Chapter 4: (P-184)

V. is called vector space if it satisfy the following

1. $\forall\, u, v \in V \Rightarrow u+v \in V$ — closure
2. $\forall\, u, v \in V \to u+v = v+u$ — commutative
3. $\forall\, u, v, w \in V \to u+(v+w) = (u+v)+w$ — associative
4. $0+u = u+0 = u$ — additive identity
5. $u+(-u) = 0$ — additive inverse

} Abellian group.

6. $\forall\, u \in V,\ ku \in V,\ (k\text{-scalar})$ — scalar multiple
7. $k(u+v) = ku+kv$
8. $(k+m)u = ku+mu$
9. $k(mu) = (km)u$
10. $1 \cdot u = u$

$\hookrightarrow$ when all true $\Rightarrow$ vector space.

Ex: 4.1: Q (1-15) examples

Q.1 V = set of all ordered pair of real numbers.

$$\left.\begin{matrix} u = (u_1, u_2) \\ v = (v_1, v_2) \end{matrix}\right\} \in V$$

$$\left.\begin{matrix} (u+v) = (u_1, u_2) + (v_1, v_2) = (u_1+v_1 = u_2+v_2) \\ ku = k(u_1, u_2) = (0, ku_2) \end{matrix}\right\} \text{given in question.}$$

$u + v = u$ — given } not commutative.
$v + u = v$ — imputed. }

For $u + v \in V$

(1) $u+v = (u_1 + v_1, u_2 + v_2)$

Note:
in proofs separate left and right & go from left to right

(2) For $u+v = v+u$

$u+v = (u_1+v_1, u_2+v_2)$

use component laws.

real number sum.

vector sum

$$
\begin{aligned}
u + v &= (v_1 + u_1, v_2 + u_2) \\
&= (v_1, v_2) + (u_1, u_2) \\
&= v + u
\end{aligned}
$$

3. For $u + (v + w) = (u+v) + w$

$w = (w_1, w_2) \in V$

$$
\begin{aligned}
u + (v + w) &= (u_1, u_2) + (v_1 + w_1, v_2 + w_2) \\
&= (u_1 + (v_1 + w_1), u_2 + (v_2 + w_2)) \\
&= (u_1 + v_1, u_2 + v_2) + (w_1, w_2) \\
&= (u + v) + w
\end{aligned}
$$

4. $0 + u = u + 0 = u$

$$0 + u = (0,0) + (u_1, u_2)$$

$((0,0)$

$$= (0 + u_1 , 0 + u_2).$$

$$= (u_1, u_2) = u$$

5. $u + (-u) = 0$

$$u = (u_1 , u_2)$$

$$-u = (-u_1 , -u_2)$$

$$u + (-u) = (u_1 - u_1 , u_2 - u_2)$$

$$= (0, 0)$$

$$= 0$$

6. $ku = k(u_1 , u_2) = (0, ku_2) \in V$

$$\cancel{0,} = ku \in V$$

7. $k(u+v) = ku + kv$

$$k(u+v) = k(u_1 + v_1 , u_2 + v_2)$$

$$= (0, k(u_2 + v_2))$$

$$= (0, ku_2) + (0, kv_2)$$

$$= k(u_1 , u_2) + k(v_1 , v_2)$$

$$= ku + kv$$

(8) $(k+m)u = ku + mu$

$$(k+m)u = (k+m)(u_1, u_2)$$
$$= (0, (k+m)u_2)$$
$$= (0, ku_2 + mu_2)$$
$$= (0, ku_2) + (0, mu_2)$$
$$= k(u_1 + u_2) + m(u_1, u_2)$$
$$= ku + mu$$

(9) $k(mu) = (km)u$

$$= k(0, m(u_1, u_2))$$
$$= k(0, mu_2)$$
$$= (0, k(mu_2))$$
$$= (0, (km)u_2)$$
$$= (km)(u_1, u_2)$$
$$= (km)u$$

(10) $1 \cdot u = u$

$$1 \cdot u = 1(u_1, u_2)$$
$$= (0, 1u_2)$$
$$= (0, u_2) \neq u$$
$$1 \cdot u \neq u$$

Hence V is not a vector space

7/4/2022. LINEAR ALGEBRA

(11) V = set of all vectors of the form $(1, x)$

with operation → no change $(1+1=1$ not $2)$

$$\left.\begin{array}{l}(1,y)+(1,y')=(1,y+y')\\ k(1,y)=(1,ky)\end{array}\right\}\text{ not standard.}$$

when $(1,y),(1,y') \in V$

(1) $u+v \in V$

take $u=(1,y)$ & $v=(1,y') \in V$

$u+v=(1,y)+(1,y')=(1,y+y') \in V_{(T)}$

(2) $u+v=v+u$

$u+v=(1,y)+(1,y')$

$=(1,y+y') \rightarrow (1,y'+y)$

$=(1,y')+(1,y) \in V_{(T)}$

(3) $u+(v+w)=(u+v)+w$

$w=(1,y'') \in V$

$u+(v+w)=(1,y)+(1,y'+y'')$

$=(1,y+(y'+y''))$

$=(1,(y+y')+y'')$

$=(1,y+y')+(1,y'')$

$=(u+v)+w$

(4) $0 = (0,0)$ — x not true.

