MTH2205 28/12/2023

Course Outline

0. Field
1. Vector Space over the real field
2. Subspace
3. Linear combination
4. Spanning set
5. Linearly dependent and independent vectors
6. Basis and Dimensions
7. linear transformation
8. Algebra of matrices.

* A vector space is a set together with operations of vectors addition and scalar multiplication satisfying the ffl axioms:

A_1- $u, v \in V \therefore u+v \in V$

A_2 $u, v \in V \therefore u+v = v+u$

A_3 $u, v, w \in V \therefore u+(v+w) = (u+v)+w$

A_4 $u \in V$ if $\exists$ $-u \in V \therefore u+(-u) = 0$

A_5 $u \in V$, there is $0 \in V \therefore u+0 = u$

M_1 $u \in V$, any scalar c, $cu \in V \therefore$

M_2 $c(u+v) = cu+cv$

M_3 $(c+k)u = cu+ku$

M_4 $c(ku) = (ck)u$

M_5 $1u = u$

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① Let V denote the set of ordered triples are element of V and define addition in V as in $\mathbb{R}^3$. For each of the fll definition of scalar multiplication, decide whether V is a vector space.

a $V = \{(x, y, z) : x, y, z \in \mathbb{R}\}$

$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$a(x, y, z) = (ax, y, az)$

V is a vector space it satisfies the fll properties:

A_1: let $u, v \in V$ $\therefore u + v \in V$

$u = x_1, y_1, z_1 \quad v = x_2, y_2, z_2$

$u + v = (x_1 + x_2), (y_1 + y_2), (z_1 + z_2) \in V$

A_2: $u, v \in V$ $\therefore u + v = v + u$

$u = x_1, y_1, z_1 \quad v = x_2, y_2, z_2$

$u + v = (x_1 + x_2), (y_1 + y_2), (z_1 + z_2) \quad v + u = (x_2 + x_1), (y_2 + y_1), (z_2 + z_1)$

