

5: ROOTS OF A QUADRATIC EQUATION

The general form of a quadratic equation

We have grown accustomed to recognising a quadratic equation in the form $ax^2 + bx + c = 0$. In this section, we will be introduced to a new format for such a quadratic equation. This format would express the quadratic in the form of its roots. It is a convenient form to know and it allows us the flexibility to switch from this form to the standard form.

Roots of a quadratic equation (α and β)

A quadratic equation in x is of the general form $ax^2 + bx + c = 0$, where a , b and c are constants.

If we divide each term by a , then the quadratic equation can be expressed in an equivalent form with the coefficient of x^2 is equal to one as shown below.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad (1)$$

Now consider α and β as the roots of the quadratic. We can now rewrite the quadratic in the form:

$$(x - \alpha)(x - \beta) = 0.$$

By expanding we get,

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0. \quad (2)$$

Equation (2) is an equivalent form of equation (1). In fact, any quadratic equation, in x , can always be expressed in the form of its roots.

We can replace $(\alpha + \beta)$ by the 'sum of the roots' and $\alpha\beta$ by the 'product of the roots', to obtain the following form for a quadratic equation.

$$x^2 - (\text{sum of roots})x + \text{product of roots} = 0$$

Sum and product of the roots of a quadratic equation

Equations (1) and (2) above are two equivalent forms of a quadratic equation.

Equating both forms we get:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 - (\alpha + \beta)x + \alpha\beta$$

When we equate coefficients, the following is obtained:

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}.$$

We can now make a general statement about the roots of a quadratic.

For the quadratic equation

$$ax^2 + bx + c = 0,$$

the sum of the roots $= -\frac{b}{a}$ and

the product of the roots $= \frac{c}{a}.$

Example 1

If α and β are the roots of the quadratic equation $x^2 - 3x + 2 = 0$, determine

- the sum of the roots and
- the product of the roots.

Solution

In the quadratic equation $x^2 - 3x + 2 = 0$
 $a = 1$, $b = -3$ and $c = 2$.

- The sum of the roots, $\alpha + \beta = -\frac{b}{a}$

$$\alpha + \beta = \frac{-(-3)}{1} = 3$$

- The product of the roots, $\alpha\beta = \frac{c}{a}$

$$\alpha\beta = \frac{2}{1} = 2.$$

Example 2

The quadratic equation $x^2 - 4x + 3 = 0$ has roots α and β .

- Obtain the equation whose roots are $\alpha + 1$ and $\beta + 1$.
- Obtain the equation whose roots are α^2 and β^2 .

Solution

If the equation $x^2 - 4x + 3 = 0$ has roots α and β , then $a = 1$, $b = -4$ and $c = 3$. Hence,

$$(\alpha + \beta) = 4 \quad \text{and} \quad \alpha\beta = 3$$

To obtain an equation whose roots are $\alpha + 1$ and

$\beta + 1$, we can substitute these roots in the following equation:

$$x^2 - (\text{sum of roots})x + \text{product of roots} = 0$$

$$x^2 - [(\alpha + 1) + (\beta + 1)]x + [(\alpha + 1)(\beta + 1)] = 0$$

$$x^2 - [(\alpha + \beta) + 2]x + [\alpha\beta + (\alpha + \beta) + 1] = 0$$

$$x^2 - (4 + 2)x + (3 + 4 + 1) = 0$$

$$x^2 - 6x + 8 = 0$$

This is the required equation.

Part b) To obtain an equation whose roots are α^2 and β^2 , we substitute these roots in:

$$x^2 - (\text{sum of roots})x + \text{product of roots} = 0$$

$$x^2 - (\alpha^2 + \beta^2)x + (\alpha^2 \times \beta^2) = 0$$

$$x^2 - (\alpha^2 + \beta^2)x + (\alpha^2 \beta^2) = 0$$

$$[\text{Recall: } (\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta]$$

$$x^2 - ((\alpha + \beta)^2 - 2\alpha\beta)x + (\alpha\beta)^2 = 0$$

$$x^2 - \{(4)^2 - 2(3)\}x + (3)^2 = 0$$

$$x^2 - 10x + 9 = 0$$

This is the required equation.

Example 3

Given that $x^2 + (k - 5)x - k = 0$ has real roots which differ by 4, determine

- the value of each root
- the value of k .

Solution

If we let α be the smaller real root, then the other will be $(\alpha + 4)$.

Then the sum of the roots is : $\alpha + (\alpha + 4) = 2\alpha + 4$

The product of the roots is $\alpha(\alpha + 4)$.

From the given equation $x^2 + (k - 5)x - k = 0$,

The sum of the roots is: $-(k - 5)$

The product of the roots is: $-k$

Equating coefficients, we have:

Sum of roots $2\alpha + 4 = -(k - 5)$ $2\alpha + 4 = -k + 5$ $k = 1 - 2\alpha$ (1)	Product of roots $\alpha(\alpha + 4) = -k$ $k = -\alpha(\alpha + 4)$ (2)
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Equating equations (1) and (2) to eliminate k , we have:

$$-\alpha^2 - 4\alpha = 1 - 2\alpha$$

$$\alpha^2 + 2\alpha + 1 = 0$$

$$(\alpha + 1)(\alpha + 1) = 0$$

$$\alpha = -1$$

The value of k : $k = 1 - 2\alpha = 1 - 2(-1) = 3$

\therefore Roots are -1 and $-1 + 4$

The roots are -1 and 3 .

Alternative Method

If we let α be the smaller real root, then the other will be $(\alpha + 4)$.

Hence the quadratic equation may be expressed as

$$(x - \alpha)(x - (\alpha + 4)) = 0$$

$$(x - \alpha)(x - \alpha - 4) = 0$$

$$x^2 - \alpha x - \alpha x + \alpha^2 - 4\alpha + 4\alpha = 0$$

$$x^2 + (-2\alpha - 4)x + (\alpha^2 + 4\alpha) = 0$$

Equating coefficient of x , we obtain

$$-2\alpha - 4 = k - 5$$

$$-2\alpha - 4 = k - 5$$

$$k = 1 - 2\alpha$$

$$\therefore \alpha = \frac{1 - k}{2}$$

Equating constant terms, we obtain

$$\alpha^2 + 4\alpha = -k$$

$$\therefore \left(\frac{1 - k}{2}\right)^2 + 4\left(\frac{1 - k}{2}\right) = -k$$

$$\frac{1 - 2k + k^2}{4} + 2 - 2k + k = 0$$

$$1 - 2k + k^2 + 8 - 8k + 4k = 0$$

$$k^2 - 6k + 9 = 0$$

$$(k - 3)^2 = 0$$

$$k = 3$$

$$\text{When, } k = 3, \alpha = \frac{1 - 3}{2} = -1$$

\therefore Roots are -1 and $-1 + 4$

The roots are -1 and 3 .