If 7 is less than 4, then 7 is not a prime number.
7 is not less than 4.

7 is a prime number.

Solution. First translate the argument into symbolic form. Let p be "7 is less than 4" and q be "7 is a prime number." Then the argument is of the form

$$p \to \sim q, \sim p \vdash q$$

Now, we construct a truth table. The above argument is shown to be a fallacy since, in the fourth line of the truth table, the premises $p \to \sim q$ and $\sim p$ are true, but the conclusion q is false.

Remark: The fact that the conclusion of the argument happens to be a true statement is irrelevant to the fact that the argument presented is a fallacy. ■

1.8 Propositional Functions, Quantifiers

Definition 13: Propositional Functions

Let A be a given set. A ***propositional function*** (or an ***open sentence*** or ***condition***) defined on A is an expression

$$p(x)$$

which has the property that $p(a)$ is true or false for each $a \in A$. That is, $p(x)$ becomes a statement (with a truth value) whenever any element $a \in A$ is substituted for the variable x. The set A is called the domain of $p(x)$, and the set T_p of all elements of A for which $p(a)$ is true is called the truth set of $p(x)$. In other words,

$$T_p = \{x \mid x \in A, p(x) \text{ is true }\} \text{ or } T_p = \{x \mid p(x)\}$$

Frequently, when A is some set of numbers, the condition $p(x)$ has the form of an equation or inequality involving the variable x.

Example 1.35. Find the truth set for each propositional function $p(x)$ defined on the set $\mathbb{N}$ of positive integers.

(a) Let $p(x)$ be "$x + 2 > 7$."

(b) Let $p(x)$ be "$x + 5 < 3$."

(c) Let $p(x)$ be "$x + 5 > 1$."

Remark: The above example shows that if $p(x)$ is a propositional function defined on a set A then $p(x)$ could be true for all $x \in A$, for some$x \in A$, or for no $x \in A$. The next two subsections discuss quantifiers related to such propositional functions.

Universal Quantifier

Definition 14: Universal Quantifier

Let $p(x)$ be a propositional function defined on a set A. Consider the expression

$$(\forall x \in A)\ p(x) \text{ or } \forall x\ p(x)$$

which reads "For every x in A, $p(x)$ is a true statement" or, simply, "For all x, $p(x)$." The symbol

$$\forall$$

which reads "for all" or "for every" is called the ***universal quantifier***.

The statement above is equivalent to the statement

$$T_p = \{x \mid x \in A, p(x)\} = A$$

that is, that the truth set of $p(x)$ is the entire set A. The expression $p(x)$ by itself is an open sentence or condition and therefore has no truth value. Specifically:

$$Q1 : If\{x \mid x \in A, p(x)\} = A \text{ then } \forall x\ p(x) \text{ is true ; otherwise }, \forall x\ p(x) \text{ is false .}$$

Example 1.36. Consider the following propositional functions

(a) The proposition $(\forall n \in \mathbb{N})\ (n + 4 > 3)$ is true since $\{n \mid n + 4 > 3\} = \{1, 2, 3, ...\} = \mathbb{N}$.

(b) The proposition $(\forall n \in \mathbb{N})\ (n + 2 > 8)$ is false since $\{n \mid n + 2 > 8\} = \{7, 8, 9, ...\} = \mathbb{N}$.

Existential Quantifier

Definition 15: Existential Quantifier

Let $p(x)$ be a propositional function defined on a set A. Consider the expression

$$(\exists x \in A)\ p(x) \text{ or } \exists x,\ p(x)$$

which reads "There exists an x in A such that $p(x)$ is a true statement" or, simply, "For some x, $p(x)$." The symbol

$$\exists$$

which reads "there exists" or "for some" or "for at least one" is called the ***existential quantifier***.

The statement above is equivalent to the statement

$$T_p = \{x \mid x \in A,\ p(x)\} \neq \emptyset$$

i.e., that the truth set of $p(x)$ is not empty. Accordingly, $\exists x\ p(x)$, that is, $p(x)$ preceded by the quantifier $\exists$, does have a truth value. Specifically:

$$Q2 : If\{x \mid p(x)\} \neq \emptyset \text{ then } \exists x\ p(x) \text{ is true ; otherwise }, \exists x\ p(x) \text{ is false .}$$

Example 1.37. Consider the following propositional functions

(a) The proposition $(\exists n \in \mathbb{N})\ (n + 4 < 7)$ is true since $\{n \mid n + 4 < 7\} = \{1, 2\} \neq \emptyset$.

