28th March 2022.

Exercise 3.2:

Q : 1, 3, 5, 7, 9, 11, 17

Dot Product

$\underline{u} = (a_1, a_2, a_3)$

$\underline{v} = (b_1, b_2, b_3)$

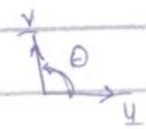
their dot (scalar) product is:-

$\underline{u} \cdot \underline{v} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

also

$\underline{u} \cdot \underline{v} = \|\underline{u}\| \|\underline{v}\| \cos\theta$

if $\underline{u} \perp \underline{v}$ $(\theta = 90^\circ)$ $\cos 90^\circ = 0$



then $\underline{u} \cdot \underline{v} = 0$

Q: 1

(a) $\underline{v} = (2, 2, 2)$

Norm

$$\|\underline{v}\| = \sqrt{2^2 + 2^2 + 2^2}$$
$$= \sqrt{12} = \sqrt{4 \times 3}$$
$$= 2\sqrt{3}$$

unit vector $\hat{v} = \dfrac{v}{\|\underline{v}\|}$

$$= \frac{(2, 2, 2)}{2\sqrt{3}}$$

$$= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Q : 7

$\|k\underline{v}\| = 5 \qquad k = ? \qquad \|k\underline{v}\| = |k|\|\underline{v}\|$

$\underline{v} = (-2, 3, 0, 6)$

$|k| \|v\| = 5$

$|k| \sqrt{(-2)^2 + 3^2 + 0^2 + 6^2} = 5.$

$|k| \, 7 = 5.$

$k = \pm \, 5/7$

Q : 11

(a) distance = ?

$\cos\theta = ?$

$\underline{u} = (3, 3, 3)$

$\underline{v} = (1, 0, 4)$

distance = ? $= \|\underline{u} - \underline{v}\|$

(a) $\underline{u} - \underline{v} = (3-1, 3-0, 3-4)$

$= (2, 3, -1)$

$\|\underline{u} - \underline{v}\| = \sqrt{2^2 + 3^2 + (-1)^2}$

$\|\underline{u} - \underline{v}\| = \sqrt{14}$

$$\cos\theta = \frac{\underline{u} \cdot \underline{v}}{\|\underline{u}\|\|\underline{v}\|} = \frac{(3, 3, 3)\cdot(1, 0, 4)}{\sqrt{3^2+3^2+3^2}\sqrt{1^2+0^2+4^2}}$$

$\cos\theta$

-ve	+ve

$$= \frac{3+0+12}{\sqrt{27}\,\sqrt{17}}$$

$$= \frac{\cancel{15}^{\,5}}{\cancel{3}\sqrt{3}\,\sqrt{17}} = \frac{5}{\sqrt{51}}$$

$$\cos\theta = \frac{5}{\sqrt{51}} \Big\} \text{ +ve (acute)}$$

$$\rightarrow \theta < 90^\circ$$

Cauchy-Schwarz inequality

$$|\underline{u}\cdot\underline{v}| \le \|\underline{u}\|\,\|\underline{v}\|$$

Q.17

(a) $\underline{u} = (-3, 1, 0)$

$\underline{v} = (2, -1, 3)$

$$|\underline{u}\cdot\underline{v}| \le \|\underline{u}\|\,\|\underline{v}\|$$

$$|-6-1+0| \le \sqrt{(-3)^2+1^2+0^2}\,\sqrt{2^2+(-1)^2+3^2}$$

$$7 \le \sqrt{10}\,\sqrt{14}$$

$$\underbrace{7 \le \sqrt{140}}_{\text{True}} \longrightarrow \text{if } 7 \le \sqrt{40} \text{ then false}$$

Exercise 3.3

Q: 1, 3, 5, 7, 9, 11, 12, 13, 15, 17, 21, 23, 25, 27, 29

Q: 1 orthogonal vector $\left(\text{i.e } \dfrac{\perp}{\text{Perpendicular vector}} \right)$ $\rightarrow$ $\underline{u} \cdot \underline{v} = 0 \quad \theta = 90^\circ$