$\therefore u + v = v + u$

A_3: $u + (v + w) = (u + v) + w$

$w = x_3, y_3, z_3$

LHS

$(x_1, y_1, z_1) + [(x_2, y_2, z_2) + (x_3, y_3, z_3)]$

$= (x_1, y_1, z_1) + [(x_2 + x_3), (y_2 + y_3), (z_2 + z_3)]$

$= ((x_1 + x_2 + x_3), (y_1 + y_2 + y_3), (z_1, z_2, z_3))$

RHS

$(u+v)+w$

$=[(x_1+x_2),(y_1+y_2),(z_1+z_2)]+(x_3+y_3+z_3)$

$=((x_1+x_2+x_3),(y_1+y_2+y_3),(z_1+z_2+z_3))$

$u+(v+w)=(u+v)+w$

A_4 for any $u \in V$, there is $-u \in V$ ∴ $u+(-u)=0$

$(x_1, y_1, z_1)+(-x_1, -y_1, -z_1)=(0,0,0)=0$ //

A_5 for any $u \in V$ there is $0 \in V$ ∴

$u+0=u$ where $0=(0,0,0)$

$u+0=(x_1, y_1, z_1)+(0,0,0)=(x_1+0, y_1+0, z_1+0)$

$=(x_1, y_1, z_1)$ // $=u$ //

$a(x,y,z)=(ax, y, az)$

M_1 $u \in V$ any scalar a $a(u) \in V$

$au=a(x_1, y_1, z_1)=(ax_1, y_1, az) \in V$

M_2 $a(u+v)=au+av$

LHS

$a(u+v)$ from above

$a(x_1+x_2, y_1+y_2, z_1+z_2)=a\cdot(x_1+x_2), y_1+y_2, a(z_1+z_2)$

$=(ax_1+ax_2),(y_1+y_2),(az_1+az_2)$

RHS from above

$au + av$

$=(ax_1, y_1, az_1) + a(x_2, y_2, z_2)$

$=(ax_1, y_1, az_1) + (ax_2, y_2, az_2)$

$=(ax_1 + ax_2, y_1 + y_2, az_1 + az_2)$ //

M_3 $(a+k)u = au + ku$

LHS

$(a+k)u = (a+k)(x_1, y_1, z_1) = (a+k)x_1, y_1, (a+k)z_1)$

$=((ax_1 + kx_1), y_1, (az_1 + kz_1))$

RHS

$au + ku$

from above

$(ax_1, y_1, az_1) + k(x_1, y_1, z_1)$

$=(ax_1, y_1, az_1) + (kx_1, y_1, kz_1)$

$=(ax_1 + kx_1, 2y_1, az_1 + kz_1)$

$\therefore (a+k)u \neq au + ku$

M_4 $a(ku) = (ak)u$

from above

LHS

$a(kx_1, y_1, kz_1)$

$=(akx_1, y_1, akz_1)$

RHS

$(ak)(x_1, y_1, z_1)$

$=(akx_1, y_1, akz_1)$

$\therefore a(ku) = (ak)u$ //

M_5- $1u = u$

$1(x_1, y_1, z_1) = x_1, y_1, z_1 = u$ //

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Subspace

Let V be a vector space, A non empty subset W of V is said to be a subspace V if it satisfy the ffl properties:-

1- $0 \in W$

2- $u, v \in W$; $u+v \in W$

3- $u \in W$ any scalar c

$cu \in W$

Example: Determine whether W is a subspace of V. If W is not a subspace which property fails?

1- $W = \{(x, 0, 0) : x \in \mathbb{R}\}$

① $0 \in W$

$0 = (0, 0, 0)$ $\therefore 0 \in W$

② $u = (x_1, 0, 0)$ $v = (x_2, 0, 0)$ $u, v \in W$, $u+v \in W$

$u+v = (x_1 + x_2, 0, 0)$ $\therefore u+v \in W$ //

③ $u \in W$, any scalar c, ; $cu \in W$ $cu = c(x_1, 0, 0)$

$= (cx_1, 0, 0)$

$\therefore cx_1 \in W$ //

Having satisfy property number 1, 2, and 3 therefore W is a subspace of V.

7. $W = \{(x+y, x^2, x-y) \; x, y \in \mathbb{R}\}$

1. $0 \in W$

$0 = (0, 0, 0)$

$= (0+0, 0^2, 0-0)$

$\therefore 0 \in W$ //

2. $u, v \in W$; $u+v \in W$

$u = (x_1 + y_1, x_1^2, x_1 - y_1) \quad v = (x_2 + y_2, x_2^2, x_2 - y_2)$

$u+v = (x_1 + y_1 + x_2 + y_2, x_1^2 + x_2^2, x_1 - y_1 + (x_2 - y_2))$

$= (x_1 + x_2) + (y_1 + y_2), x_1^2 + x_2^2, (x_1 + x_2) - (y_1 + y_2)$

$\therefore u+v \notin W$

Hence, property 2 failed then W is not a subspace of V.

3. $u \in W$, any scalar c $\quad cu \in W$

$cu = c(x_1 + y_1, x_1^2, x_1 - y_1)$

$= (cx_1 + cy_1, cx^2, cx_1 - cy_1) \in W$

5. $W = \{(0, x, x+1) : x \in \mathbb{R}\}$

1. $0 \in W$

$0 = (0,0,0)$

$= (0, 0, 0+1)$

$= (0, 0, 1) \notin W$ $\therefore W$ is not a subspace.

2. $u, v \in W$; $u+v \in W$

$u = (0, x_1, x_1+1)$ $v = (0, x_2, x_2+1)$

$u+v = (0+0, x_1+x_2, x_1+1+x_2+1)$

$= (0, (x_1+x_2), (x_1+x_2+2)) \notin W$

$\therefore W$ is not a subspace.

3. $u \in W$, any scalar c $cu \in W$

$cu = c(0, x_1, x_1+1)$

$= (0, cx_1, (cx_1+c) \notin W$

Hence, W is not a subspace since property 1-3 ~~failed~~. is not satisfied.

4. $W = \{a_1x^3 + a_2x^2 + a_3 : a_1, a_2, a_3 \in \mathbb{R}\}$

$u = (a_1x_1^3 + a_2x_1^2 + a_3)$; $v = (a_1x_2^3 + a_2x_2^2 + a_3)$

1. $0 \in W$

$0 = (0x^3 + 0x^2 + 0)$

$= (0+0+0) \in W$ property is satisfied.