(b) The proposition $(\exists n \in \mathbb{N})\ (n + 6 < 4)$ is false since $\{n \mid n + 6 < 4\} = \emptyset$.

Example 1.38. Consider the statement

$$\exists m \in \mathbb{Z}^+ \text{ such that } m^2 = m.$$

Solution. "There is at least one positive integer m such that $m^2 = m$." Observe that $1^2 = 1$. Thus "$m^2 = m$" is true for a positive integer m, and so "$\exists m \in \mathbb{Z}^+$ such that $m^2 = m$" is true. ■

Example 1.39. Let $E = \{5, 6, 7, 8\}$ and consider the statement

$$\exists m \in E \text{ such that } m^2 = m.$$

Show that this statement is false.

Solution. Note that $m^2 = m$ is not true for any integers m from 5 through 8:

$$5^2 = 25 \neq 5,\ 6^2 = 36 \neq 6,\ 7^2 = 49 \neq 7,\ 8^2 = 64 \neq 8.$$

Thus "$\exists m \in E$ such that $m^2 = m$" is false. ■

Negation of Quantified Statements

Definition 16: Negation of a Universal Statement

The negation of a statement of the form

$$\forall x \in A, P(x)$$

is logically equivalent to a statement of the form

$$\exists x \in A \text{ such that } \sim P(x).$$

Symbolically,

$$\sim (\forall x \in A, P(x)) \equiv \exists x \in A \text{ such that } \sim P(x)$$

Thus

The negation of a universal statement ("all are") is logically equivalent to an existential statement ("some are not" or "there is at least one that is not").

Definition 17: Negation of an Existential Statement

The negation of a statement of the form

$$\exists x \in A, P(x)$$

is logically equivalent to a statement of the form

$$\forall x \in A \text{ such that } \sim P(x).$$

Symbolically,

$$\sim (\exists x \in A, P(x)) \equiv \forall x \in A \text{ such that } \sim P(x)$$

Thus

The negation of an existential statement ("some are") is logically equivalent to a universal statement ("none are" or "all are not").").

Example 1.40. Write formal negations for the following statements:

(a) $\forall$ primes p, p is odd.

(b) $\exists$ a triangle T such that the sum of the angles of T equals 200^o.

Solution. Exercises ■

Example 1.41. What are the negations of the statements $\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$?

Solution. The negation of $\forall x(x^2 > x)$ is the statement $\sim (\forall x(x^2 > x))$, which is equivalent to $\exists x \sim (x^2 > x)$. This can be rewritten as $\exists x(x^2 \leq x)$. The negation of $\exists x(x^2 = 2)$ is the statement $\sim (\exists x(x^2 = 2))$, which is equivalent to $\forall x \sim (x^2 = 2)$. This can be rewritten as $\forall x(x^2 \neq 2)$. The truth values of these statements depend on the domain. ■

Counterexample

Example 1.42. Consider the statement $\forall x \in \mathbb{R}, |x| \neq 0$. The statement is false since 0 is a counterexample, that is, $|0| \neq 0$ is not true.

Example 1.43. Consider the statement $\forall x \in \mathbb{R}, x^2 \geq x$. The statement is not true since, for example, $\frac{1}{2}$ is a counterexample. Specifically, $(\frac{1}{2})^2 \geq \frac{1}{2}$ is not true, that is, $(\frac{1}{2})^2 < \frac{1}{2}$.

Propositional Functions with more than One Variable

A propositional function preceded by a quantifier for each variable, for example,

$$\forall x \exists y,\ p(x,y) \text{ or } \exists x \forall y \exists z,\ p(x,y,z)$$

denotes a statement and has a truth value.

Example 1.44. Let $B = \{1, 2, 3, ..., 9\}$ and let $p(x,y)$ denote "$x + y = 10$." Then $p(x,y)$ is a propositional function on $A = B^2 = B \times B$.