(a) $\underline{u} = (6, 1, 4)$

$\underline{v} = (2, 0, -3)$

$= 12 + (-12)$

$= 0 \, (T)$ $\underline{u}$ orthognal to $\underline{v}$

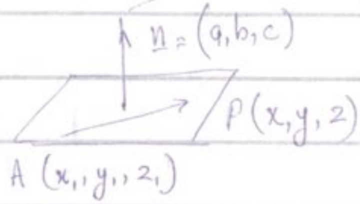
Note: Theorem on page 157

$\hookrightarrow$ Line & plane discussions.

Point normal form.

$\hookrightarrow$ plane should have a single normal only

normal to the plane



$\overrightarrow{AP} = (x - x_1, y - y_1, z - z_1)$

$\underline{n} \perp \overrightarrow{AP}$

$\underline{n} \cdot \overrightarrow{AP} = 0$

$(a, b, c) \cdot (x - x_1, y - y_1, z - z_1) = 0$

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ (Point normal form)

Note :-

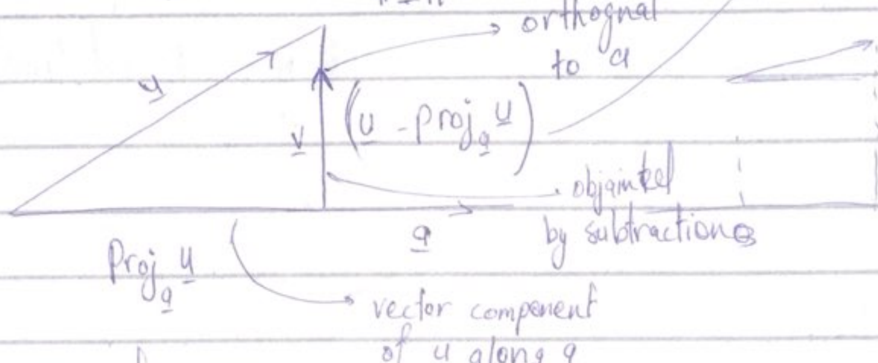
① Projection:

(Vector component of $\underline{u}$ along $\underline{a}$)

$$\boxed{\text{proj}_{\underline{a}}\, \underline{u} = \frac{\underline{u} \cdot \underline{a}}{\|\underline{a}\|^2}\, \underline{a}}$$

(vector component of $\underline{u}$ orthognal to $\underline{a}$)

$$u - \text{Proj}_{\underline{a}}\, \underline{u} = \underline{u} - \frac{\underline{u} \cdot \underline{a}}{\|\underline{a}\|^2}\, \underline{a}$$



Projection

$\text{Proj}_{\underline{a}}\, \underline{u} = \underline{u} \cdot \hat{a} \longrightarrow$ Scalar

$$\text{Proj}_{\underline{a}}\, \underline{u} = \left(\underline{u} \cdot \frac{\underline{a}}{\|\underline{a}\|} \right) \hat{a}$$

} multiply scalar with unit vector to form vector.

$$= \frac{\underline{u} \cdot \underline{a}}{\|\underline{a}\|} \; \frac{\underline{a}}{\|\underline{a}\|}$$

$$= \frac{\underline{u} \cdot \underline{a}}{\|\underline{a}\|^2}\, \underline{a}$$

$v = u - \text{proj}_{\underline{a}}\, u$

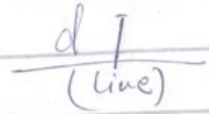
$u = v + \text{proj}_{\underline{a}}\, u$

P.no: 161

(2) Distance of $P(x_0, y_0)$ from Line

$ax + by + c = 0$ $\quad P(x_0, 0)$

$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$



(3) Distance of $P(x_0, y_0, z_0)$ from plane.

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \quad ax + by + cz + d = 0 \text{ (plane)}$$

* we can find distance of a point or straight line.

Q: 15 by these.

Q: 27

$2x - y - z = 5.$
$-4x + 2y + 2z = 12$
find distance between two planes.

