Event space $\mathscr{A}$ - (a place within the sample space - set of sets from Ω)
3. Probability Measure (assigns those events a specific number from 0-1)

Sample space example

Finite and Countable - vertices on the shapes, {4,3,4}



Infinite Uncountable - edge points around the circle



Event space (Sigma Algebra)

For an space to be called a sigma algebra it must adhere to the following:

- Has Ω and Null set.
- All other sets must be closed under complement

  for all $F \in A$, $F' \in A$.
- If $C \in A$ and $C \in A$ then $C \cup D \in A$.
- $C \cap D = (C' \cup D')'$

Event Space Example

Suppose we have the sample space $\Omega = \{00, 01, 10, 11\}$ from that $A = \{ \Omega, \phi \}$.

Now, to add $B = \{00, 01\}$ into A as per our closure properties we must also include $B' = \{10, 11\}$, hence: $A = \{ \Omega, \phi, B, B' \}$.

Adding $C = \{00\}$ to A, then gives us: $A = \{ \Omega, \phi, B, B', C, C' \}$ while $C' = \{01,10,11\}$.

Now we must account for different unions b/w both B and C.

$B \cup C = B$; $C \cup B' = \{00, 10, 11\}$; $C \cup B' = \{01\}$

Hence, $A = \{ \Omega, \phi, B, B', C \cup B', (C \cup B')' \}$

Outcomes of the Event space

After translation on a number line if Omega contains all required intervals then we can use the Boreal sigma algebra..

1.1.1 DONE

Probability Measure and KAxis

Member of A to [0,1] given following conditions are met:

- $0 \leq P(\mathrm{n}) \leq 1$ For all $\mathrm{n} \in \mathrm{A}$.
- $P(\Omega) = 1$.
- $P(\varnothing) = 0$.
- If, $\mathrm{C} \cap \mathrm{D} = \{\varnothing\}$; $P(\mathrm{C} \cup \mathrm{D}) = P(\mathrm{C}) + P(\mathrm{D})$

These give rise to a consistent probability theory such that probabilities for events gained via logical operations on other events can be calculated: e.g: one finds $P(\mathrm{C} \cup \mathrm{D}) = P(\mathrm{C}) + P(\mathrm{D}) - P(\mathrm{C} \cap \mathrm{D})$

1.1.2 DONE

Conditional probabilities and independence

- $P(A|B) = P(A \cap B) / P(B)$; $P(B|A) = P(B \cap A) / P(A)$
- E.g $P(A1|B1)$ for $0.5\,(|00\rangle + |11\rangle)$ $= 0.5$
- If A & B are statistically independent, $P(A|B) = P(A) \cdot P(B) = P(A)$

Bayes's Theorem

$$P(A|B) = P(A \cap B) \,/\, P(B) \qquad ; \qquad P(B|A) = P(B \cap A) \,/\, P(A)$$

$$P(A \cap B) = P(B \cap A)$$

So,

$$P(A|B) = P(B|A)\, P(A) \,/\, P(B)$$

1.1.3 DONE

Random Variables

Maps Ω to T

$\Omega = \{00, 01, 10, 11\}$

$X(00) = 0 \quad ; \quad X(01) = 1 \quad ; \quad X(10) = 1 \quad ; \quad X(11) = 2$

$T = \{0,1,2\}$

$$\mathscr{x} = X(\text{w})$$

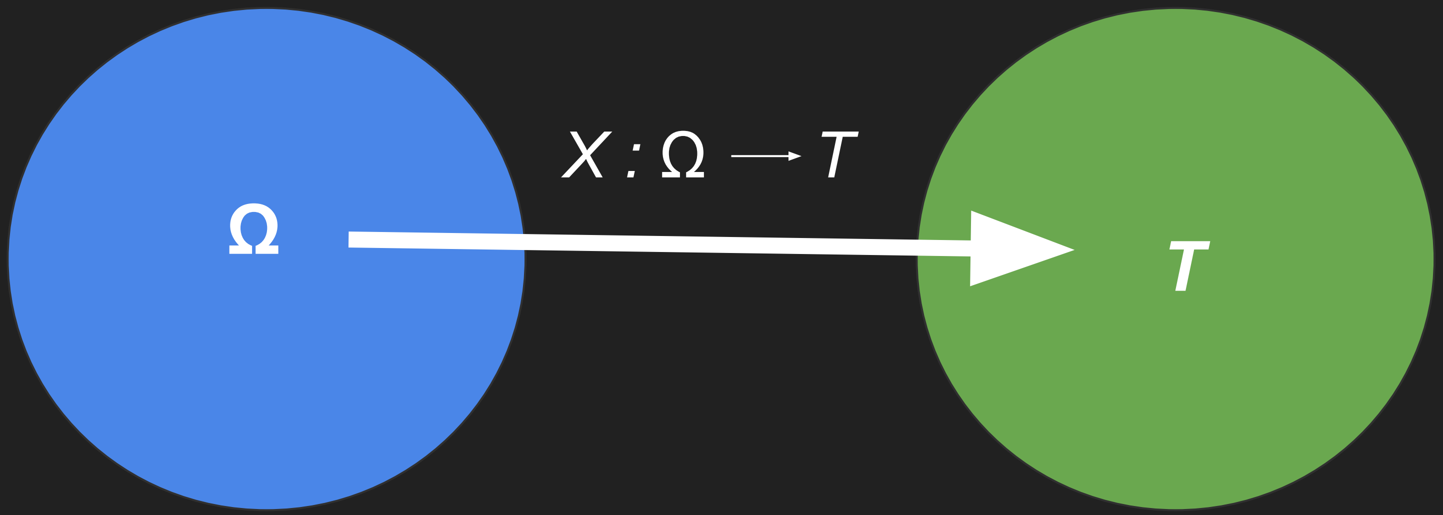
Why do this?