(a) The following is a statement since there is a quantifier for each variable:

$$\forall x \exists y,\ p(x,y), \text{ that is, "For every } x \text{, there exists a } y \text{ such that } x + y = 10\text{"}$$

This statement is true. For example, if $x = 1$, let $y = 9$; if $x = 2$, let $y = 8$, and so on.

(b) The following is also a statement:

$$\exists y \forall x,\ p(x,y), \text{ that is, "There exists a } y \text{ such that, for every } x \text{, we have } x + y = 10\text{"}$$

No such y exists; hence this statement is false.

Negating Quantified Statements with more than One Variable

Quantified statements with more than one variable may be negated by successively applying the negation of quantified statements definitions above. Thus each $\forall$ is changed to $\exists$ and each $\exists$ is changed to $\forall$ as the negation symbol $\sim$ passes through the statement from left to right. For example,

$$\sim [\forall x \exists y \exists z,\ p(x,y,z)] \equiv \exists x \forall y \forall z,\ \sim p(x,y,z)$$

Example 1.45. Express the negation of the statement $\forall x \exists y(xy = 1)$ so that no negation precedes a quantifier.

Solution. We find that $\sim \forall x \exists y(xy = 1)$ is equivalent to $\exists x \sim \exists y(xy = 1)$, which is equivalent to $\exists x \forall y \sim (xy = 1)$. Because $\sim (xy = 1)$ can be expressed more simply as $xy \neq 1$, we conclude that our negated statement can be expressed as $\exists x \forall y(xy \neq 1)$. ■

2. Elementary Number theory and Methods of Proof

Direct Proof

A direct proof of a conditional statement $p \to q$ is constructed when the first step is the assumption that p is true; subsequent steps are constructed using rules of inference, with the final step showing that q must also be true.

Definition 18: Even and Odd Integers

An integer n is ***even*** if, and only if, n equals twice some integer. An integer n is ***odd*** if, and only if, n equals twice some integer plus 1.
Symbolically, for any integer, n

$$n \text{ is even} \leftrightarrow n = 2k \text{ for some integer } k$$

$$n \text{ is odd} \leftrightarrow n = 2k + 1 \text{ for some integer } k$$

Example 2.1. Use the definitions of even and odd to justify your answers to the following questions.

(a) Is 0 even?

(b) Is -301 odd?

(c) If a and b are integers, is $6a^2b$ even?

(d) If a and b are integers, is $10a + 8b + 1$ odd?

(e) Is every integer either even or odd?

Theorem 2.1. *The sum of any two even integers is even.*

Proof. Suppose m and n are any [particular but arbitrarily chosen] even integers. [We must show that $m+n$ is even.] By definition of even, $m + 2r$ and $n = 2s$ for some integers r and s. Then

$$\begin{aligned} m + n &= 2r + 2s \\ &= 2(r + s) \end{aligned}$$

Let $t = r + s$. Note that t is an integer because it is a sum of integers. Hence

$$m + n = 2t \text{ where } t \text{ is an integer.}$$

□

Example 2.2. Show that the difference of any odd integer and any even integer is odd.

Solution. Suppose a is any odd integer and b is any even integer.[*We must show that $a - b$ is odd.*]
By definition of odd, $a = 2r + 1$ for some integer r, and $b = 2s$ for some integer s. Then

$$\begin{aligned} a - b &= (2r + 1) - 2s \\ &= 2r - 2s + 1 \\ &= 2(r - s) + 1 \\ &= 2t + 1 \text{ for some integer } t = r - s \end{aligned}$$

Hence $a - b$ is odd ■

Example 2.3. For every odd integer n, n^2 is odd

Solution. Suppose n is any odd integer. By definition of odd, $n = 2k + 1$ for some integer k. Then

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(k^2 + 2k) + 1 \\ &= 2r + 1 \text{ for some integer } r = k^2 + 2k \end{aligned}$$

Therefore n^2 is odd. ■

Example 2.4. Show that for all integers r and s, if r is even and s is odd then $3r + 2s$ is even

Solution. Exercise ■

Example 2.5. Deriving Additional Results about Even and Odd Integers

1. The sum, product, and difference of any two even integers are even.
2. The sum and difference of any two odd integers are even.
3. The product of any two odd integers is odd.
4. The product of any even integer and any odd integer is even.
5. The sum of any odd integer and any even integer is odd.
6. The difference of any odd integer minus any even integer is odd.
7. The difference of any even integer minus any odd integer is odd.

