Matrices

Frames 1 to 58

Learning outcomes

When you have completed this Programme you will be able to:

- Define a matrix
- Understand what is meant by the equality of two matrices
- Add and subtract two matrices
- Multiply a matrix by a scalar and multiply two matrices together
- Obtain the transpose of a matrix
- Recognize special types of matrix
- Obtain the determinant, cofactors and adjoint of a square matrix
- Obtain the inverse of a non-singular matrix
- Use matrices to solve a set of linear equations using inverse matrices
- Use the Gaussian elimination method to solve a set of linear equations
- Evaluate eigenvalues and eigenvectors

Matrices – definitions

1

A *matrix* is a set of real or complex numbers (or *elements*) arranged in rows and columns to form a rectangular array.

A matrix having m rows and n columns is called an $m \times n$ (i.e. 'm by n') matrix and is referred to as having *order* $m \times n$.

A matrix is indicated by writing the array within brackets

e.g. $\begin{pmatrix} 5 & 7 & 2 \\ 6 & 3 & 8 \end{pmatrix}$ is a 2×3 matrix, i.e. a '2 by 3' matrix, where

5, 7, 2, 6, 3, 8 are the elements of the matrix.

Note that, in describing the matrix, the number of rows is stated first and the number of columns second.

$\begin{pmatrix} 5 & 6 & 4 \\ 2 & -3 & 2 \\ 7 & 8 & 7 \\ 6 & 7 & 5 \end{pmatrix}$ is a matrix of order 4×3, i.e. 4 rows and 3 columns.

So the matrix $\begin{pmatrix} 6 & 4 \\ 0 & 1 \\ 2 & 3 \end{pmatrix}$ is of order ....

and the matrix $\begin{pmatrix} 2 & 5 & 3 & 4 \\ 6 & 7 & 4 & 9 \end{pmatrix}$ is of order ....

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$\boxed{3 \times 2; \quad 2 \times 4}$

A matrix is simply an array of numbers: there is no arithmetical connection between the elements and it therefore differs from a determinant in that the elements cannot be multiplied together in any way to find a numerical value of the matrix. A matrix has no numerical value. Also, in general, rows and columns cannot be interchanged as was the case with determinants.

Row matrix: A row matrix consists of 1 row only.

e.g. $(4 \quad 3 \quad 7 \quad 2)$ is a row matrix of order 1×4.

Column matrix: A column matrix consists of 1 column only.

e.g. $\begin{pmatrix} 6 \\ 3 \\ 8 \end{pmatrix}$ is a column matrix of order 3×1.

To conserve space in printing, a column matrix is sometimes written on one line but with 'curly' brackets, e.g. $\{6 \; 3 \; 8\}$ is the same column matrix of order 3×1.

Move on to the next frame

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So, from what we have already said:

(a) $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ is a .... matrix of order ....

(b) $(4 \quad 0 \quad 7 \quad 3)$ is a .... matrix of order ....

(c) $\{2 \quad 6 \quad 9\}$ is a .... matrix of order ....

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(a) column, 2×1	(b) row, 1×4	(c) column, 3×1

We use a simple row matrix in stating the x- and y-coordinates of a point relative to the x- and y-axes. For example, if P is the point (3, 5) then the 3 is the x-coordinate and the 5 the y-coordinate. In matrices generally, however, no commas are used to separate the elements.

Single element matrix: A single number may be regarded as a 1×1 matrix, i.e. having 1 row and 1 column.

Double suffix notation: Each element in a matrix has its own particular 'address' or location which can be defined by a system of double suffixes, the first indicating the row, the second the column, thus:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

$\therefore$ a_{23} indicates the element in the second row and third column.

Therefore, in the matrix

$$\begin{pmatrix} 6 & -5 & 1 & -3 \\ 2 & -4 & 8 & 3 \\ 4 & -7 & -6 & 5 \\ -2 & 9 & 7 & -1 \end{pmatrix}$$

the location of (a) the element 3 can be stated as ....
(b) the element -1 can be stated as ....
(c) the element 9 can be stated as ....

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(a) a_{24}	(b) a_{44}	(c) a_{42}

Move on

Matrix notation

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Where there is no ambiguity, a whole matrix can be denoted by a single general element enclosed in brackets, or by a single letter printed in bold type. This is a very neat shorthand and saves much space and writing. For example:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} \text{ can be denoted by } (a_{ij}) \text{ or } (a) \text{ or by } \mathbf{A}.$$

$$\text{Similarly } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ can be denoted by } (x_i) \text{ or } (x) \text{ or simply by } \mathbf{x}.$$

For an $(m \times n)$ matrix, we use a bold capital letter, e.g. $\mathbf{A}$. For a row or column matrix, we use a lower-case bold letter, e.g. $\mathbf{x}$. (In handwritten work, we can indicate bold-face type by a wavy line placed under the letter, e.g. $\underset{\sim}{A}$ or $\underset{\sim}{x}$.)

So, if $\mathbf{B}$ represents a 2×3 matrix, write out the elements b_{ij} in the matrix, using the double suffix notation. This gives ....

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$$\boxed{\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}}$$

Next frame

Equal matrices

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By definition, two matrices are said to be equal if corresponding elements throughout are equal. Thus, the two matrices must also be of the same order.

$$\text{So, if } \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 4 & 6 & 5 \\ 2 & 3 & 7 \end{pmatrix}$$

then $a_{11} = 4$; $a_{12} = 6$; $a_{13} = 5$; $a_{21} = 2$; etc.

Therefore, if $(a_{ij}) = (x_{ij})$ then $a_{ij} = x_{ij}$ for all values of i and j.

$$\text{So, if } \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} = \begin{pmatrix} 5 & -7 & 3 \\ 1 & 2 & 6 \\ 0 & 4 & 8 \end{pmatrix}$$

then $d =$ ....; $b =$ ....; $a - k =$ ....

