3.2

Definition/1

The language of first–order logic is a

$$L^{(1)} = \langle LC, Var, Con, Term, Form\rangle$$

ordered 5–tuple, where

1. $LC = \{\neg, \supset, \wedge, \vee, \equiv, =, \forall, \exists, (,)\}$: (the set of logical constants).
2. Var $(= \{x_n : n = 0, 1, 2, \dots\})$: countable infinite set of variables

Definition/2

3. $Con = \bigcup_{n=0}^{\infty}(\mathcal{F}(n) \cup \mathcal{P}(n))$ the set of non–logical constants (at best countable infinite)
   - $\mathcal{F}(0)$: the set of name parameters,
   - $\mathcal{F}(n)$: the set of n argument function parameters,
   - $\mathcal{P}(0)$: the set of prposition parameters,
   - $\mathcal{P}(n)$: the set of predicate parameters.
4. The sets LC, Var, $\mathcal{F}(n)$, $\mathcal{P}(n)$ are pairwise disjoint $(n = 0, 1, 2, \dots)$.

Definition/3

5. The set of terms, i.e. the set $Term$ is given by the following inductive definition:
   (a) $Var \cup \mathcal{F}(0) \subseteq Term$
   (b) If $f \in \mathcal{F}(n)$, $(n = 1, 2, \dots)$, s $t_1, t_2, \dots, t_n \in Term$, then $f(t_1, t_2, \dots, t_n) \in Term$.

Definition/4

6. The set of formulas, i.e. the set $Form$ is given by the following inductive definition:
   (a) $\mathcal{P}(0) \subseteq Form$
   (b) If $t_1, t_2 \in Term$, then $(t_1 = t_2) \in Form$
   (c) If $P \in \mathcal{P}(n)$, $(n = 1, 2, \dots)$, s $t_1, t_2, \dots, t_n \in Term$, then $P(t_1, t_2, \dots, t_n) \in Form$.
   (d) If $A \in Form$, then $\neg A \in Form$.
   (e) If $A, B \in Form$, then $(A \supset B)$, $(A \wedge B)$, $(A \vee B)$, $(A \equiv B) \in Form$.
   (f) If $x \in Var$, $A \in Form$, then $\forall xA$, $\exists xA \in Form$.

Definition

Let $L^{(1)} = \langle LC, Var, Con, Term, Form\rangle$ be a first order language and $A \in Form$ be a formula. The set of free variables of the formula A (in notation: $FreeVar(A)$) is given by the following inductive definition:

- If A is an atomic formula (i.e. $A \in AtForm$), then the members of the set $FreeVar(A)$ are the variables occuring in A.
- If the formula A is $\neg B$, then $FreeVar(A) = FreeVar(B)$.
- If the formula A is $(B \supset C)$, $(B \wedge C)$, $(B \vee C)$ or $(B \equiv C)$, then $FreeVar(A) = FreeVar(B) \bigcup FreeVar(C)$.
- If the formula A is $\forall xB$ or $\exists xB$, then $FreeVar(A) = FreeVar(B) \setminus \{x\}$.

Definition

Let $L^{(1)} = \langle LC, Var, Con, Term, Form\rangle$ be a first order language and $A \in Form$ be a formula. The set of bound variables of the formula A (in notation: $BoundVar(A)$) is given by the following inductive definition:

- If A is an atomic formula (i.e. $A \in AtForm$), then $BoundVar(A) = \emptyset$.
- If the formula A is $\neg B$, then $BoundVar(A) = FreeVar(B)$.
- If the formula A is $(B \supset C)$, $(B \wedge C)$, $(B \vee C)$ or $(B \equiv C)$, then $BoundVar(A) = BoundVar(B) \bigcup BoundVar(C)$.
- If the formula A is $\forall xB$ or $\exists xB$, then $BoundVar(A) = BoundVar(B) \cup \{x\}$.

Definition

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language, $A \in Form$ be a formula, and $x \in Var$ be a variable.

- A fixed occurence of the variable x in the formula A is free if it is not in the subformulas $\forall x B$ or $\exists x B$ of the formula A.
- A fixed occurence of the variable x in the formula A is bound if it is not free.

