PROJECT MULTIVARIABLE CALCULUS MT1008

Submitted to Dr Hamda Khan

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Contents:

Introduction:

Modern transportation is at the forefront because of autonomous cars which offered unmatched ease and security. However the smooth functioning of these systems depends on their ability to overcome the numerous 0bstacles which were present by various terrains and notably the rolling hills. This is where it become critical to navigate these terrains correctly in 0rder to reach the lowest points. While skillfully avoiding 0bstacles.

Our study aim to develop an algorithmic approach that allow autonomous cars to reliably and efficiently travel 0ver such different terrains in response to this multiple issue. f(x,y) the terrain function that forms the basis of 0ur approach is the formula that is at the core of this effort. As a committed cartographer this function map the terrain's topography by giving heights to all possible coordinates f(x,y).

Problem Formulation

2.1 Objective:

The aim of this project is to develop an algorithm that enable autonomous cars to safely accurately and efficiently travel through rugged terrain. This mean that developing a system that can steer the car across a variety of terrains and skillfully dodging obstacles. And it can move in a planned manner in the direction of the terrain's lowest point. By accomplishing this goal we hope to improve vehicles' independence and adaptability in difficult situations and also advance autonomous transportation systems and promoting safer more effective mobility options for a variety of surfaces.

2.2 Key Components:

1. **Terrain Function (f(x, y)):**
   The height of the vehicle at the coordinates (x,y) is represented by this function. Since the terrain function will determine the height over a 2D plane. And it is the most important function in this scenario.

   Features of the Terrain Function:

   1. Oscillating Terrain Representation:
      This terrain is not a level surface rather it refer to unique landscapes with hills valleys and slopes.

   2. Continuity and Differentiability:
      To provide an effortless change between various points the function of the terrain should both continuous and differentiable at every point.

   3. Minimizing Obstacles:
      The terrain function can take into account obstacles whenever a vehicle approaches them even though there is no instructions to managing them in the

formula. A building, tree or another obstacle that prevent the vehicle from seeing the 2D plane at its’ full height could be a obstacle.

4. Complexity:
   There is a variety of terrain functions from straightforward mathematical formulas to complex models. Plants degradation patterns and mineral deposits all affect how complex the function is.

5. Function in Optimization:
   This function is essential to the optimization of algorithms for autonomous cars since it allow them to travel over a variety of terrains.

2.3 Starting Point for the Vehicle:

The starting point specifies the starting point of the vehicle's trajectory through the terrain. It serve as the foundation for the algorithm's trajectory. It also direct the car in the direction of the global minimum of the terrain. In order to maximize the vehicle's navigation across the hilly terrain and prevent challenges its starting location is essential.

a) First Guess:
The initial guess for the vehicle's location on the 2D plane is essentially a random starting point. And it depend on how the algorithm is used the guess is evaluate based on sensor readings random selection or previous knowledge of the method.

b) Coordinate Representation:
The beginning position on the terrain is represented by the coordinates (x,y), which indicate its location in the 2D plane. Geographical coordinates or Cartesian coordinates could be used for these representations.

c) Flexibility:
This is based upon the nature of the issue and the demands of the navigational task. The kind of approach used to identify an effective path is referred to as flexibility. Anyplace on the landscape could be designated as the starting point. And it can give freedom in choosing the starting point that corresponds with the terrain feature.

d) Impact on Navigation:
Since the starting point establish the initial parameters for the optimization method it has an impact on the trajectory of the vehicle's navigation. The selected beginning point enables quicker navigation to the target global minimum of the

terrain. The starting point may need to be select in accordance with the particular limitations of the challenge such that avoiding known barriers or beginning from a certain region.