Remark: Direct proofs lead from the premises of a theorem to the conclusion. They begin with the premises, continue with a sequence of deductions, and end with the conclusion.

Proof by Contraposition

In some cases we will see that attempts at direct proofs often reach dead ends. We need other methods of proving theorems of the form $\forall x(P(x) \to Q(x))$. Proofs of theorems of this type that are not direct proofs, that is, that do not start with the premises and end with the conclusion, are called indirect proofs.

Proofs by contraposition make use of the fact that the conditional statement $p \to q$ is equivalent to its contrapositive, $\sim q \to \sim p$. This means that the conditional statement $p \to q$ can be proved by showing that its contrapositive, $\sim q \to \sim p$, is true. In a proof by contraposition of $p \to q$, we take $\sim q$ as a premise, and using axioms, definitions, and previously proven theorems, together with rules of inference, we show that $\sim p$ must follow.

Example 2.6. Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Solution. Let the conditional statement "If $3n + 2$ is odd, then n is odd" be false.
i.e. assume that n is even. Then, by the definition of an even integer, $n = 2k$ for some integer k.

$$\begin{aligned} 3n + 2 &= 3(2k) + 2 \\ &= 6k + 2 \\ &= 2(3k + 1) \end{aligned}$$

This tells us that $3n + 2$ is even (because it is a multiple of 2), and therefore not odd. This is the negation of the premise of the example. Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. ■

Example 2.7. Prove that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Solution. Because there is no obvious way of showing that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ directly from the equation $n = ab$, where a and b are positive integers, we attempt a proof by contraposition.
Let the conditional statement "If $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$" be false.

$$\text{i.e. } (a \leq \sqrt{n}) \vee (b \leq \sqrt{n}) \text{ is false.}$$

$$\begin{aligned} \sim ((a \leq \sqrt{n}) \vee (b \leq \sqrt{n})) &\equiv \sim (a \leq \sqrt{n}) \wedge \sim (b \leq \sqrt{n}) \\ &\equiv (a > \sqrt{n}) \wedge (b > \sqrt{n}) \end{aligned}$$

This implies that $a > \sqrt{n}$ and $b > \sqrt{n}$. Multiply these inequalities together we obtained

$$\begin{aligned} ab &> \sqrt{n}\sqrt{n} = n \\ &> n \end{aligned}$$

This shows that $ab \neq n$, which contradicts the statement $n = ab$.
Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. ■

Example 2.8. Prove that if n is an integer and n^2 is odd, then n is odd.

Solution. Exercise ■

Example 2.9. Show that the proposition $P(0)$ is true, where $P(n)$ is "If $n > 1$, then $n^2 > n$" and the domain consists of all integers.

Solution. Note that $P(0)$ is "If $0 > 1$, then $0^2 > 0$." We can show $P(0)$ using a vacuous proof. Indeed, the hypothesis $0 > 1$ is false. This tells us that $P(0)$ is automatically true. ■

Example 2.10. Prove that if n is an integer with $10 \leq n \leq 15$ which is a perfect square, then n is also a perfect cube.

Solution. Note that there are no perfect squares n with $10 \leq n \leq 15$, because $3^2 = 9$ and $4^2 = 16$. Hence, the statement that n is an integer with $10 \leq n \leq 15$ which is a perfect square is false for all integers n. Consequently, the statement to be proved is true for all integers n. ■

We can also quickly prove a conditional statement $p \to q$ if we know that the conclusion q is true. By showing that q is true, it follows that $p \to q$ must also be true. A proof of $p \to q$ that uses the fact that q is true is called a **trivial proof.**

Example 2.11. Let $P(n)$ be "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$," where the domain consists of all nonnegative integers. Show that $P(0)$ is true.