Elements of Ω are abstract objects.

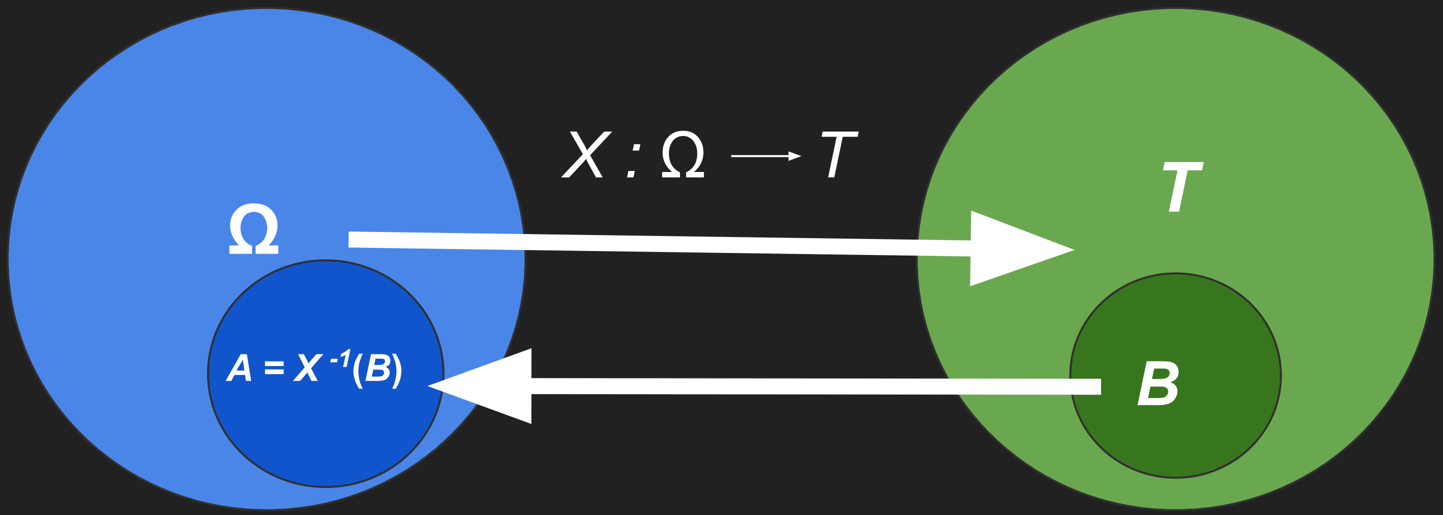
To deal with numbers (real or complex) , we assign numbers to these elements.

This gives rise to the idea of such functions (RV)

How to assign numbers to elements?



How to assign numbers to elements?



Defining random variable

$$X : \Omega \mapsto \mathbb{R},$$

***R* = *X* ⁻¹(*U*) is well defined.**

Hence, the probability distribution

$$P_X(B) = \mu\left(X^{-1}(B)\right)$$

[1]The σ-algebra of Borel sets of $\mathbb{R}$ is the smallest σ-algebra which contains all subsets of the form $(-\infty, x)$, $x \in \mathbb{R}$. In particular, it contains all open and closed intervals of the real axis.

Probability distribution from RV X

$$A_x \equiv \{\omega \in \Omega | X(\omega) \leq x\}$$

Hence we can write:

$$F_X(x) \equiv \mu(A_x) = \mu(\{\omega \in \Omega | X(\omega) \leq x\})$$

This is the Cumulative Dist. Ftn. of X.

Properties of CDF

1. $F_X(x)$ increases monotonically,

$$F_X(x_1) \leq F_X(x_2), \quad \text{for } x_1 < x_2.$$

2. $F_X(x)$ is continuous from the right,

$$\lim_{\varepsilon \to +0} F_X(x+\varepsilon) = F_X(x).$$

3. $F_X(x)$ has the following limits,

$$\lim_{x \to -\infty} F_X(x) = 0, \quad \lim_{x \to +\infty} F_X(x) = 1.$$

$F_x(x) = P(X \leq x)$ (shorthand notation)

Where X is the probability that takes a value less than or equal to x and that lies in the semi-closed interval (a,b], where a < b.

Therefore the probability within the interval is written as

$P(a < X \leq b) = F_x(b) - F_x(a)$

The CDF defined for a continuous RB is given as:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$