CPT - Info and Distinguishability

QWORLD STUDYGROUP #4
Session 4
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Properties of CDF

1. $F_X(x)$ increases monotonically,

$$F_X(x_1) \leq F_X(x_2), \quad \text{for } x_1 < x_2.$$

2. $F_X(x)$ is continuous from the right,

$$\lim_{\varepsilon \to +0} F_X(x+\varepsilon) = F_X(x).$$

3. $F_X(x)$ has the following limits,

$$\lim_{x \to -\infty} F_X(x) = 0, \quad \lim_{x \to +\infty} F_X(x) = 1.$$

$F_x(x) = P(X \leq x)$ (shorthand notation)

Where X is the probability that takes a value less than or equal to x and that lies in the semi-closed interval (a,b], where a < b.

Therefore the probability within the interval is written as

$P(a < X \leq b) = F_x(b) - F_x(a)$

The CDF defined for a continuous RB is given as:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

$F(x) = P(x \leq r)$

Sum of all prob = 1

The graph may look like a step function in discrete cases

$$\textit{probability distribution of } X$$

$$P_X(B) = \mu\left(X^{-1}(B)\right).$$

X then generates Px (B) on the Borel sets B of the real axis.

We can have multivariate random variables: X = (Xi , X2, • • • , Xn)

The joint probability density $$P_X(B) = \mu\left(X^{-1}(B)\right) = \int_B d^d x \, p_X(x)$$

Two random variables X1and X2 are statistically independent if

$$\mu(X_1 \leq x_1, X_2 \leq x_2) = \mu(X_1 \leq x_1) \cdot \mu(X_2 \leq x_2)$$

Transformation of RV

Y = g (X)

$$g : \mathbb{R}^d \longrightarrow \mathbb{R}^f$$

Given, P(x) for RV X

$$P_Y(B) = P_X(g^{-1}(B))$$

The corresponding probability densities are connected by the relation

$$p_Y(y) = \int d^d x \, \delta^{(f)}(y - g(x)) p_X(x),$$

where $\delta^{(f)}$ denotes the f-dimensional δ-function. This formula enables the determination of the density of $Y = g(X)$. For example, the sum $Y = X_1 + X_2$ of two random variables is found by taking $g(x_1, x_2) = x_1 + x_2$. If X_1 and X_2 are independent we get the formula

$$p_Y(y) = \int dx_1 \, p_{X_1}(x_1) p_{X_2}(y - x_1),$$

which shows that the density of Y is the convolution of the densities of X_1 and of X_2.

Expectation Value

$$\mathrm{E}(X) \equiv \int_{-\infty}^{+\infty} x dF_X(x) = \int_{-\infty}^{+\infty} dx \; x p_X(x).$$

Here, the quantity $dF_X(x)$ is defined as

$$dF_X(x) \equiv F_X(x + dx) - F_X(x) = \mu(x < X \leq x + dx).$$

$$\mathrm{E}(g(X)) = \int_{-\infty}^{+\infty} g(x) dF_X(x) = \int_{-\infty}^{+\infty} dx g(x) p_X(x).$$

$$\mathrm{E}(X^m) = \int_{-\infty}^{+\infty} x^m dF_X(x) = \int_{-\infty}^{+\infty} dx \; x^m p_X(x).$$

Variance

$$\mathrm{Var}(X) \equiv \mathbf{E}\left([X - \mathbf{E}(X)]^2\right) = \mathbf{E}(X^2) - \mathbf{E}(X)^2.$$

For multivariate RV X:

$$\mathrm{Cov}(X_i, X_j) \equiv \mathbf{E}\left([X_i - \mathbf{E}(X_i)]\,[X_j - \mathbf{E}(X_j)]\right)$$

If X_1 and X_2 are statistically independent the matrix would have no off-diagonal terms.

Characteristic Function

$$G(k) = \mathrm{E}\left(\exp\left[ikX\right]\right) = \int dx\, p_X(x) \exp\left(ikx\right)$$

Moments of X:

$$\mathrm{E}(X^m) = \frac{1}{i^m} \frac{d^m}{dk^m}\bigg|_{k=0} G(k)$$

G(*k*) is the generating function and for multivariate X:

$$G(k_1, k_2, \ldots, k_d) = \mathrm{E}\left(\exp\left[i \sum_{j=1}^{d} k_j X_j\right]\right)$$

For Z = X + Y (two ind. RVs) ; Prob density = convolution of densities X and Y.