Solution. The proposition $P(0)$ is "If $a \geq b$, then $a^0 \geq b^0$." Because $a^0 = b^0 = 1$, the conclusion of the conditional statement "If $a \geq b$, then$a^0 \geq b^0$" is true. Hence, this conditional statement, which is $P(0)$, is true. This is an example of a trivial proof. Note that the hypothesis, which is the statement "$a \geq b$," was not needed in this proof. ■

Definition 19

The real number r is **rational** if there exist integers p and q with $q \neq 0$ such that $r = \frac{p}{q}$. A real number that is not rational is called **irrational**.

Example 2.12. Determining whether Numbers Are Rational or Irrational

(a) Is $\frac{10}{3}$ a rational number?

(b) Is $-\frac{5}{39}$ a rational number?

(c) Is 0.281 a rational number?

(d) Is 7 a rational number?

(e) Is 0 a rational number?

(f) Is $\frac{2}{0}$ a rational number?

(g) Is $\frac{2}{0}$ a irrational number?

(h) Is 0.12121212 A a rational number (where the digits 12 are assumed to repeat forever)?

(i) If m and n are integers and neither m nor n is zero, is $\frac{(m+n)}{mn}$ a rational number?

Solution. Exercise ■

Theorem 2.2. *Every integer is a rational number.*

Example 2.13. Any Sum of Rational Numbers Is Rational

Solution. Suppose r and s are any rational numbers.
By definition $r = \frac{a}{b}$ and $s = \frac{c}{d}$ for some integers a, b, c, and d where $b \neq 0$ and $d \neq 0$.
It follows by substitution that

$$\begin{aligned} r + s &= \frac{a}{b} + \frac{c}{d} \\ &= \frac{ad}{bd} + \frac{bc}{bd} \\ &= \frac{ad+bc}{bd} \\ &= \frac{p}{q} \text{ where } p = ad + bc \text{ and } q = bd \text{ are integers and } q \neq 0. \end{aligned}$$

This means that $r + s$ is rational ■

Example 2.14. Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Solution. Suppose $\sqrt{2}$ is rational. Then

$$\begin{aligned} \sqrt{2} &= \frac{p}{q} \text{ for some integer } p \text{ and } q \neq 0. \\ 2 &= \frac{p^2}{q^2} \\ 2q^2 &= p^2 \end{aligned}$$

p^2 must be even. Since p^2 is even, then p is also even. Since p is even, it can be written as $2m$ where m is an integer.
Substituting $p = 2m$ in the above equation:

$$2q^2 = (2m)^2$$
$$2q^2 = 4m^2$$
$$q^2 = 2m^2$$

q^2 is an even number. So q is an even number. Since q is even, it can be written as $2n$ where n is an integer. Now we have $p = 2m$ and $q = 2n$ and remember we assumed that $\sqrt{2} = \frac{p}{q}$:

$$\sqrt{2} = \frac{p}{q}$$
$$\sqrt{2} = \frac{2m}{2n}$$
$$\sqrt{2} = \frac{m}{n}$$

We now have a fraction m/n that is simpler than p/q.
Hence $\sqrt{2}$ cannot be rational and so must be irrational. ■

Exhaustive Proof

Some theorems can be proved by examining a relatively small number of examples. Such proofs are called exhaustive proofs, or proofs by exhaustion because these proofs proceed by exhausting all possibilities. An exhaustive proof is a special type of proof by cases where each case involves checking a single example.

Example 2.15. Prove that $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$.

Solution. We use a proof by exhaustion.We only need verify the inequality $(n+1)^3 \geq 3^n$ when $n = 1, 2, 3,$ and 4. ■

Example 2.16. Prove that the only consecutive positive integers not exceeding 100 that are perfect powers are 8 and 9. (An integer n is a **perfect power** if it equals m^a, where m is an integer and a is an integer greater than 1.)

