UNIVERSITY OF DAR ES SALAAM

DEPARTMENT OF MATHEMATICS

MT 100: FOUNDATION OF ANALYSIS

SPECIAL TIMED TEST, 2016/2017.

Date: Saturday, February 04, 2017 **Time: 50 minutes**

Instruction: Answer **all** questions.

1. (a) Let $x, y, z \in \mathbb{R}$. Prove that if $x < y$ and $z > 0$, then $xz < yz$. - - - (2 points)

   (b) Prove that for any sets A, B, C, D, $(A \times C) \cap (B \times D) = (A \cap B) \times (C \cap D)$. - - - - - (4 points)

2. (a) Use $\epsilon - \mathbb{N}(\epsilon)$ definition to prove that $\lim_{n \to \infty} \frac{2n^2 + 1}{n^2 + 1} = 2$. - - - - (2 points)

   (b) Prove that if A is a bounded non-empty subset of $\mathbb{R}$, then $\inf A \leq \sup A$. - - - - (2 points

3. Let xRy defines the relation that $x^2 - y^2$ is even. Show that R is an equivalence relation and its equivalence classes partition $\mathbb{Z}$. - - - - (7 points)

4. (a) When do we say $f : A \longrightarrow B$ is bounded for every $x \in A$? - - - 1 point

   (b) Prove that $\forall x \forall y \exists z \ni x + y = z$. - - - - - (2 points)

Marking Scheme

MT 100: Foundations of Analysis
Special Timed Test.

1.(a) Proof

Given $x < y$ and $z > 0$, we need to show that $xz < yz$.

From hypotheses,

$x < y \Rightarrow 0 < y - x$
$\Rightarrow y - x > 0$
$\Rightarrow y - x \in \mathbb{R}^+$ } Axioms of ordering in Real numbers ----- (i)

$z > 0 \Rightarrow z \in \mathbb{R}^+$ ----- (ii) ----- ($\frac{1}{2}$ mark)

$\therefore (y-x)z \in \mathbb{R}^+$ (Closure property under multiplication in $\mathbb{R}^+$)

$\Rightarrow yz - xz \in \mathbb{R}^+$ (distributive property ----- ($\frac{1}{2}$ mark)

$\Rightarrow yz - xz > 0$ ----- ($\frac{1}{2}$ mark)

$\Rightarrow yz > xz$

$\Rightarrow xz < yz$ # ----- ($\frac{1}{2}$ mark)

1.(b) Proof:

Required to show that $(A \times C) \cap (B \times D) \subseteq (A \cap B) \times (C \cap D)$ and $(A \cap B) \times (C \cap D) \subseteq (A \times C) \cap (B \times D)$

To begin let $(x, y) \in (A \times C) \cap (B \times D)$

$\Rightarrow (x, y) \in A \times C$ and $(x, y) \in B \times D$ (Intersection of two sets)

$\Rightarrow (x \in A$ and $y \in C)$ and $(x \in B$ and $y \in D)$ (defn of cartesian product)

$\Rightarrow (x \in A$ and $x \in B)$ and $(y \in C$ and $y \in D)$ (Commutative.)

$\Rightarrow x \in (A \cap B)$ and $y \in (C \cap D)$ (defn. of intersection of two sets)

$\Rightarrow (x, y) \in (A \cap B) \times (C \cap D)$ (2 points)

$\therefore (A \times C) \cap (B \times D) \subseteq (A \cap B) \times (C \cap D)$

Conversely, suppose

$(x, y) \in (A \cap B) \times (C \cap D)$

$\Rightarrow (x \in A \cap B)$ and $(y \in C \cap D)$

$\Rightarrow (x \in A$ and $x \in B)$ and $(y \in C$ and $y \in D)$

$\Rightarrow (x \in A$ and $y \in C)$ and $(x \in B$ and $y \in D)$ — (2 marks)

$\Rightarrow (x, y) \in (A \times C)$ and $(x, y) \in (B \times D)$

$\Rightarrow (x, y) \in (A \times C) \cap (B \times D)$ #

2. (a) Proof

$\forall \varepsilon > 0, \exists N(\varepsilon)$ such that $\left| \frac{2n^2+1}{n^2+1} - 2 \right| < \varepsilon$, whenever $n > N(\varepsilon)$. — $(\frac{1}{2}$ mark$)$

$$\left| \frac{2n^2+1}{n^2+1} - 2 \right| = \left| \frac{2n^2+1-2n^2-2}{n^2+1} \right|$$

$$= \left| \frac{-1}{n^2+1} \right|$$

$$= \frac{1}{n^2+1}$$

$$< \frac{1}{n}$$ — $(\frac{1}{2}$ mark$)$

$< \varepsilon$, provided that $n > \frac{1}{\varepsilon}$ — $(\frac{1}{2}$ mark$)$