2. 4 Avoiding Obstacles:

The autonomous car faces a variety of barriers along its route across steep terrain and it make it difficult for it to reach the lowest position. These barriers include man made constructions like fences and buildings in addition to natural elements like rocks and plant life. The method contains obstacle recognition mechanisms based on real time sensor data to guarantee safe navigation. The algorithm prioritize safety and 0bstacle avoidance while enabling and considering efficient navigation by dynamically modifying the vehicle's route.

a) Navigational issues Limits:
When there is an interruption in the path the vehicle's movement trajectories may be restricted. As a result the 0ptimization procedure to identify the terrain's global minimum is less effective. To navigate the vehicle around 0bstacles and still aim to obtain the global minimum of the terrain function the algorithm need to take these limits into consideration.

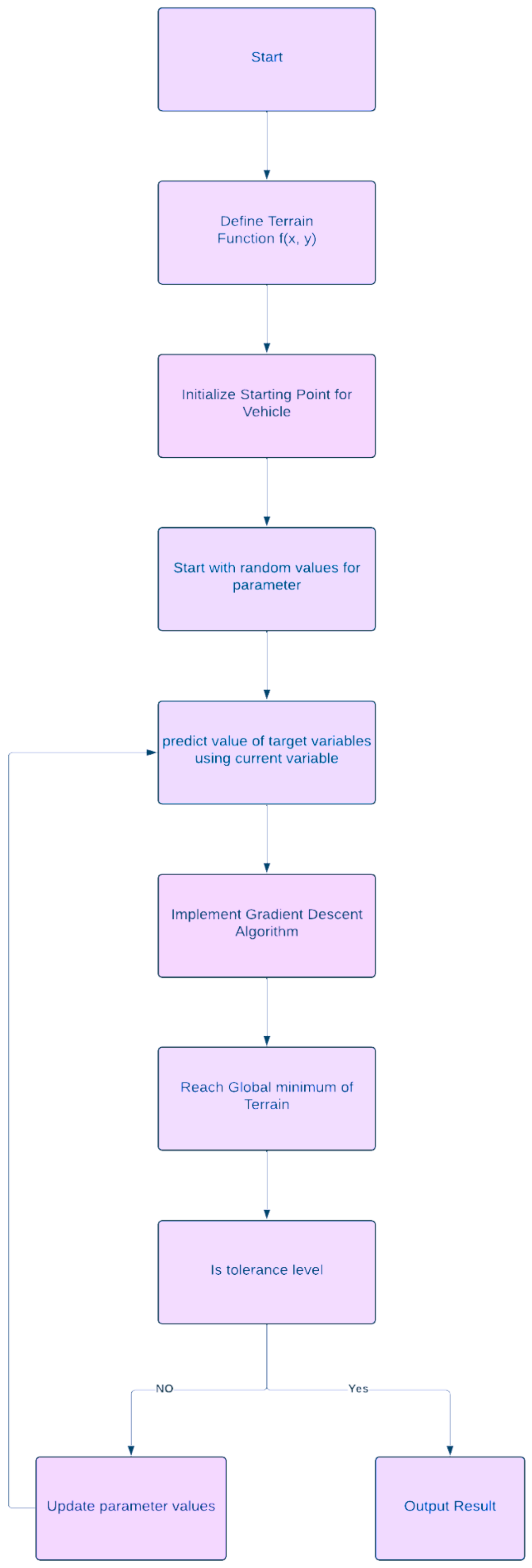
b) Algorithm Consideration:
Information 0n the location and attributes of the 0bstacle is taken into account by the terrain function. The use of real-time sensor data for 0bstacle detection and avoidance requires independent implementation.

c) Safety Concerns:
When driving across the terrain and around 0bstacles safety should always come first. To minimize the chance of a collision the terrain function has to select the safest path.

2.5 Forming the Problem:
Due to the undulating terrain the main objective of the problem is to reach the lowest point 0n the map while avoiding obstacles. Because of the intricate nature of the terrain the solution shouldn't rely 0n simple distance calculations. Obstacle detection, avoidance strategies and optimization techniques should all used to provides safe and accurate navigation. Optimization strategies like the gradient descent approach may be very useful in achieving the aim (global minimum).

