CHAPTER 12

# DIFFERENTIAL CALCULUS 2 DIFFERENTIATION BY RULE

## Differentiation by Rule

Differentiation from first principles can become tedious and difficult. Fortunately, it is not always necessary to use first principles. There are a few rules (which can be derived from first principles) which enable us to write down the derivative of a function quite easily.

**Rule 1: General Rule**

If:

$$y = x^n \quad \text{then} \quad \frac{dy}{dx} = nx^{n-1}$$
$$y = ax^n \quad \text{then} \quad \frac{dy}{dx} = nax^{n-1}$$

In words:

Multiply by the power and reduce the power by 1.

*Example*

Differentiate with respect to $x$:

**(i)** $y = x^5$ **(ii)** $y = -3x^2$ **(iii)** $y = 5x$ **(iv)** $y = \dfrac{8}{x^2}$

**(v)** $y = 6\sqrt{x}$ **(vi)** $y = \dfrac{2}{\sqrt{x}}$ **(vii)** $y = \dfrac{6}{x^{1/3}}$ **(viii)** $y = 7$

**Solution:**

**(i)** $y = x^5$ $\qquad \dfrac{dy}{dx} = 5x^{5-1} = 5x^4$

**(ii)** $y = -3x^2$ $\qquad \dfrac{dy}{dx} = 2 \times -3x^{2-1} = -6x$

**(iii)** $y = 5x = 5x^1$ $\qquad \dfrac{dy}{dx} = 1 \times 5x^{1-1} = 5x^0 = 5 \qquad (x^0 = 1)$

**(iv)** $y = \dfrac{8}{x^2} = 8x^{-2}$ $\qquad \dfrac{dy}{dx} = -2 \times 8x^{-2-1} = -16x^{-3} = -\dfrac{16}{x^3}$

| | |
|---|---|
| (v) $y = 6\sqrt{x} = 6x^{1/2}$ | $\frac{dy}{dx} = \frac{1}{2} \times 6x^{1/2-1} = 3x^{-1/2} = \frac{3}{x^{1/2}} = \frac{3}{\sqrt{x}}$ |
| (vi) $y = \frac{2}{\sqrt{x}} = 2x^{-1/2}$ | $\frac{dy}{dx} = -\frac{1}{2} \times 2x^{-1/2-1} = -1x^{-3/2} = -\frac{1}{x^{3/2}}$ |
| (vii) $y = \frac{6}{x^{1/3}} = 6x^{-1/3}$ | $\frac{dy}{dx} = -\frac{1}{3} \times 6x^{-1/3-1} = -2x^{-4/3} = -\frac{2}{x^{4/3}}$ |
| (viii) $y = 7 = 7x^0$ | $\frac{dy}{dx} = 0 \times 7x^{0-1} = 0$ |

Part (viii) leads to the rule:

> The derivative of a constant = 0.

**Note:** The line $y = 7$ is a horizontal line. Its slope is 0.
Therefore its derivative (also its slope) equals 0.
In other words, the derivative of a constant always equals zero.

**Sum or Difference**
If the expression to be differentiated contains more than one term, just differentiate, separately, eac term in the expression.

*Example*

Find $f'(x)$ for each of the following:

(i) $f(x) = x + \frac{1}{x^2}$

(ii) $f(x) = \frac{2}{\sqrt{x}} - \frac{1}{x^4} + 5$

**Solution:**

(i) $f(x) = x + \frac{1}{x^2}$

$f(x) = x + x^{-2}$

$f'(x) = 1 - 2x^{-3}$

$f'(x) = 1 - \frac{2}{x^3}$

(ii) $f(x) = \frac{2}{\sqrt{x}} - \frac{1}{x^4} + 5$

$f(x) = 2x^{-1/2} - x^{-4} + 5$

$f'(x) = -x^{-3/2} + 4x^{-5}$

$f'(x) = -\frac{1}{x^{3/2}} + \frac{4}{x^5}$

# Evaluating Derivatives

Often we have to evaluate a derivative for a particular value.

**Example**

**(i)** If $s = 3t^2 + 5t - 7$, find the value of $\frac{ds}{dt}$ when $t = 2$.

**(ii)** If $f(x) = \sqrt{x} + 3x$, evaluate $f'(4)$.

**Solution:**

**(i)**

$$s = 3t^2 + 5t - 7$$

$$\frac{ds}{dt} = 6t + 5$$

$$\left.\frac{ds}{dt}\right|_{t=2} = 6(2) + 5$$

$$= 12 + 5 = 17$$

**(ii)**

$$f(x) = \sqrt{x} + 3x$$

$$f(x) = x^{1/2} + 3x$$

$$f'(x) = \tfrac{1}{2}x^{-1/2} + 3$$

$$= \frac{1}{2\sqrt{x}} + 3$$

$$f'(4) = \frac{1}{2\sqrt{4}} + 3$$

$$= \frac{1}{4} + 3 = 3\tfrac{1}{4}$$

$\frac{ds}{dt}$ is the derivative of $s$ with respect to $t$. $\frac{dA}{dr}$ is the derivative of $A$ with respect to $r$.

# Second Derivatives

The derivative of $\frac{dy}{dx}$, that is $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, is denoted by $\frac{d^2y}{dx^2}$ and is called the

**'second derivative of $y$ with respect to $x$'.**

$\frac{d^2y}{dx^2}$ is pronounced 'dee two $y$, dee $x$ squared'.

The derivative of $f'(x)$ is denoted by $f''(x)$ and is called the

**'second derivative of $f(x)$ with respect to $x$'.**

### *Example*

**(i)** If $f(x) = x + \frac{1}{x}$, find $f''(x)$ and $f''(2)$.

**(ii)** If $h = 10 + 30t^2 - 4t^3$, evaluate $\frac{d^2h}{dt^2}$ when $t = 3$.

**Solution:**

**(i)**

$$f(x) = x + \frac{1}{x}$$
$$f(x) = x + x^{-1}$$
$$f'(x) = 1 - x^{-2}$$
$$f''(x) = 2x^{-3}$$
$$= \frac{2}{x^3}$$
$$\therefore\ f''(2) = \frac{2}{2^3} = \frac{2}{8} = \frac{1}{4}$$

**(ii)**

$$h = 10 + 30t^2 - 4t^3$$
$$\frac{dh}{dt} = 60t - 12t^2$$
$$\frac{d^2h}{dt^2} = 60 - 24t$$
$$\left.\frac{d^2h}{dt^2}\right|_{t=3} = 60 - 24(3) = -12$$

### *Example*

If $y = x^4$, show that $\frac{4y}{3}\left(\frac{d^2y}{dx^2}\right) - \left(\frac{dy}{dx}\right)^2 = 0$.

**Solution:**

$$y = x^4$$
$$\frac{dy}{dx} = 4x^3$$
$$\frac{d^2y}{dx^2} = 12x^2$$

$$\frac{4y}{3}\left(\frac{d^2y}{dx^2}\right) - \left(\frac{dy}{dx}\right)^2$$
$$= \frac{4x^4}{3}(12x^2) - (4x^3)^2$$
$$= 16x^6 - 16x^6$$
$$= 0$$

**Note:** $\left(\frac{dy}{dx}\right)^2 \neq \frac{d^2y}{dx^2}$

### *Exercise 12.1*

Differentiate each of the following with respect to $x$:

**1.** $x^3$ **2.** $3x^4$ **3.** $-5x^2$ **4.** $3x$ **5.** $-2x$

**6.** $5$ **7.** $-3$ **8.** $\frac{1}{x^2}$ **9.** $\frac{2}{x^3}$ **10.** $-\frac{2}{x^5}$

**11.** $6x^{1/3}$ **12.** $\frac{1}{x}$ **13.** $\sqrt{x}$ **14.** $\frac{4}{\sqrt{x}}$ **15.** $\frac{1}{x^{2/3}}$

**16.** $x^3 - 5x$ **17.** $1 - x^2$ **18.** $x^2 - \frac{5}{x}$ **19.** $2x^2 - \frac{3}{x^4}$ **20.** $\frac{1}{x^2} + \frac{1}{x}$

**21.** $x^4 - \frac{2}{x^2}$ **22.** $6\sqrt{x} - \frac{2}{\sqrt{x}}$ **23.** $\frac{3}{x} + \frac{2}{x^2} + \frac{6}{x^{1/3}}$ **24.** $\frac{2}{x} - \frac{1}{\sqrt{x}} + \frac{3}{x^{1/3}}$

Find $\frac{d^2y}{dx^2}$ for each of the following:

**25.** $y = 4x^3 + 6x^2$ **26.** $y = x^2 - x^4$ **27.** $y = 6x^3 - 12x^2 - 8x + 4$

**28.** $y = \frac{1}{x}$ **29.** $y = x^2 - \frac{8}{x}$ **30.** $y = \sqrt{x}$

**31.** $y = \frac{1}{\sqrt{x}} + \sqrt{x}$ **32.** $y = 8\sqrt{x} - \frac{1}{x^2}$ **33.** $y = 9x^{1/3} + \frac{18}{x^{1/3}}$

**34.** If $f(x) = 3x^2 - 4x - 7$, evaluate **(i)** $f'(2)$ **(ii)** $f''(-1)$.

**35.** If $f(x) = -4\sqrt{x}$, evaluate $f''(9)$.

**36.** If $A = 3r^2 - 5r$, find the value of $\frac{dA}{dr}$ when $r = 3$.

**37.** If $s = 3t - 2t^2$, find the value of **(i)** $\frac{ds}{dt}$ **(ii)** $\frac{d^2s}{dt^2}$ when $t = 2$.

**38.** If $V = 3h - h^2 - 3h^3$, find $\frac{dV}{dh}$ when $h = 1$.

**39.** If $A = \pi r^2$, find $\frac{dA}{dr}$ when $\frac{r}{5} = 1$.

**40.** If $V = \frac{4}{3}\pi r^3$, find $\frac{dV}{dr}$ when $2r - 5 = 0$.

**41.** $f(x) = 3x^2 - 4x$. If $f'(k) = 8$, find the value of $k$, $k \in \mathbf{R}$.

**42.** $f(x) = x^3 + 1$. If $f''(a) = 18$, find the value of $a$, $a \in \mathbf{R}$.

**43.** If $y = 3x^2 + 2x$, show that $y\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - 6x = 0$.

**44.** If $y = 4x^3 - 6x^2$, show that $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 12x = 0$.

Find the values of $x$ for which **(i)** $\frac{dy}{dx} = 0$ **(ii)** $\frac{d^2y}{dx^2} = 0$.

**45.** If $y = \frac{1}{x^2}$, show that $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 - 10y^3 = 0$.

# Product, Quotient and Chain Rules

**Rule 2: Product Rule**

Suppose $u$ and $v$ are functions of $x$.

If $\quad y = uv,$

then $\quad \dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}.$

In words:

First by the derivative of the second + second by the derivative of the first.

*Example*

If $y = (x^2 - 3x + 2)(x^2 - 2)$, find $\dfrac{dy}{dx}$.

**Solution:**

Let $\quad u = x^2 - 3x + 2 \quad$ and $\quad$ let $v = x^2 - 2$

$\dfrac{du}{dx} = 2x - 3 \quad$ and $\quad \dfrac{dv}{dx} = 2x$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \text{(product rule)}$$

$$= (x^2 - 3x + 2)(2x) + (x^2 - 2)(2x - 3)$$

$$= 2x^3 - 6x^2 + 4x + 2x^3 - 3x^2 - 4x + 6$$

$$= 4x^3 - 9x^2 + 6$$

**Rule 3: Quotient Rule**

Suppose $u$ and $v$ are functions of $x$.

If $\quad y = \dfrac{u}{v}$

then $\quad \dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$

In words:

$$\frac{\text{Bottom by the derivative of the top} - \text{Top by the derivative of the bottom}}{(\text{Bottom})^2}$$

*Example*

If $y=\dfrac{x^2}{x-2}$, find $\dfrac{dy}{dx}$.

**Solution:**

$$\text{Let } u=x^2 \qquad \text{and} \qquad \text{let} \qquad v=x-2$$

$$\frac{du}{dx}=2x \qquad \text{and} \qquad \frac{dv}{dx}=1$$

$$\frac{dy}{dx}=\frac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2} \qquad \text{(quotient rule)}$$

$$=\frac{(x-2)(2x)-(x^2)(1)}{(x-2)^2}$$

$$=\frac{2x^2-4x-x^2}{(x-2)^2}$$

$$=\frac{x^2-4x}{(x-2)^2}$$

**Note:** It is usual practice to simplify the top but **not** the bottom.

**Function of a function**

When we write, for example, $y=(x+5)^3$, we say that $y$ is a function of $x$.
If we let $u=(x+5)$, then $y=u^3$, where $u=(x+5)$.
We say that $y$ is a function $u$, and $u$ is a function of $x$.
The new variable, $u$, is the **link** between the two expressions.

**Rule 4: Chain Rule**

Suppose $u$ is a function of $x$.

If $\quad y=u^n$

then $\quad \dfrac{dy}{dx}=nu^{n-1}\dfrac{du}{dx}$.

The chain rule should be done in **one** step.

## Example

Find $\frac{dy}{dx}$ for each of the following:

**(i)** $y = (x^2 - 3x)^4$ **(ii)** $y = \frac{3}{2x+5}$

**(iii)** $y = \sqrt{4x-3}$ **(iv)** $y = \left(x^2 + \frac{1}{x}\right)^3$

**Solution:**

**(i)**
$$y = (x^2 - 3x)^4$$
$$\frac{dy}{dx} = 4(x^2 - 3x)^3(2x - 3)$$
$$= (8x - 12)(x^2 - 3x)^3$$

**(ii)**
$$y = \frac{3}{2x+5}$$
$$y = 3(2x+5)^{-1}$$
$$\frac{dy}{dx} = -3(2x+5)^{-2}(2)$$
$$= \frac{-6}{(2x+5)^2}$$

**(iii)**
$$y = \sqrt{4x-3}$$
$$y = (4x-3)^{1/2}$$
$$\frac{dy}{dx} = \tfrac{1}{2}(4x-3)^{-1/2}(4)$$
$$= \frac{2}{(4x-3)^{1/2}} = \frac{2}{\sqrt{4x-3}}$$

**(iv)**
$$y = \left(x^2 + \frac{1}{x}\right)^3$$
$$y = (x^2 + x^{-1})^3$$
$$\frac{dy}{dx} = 3(x^2 + x^{-1})^2(2x - x^{-2})$$
$$= 3\left(x^2 + \frac{1}{x}\right)^2\left(2x - \frac{1}{x^2}\right)$$

Often we have to deal with a combination of the product, quotient or chain rules.

## Example

Find $\frac{dy}{dx}$ if **(i)** $y = x\sqrt{9-x^2}$ **(ii)** $y = \sqrt{\frac{1-x}{1+x}}$

**Solution:**

**(i)**
$$y = x\sqrt{9-x^2}$$
$$y = x(9-x^2)^{1/2}$$
$$\frac{dy}{dx} = (x)\,.\underbrace{\tfrac{1}{2}(9-x^2)^{-1/2}(-2x)}_{\text{(chain rule here)}} + (9-x^2)^{1/2}(1) \qquad \text{(product rule and chain rule)}$$
$$= -x^2(9-x^2)^{-1/2} + (9-x^2)^{1/2}$$
$$= \frac{-x^2}{\sqrt{9-x^2}} + \sqrt{9-x^2}$$

**(ii)** $y = \sqrt{\dfrac{1-x}{1+x}}$

$$y = \left(\frac{1-x}{1+x}\right)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2}\left[\frac{(1+x)(1)-(1-x)(1)}{(1+x)^2}\right] \qquad \left(\begin{array}{c}\text{chain rule followed by} \\ \text{the quotient rule}\end{array}\right)$$

$$= \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{1/2}\left[\frac{-1-x-1+x}{(1+x)^2}\right] \qquad \left(\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}\right)$$

$$= \frac{(1+x)^{1/2}}{2(1-x)^{1/2}} \cdot \frac{-2}{(1+x)^2}$$

$$= \frac{-1}{(1-x)^{1/2}(1+x)^{3/2}}$$

## *Exercise 12.2*

In questions 1 to 6, use the product rule to find $\dfrac{dy}{dx}$:

**1.** $y = (2x+3)(x-4)$ **2.** $y = (x+5)(x^2-3x+2)$ **3.** $y = (3x-4)(x^2-2x+3)$

**4.** $y = (x+3)(x^2-6x+8)$ **5.** $y = (5x^2-3x)(x^2-5x)$ **6.** $y = (3x^3-2x^2+4)(2x-1)$

In questions 7 to 12, use the quotient rule to find $\dfrac{dy}{dx}$:

**7.** $y = \dfrac{3x+2}{x+1}$ **8.** $y = \dfrac{2x-1}{x+3}$ **9.** $y = \dfrac{3x-1}{x^2-2}$

**10.** $y = \dfrac{x^2-1}{x^2+1}$ **11.** $y = \dfrac{1-x}{2x-x^2}$ **12.** $y = \dfrac{x^2-x-6}{x^2+x-6}$

In questions 13–18, use the chain rule to find $\dfrac{dy}{dx}$:

**13.** $y = (3x+2)^4$ **14.** $y = (x^2+2x)^3$ **15.** $y = (2x^2+1)^5$

**16.** $y = \sqrt{4x+2}$ **17.** $y = \dfrac{1}{2x-5}$ **18.** $y = \dfrac{1}{\sqrt{2x^2-4x}}$

Find $\dfrac{dy}{dx}$ if:

**19.** $y = x^2(x+3)^4$ **20.** $y = 3x(x+2)^3$ **21.** $y = 3x^2(2x+3)^2$

**22.** $y = x^2\sqrt{2x+1}$ **23.** $y = x\sqrt{1+x^2}$ **24.** $y = \sqrt{\dfrac{x+1}{x}}$

**25.** If $f(x) = \sqrt{\dfrac{x}{x+3}}$, find the value of $f'(1)$. **26.** If $f(x) = \sqrt{\dfrac{x-1}{x+1}}$, find the value of $f'(\frac{5}{4})$.

# Differentiation of Trigonometric Functions

The rules for differentiating also apply to trigonometric functions.
The following are in the tables on page 41, but they are shown only for $x$.
The chain rule is used throughout, assuming $u$ is a function of $x$.
Therefore, if you are using the tables, replace $x$ with $u$ and **always** multiply by $\frac{du}{dx}$.

| **Basic rule** (page 41 tables) | |
|---|---|
| $f(x)$ | $f'(x)$ |
| $\cos x$ | $-\sin x$ |
| $\sin x$ | $\cos x$ |
| $\tan x$ | $\sec^2 x$ |
| $\sec x$ | $\sec x \tan x$ |
| $\text{cosec } x$ | $-\text{cosec } x \cot x$ |
| $\cot x$ | $-\text{cosec}^2 x$ |

| **Chain rule** | |
|---|---|
| $f(u)$ | $f'(u) \cdot \frac{du}{dx}$ |
| $\cos u$ | $-\sin u \cdot \frac{du}{dx}$ |
| $\sin u$ | $\cos u \cdot \frac{du}{dx}$ |
| $\tan u$ | $\sec^2 u \cdot \frac{du}{dx}$ |
| $\sec u$ | $\sec u \tan u \cdot \frac{du}{dx}$ |
| $\text{cosec } u$ | $-\text{cosec } u \cot u \cdot \frac{du}{dx}$ |
| $\cot u$ | $-\text{cosec}^2 u \cdot \frac{du}{dx}$ |

### *Example*

Find the derivatives of the functions:

**(i)** $\cos 3x$ **(ii)** $\tan^3 5x$ **(iii)** $x \sin x$ **(iv)** $\sqrt{\cos x}$

**Solution:**

**(i)**
$$y = \cos 3x$$
$$\frac{dy}{dx} = (-\sin 3x)(3)$$
$$= -3 \sin 3x$$

**(ii)**
$$t = \tan^3 5x$$
$$y = (\tan 5x)^3$$
$$\frac{dy}{dx} = 3(\tan 5x)^2(\sec^2 5x)(5)$$
[PTA: (power) (trig. function) (angle)]
$$= 15 \tan^2 5x \sec^2 5x$$

**(iii)**
$$y = x \sin x$$
(use the product rule)
$$\frac{dy}{dx} = (x)(\cos x) + (\sin x)(1)$$
$$= x \cos x + \sin x$$

**(iv)**
$$y = \sqrt{\cos x}$$
$$y = (\cos x)^{1/2}$$
(chain rule)
$$\frac{dy}{dx} = \tfrac{1}{2}(\cos x)^{-1/2}(-\sin x)$$
$$= \frac{-\sin x}{2\sqrt{\cos x}}$$

## *Example*

If $f(x) = \dfrac{x^2}{x + \cos x}$, evaluate $f'\left(\dfrac{\pi}{2}\right)$.

**Solution:**

$$f(x) = \frac{x^2}{x + \cos x}$$

$$f'(x) = \frac{(x + \cos x)(2x) - x^2(1 - \sin x)}{(x + \cos x)^2} \qquad \text{(quotient rule)}$$

$$f'\left(\frac{\pi}{2}\right) = \frac{\left(\frac{\pi}{2} + \cos\frac{\pi}{2}\right)(\pi) - \left(\frac{\pi}{2}\right)^2\left(1 - \sin\frac{\pi}{2}\right)}{\left(\frac{\pi}{2} + \cos\frac{\pi}{2}\right)^2} \qquad \left(\begin{array}{l}\text{Don't simplify:}\\ \text{put in } x = \frac{\pi}{2}\end{array}\right)$$

$$= \frac{\left(\frac{\pi}{2} + 0\right)(\pi) - \left(\frac{\pi^2}{4}\right)(1 - 1)}{\left(\frac{\pi}{2} + 0\right)^2} \qquad \left(\cos\frac{\pi}{2} = 0,\ \sin\frac{\pi}{2} = 1\right)$$

$$= \frac{\left(\frac{\pi}{2}\right)(\pi) - \left(\frac{\pi^2}{4}\right)(0)}{\left(\frac{\pi}{2}\right)^2} \quad = \frac{\frac{\pi^2}{2}}{\frac{\pi^2}{4}} \quad = \frac{2\pi^2}{\pi^2} \quad = 2$$

## *Exercise 12.3*

Find $\dfrac{dy}{dx}$ if:

1. $y = \sin 4x$
2. $y = \cos 3x$
3. $y = \tan 2x$
4. $y = \sec 5x$
5. $y = -\text{cosec } 6x$
6. $y = -2 \cot 4x$
7. $y = \sin(2x - 3)$
8. $y = \tan(3x + 2)$
9. $y = 2 \tan x + \sec x$
10. $y = x^2 \sin x$
11. $y = 3x \tan x$
12. $y = x^2 \cos 2x$
13. $y = \dfrac{\sin x}{x}$
14. $y = \dfrac{1}{1 - \sin x}$
15. $y = \dfrac{1 + \sin x}{\cos x}$
16. $y = \cos^3 x$
17. $y = \sin^2 4x$
18. $y = \tan^4 3x$
19. $y = (1 + \sin^2 x)^3$
20. $y = \sqrt{\sin x}$
21. $y = \sqrt{\cos 2x}$
22. $f(x) = \dfrac{\cos x + \sin x}{\cos x - \sin x}$. Show that $f'(x) = \dfrac{2}{1 - \sin 2x}$.

23. If $y = \cos 3x$, show that $\frac{d^2y}{dx^2} = -9y$.

24. If $y = 3\cos x + \sin x$, show that:

(i) $\cos x\left(\frac{dy}{dx}\right) + y\sin x - 1 = 0$ (ii) $\frac{d^2y}{dx^2} - 3\left(\frac{dy}{dx}\right) + 2y - 10\sin x = 0$.

25. If $f(x) = \sin x \cos x$, evaluate $f'\left(\frac{\pi}{4}\right)$.

26. If $y = \cos 2x + 2\sin x$, evaluate $\frac{dy}{dx}$ at $x = \frac{\pi}{6}$.

27. $f(x) = \frac{\sin x}{1 + \tan x}$. Evaluate $f'(0)$.

## Implicit Differentiation

If $y = f(x)$, the variable $y$ is given **explicitly** (clearly) in terms of $x$.
For example, $y = x^3 - 2x^2 + 5x - 4$ is an explicit function.
Some curves are defined by implicit functions, that is, functions which cannot be expressed in the forn $y = f(x)$.
For example, $x^2 + xy + y^3 = 7$ is an **implicit function.**
It cannot be written in the form $y = f(x)$.
It is for this reason that we must have a method for differentiating explicit functions.
An implicit function involving $x$ and $y$ can be differentiated with respect to $x$ as it stands, using th chain rule.

**Method for differentiating implicit functions:**

1. Differentiate, term by term, on both sides with respect to $x$.
2. Bring all terms with $\frac{dy}{dx}$ to the left and bring all other terms to the right.
3. Make $\frac{dy}{dx}$ the subject of the equation.

It is useful to remember that, by the chain rule,

$$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx} \quad \text{and} \quad \frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$$

as $y$ is considered as a function of $x$.

$$\boxed{\frac{d}{dx}(y^n) = ny^{n-1}\left(\frac{dy}{dx}\right)}$$

### *Example*

Given that $2x^3 + 3xy^2 - y^3 + 6 = 0$, evaluate $\dfrac{dy}{dx}$ at the point $(-1, 1)$.

**Solution:**

(use product rule here)

$$2x^3 + 3xy^2 - y^3 + 6 = 0$$

$$6x^2 + 3\left[x \,.\, 2y\frac{dy}{dx} + y^2(1)\right] - 3y^2\frac{dy}{dx} = 0$$

$$6x^2 + 6xy\frac{dy}{dx} + 3y^2 - 3y^2\frac{dy}{dx} = 0$$

(divide each term by $-3$)

$$6xy\frac{dy}{dx} - 3y^2\frac{dy}{dx} = -6x^2 - 3y^2$$

$$\frac{dy}{dx}(6xy - 3y^2) = -6x^2 - 3y^2$$

$$\frac{dy}{dx} = \frac{-6x^2 - 3y^2}{6xy - 3y^2} = \frac{2x^2 + y^2}{y^2 - 2xy} = \frac{2x^2 + y^2}{y(y - 2x)}$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=-1\\y=1}} = \frac{2(-1)^2 + (1)^2}{1(1 + 2)} = \frac{3}{3} = 1$$

Note: To evaluate $\dfrac{dy}{dx}$ we used both coordinates of the point.

### *Exercise 12.4*

For each of the following curves, express $\dfrac{dy}{dx}$ in terms of $x$ and $y$:

**1.** $x^2 + y^2 = 4$

**2.** $x^2 + 2y - y^2 = 5$

**3.** $x^2 - 6y^3 + y = 0$

**4.** $x^2 + y^2 - 4x - 6y + 9 = 0$

**5.** $x^2 + xy + y^2 = 13$

**6.** $x^2 + 3xy + 2y^2 = 6$

**7.** $x^2y - 5x = 2$

**8.** $xy^2 + x^2 = 2$

**9.** $x^2y + xy^2 = 2$

Find the value of $\dfrac{dy}{dx}$ at the point specified:

**10.** $x^2 + y^2 = 25$ at the point $(3, -4)$

**11.** $x^2 + xy + 2y^2 = 28$ at the point $(2, -4)$

**12.** $x^2 + 4xy - 2y^2 - 8 = 0$ at the point $(0, 2)$

**13.** $x^3 + y^2 + 3x^2y = 21$ at the point $(2, 1)$

**14.** Find the slope of the tangent to the curve $y^2 + 3xy + 2x^2 = 6$ at the point $(1, 1)$.

**15.** Find the slope of the tangent to the curve $x \sin y + y^2 = 1 + \dfrac{\pi^2}{4}$ at the point $\left(1, \dfrac{\pi}{2}\right)$.

Note: $\dfrac{d}{dx}(\sin y) = \cos y\dfrac{dy}{dx}$.

# Parametric Differentiation

If $x$ and $y$ are each expressed in terms of a third variable, $t$ say (or $\theta$), called the **parameter**, then $x = f(t)$ and $y = g(t)$ give the parametric forms of the equation relating to $x$ and $y$ respectively.

**To find $\frac{dy}{dx}$ do the following:**

| | |
|---|---|
| **1.** Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$, separately. | **2.** Use $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$. |

## *Example*

**(i)** If $x = 4t^3$ and $y = (1 + 3t^2)^2$, express $\frac{dy}{dx}$ in terms of $t$.

Hence, or otherwise, evaluate $\frac{dy}{dx}$ when $t = -1$.

**(ii)** Let $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$; show that $\frac{dy}{dx} = \tan\theta$.

$$\left[a \neq 0, -\pi < \theta < \pi \quad \text{and} \quad \theta \neq \pm\frac{\pi}{2}\right]$$

**Solution:**

$$x = 4t^3 \qquad\qquad y = (1 + 3t^2)^2$$

$$\frac{dx}{dt} = 12t^2 \qquad\qquad \frac{dy}{dt} = \underbrace{2(1 + 3t^2)^1(6t)}_{\text{(chain rule)}} = 12t(1 + 3t^2)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12t(1 + 3t^2)}{12t^2} = \frac{1 + 3t^2}{t}$$

$$\left.\frac{dy}{dx}\right|_{t=-1} = \frac{1 + 3(-1)^2}{-1} = \frac{1 + 3}{-1} = \frac{4}{-1} = -4$$

hen $x = f(t)$

**(ii)** $x = a(\cos\theta + \theta\sin\theta)$

$$\frac{dx}{d\theta} = a(-\sin\theta + \theta.\cos\theta + \sin\theta.1)$$
$$= a(-\sin\theta + \theta\cos\theta + \sin\theta)$$
$$= a\theta\cos\theta$$

$y = a(\sin\theta - \theta\cos\theta)$

$$\frac{dy}{d\theta} = a[\cos\theta - (\theta.-\sin\theta + \cos\theta.1)]$$
$$= a(\cos\theta + \theta\sin\theta - \cos\theta)$$
$$= a\theta\sin\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

## Exercise 12.5

Find $\frac{dy}{dx}$, in terms of $t$, if:

1. $x = 2t,\quad y = t^2$
2. $x = 2t + 3,\quad y = 2t^3$
3. $x = t^2 + 1,\quad y = t^3$
4. $x = 3t^4,\quad y = 2t^2 + 5$
5. $x = t(1 - t),\quad y = t(1 - t^2)$
6. $x = 6t + 5,\quad y = (2t - 1)^3$
7. $x = \frac{1}{t},\quad y = t^2 + 4t$
8. $x = 2\sqrt{t},\quad y = 5t + 4$
9. $x = 1 + \frac{1}{t},\quad y = t + \frac{1}{t}$
10. $x = \frac{t^2}{1 + t^3},\quad y = \frac{t^3}{1 + t^3}$
11. $x = \frac{t - 2}{t + 1}$ and $y = \frac{t + 2}{t + 1}$. If $\frac{dy}{dx} = k$, find the value of $k$.
12. If $x = \frac{2}{t}$ and $y = 3t^2 - 1$, express $\frac{dy}{dx}$ in terms of $t$. Evaluate $\frac{dy}{dx}$ at the point (2, 2).
13. If $x = \frac{3t - 1}{t}$ and $y = \frac{t^2 + 4}{t}$, express $\frac{dy}{dx}$ in terms of $t$.

    Find the values of $t$ for which $\frac{dy}{dx} = 0$.
14. If $x = 2t + \sin 2t$ and $y = \cos 2t$, show that $\frac{dy}{dx} = -\tan t$.
15. If $x = \sec\theta$ and $y = \tan\theta$, show that $\frac{dy}{dx} = \operatorname{cosec}\theta$.
16. If $x = k(\theta - \sin\theta)$ and $y = k(1 - \cos\theta)$, $k \in \mathbf{R}$, find $\frac{dy}{dx}$.
17. Given $y = \sin\theta\cos\theta - \theta$ and $x = 2\cos\theta$, show that **(i)** $\frac{dy}{d\theta} = -2\sin^2\theta$ **(ii)** $\frac{dy}{dx} = \sin\theta$.
18. If $x = 3\cos\theta - 4\sin\theta$ and $y = 4\cos\theta + 3\sin\theta$, evaluate $\frac{dy}{dx}$ at $\theta = \frac{\pi}{2}$.

**19.** If $x = \sin\theta$ and $y = \sin n\theta$, where $n \in \mathbf{R}$, show that $(1 - x^2)\left(\frac{dy}{dx}\right)^2 - n^2(1 - y^2) = 0$.

**20.** $x = k(1 + \cos\theta)$, $\quad y = 2k\sin^2\theta$, where $0 \leqslant \theta \leqslant \pi$ and $k$ is a positive constant.

**(i)** Find $\frac{dy}{dx}$ in the form $p\cos\theta$ where $p \in \mathbf{Z}$.

**(ii)** Find, in terms of $k$, the coordinates of the point $q$ where $\theta = \tan^{-1}\frac{\sqrt{7}}{3}$.

# Differentiation of Inverse Trigonometric Functions

The rules for differentiating also apply to inverse trigonometric functions.
The following are in the tables on page 41, but they are shown only for $x$.
The chain rule is used throughout, assuming $u$ is a function of $x$.

Replace $a$ with 1, $x$ with $u$, and always multiply by $\frac{du}{dx}$.

| **Basic rule** (page 41 tables) | |
|---|---|
| $f(x)$ | $f'(x)$ |
| $\sin^{-1}\frac{x}{a}$ | $\frac{1}{\sqrt{a^2 - x^2}}$ |
| $\tan^{-1}\frac{x}{a}$ | $\frac{a}{a^2 + x^2}$ |

| **Chain rule** | |
|---|---|
| $f(u)$ | $f'(u)\cdot\frac{du}{dx}$ |
| $\sin^{-1}u$ | $\frac{1}{\sqrt{1 - u^2}}\cdot\frac{du}{dx}$ |
| $\tan^{-1}u$ | $\frac{1}{1 + u^2}\cdot\frac{du}{dx}$ |

**Note:** The derivative of $\cos^{-1}u$ is **not** in the syllabus.

*Example*

**(i)** If $y = \tan^{-1}\left(\frac{x}{1+x}\right)$, show that $\frac{dy}{dx} = \frac{1}{2x^2 + 2x + 1}$, $\quad x \neq -1$.

**(ii)** Given $y = \sin^{-1}(3x - 1)$, calculate the value of $\frac{dy}{dx}$ at $x = \frac{1}{3}$.

**Solution:**

$$y = \tan^{-1}\left(\frac{x}{1+x}\right)$$

(quotient rule)

$$\frac{dy}{dx} = \frac{1}{1 + \left(\frac{x}{1+x}\right)^2}\cdot\left(\frac{(1+x)(-1) - (x)(1)}{(1+x)^2}\right)$$

$$\boxed{\begin{aligned} y &= \tan^{-1}u \\ \frac{dy}{dx} &= \frac{1}{1+u^2}\cdot\frac{du}{dx}\end{aligned}}$$

$$= \frac{1}{1+\frac{x^2}{(1+x)^2}} \cdot \left(\frac{1+x-x}{(1+x)^2}\right)$$

$$= \frac{(1+x)^2}{(1+x)^2+x^2} \cdot \frac{1}{(1+x)^2}$$ (multiply the top and bottom of the first fraction by $(1+x)^2$)

$$= \frac{1}{(1+x)^2+x^2}$$

$$= \frac{1}{1+2x+x^2+x^2}$$

$$= \frac{1}{2x^2+2x+1}$$

**(ii)** $y = \sin^{-1}(3x-1)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(3x-1)^2}} \cdot (3)$$

$$= \frac{3}{\sqrt{1-(3x-1)^2}}$$

$$\boxed{y = \sin^{-1} u \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}}$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{1}{3}} = \frac{3}{\sqrt{1-(0)^2}} = \frac{3}{\sqrt{1}} + \frac{3}{1} = 3$$

## Exercise 12.6

Find $\frac{dy}{dx}$ for each of the following:

1. $y = \sin^{-1} 2x$
2. $y = \tan^{-1} 3x$
3. $y = \sin^{-1}(x-1)$
4. $y = \tan^{-1}(2x+1)$
5. $y = \tan^{-1} x^2$
6. $y = \sin^{-1} 2x^3$
7. $y = (\sin^{-1} 5x)^2$
8. $y = \tan^{-1}\left(\frac{x}{3}\right)$
9. $y = \sin^{-1}\left(\frac{x}{2}\right)$
10. $y = \sin^{-1}(\cos x)$
11. $y = x \sin^{-1} x$
12. $y = 6x \tan^{-1} 2x$
13. Given $y = \sin^{-1}(4x-1)$, calculate the value of $\frac{dy}{dx}$ at $x = \frac{1}{4}$.
14. Given $y = \tan^{-1}\left(\frac{1}{x}\right)$, show that $\frac{dy}{dx} = -\frac{1}{1+x^2}$.
15. Given $y = \tan^{-1}(\cos x)$, calculate the value of $\frac{dy}{dx}$ at $x = \frac{\pi}{6}$.
16. If $y = \tan^{-1}\left(\frac{x}{a}\right)$, show that $\frac{dy}{dx} = \frac{a}{a^2+x^2}$.
17. Explain why $p\sqrt{1-q} = \sqrt{p^2-p^2q}$, $\quad p, q \in \mathbf{R}$.

    If $y = \sin^{-1}\left(\frac{x}{a}\right)$, show that $\frac{dy}{dx} = \frac{1}{\sqrt{a^2-x^2}}$.

18. $f(x) = \frac{1}{x}\sin^{-1}\left(\frac{1}{x}\right)$. Show that $f'(\sqrt{2}) = -\frac{1}{2} - \frac{\pi}{8}$.

19. If $y = \tan^{-1} x$, show that $\frac{d^2y}{dx^2}(1 + x^2) + 2x\frac{dy}{dx} = 0$.

20. If $u = \frac{1+x}{1-x}$, show that $\frac{du}{dx} = \frac{2}{(1-x)^2}$.

    Hence, if $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$, find $\frac{dy}{dx}$.

    Verify that $\quad 2x\left(\frac{dy}{dx}\right)^2 + \frac{d^2y}{dx^2} = 0.$

21. Explain why $\sqrt{a} = \frac{a}{\sqrt{a}}$, $a \in \mathbf{R}$, $a \neq 0$.

    Given $y = \sin^{-1} x + x\sqrt{1 - x^2}$, show that $\quad \frac{dy}{dx} = 2\sqrt{1 - x^2}$.

## Differentiation of Exponential Functions

The rules for differentiating apply also to exponential functions.
**Exponent** is another word for index. A function such as $y = 2^x$, in which the variable occurs as an index, is called 'an exponential function'.
The function $y = e^x$ is called '**the exponential function**' or '**natural exponential function**'.
$e$ is an irrational constant whose value is 2.71828 correct to six significant figures.
$e^x$ is the only basic function which is its own derivative. That is:

$$\boxed{\text{If } y = e^x, \quad \frac{dy}{dx} = e^x.}$$

**Note:** The positive number $e$ behaves just like other positive numbers such as 2 or 5.
$e^x$ obeys all the usual laws of indices or exponents.

**Using the chain rule:**

| |
|---|
| Suppose $u$ is a function of $x$. |
| If $\quad y = e^u$ |
| then $\quad \frac{dy}{dx} = e^u \cdot \frac{du}{dx}$. |

### *Example*

Find $\dfrac{dy}{dx}$ if **(i)** $y = e^{x^2-3x}$ **(ii)** $y = \dfrac{2}{e^{3x}}$ **(iii)** $y = e^{\sin 2x}$ **(iv)** $y = \dfrac{x}{e^{2x}}$

**Solution:**

**(i)**
$$\begin{aligned} y &= e^{x^2-3x} \\ \frac{dy}{dx} &= e^{x^2-3x}(2x-3) \\ &= (2x-3)e^{x^2-3x} \end{aligned}$$

**(ii)**
$$\begin{aligned} y &= \frac{2}{e^{3x}} = 2e^{-3x} \\ \frac{dy}{dx} &= 2e^{-3x}(-3) \\ &= -6e^{-3x} = -\frac{6}{e^{3x}} \end{aligned}$$

**(iii)**
$$\begin{aligned} y &= e^{\sin 2x} \\ \frac{dy}{dx} &= e^{\sin 2x}(\cos 2x)(2) \\ &= (2\cos 2x)e^{\sin 2x} \end{aligned}$$

**(iv)**
$$y = \frac{x}{e^{2x}} = xe^{-2x}$$
(use the product rule)
$$\begin{aligned} \frac{dy}{dx} &= xe^{-2x}(-2) + e^{-2x}(1) \\ &= -2xe^{-2x} + e^{-2x} \\ &= e^{-2x}(1-2x) = \frac{1-2x}{e^{2x}} \end{aligned}$$

**Note:** The quotient rule could also be used.

### *Example*

If $y = xe^{-x}$, show that $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$.

**Solution:**

$$\begin{aligned} y &= xe^{-x} \\ \frac{dy}{dx} &= x[e^{-x}(-1)] + e^{-x}(1) \qquad \text{(product rule)} \\ &= -xe^{-x} + e^{-x} \\ &= e^{-x}(1-x) \end{aligned}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= e^{-x}(-1) + (1-x)[e^{-x}(-1)] \qquad \text{(product rule, again)} \\ &= -e^{-x} + (1-x)(-e^{-x}) \\ &= -e^{-x} - e^{-x} + xe^{-x} \\ &= xe^{-x} - 2e^{-x} \\ &= e^{-x}(x-2) \end{aligned}$$

$$\begin{aligned} &\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y \\ &= e^{-x}(x-2) + 2[e^{-x}(1-x)] + (xe^{-x}) \\ &= xe^{-x} - 2e^{-x} + 2e^{-x} - 2xe^{-x} + xe^{-x} \\ &= e^{-x}(x - 2 + 2 - 2x + x) = e^{-x}(0) = 0. \end{aligned}$$

## Exercise 12.7

Find $\frac{dy}{dx}$ for each of the following:

1. $y = e^{4x}$
2. $y = 2e^{3x}$
3. $y = e^{x^2}$
4. $y = e^{x^2-5x}$
5. $y = e^{4x^2}$
6. $y = e^{-x}$
7. $y = \frac{5}{e^{2x}}$
8. $y = \frac{2}{e^{x^2}}$
9. $y = e^{\sin x}$
10. $y = e^{\cos 2x}$
11. $y = e^{4 \tan x}$
12. $y = e^{x \sin x}$
13. $y = xe^x$
14. $y = x^2 e^{5x}$
15. $y = e^{2x} \cos x$
16. $y = e^{-x^2} \sin x$
17. $y = \frac{x^2}{e^{2x}}$
18. $y = (3 + e^{x^2})^4$
19. $y = \frac{1}{3 - e^{2x^2}}$
20. $y = \sqrt{1 - 2e^{4x}}$
21. If $f(x) = \frac{1 + e^x}{1 - e^x}$, show that $f'(x) = \frac{2e^x}{(1 - e^x)^2}$.
22. If $f(\theta) = e^{1 + \sin \theta}$, evaluate **(i)** $f'(0)$ **(ii)** $f''($
23. If $y = e^{2x}$, show that $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$.
24. If $y = xe^{-2x}$, show that $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$.
25. If $y = e^x \sin x$, show that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$.
26. Given that $y = x + \sin^{-1} x$, show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + x = 0$.
27. If $y = e^{kx}$, find the values of $k \in \mathbf{R}$ for which $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$.
28. If $y = e^{2t}$ and $x = e^t$, show that $\frac{dy}{dx} = 2e^t$.
29. If $y = te^t$ and $x = t^2 e^t$, show that $\frac{dy}{dx} = \frac{t+1}{t(t+2)}$.
30. Given $y = e^\theta \cos \theta$ and $x = e^\theta \sin \theta$, where $-\frac{3\pi}{4} < \theta < \frac{\pi}{4}$, show that $\left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2 = 2e^{2\theta}$.

    Evaluate $\frac{dy}{dx}$ at $\theta = \frac{\pi}{2}$.
31. If $y = e^{-nx} \cos kx$, $n, k \in \mathbf{R}$, show that $\frac{d^2y}{dx^2} + 2n\frac{dy}{dx} + (n^2 + k^2)y = 0$.

# Differentiation of Natural Logarithmic Functions

Logarithms to the base $e$ are called '**natural logarithms**'.
The notation $\ln x$ is used as an abbreviation of $\log_e x$.
The function $y = \ln x$ is the inverse function of $y = e^x$
(exponents and logs are inverse functions of each other).

**Note:** $\log_e x$ or $\ln x$ is defined only for $x > 0$.

Natural logarithms obey the same laws as logarithms to any other base.

| **Laws of Logs:** | $\ln ab = \ln a + \ln b$ | $\ln \dfrac{a}{b} = \ln a - \ln b$ | $\ln a^n = n \ln a$ |
|---|---|---|---|

Using the laws of logs before differentiating can simplify the work.

The following is worth remembering when evaluating the derivatives of natural logarithmic functions:

$$\boxed{\ln e^k = k, \qquad \text{for any } k \in \mathbf{R}.}$$

For example,

$\ln 1 = \ln e^0 = 0,$ $\quad \ln e = \ln e^1 = 1,$ $\quad \ln e^2 = 2,$ $\quad \ln\sqrt{e} = \ln e^{1/2} = \frac{1}{2}.$

The rules for differentiating also apply to natural logarithmic functions.

Suppose $u$ is a function of $x$.

If $\quad y = \ln u$

then $\quad \dfrac{dy}{dx} = \dfrac{1}{u} \cdot \dfrac{du}{dx}.$

*Example*

Find $\dfrac{dy}{dx}$ if **(i)** $y = \ln(x^2 + 1)$ **(ii)** $y = \ln(\sin x)$ **(iii)** $y = \ln\sqrt{x^2 - 3}$ **(iv)** $y = x \ln x$.

**Solution:**

**(i)** $y = \ln(x^2 + 1)$

$$\frac{dy}{dx} = \frac{1}{x^2 + 1} \cdot 2x = \frac{2x}{x^2 + 1}$$

**(ii)** $y = \ln(\sin x)$

$$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x} = \cot x$$

**(iii)** $y = \ln\sqrt{x^2 - 3}$

$y = \ln(x^2 - 3)^{1/2} = \frac{1}{2} \ln(x^2 - 3)$

(using $\ln a^n = n \ln a$)

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{x^2 - 3} \cdot 2x = \frac{x}{x^2 - 3}$$

**(iv)** $y = x \ln x$

$$\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln(x)(1)$$

(using the product rule)

$= 1 + \ln x$

## Exercise 12.8

Find $\frac{dy}{dx}$ for each of the following:

**1.** $y = \ln 5x$

**2.** $y = \ln(2x + 3)$

**3.** $y = \ln(x^2 + 3$

**4.** $y = \ln(\cos x)$

**5.** $y = \ln\left(\frac{1}{x}\right)$

**6.** $y = \ln(e^x + 2$

**7.** $y = \ln(\sin 2x)$

**8.** $y = \ln(\tan 3x)$

**9.** $y = \ln(e^{2x})$

**10.** $y = x \ln x^2$

**11.** $y = x^3 \ln(x + 1)$

**12.** $y = x^2 \ln 4x$

Use the rules of logarithms, or otherwise, to find $\frac{dy}{dx}$ for each of the following:

**13.** $y = \ln\left(\frac{2x}{x+1}\right)$

**14.** $y = \ln(2x + 3)^2$

**15.** $y = \ln\left(\frac{1}{e^x}\right)$

**16.** $y = \ln\sqrt{1 + x^2}$

**17.** $y = \ln\sqrt{\sin x}$

**18.** $y = \ln\sqrt{\frac{x}{1+}}$

**19.** If $f(x) = \ln(e^x \cos x)$, show that $f'(x) = 1 - \tan x$.

**20.** If $y = \ln(\sec x + \tan x)$, show that $\frac{dy}{dx} = \sec x$.

**21.** If $f(x) = x^2 \ln x$, evaluate **(i)** $f'(e)$ **(ii)** $f'(1)$.

**22.** Given $f(x) = \ln\left(\frac{1 + \cos x}{1 - \cos x}\right)$, show that $f'(x) = -2 \operatorname{cosec} x$.

**23.** If $f(x) = \ln(\ln x)$, evaluate $f'(e)$.

**24.** If $f(x) = \ln\left(\frac{e^x}{1 + e^x}\right)$ evaluate $f'(0)$.

**25.** If $y = \frac{\ln x}{x}$, show that $\frac{dy}{dx} = \frac{1 - \ln x}{x^2}$.

Evaluate $\frac{d^2y}{dx^2}$ at $x = e$.

**26.** Given $f(x) = e^x \ln x$, $x > 0$, evaluate $f''(1)$.

**27.** Given $y = \ln(t + 1)$ and $x = 1 + \ln t$, express $\frac{dy}{dx}$ in terms of $t$.

**28.** If $y = e^{t+1}$ and $x = e^t$, find the value of $\ln\left(\frac{dy}{dx}\right)$.

**29.** If $y = \ln t$ and $x + \frac{1}{2}\left(t + \frac{1}{t}\right)$, show that $\frac{dy}{dx} = \frac{2t}{t^2 - 1}$.

**30.** If $x = \ln\left(\frac{e^t}{1 + e^t}\right)$ and $y = \ln\left(\frac{1 + e^t}{e^t}\right)$, $t \in \mathbf{R}$, evaluate $\frac{dy}{dx}$.

**31.** Given $y = x \ln(x^2)$, show that $x\left(\frac{dy}{dx}\right) - 2x = y$.

**32.** Using $\ln \dfrac{a}{b} = \ln a - \ln b$, or otherwise, show that if $y = \ln\left(\dfrac{1+x}{1-x}\right)$,

**(i)** $(1-x^2)\dfrac{dy}{dx} = 2$ **(ii)** $\left(\dfrac{2x}{1-x^2}\right)\dfrac{dy}{dx} - \dfrac{d^2y}{dx^2} = 0.$

**33.** Factorise $a^x + a^{2x}$. If $y = \ln(1+e^x)$, show that $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = \dfrac{dy}{dx}$.

**34.** If $y = \ln e^{-x}\sqrt{\dfrac{1+2x}{1-2x}}$, show that $\dfrac{dy}{dx} = \dfrac{1+4x^2}{1-4x^2}$.

Find the value of $\dfrac{dy}{dx}$ at $x = -1$.

(Hint: $\ln \dfrac{ab}{c} = \ln a + \ln b - \ln c$)

## Logarithmic Differentiation

Functions of the form $2^x$, $x^x$ or $3^{\sin x}$ are differentiated using '**logarithmic differentiation**'.

**The method involves three steps:**

1. Take natural logs of both sides and use the fact that $\ln a^x = x \ln a$.
2. Differentiate both sides with respect to $x$, using implicit differentiation.
3. Multiply both sides by $y$ to get $\dfrac{dy}{dx}$ on its own.

*Example*

Differentiate **(i)** $2^x$ **(ii)** $x^x$ with respect to $x$.

**Solution:**

**(i)** Let

$$y = 2^x$$
$$\ln y = \ln 2^x$$
$$\ln y = x \ln 2$$
$$\frac{1}{y}\frac{dy}{dx} = \ln 2$$
$$\frac{dy}{dx} = y \ln 2$$
$$= 2^x \ln 2$$

**(ii)** Let

$$y = x^x$$
$$\ln y = \ln x^x$$
$$\ln y = x \ln x$$
$$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x(1)$$

(using the product rule)

$$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$$
$$\frac{dy}{dx} = y(1 + \ln x)$$
$$= x^x(1 + \ln x)$$

## *Exercise 12.9*

Use logarithmic differentiation to find the derivative of each of the following:

**1.** $3^x$

**2.** $5^x$

**3.** $3^{2x}$

**4.** $4^{3x+1}$

**5.** $2^{\sin x}$

**6.** $2^{\ln x}$

**7.** $(\sin x)^x$

**8.** $2^x x^2$

**9.** If $f(x) = x4^x$, evaluate $f'(1)$.

**10.** If $y = a^x$, $a > 0$, $a \in \mathbf{R}$, show that $\frac{dy}{dx} = a^x \ln a$.

**11.** If $x^y = e^x$, show that $\frac{dy}{dx} = \frac{x - y}{x \ln x}$ or $\frac{\ln x - 1}{(\ln x)^2}$.