Worked Examples from Introductory Physics
Vol. I: Basic Mechanics

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February 24, 2005

Contents

To the Student. Yeah, *You.*

Physics is learned through problem-solving. There is no other way.

Problem–solving can be very hard to learn, and students often confuse it with the algebra with which one *finishes up* a problem. But the level of mathematics and calculator skills required in a general physics course is not very great. Any student who has difficulty solving the equations we derive in working these problems really needs to re–take some math courses! Physics is all about *finding* the right equations to solve. The rest of it ought to be easy.

My two purposes in composing this book are: (1) To summarize the principles that are absolutely essential in first–year physics. This is the material which a student *must* be familiar with before going in to take an exam. (2) To provide a set of example problems with the most complete, clearest solutions that I know how to give.

I hope I've done something useful in writing this. Of course, nowadays most physics textbooks give lots of example problems (many more than they did in years past) and even some sections on problem–solving skills, and there are study–guide–type books one can buy which have *many* worked examples in physics. But typically these books don't have enough discussion as to how to set up the problem and why one uses the particular principles to solve them; usually I find that there aren't enough *words* included between the equations that are written down. Students seem to think so too. Part of the reason for my producing this notebook is the reaction of many students to earlier example notebooks I have written up: "You put lots of words between all the equations!"

At present, most of the problems are taken from the popular calculus–based textbooks by Halliday, Resnick and Walker and by Serway. I have copied down the problems nearly verbatim from these books, except possibly to change a number here and there. There's a reason for this: Students will take their exams from individual professors who will state their problems in their own way, and they just have to get used to the professors' styles and answer the questions as their teachers have posed them. Style should not get in the way of *physics*.

Organization of the Book:

The chapters cover material *roughly* in the order that it is presented in your physics course, though there may be some differences. The chapters do *not* correspond to the same chapters in your textbook. Each chapter begins with a summary of the basic principles, where I give the most important equations that we will need in solving the problems. I have called this *The Important Stuff* because. . . well, you get the idea. In general I give no derivations of the equations though learning the derivations is an important part of an education in physics. I refer you to your textbook for those.

After that, I give worked examples. The emphasis here is to show how we try to clarify the situation presented in the problem (often with a picture), to show what principles and equations from the chapter are applicable to the situation, and finally to show how to use those equation to solve for the desired quantities.

I have not been especially careful about numerical accuracy in solving these problems; oftentimes my results will have more or fewer significant figures than they should, if one takes the data given in the problem literally. But just as often, the textbook author who stated the problem wasn't very careful about accuracy either! Questions about accuracy are very important in lab work, but the focus of *this* book is problem–solving. (After all, the problems *are* fictional!)

I am continually correcting and updating this book, and the date on the title page indicates the version of the copy you are reading. I am sure that no matter what version you are reading there will be some errors and some sections which are incomplete. I apologize in advance.

Reactions, please!

Please help me with this project: Give me your reaction to this work: Tell me what you liked, what was particularly effective, what was particularly confusing, what you'd like to see more of or less of. I can reached at `murdock@tntech.edu` or even at x-3044. If this effort is helping you to learn physics, I'll do more of it!

DPM

Chapter 1

Units and Vectors: Tools for Physics

1.1 The Important Stuff

1.1.1 The SI System

Physics is based on measurement. Measurements are made by comparisons to well–defined **standards** which define the **units** for our measurements.

The **SI system** (popularly known as the *metric system*) is the one used in physics. Its unit of length is the meter, its unit of time is the second and its unit of mass is the kilogram. Other quantities in physics are derived from these. For example the unit of energy is the joule, defined by $1\,\mathrm{J} = 1\,\frac{\mathrm{kg}\cdot\mathrm{m}^2}{\mathrm{s}^2}$.

As a convenience in using the SI system we can associate prefixes with the basic units to represent powers of 10. The most commonly used prefixes are given here:

Factor	**Prefix**	**Symbol**
10^{-12}	pico-	p
10^{-9}	nano-	n
10^{-6}	micro-	μ
10^{-3}	milli-	m
10^{-2}	centi-	c
10^{3}	kilo-	k
10^{6}	mega-	M
10^{9}	giga-	G

Other basic units commonly used in physics are:

$$\textbf{Time}: \qquad 1\,\mathrm{minute} = 60\,\mathrm{s} \qquad 1\,\mathrm{hour} = 60\,\mathrm{min} \qquad \mathrm{etc.}$$

$$\textbf{Mass}: \qquad 1 \text{ atomic mass unit} = 1\,\mathrm{u} = 1.6605 \times 10^{-27}\,\mathrm{kg}$$

1.1.2 Changing Units

In all of our mathematical operations we must *always* write down the units and we *always* treat the unit symbols as multiplicative factors. For example, if me multiply 3.0 kg by 2.0 $\frac{\mathrm{m}}{\mathrm{s}}$ we get

$$(3.0\,\mathrm{kg}) \cdot (2.0\,\tfrac{\mathrm{m}}{\mathrm{s}}) = 6.0\,\tfrac{\mathrm{kg \cdot m}}{\mathrm{s}}$$

We use the same idea in changing the units in which some physical quantity is expressed. We can multiply the original quantity by a **conversion factor**, i.e. a ratio of values for which the numerator is the same thing as the denominator. The conversion factor is then *equal to 1*, and so we *do not change* the original quantity when we multiply by the conversion factor.

Examples of conversion factors are:

$$\left(\frac{1\,\mathrm{min}}{60\,\mathrm{s}}\right) \qquad \left(\frac{100\,\mathrm{cm}}{1\,\mathrm{m}}\right) \qquad \left(\frac{1\,\mathrm{yr}}{365.25\,\mathrm{day}}\right) \qquad \left(\frac{1\,\mathrm{m}}{3.28\,\mathrm{ft}}\right)$$

1.1.3 Density

A quantity which will be encountered in your study of liquids and solids is the **density** of a sample. It is usually denoted by ρ and is defined as the ratio of mass to volume:

$$\rho = \frac{m}{V} \tag{1.1}$$

The SI units of density are $\frac{\mathrm{kg}}{\mathrm{m}^3}$ but you often see it expressed in $\frac{\mathrm{g}}{\mathrm{cm}^3}$.

1.1.4 Dimensional Analysis

Every equation that we use in physics must have *the same type of units* on both sides of the equals sign. Our basic unit types (**dimensions**) are length (L), time (T) and mass (M). When we do **dimensional analysis** we focus on the units of a physics equation without worrying about the numerical values.

1.1.5 Vectors; Vector Addition

Many of the quantities we encounter in physics have both *magnitude* ("how much") and *direction*. These are **vector** quantities.

We can represent vectors graphically as arrows and then the sum of two vectors is found (graphically) by joining the head of one to the tail of the other and then connecting head to tail for the combination, as shown in Fig. 1.1 . The sum of two (or more) vectors is often called the **resultant**.

We can add vectors in any order we want: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$. We say that vector addition is "commutative".

We express vectors in **component form** using the **unit vectors** **i**, **j** and **k**, which each have magnitude 1 and point along the x, y and z axes of the coordinate system, respectively.

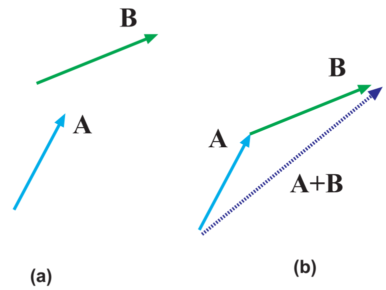

Figure 1.1: Vector addition. (a) shows the vectors **A** and **B** to be summed. (b) shows how to perform the sum graphically.

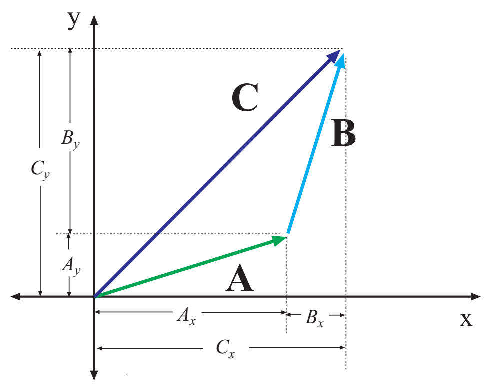

Figure 1.2: Addition of vectors by components (in two dimensions).

Any vector can be expressed as a sum of multiples of these basic vectors; for example, for the vector **A** we would write:

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \ .$$

Here we would say that A_x is the x component of the vector **A**; likewise for y and z.

In Fig. 1.2 we illustrate how we get the components for a vector which is the *sum* of two other vectors. If

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \qquad \text{and} \qquad \mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$$

then

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k} \tag{1.2}$$

Once we have found the (Cartesian) component of two vectors, addition is simple; just add the *corresponding components* of the two vectors to get the components of the resultant vector.

When we multiply a vector by a scalar, the scalar multiplies each component; If **A** is a vector and n is a scalar, then

$$c\mathbf{A} = cA_x\mathbf{i} + cA_y\mathbf{j} + cA_z\mathbf{k} \tag{1.3}$$

In terms of its components, the magnitude ("length") of a vector **A** (which we write as A) is given by:

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{1.4}$$

Many of our physics problems will be in two dimensions (x and y) and then we can also represent it in **polar** form. If **A** is a two–dimensional vector and θ as the angle that **A** makes with the $+x$ axis *measured counter-clockwise* then we can express this vector in terms of components A_x and A_y or in terms of its magnitude A and the angle θ. These descriptions are related by:

$$A_x = A\cos\theta \qquad\qquad A_y = A\sin\theta \tag{1.5}$$

$$A = \sqrt{A_x^2 + A_y^2} \qquad\qquad \tan\theta = \frac{A_y}{A_x} \tag{1.6}$$

When we use Eq. 1.6 to find θ from A_x and A_y we need to be careful because the inverse tangent operation (as done on a calculator) might give an angle in the wrong quadrant; one must think about the signs of A_x and A_y.

1.1.6 Multiplying Vectors

There are two ways to "multiply" two vectors together.

The **scalar product** (or **dot product**) of the vectors **a** and **b** is given by

$$\mathbf{a}\cdot\mathbf{b} = ab\cos\phi \tag{1.7}$$

where a is the magnitude of **a**, b is the magnitude of **b** and ϕ is the angle between **a** and **b**.

The scalar product is commutative: $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$. One can show that $\mathbf{a}\cdot\mathbf{b}$ is related to the components of **a** and **b** by:

$$\mathbf{a}\cdot\mathbf{b} = a_x b_x + a_y b_y + a_z b_z \tag{1.8}$$

If two vectors are perpendicular then their scalar product is *zero*.

The **vector product** (or **cross product**) of vectors **a** and **b** is a vector **c** whose magnitude is given by

$$c = ab\sin\phi \tag{1.9}$$

where ϕ is the *smallest* angle between **a** and **b**. The direction of **c** is perpendicular to the plane containing **a** and **b** with its orientation given by the **right–hand rule**. One way of using the right–hand rule is to let the fingers of the right hand bend (in their natural direction!) from **a** to **b**; the direction of the thumb is the direction of $\mathbf{c} = \mathbf{a}\times\mathbf{b}$. This is illustrated in Fig. 1.3.

The vector product is *anti*–commutative: $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$.

Relations among the unit vectors for vector products are:

$$\mathbf{i}\times\mathbf{j} = \mathbf{k} \qquad \mathbf{j}\times\mathbf{k} = \mathbf{i} \qquad \mathbf{k}\times\mathbf{i} = \mathbf{j} \tag{1.10}$$

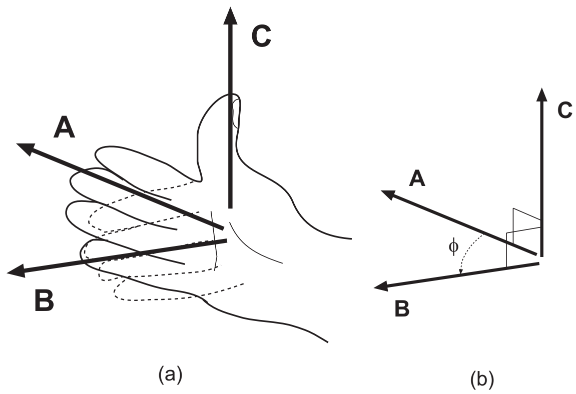

Figure 1.3: (a) Finding the direction of $\mathbf{A} \times \mathbf{B}$. Fingers of the right hand sweep from **A** to **B** in the shortest and least painful way. The extended thumb points in the direction of **C**. (b) Vectors **A**, **B** and **C**. The magnitude of **C** is $C = AB \sin\phi$.

The vector product of **a** and **b** can be computed from the components of these vectors by:

$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k} \tag{1.11}$$

which can be abbreviated by the notation of the determinant:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{1.12}$$

1.2 Worked Examples

1.2.1 Changing Units

1. The Empire State Building is $1472\,\mathrm{ft}$ high. Express this height in both meters and centimeters.

To do the first unit conversion (feet to meters), we can use the relation (see the Conversion Factors in the back of this book):

$$1\,\mathrm{m} = 3.281\,\mathrm{ft}$$

We set up the conversion factor so that "ft" cancels and leaves meters:

$$1472\,\mathrm{ft} = (1472\,\mathrm{ft})\left(\frac{1\,\mathrm{m}}{3.281\,\mathrm{ft}}\right) = 448.6\,\mathrm{m}\ .$$

So the height can be expressed as $448.6\,\mathrm{m}$. To convert this to centimeters, use:

$$1\,\mathrm{m} = 100\,\mathrm{cm}$$

and get:

$$448.6\,\mathrm{m} = (448.6\,\mathrm{m})\left(\frac{100\,\mathrm{cm}}{1\,\mathrm{m}}\right) = 4.486 \times 10^4\,\mathrm{cm}$$

The Empire State Building is 4.486×10^4 cm high!

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2. A rectangular building lot is $100.0\,\mathrm{ft}$ by $150.0\,\mathrm{ft}$. Determine the area of this lot in m^2.

The area of a rectangle is just the product of its length and width so the area of the lot is

$$A = (100.0\,\mathrm{ft})(150.0\,\mathrm{ft}) = 1.500 \times 10^4\,\mathrm{ft}^2$$

To convert this to units of m^2 we can use the relation

$$1\,\mathrm{m} = 3.281\,\mathrm{ft}$$

but the conversion factor needs to be applied *twice* so as to cancel "ft^2" and get "m^2". We write:

$$1.500 \times 10^4\,\mathrm{ft}^2 = (1.500 \times 10^4\,\mathrm{ft}^2) \cdot \left(\frac{1\,\mathrm{m}}{3.281\,\mathrm{ft}}\right)^2 = 1.393 \times 10^3\,\mathrm{m}^2$$

The area of the lot is $1.393 \times 10^3\,\mathrm{m}^2$.

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3. The Earth is approximately a sphere of radius $6.37 \times 10^6\,\mathrm{m}$. (a) What is its circumference in kilometers? (b) What is its surface area in square kilometers? (c) What is its volume in cubic kilometers?

(a) The circumference of the sphere of radius R, i.e. the distance around any "great circle" is $C = 2\pi R$. Using the given value of R we find:

$$C = 2\pi R = 2\pi(6.37 \times 10^6\,\mathrm{m}) = 4.00 \times 10^7\,\mathrm{m}\ .$$

To convert this to kilometers, use the relation $1\,\mathrm{km} = 10^3\,\mathrm{m}$ in a conversion factor:

$$C = 4.00 \times 10^7\,\mathrm{m} = (4.00 \times 10^7\,\mathrm{m}) \cdot \left(\frac{1\,\mathrm{km}}{10^3\,\mathrm{m}}\right) = 4.00 \times 10^4\,\mathrm{km}$$

The circumference of the Earth is $4.00 \times 10^4\,\mathrm{km}$.

(b) The surface area of a sphere of radius R is $A = 4\pi R^2$. So we get

$$A = 4\pi R^2 = 4\pi(6.37 \times 10^6\,\mathrm{m})^2 = 5.10 \times 10^{14}\,\mathrm{m}^2$$

Again, use $1\,\mathrm{km} = 10^3\,\mathrm{m}$ but to cancel out the units "m^2" and replace them with "km^2" it must be applied *twice*:

$$A = 5.10 \times 10^{14}\,\mathrm{m}^2 = (5.10 \times 10^{14}\,\mathrm{m}^2) \cdot \left(\frac{1\,\mathrm{km}}{10^3\,\mathrm{m}}\right)^2 = 5.10 \times 10^8\,\mathrm{km}^2$$

The surface area of the Earth is $5.10 \times 10^8\,\mathrm{km}^2$.

(c) The volume of a sphere of radius R is $V = \frac{4}{3}\pi R^3$. So we get

$$V = \tfrac{4}{3}\pi R^3 = \tfrac{4}{3}\pi(6.37 \times 10^6\,\mathrm{m})^3 = 1.08 \times 10^{21}\,\mathrm{m}^3$$

Again, use $1\,\mathrm{km} = 10^3\,\mathrm{m}$ but to cancel out the units "m^3" and replace them with "km^3" it must be applied *three times*:

$$V = 1.08 \times 10^{21}\,\mathrm{m}^3 = (1.08 \times 10^{21}\,\mathrm{m}^3) \cdot \left(\frac{1\,\mathrm{km}}{10^3\,\mathrm{m}}\right)^3 = 1.08 \times 10^{12}\,\mathrm{km}^3$$

The volume of the Earth is $1.08 \times 10^{12}\,\mathrm{km}^3$.

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4. Calculate the number of kilometers in $20.0\,\mathrm{mi}$ using only the following conversion factors: $1\,\mathrm{mi} = 5280\,\mathrm{ft}$, $1\,\mathrm{ft} = 12\,\mathrm{in}$, $1\,\mathrm{in} = 2.54\,\mathrm{cm}$, $1\,\mathrm{m} = 100\,\mathrm{cm}$, $1\,\mathrm{km} = 1000\,\mathrm{m}$.

Set up the "factors of 1" as follows:

$$\begin{aligned} 20.0\,\mathrm{mi} &= (20.0\,\mathrm{mi}) \cdot \left(\frac{5280\,\mathrm{ft}}{1\,\mathrm{mi}}\right) \cdot \left(\frac{12\,\mathrm{in}}{1\,\mathrm{ft}}\right) \cdot \left(\frac{2.54\,\mathrm{cm}}{1\,\mathrm{in}}\right) \cdot \left(\frac{1\,\mathrm{m}}{100\,\mathrm{cm}}\right) \cdot \left(\frac{1\,\mathrm{km}}{1000\,\mathrm{m}}\right) \\ &= 32.2\,\mathrm{km} \end{aligned}$$

Setting up the "factors of 1" in this way, all of the unit symbols cancel except for km (kilometers) which we keep as the units of the answer.

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5. One gallon of paint (volume $= 3.78 \times 10^{-3}\,\mathrm{m}^3$) covers an area of $25.0\,\mathrm{m}^3$. What is the thickness of the paint on the wall?

We will assume that the volume which the paint occupies while it's covering the wall is the *same* as it has when it is in the can. (There are reasons why this may not be true, but let's just do this and proceed.)

The paint on the wall covers an area A and has a thickness τ; the volume occupied is the area time the thickness:

$$V = A\tau\ .$$

We have V and A; we just need to solve for τ:

$$\tau = \frac{V}{A} = \frac{3.78 \times 10^{-3}\,\mathrm{m}^3}{25.0\,\mathrm{m}^2} = 1.51 \times 10^{-4}\,\mathrm{m}\ .$$

The thickness is $1.51 \times 10^{-4}\,\mathrm{m}$. This quantity can also be expressed as $0.151\,\mathrm{mm}$.

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6. A certain brand of house paint claims a coverage of $460\,\frac{\mathrm{ft}^2}{\mathrm{gal}}$. (a) Express this quantity in square meters per liter. (b) Express this quantity in SI base units. (c) What is the inverse of the original quantity, and what is its physical significance?

(a) Use the following relations in forming the conversion factors: $1\,\mathrm{m} = 3.28\,\mathrm{ft}$ and $1000\,\mathrm{liter} = 264\,\mathrm{gal}$. To get proper cancellation of the units we set it up as:

$$460\,\tfrac{\mathrm{ft}^2}{\mathrm{gal}} = (460\,\tfrac{\mathrm{ft}^2}{\mathrm{gal}}) \cdot \left(\frac{1\,\mathrm{m}}{3.28\,\mathrm{ft}}\right)^2 \cdot \left(\frac{264\,\mathrm{gal}}{1000\,\mathrm{L}}\right) = 11.3\,\tfrac{\mathrm{m}^2}{\mathrm{L}}$$

(b) Even though the units of the answer to part (a) are based on the metric system, they are not made from the *base* units of the SI system, which are m, s, and kg. To make the complete conversion to SI units we need to use the relation $1\,\mathrm{m}^3 = 1000\,\mathrm{L}$. Then we get:

$$11.3\,\tfrac{\mathrm{m}^2}{\mathrm{L}} = (11.3\,\tfrac{\mathrm{m}^2}{\mathrm{L}}) \cdot \left(\frac{1000\,\mathrm{L}}{1\,\mathrm{m}^3}\right) = 1.13 \times 10^4\,\mathrm{m}^{-1}$$

So the coverage can also be expressed (not so meaningfully, perhaps) as $1.13 \times 10^4\,\mathrm{m}^{-1}$.

(c) The inverse (reciprocal) of the quantity as it was *originally* expressed is

$$\left(460\,\tfrac{\mathrm{ft}^2}{\mathrm{gal}}\right)^{-1} = 2.17 \times 10^{-3}\,\tfrac{\mathrm{gal}}{\mathrm{ft}^2}\,.$$

Of course when we take the reciprocal the *units* in the numerator and denominator also switch places!

Now, the first expression of the quantity tells us that $460\,\mathrm{ft}^2$ are associated with every gallon, that is, each gallon will provide $460\,\mathrm{ft}^2$ of coverage. The new expression tells us that 2.17×10^{-3} gal are associated with every ft^2, that is, to cover one square foot of surface with paint, one needs 2.17×10^{-3} gallons of it.

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7. Express the speed of light, $3.0 \times 10^8\,\frac{\mathrm{m}}{\mathrm{s}}$ **in (a) feet per nanosecond and (b) millimeters per picosecond.**

(a) For this conversion we can use the following facts:

$$1\,\mathrm{m} = 3.28\,\mathrm{ft} \qquad \text{and} \qquad 1\,\mathrm{ns} = 10^{-9}\,\mathrm{s}$$

to get:

$$\begin{aligned} 3.0 \times 10^8\,\tfrac{\mathrm{m}}{\mathrm{s}} &= (3.0 \times 10^8\,\tfrac{\mathrm{m}}{\mathrm{s}}) \cdot \left(\frac{3.28\,\mathrm{ft}}{1\,\mathrm{m}}\right) \cdot \left(\frac{10^{-9}\,\mathrm{s}}{1\,\mathrm{ns}}\right) \\ &= 0.98\,\tfrac{\mathrm{ft}}{\mathrm{ns}} \end{aligned}$$

In these new units, the speed of light is $0.98\,\frac{\mathrm{ft}}{\mathrm{ns}}$.

(b) For this conversion we can use:

$$1\,\mathrm{mm} = 10^{-3}\,\mathrm{m} \qquad \text{and} \qquad 1\,\mathrm{ps} = 10^{-12}\,\mathrm{s}$$

and set up the factors as follows:

$$\begin{aligned} 3.0 \times 10^8\,\tfrac{\mathrm{m}}{\mathrm{s}} &= (3.0 \times 10^8\,\tfrac{\mathrm{m}}{\mathrm{s}}) \cdot \left(\frac{1\,\mathrm{mm}}{10^{-3}\,\mathrm{m}}\right) \cdot \left(\frac{10^{-12}\,\mathrm{s}}{1\,\mathrm{ps}}\right) \\ &= 3.0 \times 10^{-1}\,\tfrac{\mathrm{mm}}{\mathrm{ps}} \end{aligned}$$

In these new units, the speed of light is $3.0 \times 10^{-1}\,\frac{\mathrm{mm}}{\mathrm{ps}}$.

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8. One molecule of water (H_2O) contains two atoms of hydrogen and one atom of oxygen. A hydrogen atom has a mass of $1.0\,\mathrm{u}$ and an atom of oxygen has a mass of $16\,\mathrm{u}$, approximately. (a) What is the mass in kilograms of one molecule of water? (b) How many molecules of water are in the world's oceans, which have an estimated total mass of $1.4 \times 10^{21}\,\mathrm{kg}$?

(a) We are given the masses of the atoms of H and O in atomic mass units; using these values, one molecule of H_2O has a mass of

$$m_{\mathrm{H_2O}} = 2(1.0\,\mathrm{u}) + 16\,\mathrm{u} = 18\,\mathrm{u}$$

Use the relation between u (atomic mass units) and kilograms to convert this to kg:

$$m_{\mathrm{H_2O}} = (18\,\mathrm{u})\left(\frac{1.6605 \times 10^{-27}\,\mathrm{kg}}{1\,\mathrm{u}}\right) = 3.0 \times 10^{-26}\,\mathrm{kg}$$

One water molecule has a mass of $3.0 \times 10^{-26}\,\mathrm{kg}$.

(b) To get the number of molecules in all the oceans, divide the mass of *all* the oceans' water by the mass of *one* molecule:

$$N = \frac{1.4 \times 10^{21}\,\mathrm{kg}}{3.0 \times 10^{-26}\,\mathrm{kg}} = 4.7 \times 10^{46}\ .$$

. . . a large number of molecules!

1.2.2 Density

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9. Calculate the density of a solid cube that measures $5.00\,\mathrm{cm}$ on each side and has a mass of $350\,\mathrm{g}$.

The volume of this cube is

$$V = (5.00\,\mathrm{cm}) \cdot (5.00\,\mathrm{cm}) \cdot (5.00\,\mathrm{cm}) = 125\,\mathrm{cm}^3$$

So from Eq. 1.1 the density of the cube is

$$\rho = \frac{m}{V} = \frac{350\,\mathrm{g}}{125\,\mathrm{cm}^3} = 2.80\frac{\mathrm{g}}{\mathrm{cm}^3}$$

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10. The mass of the planet Saturn is $5.64 \times 10^{26}\,\mathrm{kg}$ and its radius is $6.00 \times 10^{7}\,\mathrm{m}$. Calculate its density.

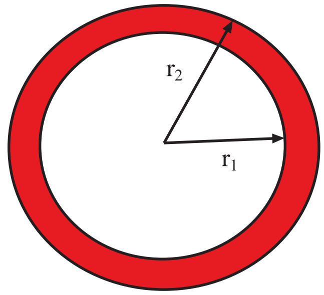

Figure 1.4: Cross–section of copper shell in Example 11.

The planet Saturn is roughly a sphere. (But only roughly! Actually its shape is rather distorted.) Using the formula for the volume of a sphere, we find the volume of Saturn:

$$V = \tfrac{4}{3}\pi R^3 = \tfrac{4}{3}\pi(6.00 \times 10^7\,\mathrm{m})^3 = 9.05 \times 10^{23}\,\mathrm{m}^3$$

Now using the definition of density we find:

$$\rho = \frac{m}{V} = \frac{5.64 \times 10^{26}\,\mathrm{kg}}{9.05 \times 10^{23}\,\mathrm{m}^3} = 6.23 \times 10^2\,\tfrac{\mathrm{kg}}{\mathrm{m}^3}$$

While this answer is correct, it is useful to express the result in units of $\frac{\mathrm{g}}{\mathrm{cm}^3}$. Using our conversion factors in the usual way, we get:

$$6.23 \times 10^2\,\tfrac{\mathrm{kg}}{\mathrm{m}^3} = (6.23 \times 10^2\,\tfrac{\mathrm{kg}}{\mathrm{m}^3}) \cdot \left(\frac{10^3\,\mathrm{g}}{1\,\mathrm{kg}}\right) \cdot \left(\frac{1\,\mathrm{m}}{100\,\mathrm{cm}}\right)^3 = 0.623\,\tfrac{\mathrm{g}}{\mathrm{cm}^3}$$

The average density of Saturn is $0.623\,\frac{\mathrm{g}}{\mathrm{cm}^3}$. Interestingly, this is less than the density of water.