$\to n_1 = (2, -1, -1)$

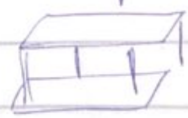
$\to ax + by + cz + \cancel{d} = 0$

$n_2 = (-4, 2, 2)$

$\to$ if multiple then parallel and planes are also parallel.

$\to$ here multiple so planes are parallel.

$\to n_2 = -2(n_1)$



$1 \parallel 2$

* find a point on one plane and get distance.

For point on 1

$y = 0 \ , \ z = 0$

$2x - 0 - 0 = 5$

$x = 5/2$

$P(5/2, 0, 0)$

$$D = \frac{\left| \cancel{4}^{2}\left(\frac{5}{\cancel{2}}\right) + 2(0) + 2(0) - 12 \right|}{\sqrt{(-4)^2 + 2^2 + 2^2}}$$

$$= \frac{|-10 - 12|}{\sqrt{24}}$$

$$= \frac{22}{\sqrt{24}} = \frac{\cancel{22}^{11}}{\cancel{2}\sqrt{6}}$$

$$= \frac{11}{\sqrt{6}}$$

*↳ We can also put x, y or x, z as 0. This depends on required point.

↳ We may need to try different points for an answer as point always exists.

- There is atleast one successive point.

Microsoft ~~visio~~ Visio, creatly / Uml modelling software.

31/3/2022 LINEAR ALGEBRA.

B.4:

Q: 1, 3, 5, 7, 9, 11, 13, 15, 17

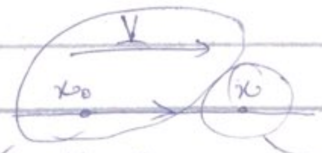
Notes (P.165)

① Plane through x_0 and parallel to $\underline{v_1}$ & $\underline{v_2}$ is:

$$\boxed{\underline{x} = \underline{x_0} + t_1 v_1 + t_2 v_2}$$

Line through x_0 and parallel to v is:

$$\underline{x} = x_0 + t\underline{v}$$



→ variable point

$\underline{x - x_0}$

↳ line

→ parallel so, $x - x_0 = tv$

$x = x_0 + tv$ ↳ a scalar is introduced

Q① Line = ?

Point $(-4,1)$, vector $\underline{v} = (0,-8)$

find equation of line & parametric

$x = x_0 + tv$

$$\boxed{\begin{gathered}(x,y) = (-4,1) + t(0,-8) \\ \text{vector equation of line}\end{gathered}}$$

Parametric equation.

$x = -4 + 0t$

$y = 1 - 8t$

for a plane.

e.g of plane is:

$x + 3y + 4z = 2$

↳ 3 var - 1 eq = 2 eq.

$y = t_1$

$z = t_2$

$\boxed{x = 2 - 4t_2 - 3t_1}$

Hence

$$\boxed{\begin{cases} x = 2 - 3t_1 - 4t_2 \\ y = t_1 \\ z = t_2 \end{cases}}$$

Parametric equation of plane

↳ vector form of equation.

$(x, y, z) = (2 - 3t_1 - 4t_2, t_1, t_2)$ ⟶ separate.

$= (2, 0, 0) + (-3t_1, t_1, 0) + (-4t_2, 0, t_2)$

$$\begin{cases} x = 2 - 3t_1 - 4t_2 \\ y = t_1 \\ z = t_2 \end{cases}$$

$(x, y, z) = (2, 0, 0) + t_1(-3, 1, 0) + t_2(-4, 0, 1)$

- We can also go to plane from vector form by reversing

(9) eq. of plane
(vector form)
(parametric form)

point $(-3,1,0)$, $V_1 = (0,-3,6)$
$V_2 = (-5,1,2)$

So

$X = x_0 + t_1 V_1 + t_2 V_2$

$(x,y,z) = (-3,1,0) + t_1(0,-3,6) + t_2(-5,1,2)$

$$\left.\begin{aligned} x &= -3 + 0t_1 - 5t_2 \\ y &= 1 - 3t_1 + t_2 \\ z &= 0 + 6t_1 + 2t_2 \end{aligned}\right\} \text{parametric form}$$