2. $u, v \in W$; $u+v \in W$

$u+v = (a_1 x_1^3 + a_1 x_2^3) + (a_2 x_1^2 + a_2 x_2^2) + (a_3 + a_3)$

$= (a_1(x_1^3 + x_2^3) + a_2(x_1^2 + x_2^2) + 2a_3) \in W$

$\therefore u+v \in W$ //

3. $u \in W$, any scalar c $\quad c.u \in W$

$cu = c(a_1 x_1^3 + a_2 x_1^2 + a_3)$

$= (ca_1 x_1^3 + ca_2 x_1^2 + ca_3) \in W$

$\therefore$ ~~$u+v \in W$~~ $cu \in W$

Therefore W is a subspace since All the properties are satisfied.

Linear Combination

let V be a vector space containing the element $v_1, v_2, v_3 \ldots v_n$ and u, then u is called the linear combination of $v_1, v_2, v_3 \ldots v_n$. If it can be expressed as $c_1 v_1 + c_2 v_2 + c_3 v_3 + \ldots c_n v_n = u$ where $c_1, c_2, c_3 \ldots c_n$ are real numbers.

Exercises

b $a(x,y,z) = (ax, 0, az)$

$u = x_1, y_1, z_1$

$v = x_2, y_2, z_2$

$A_1 - u, v \in V \therefore u+v \in V$

$u+v = x_1+x_2, y_1+y_2, z_1+z_2$

$\therefore u+v \in V$

$A_2 - u, v \in V \therefore u+v = v+u$

$u+v = x_1+x_2, y_1+y_2, z_1+z_2$

$v+u = x_2+x_1, y_2+y_1, z_2+z_1$

$\therefore u+v = v+u$

$A_3 - u, v, w \in V \therefore u+(v+w) = (u+v)+w$

$w = x_3, y_3, z_3$

$u+(v+w) = (u+v)+w$

LHS

$(x_1, y_1, z_1) + (x_2+x_3, y_2+y_3, z_2+z_3)$

$= (x_1+x_2+x_3)(y_1+y_2+y_3)(z_1+z_2+z_3)$

RHS

$(x_1+x_2, y_1+y_2, z_1+z_2) + x_3, y_3, z_3$

$= (x_1+x_2+x_3, y_1+y_2+y_3, z_1+z_2+z_3)$

$\therefore u+(v+w) = (u+v)+w$

$A_4 : u \in V, \text{if } \exists -u \in V \therefore$

$u+(-u) = 0$

$-u = -x_1, y_1, -z_1$

$(x_1, y_1, z_1) + (-x_1, y_1, z_1)$

$= (0, 0, 0) = 0$

$A_5 : u \in V, \text{if } \exists\, 0 \in V \therefore u+0 = u$

$0 = (0,0,0)$

$u+0 = u$

$(x_1, y_1, z_1) + (0,0,0) =$

$= (x_1+0, y_1+0, z_1+0)$

$= x_1, y_1, z_1 = u$

$M_1 : u \in V$ any scalar c, $\therefore$

$cu \in V$

$cu = c(x_1, y_1, z_1)$

$= (cx_1, y_1, z_1) \in V$

$M_2 : a(ku) = (ak)u$

$a(kx_1, 0, kz_1) = akx_1, 0, akz_1$

$ak(x_1, 0, z_1) = akx_1, 0, akz_1$

$\therefore a(ku) = (ak)u$

$M_3 . (c+k)u = cu + ku$

$(c+k)(x_1, 0, z_1)$

$= ((c+k)x_1, 0, (c+k)z_1)$

$cu = cx_1, 0, cz_1$

$ku = k, x_1, 0, kz_1$

$cu + ku = (x_1 + kx_1, 0+0, cz_1 + kz_1$

$= (c+k)x_1, 0, (c+k)z_1 //$

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Systems of linear equations

- consistent eqn. has a solution $<$ unique / infinite solutions
- inconsistent eqn. has no solution.

- Determine whether u is a LC of V_1, V_2, V_3 from each of the FSL.

1- $V_1 = (-1,4,1)$ $V_2 = (-5,-5,5)$ $V_3 = (-5,4,-1)$ and $u = (-2,-3,3)$

2- $V_1 = (4,2,5)$ $V_2 = (4,-4,-4)$ $V_3 = (1,1,-5)$ $u = (4,2,0)$

3- $V_1 = (5,4,-4)$ $V_2 = (5,3,3)$ $V_3 = (5,3,3)$ $u = (-20,-6,9)$

soln

1- Using back substitution.