$0 + u = u + 0 = 0$

$0 + u = (1,0) , (1+y)$

$= (1,y) = u$

(Additive identity $(1,0) \in V$) (T)

(5) $u + (-u) = 0 = (-u) + u$

$u + (-u) = (1,y) + (1,-y)$

$= (1, y-y)$

$= (1,0) = 0$ (T)

($-u = (1,-y) \in V$ additive inverse)

(6) $ku \in V$

$ku = k(1,y)$

$= (1, ky) \in V$ (T)

(7) $k(u+v) = k((1,y)+(1,y'))$

$= k(1, y+y')$

$= (1, k(y+y'))$

$= (1, ky + ky')$

$= (1,ky) + (1,ky')$

$= k(1,y) + k(1,y')$

$= ku + kv$ (T)

(8) $(k+m)u = ku + mu$

$$(k+m)u = (k+m)(1,y)$$
$$= (1,(k+m)y)$$
$$= (1, ky + my)$$
$$= (1, ky) + (1, my)$$
$$= k(1,y) + m(1,y)$$
$$= ku + mu$$

(9) $k(mu) = (km)u$

$$k(mu) = k(m(1,y))$$
$$= (1, k(my))$$
$$= (1, (km)y)$$
$$= (km)(1,y)$$
$$= (km)u \quad (T)$$

(10) $1.u = u$

$$1.u = 1(1,y)$$
$$= (1, 1y)$$
$$= (1,y)$$
$$= u \quad (T)$$

Hence vector space.

Q example 8

$u, v \in V$

$u + v = uv$

$k \cdot u = u^k$

$0 + u = 1 + u$

$= 1u$

$= u$

0 = additive identity $= 1$

8 $u + (-u) = (-u) + u = 0$

$= u + \frac{1}{u}$

$= u\frac{1}{u}$

$= 1$ is 0

$(-u)$ is $\frac{1}{u}$ additive inverse.

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9 $k(mu) = (km)u$

$k(mu) = k\, u^m$

$= (u^m)\, k$

$= u^{mk}$

$= (mk)u$

SUBSPACE

↳ subset of vector space and vector space itself.

Let V be a vector space & $W \subseteq V$ then W is called a subspace of W itself is a vector space.

Note:- (p-192)

W is a subspace of V

(i) $u+v \in W$, $\forall u, v \in W$

(ii) $ku \in W$, $\forall u \in W$ k (scalar)

} don't require 10 conditions.

(1) $u+v \in V$

(2) $u+v = v+u$

Ex: 4.2 R^3 vector space

(a) all vector of the form $(a, 0, 0)$

↳ should be 0

~~(say) W = set of~~

(say) W = set of all vector of the form $(a, 0, 0)$

(i) For $u+v \in W$

(say) $u = (a', 0, 0) \in W$

$v = (a'', 0, 0)$

$u+v = (a' + a'', 0, 0) \in W$ (i)

(ii) $kv \in W$

$kv = k(a', 0, 0)$

$= (ka', 0, 0) \in W$ (T)

Hence W is number of R

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4.2 : 1, 2, 3, 7, 9, 11, 13, 22.

To show W is a subspace of V
We show

$U + V \in W$

$\boxed{kU \in W}$

Q.1

(d) say W = set of all vector of the form (a,b,c)
when $b = a + c + 1$

For $u + w \in W$

$u = (a,b,c)$; $b = a + c + 1$ ✓

$v = (d,e,f)$ $e = d + f + 1$ ✓

$u + v = (a,b,c) + (d,e,f)$
$= (a+d, b+e, c+f)$

check $\boxed{b+e = a+d+c+f+1}$

$b + e = a + c + 1 + d + f + 1$
$= a + d + c + f + 2$ (F)

$u + v \in W$
So, w is not a subspace of R^3

Q: 3. Subspace of P_3 = ?

