Chapter – 7.1

1) System of Linear Equations and its solutions by Gaussian Elimination:

a.

Definition

The system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where the real numbers a_{ij} ($i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$) and b_k ($k \in \{1, \dots, m\}$) are known, the variables $x_1, \dots, x_n$ are unknown, is called a system of linear equations.

a_{ij}: the coefficients of the system of linear equations

b_k: the constant terms

the coefficient matrix and the augmented matrix:

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \quad \text{and} \quad A|b = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ a_{21} & \dots & a_{2n} & b_2 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right)$$

b.

Solvability of systems of linear equations

The corresponding matrix equation: $Ax = b$.

Definition

The system of linear equations is

solvable if there exists at least one solution;

- determined if there is exactly 1 solution;
- undetermined if there are more than 1 solutions;

overdetermined if it doesn't have a solution.

Definition

The rank of a matrix is the rank of the system of column vectors of the matrix. Notation: $\text{rank}(A)$.

Theorem – condition on solvability

A system of linear equation is solvable if, and only if $\text{rank}(A) = \text{rank}(A|b)$.

If it is solvable and $\text{rank}(A) = n$ (where n is the number of unknown parameters), then the system is determined, if $\text{rank}(A) < n$, then undetermined.

c.

Solutions of a system of linear equations

Definition

A system of linear equations is homogeneous if $b = 0$, thus then the matrix equation has the form $Ax = 0$. Otherwise it's called nonhomogeneous.

Remark: 0 is a solution of any homogenous system of linear equations.

Proposition – solutions of a homogeneous system of linear equations

The solutions of a homogeneous system of linear equations form a vector subspace of $\mathbb{R}^n$ with dimension $n - \mathrm{rank}(A)$.

Proposition – solutions of a nonhomogeneous system of linear equations

The solution set of a (solvable) nonhomogeneous system of linear equations $Ax = b$ is of the form $x_0 + H$, where

- x_0 is a particular solution of the system of linear equations;
- H is the solution set of the corresponding homogeneous system of linear equation, that is $Ax = 0$.

d.

Solving a system of linear equations with Gaussian elimination

The set of solutions of a system of linear equations does not change, if we

- multiply an equation by a nonzero constant;
- add a scalar multiple of an equation to another equation;
- interchange two equations;
- discard an equation which is a scalar multiple of another equation.

We annihilate the numbers under the main diagonal with the modifications above. The resulting system is easier to solve.

- If during the process we obtain a row like $(0 \dots 0| \neq 0)$, then the system of linear equations is overdetermined.
- If at the end of the process there are n number of rows, then the system is determined, if fewer number of rows remains, then undetermined. (Here n is the number of the unknown parameters.)

e. Annihilate here means make elements under main diagonal 0.