Let c_1, c_2, c_3 be scalars such that

$c_1V_1 + c_2V_2 + c_3V_3 = u$

$c_1(-1,4,1) + c_2(-5,-5,5) + c_3(-5,4,-1) = (-2,-3,3)$

$-c_1 - 5c_2 - 5c_3 = -2$ — 1

$4c_1 - 5c_2 + 4c_3 = -3$ — 2

$c_1 + 5c_2 - c_3 = 3$ — 3

Eliminate c_1 ~~frm~~ eqn 1 using eqn 3

$$
\begin{aligned}
+\ -\cancel{c_1} - \cancel{5c_2} - 5c_3 &= -2 \\
\cancel{c_1} + \cancel{5c_2} - c_3 &= 3 \\
\hline
-6c_3 &= 1 \\
c_3 &= -\frac{1}{6}
\end{aligned}
$$

Eliminate c_1 in eqn. 3 using eqn 2

$4 \times c_1 + 5c_2 - c_3 = 3$

$1 \times 4c_1 + 5c_2 + 4c_3 = -3$

$4c_1 + 20c_2 - 4c_3 = 12$

$- \quad 4c_1 - 5c_2 + 4c_3 = -3$

$25c_2 - 8c_3 = 15 \quad — (4)$

Substitute the value of c_3 in eqn 4

$25c_2 - 8\left(-\frac{1}{6}\right) = 15$

$25c_2 + \frac{8}{6} = 15 \quad = 25c_2 = \frac{15}{1} - \frac{4}{3} = \frac{45-4}{3}$

$\frac{25c_2}{25} = \frac{41}{3} \times \frac{1}{25} \quad c_2 = \frac{41}{75}$

Substitute the value of c_2, c_3 in eqn 3

$c_1 + 5\left(\frac{41}{75}\right) - \left(\frac{-1}{6}\right) = 3$

$c_1 + \frac{41}{15} + \frac{1}{6} = 3$

$c_1 = \frac{3}{1} - \frac{41}{15} - \frac{1}{6} = \frac{290 - 246 - 15}{90} = \frac{29}{90}$

Past Q

$v_1 = (1,3,7)$ $v_2 = (2,7,4)$ $v_3 = (2,6,5)$ $u = (3,7,23)$

$c_1 v_1 + c_2 v_2 + c_3 v_3 = u$

$c_1(1,3,7) + c_2(2,7,4) + c_3(2,6,5) = (3,7,23)$

$c_1 + 2c_2 + 2c_3 = 3$ — 1

$3c_1 + 7c_2 + 6c_3 = 7$ — 2

$7c_1 + 4c_2 + 5c_3 = 23$ — 3

use eqn 1 to elimatate eqn 2

$\times 1$ $3c_1 + 7c_2 + 6c_3 = 7$

$\times 3$ $c_1 + 2c_2 + 2c_3 = 3$

$-$ $\cancel{3c_1} + 7c_2 + \cancel{6c_3} = 7$

$\cancel{3c_1} + 6c_2 + \cancel{6c_3} = 9$

$c_2 = -2$ — 4

use eqn 1 to elimatate eqn 3

$\times 1$ $7c_1 + 4c_2 + 5c_3 = 23$

$\times 7$ $c_1 + 2c_2 + 2c_3 = 3$

$-$ $\cancel{7c_1} + 4c_2 + 5c_3 = 23$

$\cancel{7c_1} + 14c_2 + 14c_3 = 21$

$-10c_2 + 9c_3 = -2$ — 4

Substitute c_2 in eqn 4

$10(-2) + 9c_3 = -2$

$$-20 + 9c_3 = -2$$
$$9c_3 = -2 + 20$$
$$\frac{\cancel{9}c_3}{\cancel{9}} = \frac{18}{9} = 2$$

substitute c_3 in eqn 1

$$c_1 + 2(-2) + 2(2) = 3$$
$$c_1 - \cancel{4} + \cancel{4} = 3$$
$$c_1 = 3$$

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* Cramer's rule

for Example: $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$
$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$c_1 = \frac{\Delta_1}{\Delta} \quad c_2 = \frac{\Delta_2}{\Delta} \quad c_3 = \frac{\Delta_3}{\Delta}$$

$$\Delta = \text{Determinant of } A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{23} & a_{33} \end{vmatrix} \qquad \Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Example, past Q