Solution. We use a proof by exhaustion. In particular, we can prove this fact by examining positive integers n not exceeding 100, first checking whether n is a perfect power, and if it is, checking whether $n+1$ is also a perfect power. A quicker way to do this is simply to look at all perfect powers not exceeding 100 and checking whether the next largest integer is also a perfect power. The squares of positive integers not exceeding 100 are 1, 4, 9, 16, 25, 36, 49, 64, 81, and 100. The cubes of positive integers not exceeding 100 are 1, 8, 27, and 64. The fourth powers of positive integers not exceeding 100 are 1, 16, and 81. The fifth powers of positive integers not exceeding 100 are 1 and 32. The sixth powers of positive integers not exceeding 100 are 1 and 64. There are no powers of positive integers higher than the sixth power not exceeding 100, other than 1. Looking at this list of perfect powers not exceeding 100, we see that $n = 8$ is the only perfect power n for which $n+1$ is also a perfect power. That is, $2^3 = 8$ and $3^2 = 9$ are the only two consecutive perfect powers not exceeding 100. ■

Proof by Cases

A proof by cases must cover all possible cases that arise in a theorem. We illustrate proof by cases with a couple of examples. In each example, you should check that all possible cases are covered.

Example 2.17. Prove that if n is an integer, then $n^2 \geq n$.

Solution. We can prove that $n^2 \geq n$ for every integer by considering three cases, when $n = 0$, when $n \geq 1$, and when $n \leq -1$. We split the proof into three cases because it is straightforward to prove the result by considering zero, positive integers, and negative integers separately.
Case (i): When $n = 0$, because $0^2 = 0$, we see that $0^2 \geq 0$. It follows that $n^2 \geq n$ is true in this case.
Case (ii): When $n \geq 1$, when we multiply both sides of the inequality $n \geq 1$ by the positive integer n, we obtain $n.n \geq n.1$. This implies that $n^2 \geq n$ for $n \geq 1$.
Case (iii): In this case $n \leq -1$. However, $n^2 \geq 0$. It follows that $n^2 \geq n$.

Because the inequality $n^2 \geq n$ holds in all three cases, we can conclude that if n is an integer, then $n^2 \geq n$. ■

Example 2.18. Use a proof by cases to show that $|xy| = |x||y|$, where x and y are real numbers. **(Recall** that $|a|$, the absolute value of a, equals a when $a \geq 0$ and equals $-a$ when $a \leq 0$.) ■

Solution. Exercise

Example 2.19. Show that there are no solutions in integers x and y of $x^2 + 3y^2 = 8$. ■

Solution. Exercise

Uniqueness Proofs

Some theorems assert the existence of a unique element with a particular property. In other words, these theorems assert that there is exactly one element with this property. To prove a statement of this type we need to show that an element with this property exists and that no other element has this property. The two parts of a **uniqueness proof** are:

Existence: We show that an element x with the desired property exists.
Uniqueness: We show that if x and y both have the desired property, then $x = y$.

Example 2.20. Show that if a and b are real numbers and $a \neq 0$, then there is a unique real number r such that $ar + b = 0$.

Solution. First, note that the real number $r = -b/a$ is a solution of $ar + b = 0$ because

$$a(-b/a) + b = -b + b = 0$$

Consequently, a real number r exists for which $ar + b = 0$. This is the existence part of the proof.

Second, suppose that s is a real number such that $as + b = 0$. Then $ar + b = as + b$, where $r = -b/a$. Subtracting b from both sides, we find that$ar = as$. Dividing both sides of this last equation by a, which is nonzero, we see that $r = s$. This establishes the uniqueness part of the proof. ■

Prime and Composite Numbers

Definition 20: Prime and Composite Numbers

An integer n is **prime** if, and only if, $n > 1$ and for all positive integers r and s, if $n = rs$, then either r or s equals n. An integer n is **composite** if, and only if, $n > 1$ and $n = rs$ for some integers r and s with $1 < r < n$ and $1 < s < n$. In symbols: For each integer n with $n > 1$,

$$\begin{aligned} n \text{ is prime } &\leftrightarrow \forall \text{ positive integers } r \text{ and } s, \text{ if } n = rs \\ &\quad \text{then either } r = 1 \text{ and } s = n \text{ or } r = n \text{ and } s = 1. \\ n \text{ is composite } &\leftrightarrow \exists \text{ positive integers } r \text{ and } s, \text{ such that } n = rs \\ &\quad \text{and } 1 < r < n \text{ and } 1 < s < n. \end{aligned}$$