(a)

Saw. W = set of all polynomials

$a_0 + a_1 x + a_2 x^2 + a_3 x^3$

for which $a_0 = 0$

$u + v \in W$

taken $u = a_0 + a_1 x_1 + a_2 x^2 + a_3 x^3$ when $a_0 = 0$

$v = b_0 + b_1 x_1 + b_2 x^2 + b_3 x^3$ $b_0 = 0$

$u + v = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3$

check $(a_0 + b_0 = 0)$

$a_0 + b_0 = 0 + 0$

$= 0$ (T)

$u + v \in W$ (T)

$ku \in W$ (T)

Here W is a subspace of P_3

$ku \in W$

$ku = k(a_0 + a_1 x + a_2 x^2 + a_3 x^3)$

$= ka_0 + ka_1 x + ka_2 x^2 + ka_3 x^3$

check $ka_0 = 0$

$ka_0 = k0 = 0$ (T)

Q: 7

$u = (0, -2, 2)$

$v = (1, 3, -1)$

(a) $(2, 2, 2)$

take

$(2,2,2) = au + bv$

$(2,2,2) = a(0,-2,2) + b(1,3,-1)$ —— (1)

$2 = 0a + 1b \Rightarrow b = 2$

$2 = -2a + 3b \Rightarrow 2 = -2a + 3(2)$

$2 = 2a - 1b \Rightarrow 2a = 4$

$a = 2$

$\rightarrow$ check

$2 = 2(2) - 2$

$2 = 2$ (T)

& put $a = 2$, $b = 2$ in (1)

These vectors span this $\longrightarrow$ $(2,2,2) = 2(0,-2,2) + 2(1,3,-1)$

So give vector $(2,2,2)$ can be expressed as a linear combination & in form of u & v.

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Q: 11

$$\left.\begin{array}{l} V_1 = (2,2,2) \\ V_2 = (0,0,3) \\ V_3 = (0,1,1) \end{array}\right\} \text{Span } R^3 = ?$$

take $(x,y,z) \in R^3$

Such that

$(x,y,z) = aV_1 + bV_2 + cV_3$

find a, b, c

$(x,y,z) = a(2,2,2) + b(0,0,3) + c(0,1,1)$

$x = 2a + 0b + 0c$

$y = 2a + 0b + 1c$

$z = 2a + 3b + 1c$

$a = x/2$

$y = 2(x/2) + c$

$\boxed{c = y - x}$

$(x,y,z) = \frac{x}{2}(2,2,2) + \left(\frac{z-y}{3}\right)(0,0,3) + (y-x)(0,1,1)$

True for all $(x,y,z) \in R^3$

So vector v_1, v_2, v_3 span R^3

→ One solution.

→ when determinant becomes 0

$$|A| = \begin{vmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{vmatrix} \quad \left.\begin{array}{l} = 2(0-3) - 0 + 0 \\ = -6 \neq 0 \end{array}\right\} \text{Solution exist.}$$

So given vector span R^3.

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if $|A| = 0$

So, given vector do not span R^3

Spanning means writing in Linear combinations.

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if P_2 $\hookrightarrow$ polynomial

take $a_0 + a_1 x + a_2 x^2 \in P_2$

Question 22.

Theorem 4.2.6

Linearly independent vector

$$a_1 v_1 + a_2 v_2 + a_3 v_3 + \cdots a_n v_n = 0$$

of all $a_i^3 n = 0$

when all a are 0, we get trivial solution.

$|A| \neq 0$ (unique solution), vectors independent

if $|A| = 0$ (Non-trivial solution), Linearly dependent

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we can decide from determinant if feasible

$$\boxed{V_1 = a_2V_2 + a_3V_3 + a_4V_4}$$

$0 = (-1)V_1 + a_2V_2 + a_3V_3 + a_4V_4$

↳ dependent

↳ one coeff is not 0 so ~~independent~~ dependent.

- if 0 vector is included than still dependent as non-zero coefficient can be written.
- two equal and multiple vectors are also dependent as 1, -1 can be coefficients

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4.3 Q: 1, 2, 3, 5, 6, 7, 9, 11 (examples 2 & 3)

Q: 1

$\underline{v} = 2\underline{u}$

↳ dependent

$\underline{v} - 2\underline{u} = 0$

↳ not zero components so dependent

independent vectors
all $a_i n = 0$
(trivial Solution)
$|A| = 0$ (if feasible)
otherwise dependent vectors
($|A| \neq 0$ if feasible)
non-trivial solution
1. Multiple vectors are dependent
2. if any 1 is 0 vector then also dependent.