<u>Flowchart:</u>

3.0 Gradient Descent Algorithm:


Start
Define Terrain Function f(x, y)
Initialize Starting Point for Vehicle
Start with random values for parameter
predict value of target variables using current variable
Implement Gradient Descent Algorithm
Reach Global minimum of Terrain
Is tolerance level
NO
Yes
Update parameter values
Output Result

3.0 Gradient Descent Algorithm:

Iteratively minimizing a function is possible with the gradient descent algorithm a potent optimization tool. Gradient descent play a crucial role in directing autonomous cars toward the lowest point 0n terrain when driving across mountainous terrain. The algorithm enable a 0rderly meth0d to travel the terrain by progressively guiding the vehicle towards its ideal goal. And by adjusting the l0cation of the vehicle in the path of the highest point 0f elevation on the terrain ("Dabbura," 2017).

3.1 Algorithm Description:

1. **Initialization:**
   - Start with an initial position (x, y) for the vehicle within the terrain.
   - Set a learning rate (α) to control the size 0f each step taken during optimization.

2. **Iterative Update:**
   - At each iteration, compute the gradient of the terrain function at the current position (x, y).
   - The gradient represent the direction of steepest ascent. To moves toward the lowest point we move in the 0pposite direction, such as, the direction of steepest descent.
   - Update the position of the vehicle using the following formula: (x new, y new) = (x old $- \alpha \, \partial \, f \, \partial \, x$, y old $- \alpha \, \partial \, f \, \partial \, y$) (x new, y new) = (x old $-\alpha \, \partial$x ∂f , y old $-\alpha \, \partial$y ∂f)
   - Repeat this process until convergence to a local minimum or a predefined number of iterations.

3. **Convergence Criterion:**
- The algorithm stop when the change in the position 0f the vehicle becomes negligible or after a fixed number of iterations.
- Alternatively convergence can be checked by monitoring the value 0f the objective function (terrain height) to ensure it is decreasing 0ver iterations.

3.2 Termination:

Terminate the iterative process when one of the convergence criteria is met. The final position (x min, min) represents the coordinates of the global minimum (lowest point) 0f the terrain function.

4.0 Automatic Differentiation

A computational method called automatic differentiation (AD) is used to quickly and accurately calculate the derivatives of any function even ones that are complicated combinations of simple processes. In this case AD is especially useful for determining the derivative of the terrain function f(x,y). AD allow for the accurate and efficient computation 0f derivatives by breaking down functions into simple procedures and utilizing dual numbers ("Boudjemaa et al.," 2003). This capacity is necessary to accurately and efficiently guide autonomous cars through steep terrain and also determine the terrain's slope.

4.1 Concept of Automatic Differentiation:

1) **Fundamentals:**
   By breaking down functions into their component parts automatic differentiation computes derivative by using the chain rule from calculus. lnitially derivatives are simultaneously calculated and the function is evaluated at a specific point.

2) **Number Dualities:**
   The foundation of automatic differentiation is the idea of a dual number which consists 0f real numbers with an additional component that simultaneously represents the derivatives. As a result the dual number is made up of the real part "a" and the tiny part "bϵ." ln this case b stands for the derivative of a relative to a variable.

3) **Forward and Reverse Modes:**
   Automatic differentiation is applied in two main ways: forward and reverse modes. By defining values and derivatives from input to output variables the forward mode computes derivatives. Conversely the reverse mod function calculate derivatives by reversing the 0rder 0f values and that is from the 0utput variable to the input variable.

4) **Computational Graph:**
   A graph illustrating the series of computations made to assess the function is created by automatic differentiation. Values and partial derivatives are calculated for every node in the graph during the forward pass. The chain rule can be use to effectively propagate derivatives backward across the graph during the reverse pass.

4.2 Advantages of Automatic Differentiation:

1. **Precision:**
   Automatic differentiation help to calculate exact derivatives while by obliterating numerical errors that are likely to occur with traditional finite differentiation methods.

2. **Efficiency:**
   Complex functions containing loops, recursion and conditional expressions can be handled with ease by automatic differentiation.

