Example 1.13. Prove theorem 1.1.

Solution. Exercise! ■

Example 1.14. Use Theorem 1.1 to verify the logical equivalence $\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$

Solution. Use the laws of Theorem 1.1 to replace sections of the statement form on the left by logically equivalent expressions. Each time you do this, you obtain a logically equivalent statement form. Continue making replacements until you obtain the statement form on the right.

$$
\begin{array}{rcll}
\sim(\sim p \wedge q) \wedge (p \vee q) & \equiv & (\sim(\sim p) \vee \sim q) \wedge (p \vee q) & \text{by De Morgan's laws} \\
 & \equiv & (p \vee \sim q) \wedge (p \vee q) & \text{by the double negative law} \\
 & \equiv & p \vee (\sim q \wedge q) & \text{by the distributive law} \\
 & \equiv & p \vee (q \wedge \sim q) & \text{by the commutative law for } \wedge \\
 & \equiv & p \vee \mathbf{c} & \text{by the negation law} \\
 & \equiv & p & \text{by the identity law}
\end{array}
$$

■

Example 1.15. Show that $\sim(p \vee (\sim p \wedge q))$ and $\sim p \wedge \sim q$ are logically equivalent by developing a series of logical equivalences.

Solution. Use the laws of Theorem 1.1 to replace sections of the statement form on the left by logically equivalent expressions. Each time you do this, you obtain a logically equivalent statement form. Continue making replacements until you obtain the statement form on the right.

by the second De Morgan law

[redacted]

$$
\begin{array}{cll}
\equiv & \sim p \wedge (p \vee \sim q) & \text{by the double negative law} \\
\equiv & (\sim p \wedge p) \vee (\sim p \wedge \sim q) & \text{by the second distributive law} \\
\equiv & \mathbf{c} \vee (\sim p \wedge \sim q) & \text{because } \sim p \wedge p \equiv \mathbf{c} \\
\equiv & (\sim p \wedge \sim q) \vee \mathbf{c} & \text{by the commutative law for disjunction} \\
\equiv & (\sim p \wedge \sim q) & \text{by the identity law for } \mathbf{c}
\end{array}
$$

■

1.6 Conditional and Biconditional Statements

Definition 8: Conditional

Let p and q be propositions. The conditional statement $p \to q$ is the proposition "if p, then q." The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \to q$, p is called the **hypothesis** (or **antecedent** or **premise**) and q is called the **conclusion** (or **consequence**).

The conditional $p \to q$ is frequently read "p implies q" or "p only if q." In expressions that include $\to$ as well as other logical operators such as $\wedge$, $\vee$, and $\sim$, the **order of operations** is that $\to$ is performed last. Thus, according to the specification of order of operations, $\sim$ is performed first, then $\wedge$ and $\vee$, and finally $\to$.

Example 1.16. Construct a truth table for the statement form $p \vee \sim q \to \sim p$. ■

Solution. By the order of operations given above:

Table 12: Truth Table for $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 13: Truth Table for $p \vee \sim q \to \sim p$

p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$p \vee \sim q \to \sim p$
T	T	F	F	T	F
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Example 1.17. Show That $p \vee q \to r \equiv (p \to r) \wedge (q \to r)$

Solution. Exercise. ∎

Representation of If-Then as Or.

The truth table of $\sim p \vee q$ and $p \to q$ are identical, that is, they are both *false* only in the second case. Accordingly, $p \to q$ is logically equivalent to $\sim p \vee q$; that is,

$$p \to q \equiv \sim p \vee q$$

Example 1.18. Show that $\sim (p \to q)$ and $p \wedge \sim q$ are logically equivalent.

Solution. We could use a truth table to show that these compound propositions are equivalent. Indeed, it would not be hard to do so. However, we want to illustrate how to use logical identities that we already know to establish new logical identities, something that is of practical importance for establishing equivalences of compound propositions with a large number of variables. So, we will establish this equivalence by developing a series of logical equivalences, using one of the equivalences in Theorem 1.1

$\sim (p \to q)$	$\equiv$	$\sim (\sim p \vee q)$	by the conditional-disjunction equivalence
	$\equiv$	$\sim (\sim p) \wedge \sim q$	by the second De Morgan law
	$\equiv$	$p \wedge \sim q$	by the double negation law

∎

Example 1.19. Show that $(p \wedge q) \to (p \wedge q)$ is a tautology.