Remark

- If x is a free variable of the formula A (i.e. $x \in FreeVar(A)$), then it has at least one free occurence in A.
- If x is a bound variable of the formula A (i.e. $x \in BoundVar(A)$), then it has at least one bound occurence in A.
- A fixed occurence of a variable x in the formula A is free if
  - it does not follow a universal or an existential quantifier, or
  - it is not in a scope of a $\forall x$ or a $\exists x$ quantification.
- A variable x may be a free and a bound variable of the formula A: $(P(x) \wedge \exists x R(x))$

Definition

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order languuage and $A \in Form$ be a formula.

- If $FreeVar(A) \neq \emptyset$, then the formula A is an open formula.
- If $FreeVar(A) = \emptyset$, then the formula A is a closed formula.

Remark:
The formula A is open if there is at least one variable which has at least one free occurence in A.
The formula A is closed if there is no variable which has a free occurence in A.

Definition (interpretation)

The ordered pair $\langle U, \varrho \rangle$ is an interpretation of the language $L^{(1)}$ if

- $U \neq \emptyset$ (i.e. U is a nonempty set);
- $Dom(\varrho) = Con$
  - If $a \in \mathcal{F}(0)$, then $\varrho(a) \in U$;
  - If $f \in \mathcal{F}(n)$ $(n \neq 0)$, then $\varrho(f) \in U^{U^{(n)}}$
  - If $p \in \mathcal{P}(0)$, then $\varrho(p) \in \{0, 1\}$;
  - If $P \in \mathcal{P}(n)$ $(n \neq 0)$, then $\varrho(P) \subseteq U^{(n)}$ $(\varrho(P) \in \{0, 1\}^{U^{(n)}})$.

Definition (assignment)

The function v is an assignment relying on the interpretation $\langle U, \varrho \rangle$ if the followings hold:

- $Dom(v) = Var$;
- If $x \in Var$, then $v(x) \in U$.

Definition (modified assignment)

Let v be an assignment relying on the interpretation $\langle U, \varrho \rangle$, $x \in Var$ and $u \in U$.

$$v[x : u](y) = \begin{cases} u, & \text{if } y = x; \\ v(y), & \text{otherwise.} \end{cases}$$

for all $y \in Var$.

Definition (Semantic rules/1)

Let $\langle U, \varrho\rangle$ be a given interpretation and v be an assignment relying on $\langle U, \varrho\rangle$.

- If $a \in \mathcal{F}(0)$, then $|a|_v^{\langle U,\varrho\rangle} = \varrho(a)$.
- If $x \in Var$, then $|x|_v^{\langle U,\varrho\rangle} = v(x)$.
- If $f \in \mathcal{F}(n)$, $(n = 1, 2, \dots)$, and $t_1, t_2, \dots, t_n \in Term$, then $|f(t_1)(t_2)\dots(t_n)|_v^{\langle U,\varrho\rangle} = \varrho(f)(\langle |t_1|_v^{\langle U,\varrho\rangle}, |t_2|_v^{\langle U,\varrho\rangle}, \dots, |t_n|_v^{\langle U,\varrho\rangle}\rangle)$
- If $p \in \mathcal{P}(0)$, then $|p|_v^{\langle U,\varrho\rangle} = \varrho(p)$
- If $t_1, t_2 \in Term$, then

$$|(t_1 = t_2)|_v^{\langle U,\varrho\rangle} = \begin{cases} 1, & \text{if } |t_1|_v^{\langle U,\varrho\rangle} = |t_2|_v^{\langle U,\varrho\rangle} \\ 0, & \text{otherwise.} \end{cases}$$

Definition (Semantic rules/2)

- If $P \in \mathcal{P}(n)$ $(n \neq 0)$, $t_1, \dots, t_n \in Term$, then

$$|P(t_1)\dots(t_n)|_v^{\langle U,\varrho\rangle} = \begin{cases} 1, & \text{if } \langle |t_1|_v^{\langle U,\varrho\rangle}, \dots, |t_n|_v^{\langle U,\varrho\rangle}\rangle \in \varrho(P); \\ 0, & \text{otherwise.} \end{cases}$$

Definition (Semantic rules/3)