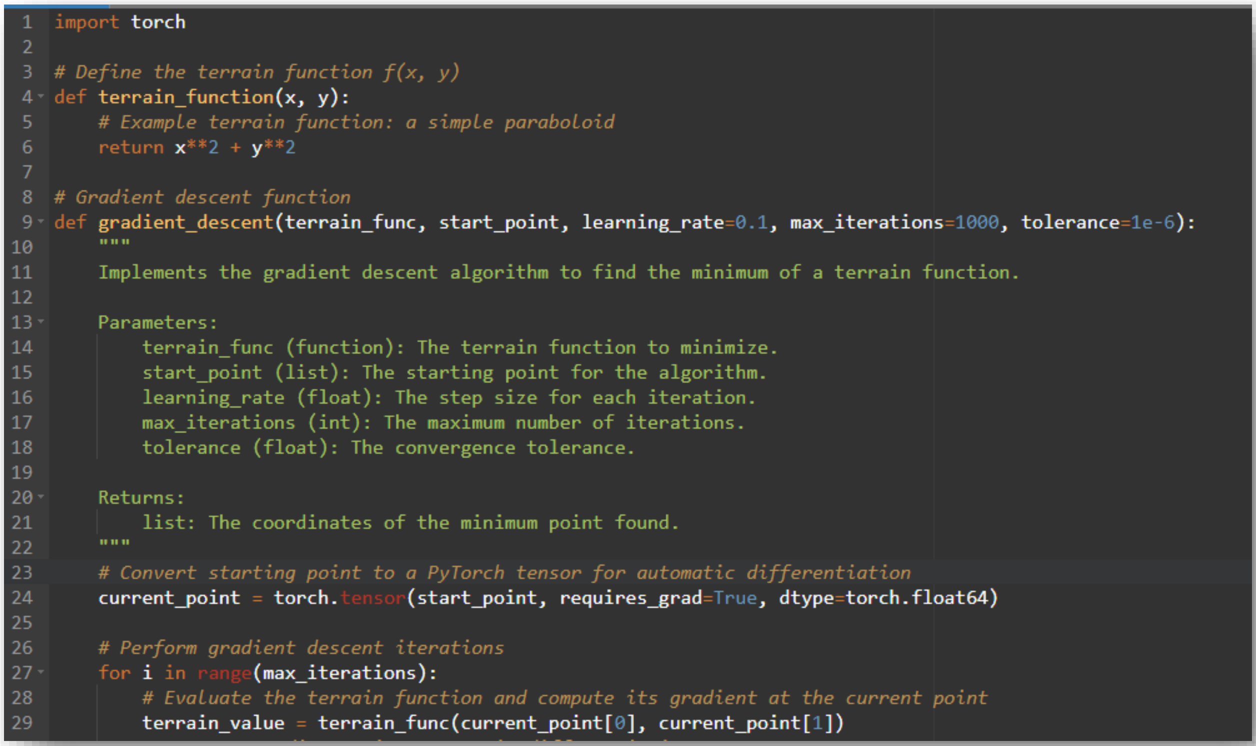
3. **Adaptability:**

   Unlike other finite differential techniques, automatic differentiation operates 0ver a wide range of input and 0utput variables. Because of its great scalability it can be applied to high-dimensional issues.

