Chapter – 2.1

1) Basic concepts of probability and combinatorics (permutation, variation, combination):

A probability P is a rule (function) which assigns a positive number to each event, and which satisfies the following axioms:
Axiom 1: $P(A) \geq 0$.
Axiom 2: $P(\Omega) = 1$.
Axiom 3: For any sequence A1, A2, … of disjoint events we have

$$\boxed{\mathbb{P}(\bigcup_i A_i) = \sum_i \mathbb{P}(A_i)}.$$

Permutations:
Permutations are arrangements of objects (with or without repetition), order does matter.

The number of permutations of n objects, without repetition, is:

$$P_n = P_n^n = n!.$$

Combinations:
Combinations are selections of objects, with or without repetition, order does not matter.

The number of k-element combinations of n objects, without repetition is:

$$C_{n,k} = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The number of k-element combinations of n objects, with repetition is:

$$\overline{C}_{n,k} = C_{n+k-1,k} = \binom{n+k-1}{k} = \left(\binom{n}{k}\right).$$

Variations:
Variations are arrangements of selections of objects, where the order of the selected objects matters.

The number of k-element variations of n elements with repetition not allowed is:

$$V_{n,k} = P_{n,k} = \binom{n}{k} \cdot k! = (n)_k.$$

The number of k-element variations of n-elements with repetition allowed, is:

$$V_{n,k} = n^k.$$

2) Conditional Probability and Independence

How do probabilities change when we know some event $B \subset \Omega$ has occurred? Suppose B has occurred. Thus, we know that the outcome lies in B. Then A will occur if and only if $A \cap B$ occurs, and the relative chance of A occurring is therefore:

$$P(A \cap B)/P(B)$$

This leads to the definition of the conditional probability of A given B:

$$P(A|B) = P(A \cap B)/P(B)$$

We throw two dice. Given that the sum of the eyes is 10, what is the probability that one 6 is cast?
2/3.

Product Rule:
By the definition of conditional probability, we have:

$$P(A \cap B) = P(A|B) * P(B)$$

We can generalise this to n intersections $A_1, A_2, \ldots, A_n$, which we abbreviate as $A_1, A_2, \ldots, A_n$. This gives the product rule of probability (also called chain rule).

(Product rule) Let $A_1, A_2, \ldots, A_n$ be a sequence of events with $P(A_1, \ldots, A_{n-1}) > 0$. Then,

$$\boxed{\mathbb{P}(A_1 \cdots A_n) = \mathbb{P}(A_1)\,\mathbb{P}(A_2 \mid A_1)\,\mathbb{P}(A_3 \mid A_1 A_2) \cdots \mathbb{P}(A_n \mid A_1 \cdots A_{n-1}).}$$

3) Independence

Independence is a very important concept in probability and statistics. Loosely speaking it models the lack of information between events. We say A and B are independent if the knowledge that A has occurred does not change the probability that B occurs. That is

$$A,\, B \text{ independent} \Leftrightarrow \mathbb{P}(A|B) = \mathbb{P}(A)$$

Since, $P(A|B) = P(A \cap B)/P(B)$ an alternative definition of independence is:

$$\boxed{A,\, B \text{ independent} \Leftrightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)}$$

4) Bayes theorem

Suppose $B_1, B_2, \ldots, B_n$ is a partition of. That is, $B_1, B_2, \ldots, B_n$ are disjoint and their union is, see below:

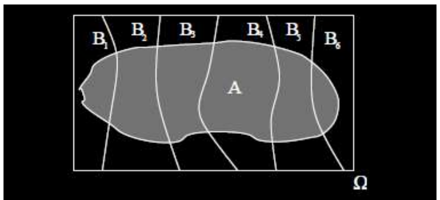

Figure 1.16: A partition of the sample space

Then, by the sum rule, $\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A \cap B_i)$ and hence, by the definition of conditional probability we have

$$\boxed{\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A|B_i)\,\mathbb{P}(B_i)}$$

This is called Law of total probability.

Combining the Law of Total Probability with the de_nition of conditional probability gives Bayes' Rule:

$$\boxed{\mathbb{P}(B_j|A) = \frac{\mathbb{P}(A|B_j)\,\mathbb{P}(B_j)}{\sum_{i=1}^{n} \mathbb{P}(A|B_i)\mathbb{P}(B_i)}}$$