The generating ftn.: product of the characteristic functions of X and Y

Stochastic Process / Random Process

Synonyms

Name for probability models

Think of RV + Time = Stochastic process

A stochastic process is a map such that: $X : \Omega \times T \longrightarrow \mathbb{R},$

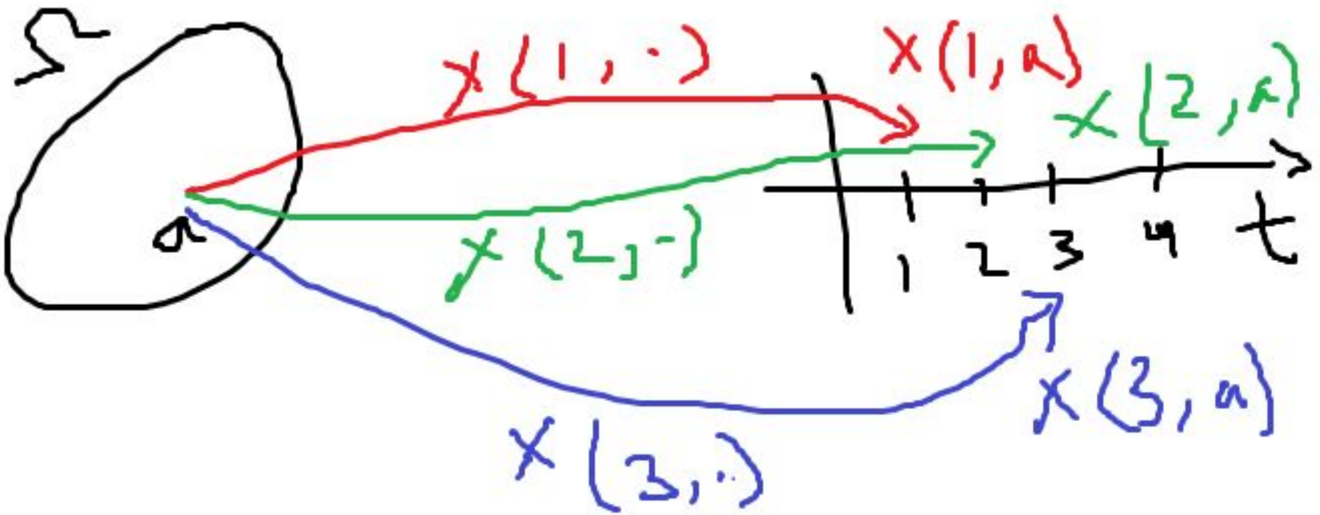
Keeping w fixed then map: $t \mapsto X(\omega, t), \quad t \in T$

For multivariate; $X(t) = (X_1(t), X_2(t), ..., X_d(t))$:

$$X : \Omega \times T \longrightarrow \mathbb{R}^d$$

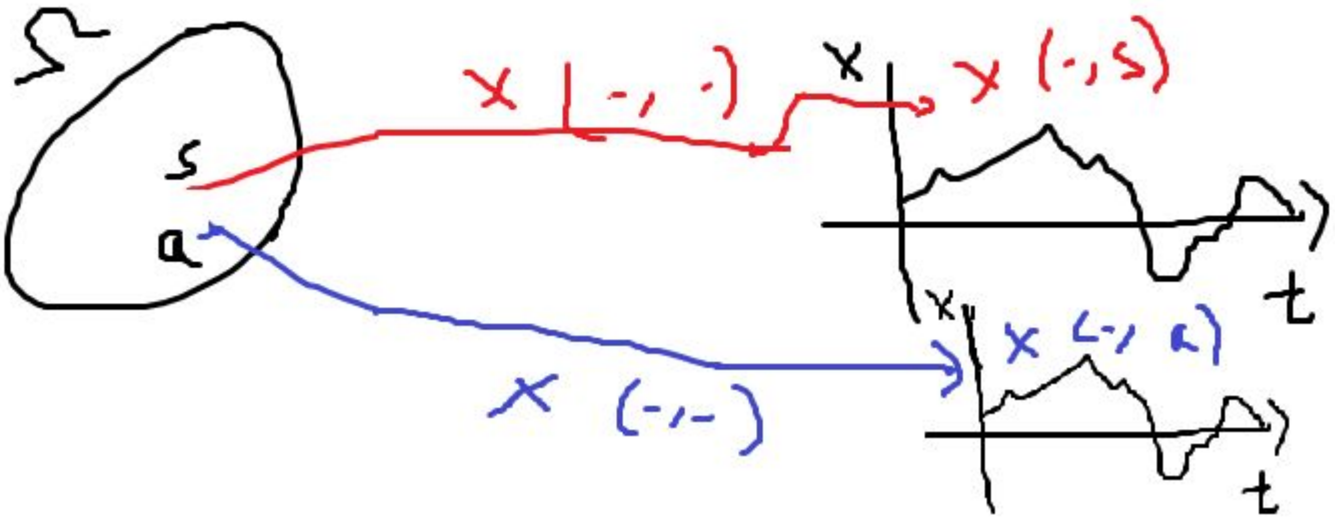
Two definitions:

1 A Stochastic processes is a collection **{X(t ; .) : t ε T}** of RVs indexed by t.



Two definitions:

2 A Stochastic processes is a collection **{X(. ; s) : t ε T}** of deterministic ftn.s of time indexed by outcome s.



4 cases of notaion

Four different cases:

- $\{X(t;\cdot) : t \in \mathbb{T}\}$ or $\{X(\cdot;s) : s \in \mathbb{S}\}$ - a stochastic/random process
- $X(t;\cdot)$ - a random variable (time is fixed)
- $X(\cdot;s)$ - a deterministic function of time (outcome fixed)
- $X(t;s)$ - a numerical value (both time and outcome are fixed)

For all these X(t)