point v_1 v_2 t_2

Now plane equation

$t_2 = \frac{x+3}{-5}$

$y = 1 - 3t_1 + \left(\frac{x+3}{-5}\right)$

$y - 1 + \left(\frac{x+3}{5}\right) = -3t$

$t_1 = \frac{y}{-3} + \frac{-1}{-3} + \frac{x+3}{-15}$

$t_1 = \frac{5y - 5 + x + 3}{-15}$

$z = 0 + 6($

$$z = \cancel{10} + \overset{2}{\cancel{6}}\left(\frac{x+5y-2}{-\underset{5}{\cancel{15}}}\right) + 2\left(\frac{x+3}{-5}\right)$$

$$z = \frac{2x+10y-4+2x+6}{-5}$$

$$z = \frac{4x+10y+2}{-5}$$

$$-5z = 4x+10y+2$$

$$\boxed{\begin{array}{c} 4x+10y+5z+2=0 \\ \text{eq of plane} \end{array}}$$

---

(17) $x_1 + x_2 + x_3 = 0$

$2x_1 + 2x_2 + 2x_3 = 0$

$3x_1 + 3x_2 + 3x_3 = 0$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

$r_1 = (1,1,1) = [1\ 1\ 1]$

$r_2 = (2,2,2) = [2\ 2\ 2]$

$r_3 = (3,3,3) = [3\ 3\ 3]$

goal is solution orthognal to these 3 rows.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

echelon.

So system is equivalent to 1 eq
3 var − 1 eq = 2 params.

$x_2 = t_1$

$x_3 = t_2$

$\cancel{x_1 = -t_1 - t_2}$

$x_1 + t_1 + t_2 = 0$

$\boxed{x_1 = -t_1 - t_2}$

$\boxed{(x_1, x_2, x_3) = (-t_1 - t_2, t_1, t_2)} = X$

↳ is this orthognal to rows (r_1, r_2, r_3)

$x \cdot r_1 = 0$

$= (-t_1 - t_2, t_1, t_2) \cdot (1, 1, 1)$

$= \cancel{-t_1} \cancel{-t_2} + \cancel{t_1} + \cancel{t_2}$

$\left.\begin{array}{l} \boxed{x \cdot r_1 = 0} \\ \\ x_2 \cdot r_2 = 0 \\ x \cdot r_3 = 0 \end{array}\right\}$ (T) orthognal.

3.6 :

Q : 1, 5, 7, 9, 11, 13, 15, 17, 19, 21, 25, 29.

Cross - Product. (vector product)

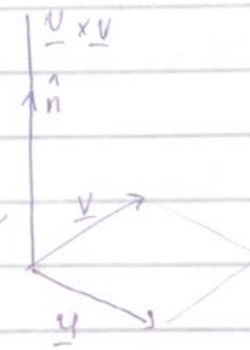
$\underline{u} = (u_1, u_2, u_3)$
$\underline{v} = (v_1, v_2, v_3)$

$$\underline{u} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\underline{u} \times \underline{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

also,

$\underline{u} \times \underline{v} = \|\underline{u}\|\|\underline{v}\| \sin\theta \ \hat{n}$



moving from $\underline{u}$ to $\underline{v}$ by right hand rule, thumb gives $\underline{u} \times \underline{v}$

$(\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u})$ as cos is commutative.

$(\underline{u} \times \underline{v} \neq \underline{v} \times \underline{u})$ as $\cos\theta$ is not commutative.

but

$(\underline{u} \times \underline{v} = - \underline{v} \times \underline{u}$

$$\hat{n} = \frac{\underline{u} \times \underline{v}}{\|\underline{u} \times \underline{v}\|}$$

$$\sin\theta = \frac{\underline{u} \times \underline{v}}{\|\underline{u}\|\|\underline{v}\| \hat{n}}$$