Solution. To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to **t**. (Note: This could be done using a truth table.)

$(p \wedge q) \to (p \vee q)$	$\equiv$	$\sim (p \wedge q) \vee (p \vee q)$	by the conditional-disjunction equivalence
	$\equiv$	$(\sim p \vee \sim q) \vee (p \vee q)$	by the first De Morgan law
	$\equiv$	$(\sim p \vee p) \vee (\sim q \vee q)$	by the associative and commutative laws for disjunc[tion]
	$\equiv$	$\mathbf{t} \vee \mathbf{t}$	by Example 1 and the commutative law for disjuncti[on]
	$\equiv$	$\mathbf{t}$	by the domination law

∎

The Contrapositive of a Conditional Statement

Definition 9: Contrapositive

The **contrapositive** of a conditional statement of the form "If p then q" is

$$\text{If } \sim q \text{ then } \sim q.$$

Symbolically,

$$\text{The contrapositive of } p \to q \text{ is } \sim q \to \sim p.$$

Example 1.20. Write each of the following statements in its equivalent contrapositive form:

a. If John can swim across the lake, then John can swim to the island.

b. If today is Sunday, then tomorrow is Monday.

Solution. a. If John cannot swim to the island, then John cannot swim across the lake.

b. If tomorrow is not Monday, then today is not Sunday. ∎

When you are trying to solve certain problems, you may find that the contrapositive form of a conditional statement is easier to work with than the original statement. Replacing a statement by its contrapositive may give the extra push that helps you over the top in your search for a solution. This logical equivalence is also the basis for the contrapositive method of proof.

Definition 10: Converse and Inverse

Suppose a conditional statement of the form "If p then q" is given.

1. The **converse** is "If q then p."
2. The **inverse** is "If $\sim p$ then $\sim q$."

Symbolically,

$$\text{The converse of } p \to q \text{ is } q \to p,$$

and

$$\text{The inverse of } p \to q \text{ is } \sim p \to \sim q.$$

Example 1.21. Write the converse and inverse of each of the following statements:

a. If John can swim across the lake, then John can swim to the island.

b. If today is Sunday, then tomorrow is Monday.

Solution. a. *Converse:* If Howard can swim to the island, then Howard can swim across the lake.
Inverse: If John cannot swim across the lake, then John cannot swim to the island.

b. *Converse:* If tomorrow is Monday, then today is Sunday.
Inverse: If today is not Sunday, then tomorrow is not Monday. ∎

1. A conditional statement and its converse are not logically equivalent.
2. A conditional statement and its inverse are not logically equivalent.
3. The converse and the inverse of a conditional statement are logically equivalent to each other.

Biconditional

Definition 11: Biconditional

Given statement variables p and q, the biconditional of p and q is "p if, and only if, q" and is denoted $p \leftrightarrow q$. It is true if both p and q have the same truth values and is false if p and q have opposite truth values. The words if and only if are sometimes abbreviated *iff*.

The biconditional has the following truth table:

Table 14: Truth Table for $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Order of Operations for Logical Operators

1. $\sim$ Evaluate negations first.
2. $\wedge, \vee$ Evaluate $\wedge$ and $\vee$ second. When both are present.
3. $\rightarrow, \leftrightarrow$ Evaluate $\rightarrow$ and $\leftrightarrow$ third. When both are present.

According to the separate definitions of *if* and only *if*, saying "p if, and only if, q" should mean the same as saying both "p if q" and "p only if q." The following annotated truth table shows that this is the case:

Example 1.22. Rewrite the following statement as a conjunction of two if-then statements:

> This computer program is correct if, and only if, it produces correct answers for all possible sets of input data.

Solution. If this program is correct, then it produces the correct answers for all possible sets of input data; and if this program produces the correct answers for all possible sets of input data, then it is correct. ■

1.7 Arguments

Definition 42: Arguments

An *argument* is an assertion that a given set of propositions $P_1, P_2, ..., P_n$, called *premises*, yields (has a consequence) another proposition Q, called the *conclusion*. Such an argument is denoted by

$$P_1, P_2, ..., P_n \vdash Q$$

An argument $P_1, P_2, ..., P_n \vdash Q$ is said to be valid if Q is true whenever all the premises $P_1, P_2, ..., P_n$ are true.
An argument which is not valid is called *fallacy*.