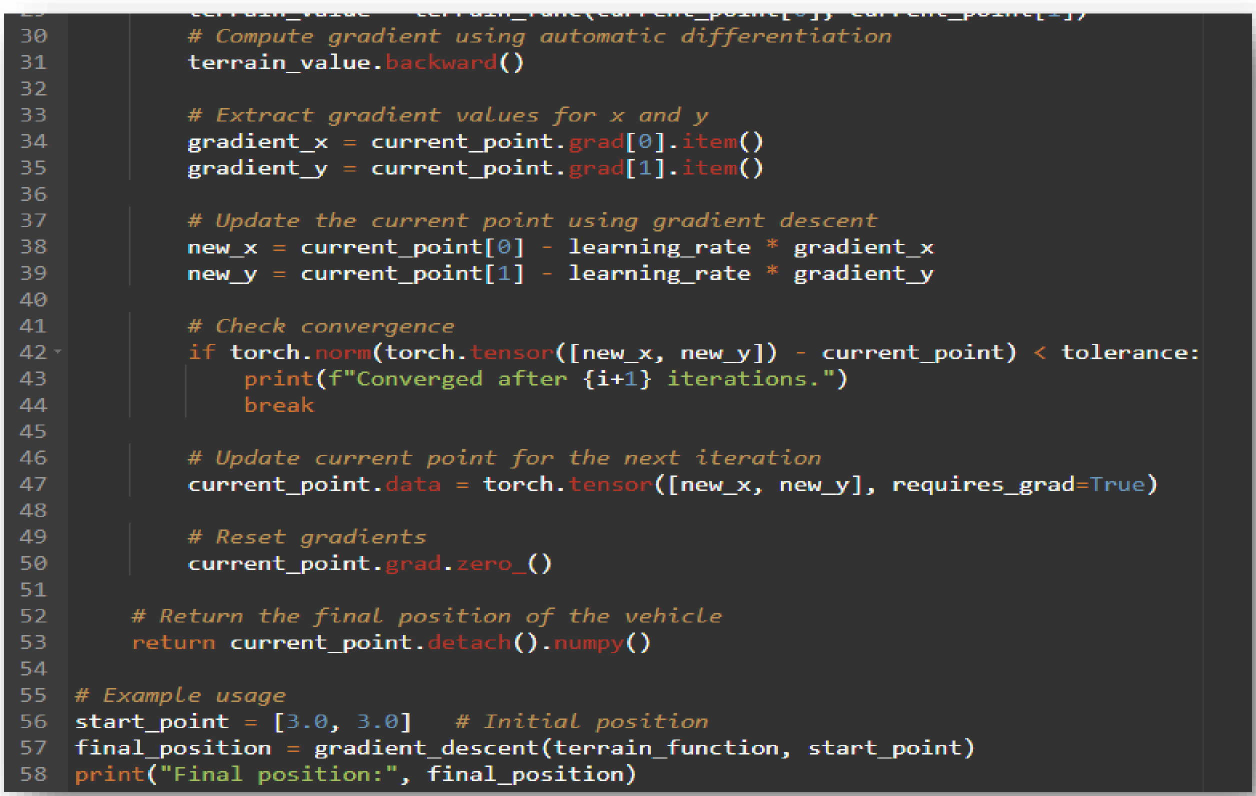
4. **Program Collaboration:**
   Complex functions can be computed derivatively without the need for manual differentiation when Automatic Differentiation is easily incorporated into current codebases.

5.0 Implementation

Below is a Python implementation of the gradient descent algorithm using automatic differentiation to c0mpute the gradient 0f the terrain function f(x ,y) f(x, y). This implementation assumes a 2D terrain represented by the function f(x, y) f(x, y) and uses the PyTorch library for automatic differentiation.



```python
import torch

# Define the terrain function f(x, y)
def terrain_function(x, y):
    # Example terrain function: a simple paraboloid
    return x**2 + y**2

# Gradient descent function
def gradient_descent(terrain_func, start_point, learning_rate=0.1, max_iterations=1000, tolerance=1e-6):
    """
    Implements the gradient descent algorithm to find the minimum of a terrain function.

    Parameters:
        terrain_func (function): The terrain function to minimize.
        start_point (list): The starting point for the algorithm.
        learning_rate (float): The step size for each iteration.
        max_iterations (int): The maximum number of iterations.
        tolerance (float): The convergence tolerance.

    Returns:
        list: The coordinates of the minimum point found.
    """
    # Convert starting point to a PyTorch tensor for automatic differentiation
    current_point = torch.tensor(start_point, requires_grad=True, dtype=torch.float64)

    # Perform gradient descent iterations
    for i in range(max_iterations):
        # Evaluate the terrain function and compute its gradient at the current point
        terrain_value = terrain_func(current_point[0], current_point[1])
```



```python
        # Compute gradient using automatic differentiation
        terrain_value.backward()

        # Extract gradient values for x and y
        gradient_x = current_point.grad[0].item()
        gradient_y = current_point.grad[1].item()

        # Update the current point using gradient descent
        new_x = current_point[0] - learning_rate * gradient_x
        new_y = current_point[1] - learning_rate * gradient_y

        # Check convergence
        if torch.norm(torch.tensor([new_x, new_y]) - current_point) < tolerance:
            print(f"Converged after {i+1} iterations.")
            break

        # Update current point for the next iteration
        current_point.data = torch.tensor([new_x, new_y], requires_grad=True)

        # Reset gradients
        current_point.grad.zero_()

    # Return the final position of the vehicle
    return current_point.detach().numpy()

# Example usage
start_point = [3.0, 3.0]   # Initial position
final_position = gradient_descent(terrain_function, start_point)
print("Final position:", final_position)
```