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$$\boxed{d = 1; \quad b = -7; \quad a - k = -3}$$

Addition and subtraction of matrices

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To be added or subtracted, two matrices must be of the *same order*. The sum or difference is then determined by adding or subtracting corresponding elements.

e.g. $\begin{pmatrix} 4 & 2 & 3 \\ 5 & 7 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 8 & 9 \\ 3 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 4+1 & 2+8 & 3+9 \\ 5+3 & 7+5 & 6+4 \end{pmatrix}$

$= \begin{pmatrix} 5 & 10 & 12 \\ 8 & 12 & 10 \end{pmatrix}$

and $\begin{pmatrix} 6 & 5 & 12 \\ 9 & 4 & 8 \end{pmatrix} - \begin{pmatrix} 3 & 7 & 1 \\ 2 & 10 & -5 \end{pmatrix} = \begin{pmatrix} 6-3 & 5-7 & 12-1 \\ 9-2 & 4-10 & 8+5 \end{pmatrix}$

$= \begin{pmatrix} 3 & -2 & 11 \\ 7 & -6 & 13 \end{pmatrix}$

So, (a) $\begin{pmatrix} 6 & 5 & 4 & 1 \\ 2 & 3 & -7 & 8 \end{pmatrix} + \begin{pmatrix} 1 & 4 & 2 & 3 \\ 6 & -1 & 0 & 5 \end{pmatrix} =$ ....

(b) $\begin{pmatrix} 8 & 3 & 6 \\ 5 & 2 & 7 \\ 1 & 0 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} =$ ....

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(a) $\begin{pmatrix} 7 & 9 & 6 & 4 \\ 8 & 2 & -7 & 13 \end{pmatrix}$ (b) $\begin{pmatrix} 7 & 1 & 3 \\ 1 & -3 & 1 \\ -6 & -8 & -5 \end{pmatrix}$

Multiplication of matrices

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1 Scalar multiplication

To multiply a matrix by a single number (i.e. a scalar), each individual element of the matrix is multiplied by that factor:

e.g. $4 \times \begin{pmatrix} 3 & 2 & 5 \\ 6 & 1 & 7 \end{pmatrix} = \begin{pmatrix} 12 & 8 & 20 \\ 24 & 4 & 28 \end{pmatrix}$

i.e. in general, $k(a_{ij}) = (ka_{ij})$.

It also means that, in reverse, we can take a common factor out of every element – not just one row or one column as in determinants.

Therefore, $\begin{pmatrix} 10 & 25 & 45 \\ 35 & 15 & 50 \end{pmatrix}$ can be written ....

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$$\boxed{5 \times \begin{pmatrix} 2 & 5 & 9 \\ 7 & 3 & 10 \end{pmatrix}}$$

2 Multiplication of two matrices

Two matrices can be multiplied together only when the number of columns in the first is equal to the number of rows in the second.

$$\text{e.g. if } \mathbf{A} = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \text{ and } \mathbf{b} = (b_i) = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\text{then} \quad \mathbf{A}.\mathbf{b} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} . \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3 \end{pmatrix}$$

i.e. each element in the top row of **A** is multiplied by the corresponding element in the first column of **b** and the products added. Similarly, the second row of the product is found by multiplying each element in the second row of **A** by the corresponding element in the first column of **b**.

Example 1

$$\begin{pmatrix} 4 & 7 & 6 \\ 2 & 3 & 1 \end{pmatrix} . \begin{pmatrix} 8 \\ 5 \\ 9 \end{pmatrix} = \begin{pmatrix} 4 \times 8 + 7 \times 5 + 6 \times 9 \\ 2 \times 8 + 3 \times 5 + 1 \times 9 \end{pmatrix} = \begin{pmatrix} 32 + 35 + 54 \\ 16 + 15 + 9 \end{pmatrix} = \begin{pmatrix} 121 \\ 40 \end{pmatrix}$$

$$\text{Similarly} \begin{pmatrix} 2 & 3 & 5 & 1 \\ 4 & 6 & 0 & 7 \end{pmatrix} . \begin{pmatrix} 3 \\ 4 \\ 2 \\ 9 \end{pmatrix} = \dots\dots\dots\dots$$

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$$\boxed{\begin{pmatrix} 6 + 12 + 10 + 9 \\ 12 + 24 + 0 + 63 \end{pmatrix} = \begin{pmatrix} 37 \\ 99 \end{pmatrix}}$$

In just the same way, if $\mathbf{A} = \begin{pmatrix} 3 & 6 & 8 \\ 1 & 0 & 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 7 \\ 4 \\ 5 \end{pmatrix}$ then

$\mathbf{A}.\mathbf{b} = \dots\dots\dots\dots$

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$$\boxed{\begin{pmatrix} 85 \\ 17 \end{pmatrix}}$$

The same process is carried out for each row and column.

Example 2

If $\mathbf{A} = (a_{ij}) = \begin{pmatrix} 1 & 5 \\ 2 & 7 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = (b_{ij}) = \begin{pmatrix} 8 & 4 & 3 & 1 \\ 2 & 5 & 8 & 6 \end{pmatrix}$

$$\text{then}\quad \mathbf{A}.\mathbf{B} = \begin{pmatrix} 1 & 5 \\ 2 & 7 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 8 & 4 & 3 & 1 \\ 2 & 5 & 8 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times 8+5\times 2 & 1\times 4+5\times 5 & 1\times 3+5\times 8 & 1\times 1+5\times 6 \\ 2\times 8+7\times 2 & 2\times 4+7\times 5 & 2\times 3+7\times 8 & 2\times 1+7\times 6 \\ 3\times 8+4\times 2 & 3\times 4+4\times 5 & 3\times 3+4\times 8 & 3\times 1+4\times 6 \end{pmatrix}$$

$$= \begin{pmatrix} 8+10 & 4+25 & 3+40 & 1+30 \\ 16+14 & 8+35 & 6+56 & 2+42 \\ 24+8 & 12+20 & 9+32 & 3+24 \end{pmatrix}$$

$$= \begin{pmatrix} 18 & 29 & 43 & 31 \\ 30 & 43 & 62 & 44 \\ 32 & 32 & 41 & 27 \end{pmatrix}$$

Note that multiplying a (3×2) matrix and a (2×4) matrix gives a product matrix of order (3×4)

i.e. order $(3\times 2) \times$ order $(2\times 4) \rightarrow$ order (3×4).

(same)

In general then, the product of an $(l\times m)$ matrix and an $(m\times n)$ matrix has order $(l\times n)$.

If $\mathbf{A} = \begin{pmatrix} 2 & 4 & 6 \\ 3 & 9 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 7 & 1 \\ -2 & 9 \\ 4 & 3 \end{pmatrix}$

then $\mathbf{A}.\mathbf{B} = \dots\dots\dots\dots$

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$$\boxed{\begin{pmatrix} 30 & 56 \\ 23 & 99 \end{pmatrix}}$$

$$\text{since } \mathbf{A}.\mathbf{B} = \begin{pmatrix} 2 & 4 & 6 \\ 3 & 9 & 5 \end{pmatrix} \cdot \begin{pmatrix} 7 & 1 \\ -2 & 9 \\ 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 14-8+24 & 2+36+18 \\ 21-18+20 & 3+81+15 \end{pmatrix} = \begin{pmatrix} 30 & 56 \\ 23 & 99 \end{pmatrix}$$

▶

Example 3

It follows that a matrix can be squared only if it is itself a square matrix, i.e. the number of rows equals the number of columns.

$$\text{If } \mathbf{A} = \begin{pmatrix} 4 & 7 \\ 5 & 2 \end{pmatrix}$$

$$\mathbf{A}^2 = \begin{pmatrix} 4 & 7 \\ 5 & 2 \end{pmatrix} \cdot \begin{pmatrix} 4 & 7 \\ 5 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 16+35 & 28+14 \\ 20+10 & 35+4 \end{pmatrix} = \begin{pmatrix} 51 & 42 \\ 30 & 39 \end{pmatrix}$$

Remember that multiplication of matrices is defined only when ....

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$$\boxed{\begin{array}{c}\text{the number of columns in the first} \\ = \text{the number of rows in the second}\end{array}}$$

That is correct. $\begin{pmatrix} 1 & 5 & 6 \\ 4 & 9 & 7 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 & 5 \\ 8 & 7 & 1 \end{pmatrix}$ has no meaning.

$$\left.\begin{array}{r}\text{If } \mathbf{A} \text{ is an } (m \times n) \text{ matrix} \\ \text{and } \mathbf{B} \text{ is an } (n \times m) \text{ matrix}\end{array}\right\} \text{then products } \mathbf{A}.\mathbf{B} \text{ and } \mathbf{B}.\mathbf{A} \text{ are possible.}$$

Example

$$\text{If } \mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{pmatrix}$$

$$\text{then } \mathbf{A}.\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} 7+16+27 & 10+22+36 \\ 28+40+54 & 40+55+72 \end{pmatrix} = \begin{pmatrix} 50 & 68 \\ 122 & 167 \end{pmatrix}$$

$$\text{and } \mathbf{B}.\mathbf{A} = \begin{pmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 7+40 & 14+50 & 21+60 \\ 8+44 & 16+55 & 24+66 \\ 9+48 & 18+60 & 27+72 \end{pmatrix} = \begin{pmatrix} 47 & 64 & 81 \\ 52 & 71 & 90 \\ 57 & 78 & 99 \end{pmatrix}$$

Note that, in matrix multiplication, $\mathbf{A}.\mathbf{B} \neq \mathbf{B}.\mathbf{A}$, i.e. multiplication is not commutative. The order of the factors is important!

In the product **A**.**B**, **B** is *pre-multiplied* by **A**
and **A** is *post-multiplied* by **B**.

▶

So, if $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 7 & 4 \\ 3 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 9 & 2 & 4 \\ -2 & 3 & 6 \end{pmatrix}$

then $\mathbf{A}.\mathbf{B} = \ldots\ldots\ldots\ldots$ and $\mathbf{B}.\mathbf{A} = \ldots\ldots\ldots\ldots$

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$$\mathbf{A}.\mathbf{B} = \begin{pmatrix} 41 & 16 & 32 \\ 55 & 26 & 52 \\ 25 & 9 & 18 \end{pmatrix}; \quad \mathbf{B}.\mathbf{A} = \begin{pmatrix} 71 & 30 \\ 29 & 14 \end{pmatrix}$$

Transpose of a matrix

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If the rows and columns of a matrix are interchanged:

i.e. the first row becomes the first column,
the second row becomes the second column,
the third row becomes the third column, etc.,

then the new matrix so formed is called the *transpose* of the original matrix. If $\mathbf{A}$ is the original matrix, its transpose is denoted by $\tilde{\mathbf{A}}$ or $\mathbf{A}^{\mathrm{T}}$. We shall use the latter.

$$\therefore \text{ If } \mathbf{A} = \begin{pmatrix} 4 & 6 \\ 7 & 9 \\ 2 & 5 \end{pmatrix}, \text{ then } \mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 4 & 7 & 2 \\ 6 & 9 & 5 \end{pmatrix}$$

Therefore, given that

$$\mathbf{A} = \begin{pmatrix} 2 & 7 & 6 \\ 3 & 1 & 5 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 4 & 0 \\ 3 & 7 \\ 1 & 5 \end{pmatrix}$$

then $\mathbf{A}.\mathbf{B} = \ldots\ldots\ldots\ldots$ and $(\mathbf{A}.\mathbf{B})^{\mathrm{T}} = \ldots\ldots\ldots\ldots$

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$$\mathbf{A}.\mathbf{B} = \begin{pmatrix} 35 & 79 \\ 20 & 32 \end{pmatrix}; \quad \mathbf{A}.\mathbf{B}^{\mathrm{T}} = \begin{pmatrix} 35 & 20 \\ 79 & 32 \end{pmatrix}$$

Special matrices

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(a) *Square matrix* is a matrix of order $m \times m$.

e.g. $\begin{pmatrix} 1 & 2 & 5 \\ 6 & 8 & 9 \\ 1 & 7 & 4 \end{pmatrix}$ is a 3×3 matrix

A square matrix (a_{ij}) is *symmetric* if $a_{ij} = a_{ji}$, e.g. $\begin{pmatrix} 1 & 2 & 5 \\ 2 & 8 & 9 \\ 5 & 9 & 4 \end{pmatrix}$

i.e. it is symmetrical about the leading diagonal.

Note that $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$.

A square matrix (a_{ij}) is skew-symmetric if $a_{ij} = -a_{ji}$ e.g. $\begin{pmatrix} 0 & 2 & 5 \\ -2 & 0 & 9 \\ -5 & -9 & 0 \end{pmatrix}$

In that case, $\mathbf{A} = -\mathbf{A}^{\mathrm{T}}$.

(b) *Diagonal matrix* is a square matrix with all elements zero except those on the leading diagonal, thus $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$

(c) *Unit matrix* is a diagonal matrix in which the elements on the leading diagonal are all unity, i.e. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The unit matrix is denoted by **I**.

If $\mathbf{A} = \begin{pmatrix} 5 & 2 & 4 \\ 1 & 3 & 8 \\ 7 & 9 & 6 \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ then $\mathbf{A.I} = \ldots\ldots\ldots\ldots$

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$$\boxed{\begin{pmatrix} 5 & 2 & 4 \\ 1 & 3 & 8 \\ 7 & 9 & 6 \end{pmatrix} \text{ i.e. } \mathbf{A.I} = \mathbf{A}}$$

Similarly, if we form the product **I.A** we obtain:

$$\mathbf{I.A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 5 & 2 & 4 \\ 1 & 3 & 8 \\ 7 & 9 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 5+0+0 & 2+0+0 & 4+0+0 \\ 0+1+0 & 0+3+0 & 0+8+0 \\ 0+0+7 & 0+0+9 & 0+0+6 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 4 \\ 1 & 3 & 8 \\ 7 & 9 & 6 \end{pmatrix} = \mathbf{A}$$

$$\mathbf{A.I} = \mathbf{I.A} = \mathbf{A}$$

Therefore, the unit matrix **I** behaves very much like the unit factor in ordinary algebra and arithmetic.

(d) *Null matrix*: A null matrix is one whose elements are all zero.

i.e. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and is denoted by **0**.

If $\mathbf{A.B} = \mathbf{0}$, we cannot say that therefore $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$

for if $\mathbf{A} = \begin{pmatrix} 2 & 1 & -3 \\ 6 & 3 & -9 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 9 \\ 4 & -6 \\ 2 & 4 \end{pmatrix}$

then $$\mathbf{A.B} = \begin{pmatrix} 2 & 1 & -3 \\ 6 & 3 & -9 \end{pmatrix} \cdot \begin{pmatrix} 1 & 9 \\ 4 & -6 \\ 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2+4-6 & 18-6-12 \\ 6+12-18 & 54-18-36 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

That is, $\mathbf{A.B} = \mathbf{0}$, but clearly $\mathbf{A} \neq \mathbf{0}$ and $\mathbf{B} \neq \mathbf{0}$.

Now a short revision exercise. Do these without looking back.

1 If $\mathbf{A} = \begin{pmatrix} 4 & 6 & 5 & 7 \\ 3 & 1 & 9 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 8 & 3 & -1 \\ 5 & 2 & -4 & 6 \end{pmatrix}$

determine (a) $\mathbf{A} + \mathbf{B}$ and (b) $\mathbf{A} - \mathbf{B}$.

2 If $\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 2 & 7 \\ 6 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 9 & 2 \\ 4 & 0 & 8 \end{pmatrix}$

determine (a) $5\mathbf{A}$; (b) $\mathbf{A.B}$; (c) $\mathbf{B.A}$.

3 If $\mathbf{A} = \begin{pmatrix} 2 & 6 \\ 5 & 7 \\ 4 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 2 \\ 0 & 7 \\ 2 & 3 \end{pmatrix}$ then $\mathbf{A.B} = \ldots\ldots\ldots\ldots$

4 Given that $\mathbf{A} = \begin{pmatrix} 4 & 2 & 6 \\ 1 & 8 & 7 \end{pmatrix}$ determine (a) $\mathbf{A}^{\mathrm{T}}$ and (b) $\mathbf{A.A}^{\mathrm{T}}$.

When you have completed them, check your results with the next frame

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Here are the solutions. Check your results.

1 (a) $\mathbf{A}+\mathbf{B}=\begin{pmatrix}6 & 14 & 8 & 6\\ 8 & 3 & 5 & 10\end{pmatrix}$; (b) $\mathbf{A}-\mathbf{B}=\begin{pmatrix}2 & -2 & 2 & 8\\ -2 & -1 & 13 & -2\end{pmatrix}$

2 (a) $5\mathbf{A}=\begin{pmatrix}20 & 15\\ 10 & 35\\ 30 & 5\end{pmatrix}$ (b) $\mathbf{A}.\mathbf{B}=\begin{pmatrix}32 & 36 & 32\\ 38 & 18 & 60\\ 34 & 54 & 20\end{pmatrix}$ (c) $\mathbf{B}.\mathbf{A}=\begin{pmatrix}50 & 80\\ 64 & 20\end{pmatrix}$

3 $\mathbf{A}.\mathbf{B}=\begin{pmatrix}2 & 6\\ 5 & 7\\ 4 & 1\end{pmatrix}\cdot\begin{pmatrix}3 & 2\\ 0 & 7\\ 2 & 3\end{pmatrix}$ is not possible since the number of columns in the first must be equal to the number of rows in the second.

4 $\mathbf{A}=\begin{pmatrix}4 & 2 & 6\\ 1 & 8 & 7\end{pmatrix}$ $\therefore$ $\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}4 & 1\\ 2 & 8\\ 6 & 7\end{pmatrix}$

$$\mathbf{A}.\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}4 & 2 & 6\\ 1 & 8 & 7\end{pmatrix}\cdot\begin{pmatrix}4 & 1\\ 2 & 8\\ 6 & 7\end{pmatrix}=\begin{pmatrix}16+4+36 & 4+16+42\\ 4+16+42 & 1+64+49\end{pmatrix}$$
$$=\begin{pmatrix}56 & 62\\ 62 & 114\end{pmatrix}$$

Now move on to the next frame

Determinant of a square matrix

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The determinant of a square matrix is the determinant having the same elements as those of the matrix. For example:

the determinant of $\begin{pmatrix}5 & 2 & 1\\ 0 & 6 & 3\\ 8 & 4 & 7\end{pmatrix}$ is $\begin{vmatrix}5 & 2 & 1\\ 0 & 6 & 3\\ 8 & 4 & 7\end{vmatrix}$ and the value of this

determinant is $5(42-12)-2(0-24)+1(0-48)$
$$=5(30)-2(-24)+1(-48)=150+48-48=150$$

Note that the transpose of the matrix is $\begin{pmatrix}5 & 0 & 8\\ 2 & 6 & 4\\ 1 & 3 & 7\end{pmatrix}$ and the

determinant of the transpose is $\begin{vmatrix}5 & 0 & 8\\ 2 & 6 & 4\\ 1 & 3 & 7\end{vmatrix}$ the value of which is

$5(42-12)-0(14-4)+8(6-6)=5(30)=150.$

That is, the determinant of a square matrix has the same value as that of the determinant of the transposed matrix.

A matrix whose determinant is zero is called a *singular* matrix.

▶

The determinant of the matrix $\begin{pmatrix} 3 & 2 & 5 \\ 4 & 7 & 9 \\ 1 & 8 & 6 \end{pmatrix}$ has the value ....

and the determinant value of the diagonal matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ has the value ....

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$$\begin{vmatrix} 3 & 2 & 5 \\ 4 & 7 & 9 \\ 1 & 8 & 6 \end{vmatrix} = 3(-30) - 2(15) + 5(25) = 5$$

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{vmatrix} = 2(20) + 0 + 0 = 40$$

Cofactors

If $\mathbf{A} = (a_{ij})$ is a square matrix, we can form a determinant of its elements:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}$$

Each element gives rise to a *cofactor*, which is simply the minor of the element in the determinant together with its 'place sign', which was described in detail in the previous programme.

For example, the determinant of the matrix $\mathbf{A} = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{pmatrix}$ is

$\det \mathbf{A} = |\mathbf{A}| = \begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{vmatrix}$ which has a value of 45.

The minor of the element 2 is $\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} = 0 - 24 = -24$.

The place sign is $+$. Therefore the cofactor of the element 2 is $+(-24)$ i.e. -24.

Similarly, the minor of the element 3 is $\begin{pmatrix} 4 & 6 \\ 1 & 0 \end{pmatrix} = 0 - 6 = -6$.

The place sign is $-$. Therefore the cofactor of the element 3 is $-(-6) = 6$.

In each case the minor is found by striking out the line and column containing the element in question and forming a determinant of the remaining elements. The appropriate place signs are given by:

$$\begin{pmatrix} + & - & + & - & \dots \\ - & + & - & + & \\ + & - & + & & \\ \vdots & & & & \end{pmatrix}$$

alternate plus and minus from the top left-hand corner which carries a $+$

Therefore, in the example above, the minor of the element 6 is $\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$ i.e. $8 - 3 = 5$. The place sign is $-$. Therefore the cofactor of the element 6 is -5.

So, for the matrix $\begin{pmatrix} 7 & 1 & -2 \\ 6 & 5 & 4 \\ 3 & 8 & 9 \end{pmatrix}$, the cofactor of the element 3 is ....
and that of the element 4 is ....

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Cofactor of 3 is $4 - (-10) = 14$
Cofactor of 4 is $-(56 - 3) = -53$

Adjoint of a square matrix

If we start afresh with $\mathbf{A} = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{pmatrix}$, its determinant

det $\mathbf{A} = |\mathbf{A}| = \begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{vmatrix}$ from which we can form a new matrix $\mathbf{C}$ of the cofactors.

$\mathbf{C} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$ where A_{11} is the cofactor of a_{11}; A_{ij} is the cofactor of a_{ij} etc.

$$A_{11} = +\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} = +(0 - 24) = -24 \qquad A_{12} = -\begin{vmatrix} 4 & 6 \\ 1 & 0 \end{vmatrix} = -(0 - 6) = 6$$

$$A_{13} = +\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} = +(16 - 1) = 15$$

$$A_{21} = -\begin{vmatrix} 3 & 5 \\ 4 & 0 \end{vmatrix} = -(0 - 20) = 20 \qquad A_{22} = +\begin{vmatrix} 2 & 5 \\ 1 & 0 \end{vmatrix} = +(0 - 5) = -5$$

$$A_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = -(8 - 3) = -5$$

$$A_{31} = +\begin{vmatrix} 3 & 5 \\ 1 & 6 \end{vmatrix} = +(18 - 5) = 13 \qquad A_{32} = -\begin{vmatrix} 2 & 5 \\ 4 & 6 \end{vmatrix} = -(12 - 20) = 8$$

$$A_{33} = +\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = +(2 - 12) = -10$$

$\therefore$ The matrix of cofactors is $\mathbf{C} = \begin{pmatrix} -24 & 6 & 15 \\ 20 & -5 & -5 \\ 13 & 8 & -10 \end{pmatrix}$

and the transpose of $\mathbf{C}$, i.e. $\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{pmatrix}$

This is called the *adjoint* of the original matrix $\mathbf{A}$ and is written adj $\mathbf{A}$.

Therefore, to find the adjoint of a square matrix $\mathbf{A}$:

(a) we form the matrix $\mathbf{C}$ of cofactors,

(b) we write the transpose of $\mathbf{C}$, i.e. $\mathbf{C}^T$.

Hence the adjoint of $\begin{pmatrix} 5 & 2 & 1 \\ 3 & 1 & 4 \\ 4 & 6 & 3 \end{pmatrix}$ is ....

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$$\text{adj } \mathbf{A} = \mathbf{C}^T = \begin{pmatrix} -21 & 0 & 7 \\ 7 & 11 & -17 \\ 14 & -22 & -1 \end{pmatrix}$$

Inverse of a square matrix

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The adjoint of a square matrix is important, since it enables us to form the inverse of the matrix. If each element of the adjoint of $\mathbf{A}$ is divided by the value of the determinant of $\mathbf{A}$, i.e. $|\mathbf{A}|$, (provided $|\mathbf{A}| \neq 0$), the resulting matrix is called the *inverse* of $\mathbf{A}$ and is denoted by $\mathbf{A}^{-1}$.

For the matrix which we used in the last frame, $\mathbf{A} = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{pmatrix}$

$$\det \mathbf{A} = |\mathbf{A}| = \begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{vmatrix} = 2(0-24) - 3(0-6) + 5(16-1) = 45,$$

$$\text{the matrix of cofactors } \mathbf{C} = \begin{pmatrix} -24 & 6 & 15 \\ 20 & -5 & -5 \\ 13 & 8 & -10 \end{pmatrix}$$

$$\text{and the adjoint of } \mathbf{A}, \text{ i.e. } \mathbf{C}^T = \begin{pmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{pmatrix}$$

Then the inverse of $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \begin{pmatrix} -\frac{24}{45} & \frac{20}{45} & \frac{13}{45} \\ \frac{6}{45} & -\frac{5}{45} & \frac{8}{45} \\ \frac{15}{45} & -\frac{5}{45} & -\frac{10}{45} \end{pmatrix} = \frac{1}{45}\begin{pmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{pmatrix}$$

▶

Therefore, to form the inverse of a square matrix **A**:

(a) Evaluate the determinant of **A**, i.e. $|\mathbf{A}|$

(b) Form a matrix **C** of the cofactors of the elements of $|\mathbf{A}|$

(c) Write the transpose of **C**, i.e. $\mathbf{C}^T$, to obtain the adjoint of **A**

(d) Divide each element of $\mathbf{C}^T$ by $|\mathbf{A}|$

(e) The resulting matrix is the inverse $\mathbf{A}^{-1}$ of the original matrix **A**.

Let us work through an example in detail:

To find the inverse of $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{pmatrix}$

(a) Evaluate the determinant of **A**, i.e. $|\mathbf{A}|$. $|\mathbf{A}| = \ldots\ldots\ldots\ldots$

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$$\boxed{|\mathbf{A}| = 28}$$

Because

$$|\mathbf{A}| = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{pmatrix} = 1(2-0) - 2(8-30) + 3(0-6) = 28$$

(b) Now form the matrix of the cofactors. $\mathbf{C} = \ldots\ldots\ldots\ldots$

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$$\boxed{\mathbf{C} = \begin{pmatrix} 2 & 22 & -6 \\ -4 & -16 & 12 \\ 7 & 7 & -7 \end{pmatrix}}$$

Because

$A_{11} = +(2-0) = 2;$ $\quad A_{12} = -(8-30) = 22;$ $\quad A_{13} = +(0-6) = -6$

$A_{21} = -(4-0) = -4;$ $\quad A_{22} = +(2-18) = -16;$ $\quad A_{23} = -(0-12) = 12$

$A_{31} = +(10-3) = 7;$ $\quad A_{32} = -(5-12) = 7;$ $\quad A_{33} = +(1-8) = -7$

(c) Next we have to write down the transpose of **C** to obtain the adjoint of **A**.

adj $\mathbf{A} = \mathbf{C}^T = \ldots\ldots\ldots\ldots$

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$$\boxed{\text{adj } \mathbf{A} = \mathbf{C}^T = \begin{pmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{pmatrix}}$$

(d) Finally, we divide the elements of adj **A** by the value of $|\mathbf{A}|$, i.e. 28, to arrive at $\mathbf{A}^{-1}$, the inverse of **A**.

$\therefore\ \mathbf{A}^{-1} = \ldots\ldots\ldots\ldots$

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$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{2}{28} & -\frac{4}{28} & \frac{7}{28} \\ \frac{22}{28} & -\frac{16}{28} & \frac{7}{28} \\ -\frac{6}{28} & \frac{12}{28} & -\frac{7}{28} \end{pmatrix} = \frac{1}{28}\begin{pmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{pmatrix}$$

Every one is done in the same way. Work the next one right through on your own.

Determine the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 2 & 7 & 4 \\ 3 & 1 & 6 \\ 5 & 0 & 8 \end{pmatrix}$

$\mathbf{A}^{-1} = \dots\dots\dots\dots$

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$$\mathbf{A}^{-1} = \frac{1}{38}\begin{pmatrix} 8 & -56 & 38 \\ 6 & -4 & 0 \\ -5 & 35 & -19 \end{pmatrix}$$

Here are the details:

$$\det \mathbf{A} = |\mathbf{A}| = \begin{vmatrix} 2 & 7 & 4 \\ 3 & 1 & 6 \\ 5 & 0 & 8 \end{vmatrix} = 2(8) - 7(-6) + 4(-5) = 38$$

Cofactors:

$A_{11} = +(8-0) = 8$; $\quad A_{12} = -(24-30) = 6$; $\quad A_{13} = +(0-5) = -5$

$A_{21} = -(56-0) = -56$; $\quad A_{22} = +(16-20) = -4$; $\quad A_{23} = -(0-35) = 35$

$A_{31} = +(42-4) = 38$; $\quad A_{32} = -(12-12) = 0$; $\quad A_{33} = +(2-21) = -19$

$$\therefore\ \mathbf{C} = \begin{pmatrix} 8 & 6 & -5 \\ -56 & -4 & 35 \\ 38 & 0 & -19 \end{pmatrix} \quad \therefore\ \mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 8 & -56 & 38 \\ 6 & -4 & 0 \\ -5 & 35 & -19 \end{pmatrix}$$

$$\text{then } \mathbf{A}^{-1} = \frac{1}{38}\begin{pmatrix} 8 & -56 & 38 \\ 6 & -4 & 0 \\ -5 & 35 & -19 \end{pmatrix}$$

Now let us find some uses for the inverse.

▶

Product of a square matrix and its inverse

From a previous example, we have seen that when $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{pmatrix}$

$$\mathbf{A}^{-1} = \frac{1}{28}\begin{pmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{pmatrix}$$

$$\text{Then } \mathbf{A}^{-1}.\mathbf{A} = \frac{1}{28}\begin{pmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{28}\begin{pmatrix} 2-16+42 & 4-4+0 & 6-20+14 \\ 22-64+42 & 44-16+0 & 66-80+14 \\ -6+48-42 & -12+12+0 & -18+60-14 \end{pmatrix}$$

$$= \frac{1}{28}\begin{pmatrix} 28 & 0 & 0 \\ 0 & 28 & 0 \\ 0 & 0 & 28 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \qquad \therefore\ \mathbf{A}^{-1}.\mathbf{A} = \mathbf{I}$$

$$\text{Also } \mathbf{A}.\mathbf{A}^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{pmatrix} \times \frac{1}{28}\begin{pmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{pmatrix}$$

$$= \frac{1}{28}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{pmatrix} = \ldots\ldots\ldots\ldots$$

Finish it off

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$$\boxed{\mathbf{A}.\mathbf{A}^{-1} = \frac{1}{28}\begin{pmatrix} 28 & 0 & 0 \\ 0 & 28 & 0 \\ 0 & 0 & 28 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}}$$

$$\therefore\ \mathbf{A}.\mathbf{A}^{-1} = \mathbf{A}^{-1}.\mathbf{A} = \mathbf{I}$$

That is, the product of a square matrix and its inverse, in whatever order the factors are written, is the unit matrix of the same matrix order.

Solution of a set of linear equations

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Consider the set of linear equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots\ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots\ldots + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots \qquad\qquad\qquad \vdots \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots\ldots + a_{nn}x_n = b_n$$

From our knowledge of matrix multiplication, this can be written in matrix form:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \quad \text{i.e. } \mathbf{A}.\mathbf{x} = \mathbf{b}$$

$$\text{where } \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}; \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}; \quad \text{and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

If we multiply both sides of the matrix equation by the inverse of **A**, we have:

$$\mathbf{A}^{-1}.\mathbf{A}.\mathbf{x} = \mathbf{A}^{-1}.\mathbf{b}$$

$$\text{But } \mathbf{A}^{-1}.\mathbf{A} = \mathbf{I} \quad \therefore \ \mathbf{I}.\mathbf{x} = \mathbf{A}^{-1}.\mathbf{b} \quad \text{i.e. } \mathbf{x} = \mathbf{A}^{-1}.\mathbf{b}$$

Therefore, if we form the inverse of the matrix of coefficients and pre-multiply matrix **b** by it, we shall determine the matrix of the solutions of **x**.

Example

To solve the set of equations:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ 3x_1 - 4x_2 - 2x_3 &= 2 \\ 5x_1 + 3x_2 + 5x_3 &= -1 \end{aligned}$$

First write the set of equations in matrix form, which gives ....

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$$\boxed{\begin{pmatrix} 1 & 2 & 1 \\ 3 & -4 & -2 \\ 5 & 3 & 5 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}}$$

$$\text{i.e. } \mathbf{A}.\mathbf{x} = \mathbf{b} \qquad \therefore \ \mathbf{x} = \mathbf{A}^{-1}.\mathbf{b}$$

So the next step is to find the inverse of **A** where **A** is the matrix of the coefficients of **x**. We have already seen how to determine the inverse of a matrix, so in this case $\mathbf{A}^{-1} = \dots\dots\dots\dots$

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$$\boxed{\mathbf{A}^{-1} = -\frac{1}{35}\begin{pmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{pmatrix}}$$

Because

$$|\mathbf{A}| = \begin{vmatrix} 1 & 2 & 1 \\ 3 & -4 & -2 \\ 5 & 3 & 5 \end{vmatrix} = -14 - 50 + 29 = 29 - 64 \quad \therefore \quad |\mathbf{A}| = -35$$

Cofactors:

$$\begin{array}{lll} A_{11} = +(-20+6) = -14; & A_{12} = -(15+10) = -25; & A_{13} = +(9+20) = 29 \\ A_{21} = -(10-3) = -7; & A_{22} = +(5-5) = 0; & A_{23} = -(3-10) = 7 \\ A_{31} = +(-4+4) = 0; & A_{32} = -(-2-3) = 5; & A_{33} = +(-4-6) = -10 \end{array}$$

$$\therefore \ \mathbf{C} = \begin{pmatrix} -14 & -25 & 29 \\ -7 & 0 & 7 \\ 0 & 5 & -10 \end{pmatrix} \quad \therefore \ \text{adj } \mathbf{A} = \mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{pmatrix}$$

$$\text{Now } |\mathbf{A}| = -35 \quad \therefore \ \mathbf{A}^{-1} = \frac{\text{adj } \mathbf{A}}{|\mathbf{A}|} = -\frac{1}{35}\begin{pmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{pmatrix}$$

$$\therefore \ \mathbf{x} = \mathbf{A}^{-1}.\mathbf{b} = -\frac{1}{35}\begin{pmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} = \ldots\ldots\ldots\ldots$$

Multiply it out

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$$\boxed{\mathbf{x} = -\frac{1}{35}\begin{pmatrix} -70 \\ -105 \\ 140 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}}$$

$$\text{So finally } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \quad \therefore \ x_1 = 2; \ x_2 = 3; \ x_3 = -4$$