Example 1.23. Determine the validity of the following argument:

If Socrates is a man, then Socrates is mortal.
Socrates is a man.
$\therefore$ Socrates is mortal.

Solution. The argument has the abstract form

If p the q
p
$\therefore q$

Table 15: Truth Table for the argument

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 1.24. Determine the validity of the following argument:

$p \to q \vee \sim r$
$q \to p \wedge r$
$\therefore p \to r$

Solution. The truth table shows that even though there are several situations in which the premises and the conclusion are all true (rows 1, 7, and 8), there is one situation (row 4) where the premises are true and the conclusion is false.

p	q	r	$\sim r$	$q \vee \sim r$	$p \wedge r$	$p \to q \vee \sim r$	$q \to p \wedge r$	$p \to r$
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	
T	F	T	F	F	T	F	T	
T	F	F	T	T	F	T	T	T
F	T	T	F	T	F	T	F	
F	T	F	T	T	F	T	F	
F	F	T	F	F	F	T	T	T
F	F	F	T	T	F	T	T	T

Example 1.25. Determine the validity of the following argument:

If Zeus is human, then Zeus is mortal.
Zeus is not mortal.
$\therefore$ Zeus is not human.

Solution. Exercise

Example 1.26. Generalization: The following argument forms are valid:

(a)

p
$\therefore p \vee q$

(b)

q
$\therefore p \vee q$

Example 1.27. Specialization: The following argument forms are valid:

(a)

$p \wedge q$
$\therefore p$

(b)

$p \wedge q$
$\therefore q$

Example 1.28. Elimination: The following argument forms are valid:

(a)

$p \vee q$
$\sim q$
$\therefore p$

(b)

$p \vee q$
$\sim p$
$\therefore q$

Example 1.29. Transitivity: The following argument forms are valid:

$p \to q$
$q \to r$
$\therefore p \to r$

If n is divisible by 18, then n is divisible by 9.
If n is divisible by 9, then the sum of the digits of n is divisible by 9.
$\therefore$ If n is divisible by 18, then the sum of the digits of n is divisible by 9.

Example 1.30. Proof by Division into Cases: The following argument forms are valid:

$p \vee q$
$p \to r$
$q \to r$
$\therefore r$

x is positive or x is negative.
If x is positive, then $x^2 > 0$.
If x is negative, then $x^2 > 0$.
$\therefore x^2 > 0$

Example 1.31. Show that the following argument is invalid:

$p \rightarrow q$
q
$\therefore p$

If Zeke is a cheater, then Zeke sits in the back row.
Zeke sits in the back row.
$\therefore$ Zeke is a cheater.

Theorem 1.2. *The argument $P_1, P_2, \ldots, P_n \vdash Q$ is valid if and only if the proposition $(P_1 \wedge P_2 \ldots \wedge P_n) \rightarrow Q$ is a tautology.*

Example 1.32. A fundamental principle of logical reasoning states:

"If p implies q and q implies r, then p implies r"

Show that the above argument is valid

Solution. Construct the truth table for "If p implies q and q implies r, then p implies r" ■

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$p \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r)$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
[illegible]	[illegible]	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Example 1.33. Show that the following argument is a fallacy: $p \rightarrow q, \sim p \vdash \sim q$.

Solution. Construct the truth table for $[(p \rightarrow q) \wedge \sim p] \rightarrow \sim q$ as in Fig. below Since the proposition $[(p \rightarrow q) \wedge \sim p] \rightarrow \sim q$ is not a tautology, the argument is a fallacy. Equivalently, the argument is a fallacy since in the third line of the truth table $p \rightarrow q$ and $\sim p$ are true but $\sim q$ is false. ■

p	q	$p \rightarrow q$	$\sim p$	$(p \rightarrow q) \wedge \sim p$	$\sim q$	$[(p \rightarrow q) \wedge \sim p] \rightarrow \sim q$
T	T	T	F	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

Example 1.34. Determine the validity of the following argument: