Chapter – 5.1

1) Concept of Matrix

a.

Definition

A matrix is a rectangular array of numbers. An $m \times n$ (m-by-n) matrix has m rows and n columns.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad \text{elements of } A: \ a_{ij} \qquad A = (a_{ij})$$

The set of all $m \times n$ matrices is denoted by $\mathcal{M}_{m\times n}$.

Definition

If $n = m$, then the matrix is a square matrix.

The main diagonal of a matrix is formed by the elements $(a_{11}, a_{22}, a_{33}, \dots)$.

The identity matrix of size n is the $n \times n$ matrix such that the elements on the main diagonal are equal to 1 and all other elements are zero. Notation: I_n.

2) Matrix Operations:

a.

1. Matrix addition

We can only add matrices of the same type.
If $A = (a_{ij})$ and $B = (b_{ij})$ are $m \times n$ matrices, then we calculate the sum entrywise: $C = A + B$, where $c_{ij} = a_{ij} + b_{ij}$; $i = 1, \dots, m$, $j = 1, \dots, n$.

2. Scalar multiplication

We do the scalar multiplication entrywise. That is, let $\lambda \in \mathbb{R}$, $A = (a_{ij}) \in \mathcal{M}_{m\times n}$, $\lambda A = (\lambda a_{ij}) \in \mathcal{M}_{m\times n}$.

3. Matrix multiplication

Let $A = (a_{ij})$ be an $m \times k$ and $B = (b_{ij})$ be a $k \times n$ matrix. Then the product of A and B is the $m \times n$ matrix $C = (c_{ij})$, such that

$$c_{ij} = \sum_{r=1}^{k} a_{ir} b_{rj}.$$

b. **Theorem – properties of matrix multiplication**

- If A is an $m \times n$ matrix, then $I_m \cdot A = A$ and $A \cdot I_n = A$.
- If A, B, C are such matrices that AB and BC exist, then $(AB)C = A(BC)$. The matrix multiplication is associative.
- If A and B are of the same size and AC exists, then BC exists as well and $(A+B)C = AC + BC$.
- Matrix multiplication is not commutative, that is, in general $AB \neq BA$.

Definition

Let A be an $m \times n$ matrix. The $n \times m$ matrix denoted by A^T is called the transpose of A.

Proposition – properties of transposition

- $(A^T)^T = A$
- Transposition and matrix multiplication: $(AB)^T = B^T \cdot A^T$.

c. **Definition**

Let A be a square matrix of order n.

- A is symmetric if $A^T = A$,
- A is skew-symmetric if $A^T = -A$.

3) Determinant

a. **Definition**

Let $A = (a_{ij})$ be a square matrix. Let us choose n elements of A such that we choose exactly one element from each row and each column. The chosen elements:

$$a_{1\sigma(1)}, a_{2\sigma(2)}, \ldots, a_{n\sigma(n)}.$$

The determinant of A is

$$\det(A) = |A| = \sum_{\sigma} (-1)^{I(\sigma)} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

- There are $n!$ terms in the sum above.
- Other notation: $\varepsilon(\sigma) = (-1)^{I(\sigma)}$.

Example:

- $n = 2$: $\det(A) = |A| = a_{11}a_{22} - a_{12}a_{21}$.
- $n = 3$: $\det(A) = \ldots$.

Theorem

If A and B are square matrices of the same size, then

$$\det(AB) = \det(A) \cdot \det(B).$$

b. The **determinant** is a scalar value that can be computed from the elements of a square matrix and encodes certain properties of the linear transformation described by the matrix.

c. The determinant of a matrix A is denoted $\det(A)$, $\det A$, or $|A|$.

d.

Determinant of matrices of special form

Proposition

For any $n \in \mathbb{N}$ the determinant of the identity matrix is 1.

$$\det(I_n) = 1$$

Proposition

Let A be an upper triangular matrix, that is, a square matrix with zeros underneath its main diagonal:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix}.$$

Then the determinant of A is the product of the elements in the main diagonal.

e.

Proposition – properties of the determinant

- $\det(A) = \det(A^T)$
- If A has a column full of zeros, then $\det(A) = 0$.
- If we interchange 2 columns of A, then the sign of the determinant changes.
- If one column of A is a scalar multiple of another column, then $\det(A) = 0$.
- If we multiply a column of A by a real number λ, then the obtained matrix has determinant $\lambda \cdot \det(A)$.
- The determinant doesn't change if we add a scalar multiple of a column to another column.
- If a column of A is the linear combination of the other columns, then $\det(A) = 0$.
- The properties above are true if we consider rows instead of columns.

Corollary

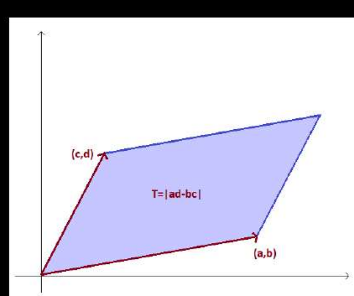
If $\det(A) \neq 0$, then the columns (or rows) of A are linearly independent vectors. Then is A is of size $n \times n$: its columns form a basis of $\mathbb{R}^n$.

f.

Geometric meaning of the determinant

- 2-by-2 determinants: it's absolute value is the area of the parallelogram determined by the rows of the determinant, as vectors

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$



- 3-by-3 determinants: it's absolute value is the volume of the parallelepiped determined by the rows of the determinant, as vectors

4) Rank

a. The **rank** of a matrix is defined as (a) the maximum number of [linearly independent] *column* vectors in the matrix or (b) the maximum number of linearly independent *row* vectors in the matrix. Both definitions are equivalent.

b. For an r x c matrix,

   i. If r is less than c, then the maximum rank of the matrix is r.

   ii. If r is greater than c, then the maximum rank of the matrix is c.

c. The rank of a matrix would be zero only if the matrix had no elements. If a matrix had even one element, its minimum rank would be one.

d. Linearly independent also check if any linear multiplication of a row plus any linear multiplication of other row is not equal to any row. If it is equal that row is linearly dependent.

e. The above rule also applies to columns.

5) Matrix Classes

6) Inverse Matrix:

a.

Definition

We say that a square matrix A of order n is invertible or that it has an inverse if there exists a square matrix B of order n, such that

$$AB = BA = I_n.$$

Theorem

If A is invertible, then its inverse is uniquely determined. Notation: A^{-1}.

Example:

$$A = \begin{pmatrix} 4 & 3 \\ 7 & 5 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} -5 & 3 \\ 7 & -4 \end{pmatrix}$$

Proposition – properties of matrix inverse

If A is invertible, then so is A^{-1}, and $(A^{-1})^{-1} = A$.

If A and B are invertible and AB exists, then $(AB)^{-1} = B^{-1}A^{-1}$.

If A is invertible, then so is A^T, and $(A^{-1})^T = (A^T)^{-1}$.

b.

Determinant and matrix inverse

Definition

We say that the square matrix A is regular if $\det(A) \neq 0$.
Otherwise A is said to be singular.

Theorem

A matrix is invertible if, and only if, it is regular. (That is, it's determinant is non-zero.)

Proposition

If A is invertible, then

$$\det(A)^{-1} = \det(A^{-1}).$$

7) Linear Transformations:

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8) Eigenvectors and eigenvalues:

a. For a square matrix **A**, an Eigenvector and Eigenvalue make this equation true:

$$\underset{\text{Matrix}}{A}v = \underset{\text{Eigenvalue}}{\lambda}v$$

Eigenvector

How do we find these eigen things?

We start by finding the **eigenvalue**: we know this equation must be true:

$$Av = \lambda v$$

Now let us put in an identity matrix so we are dealing with matrix-vs-matrix:

$$Av = \lambda I v$$

Bring all to left hand side:

$$Av - \lambda I v = 0$$

If **v** is non-zero then we can solve for λ using just the determinant:

b. $$| A - \lambda I | = 0$$

c. Now, we know eigenvalue, we find eigenvector with this equation: $Av = \lambda v$.

d. Put in the values we know:

$$\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 6\begin{bmatrix} x \\ y \end{bmatrix}$$

After multiplying we get these two equations:

$$\begin{aligned} -6x + 3y &= 6x \\ 4x + 5y &= 6y \end{aligned}$$

Bringing all to left hand side:

$$\begin{aligned} -12x + 3y &= 0 \\ 4x - 1y &= 0 \end{aligned}$$

e. For $A \in R$ n×n the value $\lambda \in C$ is an eigenvalue and the vector $v \neq 0$ is the eigenvector belonging to λ if:

    i. $Av = \lambda v$

f. After rearrangement: $(A - \lambda I)v = 0$.

g. I is identity matrix

h. When λ is given, it is a homogeneous system of linear equations for the vector v.

i. There are non-trivial solutions only for those λ values for which eigenvector $(v) = 0$.