Once you have found the inverse, the rest is simply $\mathbf{x} = \mathbf{A}^{-1}.\mathbf{b}$. Here is another example to solve in the same way:

$$\begin{array}{lllll} \text{If} & 2x_1 & -x_2 & +3x_3 & = 2 \\ & x_1 & +3x_2 & -x_3 & = 11 \\ & 2x_1 & -2x_2 & +5x_3 & = 3 \end{array}$$

then $\quad x_1 = \ldots\ldots\ldots\ldots; \ x_2 = \ldots\ldots\ldots\ldots; \ x_3 = \ldots\ldots\ldots\ldots$

$$\boxed{x_1 = -1;\; x_2 = 5;\; x_3 = 3}$$

The essential intermediate results are as follows:

$$\begin{pmatrix} 2 & -1 & 3 \\ 1 & 3 & -1 \\ 2 & -2 & 5 \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 11 \\ 3 \end{pmatrix} \text{ i.e. } \mathbf{A}.\mathbf{x} = \mathbf{b} \quad \therefore\; \mathbf{x} = \mathbf{A}^{-1}.\mathbf{b}$$

$$\det \mathbf{A} = |\mathbf{A}| = 9$$

$$\mathbf{C} = \begin{pmatrix} 13 & -7 & -8 \\ -1 & 4 & 2 \\ -8 & 5 & 7 \end{pmatrix} \qquad \therefore\; \text{adj } \mathbf{A} = \mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 13 & -1 & -8 \\ -7 & 4 & 5 \\ -8 & 2 & 7 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{\mathbf{C}^{\mathrm{T}}}{|\mathbf{A}|} = \frac{1}{9}\begin{pmatrix} 13 & -1 & -8 \\ -7 & 4 & 5 \\ -8 & 2 & 7 \end{pmatrix}$$

$$\mathbf{x} = \mathbf{A}^{-1}.\mathbf{b} = \frac{1}{9}\begin{pmatrix} 13 & -1 & -8 \\ -7 & 4 & 5 \\ -8 & 2 & 7 \end{pmatrix} . \begin{pmatrix} 2 \\ 11 \\ 3 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} -9 \\ 45 \\ 27 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}$$

$$\therefore\; \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} \qquad \therefore\; x_1 = -1;\; x_2 = 5;\; x_3 = 3$$

Gaussian elimination method for solving a set of linear equations

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \text{ i.e. } \mathbf{A}.\mathbf{x} = \mathbf{b}$$

All the information for solving the set of equations is provided by the matrix of coefficients **A** and the column matrix **b**. If we write the elements of **b** within the matrix **A**, we obtain the *augmented matrix* **B** of the given set of equations.

$$\text{i.e. } \mathbf{B} = \left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} & b_n \end{array}\right)$$

(a) We then eliminate the elements other than a_{11} from the first column by subtracting a_{21}/a_{11} times the first row from the second row and a_{31}/a_{11} times the first row from the third row, etc.

▶

(b) This gives a new matrix of the form:

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ 0 & c_{22} & c_{23} & \dots & c_{2n} & d_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & c_{n2} & c_{n3} & \dots & c_{nn} & d_n \end{array}\right)$$

The process is then repeated to eliminate c_{i2} from the third and subsequent rows.

A specific example will explain the method, so move on to the next frame

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To solve
$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 3 \\ 2x_1 - x_2 - x_3 &= 11 \\ 3x_1 + 2x_2 + x_3 &= -5 \end{aligned}$$

This can be written $\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ 3 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ -5 \end{pmatrix}$

The augmented matrix becomes $\left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 2 & -1 & -1 & 11 \\ 3 & 2 & 1 & -5 \end{array}\right)$

Now subtract $\dfrac{2}{1}$ times the first row from the second row

and $\dfrac{3}{1}$ times the first row from the third row.

This gives $\left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & -5 & 5 & 5 \\ 0 & -4 & 10 & -14 \end{array}\right)$

Now subtract $\dfrac{-4}{-5}$, i.e. $\dfrac{4}{5}$, times the second row from the third row.

The matrix becomes $\left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & -5 & 5 & 5 \\ 0 & 0 & 6 & -18 \end{array}\right)$

Note that as a result of these steps, the matrix of coefficients of x has been reduced to a triangular matrix.

Finally, we detach the right-hand column back to its original position:

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -5 & 5 \\ 0 & 0 & 6 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -18 \end{pmatrix}$$

Then, by 'back-substitution', starting from the bottom row we get:

$$\begin{aligned} 6x_3 &= -18 \quad \therefore \quad x_3 = -3 \\ -5x_2 + 5x_3 &= 5 \quad \therefore \quad -5x_2 = 5 + 15 = 20 \quad \therefore \quad x_2 = -4 \\ x_1 + 2x_2 - 3x_3 &= 3 \quad \therefore \quad x_1 - 8 + 9 = 3 \quad \therefore \quad x_1 = 2 \end{aligned}$$

$$\therefore \quad x_1 = 2; \; x_2 = -4; \; x_3 = -3$$

▶

Note that when dealing with the augmented matrix, we may, if we wish:

(a) interchange two rows

(b) multiply any row by a non-zero factor

(c) add (or subtract) a constant multiple of any one row to (or from) another.

These operations are permissible since we are really dealing with the coefficients of both sides of the equations.

Now for another example: move on to the next frame

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Solve the following set of equations:

$$x_1 - 4x_2 - 2x_3 = 21$$
$$2x_1 + x_2 + 2x_3 = 3$$
$$3x_1 + 2x_2 - x_3 = -2$$

First write the equations in matrix form, which is ....

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$$\boxed{\begin{pmatrix} 1 & -4 & -2 \\ 2 & 1 & 2 \\ 3 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 21 \\ 3 \\ -2 \end{pmatrix}}$$

The augmented matrix is then ....

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$$\boxed{\left(\begin{array}{ccc:c} 1 & -4 & -2 & 21 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & -1 & -2 \end{array}\right)}$$

We can now eliminate the x_1 coefficients from the second and third rows by .... and ....

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subtracting	2 times the first row from the second row
and	3 times the first row from the third row.

So the matrix now becomes $\left(\begin{array}{ccc:c} 1 & -4 & -2 & 21 \\ 0 & 9 & 6 & -39 \\ 0 & 14 & 5 & -65 \end{array}\right)$

and the next stage is to subtract from the third row .... times the second row.