So

$$\sin\theta = \frac{\cancel{\underline{u} \times \underline{v}}}{\|\underline{u}\|\|\underline{v}\| \frac{\underline{u} \times \underline{v}}{\|\underline{u} \times \underline{v}\|}}$$

$$\sin\theta = \frac{\|\underline{u} \times \underline{v}\|}{\|\underline{u}\| \cdot \|\underline{v}\|}$$

Note

if $\underline{u} \parallel \underline{v}$ → parallel $\qquad \theta = 0°$, $\sin\theta = 0$

$\underline{u} \times \underline{v} = 0$

(2) area of parallelogram $= \|\underline{u} \times \underline{v}\|$

area of triangle $= \frac{1}{2} \|\underline{u} \times \underline{v}\|$

(3) Scalar triple product

$$\underline{u} \cdot \underline{v} \times \underline{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \longrightarrow$$

5/4/2022

Ex: 3.5

Q:1:-

$u = (3, 2, -1)$
$v = (0, 2, -3)$
$w = (2, 6, 7)$

(a) $v \times w = \begin{vmatrix} i & j & k \\ 0 & 2 & -3 \\ 2 & 6 & 7 \end{vmatrix}$

$= i(14+18) - j(0+6) + k(0-4)$
$= 32i - 6j - 4k$
$= (32, -6, -4)$

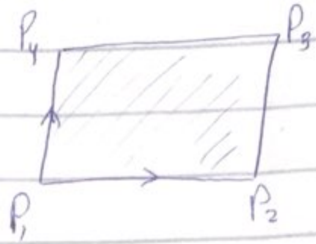
Q:6

$(\underline{u} \times \underline{v}) \times \underline{w} = (\underline{w} \cdot \underline{u})\underline{v} - (\underline{w} \cdot \underline{v})\underline{u}$
$= (6+12-7)(0, 2, -3) - (0+12-21)(3, 2, -1)$
$= 11(0, 2, -3) - (-9)(3, 2, -1)$
$= (0, 22, -33) + (27, 18, -9)$
$= (0+27, 22+18, -33-9)$
$= (27, 40, -42)$

Q: 9:-

Area of $\parallel$gram = ?
$\| \underline{u} \times \underline{v} \| = ?$

Q: 11: $P_1(1,2)$, $P_2(4,4)$, $P_3(7,5)$, $P_4(4,3)$



$\overrightarrow{P_1P_2} = (4-1, 4-2) = (3,2)$
$\overrightarrow{P_1P_4} = (4-1, 3-2) = (3,1)$

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 2 & 0 \\ 3 & 1 & 0 \end{vmatrix}$$

$= i(0-0) - j(0-0) + \underline{k}(3-6)$
$= 0\underline{i} - 0\underline{j} - 3\underline{k}$
$= (0, 0, -3)$

Area of ||gram $= \| \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} \|$
$= \sqrt{0^2 + 0^2 + (-3)^2}$
$= 3$ Ans.

Q: 12
Volume of parallelepiped = ?
$\underline{u} = (2, -6, 2)$
$\underline{v} = (0, 4, -2)$
$\underline{w} = (2, 2, -4)$

$= \underline{u} \cdot (\underline{v} \times \underline{w})$

$$= \begin{vmatrix} 2 & -6 & 2 \\ 0 & 4 & -2 \\ 2 & 2 & -4 \end{vmatrix} =$$

3

(b) $\underline{u} \cdot (\underline{w} \times \underline{v}) = -\underline{u} \cdot (\underline{v} \times \underline{w})$

$\boxed{= -3}$

(c) $\underline{w} \cdot (\underline{u} \times \underline{v}) = \underline{u} \cdot (\underline{v} \times \underline{w})$

$= 3$

Q: 29:

$(\underline{u} + \underline{v}) \times (\underline{u} - \underline{v})$

$= \underline{u} \times \underline{u} - \underline{u} \times \underline{v} + \underline{v} \times \underline{u} - \underline{v} \times \underline{v}$

$= 0 + \underline{v} \times \underline{u} + \underline{v} \times \underline{u} + 0$

$= 2(\underline{v} \times \underline{u})$