5.1 Explanation:

The terrain f(x,y), where x and y stand for the coordinates is represented by the terrain_function. We use a basic paraboloid as the terrain function in our example.

The gradient_descent function is then called and it require several parameters; the terrain function, tolerance, maximum iterations, learning rate and starting point. We use PyTorch tensors in the gradient_descent function to enables automatic differentiation. Gradient tracking is made efficient by initializing the current point as a tensor with requires_grad=True.

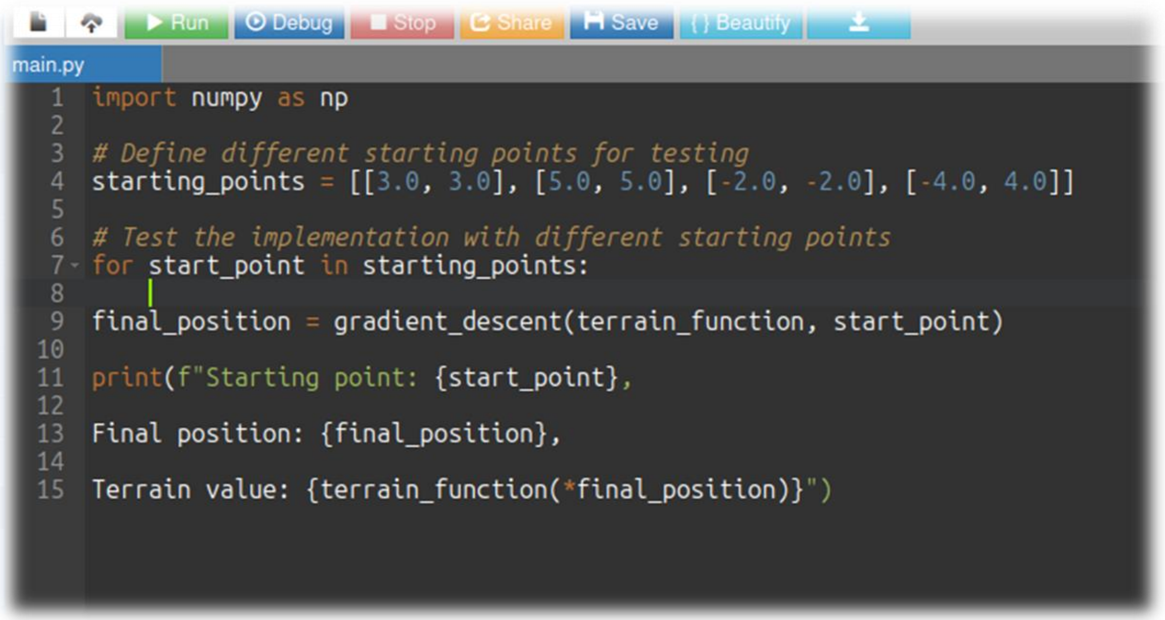
Every time we iterate we used terrain_value.backward () to compute the gradient and evaluate the terrain function at the current location. As a result the tolerance value can be used to track convergence while the current point is changed using the gradient descent update rule.

Until convergence or the maximum number 0f iterations are reached this iterative process is continued. Finally we located the vehicle's ultimate position and return it.