$v_1 = (1,3,7) \quad v_2 = (2,7,4) \quad v_3 = (2,6,5) \quad u = (3,7,23)$

$c_1 v_1 + c_2 v_2 + c_3 v_3 = u$

$c_1(1,3,7) + c_2(2,7,4) + c_3(2,6,5) = (3,7,23)$

$c_1 + 2c_2 + 2c_3 = 3 \quad -1$

$3c_1 + 7c_2 + 6c_3 = 7 \quad -2$

$7c_1 + 4c_2 + 5c_3 = 23 \quad -3$

$$\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 6 \\ 7 & 4 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 23 \end{bmatrix}$$

Δ = Determinant of A = $\begin{vmatrix} 1 & 2 & 2 \\ 3 & 7 & 6 \\ 7 & 4 & 5 \end{vmatrix} =$

Linear Spanning

Def: If every vector in a vectorspace V can be [written] as a linear combination of the vectors $v_1, v_2, v_3 --- v_k$ then V is spanned or generated by the vectors $v_1, v_2, v_3 --- v_k$ and called the sets of the vectors $v_1, v_2, v_3 --- v_k$ a spanning set of V.

$V = \{v_1, v_2, v_3, u\}$

$c_1v_1 + c_2v_2 + c_3v_3 = u$

$c_1v_2 + c_2v_3 + c_3u = v_1$

$c_1v_1 + c_2v_3 + c_3u = v_2$

$c_1v_1 + c_2v_2 + c_3u = v_3$

Example: past Q

$S = \{v_1, v_2, v_3\}$ $V = \mathbb{R}^3$

$v_1 = (1,2,3)$. $v_2 = (3,4,5)$ $v_3 = (4,5,6)$

Soln

$c_1v_1 + c_2v_2 + c_3v_3 = u$

$c_1(1,2,3) + c_2(3,4,5) + c_3(4,5,6) = (p,q,r)$

$c_1 + 3c_2 + 4c_3 = p$ — (1)

$2c_1 + 4c_2 + 5c_3 = q$ — (2)

$3c_1 + 5c_2 + 6c_3 = r$ — (3)

$$c_1 + 3c_2 + 4c_3 = p \quad (1)$$
$$2c_2 + 3c_3 = 2p - q \quad (4)$$
$$4c_2 + 6c_3 = 3p - r \quad (5)$$

$$\times 2 \quad c_1 + 3c_2 + 4c_3 = p$$
$$\times 1 \quad 2c_1 + 4c_2 + 5c_3 = q$$

$$2c_1 + 6c_2 + 8c_3 = 2p$$
$$- \quad 2c_1 + 4c_2 + 5c_3 = q$$
$$2c_2 + 3c_3 = 2p - q \quad (4)$$

$$\times 3 \quad c_1 + 3c_2 + 4c_3 = p$$
$$\times 1 \quad 3c_1 + 5c_2 + 6c_3 = r$$

$$3c_1 + 9c_2 + 12c_3 = 3p$$
$$- \quad 3c_1 + 5c_2 + 6c_3 = r$$
$$4c_2 + 6c_3 = 3p - r \quad (5)$$

$$\frac{\cancel{2}c_2}{\cancel{2}} = \frac{2p - q - 3c_3}{2}$$

$$\overset{2}{\cancel{4}}\left(\frac{2p - q - 3c_3}{\cancel{2}_1}\right) + 6c_3 = 3p - r$$

$$4p - 2q - \cancel{6c_3} + \cancel{6c_3} = 3p - r$$
$$4p - 2q = 3p - r$$
$$p - 2q + r = 0$$

$\therefore$ $0 \neq$ a number, there is no solution

Hence, the vector is not a linear spanning.

V_1, V_2, V_3 does not span $\mathbb{R}^3$

if $0 = 0$, there is solution with infinite.
if $c_1, c_2, c_3 =$ number there is unique soln

Exercise : ②

- Determine whether vectors $v_1 = (1, 1, 4)$. $v_2 = (2, 1, 3)$

$v_3 = (4, -3, 5)$ spanned $\mathbb{R}^3$