Thus $\left| \frac{2n^2+1}{n^2+1} - 2 \right| < \varepsilon$

$\therefore \lim_{n \to \infty} \frac{2n^2+1}{n^2+1} = 2$ # — $(\frac{1}{2}$ mark$)$

2 (b) Proof

Since A is non-empty, then $x \in A$. ---- ($\frac{1}{2}$ mark)

$\Rightarrow \inf A \leq x$ --- (i) --- ($\frac{1}{2}$ mark)

$\Rightarrow x \leq \sup A$ --- (ii) --- ($\frac{1}{2}$ mark)

From (i) and (ii)

$\Rightarrow \inf A \leq \sup(A)$ (transitivity property of relations) ## ($\frac{1}{2}$ mark)

3. $xRy \Rightarrow x^2 - y^2$ is even.

$\Rightarrow x^2 - y^2 = 2k, k \in \mathbb{Z}$.

$\boxed{xRy \Rightarrow \frac{x^2 - y^2}{2} = k, k \in \mathbb{Z}}$ --- ($\frac{1}{2}$ mark)

$xRx \Rightarrow \frac{x^2 - x^2}{2} = k = 0 \in \mathbb{Z}$.

$\therefore$ R is reflexive. --- (1 mark)

$xRy \Rightarrow \frac{x^2 - y^2}{2} = k, k \in \mathbb{Z}$.

$\Rightarrow \frac{-(y^2 - x^2)}{2} = k, k \in \mathbb{Z}$.

$\Rightarrow \frac{y^2 - x^2}{2} = -k, -k \in \mathbb{Z}$. --- (1 mark)

$\therefore xRy \Rightarrow yRx$.

$\Rightarrow$ R is symmetric.

$$x R y \Rightarrow \frac{x^2 - y^2}{2} = k_1,\ k_1 \in \mathbb{Z}.$$

$$y R z \Rightarrow \frac{y^2 - z^2}{2} = k_2,\ k_2 \in \mathbb{Z}.$$

$$x R y \wedge y R z \Rightarrow \frac{x^2 - y^2}{2} + \frac{y^2 - z^2}{2} = k_1 + k_2 = k \in \mathbb{Z}.$$

(Closure property of addition in $\mathbb{Z}$)

$$\Rightarrow \frac{x^2 - z^2}{2} = k,\ k \in \mathbb{Z}.$$ (1 mark)

$$\Rightarrow x R y$$

$\therefore$ R is transitive.

$\therefore$ Since R is reflexive, symmetric and transitive then R is an equivalence relation. (0 1/2 mark)

Let $[a]$ denotes an equivalence class.

$$[a] = \{x \in \mathbb{Z} : x R a\}.$$ (1/2 mark)

$$= \{x \in \mathbb{Z} : \frac{x^2 - a^2}{2} = k,\ k \in \mathbb{Z}\}.$$

$$= \{x \in \mathbb{Z} : x^2 = a^2 + 2k,\ k \in \mathbb{Z}\}.$$

$$[0] = \{x \in \mathbb{Z} : x^2 = 2k,\ k \in \mathbb{Z}\}$$

$$\Rightarrow [0] = \{\ldots, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10, 12, \ldots\}$$

$$[1] = \{x \in \mathbb{Z} : x^2 = 1 + 2k,\ k \in \mathbb{Z}\}$$ (0 1/2 mark)

$$= \{\ldots, -7, -5, -3, -1, 1, 3, 5, 7, 9, 11, 13, \ldots\}$$

$$[2] = \{x \in \mathbb{Z} : x^2 = 4 + 2k\}$$

$$= \{\ldots, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10, 12, \ldots\} = [0].$$

$$[3] = \{x \in \mathbb{Z} : x^2 = 9 + 2k\}$$

$$= \{\ldots, -7, -5, -3, -1, 1, 3, 5, 7, 9, 11, 13, \ldots\} = [1].$$

$\therefore$ Distinct equivalence classes are $[0]$ and $[1]$ (1/2 mark)

Now it follows that

$$[0] \cap [1] = \phi$$ and (1 mark)

$$[0] \cup [1] = \mathbb{Z}$$

$\therefore$ $[0]$ and $[1]$ partition $\mathbb{Z}$. ---- ($\frac{1}{2}$ mark)

4 (a) The function $f: A \to B$ is said to be bounded for every $x \in A$ provided that there is a positive real number, M Such that

$$|f(x)| \leq M$$ fore every $x \in A$.

1 Point

4 (b) $\forall x \, \forall y \, \exists z \ni x + y = z$.

<u>Proof</u>: Let existence of $z = y + x$ ---- (1 point)

$\Rightarrow$ $x + y = y + x$ is TRUE for every $x, y \in \mathbb{R}$ ---- ($\frac{1}{2}$ point)

(Commutative property of addition in Real numbers) #

---- ($\frac{1}{2}$ point)