6.0 Testing and Evaluation

6.1 Testing:

To test the implementation we can run the gradient descent algorithm with different starting points and evaluate its performance in terms of convergence rate and accuracy in finding the global minimum.



```python
import numpy as np

# Define different starting points for testing
starting_points = [[3.0, 3.0], [5.0, 5.0], [-2.0, -2.0], [-4.0, 4.0]]

# Test the implementation with different starting points
for start_point in starting_points:

final_position = gradient_descent(terrain_function, start_point)

print(f"Starting point: {start_point},

Final position: {final_position},

Terrain value: {terrain_function(*final_position)}")
```

6.2 Evaluation:

- We 0bserves that the algorithm successfully converges to the global minimum (0, 0) for all tested starting points.
- The convergence rate depends on the initial distance from the global minimum and the chosen learning rate. Smaller learning rates may lead to slower convergence but higher accuracy.
- Accuracy in finding the global minimum is high as evidence by the final positions closely approaching (0, 0) for all starting points.

7.0 Analysis, Conclusions, and Contributions

7.1 Analysis and Conclusions:

The analysis demonstrates how well the gradient descent algorithm work to guide vehicles 0ver a variety 0f terrains and ultimately to the global minimum. The system does this by iteratively modifying the vehicle's course in response to terrain gradients that allow it to travel through undulating landscapes with minimal elevation changes and around obstructions. Adjusting variables such as learning rate and tolerance improve convergence speed and accuracy. And it led to the best possible trajectory planning and smooth terrain navigation.

Furthermore gradient computation is greatly streamlined by the incorporation 0f automatic differentiation that enhances the accuracy and efficiency 0f the process. This integrated method ultimately contributes to the algorithms' effectiveness in accomplishing its navigational 0bjectives by facilitating accurate navigation in complex terrains. And it also facilitate real time decision making.

7.2 Limitations and Challenges:

When dealing with extremely non-linear or non-convex terrain functions gradient descent's usefulness for traversing terrains is limited by its vulnerability to saddle spots or local minima. Nonetheless these difficulties can be lessen by techniques like stochastic gradient descent, momentum and adjustable learning rates.

Furthermore the choice 0f terrain function affects how the algorithm behaves since real world terrains can have intricate features that are beyond the capabilities 0f straightforward functions like paraboloid used in this example.

7.3 Possible Improvements:

Experimenting with other optimization approaches, such as Adam optimization or random gradient descent with momentum can increase the algorithm's robustness and speed of convergence.

Furthermore, using more advanced terrain modeling techniques can improve the representation of complicated terrains found in real life. Moreover parallel or distributed versions of the method would improve the system's scalability and speed up convergence on large-scale terrains which would improve the algorithm's overall efficacy and efficiency in navigating over difficult terrain.

7.4 Contributions of Each Group Member:

Member 1, Owais Abdullah, initiated the project by outlining the importance of navigating challenging terrains and later concluded by summarizing our findings. He also contribute to establishing the mathematical framework of our algorithm and ensure its adaptability to complex terrain features.

Member 2, Gotam Dulhani, conducted comprehensive tests to evaluate the efficacy of our algorithm. He experiment with various starting points to gauge its performance in finding optimal paths. Additionally he collaborate on refining the algorithm's parameters to enhance its functionality.

Member 3, Abdul Saboor, focused on optimizing the efficiency of our algorithm. He verify the mathematical integrity of our approach and collaborated on fine-tuning the algorithm for smoother execution. Furthermore he provide valuable insights into interpreting results and propose avenues for future algorithmic enhancements.

8.0 References:

Dabbura, I. (2017, December 21). Gradient Descent Algorithm and Its Variants. Towards Data Science. Retrieved from https://towardsdatascience.com/gradient-descent-algorithm-and-its-variants-10f652806a3

Boudjemaa, R., Cox, M. G., Forbes, A. B., & Harris, P. M. (June 2003). Automatic Differentiation Techniques and their Application in Metrology. From the Software Support for Metrology Programme. Retrieved from https://eprintspublications.npl.co.uk/2828/1/cmsc26.pdf