(Exercise 8.3)

Q1, 3, 5, 11

$T: V \to W$ is called isomorphic if T is both one-to-one and onto.

Note for isomorphic:

$\Rightarrow$ In case of system trivial solution given $kr(T) = 0$

$\Rightarrow$ For non-trivial solution, $kr(T) \neq 0$, nullity exists.

Question 1) State whether the transformation is an isomorphism.

$c_0 + c_1 x \to (c_0 - c_1, c_1)$ from $P_1 \to R^2$

$\Rightarrow T(c_0 + c_1 x)$

$\Rightarrow u = c_0 + c_1 x$

$\Rightarrow (c_0 - c_1, c_1) = (0, 0)$

$\Rightarrow c_0 - c_1 = 0$

$\Rightarrow \boxed{c_1 = 0}$

$\Rightarrow \boxed{c_0 = 0}$

$\Rightarrow u = 0 + (0)x = 0$

$\Rightarrow T(u) = 0$

$\Rightarrow T(0) = 0$

$\Rightarrow kr(T) = 0$

$\Rightarrow$ As $kr(T) = 0$, T is both one-to-one and onto

Question 3) State whether the transformation is an isomorphic.

$$a+bx+cx^2+dx^3 \rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$ from

P_3 to M_{22}

$\Rightarrow T(a+bx+cx^2+dx^3)$

$\Rightarrow \quad u = a+bx+cx^2+dx^3$

$$\Rightarrow \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad a=0, \; b=0, \; c=0, \; d=0$

$\Rightarrow \quad u = 0+(0)x+(0)x^2+(0)x^3$

$\Rightarrow \quad u = 0$

$\Rightarrow \quad T(0) = 0$

$\Rightarrow \quad \ker(T) = 0$

$\Rightarrow$ As $\ker(T) = 0$, T is both one-to-one and onto. Isomorphic.

Exercise 8.3 continued.

Question 5) state whether the transformation is an isomorphism.

$(a,b,c,d) \longrightarrow a+bx+cx^2+(d+1)x^3$
from R^4 to P_3.

$\Rightarrow \quad u=(a,b,c,d)$

$\Rightarrow \quad T(u) = 0$

$\Rightarrow \quad T(a,b,c,d) = 0$

$\Rightarrow \quad a+bx+cx^2+(d+1)x^3 = 0$

$\Rightarrow \quad 0+(0)x+(0)x^2+(0+1)x^3$

$\Rightarrow \quad x^3 = a+bx+cx^2+(d+1)x^3$

$\Rightarrow \quad a=0, \; b=0, \; c=0$

$d+1=0 \Rightarrow d=-1$

$\Rightarrow \quad u(a,b,c,d) = (0,0,0,-1) \neq 0$

$\Rightarrow \quad T(u) \neq 0$

$\Rightarrow \quad \ker(T) \neq 0$

$\Rightarrow$ Thus transformation T is not isomorphic.

Question 11) determine whether the matrix transformation $T_A : R^3 \longrightarrow R^3$ is an isomorphism.

$u=(x_1,x_2,x_3) \in R^3$

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

$\Rightarrow \quad T(u) = Au$

$$= \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 + x_2 - x_3 \\ x_1 + 0 + 2x_3 \\ -x_1 + x_2 + 0 \end{bmatrix}$$

$T(x_1, x_2, x_3) = (x_2 - x_3,\ x_1 + 2x_3,\ -x_1 + x_2)$

$T(u) = 0$

$\Rightarrow (x_2 - x_3,\ x_1 + 2x_3,\ -x_1 + x_2) = (0, 0, 0)$

$\Rightarrow \quad x_2 - x_3 = 0 \Rightarrow x_2 = -x_3$

$\quad x_1 + 2x_3 = 0 \Rightarrow x_1 = 2x_3$

$\quad -x_1 + x_2 = 0 \Rightarrow x_2 = x_1$

$\Rightarrow \quad x_1 = -x_3 \Rightarrow x_3 = -x_1$

$\Rightarrow \quad x_1 = 2(-x_1)$

$\Rightarrow \quad x_1 + 2x_1 = 0 \Rightarrow x_1 = 0$

$\Rightarrow \quad x_3 = 0, \quad x_2 = 0$

$\Rightarrow$ Hence kr$(T) = 0$, so T is isomorphic.