- If $A \in Form$, then $|\neg A|_v^{\langle U,\varrho\rangle} = 1 - |A|_v^{\langle U,\varrho\rangle}$.
- If $A, B \in Form$, then

$$|(A \supset B)|_v^{\langle U,\varrho\rangle} = \begin{cases} 0 & \text{if } |A|_v^{\langle U,\varrho\rangle} = 1, \text{ and } |B|_v^{\langle U,\varrho\rangle} = 0; \\ 1, & \text{otherwise.} \end{cases}$$

$$|(A \wedge B)|_v^{\langle U,\varrho\rangle} = \begin{cases} 1 & \text{if } |A|_v^{\langle U,\varrho\rangle} = 1, \text{ and } |B|_v^{\langle U,\varrho\rangle} = 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$|(A \vee B)|_v^{\langle U,\varrho\rangle} = \begin{cases} 0 & \text{if } |A|_v^{\langle U,\varrho\rangle} = 0, \text{ and } |B|_v^{\langle U,\varrho\rangle} = 0; \\ 1, & \text{otherwise.} \end{cases}$$

$$|(A \equiv B)|_v^{\langle U,\varrho\rangle} = \begin{cases} 1 & \text{if } |A|_v^{\langle U,\varrho\rangle} = |B|_v^{\langle U,\varrho\rangle} = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Upside down A means for all. Backwards E means there exist

Definition (Semantic rules/4)

- If $A \in Form, x \in Var$, then

$$|\forall x A|_v^{\langle U,\varrho\rangle} = \begin{cases} 0, & \text{if there is an } u \in U \text{ such that } |A|_{v[x:u]}^{\langle U,\varrho\rangle} = 0; \\ 1, & \text{otherwise.} \end{cases}$$

$$|\exists x A|_v^{\langle U,\varrho\rangle} = \begin{cases} 1, & \text{if there is an } u \in U \text{ such that } |A|_{v[x:u]}^{\langle U,\varrho\rangle} = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Definition – satisfiable a set of formulas

The set of formulas $\Gamma \subseteq Form$ is satisfiable if it has a model.
(If there is an interpretation in which all members of the set Γ are ture.)

Definition – satisfiable a formula

A formula $A \in Form$ is satisfiable, if the singleton $\{A\}$ is satisfiable.

Remark

- A satisfiable set of formulas does not involve a logical contradiction; its formulas may be true together.
- A safisfiable formula may be true.
- If a set of formulas is satisfiable, then its members are satisfiable.
- But: all members of the set $\{p, \neg p\}$ are satisfiable, and the set is not satisfiable.

Definition

The formula A is valid if $\emptyset \vDash A$. (Notation: $\vDash A$)

The formulas A and B are logically equivalent if $A \vDash B$ and $B \vDash A$. (Notation: $A \Leftrightarrow B$)

Weird = is logical consequence

Theorem

If Γ is unsatisfiable, then $\Gamma \vDash A$ for all A. (All formulas are the consequences of an unsatisfiable set of formulas.)

Proof

- According to a proved theorem: If Γ is unsatisfiable, the all expansions of Γ are unsatisfiable.
- $\Gamma \cup \{\neg A\}$ is an expansion of Γ, and so it is unsatisfiable, i.e. $\Gamma \vDash A$.

Definition

A formula A is the logical consequence of the set of formulas Γ if the set $\Gamma \cup \{\neg A\}$ is unsatifiable. $(Notation : \Gamma \vDash A)$

Definition

A disjunction of elementary conjunctions is a disjunctive normal form.

Definition

A conjunction of elementary disjunctions is a conjunctive normal form.

Theorem

There is a normal form of any formula of proposition logic, i. e. if $A \in Form$, then there is a formula B such that B is a normal form and $A \Leftrightarrow B$

Definition

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language and $A \in Form$ be a formula.
The formula A is prenex if

- there is no quantifier in A or
- the formula A is in the form $Q_1x_1Q_2x_2 \dots Q_nx_nB$ $(n = 1, 2, \dots)$, where
  - there is no quantifier in the formula $B \in Form$;
  - $x_1, x_2 \dots x_n \in Var$ are diffrent variables;
  - $Q_1, Q_2, \dots, Q_n \in \{\forall, \exists\}$ are quantifiers.

Theorem

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language and $A \in Form$ be a formula.
Then there is a formula $B \in Form$ such that

- the formula B is prenex;
- $A \Leftrightarrow B$.