Q: 2

$$\begin{array}{ccc} v_1 & v_2 & v_3 \\ (-3,0,4), & (5,-1,2), & (1,1,3) \end{array}$$

(a)

take

$a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$

$a(-3,0,4) + b(5,-1,2) + c(1,1,3) = 0$

$-3a + 5b + 1c = 0$
$0a + 1b + 1c = 0$
$4a + 2b + 3c = 0$

determinant is feasible

$$\begin{bmatrix} -3 & 5 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 4 & 2 & 3 & 0 \end{bmatrix}$$

echelon form.

$R_1 + (1)R_3$ $\begin{bmatrix} 1 & 7 & 4 & 0 \\ 0 & -1 & 1 & 0 \\ 4 & 2 & 3 & 0 \end{bmatrix}$

$R_3 + (-4)R_1$ $\begin{bmatrix} 1 & 7 & 4 \\ 0 & -1 & 1 \\ 0 & -26 & -13 \end{bmatrix}$

$(-1)R_2$ $\begin{bmatrix} 1 & 7 & 4 \\ 0 & 1 & -1 \\ 0 & -26 & -13 \end{bmatrix}$

$$R_3 + (26)R_2 \begin{bmatrix} 1 & 7 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & -39 \end{bmatrix}$$

$$-\frac{1}{39} R_3 \quad \begin{bmatrix} 1 & 7 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad 1a_1 + 7a_2 - 4a_3 = 0$$
$$1a_2 - 1a_3 = 0$$
$$1a_3 = 0$$

$a_3 = 0$

$a_2 - 0 = 0$, $a_2 = 0$

$a_1 + 0 - 0 = 0$, $a_1 = 0$

all

$$\left.\begin{aligned} a_1 &= 0 \\ a_2 &= 0 \\ a_3 &= 0 \end{aligned}\right]$$ trivial solution.
so vectors are independent

$$A^T = \begin{bmatrix} -3 & 5 & 1 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{bmatrix} \qquad |A| = 0$$

$|A^T| = |A|$, So no difference.

Q:2 (b) $\in R^3$.

$(-2,0,1), (3,2,5), (6,-1,1), (7,0,-2)$

$a(-2,0,1) + b(3,2,5) + c(6,-1,1) + d(7,0,-2) = 0$

↳ 3 eq & 4 vars have never trivial solution

$-2a + 3b + 6c + 7d = 0$

$0a + 2b - c + 0d = 0$

$1a + 5b + c + 2d = 0$

$$\left[\begin{array}{cccc|c} -2 & 3 & 6 & 7 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 1 & 5 & 1 & -2 & 0 \end{array}\right]$$

echelon form.

R_{13} $\begin{bmatrix} 1 & 5 & 1 & -2 \\ 0 & 2 & -1 & 0 \\ -2 & 3 & 6 & 7 \end{bmatrix}$

$R_3 + (2)R_1$ $\begin{bmatrix} 1 & 5 & 1 & -2 \\ 0 & 2 & -1 & 0 \\ 0 & 13 & 8 & 3 \end{bmatrix}$

$\frac{1}{2}R_1$ $\begin{bmatrix} 1 & 5 & 1 & -2 \\ 0 & 1 & -1/2 & 0 \\ 0 & 13 & 8 & 3 \end{bmatrix}$

$R_3 + (-13)R_2$ $\begin{bmatrix} 1 & 5 & 1 & -2 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 29/2 & 3 \end{bmatrix}$

$\frac{2}{29}R_3$ $\begin{bmatrix} 1 & 5 & 1 & -2 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & 6/29 \end{bmatrix}$

Rank $A = 3 =$ Rank $A_b \neq 3 <$ No of variables.

(non-trivial solution)

→ because 4 variables with 3 equation, trivial solution is not possible.

determinant of co-planar vectors is 0.

$(a \cdot b) \times c$

Q:11

$$\left.\begin{array}{l} V_1 = (\lambda, -1/2, -1/2) \\ V_2 = (-1/2, \lambda, -1/2) \\ V_3 = (-1/2, -1/2, \lambda) \end{array}\right]$$

$\lambda = ?$

Linearly dependent vector

$|A| = 0$

$\underline{a} \cdot \underline{b} \times \underline{c} = 0$

↳ coplanar vectors.

$\underline{a}, \underline{b}, \underline{c}$

$$\left| \quad \right| = 0$$

$$= \begin{vmatrix} \lambda & -1/2 & -1/2 \\ -1/2 & \lambda & -1/2 \\ -1/2 & -1/2 & \lambda \end{vmatrix} = 0$$

$$\lambda\left(\lambda^2 - 1/4\right) + \frac{1}{2}\left(-\lambda/2 - 1/4\right) - 1/2\left(1/4 + \frac{\lambda}{2}\right) = 0$$

$$\lambda^3 + \frac{\lambda}{4} - \frac{\lambda}{4} - \frac{1}{8} - \frac{1}{8} - \frac{\lambda}{4} = 0$$

$$\lambda^3 - \frac{3\lambda}{4} - \frac{1}{4} = 0$$

$$\frac{4\lambda^3 - 3\lambda - 1}{4} = 0$$

$$4\lambda^3 - 3\lambda - 1 = 0$$

$4\lambda^3 - 3\lambda - 1 = 0$ (synthetic division)

$\left.\begin{array}{l} \pm 1 \\ \pm 2 \\ \pm 4 \end{array}\right]$ divisors of 1st & last

$$\begin{array}{c|cccc} 1 & 4 & 0 & -3 & -1 \\ & \cancel{8} & 4 & 4 & 1 \\ \hline & 4 & 4 & 1 & \big/ \; 0 \end{array}$$

1 is must as coeff becomes 0

Depressed eq

$4\lambda^2 + 4\lambda + 1 = 0$

$(2\lambda + 1)^2 = 0$

$(2\lambda + 1)(2\lambda + 1) = 0$

$2\lambda + 1 = 0 \quad 2\lambda + 1 = 0$

$\left(\lambda = -\frac{1}{2}\right), \left(\lambda = -1/2\right)$

$\left.\lambda = 1, -1/2, \frac{1}{2}\right\}$ 3 roots

Ch: 3 — 4.8 (26/4/2022)

19/4/2022 LA

4.4 :-

Q: 1, 2, 3, 5, 7, 11, 13 examples (9), examples (10)

Basis (for a vector)

→ show independent & spanning vector

→ we can tell by determinant only

→ write as Linear combo

→ unique solu, det ≠ 0

for independent (det = 0)

Q: 2

Basis for R^3

$\{(3,1,-4), (2,5,6), (1,4,8)\}$

For spanning R^3

take $(x,y,z) \in R^3$

$(x,y,z) = a(3,1,-4) + b(2,5,6) + c(1,4,8)$

$$\left.\begin{aligned} x &= 3a + 2b + 1c \\ y &= 1a + 5b + 4c \\ z &= -4a + 6b + 8c \end{aligned}\right\} \text{— (1)}$$

→ unique sol if det ≠ 0

For independent

$a(3,1,-4) + b(2,5,6) + c(1,4,8) = (0,0,0)$

$$\left.\begin{aligned} 3a + 2b + 1c &= 0 \\ 1a + 5b + 4c &= 0 \\ -4a + 6b + 8c &= 0 \end{aligned}\right\} \text{— (2)}$$

trivial sol if det = 0

$\circ$ 1 output

1 2

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 8 & 4 \\ -4 & 6 & 8 \end{bmatrix}$$

$|A| = 3(40-24) - 2(8+16) + 1(6+20)$

$= 48 - 48 + 26$

$|A| = 26 \neq 0$

spanning possible

So, for 1 unique solution & for 2 trivial sol

independent vector

Hence, given set of vectors are independent & span R^3

Hence $\{(3,1,-4), (2,8,6), (1,4,8)\}$ is a basis for R^3.

★ |A| is easy approach if possible.

★ ~~Dimens~~ Dimension is number of vectors in the set of basis.

↳ Above's dimension is 3.

Note :-

Coordinate vector of v ($v \in V$) relative to the basis $S = \{v_1, v_2 \dots v\}$.

$v = c_1 v$

$(v)_S = (c_1, c_2 \dots c_n)$

↳ coordinate vector v, relative to basis S.

vector of coefficients is coordinate vector.

Q: 11

Coordinate vector of w relative to the basis $S = \{u_1, u_2\}$ for R^2

(a) $u_1 = (2, -4)$,

$u_2 = (3, 8)$,

$w = (1, 1)$

$w = c_1 u_1 + c_2 u_2$

$(w)_S = ?$

$(1,1) = c_1(2,-4) + c_2(3,8)$

$1 = 2c_1 + 3c_2$ —— I

$1 = -4c_1 + 8c_2$ —— II

we $2 \times (\mathrm{I}) + (\mathrm{II})$

$2 = 4c_1 + 6c_2$

$1 = -4c_1 + 8c_2$

$3 = 14c_2$

$c_2 = 3/14$

Put in I

$1 = 2c_1 + 3(3/14)$

$1 - \frac{9}{14} = 2c_1$

$\frac{5}{28} = c_1$

$(w)_S = (c_1, c_2)$

$= (5/28, 3/14)$