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11. How many grams of copper are required to make a hollow spherical shell with an inner radius of $5.70\,\mathrm{cm}$ and an outer radius of $5.75\,\mathrm{cm}$? The density of copper is $8.93\,\mathrm{g/cm}^3$.

A cross–section of the copper sphere is shown in Fig. 1.4. The outer and inner radii are noted as r_2 and r_1, respectively. We must find the volume of space *occupied by the copper metal*; this volume is the difference in the volumes of the two spherical surfaces:

$$V_{\mathrm{copper}} = V_2 - V_1 = \tfrac{4}{3}\pi r_2^3 - \tfrac{4}{3}\pi r_1^3 = \tfrac{4}{3}\pi(r_2^3 - r_1^3)$$

With the given values of the radii, we find:

$$V_{\mathrm{copper}} = \tfrac{4}{3}\pi((5.75\,\mathrm{cm})^3 - (5.70\,\mathrm{cm})^3) = 20.6\,\mathrm{cm}^3$$

Now use the definition of density to find the mass of the copper contained in the shell:

$$\rho = \frac{m_{\mathrm{copper}}}{V_{\mathrm{copper}}} \qquad \Longrightarrow \qquad m_{\mathrm{copper}} = \rho V_{\mathrm{copper}} = \left(8.93\,\tfrac{\mathrm{g}}{\mathrm{cm}^3}\right)(20.6\,\mathrm{cm}^3) = 184\,\mathrm{g}$$

184 grams of copper are required to make the spherical shell of the given dimensions.

---

12. One cubic meter ($1.00\,\mathrm{m}^3$) of aluminum has a mass of $2.70 \times 10^3\,\mathrm{kg}$, and $1.00\,\mathrm{m}^3$ of iron has a mass of $7.86 \times 10^3\,\mathrm{kg}$. Find the radius of a solid aluminum sphere that will balance a solid iron sphere of radius $2.00\,\mathrm{cm}$ on an equal–arm balance.

In the statement of the problem, we are given the densities of aluminum and iron:

$$\rho_{\mathrm{Al}} = 2.70 \times 10^3\,\tfrac{\mathrm{kg}}{\mathrm{m}^3} \qquad \text{and} \qquad \rho_{\mathrm{Fe}} = 7.86 \times 10^3\,\tfrac{\mathrm{kg}}{\mathrm{m}^3}\ .$$

A solid iron sphere of radius $R = 2.00\,\mathrm{cm} = 2.00 \times 10^{-2}\,\mathrm{m}$ has a volume

$$V_{\mathrm{Fe}} = \tfrac{4}{3}\pi R^3 = \tfrac{4}{3}\pi(2.00 \times 10^{-2}\,\mathrm{m})^3 = 3.35 \times 10^{-5}\,\mathrm{m}^3$$

so that from $M_{\mathrm{Fe}} = \rho_{\mathrm{Fe}} V_{\mathrm{Fe}}$ we find the mass of the iron sphere:

$$M_{\mathrm{Fe}} = \rho_{\mathrm{Fe}} V_{\mathrm{Fe}} = \left(7.86 \times 10^3\,\tfrac{\mathrm{kg}}{\mathrm{m}^3}\right)(3.35 \times 10^{-5}\,\mathrm{m}^3) = 2.63 \times 10^{-1}\,\mathrm{kg}$$

If this sphere balances one made from aluminum in an "equal–arm balance", then they have the same mass. So $M_{\mathrm{Al}} = 2.63 \times 10^{-1}\,\mathrm{kg}$ is the mass of the aluminum sphere. From $M_{\mathrm{Al}} = \rho_{\mathrm{Al}} V_{\mathrm{Al}}$ we can find its volume:

$$V_{\mathrm{Al}} = \frac{M_{\mathrm{Al}}}{\rho_{\mathrm{Al}}} = \frac{2.63 \times 10^{-1}\,\mathrm{kg}}{2.70 \times 10^3\,\frac{\mathrm{kg}}{\mathrm{m}^3}} = 9.76 \times 10^{-5}\,\mathrm{m}^3$$

Having the volume of the sphere, we can find its radius:

$$V_{\mathrm{Al}} = \tfrac{4}{3}\pi R^3 \qquad \Longrightarrow \qquad R = \left(\frac{3V_{\mathrm{Al}}}{4\pi}\right)^{\frac{1}{3}}$$

This gives:

$$R = \left(\frac{3(9.76 \times 10^{-5}\,\mathrm{m}^3)}{4\pi}\right)^{\frac{1}{3}} = 2.86 \times 10^{-2}\,\mathrm{m} = 2.86\,\mathrm{cm}$$

The aluminum sphere must have a radius of $2.86\,\mathrm{cm}$ to balance the iron sphere.

1.2.3 Dimensional Analysis

---

13. The period T of a simple pendulum is measured in time units and is

$$T = 2\pi\sqrt{\frac{\ell}{g}}\ .$$

where ℓ is the length of the pendulum and g is the free–fall acceleration in units of length divided by the square of time. Show that this equation is dimensionally correct.

The period (T) of a pendulum is the amount of time it takes to makes one full swing back and forth. It is measured in units of *time* so its dimensions are represented by T.

On the right side of the equation we have the length ℓ, whose dimensions are represented by L. We are told that g is a length divided by the square of a time so its dimensions must be L/T^2. There is a factor of 2π on the right side, but this is a pure number and has no units. So the dimensions of the right side are:

$$\sqrt{\frac{L}{\left(\frac{L}{T^2}\right)}} = \sqrt{T^2} = T$$

so that the right hand side must also have units of time. Both sides of the equation agree in their units, which must be true for it to be a valid equation!

---

14. The volume of an object as a function of time is calculated by $V = At^3 + B/t$, where t is time measured in seconds and V is in cubic meters. Determine the dimension of the constants A and B.

Both sides of the equation for volume must have the same dimensions, and those must be the dimensions of volume where are L^3 (SI units of m^3). Since we can only add terms with the same dimensions, each of the terms on right side of the equation (At^3 and B/t) must have the same dimensions, namely L^3.

Suppose we denote the units of A by $[A]$. Then our comment about the dimensions of the first term gives us:

$$[A]T^3 = L^3 \qquad \Longrightarrow \qquad [A] = \frac{L^3}{T^3}$$

so A has dimensions L^3/T^3. In the SI system, it would have units of $\mathrm{m}^3/\mathrm{s}^3$.

Suppose we denote the units of B by $[B]$. Then our comment about the dimensions of the second term gives us:

$$\frac{[B]}{T} = L^3 \qquad \Longrightarrow \qquad [B] = L^3T$$

so B has dimensions L^3T. In the SI system, it would have units of $\mathrm{m}^3\,\mathrm{s}$.

---

15. Newton's law of universal gravitation is

$$F = G\frac{Mm}{r^2}$$

Here F is the force of gravity, M and m are masses, and r is a length. Force has the SI units of $\mathrm{kg}\cdot\mathrm{m}/\mathrm{s}^2$. What are the SI units of the constant G?

If we denote the *dimensions* of F by $[F]$ (and the same for the other quantities) then then dimensions of the quantities in Newton's Law are:

$$[M] = M \text{ (mass)} \quad [m] = M \quad [r] = L \qquad [F] : \frac{ML}{T^2}$$

What we don't know (yet) is $[G]$, the dimensions of G. Putting the known dimensions into Newton's Law, we must have:

$$\frac{ML}{T^2} = [G]\frac{M \cdot M}{L^2}$$

since the dimensions must be the same on both sides. Doing some algebra with the dimensions, this gives:

$$[G] = \left(\frac{ML}{T^2}\right)\frac{L^2}{M^2} = \frac{L^3}{MT^2}$$

so the dimensions of G are $L^3/(MT^2)$. In the SI system, G has *units* of

$$\frac{\mathrm{m}^3}{\mathrm{kg} \cdot \mathrm{s}^3}$$

---

16. In quantum mechanics, the fundamental constant called Planck's constant, h, has dimensions of $[ML^2T^{-1}]$. Construct a quantity with the dimensions of length from h, a mass m, and c, the speed of light.

The problem suggests that there is some product of powers of h, m and c which has dimensions of length. If these powers are r, s and t, respectively, then we are looking for values of r, s and t such that

$$h^r m^s c^t$$

has dimensions of length.

What are the dimensions of this product, as written? We were given the dimensions of h, namely $[ML^2T^{-1}]$; the dimensions of m are M, and the dimensions of c are $\frac{L}{T} = LT^{-1}$ (it is a speed). So the dimensions of $h^r m^s c^t$ are:

$$[ML^2T^{-1}]^r[M]^s[LT^{-1}]^t = M^{r+s}L^{2r+t}T^{-r-t}$$

where we have used the laws of combining exponents which we all remember from algebra.

Now, since this is supposed to have dimensions of length, the power of L must be 1 but the other powers are zero. This gives the equations:

$$\begin{aligned} r + s &= 0 \\ 2r + t &= 1 \\ -r - t &= 0 \end{aligned}$$

which is a set of three equations for three unknowns. Easy to solve!

The last of them gives $r = -t$. Substituting this into the second equation gives

$$2r + t = 2(-t) + t = -t = 1 \qquad \Longrightarrow \qquad t = -1$$

Then $r = +1$ and the first equation gives us $s = -1$. With these values, we can confidently say that

$$h^r m^s c^t = h^1 m^{-1} c^{-1} = \frac{h}{mc}$$

has units of length.

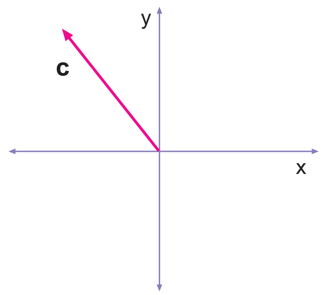

Figure 1.5: Vector **c**, found in Example 17. With $c_x = -9.0$ and $c_y = +10.0$, the direction of **c** is in the second quadrant.

1.2.4 Vectors; Vector Addition

---

17. (a) What is the sum in unit–vector notation of the two vectors a $= 4.0\mathbf{i} + 3.0\mathbf{j}$ **and b** $= -13.0\mathbf{i} + 7.0\mathbf{j}$**? (b) What are the magnitude and direction of a** $+$ **b?**

(a) Summing the corresponding components of vectors **a** and **b** we find:

$$\begin{aligned}\mathbf{a}+\mathbf{b} &= (4.0 - 13.0)\mathbf{i} + (3.0 + 7.0)\mathbf{j} \\ &= -9.0\mathbf{i} + 10.0\mathbf{j}\end{aligned}$$

This is the sum of the two vectors is unit–vector form.

(b) Using our results from (a), the magnitude of $\mathbf{a}+\mathbf{b}$ is

$$|\mathbf{a}+\mathbf{b}| = \sqrt{(-9.0)^2 + (10.0)^2} = 13.4$$

and if $\mathbf{c} = \mathbf{a} + \mathbf{b}$ points in a direction θ as measured from the positive x axis, then the tangent of θ is found from

$$\tan\theta = \left(\frac{c_y}{c_x}\right) = -1.11$$

If we naively take the arctangent using a calculator, we are told:

$$\theta = \tan^{-1}(-1.11) = -48.0^\circ$$

which is not correct because (as shown in Fig. 1.5), with c_x negative, and c_y positive, the correct angle must be in the *second* quadrant. The calculator was fooled because angles which differ by multiples of 180° have the same tangent. The direction we really want is

$$\theta = -48.0^\circ + 180.0^\circ = 132.0^\circ$$

---

18. Vector a has magnitude $5.0\,\mathrm{m}$ **and is directed east. Vector b has magnitude** $4.0\,\mathrm{m}$ **and is directed** 35° **west of north. What are (a) the magnitude and (b) the**

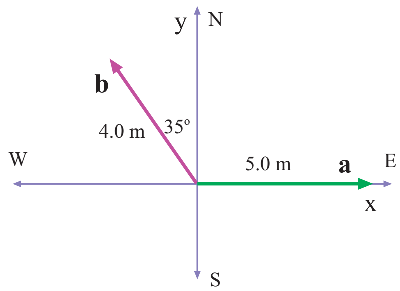

Figure 1.6: Vectors **a** and **b** as given in Example 18.

direction of a + b? What are (c) the magnitude and (d) the direction of b – a? Draw a vector diagram for each combination.

(a) The vectors are shown in Fig. 1.6. (On the axes are shown the common directions N, S, E, W and also the x and y axes; "North" is the positive y direction, "East" is the positive x direction, etc.) Expressing the vectors in **i**, **j** notation, we have:

$$\mathbf{a} = (5.00\,\mathrm{m})\mathbf{i}$$

and

$$\begin{aligned}\mathbf{b} &= -(4.00\,\mathrm{m})\sin 35^\circ + (4.00\,\mathrm{m})\cos 35^c irc \\ &= (-2.29\,\mathrm{m})\mathbf{i} + (3.28\,\mathrm{m})\mathbf{j}\end{aligned}$$

So if vector **c** is the sum of vectors **a** and **b** then:

$$c_x = a_x + b_x = (5.00\,\mathrm{m}) + (-2.29\,\mathrm{m}) = 2.71\,\mathrm{m}$$
$$c_y = a_y + b_y = (0.00\,\mathrm{m}) + (3.28\,\mathrm{m}) = 3.28\,\mathrm{m}$$

The magnitude of **c** is

$$c = \sqrt{c_x^2 + c_y^2} = \sqrt{(2.71\,\mathrm{m})^2 + (3.28\,\mathrm{m})^2} = 4.25\,\mathrm{m}$$

(b) If the direction of **c**, as measured counterclockwise from the $+x$ axis is θ then

$$\tan\theta = \frac{c_y}{c_x} = \frac{3.28\,\mathrm{m}}{2.71\,\mathrm{m}} = 1.211$$

then the `tan`$^{-1}$ operation on a calculator gives

$$\theta = \tan^{-1}(1.211) = 50.4^\circ$$

and since vector **c** must lie in the first quadrant this angle is correct. We note that this angle is

$$90.0^\circ - 50.4^\circ = 39.6^\circ$$

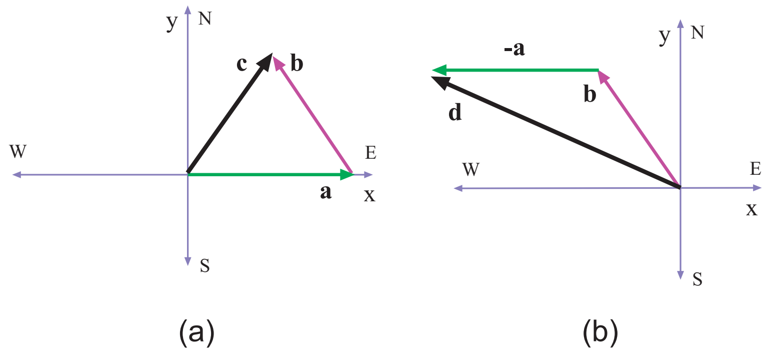

Figure 1.7: (a) Vector diagram showing the addition $\mathbf{a}+\mathbf{b}$. (b) Vector diagram showing $\mathbf{b}-\mathbf{a}$.

just shy of the $+y$ axis (the "North" direction). So we can also express the direction by saying it is "39.6° East of North".

A vector diagram showing **a**, **b** and **c** is given in Fig. 1.7(a).

(c) If the vector **d** is given by $\mathbf{d} = \mathbf{b} - \mathbf{a}$ then the components of **d** are given by

$$d_x = b_x - a_x = (-2.29\,\mathrm{m}) - (5.00\,\mathrm{m}) = -7.29\,\mathrm{m}$$
$$c_y = a_y + b_y = (3.28\,\mathrm{m}) - (0.00\,\mathrm{m}) + (3.28\,\mathrm{m}) = 3.28\,\mathrm{m}$$

The magnitude of **c** is

$$d = \sqrt{d_x^2 + d_y^2} = \sqrt{(-7.29\,\mathrm{m})^2 + (3.28\,\mathrm{m})^2} = 8.00\,\mathrm{m}$$

(d) If the direction of **d**, as measured counterclockwise from the $+x$ axis is θ then

$$\tan\theta = \frac{d_y}{d_x} = \frac{3.28\,\mathrm{m}}{-7.29\,\mathrm{m}} = -0.450$$

Naively pushing buttons on the calculator gives

$$\theta = \tan^{-1}(-0.450) = -24.2^\circ$$

which can't be right because from the signs of its components we know that **d** must lie in the second quadrant. We need to add 180° to get the correct answer for the `tan`$^{-1}$ operation:

$$\theta = -24.2^\circ + 180.0^\circ = 156^\circ$$

But we note that this angle is

$$180^\circ - 156^\circ = 24^\circ$$

shy of the $-y$ axis, so the direction can also be expressed as "24° North of West".

A vector diagram showing **a**, **b** and **d** is given in Fig. 1.7(b).

---

19. The two vectors a and b in Fig. 1.8 have equal magnitudes of $10.0\,\mathrm{m}$**. Find**

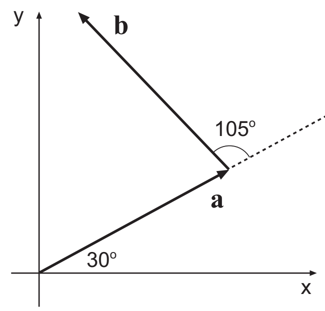

Figure 1.8: Vectors for Example 19.

(a) the x component and (b) the y component of their vector sum r, (c) the magnitude of r and (d) the angle r makes with the positive direction of the x axis.

(a) First, find the x and y components of the vectors **a** and **b**. The vector **a** makes an angle of 30° with the $+x$ axis, so its components are

$$\begin{aligned} a_x &= a\cos 30^\circ = (10.0\,\mathrm{m})\cos 30^\circ = 8.66\,\mathrm{m} \\ a_y &= a\sin 30^\circ = (10.0\,\mathrm{m})\sin 30^\circ = 5.00\,\mathrm{m} \end{aligned}$$

The vector **b** makes an angle of 135° with the $+x$ axis (30° plus 105° more) so its components are

$$\begin{aligned} b_x &= b\cos 135^\circ = (10.0\,\mathrm{m})\cos 135^\circ = -7.07\,\mathrm{m} \\ b_y &= b\sin 135^\circ = (10.0\,\mathrm{m})\sin 135^\circ = 7.07\,\mathrm{m} \end{aligned}$$

Then if $\mathbf{r} = \mathbf{a} + \mathbf{b}$, the x and y components of the vector **r** are:

$$\begin{aligned} r_x &= a_x + b_x = 8.66\,\mathrm{m} - 7.07\,\mathrm{m} = 1.59\,\mathrm{m} \\ r_y &= a_y + b_y = 5.00\,\mathrm{m} + 7.07\,\mathrm{m} = 12.07\,\mathrm{m} \end{aligned}$$

So the x component of the sum is $r_x = 1.59\,\mathrm{m}$, and...

(b) ...the y component of the sum is $r_y = 12.07\,\mathrm{m}$.

(c) The magnitude of the vector **r** is

$$r = \sqrt{r_x^2 + r_y^2} = \sqrt{(1.59\,\mathrm{m})^2 + (12.07\,\mathrm{m})^2} = 12.18\,\mathrm{m}$$

(d) To get the direction of the vector **r** expressed as an angle θ measured from the $+x$ axis, we note:

$$\tan\theta = \frac{r_y}{r_x} = 7.59$$

and then take the inverse tangent of 7.59:

$$\theta = \tan^{-1}(7.59) = 82.5^\circ$$

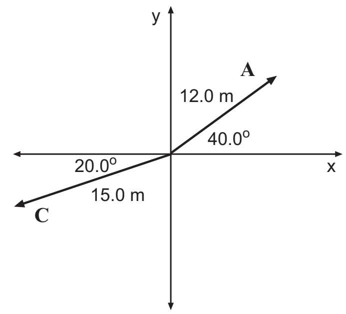

Figure 1.9: Vectors **A** and **C** as described in Example 20.

Since the components of **r** are both positive, the vector *does* lie in the first quadrant so that the inverse tangent operation has (this time) given the correct answer. So the direction of **r** is given by $\theta = 82.5^\circ$.

---

20. In the sum $\mathbf{A}+\mathbf{B} = \mathbf{C}$, vector A has a magnitude of $12.0\,\mathrm{m}$ and is angled 40.0° counterclockwise from the $+x$ direction, and vector C has magnitude of $15.0\,\mathrm{m}$ and is angled 20.0° counterclockwise from the $-x$ direction. What are (a) the magnitude and (b) the angle (relative to $+x$) of B?

(a) Vectors **A** and **C** are diagrammed in Fig. 1.9. From these we can get the components of **A** and **C** (watch the signs on vector **C** from the odd way that its angle is given!):

$$A_x = (12.0\,\mathrm{m})\cos(40.0^\circ) = 9.19\,\mathrm{m} \qquad A_y = (12.0\,\mathrm{m})\sin(40.0^\circ) = 7.71\,\mathrm{m}$$

$$C_x = -(15.0\,\mathrm{m})\cos(20.0^\circ) = -14.1\,\mathrm{m} \qquad C_y = -(15.0\,\mathrm{m})\sin(20.0^\circ) = -5.13\,\mathrm{m}$$

(Note, the vectors in this problem have *units* to go along with their magnitudes, namely m (meters).) Then from the relation $\mathbf{A}+\mathbf{B} = \mathbf{C}$ it follows that $\mathbf{B} = \mathbf{C} - \mathbf{A}$, and from this we find the components of **B**:

$$B_x = C_x - A_x = -14.1\,\mathrm{m} - 9.19\,\mathrm{m} = -23.3\,\mathrm{m}$$

$$B_y = C_y - A_y = -5.13\,\mathrm{m} - 7.71\,\mathrm{m} = -12.8\,\mathrm{m}$$

Then we find the magnitude of vector **B**:

$$B = \sqrt{B_x^2 + B_y^2} = \sqrt{(-23.3)^2 + (-12.8)^2}\,\mathrm{m} = 26.6\,\mathrm{m}$$

(b) We find the direction of **B** from:

$$\tan\theta = \left(\frac{B_y}{B_x}\right) = 0.551$$

If we *naively* press the "`atan`" button on our calculators to get θ, we are told:

$$\theta = \tan^{-1}(0.551) = 28.9^\circ \qquad (?)$$

which cannot be correct because from the components of **B** (both negative) we know that vector **B** lies in the third quadrant. So we need to ad 180° to the naive result to get the *correct* answer:

$$\theta = 28.9^\circ + 180.0^\circ = 208.9^\circ \ .$$

This is the angle of **B**, measured counterclockwise from the $+x$ axis.

---

21. If $\mathbf{a} - \mathbf{b} = 2\mathbf{c}$, $\mathbf{a} + \mathbf{b} = 4\mathbf{c}$ and $\mathbf{c} = 3\mathbf{i} + 4\mathbf{j}$, then what are a and b?

We notice that if we add the first two relations together, the vector **b** will cancel:

$$(\mathbf{a} - \mathbf{b}) + (\mathbf{a} + \mathbf{b}) = (2\mathbf{c}) + (4\mathbf{c})$$

which gives:

$$2\mathbf{a} = 6\mathbf{c} \qquad \Longrightarrow \qquad \mathbf{a} = 3\mathbf{c}$$

and we can use the last of the given equations to substitute for **c**; we get

$$\mathbf{a} = 3\mathbf{c} = 3(3\mathbf{i} + 4\mathbf{j}) = 9\mathbf{i} + 12\mathbf{j}$$

Then we can rearrange the first of the equations to solve for **b**:

$$\begin{aligned} \mathbf{b} &= \mathbf{a} - 2\mathbf{c} = (9\mathbf{i} + 12\mathbf{j}) - 2(3\mathbf{i} + 4\mathbf{j}) \\ &= (9-6)\mathbf{i} + (12-8)\mathbf{j} \\ &= 3\mathbf{i} + 4\mathbf{j} \end{aligned}$$

So we have found:

$$\mathbf{a} = 9\mathbf{i} + 12\mathbf{j} \qquad \text{and} \qquad \mathbf{b} = 3\mathbf{i} + 4\mathbf{j}$$

---

22. If $\mathbf{A} = (6.0\mathbf{i} - 8.0\mathbf{j})$ units, $\mathbf{B} = (-8.0\mathbf{i} + 3.0\mathbf{j})$ units, and $\mathbf{C} = (26.0\mathbf{i} + 19.0\mathbf{j})$ units, determine a and b so that $a\mathbf{A} + b\mathbf{B} + \mathbf{C} = 0$.

The condition on the vectors given in the problem:

$$a\mathbf{A} + b\mathbf{B} + \mathbf{C} = 0$$

is a condition on the *individual components* of the vectors. It implies:

$$aA_x + bB_x + C_x = 0 \qquad \text{and} \qquad aA_y + bB_y + C_y = 0 \ .$$

So that we have the equations:

$$\begin{aligned} 6.0a - 8.0b + 26.0 &= 0 \\ -8.0a + 3.0b + 19.0 &= = 0 \end{aligned}$$

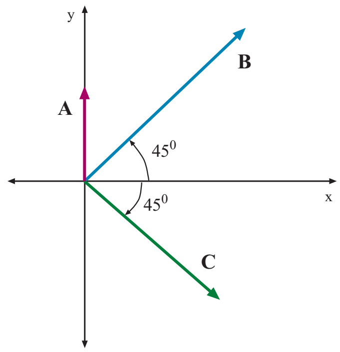

Figure 1.10: Vectors for Example 23

We have two equations for two unknowns so we *can* find a and b. The are lots of ways to do this; one could multiply the first equation by 4 and the second equation by 3 to get:

$$\begin{aligned} 24.0a - 32.0b + 104.0 &= 0 \\ -24.0a + 9.0b + 57.0 &= = 0 \end{aligned}$$

Adding these gives

$$-23.0b + 161 = 0 \qquad \Longrightarrow \qquad b = \frac{-161.0}{-23.0} = 7.0$$

and then the first of the original equations gives us a:

$$6.0a = 8.0b - 26.0 = 8.0(7.0) - 26.0 = 30.0 \qquad \Longrightarrow \qquad a = \frac{30.0}{6.0} = 5.0$$

and our solution is

$$a = 7.0 \qquad\qquad b = 5.0$$

---

23. Three vectors are oriented as shown in Fig. 1.10, where $|\mathbf{A}| = 20.0\,\mathrm{units}$, $|\mathbf{B}| = 40.0\,\mathrm{units}$, and $|\mathbf{C}| = 30.0\,\mathrm{units}$. Find (a) the x and y components of the resultant vector and (b) the magnitude and direction of the resultant vector.

(a) Let's first put these vectors into "unit–vector notation":

$$\begin{aligned} \mathbf{A} &= 20.0\mathbf{j} \\ \mathbf{B} &= (40.0\cos 45^\circ)\mathbf{i} + (40.0\sin 45^\circ)\mathbf{j} = 28.3\mathbf{i} + 28.3\mathbf{j} \\ \mathbf{C} &= (30.0\cos(-45^\circ))\mathbf{i} + (30.0\sin(-45^\circ))\mathbf{j} = 21.2\mathbf{i} - 21.2\mathbf{j} \end{aligned}$$

Adding the components together, the resultant (total) vector is:

$$\begin{aligned} \text{Resultant} &= \mathbf{A} + \mathbf{B} + \mathbf{C} \\ &= (28.3 + 21.2)\mathbf{i} + (20.0 + 28.3 - 21.2)\mathbf{j} \\ &= 49.5\mathbf{i} + 27.1\mathbf{j} \end{aligned}$$

So the x component of the resultant vector is 49.5 and the y component of the resultant is 27.1.

(b) If we call the resultant vector $\mathbf{R}$, then the magnitude of $\mathbf{R}$ is given by

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(49.5)^2 + (27.1)^2} = 56.4$$

To find its direction (given by θ, measured counterclockwise from the x axis), we find:

$$\tan\theta = \frac{R_y}{R_x} = \frac{27.1}{49.5} = 0.547$$

and then taking the inverse tangent gives a *possible* answer for θ:

$$\theta = \tan^{-1}(0.547) = 28.7^\circ \ .$$

Is this the right answer for θ? Since *both* components of $\mathbf{R}$ are *positive*, it must lie in the first quadrant and so θ must be between 0° and 90°. So the direction of $\mathbf{R}$ is given by 28.7°.

---

24. A vector B, when added to the vector $\mathbf{C} = 3.0\mathbf{i} + 4.0\mathbf{j}$, yields a resultant vector that is in the positive y direction and has a magnitude equal to that of C. What is the magnitude of B?

If the vector $\mathbf{B}$ is denoted by $\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j}$ then the resultant of $\mathbf{B}$ and $\mathbf{C}$ is

$$\mathbf{B} + \mathbf{C} = (B_x + 3.0)\mathbf{i} + (B_y + 4.0)\mathbf{j} \ .$$

We are told that the resultant points in the positive y direction, so its x component must be *zero*. Then:

$$B_x + 3.0 = 0 \qquad \Longrightarrow \qquad B_x = -3.0 \ .$$

Now, the magnitude of $\mathbf{C}$ is

$$C = \sqrt{C_x^2 + C_y^2} = \sqrt{(3.0)^2 + (4.0)^2} = 5.0$$

so that if the magnitude of $\mathbf{B} + \mathbf{C}$ is also 5.0 then we get

$$|\mathbf{B} + \mathbf{C}| = \sqrt{(0)^2 + (B_y + 4.0)^2} = 5.0 \qquad \Longrightarrow \qquad (B_y + 4.0)^2 = 25.0 \ .$$

The last equation gives $(B_y + 4.0) = \pm 5.0$ and apparently there are *two* possible answers

$$B_y = +1.0 \qquad \text{and} \qquad B_y = -9.0$$

but the second case gives a resultant vector $\mathbf{B} + \mathbf{C}$ which points in the *negative* y direction so we omit it. Then with $B_y = 1.0$ we find the magnitude of $\mathbf{B}$:

$$B = \sqrt{(B_x)^2 + (B_y)^2} = \sqrt{(-3.0)^2 + (1.0)^2} = 3.2$$

The magnitude of vector $\mathbf{B}$ is 3.2.

1.2.5 Multiplying Vectors

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25. Vector A extends from the origin to a point having polar coordinates $(7, 70^\circ)$ **and vector B extends from the origin to a point having polar coordinates** $(4, 130^\circ)$**. Find** $\mathbf{A} \cdot \mathbf{B}$**.**

We can use Eq. 1.7 to find $\mathbf{A} \cdot \mathbf{B}$. We have the magnitudes of the two vectors (namely $A = 7$ and $B = 4$) and the angle ϕ between the two is

$$\phi = 130^\circ - 70^\circ = 60^\circ \ .$$

Then we get:

$$\mathbf{A} \cdot \mathbf{B} = AB \cos\phi = (7)(4)\cos 60^\circ = 14$$

---

26. Find the angle between $\mathbf{A} = -5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ **and** $\mathbf{B} = -2\mathbf{j} - 2\mathbf{k}$**.**

Eq. 1.7 allows us to find the cosine of the angle between two vectors as long as we know their magnitudes and their dot product. The magnitudes of the vectors $\mathbf{A}$ and $\mathbf{B}$ are:

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{(-5)^2 + (-3)^2 + (2)^2} = 6.164$$

$$B = \sqrt{B_x^2 + B_y^2 + B_z^2} = \sqrt{(0)^2 + (-2)^2 + (-2)^2} = 2.828$$

and their dot product is:

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z = (-5)(0) + (-3)(-2) + (2)(-2) = 2$$

Then from Eq. 1.7, if ϕ is the angle between $\mathbf{A}$ and $\mathbf{B}$, we have

$$\cos\phi = \frac{\mathbf{A} \cdot \mathbf{B}}{AB} = \frac{2}{(6.164)(2.828)} = 0.114$$

which then gives

$$\phi = 83.4^\circ \ .$$

---

27. Two vectors a and b have the components, in arbitrary units, $a_x = 3.2$**,** $a_y = 1.6$**,** $b_x = 0.50$**,** $b_y = 4.5$**. (a) Find the angle between the directions of a and b. (b) Find the components of a vector c that is perpendicular to a, is in the** xy **plane and has a magnitude of** 5.0 **units.**

(a) The scalar product has something to do with the angle between two vectors... if the angle between $\mathbf{a}$ and $\mathbf{b}$ is ϕ then from Eq. 1.7 we have:

$$\cos\phi = \frac{\mathbf{a} \cdot \mathbf{b}}{ab} \ .$$

We can compute the right–hand–side of this equation since we know the components of **a** and **b**. First, find $\mathbf{a}\cdot\mathbf{b}$. Using Eq. 1.8 we find:

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= a_x b_x + a_y b_y \\ &= (3.2)(0.50) + (1.6)(4.5) \\ &= 8.8\end{aligned}$$

Now find the magnitudes of **a** and **b**:

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(3.2)^2 + (1.6)^2} = 3.6$$

$$b = \sqrt{b_x^2 + b_y^2} = \sqrt{(0.50)^2 + (4.5)^2} = 4.5$$

This gives us:

$$\cos\phi = \frac{\mathbf{a}\cdot\mathbf{b}}{ab} = \frac{8.8}{(3.6)(4.5)} = 0.54$$

From which we get ϕ by:

$$\phi = \cos^{-1}(0.54) = 57^\circ$$

(b) Let the components of the vector **c** be c_x and c_y (we are told that it lies in the xy plane). If **c** is perpendicular to **a** then the dot product of the two vectors must give zero. This tells us:

$$\mathbf{a}\cdot\mathbf{c} = a_x c_x + a_y c_y = (3.2)c_x + (1.6)c_y = 0$$

This equation doesn't allow us to solve for the components of **c** but it does give us:

$$c_x = -\frac{1.6}{3.2}c_y = -0.50 c_y$$

Since the vector **c** has magnitude 5.0, we know that

$$c = \sqrt{c_x^2 + c_y^2} = 5.0$$

Using the previous equation to substitute for c_x gives:

$$\begin{aligned} c &= \sqrt{c_x^2 + c_y^2} \\ &= \sqrt{(-0.50\, c_y)^2 + c_y^2} \\ &= \sqrt{1.25\, c_y^2} = 5.0\end{aligned}$$

Squaring the last line gives

$$1.25 c_y^2 = 25 \qquad \Longrightarrow \qquad c_y^2 = 20. \qquad \Longrightarrow \qquad c_y = \pm 4.5$$

One must be careful... there are *two* possible solutions for c_y here. If $c_y = 4.5$ then we have

$$c_x = -0.50\, c_y = (-0.50)(4.5) = -2.2$$

But if $c_y = -4.5$ then we have

$$c_x = -0.50\, c_y = (-0.50)(-4.5) = 2.2$$

So the two possibilities for the vector $\mathbf{c}$ are

$$c_x = -2.2 \qquad c_y = 4.5$$

and

$$c_x = 2.2 \qquad c_y = -4.5$$

---

28. Two vectors are given by $\mathbf{A} = -3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{B} = 2\mathbf{i} + 3\mathbf{j}$. Find (a) $\mathbf{A} \times \mathbf{B}$ and (b) the angle between $\mathbf{A}$ and $\mathbf{B}$.

(a) Setting up the determinant in Eq. 1.12 (or just using Eq. 1.11 for the cross product) we find:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 4 & 0 \\ 2 & 3 & 0 \end{vmatrix} = (0-0)\mathbf{i} + (0-0)\mathbf{j} + ((-9) - (8))\mathbf{k} = -17\mathbf{k}$$

(b) To get the angle between $\mathbf{A}$ and $\mathbf{B}$ it is easiest to use the dot product and Eq. 1.7. The magnitudes of $\mathbf{A}$ and $\mathbf{B}$ are:

$$A = \sqrt{A_x^2 + A_y^2} = \sqrt{(-3)^2 + (4)^2} = 5 \qquad B = \sqrt{B_x^2 + B_y^2} = \sqrt{(2)^2 + (3)^2} = 3.61$$

and the dot product of the two vectors is

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z = (-3)(2) + (4)(3) = 6$$

so then if ϕ is the angle between $\mathbf{A}$ and $\mathbf{B}$ we get:

$$\cos\phi = \frac{\mathbf{A} \cdot \mathbf{B}}{AB} = \frac{6}{(5)(3.61)} = 0.333$$

which gives

$$\phi = 70.6^\circ \; .$$

---

29. Prove that two vectors must have equal magnitudes if their sum is perpendicular to their difference.

Suppose the condition stated in this problem holds for the two vectors $\mathbf{a}$ and $\mathbf{b}$. If the sum $\mathbf{a} + \mathbf{b}$ is perpendicular to the difference $\mathbf{a} - \mathbf{b}$ then the dot product of these two vectors is zero:

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = 0$$

Use the distributive property of the dot product to expand the left side of this equation. We get:

$$\mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b}$$

But the dot product of a vector with itself gives the magnitude squared:

$$\mathbf{a}\cdot\mathbf{a}=a_x^2+a_y^2+a_z^2=a^2$$

(likewise $\mathbf{b}\cdot\mathbf{b}=b^2$) and the dot product is commutative: $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$. Using these facts, we then have

$$a^2-\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{b}+b^2=0\ ,$$

which gives:

$$a^2-b^2=0 \qquad \Longrightarrow \qquad a^2=b^2$$

Since the magnitude of a vector must be a positive number, this implies $a=b$ and so vectors $\mathbf{a}$ and $\mathbf{b}$ have the same magnitude.

---

30. For the following three vectors, what is $3\mathbf{C}\cdot(2\mathbf{A}\times\mathbf{B})$?

$$\mathbf{A}=2.00\mathbf{i}+3.00\mathbf{j}-4.00\mathbf{k}$$

$$\mathbf{B}=-3.00\mathbf{i}+4.00\mathbf{j}+2.00\mathbf{k} \qquad\qquad \mathbf{C}=7.00\mathbf{i}-8.00\mathbf{j}$$

Actually, from the properties of scalar multiplication we can combine the factors in the desired vector product to give:

$$3\mathbf{C}\cdot(2\mathbf{A}\times\mathbf{B})=6\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})\ .$$

Evaluate $\mathbf{A}\times\mathbf{B}$ first:

$$\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ 2.0&3.0&-4.0\\ -3.0&4.0&2.0\end{vmatrix}=(6.0+16.0)\mathbf{i}+(12.0-4.0)\mathbf{j}+(8.0+9.0)\mathbf{k}$$

$$=22.0\mathbf{i}+8.0\mathbf{j}+17.0\mathbf{k}$$

Then:

$$\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})=(7.0)(22.0)-(8.0)(8.0)+(0.0)(17.0)=90$$

So the answer we want is:

$$6\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})=(6)(90.0)=540$$

---

31. A student claims to have found a vector A such that

$$(2\mathbf{i}-3\mathbf{j}+4\mathbf{k})\times\mathbf{A}=(4\mathbf{i}+3\mathbf{j}-\mathbf{k})\ .$$

Do you believe this claim? Explain.

Frankly, I've been in this teaching business so long and I've grown so cynical that I don't believe *anything* any student claims anymore, and this case is no exception. But enough about *me*; let's see if we can provide a mathematical answer.

We might try to work out a solution for $\mathbf{A}$, but let's think about some of the basic properties of the cross product. We know that the cross product of two vectors must be perpendicular to *each* of the "multiplied" vectors. So if the student is telling the truth, it must be true that $(4\mathbf{i} + 3\mathbf{j} - \mathbf{k})$ is perpendicular to $(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$. Is it?

We can test this by taking the dot product of the two vectors:

$$(4\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) = (4)(2) + (3)(-3) + (-1)(4) = -5 \ .$$

The dot product does *not* give zero as it must if the two vectors are perpendicular. So we have a contradiction. There can't be *any* vector $\mathbf{A}$ for which the relation is true.

Chapter 2

Motion in One Dimension

2.1 The Important Stuff

2.1.1 Position, Time and Displacement

We begin our study of motion by considering objects which are very small in comparison to the size of their movement through space. When we can deal with an object in this way we refer to it as a **particle**. In this chapter we deal with the case where a particle moves along a straight line.

The particle's location is specified by its **coordinate**, which will be denoted by x or y. As the particle moves, its coordinate changes with the time, t. The change in position from x_1 to x_2 of the particle is the **displacement** Δx, with $\Delta x = x_2 - x_1$.

2.1.2 Average Velocity and Average Speed

When a particle has a displacement Δx in a change of time Δt, its **average velocity** *for that time interval* is

$$\overline{v} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \tag{2.1}$$

The **average speed** of the particle is absolute value of the average velocity and is given by

$$\overline{s} = \frac{\text{Distance travelled}}{\Delta t} \tag{2.2}$$

In general, the value of the average velocity for a moving particle depends on the initial and final times for which we have found the displacements.

2.1.3 Instantaneous Velocity and Speed

We can answer the question "how fast is a particle moving at a particular time t?" by finding the **instantaneous velocity**. This is the limiting case of the average velocity when the time interval Δt include the time t and is as small as we can imagine:

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \tag{2.3}$$

The **instantaneous speed** is the absolute value (magnitude) of the instantaneous velocity.

If we make a plot of x vs. t for a moving particle the instantaneous velocity is the slope of the tangent to the curve at any point.

2.1.4 Acceleration

When a particle's velocity changes, then we way that the particle undergoes an **acceleration**.

If a particle's velocity changes from v_1 to v_2 during the time interval t_1 to t_2 then we define the **average acceleration** as

$$\overline{v} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \tag{2.4}$$

As with velocity it is usually more important to think about the **instantaneous acceleration**, given by

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \tag{2.5}$$

If the acceleration a is positive it means that the velocity is instantaneously *increasing*; if a is negative, then v is instantaneously *decreasing*. Oftentimes we will encounter the word **deceleration** in a problem. This word is used when the sense of the acceleration is opposite that of the instantaneous velocity (the motion). Then the *magnitude* of acceleration is given, with its direction being understood.

2.1.5 Constant Acceleration

A very useful *special case* of accelerated motion is the one where the acceleration a is constant. For this case, one can show that the following are true:

$$v = v_0 + at \tag{2.6}$$

$$x = x_0 + v_0 t + \tfrac{1}{2}at^2 \tag{2.7}$$

$$v^2 = v_0^2 + 2a(x - x_0) \tag{2.8}$$

$$x = x_0 + \tfrac{1}{2}(v_0 + v)t \tag{2.9}$$

In these equations, we mean that the particle has position x_0 and velocity v_0 at time $t = 0$; it has position x and velocity v at time t.

These equations are valid *only* for the case of constant acceleration.

2.1.6 Free Fall

An object tossed up or down near the surface of the earth has a constant downward acceleration of magnitude $9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$. This number is always denoted by g. Be very careful about the sign; in a coordinate system where the y axis points straight up, the acceleration of a freely–falling object is

$$a_y = -9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2} = -g \tag{2.10}$$

Here we are assuming that the air has no effect on the motion of the falling object. For an object which falls for a long distance this can be a bad assumption.

Remember that an object in free–fall has an acceleration equal to $-9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$ while it is moving up, while it is moving down, while it is at maximum height... always!

2.2 Worked Examples

2.2.1 Average Velocity and Average Speed

---

1. Boston Red Sox pitcher Roger Clemens could routinely throw a fastball at a horizontal speed of $160\,\frac{\mathrm{km}}{\mathrm{hr}}$. How long did the ball take to reach home plate $18.4\,\mathrm{m}$ away?

We assume that the ball moves in a horizontal straight line with an average speed of 160 km/hr. Of course, in reality this is not quite true for a thrown baseball.

We are given the average velocity of the ball's motion and also a particular displacement, namely $\Delta x = 18.4\,\mathrm{m}$. Equation 2.1 gives us:

$$\overline{v} = \frac{\Delta x}{\Delta t} \qquad \Longrightarrow \qquad \Delta t = \frac{\Delta x}{\overline{v}}$$

But before using it, it might be convenient to change the units of $\overline{v}$. We have:

$$\overline{v} = 160\,\frac{\mathrm{km}}{\mathrm{hr}} \cdot \left(\frac{1000\,\mathrm{m}}{1\,\mathrm{km}}\right) \cdot \left(\frac{1\,\mathrm{hr}}{3600\,\mathrm{s}}\right) = 44.4\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

Then we find:

$$\Delta t = \frac{\Delta x}{\overline{v}} = \frac{18.4\,\mathrm{m}}{44.4\,\frac{\mathrm{m}}{\mathrm{s}}} = 0.414\,\mathrm{s}$$

The ball takes 0.414 seconds to reach home plate.

2.2.2 Acceleration

---

2. An electron moving along the x axis has a position given by $x = (16te^{-t})\,\mathrm{m}$, where t is in seconds. How far is the electron from the origin when it momentarily stops?

To find the velocity of the electron as a function of time, take the first derivative of $x(t)$:

$$v = \frac{dx}{dt} = 16e^{-t} - 16te^{-t} = 16e^{-t}(1-t)\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

again where t is in seconds, so that the units for v are $\frac{\mathrm{m}}{\mathrm{s}}$.

Now the electron "momentarily stops" when the velocity v is zero. From our expression for v we see that this occurs at $t = 1\,\mathrm{s}$. At this particular time we can find the value of x:

$$x(1\,\mathrm{s}) = 16(1)e^{-1}\,\mathrm{m} = 5.89\,\mathrm{m}$$

The electron was 5.89 m from the origin when the velocity was zero.

---

3. (a) If the position of a particle is given by $x = 20t - 5t^3$, where x is in meters and t is in seconds, when if ever is the particle's velocity zero? (b) When is its acceleration a zero? (c) When is a negative? Positive? (d) Graph $x(t)$, $v(t)$, and $a(t)$.

(a) From Eq. 2.3 we find $v(t)$ from $x(t)$:

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}(20t - 5t^3) = 20 - 15t^2$$

where, if t is in seconds then v will be in $\frac{\mathrm{m}}{\mathrm{s}}$. The velocity v will be zero when

$$20 - 15t^2 = 0$$

which we can solve for t:

$$15t^2 = 20 \qquad \Longrightarrow \qquad t^2 = \frac{20}{15} = 1.33\,\mathrm{s}^2$$

(The units s^2 were inserted since we know t^2 must have these units.) This gives:

$$t = \pm 1.15\,\mathrm{s}$$

(We should be careful... t may be meaningful for negative values!)

(b) From Eq. 2.5 we find $a(t)$ from $v(t)$:

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}(20 - 15t^2) = -30t$$

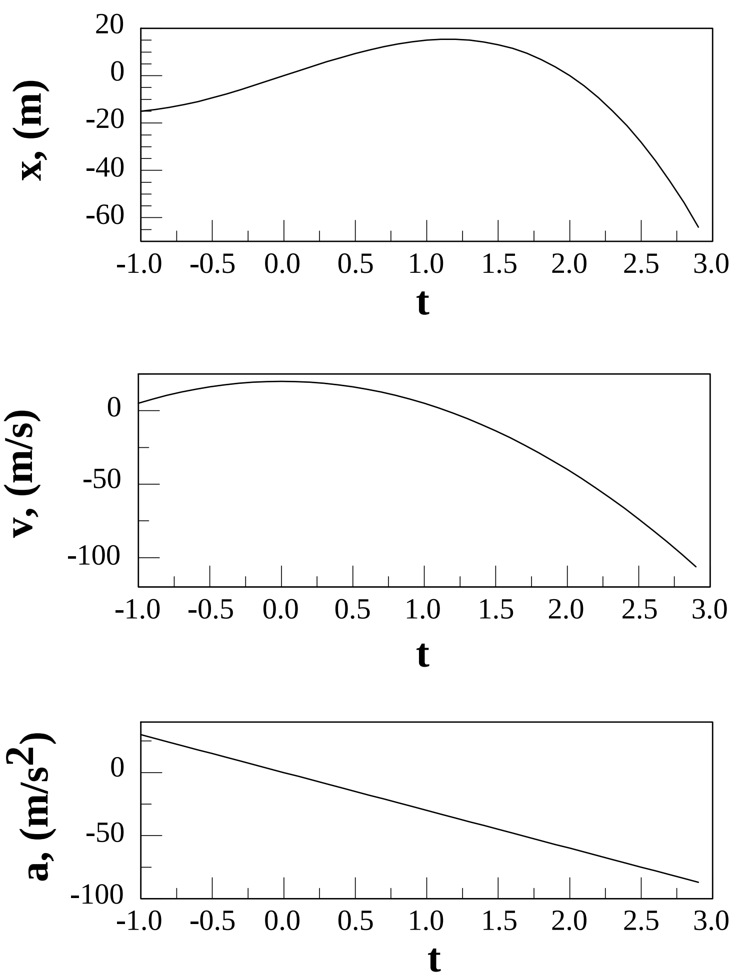

Figure 2.1: Plot of $x(t)$, $v(t)$, and $a(t)$ for Example 3.

where we mean that if t is given in seconds, a is given in $\frac{\mathrm{m}}{\mathrm{s}^2}$. From this, we see that a can be zero only at $t = 0$.

(c) From the result is part (b) we can also see that a is negative whenever t is positive. a is positive whenever t is negative (again, assuming that $t < 0$ has meaning for the motion of this particle).

(d) Plots of $x(t)$, $v(t)$, and $a(t)$ are given in Fig. 2.1.

---

4. In an arcade video game a spot is programmed to move across the screen according to $x = 9.00t - 0.750t^3$, where x is distance in centimeters measured from the left edge of the screen and t is time in seconds. When the spot reaches a screen edge, at either $x = 0$ or $x = 15.0\,\mathrm{cm}$, t is reset to 0 and the spot starts moving again according to $x(t)$. (a) At what time after starting is the spot instantaneously at rest? (b) Where does this occur? (c) What is its acceleration when this occurs? (d) In what direction is it moving just prior to coming to rest? (e) Just after? (f) When does it first reach an edge of the screen after $t = 0$?

(a) This is a question about the instantaneous velocity of the spot. To find $v(t)$ we calculate:

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}(9.00t - 0.750t^3) = 9.00 - 2.25t^2$$

where this expression will give the value of v in $\frac{\mathrm{cm}}{\mathrm{s}}$ when t is given in seconds.

We want to know the value of t for which v is zero, i.e. the spot is instantaneously at rest. We solve:

$$9.00 - 2.25t^2 = 0 \qquad \Longrightarrow \qquad t^2 = \frac{9.00}{2.25} = 4.00\,\mathrm{s}^2$$

(Here we have filled in the proper units for t^2 since by laziness they were omitted from the first equations!) The solutions to this equation are

$$t = \pm 2.00\,\mathrm{s}$$

but since we are only interested in times after the clock starts at $t = 0$, we choose $t = 2.00\,\mathrm{s}$.

(b) In this part we are to find the value of x at which the instantaneous velocity is zero. In part (a) we found that this occurred at $t = 3.00\,\mathrm{s}$ so we calculate the value of x at $t = 2.00\,\mathrm{s}$:

$$x(2.00\,\mathrm{s}) = 9.00 \cdot (2.00) - 0.750 \cdot (2.00)^3 = 12.0\,\mathrm{cm}$$

(where we have filled in the units for x since *centimeters* are implied by the equation). The dot is located at $x = 12.0\,\mathrm{cm}$ at this time. (And recall that the width of the screen is 15.0 cm.)

(c) To find the (instantaneous) acceleration at all times, we calculate:

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}(9.00 - 2.25t^2) = -4.50t$$

where we mean that if t is given in seconds, a will be given in $\frac{\mathrm{m}}{\mathrm{s}^2}$. At the time in question ($t = 2.00\,\mathrm{s}$) the acceleration is

$$a(t = 2.00\,\mathrm{s}) = -4.50 \cdot (2.00) = -9.00$$

that is, the acceleration at this time is $-9.00\,\frac{\mathrm{m}}{\mathrm{s}^2}$.

(d) From part (c) we note that at the time that the velocity was instantaneously zero the acceleration was *negative*. This means that the velocity was *decreasing* at the time. If the velocity was decreasing yet instantaneously equal to zero then it had to be going from positive to negative values at $t = 2.00\,\mathrm{s}$. So just before this time its velocity was positive.

(e) Likewise, from our answer to part (d) just after $t = 2.00\,\mathrm{s}$ the velocity of particle had to be negative.

(f) We have seen that the dot never gets to the right edge of the screen at $x = 15.0\,\mathrm{cm}$. It will not reverse its velocity again since $t = 2.00\,\mathrm{s}$ is the only positive time at which $v = 0$. So it will keep moving to back to the left, and the coordinate x will equal zero when we have:

$$x = 0 = 9.00t - 0.750t^3$$

Factor out t to solve:

$$t(9.00 - 0.750t^2) = 0 \qquad \Longrightarrow \qquad \begin{cases} t = 0 & \text{or} \\ (9.00 - 0.750t^2) = 0 & \text{otherwise.} \end{cases}$$

The first solution is the time that the dot started moving, so that is not the one we want. The second case gives:

$$(9.00 - 0.750t^2) = 0 \qquad \Longrightarrow \qquad t^2 = \frac{9.00}{0.750} = 12.0\,\mathrm{s}^2$$

which gives

$$t = 3.46\,\mathrm{s}$$

since we only want the positive solution. So the dot returns to $x = 0$ (the left side of the screen) at $t = 3.46\,\mathrm{s}$.

If we plot the original function $x(t)$ we get the curve given in Fig. 2.2 which shows that the spot does not get to $x = 15.0\,\mathrm{cm}$ before it turns around. (However as explained in the problem, the curve does *not* extend to negative values as the graph indicates.)

2.2.3 Constant Acceleration

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5. The head of a rattlesnake can accelerate $50\,\frac{\mathrm{m}}{\mathrm{s}^2}$ in striking a victim. If a car could do as well, how long would it take to reach a speed of $100\,\frac{\mathrm{km}}{\mathrm{hr}}$ from rest?

First, convert the car's final speed to SI units to make it easier to work with:

$$100\,\frac{\mathrm{km}}{\mathrm{hr}} = \left(100\,\frac{\mathrm{km}}{\mathrm{hr}}\right) \cdot \left(\frac{1000\,\mathrm{m}}{1\,\mathrm{km}}\right) \cdot \left(\frac{1\,\mathrm{hr}}{3600\,\mathrm{s}}\right) = 27.8\,\frac{\mathrm{m}}{\mathrm{s}}$$

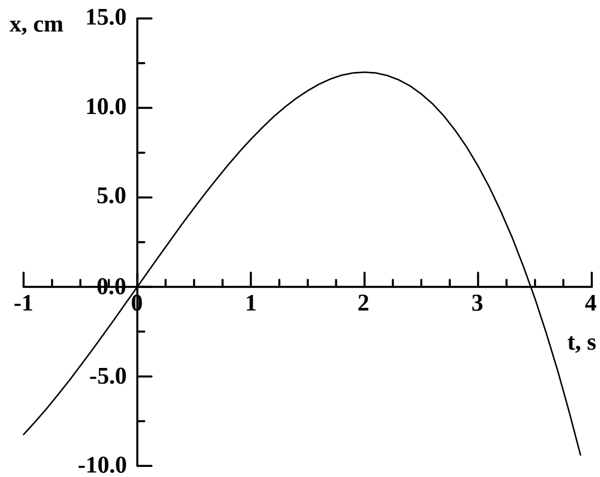

Figure 2.2: Plot of x vs t for moving spot. Ignore the parts where x is negative!

The acceleration of the car is $50\ \frac{\mathrm{m}}{\mathrm{s}^2}$ and it starts from rest which means that $v_0 = 0$. As we've found, the final velocity v of the car is $27.8\ \frac{\mathrm{m}}{\mathrm{s}}$. (The problem actually that this is final *speed* but if our coordinate system points in the same direction as the car's motion, these are the same thing.) Equation 2.6 lets us solve for the time t:

$$v = v_0 + at \qquad \Longrightarrow \qquad t = \frac{v - v_0}{a}$$

Substituting, we find

$$t = \frac{27.8\ \frac{\mathrm{m}}{\mathrm{s}} - 0}{50\ \frac{\mathrm{m}}{\mathrm{s}^2}} = 0.55\,\mathrm{s}$$

If a car had such a large acceleration, it would take $0.55\,\mathrm{s}$ to attain the given speed.

---

6. A body moving with uniform acceleration has a velocity of $12.0\ \frac{\mathrm{cm}}{\mathrm{s}}$ when its x coordinate is $3.00\,\mathrm{cm}$. If its x coordinate $2.00\,\mathrm{s}$ later is $-5.00\,\mathrm{cm}$, what is the magnitude of its acceleration?

In this problem we are given the initial coordinate ($x = 3.00\,\mathrm{cm}$), the initial velocity ($v_0 = 12.0\ \frac{\mathrm{cm}}{\mathrm{s}}$), the final x coordinate ($x = -5.00\,\mathrm{cm}$) and the elapsed time ($2.00\,\mathrm{s}$). Using Eq. 2.7 (since we are told that the acceleration *is* constant) we can solve for a. We find:

$$x = x_0 + v_0 t + \tfrac{1}{2}at^2 \qquad \Longrightarrow \qquad \tfrac{1}{2}at^2 = x - x_0 - v_0 t$$

Substitute things:

$$\tfrac{1}{2}at^2 = -5.00\,\mathrm{cm} - 3.00\,\mathrm{cm} - \left(12.0\ \tfrac{\mathrm{cm}}{\mathrm{s}}\right)(2.00\,\mathrm{s}) = -32.0\,\mathrm{cm}$$

Solve for a:

$$a = \frac{2(-32.0\,\mathrm{cm})}{t^2} = \frac{2(-32.0\,\mathrm{cm})}{(2.00\,\mathrm{s})^2} = -16.0\frac{\mathrm{cm}}{\mathrm{s}^2}$$

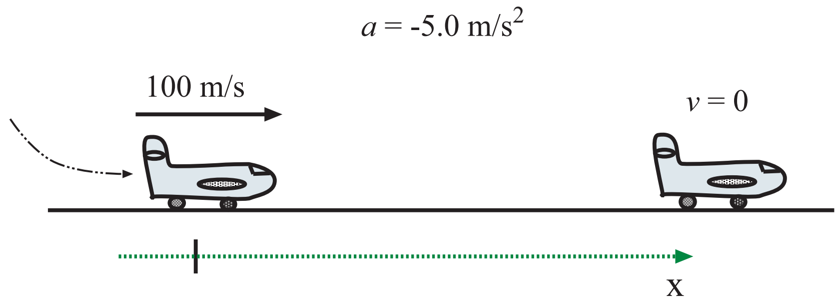

Figure 2.3: Plane touches down on runway at $100\,\frac{\mathrm{m}}{\mathrm{s}}$ and comes to a halt.

The x acceleration of the object is $-16.\,\frac{\mathrm{cm}}{\mathrm{s}^2}$. (The magnitude of the acceleration is $16.0\,\frac{\mathrm{cm}}{\mathrm{s}^2}$.)

---

7. A jet plane lands with a velocity of $100\,\frac{\mathrm{m}}{\mathrm{s}}$ and can accelerate at a maximum rate of $-5.0\,\frac{\mathrm{m}}{\mathrm{s}^2}$ as it comes to rest. (a) From the instant it touches the runway, what is the minimum time needed before it stops? (b) Can this plane land at a small airport where the runway is $0.80\,\mathrm{km}$ long?

(a) The data given in the problem is illustrated in Fig. 2.3. The minus sign in the acceleration indicates that the sense of the acceleration is opposite that of the motion, that is, the plane is decelerating.

The plane will stop as quickly as possible if the acceleration *does* have the value $-5.0\,\frac{\mathrm{m}}{\mathrm{s}^2}$, so we use this value in finding the time t in which the velocity changes from $v_0 = 100\,\frac{\mathrm{m}}{\mathrm{s}}$ to $v = 0$. Eq. 2.6 tells us:

$$t = \frac{v - v_0}{a}$$

Substituting, we find:

$$t = \frac{(0 - 100\,\frac{\mathrm{m}}{\mathrm{s}})}{(-5.0\,\frac{\mathrm{m}}{\mathrm{s}^2})} = 20\,\mathrm{s}$$

The plane needs $20\,\mathrm{s}$ to come to a halt.

(b) The plane also travels the shortest distance in stopping if its acceleration is $-5.0\,\frac{\mathrm{m}}{\mathrm{s}^2}$. With $x_0 = 0$, we can find the plane's final x coordinate using Eq. 2.9, using $t = 20\,\mathrm{s}$ which we got from part (a):

$$x = x_0 + \tfrac{1}{2}(v_0 + v)t = 0 + \tfrac{1}{2}(100\,\tfrac{\mathrm{m}}{\mathrm{s}} + 0)(20\,\mathrm{s}) = 1000\,\mathrm{m} = 1.0\,\mathrm{km}$$

The plane must have at least $1.0\,\mathrm{km}$ of runway in order to come to a halt safely. $0.80\,\mathrm{km}$ is *not* sufficient.

---

8. A drag racer starts her car from rest and accelerates at $10.0\,\frac{\mathrm{m}}{\mathrm{s}^2}$ for the entire distance of $400\,\mathrm{m}$ ($\frac{1}{4}$ mile). (a) How long did it take the car to travel this distance? (b) What is the speed at the end of the run?

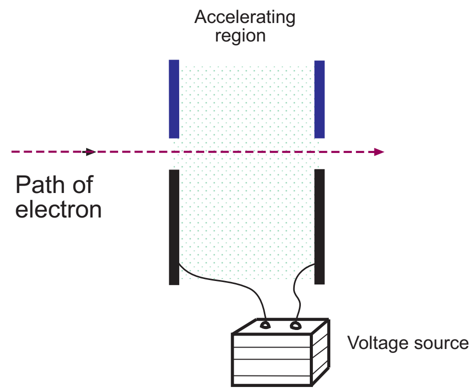

Figure 2.4: Electron is accelerated in a region between two plates, in Example 9.

(a) The racer moves in one dimension (along the x axis, say) with constant acceleration $a = 10.0\,\frac{\mathrm{m}}{\mathrm{s}^2}$. We can take her initial coordinate to be $x_0 = 0$; she starts from rest, so that $v_0 = 0$. Then the location of the car (x) is given by:

$$\begin{aligned} x &= x_0 + v_0 t + \tfrac{1}{2}at^2 \\ &= = 0 + 0 + \tfrac{1}{2}at^2 = \tfrac{1}{2}(10.0\,\tfrac{\mathrm{m}}{\mathrm{s}^2})t^2 \end{aligned}$$

We want to know the time at which $x = 400\,\mathrm{m}$. Substitute and solve for t:

$$400\,\mathrm{m} = \tfrac{1}{2}(10.0\,\tfrac{\mathrm{m}}{\mathrm{s}^2})t^2 \qquad \Longrightarrow \qquad t^2 = \frac{2(400\,\mathrm{m})}{(10.0\,\frac{\mathrm{m}}{\mathrm{s}^2})} = 80.0\,\mathrm{s}^2$$

which gives

$$t = 8.94\,\mathrm{s}\ .$$

The car takes 8.94 s to travel this distance.

(b) We would like to find the velocity at the end of the run, namely at $t = 8.94\,\mathrm{s}$ (the time we found in part (a)). The velocity is:

$$\begin{aligned} v &= v_0 + at \\ &= 0 + (10.0\,\tfrac{\mathrm{m}}{\mathrm{s}^2})t = (10.0\,\tfrac{\mathrm{m}}{\mathrm{s}^2})t \end{aligned}$$

At $t = 8.94\,\mathrm{s}$, the velocity is

$$v = (10.0\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(8.94\,\mathrm{s}) = 89.4\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

The speed at the end of the run is $89.4\,\frac{\mathrm{m}}{\mathrm{s}}$.

---

9. An electron with initial velocity $v_0 = 1.50 \times 10^5\,\frac{\mathrm{m}}{\mathrm{s}}$ enters a region $1.0\,\mathrm{cm}$ long where it is electrically accelerated, as shown in Fig. 2.4. It emerges with velocity $v = 5.70 \times 10^6\,\frac{\mathrm{m}}{\mathrm{s}}$. What was its acceleration, assumed constant? (Such a process

occurs in the electron gun in a cathode–ray tube, used in television receivers and oscilloscopes.)

We are told that the acceleration of the electron is constant, so that Eqs. 2.6–2.9 can be used.

Here we know the initial and final velocities of the electron (v_0 and v). If we let its initial coordinate be $x_0 = 0$ then the final coordinate is $x = 1.0\,\mathrm{cm} = 1.0 \times 10^{-2}\,\mathrm{m}$. We don't know the time t for its travel through the accelerating region and of course we don't know the (constant) acceleration, which is what we're being asked in this problem.

We see that we can solve for a if we use Eq. 2.8:

$$v^2 = v_0^2 + 2a(x - x_0) \qquad \Longrightarrow \qquad a = \frac{v^2 - v_0^2}{2(x - x_0)}$$

Substitute and get:

$$\begin{aligned} a &= \frac{(5.70 \times 10^6\,\frac{\mathrm{m}}{\mathrm{s}})^2 - (1.50 \times 10^5\,\frac{\mathrm{m}}{\mathrm{s}})^2}{2(1.0 \times 10^{-2}\,\mathrm{m})} \\ &= 1.62 \times 10^{15}\,\frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

The acceleration of the electron is $1.62 \times 10^{15}\,\frac{\mathrm{m}}{\mathrm{s}^2}$ (while it is in the accelerating region).

---

10. A world's land speed record was set by Colonel John P. Stapp when on March 19, 1954 he rode a rocket–propelled sled that moved down a track at $1020\,\frac{\mathrm{km}}{\mathrm{h}}$. He and the sled were brought to a stop in $1.4\,\mathrm{s}$. What acceleration did he experience? Express your answer in g units.

For the period of deceleration of the rocket sled (which lasts for $1.4\,\mathrm{s}$) were are given the initial velocity and the final velocity, which is *zero* since the sled comes to rest at the end.

First, convert his initial velocity to SI units:

$$v_0 = 1020\,\frac{\mathrm{km}}{\mathrm{h}} = (1020\,\frac{\mathrm{km}}{\mathrm{h}})\left(\frac{10^3\,\mathrm{m}}{1\,\mathrm{km}}\right)\left(\frac{1\,\mathrm{h}}{3600\,\mathrm{s}}\right) = 283.3\,\frac{\mathrm{m}}{\mathrm{s}}$$

The Eq. 2.6 gives us the acceleration a:

$$v = v_0 + at \qquad \Longrightarrow \qquad a = \frac{v - v_0}{t}$$

Substitute:

$$a = \frac{0 - 283.3\,\frac{\mathrm{m}}{\mathrm{s}}}{1.4\,\mathrm{s}} = -202.4\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

The acceleration is a *negative* number since it is opposite to the sense of the motion; it is a *deceleration*. The *magnitude* of the sled's acceleration is $202.4\,\frac{\mathrm{m}}{\mathrm{s}^2}$.

To express this as a multiple of g, we note that

$$\frac{|a|}{g} = \frac{202.4\,\frac{\mathrm{m}}{\mathrm{s}^2}}{9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}} = 20.7$$

so the magnitude of the acceleration was $|a| = 20.7\,g$. That's a lotta g's!

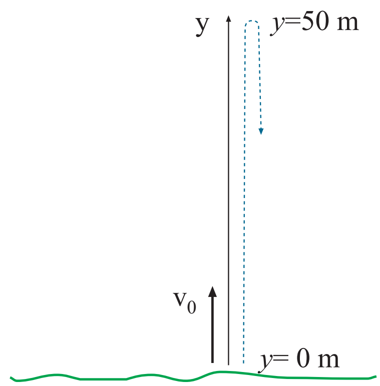

Figure 2.5: Object thrown upward reaches height of 50 m.

2.2.4 Free Fall

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11. (a) With what speed must a ball be thrown vertically from ground level to rise to a maximum height of 50 m? (b) How long will it be in the air?

(a) First, we decide on a coordinate system. I will use the one shown in Fig. 2.5, where the y axis points upward and the origin is at ground level. The ball starts its flight from ground level so its initial position is $y_0 = 0$. When the ball is at maximum height its coordinate is $y = 50\,\mathrm{m}$, but *we also know its velocity at this point*. At maximum height the instantaneous velocity of the ball is *zero*. So if our "final" point is the time of maximum height, then $v = 0$.

So for the trip from ground level to maximum height, we know y_0, y, v and the acceleration $a = -9.8\,\frac{\mathrm{m}}{\mathrm{s}^2} = -g$, but we *don't* know v_0 or the time t to get to maximum height.

From our list of constant–acceleration equations, we see that Equation 2.8 will give us the initial velocity v_0:

$$v^2 = v_0^2 + 2a(y - y_0) \qquad \Longrightarrow \qquad v_0^2 = v^2 - 2a(y - y_0)$$

Substitute, and get:

$$v_0^2 = (0)^2 - 2(-9.8\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(50\,\mathrm{m} - 0) = 980\,\tfrac{\mathrm{m}^2}{\mathrm{s}^2}$$

The next step is to "take the square root". Since we know that v_0 must be a *positive* number, we know that we should take the positive square root of $980\,\frac{\mathrm{m}^2}{\mathrm{s}^2}$. We get:

$$v_0 = +31\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

The initial speed of the ball is $31\,\frac{\mathrm{m}}{\mathrm{s}}$

(b) We want to find the total time that the ball is in flight. What do we know about the ball when it returns to earth and hits the ground? *We know that its y coordinate is equal to zero.* (So far, we don't know anything about the ball's *velocity* at the the time it returns to ground level.) If we consider the time between throwing and impact, then we do know y_0, y, v_0 and of course a. If we substitute into Eq. 2.7 we find:

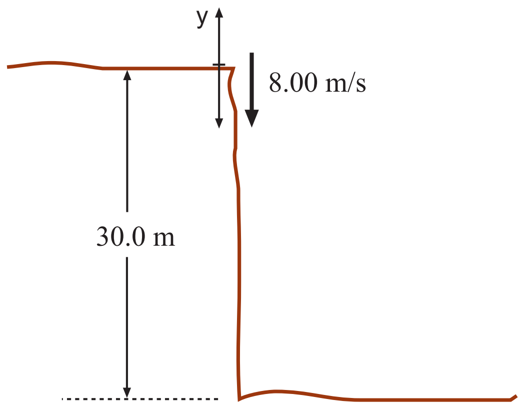

Figure 2.6: Ball is thrown straight down with speed of $8.00\,\frac{\mathrm{m}}{\mathrm{s}}$, in Example 12.

$$0 = 0 + (31\,\tfrac{\mathrm{m}}{\mathrm{s}})t + \tfrac{1}{2}(-9.8\,\tfrac{\mathrm{m}}{\mathrm{s}^2})t^2$$

It is not hard to solve this equation for t. We can factor it to give:

$$t[(31\,\tfrac{\mathrm{m}}{\mathrm{s}}) + \tfrac{1}{2}(-9.8\,\tfrac{\mathrm{m}}{\mathrm{s}^2})t] = 0$$

which has two solutions. One of them is simply $t = 0$. This solution *is* an answer to the question we are asking, namely "When does $y = 0$?" because the ball was at ground level at $t = 0$. But it is not the solution we want. For the other solution, we must have:

$$(31\,\tfrac{\mathrm{m}}{\mathrm{s}}) + \tfrac{1}{2}(-9.8\,\tfrac{\mathrm{m}}{\mathrm{s}^2})t = 0$$

which gives

$$t = \frac{2(31\,\frac{\mathrm{m}}{\mathrm{s}})}{9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}} = 6.4\,\mathrm{s}$$

The ball spends a total of 6.4 seconds in flight.

---

12. A ball is thrown directly downward with an initial speed of $8.00\,\frac{\mathrm{m}}{\mathrm{s}}$ from a height of $30.0\,\mathrm{m}$. When does the ball strike the ground?

We diagram the problem as in Fig. 2.6. We have to choose a coordinate system, and here I will put the let the origin of the y axis be at the place where the ball starts its motion (at the top of the 30 m height). With this choice, the ball starts its motion at $y = 0$ and strikes the ground when $y = -30\,\mathrm{m}$.

We can now see that the problem is asking us: At what time does $y = -30.0\,\mathrm{m}$? We have $v_0 = -8.00\,\frac{\mathrm{m}}{\mathrm{s}}$ (*minus* because the ball is thrown *downward*!) and the acceleration of the the ball is $a = -g = -9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}$, so at any time t the y coordinate is given by

$$y = y_0 + v_0 t + \tfrac{1}{2}at^2 = (-8.00\,\tfrac{\mathrm{m}}{\mathrm{s}})t - \tfrac{1}{2}gt^2$$

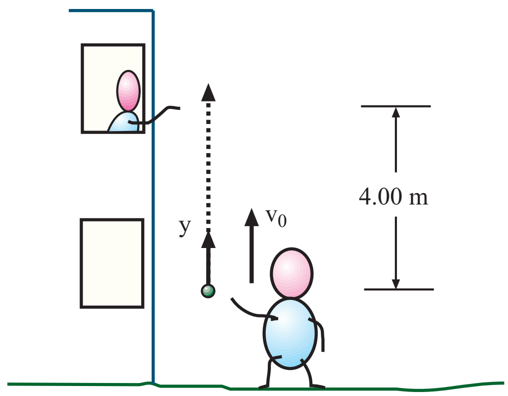

Figure 2.7: Student throws her keys into the air, in Example 13.

But at the time of impact we have

$$y = -30.0\,\mathrm{m} = (-8.00\,\tfrac{\mathrm{m}}{\mathrm{s}})t - \tfrac{1}{2}gt^2 = (-8.00\,\tfrac{\mathrm{m}}{\mathrm{s}})t - (4.90\,\tfrac{\mathrm{m}}{\mathrm{s}^2})t^2 \ ,$$

an equation for which we can solve for t. We rewrite it as:

$$(4.90\,\tfrac{\mathrm{m}}{\mathrm{s}^2})t^2 + (8.00\,\tfrac{\mathrm{m}}{\mathrm{s}})t - 30.0\,\mathrm{m} = 0$$

which is just a quadratic equation in t. From our algebra courses we know how to solve this; the solutions are:

$$t = \frac{-(8.00\,\frac{\mathrm{m}}{\mathrm{s}}) \pm \sqrt{(8.00\,\frac{\mathrm{m}}{\mathrm{s}})^2 - 4(4.90\,\frac{\mathrm{m}}{\mathrm{s}^2})(-30.0\,\mathrm{m})}}{2(4.90\,\frac{\mathrm{m}}{\mathrm{s}^2})}$$

and a little calculator work finally gives us:

$$t = \begin{cases} -3.42\,\mathrm{s} \\ 1.78\,\mathrm{s} \end{cases}$$

Our answer is one of these ... which one? Obviously the ball had to strike the ground at some *positive* value of t, so the answer is $t = 1.78\,\mathrm{s}$.

The ball strikes the ground 1.78 s after being thrown.

---

13. A student throws a set of keys vertically upward to her sorority sister in a window $4.00\,\mathrm{m}$ **above. The keys are caught** $1.50\,\mathrm{s}$ **later by the sister's outstretched hand. (a) With what initial velocity were the keys thrown? (b) What was the velocity of the keys just before they were caught?**

(a) We draw a simple picture of the problem; such a simple picture is given in Fig. 2.7. Having a picture is important, but we should be careful not to put *too much* into the picture; the problem did not say that the keys were caught while they were going up or going down. For all we know at the moment, it could be either one!

We will put the origin of the y axis at the point where the keys were thrown. This simplifies things in that the initial y coordinate of the keys is $y_0 = 0$. Of course, since this is a problem about free–fall, we know the acceleration: $a = -g = -9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$.

What mathematical information does the problem give us? We are told that when $t = 1.50\,\mathrm{s}$, the y coordinate of the keys is $y = 4.00\,\mathrm{m}$. Is this enough information to solve the problem? We write the equation for $y(t)$:

$$\begin{aligned} y &= y_0 + v_0 t + \tfrac{1}{2}at^2 \\ &= v_0 t - \tfrac{1}{2}gt^2 \end{aligned}$$

where v_0 is presently unknown. At $t = 1.50\,\mathrm{s}$, $y = 4.00\,\mathrm{m}$, so:

$$4.00\,\mathrm{m} = v_0(1.50\,\mathrm{s}) - \tfrac{1}{2}(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(1.50\,\mathrm{s})^2 \ .$$

Now we can solve for v_0. Rearrange this equation to get:

$$v_0(1.50\,\mathrm{s}) = 4.00\,\mathrm{m} + \tfrac{1}{2}(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(1.50\,\mathrm{s})^2 = 15.0\,\mathrm{m} \ .$$

So:

$$v_0 = \frac{15.0\,\mathrm{m}}{1.50\,\mathrm{s}} = 10.0\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

(b) We want to find the velocity of the keys at the time they were caught, that is, at $t = 1.50\,\mathrm{s}$. We know v_0; the velocity of the keys at all times follows from Eq. 2.6,

$$v = v_0 + at = 10.0\,\tfrac{\mathrm{m}}{\mathrm{s}} - 9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2}t$$

So at $t = 1.50\,\mathrm{s}$,

$$v = 10.0\,\tfrac{\mathrm{m}}{\mathrm{s}} - 9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2}(1.50\,\mathrm{s}) = -4.68\,\tfrac{\mathrm{m}}{\mathrm{s}} \ .$$

So the velocity of the keys when they were caught was $-4.68\,\frac{\mathrm{m}}{\mathrm{s}}$. Note that the keys had a *negative* velocity; this tells us that the keys were moving *downward* at the time they were caught!

---

14. A ball is thrown vertically upward from the ground with an initial speed of $15.0\,\frac{\mathrm{m}}{\mathrm{s}}$. (a) How long does it take the ball to reach its maximum altitude? (b) What is its maximum altitude? (c) Determine the velocity and acceleration of the ball at $t = 2.00\,\mathrm{s}$.

(a) An illustration of the data given in this problem is given in Fig. 2.8. We measure the coordinate y upward from the place where the ball is thrown so that $y_0 = 0$. The ball's acceleration while in flight is $a = -g = -9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$. We are given that $v_0 = +15.0\,\frac{\mathrm{m}}{\mathrm{s}}$.

The ball is at maximum altitude when its (instantaneous) velocity v is *zero* (it is neither going up nor going down) and we can use the expression for v to solve for t:

$$v = v_0 + at \qquad \Longrightarrow \qquad t = \frac{v - v_0}{a}$$

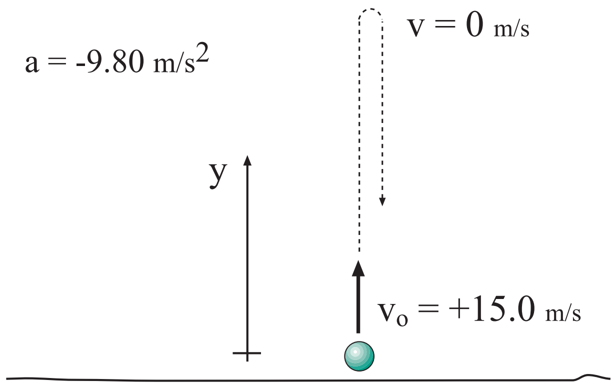

Figure 2.8: Ball is thrown straight up with initial speed $15.0\,\frac{\mathrm{m}}{\mathrm{s}}$.

Plug in the values for the *top* of the ball's flight and get:

$$t = \frac{(0) - (15.0\,\frac{\mathrm{m}}{\mathrm{s}})}{(-9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})} = 1.53\,\mathrm{s}\ .$$

The ball takes $1.53\,\mathrm{s}$ to reach maximum height.

(b) Now that we have the value of t when the ball is at maximum height we can plug it into Eq. 2.7 and find the value of y at this time and that will be the *value* of the maximum height. But we can also use Eq. 2.8 since we know all the values except for y. Solving for y we find:

$$v^2 = v_0^2 + 2ay \qquad \Longrightarrow \qquad y = \frac{v^2 - v_0^2}{2a}$$

Plugging in the numbers, we get

$$y = \frac{(0)^2 - (15.0\,\frac{\mathrm{m}}{\mathrm{s}})^2}{2(-9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})} = 11.5\,\mathrm{m}$$

The ball reaches a maximum height of $11.5\,\mathrm{m}$.

(c) At $t = 2.00\,\mathrm{s}$ (that is, 2.0 seconds after the ball was thrown) we use Eq. 2.6 to find:

$$v = v_0 + at = (15.0\,\tfrac{\mathrm{m}}{\mathrm{s}}) + (-9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(2.00\,\mathrm{s}) = -4.60\,\tfrac{\mathrm{m}}{\mathrm{s}}\ .$$

so at $t = 2.00\,\mathrm{s}$ the ball is on its way back *down* with a speed of $4.60\,\frac{\mathrm{m}}{\mathrm{s}}$.

As for the next part, the acceleration of the ball is *always* equal to $-9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$ while it is in flight.

---

15. A baseball is hit such that it travels straight upward after being struck by the bat. A fan observes that it requires $3.00\,\mathrm{s}$ for the ball to reach its maximum height. Find (a) its initial velocity and (b) its maximum height. Ignore the effects of air resistance.

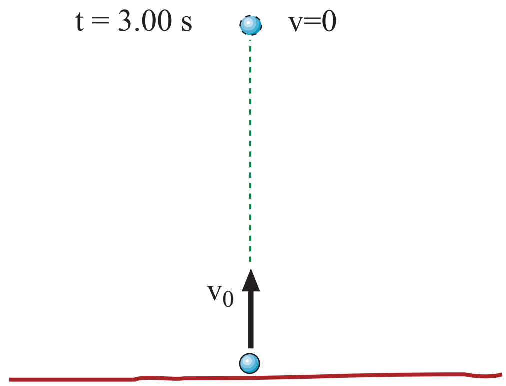

Figure 2.9: Ball is hit straight up; reaches maximum height 3.00 s later.

(a) An illustration of the data given in the problem is given in Fig. 2.9.

For the period from when the ball is hit to the time it reaches maximum height, we know the time interval, the acceleration ($a = -g$) and also the final velocity, since at maximum height the velocity of the ball is *zero*. Then Eq. 2.6 gives us v_0:

$$v = v_0 + at \qquad \Longrightarrow \qquad v_0 = v - at$$

and we get:

$$v_0 = 0 - (-9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(3.00\,\mathrm{s}) = 29.4\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

The initial velocity of the ball was $+29.4\,\frac{\mathrm{m}}{\mathrm{s}}$.

(b) To find the value of the maximum height, we need to find the value of the y coordinate at time $t = 3.00\,\mathrm{s}$. We can use either Eq. 2.7 or Eq. 2.8. the latter gives:

$$v^2 = v_0^2 + 2a(y - y_0) \qquad \Longrightarrow \qquad (y - y_0) = \frac{v^2 - v_0^2}{2a}$$

Plugging in the numbers we find that the change in y coordinate for the trip up was:

$$y - y_0 = \frac{0^2 - (29.4\,\frac{\mathrm{m}}{\mathrm{s}})^2}{2(-9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})} = 44.1\,\mathrm{m}\ .$$

The ball reached a maximum height of 44.1 m.

---

16. A parachutist bails out and freely falls $50\,\mathrm{m}$. Then the parachute opens, and thereafter she decelerates at $2.0\,\frac{\mathrm{m}}{\mathrm{s}^2}$. She reaches the ground with a speed of $3.0\,\frac{\mathrm{m}}{\mathrm{s}}$. (a) How long was the parachutist in the air? (b) At what height did the fall begin?

(a) This problem gives several odd bits of information about the motion of the parachutist! We organize the information by *drawing a diagram*, like the one given in Fig. 2.10. It is very important to organize our work in this way!

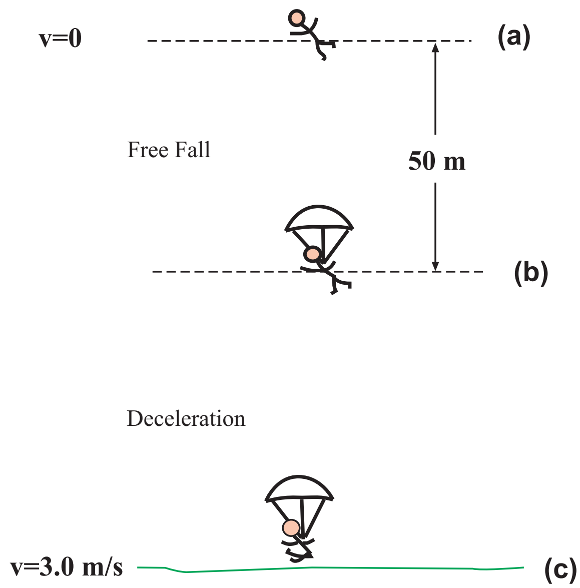

Figure 2.10: Diagram showing motion of parachutist in Example 16.

At the height indicated by (a) in the figure, the skydiver has *zero* initial speed. As she falls from (a) to (b) her acceleration is that of gravity, namely $9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$ *downward*. We know that (b) is 50 m lower than (a) but we don't yet know the skydiver's speed at (b). Finally, at point (c) her speed is $3.0\,\frac{\mathrm{m}}{\mathrm{s}}$ and between (b) and (c) her "deceleration" was $2.0\,\frac{\mathrm{m}}{\mathrm{s}^2}$, but we don't know the difference in height between (b) and (c).

How can we start to fill in the gaps in our knowledge?

We note that on the trip from (a) to (b) we do know the starting velocity, the distance travelled and the acceleration. From Eq. 2.8 we can see that this is enough to find the final velocity, that is, the velocity at (b).

Use a coordinate system (y) which has its origin at level (b), and the y axis pointing upward. Then the initial y coordinate is $y_0 = 50\,\mathrm{m}$ and the the initial velocity is $v_0 = 0$. The final y coordinate is $y = 0$ and the acceleration is constant at $a = -9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$. Then using Eq. 2.8 we have:

$$v^2 = v_0^2 + 2a(y - y + 0) = 0 + 2(-9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(0 - 50\,\mathrm{m}) = 980\,\tfrac{\mathrm{m}^2}{\mathrm{s}^2}$$

which has the solutions

$$v = \pm 31.3\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

but here the skydiver is obviously moving *downward* at (b), so we must pick

$$v = -31.3\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

for the velocity at (b).

While we're at it, we can find the time it took to get from (a) to (b) using Eq. 2.6, since we know the velocities and the acceleration for the motion. We find:

$$v = v_0 + at \qquad \Longrightarrow \qquad t = \frac{v - v_0}{a}$$

Substitute:

$$t = \frac{(-31.3\,\frac{\mathrm{m}}{\mathrm{s}} - 0)}{-9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}} = 3.19\,\mathrm{s}$$

The skydiver takes 3.19 s to fall from (a) to (b).

Now we consider the motion from (b) to (c). For this part of the motion we know the initial and final velocities. We also know the acceleration, but we must be careful about how it is expressed. During this part of the trip, the skydiver's motion is always downward (velocity is always negative) but her speed decreases from $31.9\,\frac{\mathrm{m}}{\mathrm{s}}$ to $3.0\,\frac{\mathrm{m}}{\mathrm{s}}$. The *velocity* changes from $-31.3\,\frac{\mathrm{m}}{\mathrm{s}}$ to $-3.0\,\frac{\mathrm{m}}{\mathrm{s}}$ so that the velocity has *increased*. The acceleration is *positive*; it is in the opposite sense as the motion and thus it was rightly called a "deceleration" in the problem. So for the motion from (b) to (c), we have

$$a = +2.0\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

We have the starting and final velocities for the trip from (b) to (c) so Eq. 2.6 lets us solve for the time t:

$$v = v_0 + at \qquad \Longrightarrow \qquad t = \frac{v - v_0}{a}$$

Substitute:

$$t = \frac{-3.0\,\frac{\mathrm{m}}{\mathrm{s}} - (-31.3\,\frac{\mathrm{m}}{\mathrm{s}})}{+2.0\,\frac{\mathrm{m}}{\mathrm{s}^2}} = 14.2\,\mathrm{s}$$

Now we are prepared to answer part (a) of the problem. The time of the travel from (a) was 3.19 s; the time of travel from (b) to (c) was 14.2 s. The total time in the air was

$$t_{\mathrm{Total}} = 3.19\,\mathrm{s} + 14.2\,\mathrm{s} = 17.4\,\mathrm{s}$$

(b) Let's keep thinking about the trip from (b) to (c); we'll keep the origin at the same place as before (at (b)). Then for the trip from (b) to (c) the initial coordinate is $y_0 = 0$. The initial velocity is $v_0 = -31.9\,\frac{\mathrm{m}}{\mathrm{s}}$ and the final velocity is $v = -3.0\,\frac{\mathrm{m}}{\mathrm{s}}$. We have the acceleration, so Eq. 2.8 gives us the final coordinate y:

$$v^2 = v_0^2 + 2a(y - y_0) \qquad \Longrightarrow \qquad y - y_0 = \frac{v^2 - v_0^2}{2a}$$

Substitute:

$$y - y_0 = \frac{(-3.0\,\frac{\mathrm{m}}{\mathrm{s}})^2 - (-31.3\,\frac{\mathrm{m}}{\mathrm{s}})^2}{2(+2.0\,\frac{\mathrm{m}}{\mathrm{s}^2})} = -243\,\mathrm{m}$$

Since we chose $y_0 = 0$, the final coordinate of the skydiver is $y = -243\,\mathrm{m}$.

We have used the same coordinate system in both parts, so overall the skydiver has gone from $y = +50\,\mathrm{m}$ to $y = -243\,\mathrm{m}$. The change in height was

$$\Delta y = -243\,\mathrm{m} - 50\,\mathrm{m} = -293\,\mathrm{m}$$

So the parachutist's fall began at a height of 293 m above the ground.

17. A stone falls from rest from the top of a high cliff. a second stone is thrown downward from the same height $2.00\,\mathrm{s}$ later with an initial speed of $30.0\,\frac{\mathrm{m}}{\mathrm{s}}$. If both stones hit the ground simultaneously, how high is the cliff?

This is a "puzzle"–type problem which goes beyond the normal substitute–and–solve type; it involves more organization of our work and a clear understanding of our equations. Here's the way I would attack it.

We have two different falling objects here with their own coordinates; we'll put our origin at the top of the cliff and call the y coordinate of the first stone y_1 and that of the second stone y_2. Each has a different dependence on the time t.

For the first rock, we have $v_0 = 0$ since it falls from rest and of course $a = -g$ so that its position is given by

$$y_1 = y_0 + v_0 t + \tfrac{1}{2}at^2 = -\tfrac{1}{2}gt^2$$

This is simple enough but we need to remind ourselves that here t is the time since the *first* stone started its motion. It is *not* the same as the time since the *second* stone starts its motion. To be clear, let's call this time t_1. So we have:

$$y_1 = -\tfrac{1}{2}gt_1^2 = -4.90\,\tfrac{\mathrm{m}}{\mathrm{s}^2}t_1^2$$

Now, for the motion of the second stone, if we write t_2 for the time since *it* started its motion, the facts stated in the problem tell us that its y coordinate is given by:

$$y_2 = y_0 + v_0 t_2 + \tfrac{1}{2}at_2^2 = (-30.0\,\tfrac{\mathrm{m}}{\mathrm{s}})t_2 - \tfrac{1}{2}gt_2^2$$

So far, so good. The problem tells us that the first stone has been falling for $2.0\,\mathrm{s}$ longer than the second one. This means that t_1 is $2.0\,\mathrm{s}$ larger than t_2. So:

$$t_1 = t_2 + 2.0\,\mathrm{s} \qquad \Longrightarrow \qquad t_2 = t_1 - 2.0\,\mathrm{s}$$

(We will use t_1 as our one time variable.) Putting this into our last equation and doing some algebra gives

$$\begin{aligned} y_2 &= (-30.0\,\tfrac{\mathrm{m}}{\mathrm{s}})(t_1 - 2.0\,\mathrm{s}) - \tfrac{1}{2}(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(t_1 - 2.0\,\mathrm{s})^2 \\ &= (-30.0\,\tfrac{\mathrm{m}}{\mathrm{s}})(t_1 - 2.0\,\mathrm{s}) - (4.90\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(t_1^2 - 4.0\,\mathrm{s}t_1 + 4.0\,\mathrm{s}^2)^2 \\ &= (-4.90\,\tfrac{\mathrm{m}}{\mathrm{s}^2})t_1^2 + (-30.0\,\tfrac{\mathrm{m}}{\mathrm{s}} + 19.6\,\tfrac{\mathrm{m}}{\mathrm{s}})t_1 + (60.0\,\mathrm{m} - 19.6\,\mathrm{m}) \\ &= (-4.90\,\tfrac{\mathrm{m}}{\mathrm{s}^2})t_1^2 + (-10.4\,\tfrac{\mathrm{m}}{\mathrm{s}})t_1 + (40.4\,\mathrm{m}) \end{aligned}$$

We need to remember that this expression for y_2 will be meaningless for values of t_1 which are less than $2.0\,\mathrm{s}$. With this expression we can find values of y_1 and y_2 using the *same* time coordinate, t_1.

Now, the problem tells us that at some time (t_1) the coordinates of the two stones are *equal*. We don't yet yet know what that time or coordinate *is* but that is the information contained in the statement "both stones hit the ground simultaneously". We can find this time by setting y_1 equal to y_2 and solving:

$$(-4.90\,\tfrac{\mathrm{m}}{\mathrm{s}^2})t_1^2 = (-4.90\,\tfrac{\mathrm{m}}{\mathrm{s}^2})t_1^2 + (-10.4\,\tfrac{\mathrm{m}}{\mathrm{s}})t_1 + (40.4\,\mathrm{m})$$

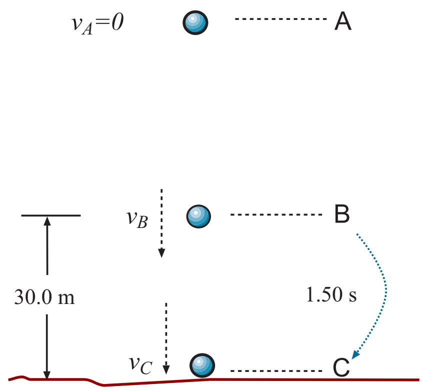

Figure 2.11: Diagram for the falling object in Example 18.

Fortunately the t^2 term cancels in this equation making it a lot easier. We get:

$$(-10.4\,\frac{\mathrm{m}}{\mathrm{s}})t_1 + (40.4\,\mathrm{m}) = 0$$

which has the solution

$$t_1 = \frac{40.4\,\mathrm{m}}{10.4\,\frac{\mathrm{m}}{\mathrm{s}}} = 3.88\,\mathrm{s}$$

So the rocks will have the same location at $t_1 = 4.88\,\mathrm{s}$, that is, 4.88 s after the first rock has been dropped.

What *is* that location? We can find this by using our value of t_1 to get either y_1 or y_2 (the answer will be the same). Putting it into the expression for y_1 we get:

$$y_1 = -4.90\,\frac{\mathrm{m}}{\mathrm{s}^2}t^2 = (-4.90\,\frac{\mathrm{m}}{\mathrm{s}^2})(4.88\,\mathrm{s})^2 = -117\,\mathrm{m}$$

So *both* stones were 117 m below the initial point at the time of impact. The cliff is 117 m high.

---

18. A falling object requires 1.50 s to travel the last 30.0 m before hitting the ground. From what height above the ground did it fall?

This is an intriguing sort of problem... very easy to state, but not so clear as to where to begin in setting it up!

The first thing to do is *draw a diagram*. We draw the important points of the object's motion, as in Fig. 2.11. The object has zero velocity at A; at B it is at a height of 30.0 m above the ground with an unknown velocity. At C it is at ground level, the time is 1.50 s later than at B and we also don't know the velocity here. Of course, we know the acceleration: $a = -9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$!!

We are given all the information about the trip from B to C, so why not try to fill in our knowledge about this part? We know the final and initial coordinates, the acceleration and the time so we can find the initial velocity (that is, the velocity at B). Let's put the origin at ground level; then, $y_0 = 30.0\,\mathrm{m}$, $y = 0$ and $t = 1.50\,\mathrm{s}$, and using

$$y = y_0 + v_0 t + \tfrac{1}{2}at^2$$

we find:

$$v_0 t = (y - y_0) - \tfrac{1}{2}at^2 = (0 - (30.0\,\mathrm{m})) - \tfrac{1}{2}(-9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(1.50)^2 = -19.0\,\mathrm{m}$$

so that

$$v_0 = \frac{(-19.0\,\mathrm{m})}{t} = \frac{(-19.0\,\mathrm{m})}{(1.50\,\mathrm{s})} = -12.5\,\tfrac{\mathrm{m}}{\mathrm{s}}\ .$$

This is the velocity at point B; we can also find the velocity at C easily, since that is the final velocity, v:

$$v = v_0 + at = (-12.5\,\tfrac{\mathrm{m}}{\mathrm{s}}) + (-9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(1.50\,\mathrm{s}) = -27.3\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

Now we can consider the trip from the starting point, A to the point of impact, C. We don't know the initial y coordinate, but we *do* know the final and initial velocities: The initial velocity is $v_0 = 0$ and the final velocity is $v = -27.3\,\frac{\mathrm{m}}{\mathrm{s}}$, as we just found. With the origin set at ground level, the final y coordinate is $y = 0$. We don't know the *time* for the trip, but if we use:

$$v^2 = v_0^2 + 2a(y - y_0)$$

we find:

$$(y - y_0) = \frac{(v^2 - v_0^2)}{2a} = \frac{(-27.3\,\frac{\mathrm{m}}{\mathrm{s}})^2 - (0)^2}{2(-9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})} = -38.2\,\mathrm{m}$$

and we can rearrange this to get:

$$y_0 = y + 38.2\,\mathrm{m} = 0 + 38.2\,\mathrm{m} = 38.2\,\mathrm{m}$$

and the so the object started falling from a height of 38.2 m.

There are probably cleverer ways to do this problem, but here I wanted to give you the slow, patient approach!

Chapter 3

Motion in Two and Three Dimensions

3.1 The Important Stuff

3.1.1 Position

In three dimensions, the location of a particle is specified by its **location vector, r**:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{3.1}$$

If during a time interval Δt the position vector of the particle changes from $\mathbf{r}_1$ to $\mathbf{r}_2$, the displacement $\Delta\mathbf{r}$ for that time interval is

$$\begin{aligned} \Delta\mathbf{r} &= \mathbf{r}_1 - \mathbf{r}_2 &(3.2)\\ &= (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k} &(3.3) \end{aligned}$$

3.1.2 Velocity

If a particle moves through a displacement $\Delta\mathbf{r}$ in a time interval Δt then its average velocity for that interval is

$$\overline{v} = \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\mathbf{i} + \frac{\Delta y}{\Delta t}\mathbf{j} + \frac{\Delta z}{\Delta t}\mathbf{k} \tag{3.4}$$

As before, a more interesting quantity is the *instantaneous* velocity $\mathbf{v}$, which is the limit of the average velocity when we shrink the time interval Δt to zero. It is the time derivative of the position vector $\mathbf{r}$:

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt} &(3.5)\\ &= \frac{d}{dt}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) &(3.6)\\ &= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} &(3.7) \end{aligned}$$

can be written:

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} \tag{3.8}$$

where

$$v_x = \frac{dx}{dt} \qquad v_y = \frac{dy}{dt} \qquad v_z = \frac{dz}{dt} \tag{3.9}$$

The instantaneous velocity $\mathbf{v}$ of a particle is always tangent to the path of the particle.

3.1.3 Acceleration

If a particle's velocity changes by $\Delta\mathbf{v}$ in a time period Δt, the average acceleration $\overline{\mathbf{a}}$ for that period is

$$\overline{\mathbf{a}} = \frac{\Delta\mathbf{v}}{\Delta t} = \frac{\Delta v_x}{\Delta t}\mathbf{i} + \frac{\Delta v_y}{\Delta t}\mathbf{j} + \frac{\Delta v_z}{\Delta t}\mathbf{k} \tag{3.10}$$

but a much more interesting quantity is the result of shrinking the period Δt to zero, which gives us the instantaneous acceleration, $\mathbf{a}$. It is the time derivative of the velocity vector $\mathbf{v}$:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \tag{3.11}$$

$$= \frac{d}{dt}(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}) \tag{3.12}$$

$$= \frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} + \frac{dv_z}{dt}\mathbf{k} \tag{3.13}$$

which can be written:

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \tag{3.14}$$

where

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \qquad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} \qquad a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2} \tag{3.15}$$

3.1.4 Constant Acceleration in Two Dimensions

When the acceleration $\mathbf{a}$ (for motion in two dimensions) is constant we have two sets of equations to describe the x and y coordinates, each of which is similar to the equations in Chapter 2. (Eqs. 2.6—2.9.) In the following, motion of the particle begins at $t = 0$; the initial position of the particle is given by

$$\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j}$$

and its initial velocity is given by

$$\mathbf{v}_0 = v_{0x}\mathbf{i} + v_{0y}\mathbf{j}$$

and the vector $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j}$ is *constant.*

$$v_x = v_{0x} + a_x t \qquad v_y = v_{0y} + a_y t \tag{3.16}$$

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2 \qquad y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2 \tag{3.17}$$

$$v_x^2 = v_{0x}^2 + 2a_x(x - x_0) \qquad v_y^2 = v_{0y}^2 + 2a_y(y - y_0) \tag{3.18}$$

$$x = x_0 + \tfrac{1}{2}(v_{0x} + v_x)t \qquad y = y_0 + \tfrac{1}{2}(v_{0y} + v_y)t \tag{3.19}$$

Though the equations in each pair have the same *form* they are not identical because the components of $\mathbf{r}_0$, $\mathbf{v}_0$ and $\mathbf{a}$ are not the same.

3.1.5 Projectile Motion

When a particle moves in a vertical plane during free–fall its acceleration is constant; the acceleration has magnitude $9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$ and is directed downward. If its coordinates are given by a horizontal x axis and a vertical y axis which is directed upward, then the acceleration of the **projectile** is

$$a_x = 0 \qquad a_y = -9.80\,\frac{\mathrm{m}}{\mathrm{s}^2} = -g \tag{3.20}$$

For a projectile, the horizontal acceleration a_x is zero!!!

Projectile motion is a special case of constant acceleration, so we simply use Eqs. 3.16–3.19, with the proper values of a_x and a_y.

3.1.6 Uniform Circular Motion

When a particle is moving in a circular path (or part of one) at *constant speed* we say that the particle is in **uniform circular motion**. Even though the speed is not changing, *the particle is accelerating* because its velocity $\mathbf{v}$ is changing *direction*.

The acceleration of the particle is directed toward the center of the circle and has magnitude

$$a = \frac{v^2}{r} \tag{3.21}$$

where r is the radius of the circular path and v is the (constant) speed of the particle. Because of the direction of the acceleration (i.e. toward the center), we say that a particle in uniform circular motion has a **centripetal acceleration**.

If the particle repeatedly makes a complete circular path, then it is useful to talk about the time T that it takes for the particle to make one complete trip around the circle. This is called the **period** of the motion. The period is related to the speed of the particle and radius of the circle by:

$$T = \frac{2\pi r}{v} \tag{3.22}$$

3.1.7 Relative Motion

The velocity of a particle depends on who is doing the measuring; as we see later on it is perfectly valid to consider "moving" observers who carry their own clocks and coordinate systems with them, i.e. they make measurements according to their own **reference frame**; that is to say, a set of Cartesian coordinates which may be in motion with respect to another set of coordinates. Here we will assume that the axes in the different system remain parallel to one another; that is, one system can move (translate) but not *rotate* with respect to another one.

Suppose observers in frames A and B measure the position of a point P. Then then if we have the definitions:

$$\mathbf{r}_{PA} = \text{ position of } P \text{ as measured by } A$$

$$\mathbf{r}_{PB} = \text{ position of } P \text{ as measured by } B$$

$$\mathbf{r}_{BA} = \text{ position of } B\text{'s origin, as measured by } A$$

with $\mathbf{v}$'s and $\mathbf{a}$'s standing for the appropriate time derivatives, then we have the relations:

$$\mathbf{r}_{PA} = \mathbf{r}_{PB} + \mathbf{r}_{BA} \tag{3.23}$$

$$\mathbf{v}_{PA} = \mathbf{v}_{PB} + \mathbf{v}_{BA} \tag{3.24}$$

For the purposes of doing physics, it is important to consider reference frames which move at *constant velocity* with respect to one another; for these cases, $\mathbf{v}_{BA} = 0$ and then we find that point P has the same acceleration in these reference frames:

$$\mathbf{a}_{PA} = \mathbf{a}_{PB}$$

Newton's Laws (next chapter!) apply to such a set of **inertial reference frames**. Observers in each of these frames agree on the value of a particle's acceleration.

Though the above rules for translation between reference frames seem very reasonable, it was the great achievement of Einstein with his theory of **Special Relativity** to understand the more subtle ways that we must relate measured quantities between reference frames. The trouble comes about because time (t) is *not* the same absolute quantity among the different frames.

Among other places, Eq. 3.24 is used in problems where an object like a plane or boat has a known velocity in the frame of (with respect to) a medium like air or water which *itself* is moving with respect to the stationary ground; we can then find the velocity of the plane or boat with respect to the *ground* from the *vector sum* in Eq. 3.24.

3.2 Worked Examples

3.2.1 Velocity

1. The position of an electron is given by $\mathbf{r} = 3.0t\mathbf{i} - 4.0t^2\mathbf{j} + 2.0\mathbf{k}$ (where t is in seconds and the coefficients have the proper units for r to be in meters). (a) What is $\mathbf{v}(t)$ for the electron? (b) In unit–vector notation, what is v at $t = 2.0\,\mathrm{s}$? (c) What are the magnitude and direction of v just then?

(a) The velocity vector $\mathbf{v}$ is the time–derivative of the position vector $\mathbf{r}$:

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(3.0t\mathbf{i} - 4.0t^2\mathbf{j} + 2.0\mathbf{k}) \\ &= 3.0\mathbf{i} - 8.0t\mathbf{j} \end{aligned}$$

where we mean that when t is in seconds, $\mathbf{v}$ is given in $\frac{\mathrm{m}}{\mathrm{s}}$.

(b) At $t = 2.00\,\mathrm{s}$, the value of $\mathbf{v}$ is

$$\mathbf{v}(t = 2.00\,\mathrm{s}) = 3.0\mathbf{i} - (8.0)(2.0)\mathbf{j} = 3.0\mathbf{i} - 16.\mathbf{j}$$

that is, the velocity is $(3.0\mathbf{i} - 16.\mathbf{j})\,\frac{\mathrm{m}}{\mathrm{s}}$.

(c) Using our answer from (b), at $t = 2.00\,\mathrm{s}$ the magnitude of $\mathbf{v}$ is

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{(3.00\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 + (-16.\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 + (0)^2} = 16.\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

we note that the velocity vector lies in the xy plane (even though this is a three–dimensional problem!) so that we can express its direction with a single angle, the usual angle θ measured anti-clockwise in the xy plane from the x axis. For this angle we get:

$$\tan\theta = \frac{v_y}{v_x} = -5.33 \qquad \Longrightarrow \qquad \theta = \tan^{-1}(-5.33) = -79^\circ \ .$$

When we take the inverse tangent, we should always check and see if we have chosen the right quadrant for θ. In this case -79° *is* correct since v_y is negative and v_x is positive.

3.2.2 Acceleration

---

2. A particle moves so that its position as a function of time in SI units is $\mathbf{r} = \mathbf{i} + 4t^2\mathbf{j} + t\mathbf{k}$. Write expressions for (a) its velocity and (b) its acceleration as functions of time.

(a) To clarify matters, what we mean here is that when we use the numerical value of t in *seconds*, we will get the values of $\mathbf{r}$ in *meters*. Since the velocity vector is the time–derivative of the position vector $\mathbf{r}$, we have:

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt} \\ &= \frac{d}{dt}(\mathbf{i} + 4t^2\mathbf{j} + t\mathbf{k}) \\ &= 0\mathbf{i} + 8t\mathbf{j} + \mathbf{k} \end{aligned}$$

That is, $\mathbf{v} = 8t\mathbf{j} + \mathbf{k}$. Here, we mean that when we use the numerical value of t in seconds, we will get the value of $\mathbf{v}$ in $\frac{\mathrm{m}}{\mathrm{s}}$.

(b) The acceleration $\mathbf{a}$ is the time–derivative of $\mathbf{v}$, so using our result from part (a) we have:

$$\begin{aligned} \mathbf{a} &= \frac{d\mathbf{v}}{dt} \\ &= \frac{d}{dt}(8t\mathbf{j} + \mathbf{k}) \\ &= 8\mathbf{j} \end{aligned}$$

So $\mathbf{a} = 8\mathbf{j}$, where we mean that the value of $\mathbf{a}$ is in units of $\frac{\mathrm{m}}{\mathrm{s}^2}$. In fact, we should really include the units *here* and write:

$$\mathbf{a} = \left(8\,\tfrac{\mathrm{m}}{\mathrm{s}^2}\right)\mathbf{j}$$

---

3. A particle moving with an initial velocity $\mathbf{v} = (50\,\frac{\mathrm{m}}{\mathrm{s}})\mathbf{j}$ undergoes an acceleration $\mathbf{a} = [35\,\mathrm{m/s^2} + (2\,\mathrm{m/s^5})t^3]\mathbf{i} + [4\,\mathrm{m/s^2} - (1\,\mathrm{m/s^4})t^2]\mathbf{j}$. What are the particle's position and velocity after $3.0\,\mathrm{s}$, assuming that it starts at the origin?

In the problem we are given the acceleration at *all* times, the *initial* velocity and also the *initial* position. We know that at $t = 0$, the velocity components are $v_x = 0$ and $v_y = 50\,\frac{\mathrm{m}}{\mathrm{s}}$ and the coordinates are $x = 0$ and $y = 0$.

From the acceleration $\mathbf{a}$ we do know *something* about the velocity. Since the acceleration is the time derivative of the velocity:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \ ,$$

the velocity is the *anti-derivative* (or “indefinite integral”, “primitive”...) of the acceleration. Having learned our calculus well, we immediately write:

$$\mathbf{v} = \left[35t + \frac{1}{2}t^4 + C_1\right]\mathbf{i} + \left[4t - \frac{1}{3}t^3 + C_2\right]\mathbf{j}$$

Here, for simplicity, I have omitted the units that are supposed to go with the coefficients. (I'm not supposed to do that!) Just keep in mind that time is supposed to be in *seconds*, length is in *meters*...

Of course, when we do the integration, we get constants C_1 and C_2 which (so far) have not been determined. We can determine them using the rest of the information in the problem. Since $v_x = 0$ at $t = 0$ we get:

$$35(0) + \tfrac{1}{2}(0)^4 + C_1 = 0 \qquad \Longrightarrow \qquad C_1 = 0$$

and

$$4(0) - \tfrac{1}{3}(0)^3 + C_2 = 50 \qquad \Longrightarrow \qquad C_2 = 50$$

so the velocity as a function of time is

$$\mathbf{v} = \left[35t + \frac{1}{2}t^4\right]\mathbf{i} + \left[4t - \frac{1}{3}t^3 + 50\right]\mathbf{j}$$

where t is in seconds and the result is in $\frac{\mathrm{m}}{\mathrm{s}}$.

We can find $\mathbf{r}$ as a function of time in the same way. Since

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

then $\mathbf{r}$ is the anti-derivative of $\mathbf{v}$. We get:

$$\mathbf{r} = \left[\frac{35}{2}t^2 + \frac{1}{10}t^5 + C_3\right]\mathbf{i} + \left[2t^2 - \frac{1}{12}t^4 + 50t + C_4\right]\mathbf{j}$$

and once again we need to solve for the constants. $x = 0$ at $t = 0$, so

$$\frac{35}{2}(0)^2 + \frac{1}{10}(0)^5 + C_3 = 0 \qquad \Longrightarrow \qquad C_3 = 0$$

and $y = 0$ at $t = 0$, so

$$2(0)^2 - \frac{1}{12}(0)^4 + 50(0) + C_4 = 0 \qquad \Longrightarrow \qquad C_4 = 0$$

and so $\mathbf{r}$ is fully determined:

$$\mathbf{r} = \left[\frac{35}{2}t^2 + \frac{1}{10}t^5\right]\mathbf{i} + \left[2t^2 - \frac{1}{12}t^4 + 50t\right]\mathbf{j}$$

Now we can answer the questions.

We want to know the value of $\mathbf{r}$ (the particle's position) at $t = 3.0\,\mathrm{s}$. Just plug in!

$$x(t = 3.0\,\mathrm{s}) = \frac{35}{2}(3.0)^2 + \frac{1}{10}(3.0)^5 = 181\,\mathrm{m}$$

and

$$y(t = 3.0\,\mathrm{s}) = 2(3.0)^2 - \frac{1}{12}(3.0)^4 + 50(3.0) = 161\,\mathrm{m}\ .$$

The components of the velocity at $t = 3.0\,\mathrm{s}$ are

$$v_x(t = 3.0\,\mathrm{s}) = 35(3.0) + \tfrac{1}{2}(3.0)^4 = 146\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

and

$$v_y(t = 3.0\,\mathrm{s}) = 4(3.0) - \tfrac{1}{3}(3.0)^3 + 50 = 53\,\tfrac{\mathrm{m}}{\mathrm{s}}\ .$$

Here we have been careful to include the proper (SI) units in the final answers because coordinates and velocities must have *units*.

3.2.3 Constant Acceleration in Two Dimensions

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4. A fish swimming in a horizontal plane has a velocity $\mathbf{v}_0 = (4.0\mathbf{i} + 1.0\mathbf{j})\,\frac{\mathrm{m}}{\mathrm{s}}$ at a point in the ocean whose position vector is $\mathbf{r}_0 = (10.0\mathbf{i} - 4.0\mathbf{j})\,\mathrm{m}$ relative to a stationary rock at the shore. After the fish swims with constant acceleration for $20.0\,\mathrm{s}$, its velocity is $\mathbf{v} = (20.0\mathbf{i} - 5.0\mathbf{j})\,\frac{\mathrm{m}}{\mathrm{s}}$. (a) What are the components of the acceleration? (b) What is the direction of the acceleration with respect to the fixed x axis? (c) Where is the fish at $t = 25\,\mathrm{s}$ and in what direction is it moving?

(a) Since we are given that the acceleration is *constant*, we can use Eqs. 3.16:

$$v_x = v_{0x} + a_x t \qquad\qquad v_y = v_{0y} + a_y t$$

to get:

$$a_x = \frac{v_x - v_{0x}}{t} = \frac{(20.0\,\frac{\mathrm{m}}{\mathrm{s}} - 4.0\,\frac{\mathrm{m}}{\mathrm{s}})}{20.0\,\mathrm{s}} = 0.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

and

$$a_y = \frac{v_y - v_{0y}}{t} = \frac{(-5.0\,\frac{\mathrm{m}}{\mathrm{s}} - 1.0\,\frac{\mathrm{m}}{\mathrm{s}})}{20.0\,\mathrm{s}} = -0.30\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

and the acceleration *vector* of the fish is

$$\mathbf{a} = (0.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})\mathbf{i} - (0.30\,\tfrac{\mathrm{m}}{\mathrm{s}^2})\mathbf{j}\ .$$

(b) With the angle θ measured counterclockwise from the $+x$ axis, the direction of the acceleration $\mathbf{a}$ is:

$$\tan\theta = \frac{a_y}{a_x} = \frac{-0.30}{0.80} = -0.375$$

A calculator gives us:

$$\theta = \tan^{-1}(-0.375) = -20.6^\circ$$

Since the y component of the acceleration is negative, this angle *is* in the proper quadrant. The direction of the acceleration is given by $\theta = -20.6^\circ$. (The same as $\theta = 360^\circ - 20.6^\circ = 339.4^\circ$.

(c) We can use Eq. 3.17 to find the values of x and y at $t = 25\,\mathrm{s}$:

$$\begin{aligned} x &= x_0 + v_{0x} + \tfrac{1}{2}a_x t^2 \\ &= 10\,\mathrm{m} + 4.0\,\tfrac{\mathrm{m}}{\mathrm{s}}(25\,\mathrm{s}) + \tfrac{1}{2}(0.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(25\,\mathrm{s})^2 \\ &= 360\,\mathrm{m} \end{aligned}$$

and

$$\begin{aligned} y &= y_0 + v_{0y} + \tfrac{1}{2}a_y t^2 \\ &= -4.0\,\mathrm{m} + 1.0\,\tfrac{\mathrm{m}}{\mathrm{s}}(25\,\mathrm{s}) + \tfrac{1}{2}(-0.30\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(25\,\mathrm{s})^2 \\ &= -72.8\,\mathrm{m} \end{aligned}$$

At $t = 25\,\mathrm{s}$ the velocity components of the fish are given by:

$$\begin{aligned} v_x &= v_{0x} + a_x t \\ &= 4.0\,\tfrac{\mathrm{m}}{\mathrm{s}} + (0.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(25\,\mathrm{s}) = 24\,\tfrac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

and

$$\begin{aligned} v_y &= v_{0y} + a_y t \\ &= 1.0\,\tfrac{\mathrm{m}}{\mathrm{s}} + (-0.30\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(25\,\mathrm{s}) = -6.5\,\tfrac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

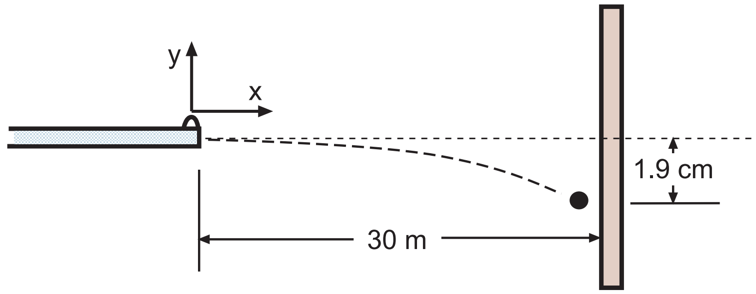

Figure 3.1: Bullet hits target 1.9 cm below the aiming point.

so that at that time the speed of the fish is

$$\begin{aligned} v &= \sqrt{v_x^2 + v_y^2} \\ &= \sqrt{(24\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 + (-6.5\,\tfrac{\mathrm{m}}{\mathrm{s}})^2} = 24.9\,\tfrac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

and the direction of its motion θ is found from:

$$\tan\theta = \frac{v_y}{v_x} = \frac{-6.5}{24} = -0.271$$

so that

$$\theta = -15.2^\circ \ .$$

Again, since v_y is negative and v_x is positive, this is the correct choice for θ. So the direction of the fish's motion is -15.2° from the $+x$ axis.

3.2.4 Projectile Motion

---

5. A rifle is aimed horizontally at a target $30\,\mathrm{m}$ **away. The bullet hits the target** $1.9\,\mathrm{cm}$ **below the aiming point. (a) What is the bullet's time of flight? (b) What is the muzzle velocity?**

(a) First, we define our coordinates. I will use the coordinate system indicated in Fig. 3.1, where the origin is placed at the tip of the gun. Then we have $x_0 = 0$ and $y_0 = 0$. We also know the acceleration:

$$a_x = 0 \qquad \text{and} \qquad a_y = -9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2} = -g$$

What else do we know? The gun is fired *horizontally* so that $v_{0y} = 0$, but we don't know v_{0x}. We don't know the time of flight but we do know that when x has the value $30\,\mathrm{m}$ then y has the value $-1.9 \times 10^{-2}\,\mathrm{m}$. (Minus!)

Our equation for the y coordinate is

$$\begin{aligned} y &= y_0 + y_{0y}t + \tfrac{1}{2}a_y t^2 \\ &= 0 + 0 + \tfrac{1}{2}(-g)t^2 \\ &= -\tfrac{1}{2}gt^2 \end{aligned}$$

We can now ask: "At what time t does y equal $-1.9 \times 10^{-2}\,\mathrm{m}$?" . Substitute $y = -1.9 \times 10^{-2}\,\mathrm{m}$ and solve:

$$t^2 = -\frac{2y}{g} = -\frac{2(-1.9 \times 10^{-2}\,\mathrm{m})}{9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}} = 3.9 \times 10^{-3}\,\mathrm{s}^2$$

which gives:

$$t = 6.2 \times 10^{-2}\,\mathrm{s}$$

Since this is the time of impact with the target, the time of flight of the bullet is $t = 6.2 \times 10^{-2}\,\mathrm{s}$.

(b) The equation for $x-$motion is

$$\begin{aligned} x &= x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2 \\ &= 0 + v_{0x}t + 0 \\ &= v_{0x}t \end{aligned}$$

From part (a) we know that when $t = 6.2 \times 10^{-2}\,\mathrm{s}$ then $x = 30\,\mathrm{m}$. This allows us to solve for v_{0x}:

$$v_{0x} = \frac{x}{t} = \frac{30\,\mathrm{m}}{6.2 \times 10^{-2}\,\mathrm{s}} = 480\,\frac{\mathrm{m}}{\mathrm{s}}$$

The muzzle velocity of the bullet is $480\,\frac{\mathrm{m}}{\mathrm{s}}$.

---

6. In a local bar, a customer slides an empty beer mug on the counter for a refill. The bartender does not see the mug, which slides off the counter and strikes the floor $1.40\,\mathrm{m}$ from the base of the counter. If the height of the counter is $0.860\,\mathrm{m}$, (a) with what speed did the mug leave the counter and (b) what was the direction of the mug's velocity just before it hit the floor?

(a) The motion of the beer mug is shown in Fig. 3.2(a). We choose the origin of our xy coordinate system as being at the point where the mug leaves the counter. So the mug's initial coordinates for its flight are $x_0 = 0$, $y_0 = 0$.

At the very beginning of its motion through the air, the velocity of the mug is *horizontal*. (This is because its velocity was horizontal all the time it was sliding on the counter.) So we know that $v_{0y} = 0$ but we don't know the value of v_{0x}. (In fact, that's what we're trying to figure out!)

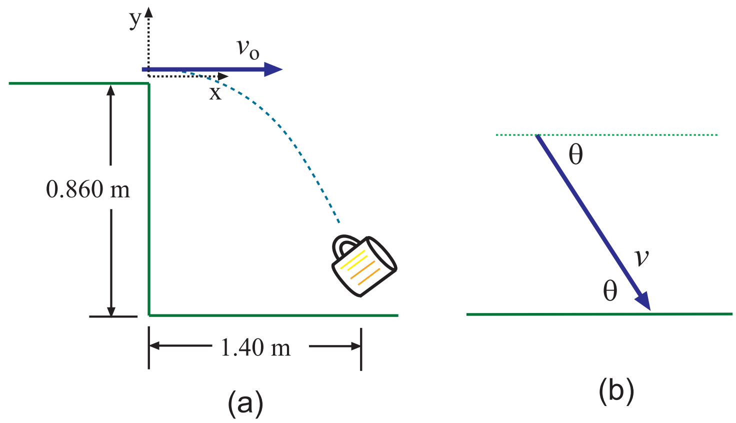

Figure 3.2: (a) Beer mug slides off counter and strikes floor! (b) Velocity vector of the beer mug at the time of impact.

We might begin by finding the time t at which the mug hit the floor. This is the time t at which $y = -0.860\,\mathrm{m}$ (recall how we chose the coordinates!), and we will need the y equation of motion for this; since $v_{0y} = 0$ and $a_y = -g$, we get:

$$y = v_{0y}t + \tfrac{1}{2}a_y t^2 = -\tfrac{1}{2}gt^2$$

So we solve

$$-0.860\,\mathrm{m} = -\tfrac{1}{2}gt^2$$

which gives

$$t^2 = \frac{2(0.860\,\mathrm{m})}{g} = \frac{2(0.860\,\mathrm{m})}{(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})} = 0.176\,\mathrm{s}^2$$

so then

$$t = 0.419\,\mathrm{s}$$

is the time of impact.

To find v_{0x} we consider the x equation of motion; $x_0 = 0$ and $a_x = 0$, so we have

$$x = v_{0x}t \ .$$

At $t = 0.419\,\mathrm{s}$ we know that the x coordinate was equal to $1.40\,\mathrm{m}$. So

$$1.40\,\mathrm{m} = v_{0x}(0.419\,\mathrm{s})$$

Solve for v_{0x}:

$$v_{0x} = \frac{1.40\,\mathrm{m}}{0.419\,\mathrm{s}} = 3.34\,\frac{\mathrm{m}}{\mathrm{s}}$$

which tells us that the initial speed of the mug was $v_0 = 3.34\,\frac{\mathrm{m}}{\mathrm{s}}$.

(b) We want to find the components of the mug's velocity at the time of impact, that is, at $t = 0.419\,\mathrm{s}$. Substitute into our expressions for v_x and v_y:

$$v_x = v_{0x} + a_x t = v_{0x} = 3.34\,\frac{\mathrm{m}}{\mathrm{s}}$$

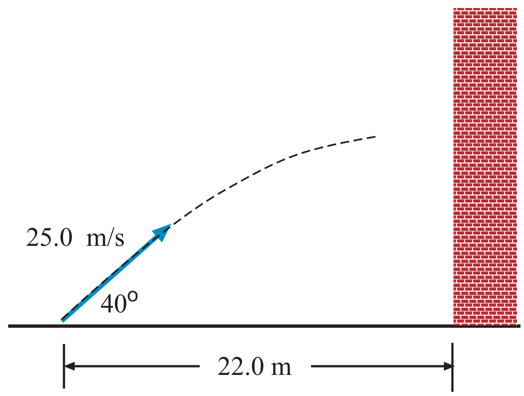

Figure 3.3: Ball is thrown toward wall at 40° above horizontal, in Example 7.

and

$$v_y = v_{0y} + a_y t = 0 + (-9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(0.419\,\mathrm{s}) = -4.11\,\tfrac{\mathrm{m}}{\mathrm{s}} \ .$$

So at the time of impact, the *speed* of the mug was

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(3.34\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 + (-4.11\,\tfrac{\mathrm{m}}{\mathrm{s}})^2} = 5.29\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

and, if as in Fig. 3.2(b) we let θ be the angle *below the horizontal* at which the velocity vector is pointing, we see that

$$\tan\theta = \frac{4.11}{3.34} = 1.23 \qquad \Longrightarrow \qquad \theta = \tan^{-1}(1.23) = 50.9^\circ \ .$$

At the time of impact, the velocity of the mug was directed at 50.9° below the horizontal.

---

7. You throw a ball with a speed of $25.0\,\frac{\mathrm{m}}{\mathrm{s}}$ at an angle of 40.0° above the horizontal directly toward a wall, as shown in Fig. 3.3. The wall is $22.0\,\mathrm{m}$ from the release point of the ball. (a) How long does the ball take to reach the wall? (b) How far above the release point does the ball hit the wall? (c) What are the horizontal and vertical components of its velocity as it hits the wall? (d) When it hits, has it passed the highest point on its trajectory?

(a) We will use a coordinate system which has its origin at the point of firing, which we take to be at ground level.

What is the mathematical condition which determines when the ball hits the wall? It is when the x coordinate of the ball is equal to 22.0 m. Then let's write out the $x-$equation of motion for the ball. The ball's initial $x-$ velocity is

$$v_{0x} = v_0\cos\theta_0 = (25.0\,\tfrac{\mathrm{m}}{\mathrm{s}})\cos 40.0^\circ = 19.2\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

and of course $a_x = 0$, so that the x motion is given by

$$x = x_0 + v_{ox}t + \tfrac{1}{2}a_x t^2 = 19.2\,\tfrac{\mathrm{m}}{\mathrm{s}}t$$

We solve for the time at which $x = 22.0\,\mathrm{m}$:

$$x = 19.2\,\tfrac{\mathrm{m}}{\mathrm{s}}t = 22.0\,\mathrm{m} \qquad \Longrightarrow \qquad t = \frac{22.0\,\mathrm{m}}{19.2\,\frac{\mathrm{m}}{\mathrm{s}}} = 1.15\,\mathrm{s}$$

The ball hits the wall 1.15 s after being thrown.

(b) We will be able to answer this question if we can find the y *coordinate* of the ball at the time that it hits the wall, namely at $t = 1.15\,\mathrm{s}$.

We need the y equation of motion. The initial y velocity of the ball is

$$v_{0y} = v_0 \sin\theta_0 = \left(25.0\,\tfrac{\mathrm{m}}{\mathrm{s}}\right)\sin 40.0^\circ = 16.1\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

and the y acceleration of the ball is $a_y = -g$ giving:

$$y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2 = \left(16.1\,\tfrac{\mathrm{m}}{\mathrm{s}}\right)t - \tfrac{1}{2}gt^2$$

which we use to find the y coordinate at $t = 1.15\,\mathrm{s}$:

$$y = (16.1\,\tfrac{\mathrm{m}}{\mathrm{s}})(1.15\,\mathrm{s}) - \tfrac{1}{2}(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(1.15\,\mathrm{s})^2 = 12.0\,\mathrm{m}$$

which tells us that the ball hits the wall at 12.0 m above the ground level (above the release point).

(c) The x and y components of the balls's velocity at the time of impact, namely at $t = 1.15\,\mathrm{s}$ are found from Eqs. 3.16:

$$v_x = v_{0x} + a_x t = 19.2\,\tfrac{\mathrm{m}}{\mathrm{s}} + 0 = 19.2\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

and

$$v_y = v_{0y} + a_y t = 16.1\,\tfrac{\mathrm{m}}{\mathrm{s}} + (-9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(1.15\,\mathrm{s}) = +4.83\,\tfrac{\mathrm{m}}{\mathrm{s}}\ .$$

(d) Has the ball already passed the highest point on its trajectory? Suppose the ball was on its way *downward* when it struck the wall. Then the y component of the velocity would be *negative*, since it is always decreasing and at the trajectory's highest point it is zero. (Of course, the x component of the velocity stays the same while the ball is in flight.) Here we see that the y component of the ball's velocity is still *positive* at the time of impact. So the ball was still climbing when it hit the wall; it had *not* reached the highest point of its (free) trajectory.

---

8. The launching speed of a certain projectile is five times the speed it has at its maximum height. Calculate the elevation angle at launching.

We make a diagram of the projectile's motion in Fig. 3.4. The launch it speed is v_0, and the projectile is launched at an angle θ_0 upward from the horizontal.

We might start this problem by solving for the time it takes the projectile to get to maximum height, but we can note that at maximum height, there is no y velocity component, and

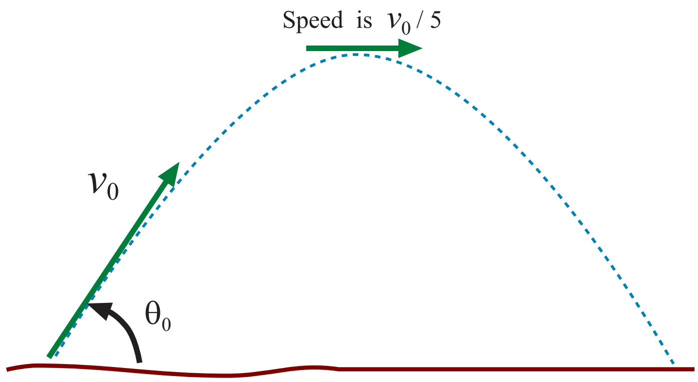

Figure 3.4: Motion of projectile in Example 8.

the x velocity component is *the same as it was when the projectile was launched*. Therefore at maximum height the velocity components are

$$v_x = v_0 \cos\theta_0 \qquad \text{and} \qquad v_y = 0$$

and so the speed of the projectile at maximum height is $v_0 \cos\theta_0$.

Now, we are told that the launching speed (v_0) is five times the speed at maximum height. This gives us:

$$v_0 = 5v_0 \cos\theta_0 \qquad \Longrightarrow \qquad \cos\theta_0 = \frac{1}{5}$$

which has the solution

$$\theta_0 = 78.5^\circ$$

So the elevation angle at launching is $\theta_0 = 78.5^\circ$.

---

9. A projectile is launched from ground level with speed v_0 at an angle of θ_0 above the horizontal. Find: (a) the maximum height H attained by the projectile, and (b) the distance from the starting point at which the projectile strikes the ground; this is called the range R of the projectile.

Comment: This problem is worked in virtually every physics text, and it is sometimes simply called "The Projectile Problem". I include it in this book for the sake of completeness and so that we can use the results if we need them later on. I do *not* treat it as part of the fundamental material of this chapter because it is a very particular application of free–fall motion. In this problem, the projectile impacts at the *same height* as the one from which it started, and that is *not* always the case. We must think about all projectile problems *individually* and not rely on simple formulae to plug numbers into!

The path of the projectile is shown in Fig. 3.5. The initial coordinates of the projectile are

$$x_0 = 0 \qquad \text{and} \qquad y_0 = 0\ ,$$

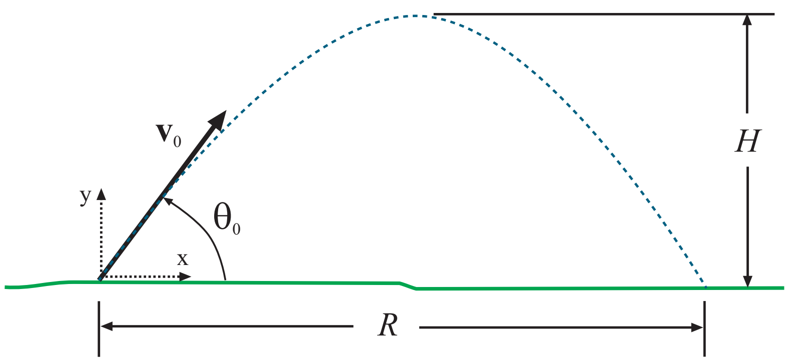

Figure 3.5: The common projectile problem; projectile is shot from ground level with speed v_0 and angle θ_0 above the horizontal.

the components of the initial velocity are

$$v_{0x} = v_0 \cos\theta_0 \qquad \text{and} \qquad v_{0y} = v_0 \sin\theta_0$$

and of course the (constant) acceleration of the projectile is

$$a_x = 0 \qquad \text{and} \qquad a_y = -g = -9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2}$$

Then our equations for $x(t)$, $v_x(t)$, $y(t)$ and $v_y(t)$ are

$$\begin{aligned} v_x &= v_0 \cos\theta_0 \\ x &= v_0 \cos\theta_0\, t \\ v_y &= v_0 \sin\theta_0 - gt \\ y &= v_0 \sin\theta_0\, t - \tfrac{1}{2}gt^2 \end{aligned}$$

(a) What does it mean for the projectile to get to "maximum height"? This is when it is neither increasing in height (rising) nor decreasing in height (falling); the vertical component of the velocity at this point is *zero*. At this particular time then,

$$v_y = v_0 \sin\theta_0 - gt = 0$$

so solving this equation for t, the projectile reaches maximum height at

$$t = \frac{v_0 \sin\theta_0}{g} \ .$$

How high is the projectile at this time? To answer this, substitute this value of t into the equation for y and get:

$$\begin{aligned} y &= v_0 \sin\theta_0 \left(\frac{v_0 \sin\theta_0}{g}\right) - \tfrac{1}{2}g\left(\frac{v_0 \sin\theta_0}{g}\right)^2 \\ &= \frac{v_0^2 \sin^2\theta_0}{g} - \frac{v_0^2 \sin^2\theta_0}{2g} \\ &= \frac{v_0^2 \sin^2\theta_0}{2g} \end{aligned}$$

This is the maximum height attained by the projectile:

$$H = \frac{v_0^2 \sin^2 \theta_0}{2g}$$

(b) What is the *mathematical* condition for when the projectile strikes the ground (since that is how we will find the range R)? We know that at this point, the projectile's y coordinate is zero:

$$y = v_0 \sin\theta_0 \, t - \tfrac{1}{2}gt^2 = 0$$

We want to solve this equation for t; we can factor out t in this expression to get:

$$t(v_0 \sin\theta_0 - \tfrac{1}{2}gt) = 0$$

which has two solutions:

$$t = 0 \qquad \text{and} \qquad t = \frac{2v_0 \sin\theta_0}{g}$$

The first of these is just the time when the projectile was fired; yes, y *was* equal to zero then, but that's not what we want! The time at which the projectile strikes the ground is

$$t = \frac{2v_0 \sin\theta_0}{g} \ .$$

We want to find the value of x at the time of impact. Substituting this value of t into our equation for $x(t)$, we find:

$$\begin{aligned} x &= v_0 \cos\theta_0 \left(\frac{2v_0 \sin\theta_0}{g} \right) \\ &= \frac{2v_0^2 \sin\theta_0 \cos\theta_0}{g} \end{aligned}$$

This value of x is the range R of the projectile.

We can make this result a little simpler by recalling the trig relation:

$$\sin 2\theta_0 = 2 \sin\theta_0 \cos\theta_0 \ .$$

Using this in our result for the range gives:

$$R = \frac{2v_0^2 \sin\theta_0 \cos\theta_0}{g} = \frac{v_0^2 \sin 2\theta_0}{g}$$

---

10. A projectile is fired in such a way that its horizontal range is equal to three times its maximum height. What is the angle of projection?

Now, this problem *does* deal with a projectile which starts and ends its flight at the same height, just as we calculated in the previous example. So we *can* use the results for the range R and maximum height H that we found there.

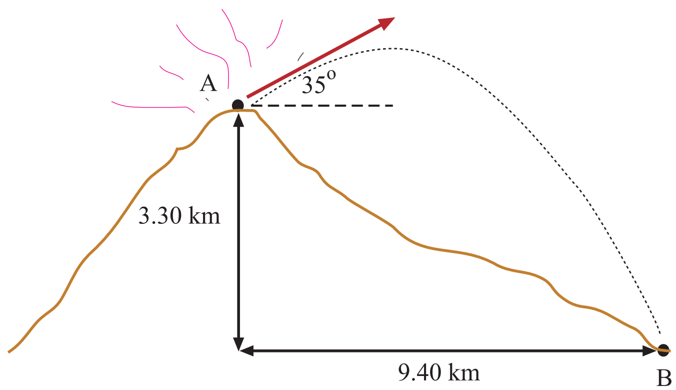

Figure 3.6: Volcanic bombs away!

The problem tells us that $R = 3H$. Substituting the expressions for H and R that we found in the last example (we pick the first expression we got for R), we get:

$$R = \frac{2v_0^2 \sin\theta_0 \cos\theta_0}{g} = 3H = 3\left(\frac{v_0^2 \sin^2\theta_0}{2g}\right)$$

Cancelling stuff, we get:

$$2\cos\theta_0 = \frac{3}{2}\sin\theta_0 \qquad \Longrightarrow \qquad \tan\theta_0 = \frac{4}{3}$$

The solution is:

$$\theta_0 = \tan^{-1}(4/3) = 53.1^\circ$$

The projectile was fired at 53.1° above the horizontal.

---

11. During volcanic eruptions, chunks of solid rock can be blasted out of a volcano; these projectiles are called *volcanic bombs*. Fig. 3.6 shows a cross section of Mt. Fuji in Japan. (a) At what initial speed would the bomb have to be ejected, at 35° to the horizontal, from the vent at A in order to fall at the foot of the volcano at B? (Ignore the effects of air on the bomb's travel.) (b) What would be the time of flight?

(a) We use a coordinate system with its origin at point A (the volcano "vent"); then for the flight from the vent at A to point B, the initial coordinates are $x_0 = 0$ and $y_0 = 0$, and the final coordinates are $x = 9.40\,\mathrm{km}$ and $y = -3.30\,\mathrm{km}$. Aside from this, we don't know the initial speed of the rock (that's what we're trying to find) or the time of flight from A to B. Of course, the acceleration of the rock is given by $a_x = 0$, $a_y = -g$.

We start with the x equation of motion. The initial $x-$velocity is

$$v_{0x} = v_0 \cos\theta$$

where $\theta = 35^\circ$ so the function $x(t)$ is

$$\begin{aligned} x &= x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2 \\ &= 0 + v_0 \cos\theta t + 0 \\ &= v_0 \cos\theta \, t \end{aligned}$$

Now, we *do* know that at the time of impact x had the value $x = 9.40\,\mathrm{km}$ so if we now let t be the time of flight, then

$$(9.40\,\mathrm{km}) = v_0 \cos\theta t \qquad \text{or} \qquad t = \frac{(9.40\,\mathrm{km})}{v_0 \cos\theta} \tag{3.25}$$

Next we look at the y equation of motion. Since $v_{0y} = v_0 \sin\theta$ we get:

$$\begin{aligned} y &= y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2 \\ &= 0 + v_0 \sin\theta t - \tfrac{1}{2}gt^2 \\ &= v_0 \sin\theta \, t - \tfrac{1}{2}gt^2 \end{aligned}$$

But at the time t of impact the y coordinate had the value $y = -3.30\,\mathrm{km}$. If we also substitute for t in this expression using Eq. 3.25 we get:

$$\begin{aligned} -3.30\,\mathrm{km} &= v_0 \sin\theta \left(\frac{9.40\,\mathrm{km}}{v_0 \cos\theta}\right) - \tfrac{1}{2}g\left(\frac{9.40\,\mathrm{km}}{v_0 \cos\theta}\right)^2 \\ &= (9.40\,\mathrm{km}) \tan\theta - \frac{g(9.40\,\mathrm{km})^2}{2v_0^2 \cos^2\theta} \end{aligned}$$

At this point we are done with the *physics* problem. The *only* unknown in this equation is v_0, which we can find by doing a little algebra:

$$\begin{aligned} \frac{g(9.40\,\mathrm{km})^2}{2v_0^2 \cos^2\theta} &= (9.40\,\mathrm{km}) \tan\theta + 3.30\,\mathrm{km} \\ &= 9.88\,\mathrm{km} \end{aligned}$$

which gives:

$$\begin{aligned} v_0^2 &= \frac{g(9.40\,\mathrm{km})^2}{\cos^2\theta(9.88\,\mathrm{km})} \\ &= \frac{g(0.951\,\mathrm{km})}{\cos^2\theta} \\ &= \frac{(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})(951\,\mathrm{m})}{\cos^2 35^\circ} \\ &= 1.39 \times 10^4\,\tfrac{\mathrm{m}^2}{\mathrm{s}^2} \end{aligned}$$

and finally

$$v_0 = 118\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

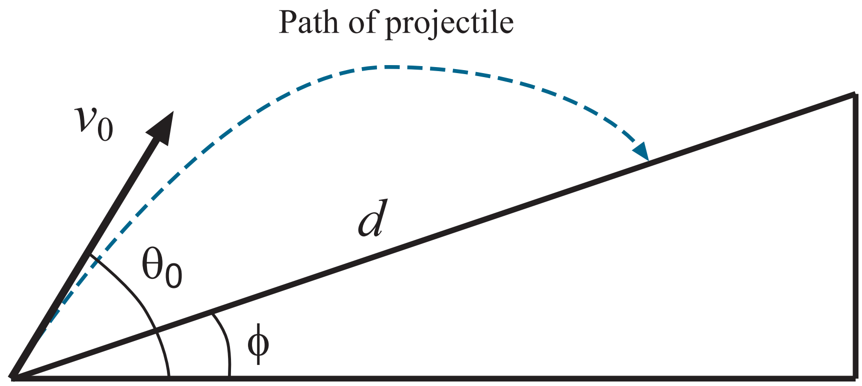

Figure 3.7: Projectile is fired up an incline, as described in Example 12

(b) Having v_0 in hand, finding t is easy. Using our result from part(a) and Eq. 3.25 we find:

$$t = \frac{(9.40\,\mathrm{km})}{v_0 \cos\theta} = \frac{(9400\,\mathrm{m})}{(118\,\frac{\mathrm{m}}{\mathrm{s}})\cos 35^\circ} = 97.2\,\mathrm{s}$$

The time of flight is 97.2 s.

---

12. A projectile is fired up an incline (incline angle ϕ) with an initial speed v_0 at an angle θ_0 with respect to the horizontal ($\theta_0 > \phi$) as shown in Fig. 3.7. (a) Show that the projectile travels a distance d up the incline, where

$$d = \frac{2v_0^2 \cos\theta_0 \sin(\theta_0 - \phi)}{g\cos^2\phi}$$

(b) For what value of θ_0 is d a maximum, and what is the maximum value?

(a) This is a relatively challenging problem, and of course it is completely analytic.

We can start by writing down equations for x and y as functions of time. By now we can easily see that we have:

$$x = v_0 \cos\theta_0\, t$$

$$y = v_0 \sin\theta_0\, t - \tfrac{1}{2}gt^2$$

We can combine these equations to get a relation between x and y for points on the trajectory; from the first, we have $t = x/(v_0 \cos\theta_0)$, and putting this into the second one gives:

$$\begin{aligned} y &= v_0 \sin\theta_0 \left(\frac{x}{v_0\cos\theta_0}\right) - \tfrac{1}{2}g\left(\frac{x}{v_0\cos\theta_0}\right)^2 \\ &= (\tan\theta_0)x - \frac{g}{2}\frac{x^2}{v_0^2\cos^2\theta_0} \end{aligned}$$

What is the condition for the time that the projectile hits the slope? Unlike the problems where a projectile impacts with the flat ground or a wall, we don't know the value of x or y

at impact. But since the incline has a slope of $\tan\phi$, the relation between x and y for points on the slope is

$$y = (\tan\phi)x \ .$$

These two relations between x and y allow us to solve for the values of x and y where the impact occurs. Substituting for y above, we find:

$$(\tan\phi)x = (\tan\theta_0)x - \frac{g}{2}\frac{x^2}{v_0^2\cos^2\theta_0}$$

A little rearranging gives:

$$\frac{g}{2}\frac{x}{v_0^2\cos^2\theta_0} + (\tan\phi - \tan\theta_0) = 0$$

and the solution for x is:

$$x = \frac{2v_0^2\cos^2\theta_0(\tan\theta_0 - \tan\phi)}{g}$$

The problem has us solve for the distance d *up the slope*; this distance is related to the impact value of x by:

$$d = \frac{x}{\cos\phi}$$

and this gives us:

$$d = \frac{x}{\cos\phi} = \frac{2v_0^2\cos^2\theta_0(\tan\theta_0 - \tan\phi)}{g\cos\phi} \ .$$

Although this is a perfectly good expression for d, it is not the one presented in the problem. (Among other things, it has another factor of $\cos\phi$ downstairs.) If we multiply top and bottom by $\cos\phi$ we find:

$$\begin{aligned} d &= \frac{2v_0^2\cos^2\theta_0\cos\phi(\tan\theta_0 - \tan\phi)}{g\cos^2\phi} \\ &= \frac{2v_0^2\cos\theta_0(\cos\theta_0\cos\phi\tan\theta_0 - \cos\theta_0\cos\phi\tan\phi)}{g\cos^2\phi} \\ &= \frac{2v_0^2\cos\theta_0(\cos\phi\sin\theta_0 - \cos\theta_0\sin\phi)}{g\cos^2\phi} \end{aligned}$$

And now using an angle–addition identity from trigonometry in the numerator, we arrive at

$$d = \frac{2v_0^2\cos\theta_0\sin(\theta_0 - \phi)}{g\cos^2\phi}$$

which is the preferred expression for d.

(b) In part (a) we found the up–slope impact distance as a function of launch angle θ_0. (The launch speed v_0 and the slope angle ϕ are taken to be fixed.) For a certain value of $theta_0$, this function $d(\theta_0)$ will take on a maximum value. To find this value, we differentiate the function $d(\theta_0)$ and set the derivative equal to zero. We find:

$$\begin{aligned}
d'(\theta_0) &= \frac{2v_0^2}{g\cos^2\phi}\frac{d}{d\theta_0}[\cos\theta_0\sin(\theta_0-\phi)] \\
&= \frac{2v_0^2}{g\cos^2\phi}[-\sin\theta_0\sin(\theta_0-\phi)+\cos\theta_0\cos(\theta_0-\phi)] \\
&= \frac{2v_0^2}{g\cos^2\phi}\cos(2\theta_0-\phi)
\end{aligned}$$

where in the last step we used the trig identity $\cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha+\beta)$.

Now, to satisfy $d'(\theta_0) = 0$ we must have $\cos(2\theta_0 - \phi) = 0$. While this equation has infinitely many solutions for θ_0, considering the values that θ_0 and ϕ may take on, we see that we need only look at the case where

$$2\theta_0 - \phi = \frac{\pi}{2}$$

which of course, *does* solve the equation. This gives us:

$$\theta_0 = \frac{\pi}{4} + \frac{\phi}{2}$$

for the value of θ which makes the projectile go the farthest distance d up the slope.

To find what this value of d is, we substitute for θ_0 in our function $d(\theta_0)$. We find:

$$\begin{aligned}
d_{\max} &= \frac{2v_0^2}{g\cos^2\phi}\cos\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\sin\left(\frac{\pi}{4}+\frac{\phi}{2}-\phi\right) \\
&= \frac{2v_0^2}{g\cos^2\phi}\cos\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)
\end{aligned}$$

This expression is *correct* but it can be simplified. We use the trig identity which states:

$$\sin\alpha\cos\beta = \tfrac{1}{2}\sin(\alpha+\beta) + \tfrac{1}{2}\sin(\alpha-\beta)$$

this gives us:

$$\begin{aligned}
\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\cos\left(\frac{\pi}{4}+\frac{\phi}{2}\right) &= \tfrac{1}{2}\sin\left(\frac{\pi}{2}\right) + \tfrac{1}{2}\sin(-\phi) \\
&= \tfrac{1}{2} - \tfrac{1}{2}\sin\phi \\
&= \tfrac{1}{2}(1-\sin\phi)
\end{aligned}$$

which *is* a lot simpler. Using this result in our expression for $d_{\max}$ gives:

$$d_{\max} = \frac{2v_0^2}{g\cos^2\phi}\frac{(1-\sin\phi)}{2} = \frac{v_0^2(1-\sin\phi)}{g(1-\sin^2\phi)} = \frac{v_0^2}{g(1+\sin\phi)}$$

which is a simple as it's going to get!

We can *check* result for a couple well–known cases. If $\phi = 0$ we are dealing with the common projectile problem on level ground for which we know we get maximum range when $\theta_0 = 45^\circ$ and from our solution for that problem we get $R = \frac{v_0^2}{g}$. If $\phi = 90^\circ$ we have the problem of a projectile fired straight up; one can show that the maximum height reached is $H = \frac{v_0^2}{2g}$ which again agrees with the formula we've derived.

3.2.5 Uniform Circular Motion

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13. In one model of the hydrogen atom, an electron orbits a proton in a circle of radius $5.28 \times 10^{-11}\,\mathrm{m}$ with a speed of $2.18 \times 10^{6}\,\frac{\mathrm{m}}{\mathrm{s}}$. (a) What is the acceleration of the electron in this model? (b) What is the period of the motion?

(a) The electron moves in a circle with constant speed. It is accelerating *toward the center of the circle* and the acceleration has magnitude $a_{\mathrm{cent}} = \frac{v^2}{r}$. Substituting the given values, we have:

$$a_{\mathrm{cent}} = \frac{v^2}{r} = \frac{(2.18 \times 10^{6}\,\frac{\mathrm{m}}{\mathrm{s}})^2}{(5.28 \times 10^{-11}\,\mathrm{m})} = 9.00 \times 10^{22}\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

The acceleration has magnitude $9.00 \times 10^{22}\,\frac{\mathrm{m}}{\mathrm{s}^2}$.

(b) As the electron makes one trip around the circle of radius r, it moves a distance $2\pi r$ (the circumference of the circle). If T is the period of the motion, then the speed of the electron is given by the ratio of distance to time,

$$v = \frac{2\pi r}{T} \qquad \text{which gives...} \qquad T = \frac{2\pi r}{v}$$

which shows why Eq. 3.22 is true. Substituting the given values, we get:

$$T = \frac{2\pi(5.28 \times 10^{-11}\,\mathrm{m})}{(2.18 \times 10^{6}\,\frac{\mathrm{m}}{\mathrm{s}})} = 1.52 \times 10^{-16}\,\mathrm{s}$$

The period of the electron's motion is $1.52 \times 10^{-16}\,\mathrm{s}$.

---

14. A rotating fan completes 1200 revolutions every minute. Consider a point on the tip of a blade, at a radius of 0.15 m. (a) Through what distance does the point move in one revolution? (b) What is the speed of the point? (c) What is its acceleration? (d) What is the period of the motion?

(a) As the fan makes one revolution, the point in question moves through a circle of radius 0.15 m so the distance it travels is the circumference of that circle, i.e.

$$d = 2\pi r = 2\pi(0.15\,\mathrm{m}) = 0.94\,\mathrm{m}$$

The point travels 0.94 m.

(b) If in one minute (60 s) the fan makes 1200 revolutions, the time to make *one* revolution must be

$$\text{Time for one rev} = T = \frac{1}{1200} \cdot (1.00\,\mathrm{min}) = \frac{1}{1200} \cdot (60.0\,\mathrm{s}) = 5.00 \times 10^{-2}\,\mathrm{s}$$

Using our answer from part (a), we know that the point travels 0.94 m in $5.000 \times 10^{-2}\,\mathrm{s}$, moving at constant speed. Therefore that speed is:

$$v = \frac{d}{T} = \frac{0.94\,\mathrm{m}}{5.000 \times 10^{-2}\,\mathrm{s}} = 19\,\frac{\mathrm{m}}{\mathrm{s}}$$

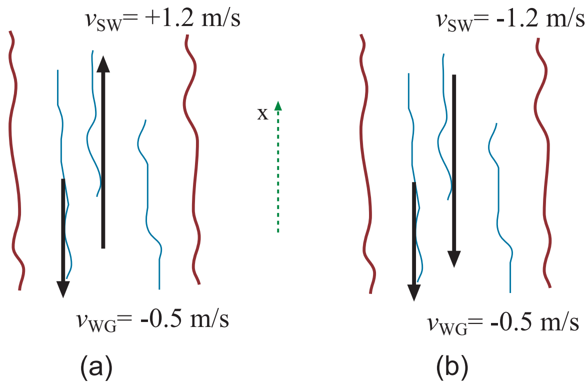

Figure 3.8: (a) Velocities for case where swimmer swims upstream. (b) Velocities for case where swimmer swims downstream.

(c) The point is undergoing uniform circular motion; its acceleration is always *toward the center* and has magnitude $a_{\rm cent} = \frac{v^2}{r}$. Substituting,

$$a_{\rm cent} = \frac{v^2}{r} = \frac{(19\,\frac{\rm m}{\rm s})^2}{(0.15\,{\rm m})} = 2.4 \times 10^3\,\frac{\rm m}{\rm s^2}$$

(d) The period of the motion is the time for the fan to make one revolution. And we already found this in part (b)! It is:

$$T = 5.00 \times 10^{-2}\,{\rm s}$$

3.2.6 Relative Motion

---

15. A river has a steady speed of $0.500\,\frac{\rm m}{\rm s}$. A student swims upstream a distance of $1.00\,{\rm km}$ and returns to the starting point. If the student can swim at a speed of $1.20\,\frac{\rm m}{\rm s}$ in still water, how long does the trip take? Compare this with the time the trip would take if the water were still.

What happens if the water *is* still? The student swims a distance of 1.00 km "upstream" at a speed of $1.20\,\frac{\rm m}{\rm s}$; using the simple distance/time formula $d = vt$ the time for the trip is

$$t = \frac{d}{v} = \frac{1.0 \times 10^3\,{\rm m}}{1.20\,\frac{\rm m}{\rm s}} = 833\,{\rm s}$$

and the same is true for the trip back "downstream". So the total time for the trip is

$$833\,{\rm s} + 833\,{\rm s} = 1.67 \times 10^3\,{\rm s} = 27.8\,{\rm min}$$

Good enough, but what about the case where the water is *not* still? And what does that have to do with relative velocities? In Fig. 3.8, the river is shown; it flows in the $-x$

direction. At all times, the velocity of the water *with respect to the ground* is

$$v_{\rm WG} = -0.500\,\tfrac{\rm m}{\rm s}\ .$$

When the student swims upstream, as represented in Fig. 3.8(a), his velocity *with respect to the water* is

$$v_{\rm SW} = +1.20\,\tfrac{\rm m}{\rm s}\ .$$

We know this because we are given his swimming speed for *still* water.

Now we are interested in the student's velocity *with respect to the ground*, which we will call $v_{\rm SG}$. It is given by the sum of his velocity with respect to the water and the water's velocity with respect to the ground:

$$v_{\rm SG} = v_{\rm SW} + v_{\rm WG} = +1.20\,\tfrac{\rm m}{\rm s} - 0.500\,\tfrac{\rm m}{\rm s} = 0.70\,\tfrac{\rm m}{\rm s}$$

and so to cover a displacement of $\Delta x = 1.00\,{\rm km}$ (measured along the ground!) requires a time

$$\Delta t = \frac{\Delta x}{v_{\rm SG}} = \frac{1.00\times 10^3\,{\rm m}}{0.70\,\frac{\rm m}{\rm s}} = 1.43\times 10^3\,{\rm s}$$

Then the student swims downstream (Fig. 3.8(b)) and his velocity with respect to the water is

$$v_{\rm SW} = -1.20\,\tfrac{\rm m}{\rm s}$$

giving him a velocity with respect to the ground of

$$v_{\rm SG} = v_{\rm SW} + v_{\rm WG} = -1.20\,\tfrac{\rm m}{\rm s} - 0.500\,\tfrac{\rm m}{\rm s} = 1.70\,\tfrac{\rm m}{\rm s}$$

so that the time to cover a displacement of $\Delta x = -1.00\,{\rm km}$ is

$$\Delta t = \frac{\Delta x}{v_{\rm SG}} = \frac{(-1.00\times 10^3\,{\rm m})}{(-1.70\,\frac{\rm m}{\rm s})} = 5.88\times 10^2\,{\rm s}$$

The *total* time to swim upstream and then downstream is

$$\begin{aligned} t_{\rm Total} &= t_{\rm up} + t_{\rm down} \\ &= 1.43\times 10^3\,{\rm s} + 5.88\times 10^2\,{\rm s} = 2.02\times 10^3\,{\rm s} = 33.6\,{\rm min}\ . \end{aligned}$$

---

16. A light plane attains an airspeed of $500\,{\rm km/hr}$. The pilot sets out for a destination $800\,{\rm km}$ to the north but discovers that the plane must be headed 20.0° east of north to fly there directly. The plane arrives in $2.00\,{\rm hr}$. What was the wind velocity vector?

Whoa! What the Hell is this problem talking about???

When a plane flies in air which *itself* is moving (i.e. there is a wind velocity) there are three (vector) velocities we need to think about; I will refer to them as:

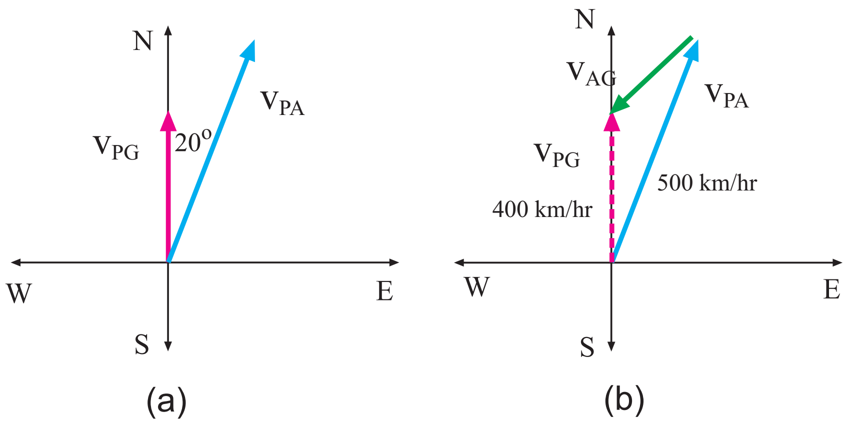

Figure 3.9: (a) Vectors for the planes velocity with respect to the ground ($\mathbf{v}_{PG}$) and with respect to the moving air ($\mathbf{v}_{PA}$). (b) The sum of the plane's velocity relative to the air and the wind velocity gives the plane's velocity with respect to the ground, $\mathbf{v}_{PG}$.

$\mathbf{v}_{PA}$: Velocity of the plane with respect to the air. The magnitude of this vector is the "airspeed" of the plane. (This is the only thing that a plane's "speedometer" can really measure.)

$\mathbf{v}_{AG}$: Velocity of the air with respect to the ground. This is the wind velocity.

$\mathbf{v}_{PG}$: Velocity of the plane with respect to the ground. This is the quantity which tells us the rate of (ground!) travel of the plane.

These three vectors are related via:

$$\mathbf{v}_{PG} = \mathbf{v}_{PA} + \mathbf{v}_{AG}$$

The first thing we are given in this problem is that the magnitude of $\mathbf{v}_{PA}$ is $500\,\mathrm{km/hr}$. The plane needs to fly *due north* and this tells us that $\mathbf{v}_{PG}$ (the real direction of motion of the plane) points north (along the y axis). We are then told that the plane's "heading" is 20.0° east of north, which tells us that the direction of $\mathbf{v}_{PA}$ lies in this direction. These facts are illustrated in Fig. 3.9(a).

Now if the plane travels 800 km in 2.00 hr then its speed (with respect to the ground!) is

$$v_{PG} = \frac{800\,\mathrm{km}}{2.00\,\mathrm{hr}} = 400\,\tfrac{\mathrm{km}}{\mathrm{hr}}\ .$$

which we note in Fig. 3.9(b). Since we now have the magnitudes and and directions of $\mathbf{v}_{PA}$ and $\mathbf{v}_{PG}$ we can compute the wind velocity,

$$\mathbf{v}_{AG} = \mathbf{v}_{PG} - \mathbf{v}_{PA}$$

The x component of this vector is

$$\mathbf{v}_{AG,x} = 0 - 500\,\tfrac{\mathrm{km}}{\mathrm{hr}}\sin 20.0^\circ = -171\,\tfrac{\mathrm{km}}{\mathrm{hr}}$$

and its y component is

$$\mathbf{v}_{AG,y} = 400 - 500\,\tfrac{\mathrm{km}}{\mathrm{hr}}\cos 20.0^\circ = -69.8\,\tfrac{\mathrm{km}}{\mathrm{hr}}$$

So the wind velocity is

$$\mathbf{v}_{AG} = -171\,\tfrac{\mathrm{km}}{\mathrm{hr}}\mathbf{i} - 69.8\,\tfrac{\mathrm{km}}{\mathrm{hr}}\mathbf{j}$$

If we want to express the velocity as a magnitude and direction, we find:

$$v_{\mathrm{AG}} = \sqrt{(-171\,\tfrac{\mathrm{km}}{\mathrm{hr}})^2 + (-69.8\,\tfrac{\mathrm{km}}{\mathrm{hr}})^2} = 185\,\tfrac{\mathrm{km}}{\mathrm{hr}}$$

so the wind speed is $185\,\frac{\mathrm{km}}{\mathrm{hr}}$. The direction of the wind, measured as an angle θ counter-clockwise from the east is found from its components:

$$\tan\theta = \frac{-69.8}{-171} = 0.408 \qquad \Longrightarrow \qquad \theta = \tan^{-1}(0.408) = 202^\circ$$

(Here we have made sure to get the angle right! Since both components are negative, θ lies in the third quadrant!) Since 180° would be *due West* and the wind direction is 22° larger than that, we can also say that the wind direction is "22° south of west".

Chapter 4

Forces I

4.1 The Important Stuff

4.1.1 Newton's First Law

With Newton's Laws we begin the study of *how* motion occurs in the real world. The study of the *causes* of motion is called **dynamics**, or **mechanics**. The relation between force and acceleration was given by Isaac Newton in his three laws of motion, which form the basis of elementary physics. Though Newton's formulation of physics had to be replaced later on to deal with motion at speeds comparable to the speed light and for motion on the scale of atoms, it is applicable to everyday situations and is still the best introduction to the fundamental laws of nature. The study of Newton's laws and their implications is often called **Newtonian** or **classical mechanics**.

Particles accelerate because they are being acted on by **forces**. In the absence of forces, a particle will not accelerate, that is, it will move at constant velocity.

The user–friendly way of stating Newton's First Law is:

Consider a body on which no force is acting. Then if it is at rest it will remain at rest, and if it is moving with constant velocity it will continue to move at that velocity.

Forces serve to *change* the velocity of an object, not to maintain its motion (contrary to the ideas of philosophers in ancient times).

4.1.2 Newton's Second Law

Experiments show that objects have a property called **mass** which measures how their motion is influenced by forces. Mass is measured in kilograms in the SI system.

Newton's Second Law is a relation between the **net force** (**F**) acting on a mass m and its acceleration **a**. It says:

$$\boxed{\sum \mathbf{F} = m\mathbf{a}}$$

In words, Newton's Second Law tells us to add up the forces acting on a mass m; this sum, $\sum \mathbf{F}$ (or, $\mathbf{F}_{\rm net}$) is equal to the mass m times its acceleration $\mathbf{a}$.

This is a *vector* relation; if we are working in two dimensions, this equation implies *both* of the following:

$$\sum F_x = ma_x \qquad \text{and} \qquad \sum F_y = ma_y \tag{4.1}$$

The units of force must be $\mathrm{kg} \cdot \frac{\mathrm{m}}{\mathrm{s}^2}$, which is abbreviated 1 newton (N), to honor Isaac Newton (1642–1727), famous physicist and smart person. Thus:

$$1\,\text{newton} = 1\,\mathrm{N} = 1\,\tfrac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}^2} \tag{4.2}$$

Two other units of force which we encounter sometimes are:

$$1\,\text{dyne} = 1\,\tfrac{\mathrm{g}\cdot\mathrm{cm}}{\mathrm{s}^2} = 10^{-5}\,\mathrm{N} \qquad \text{and} \qquad 1\,\text{pound} = 1\,\mathrm{lb} = 4.45\,\mathrm{N}$$

4.1.3 Examples of Forces

To begin our study of dynamics we consider problems involving simple objects in simple situations. Our first problems will involve little more than small masses, hard, smooth surfaces and ideal strings, or objects that can be treated as such.

For all masses near the earth's surface, the earth exerts a downward gravitational force which is known as the **weight** of the mass and has a magnitude given by

$$W = mg$$

A taught string (a string "under tension") exerts forces on the objects which are attached to either end. The forces are directed *inward* along the length of the string.) In our first problems we will make the approximation that the string has no mass, and when it passes over any pulley, the pulley's mass can also be ignored. In that case, the magnitude of the string's force on either end is *the same* and will usually be called T, the string's **tension**.

A solid surface will exert forces on a mass which is in contact with it. In general the force from the surface will have a perpendicular (normal) component which we call the **normal force** of the surface. The surface can also exert a force which is parallel; this is a friction force and will be covered in the next chapter.

4.1.4 Newton's Third Law

> Consider two objects A and B. The force which object A exerts on object B is equal and opposite to the force which object B exerts on object A: $\mathbf{F}_{AB} = -\mathbf{F}_{BA}$

This law is popularly stated as the "law of action and reaction", but in fact it deals with the *forces* between two objects.

4.1.5 Applying Newton's Laws

In this chapter we will look at some applications of Newton's law to simple systems involving small blocks, surfaces and strings. (In the next chapter we'll deal with more complicated examples.)

A useful practice for problems involving more than one force is to draw a diagram showing the individual masses in the problem along with the vectors showing the directions and magnitudes of the individual forces. It is so important to do this that these diagrams are given a special name, **free–body diagrams**.

4.2 Worked Examples

4.2.1 Newton's Second Law

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1. A $3.0\,\mathrm{kg}$ **mass undergoes an acceleration given by** $\mathbf{a} = (2.0\mathbf{i} + 5.0\mathbf{j})\,\frac{\mathrm{m}}{\mathrm{s}^2}$**. Find the resultant force F and its magnitude.**

Newton's Second Law tells us that the resultant (net) force on a mass m is $\sum \mathbf{F} = m\mathbf{a}$. So here we find:

$$\begin{aligned} \mathbf{F}_{\mathrm{net}} &= m\mathbf{a} \\ &= (3.0\,\mathrm{kg})(2.0\mathbf{i} + 5.0\mathbf{j})\,\tfrac{\mathrm{m}}{\mathrm{s}^2} \\ &= (6.0\mathbf{i} + 15.\mathbf{j})\,\mathrm{N} \end{aligned}$$

The *magnitude* of the resultant force is

$$F_{\mathrm{net}} = \sqrt{(6.0\,\mathrm{N})^2 + (15.\,\mathrm{N})^2} = 16.\,\mathrm{N}$$

---

2. While two forces act on it, a particle of mass $m = 3.2\,\mathrm{kg}$ **is to move continuously with velocity** $(3\,\frac{\mathrm{m}}{\mathrm{s}})\mathbf{i} - (4\,\frac{\mathrm{m}}{\mathrm{s}})\mathbf{j}$**. One of the forces is** $\mathbf{F}_1 = (2\,\mathrm{N})\mathbf{i} + (-6\,\mathrm{N})\mathbf{j}$**. What is the other force?**

Newton's Second Law tells us that if **a** is the acceleration of the particle, then (as there are only two forces acting on it) we have:

$$\mathbf{F}_1 + \mathbf{F}_2 = m\mathbf{a}$$

But here the *acceleration* of the particle is *zero*!! (Its velocity does not change.) This tells us that

$$\mathbf{F}_1 + \mathbf{F}_2 = 0 \qquad \Longrightarrow \qquad \mathbf{F}_2 = -\mathbf{F}_1$$

and so the second force is

$$\mathbf{F}_2 = -\mathbf{F}_1 = (-2\,\mathrm{N})\mathbf{i} + (6\,\mathrm{N})\mathbf{j}$$

This was a simple problem just to see if you're paying attention!

---

3. A $4.0\,\mathrm{kg}$ object has a velocity of $3.0\mathbf{i}\,\frac{\mathrm{m}}{\mathrm{s}}$ at one instant. Eight seconds later, its velocity is $(8.0\mathbf{i} + 10.0\mathbf{j})\,\frac{\mathrm{m}}{\mathrm{s}}$. Assuming the object was subject to a constant net force, find (a) the components of the force and (b) its magnitude.

(a) We are told that the (net) force acting on the mass was *constant*. Then we know that its acceleration was also constant, and we can use the constant–acceleration results from the previous chapter. We are given the initial and final velocities so we can compute the components of the acceleration:

$$a_x = \frac{\Delta v_x}{\Delta t} = \frac{[(8.0\,\frac{\mathrm{m}}{\mathrm{s}}) - (3.0\,\frac{\mathrm{m}}{\mathrm{s}})]}{(8.0\,\mathrm{s})} = 0.63\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

and

$$a_y = \frac{\Delta v_y}{\Delta t} = \frac{[(10.0\,\frac{\mathrm{m}}{\mathrm{s}}) - (0.0\,\frac{\mathrm{m}}{\mathrm{s}})]}{(8.0\,\mathrm{s})} = 1.3\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

We have the mass of the object, so from Newton's Second Law we get the components of the force:

$$F_x = ma_x = (4.0\,\mathrm{kg})(0.63\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 2.5\,\mathrm{N}$$

$$F_y = ma_y = (4.0\,\mathrm{kg})(1.3\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 5.0\,\mathrm{N}$$

(b) The magnitude of the (net) force is

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{(2.5\,\mathrm{N})^2 + (5.0\,\mathrm{N})^2} = 5.6\,\mathrm{N}$$

and its direction θ is given by

$$\tan\theta = \frac{F_y}{F_x} = \frac{5.0}{2.5} = 2.0 \qquad \Longrightarrow \qquad \theta = \tan^{-1}(2.0) = 63.4^\circ$$

(The question didn't ask for the direction but there it is anyway!)

---

4. Five forces pull on the $4.0\,\mathrm{kg}$ box in Fig. 4.1. Find the box's acceleration (a) in unit–vector notation and (b) as a magnitude and direction.

(a) Newton's Second Law will give the box's acceleration, but we must first find the sum of the forces on the box. Adding the x components of the forces gives:

$$\begin{aligned}\sum F_x &= -11\,\mathrm{N} + 14\,\mathrm{N}\cos 30^\circ + 3.0\,\mathrm{N} \\ &= 4.1\,\mathrm{N}\end{aligned}$$

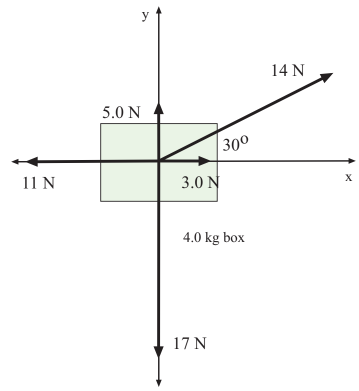

Figure 4.1: Five forces pull on a box in Example 4

(two of the forces have only y components). Adding the y components of the forces gives:

$$\begin{aligned}\sum F_y &= +5.0\,\mathrm{N} + 14\,\mathrm{N}\sin 30^\circ - 17\,\mathrm{N} \\ &= -5.0\,\mathrm{N}\end{aligned}$$

So the net force on the box (in unit-vector notation) is

$$\sum \mathbf{F} = (4.1\,\mathrm{N})\mathbf{i} + (-5.0\,\mathrm{N})\mathbf{j}\ .$$

Then we find the x and y components of the box's acceleration using $\mathbf{a} = \sum \mathbf{F}/m$:

$$\begin{aligned}a_x &= \frac{\sum F_x}{m} = \frac{(4.1\,\mathrm{N})}{(4.0\,\mathrm{kg})} = 1.0\,\tfrac{\mathrm{m}}{\mathrm{s}^2} \\ a_y &= \frac{\sum F_y}{m} = \frac{(-5.0\,\mathrm{N})}{(4.0\,\mathrm{kg})} = -1.2\,\tfrac{\mathrm{m}}{\mathrm{s}^2}\end{aligned}$$

So in unit–vector form, the acceleration of the box is

$$\mathbf{a} = (1.0\,\tfrac{\mathrm{m}}{\mathrm{s}^2})\mathbf{i} + (-1.2\,\tfrac{\mathrm{m}}{\mathrm{s}^2})\mathbf{j}$$

(b) The acceleration found in part (a) has a magnitude of

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(1.0\,\tfrac{\mathrm{m}}{\mathrm{s}^2})^2 + (-1.2\,\tfrac{\mathrm{m}}{\mathrm{s}^2})^2} = 1.6\,\tfrac{\mathrm{m}}{\mathrm{s}^2}$$

and to find its direction θ we calculate

$$\tan\theta = \frac{a_y}{a_x} = \frac{-1.2}{1.0} = -1.2$$

which gives us:

$$\theta = \tan^{-1}(-1.2) = -50^\circ$$

Here, since a_y is negative and a_x is positive, this choice for θ lies in the proper quadrant.

4.2.2 Examples of Forces

---

5. What are the weight in newtons and the mass in kilograms of (a) a $5.0\,\mathrm{lb}$ **bag of sugar, (b) a** $240\,\mathrm{lb}$ **fullback, and (c) a** $1.8\,\mathrm{ton}$ **automobile? (**$1\,\mathrm{ton} = 2000\,\mathrm{lb}$**.)**

(a) The bag of sugar has a *weight* of 5.0 lb. ("Pound" is a unit of force, or weight.) Then its weight in newtons is

$$5.0\,\mathrm{lb} = (5.0\,\mathrm{lb}) \cdot \left(\frac{4.45\,\mathrm{N}}{1\,\mathrm{lb}}\right) = 22\,\mathrm{N}$$

Then from $W = mg$ we calculate the *mass* of the bag,

$$m = \frac{W}{g} = \frac{22\,\mathrm{N}}{9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}} = 2.3\,\mathrm{kg}$$

(b) Similarly, the weight of the fullback in newtons is

$$240\,\mathrm{lb} = (240\,\mathrm{lb}) \cdot \left(\frac{4.45\,\mathrm{N}}{1\,\mathrm{lb}}\right) = 1070\,\mathrm{N}$$

and then his (her) mass is

$$m = \frac{W}{g} = \frac{1070\,\mathrm{N}}{9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}} = 109\,\mathrm{kg}$$

(c) The automobile's weight is given in tons; express it in newtons:

$$1.8\,\mathrm{ton} = (1.8\,\mathrm{ton}) \left(\frac{2000\,\mathrm{lb}}{1\,\mathrm{ton}}\right) \left(\frac{4.45\,\mathrm{N}}{1\,\mathrm{lb}}\right) = 1.6 \times 10^4\,\mathrm{N}\ .$$

Then its mass is

$$m = \frac{W}{g} = \frac{1.6 \times 10^4\,\mathrm{N}}{9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}} = 1.6 \times 10^3\,\mathrm{kg}$$

---

6. If a man weighs $875\,\mathrm{N}$ **on Earth, what would he weigh on Jupiter, where the free–fall acceleration is** $25.9\,\frac{\mathrm{m}}{\mathrm{s}^2}$**?**

The weight of a mass m on the earth is $W = mg$ where g is the free–fall acceleration *on Earth*. The mass of the man is:

$$m = \frac{W}{g} = \frac{875\,\mathrm{N}}{9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}} = 89.3\,\mathrm{kg}$$

His weight *on Jupiter* is found using g_{Jupiter} instead of g:

$$W_{\mathrm{Jupiter}} = mg_{\mathrm{Jupiter}} = (89.3\,\mathrm{kg})(25.9\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 2.31 \times 10^3\,\mathrm{N}$$

The man's weight on Jupiter is $2.31 \times 10^3\,\mathrm{N}$.

(The statement of the problem is a little deceptive; Jupiter has no solid surface! The planet will indeed pull on this man with a force of $2.31 \times 10^3\,\mathrm{N}$, but there is no "ground" to push back!)

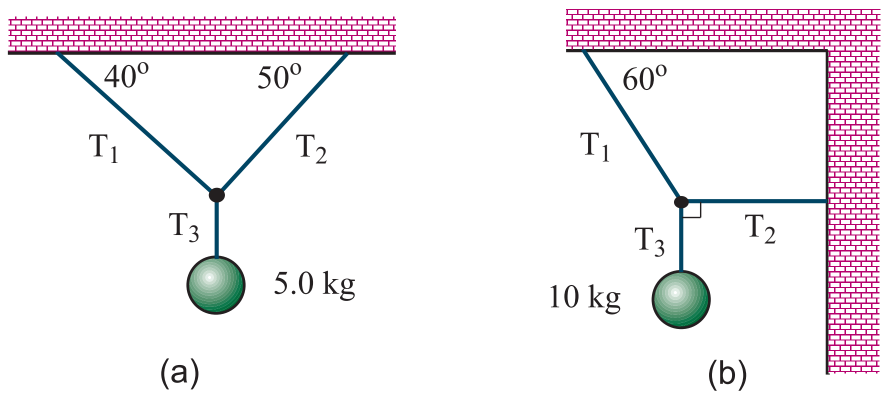

Figure 4.2: Masses suspended by strings, for Example 7.

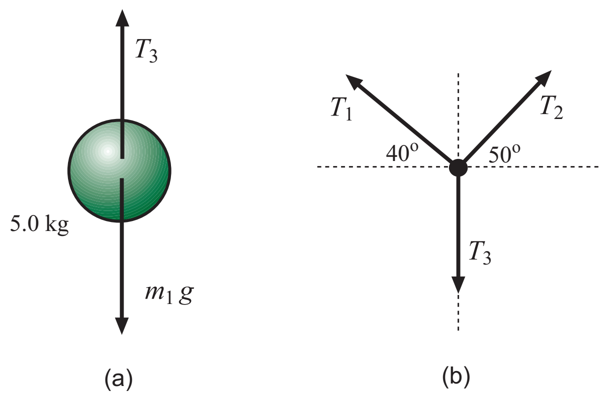

Figure 4.3: Force diagrams for part (a) in Example 7.

4.2.3 Applying Newton's Laws

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7. Find the tension in each cord for the systems shown in Fig. 4.2. (Neglect the mass of the cords.)

(a) In this part, we solve the system shown in Fig. 4.2(a).

Think of the forces acting on the 5.0 kg mass (which we'll call m_1). Gravity pulls downward with a force of magnitude m_g. The vertical string pulls upward with a force of magnitude T_3. (These forces are shown in Fig. 4.3(a).) Since the hanging mass has *no* acceleration, it must be true that $T_3 = m_1 g$. This gives us the value of T_3:

$$T_3 = m_1 g = (5.0\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 49\,\mathrm{N}\ .$$

Next we look at the forces which act at the point where all three strings join; these are shown in Fig. 4.3(b). The force which the strings exert all point *outward* from the joining point and from simple geometry they have the directions shown

Now this point is not accelerating either, so the forces on *it* must all sum to zero. The horizontal components and the vertical components of these forces *separately* sum to zero.

The horizontal components give:

$$-T_1 \cos 40^\circ + T_2 \cos 50^\circ = 0$$

This equation by itself does not let us solve for the tensions, but it does give us:

$$T_2 \cos 50^\circ = T_1 \cos 40^\circ \qquad \Longrightarrow \qquad T_2 = \frac{\cos 40^\circ}{\cos 50^\circ} T_2 = 1.19 T_1$$

The vertical forces sum to zero, and this gives us:

$$T_1 \sin 40^\circ + T_2 \sin 50^\circ - T_3 = 0$$

We already know the value of T_3. Substituting this and also the expression for T_2 which we just found, we get:

$$T_1 \sin 40^\circ + (1.19 T_1) \sin 50^\circ - 49\,\mathrm{N} = 0$$

and now we can solve for T_1. A little rearranging gives:

$$(1.56) T_1 = 49\,\mathrm{N}$$

which gives

$$T_1 = \frac{49\,\mathrm{N}}{(1.56)} = 31.5\,\mathrm{N} \ .$$

Now with T_1 in hand we get T_2:

$$T_2 = (1.19) T_1 = (1.19)(31.5\,\mathrm{N}) = 37.5\,\mathrm{N} \ .$$

Summarizing, the tensions in the three strings for this part of the problem are

$$T_1 = 31.5\,\mathrm{N} \qquad T_2 = 37.5\,\mathrm{N} \qquad T_3 = 49\,\mathrm{N} \ .$$

(b) Now we study the system shown in Fig. 4.2(b).

Once again, the net force on the hanging mass (which we call m_2) must be zero. Since gravity pulls down with a a force $m_2 g$ and the vertical string pulls upward with a force T_3, we know that we just have $T_3 = m_2 g$, so:

$$T_3 = m_2 g = (10\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 98\,\mathrm{N} \ .$$

Now consider the forces which act at the place where all the strings meet. We do as in part (a); the horizontal forces sum to zero, and this gives:

$$-T_1 \cos 60^\circ + T_2 = 0 \qquad \Longrightarrow \qquad T_2 = T_1 \cos 60^\circ$$

The vertical forces sum to zero, giving us:

$$T_1 \sin 60^\circ - T_3 = 0$$

But notice that since we know T_3, this equation has only *one* unknown. We find:

$$T_1 = \frac{T_3}{\sin 60^\circ} = \frac{98\,\mathrm{N}}{\sin 60^\circ} = 113\,\mathrm{N}\ .$$

Using this is our expression for T_2 gives:

$$T_2 = T_1 \cos 60^\circ = (113\,\mathrm{N}) \cos 60^\circ = 56.6\,\mathrm{N}$$

Summarizing, the tensions in the three strings for this part of the problem are

$$T_1 = 113\,\mathrm{N} \qquad T_2 = 56.6\,\mathrm{N} \qquad T_3 = 98\,\mathrm{N}\ .$$

---

8. A $210\,\mathrm{kg}$ motorcycle accelerates from 0 to $55\,\frac{\mathrm{mi}}{\mathrm{hr}}$ in $6.0\,\mathrm{s}$. (a) What is the magnitude of the motorcycle's constant acceleration? (b) What is the magnitude of the net force causing the acceleration?

(a) First, let's convert some units:

$$55\,\tfrac{\mathrm{mi}}{\mathrm{hr}} = (55\,\tfrac{\mathrm{mi}}{\mathrm{hr}})\left(\frac{1609\,\mathrm{m}}{1\,\mathrm{mi}}\right)\left(\frac{1\,\mathrm{hr}}{3600\,\mathrm{s}}\right) = 24.6\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

so that the acceleration of the motorcycle is

$$a = \frac{v_x - v_{x0}}{t} = \frac{24.6\,\frac{\mathrm{m}}{\mathrm{s}} - 0}{6.0\,\mathrm{s}} = 4.1\,\tfrac{\mathrm{m}}{\mathrm{s}^2}$$

(b) Now that we know the acceleration of the motorcycle (and its mass) we know the net horizontal force, because Newton's Law tells us:

$$\sum F_x = ma_x = (210\,\mathrm{kg})(4.1\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 8.6 \times 10^2\,\mathrm{N}$$

The magnitude of the net force on the motorcycle is $8.6 \times 10^2\,\mathrm{N}$.

---

9. A rocket and its payload have a total mass of $5.0 \times 10^4\,\mathrm{kg}$. How large is the force produced by the engine (the thrust) when (a) the rocket is "hovering" over the launchpad just after ignition, and (b) when the rocket is accelerating upward at $20\,\frac{\mathrm{m}}{\mathrm{s}^2}$?

(a) First thing: draw a diagram of the forces acting on the rocket! This is done in Fig. 4.4. If the mass of the rocket is M then we know that gravity will be exerting a force Mg downward. The engines (actually, the gas rushing out of the rocket) exerts a force of magnitude F_{thrust} upward on the rocket.

If the rocket is hovering, i.e. it is motionless but off the ground then it has no acceleration; so, here, a_y=0. Newton's Second Law then says:

$$\sum F_y = F_{\mathrm{thrust}} - Mg = Ma_y = 0$$

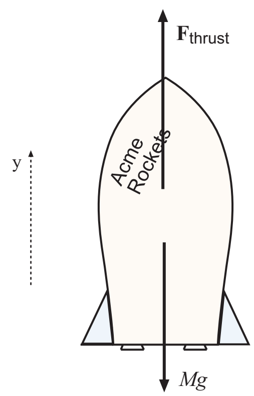

Figure 4.4: Forces acting on the rocket in Example 9

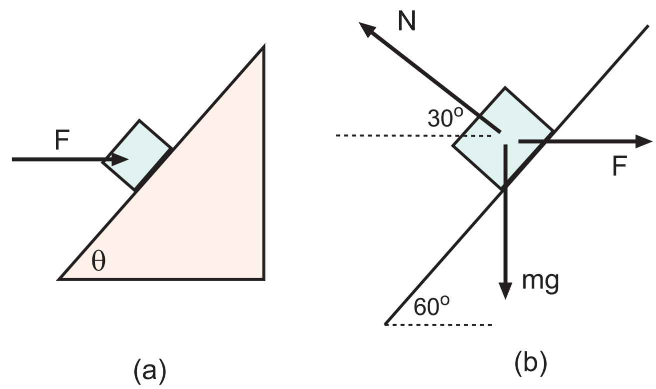

Figure 4.5: (a) Block held in place on a smooth ramp by a horizontal force. (b) Forces acting on the block.

which gives

$$F_{\text{thrust}} = Mg = (5.0 \times 10^4\,\text{kg})(9.80\,\tfrac{\text{m}}{\text{s}^2}) = 4.9 \times 10^5\,\text{N}$$

The engines exert an upward force of $4.9 \times 10^5\,\text{N}$ on the rocket.

(b) As in part (a), gravity and thrust are the only forces acting on the rocket, but now it has an acceleration of $a_y = 20\,\frac{\text{m}}{\text{s}^2}$. So Newton's Second Law now gives

$$\sum F_y = F_{\text{thrust}} - Mg = Ma_y$$

so that the force of the engines is

$$F_{\text{thrust}} = Mg + Ma_y = M(g + a_y) = (5.0 \times 10^4\,\text{kg})(9.80\,\tfrac{\text{m}}{\text{s}^2} + 20\,\tfrac{\text{m}}{\text{s}^2}) = 1.5 \times 10^6\,\text{N}$$

---

10. A block of mass $m = 2.0\,\text{kg}$ is held in equilibrium on an incline of angle $\theta = 60^\circ$ by the horizontal force F, as shown in Fig. 4.5(a). (a) Determine the value of F, the magnitude of F. (b) Determine the normal force exerted by the incline on the block (ignore friction).

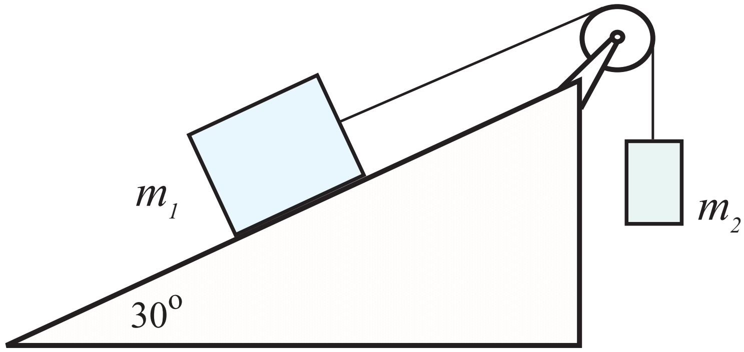

Figure 4.6: Masses m_1 and m_2 are connected by a cord; m_1 slides on frictionless slope.

(a) The first thing to do is to *draw a diagram* of the forces acting on the block, which we do in Fig. 4.5(b). Gravity pulls downward with a force mg. The applied force, of magnitude F, is horizontal. The surface exerts a normal force N on the block; using a little geometry, we see that if the angle of the incline is 60°, then the normal force is directed at 30° above the horizontal, as shown in Fig. 4.5(b). There is no friction force from the surface, so we have shown *all* the forces acting on the block.

Oftentimes for problems involving a block on a slope it is easiest to use the components of the gravity force along the slope and perpendicular to it. For this problem, this would not make things any easier since there is no motion along the slope.

Now, the block is in equilibrium, meaning that it has no acceleration and the forces sum to zero. The fact that the vertical components of the forces sum to zero gives us:

$$N \sin 30^{\circ} - mg = 0 \qquad \Longrightarrow \qquad N = \frac{mg}{\sin 30^{\circ}}$$

Substitute and get:

$$N = \frac{(2.0\,\mathrm{kg})(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})}{\sin 30^{\circ}} = 39.2\,\mathrm{N}\ .$$

The horizontal forces also sum to zero, giving:

$$F - N \cos 30^{\circ} = 0 \qquad \Longrightarrow \qquad F = N \cos 30^{\circ} = (39.2\,\mathrm{N}) \cos 30^{\circ} = 33.9\,\mathrm{N}\ .$$

The applied force F is 33.9 N.

(b) The magnitude of the normal force was found in part (a); there we found:

$$N = 39.2\,\mathrm{N}\ .$$

---

11. A block of mass $m_1 = 3.70\,\mathrm{kg}$ on a frictionless inclined plane of angle $\theta = 30.0^{\circ}$ is connected by a cord over a massless, frictionless pulley to a second block of mass $m_2 = 2.30\,\mathrm{kg}$ hanging vertically, as shown in Fig. 4.6. What are (a) the magnitude of the acceleration of each block and (b) the direction of the acceleration of m_2? (c) What is the tension in the cord?

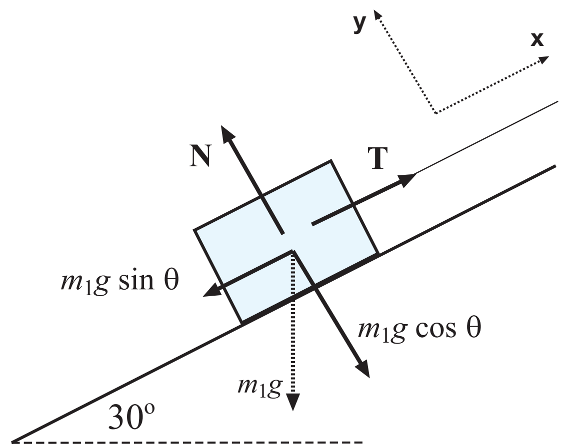

Figure 4.7: The forces acting on m_1

(a) Before thinking about the forces acting on these blocks, we can think about their motion. m_1 is constrained to move along the slope and m_2 must move vertically. Because the two masses are joined by a string, the distance by which m_1 moves up the slope is equal to the distance which m_2 moves downward, and the amount by which m_1 moves down the slope is the amount by which m_2 moves upward. The same is true of their *accelerations*; if it turns out that m_1 is accelerating up the slope, that will be the same as m_2's downward acceleration.

Now we draw "free–body diagrams" and invoke Newton's Second Law for each mass. Consider all the forces acting on m_1. These are shown in Fig. 4.7The force of gravity, with magnitude m_1 pulls straight down. Here, looking ahead to the fact that motion can only occur along the slope it has decomposed into its components perpendicular to the surface (with magnitude $m_1 \cos\theta$) and down the slope (with magnitude $m_1 \sin\theta$). The normal force of the surface has magnitude N and points... normal to the surface! Finally the string pulls *up* with slope with a force of magnitude T, the tension in the string.

Suppose we let x be a coordinate which measures movement *up* the slope. (Note, we are not saying that the block *will move up the slope, this is just a choice of coordinate.* Let y be a coordinate going perpendicular to the slope. We know that there is no y acceleration so the components of the forces in the y direction must add to zero. This gives:

$$N - m_1 g \cos\theta = 0 \qquad \Longrightarrow \qquad N = m_1 g \cos\theta$$

which gives the normal force should we ever need it. (We won't.) Next, the sum of the x forces gives $m_1 a_x$, which will *not* be zero. We get:

$$T - m_1 g \sin\theta = m_1 a_x \tag{4.3}$$

Here there are two unknowns, T and a_x.

The free–body diagram for mass m_2 is shown in Fig. 4.8. The force of gravity, $m_2 g$ pulls downward and the string tension T pulls upward. Suppose we use a coordinate x' which points straight *down*. (This is a little unconventional, but you can see that there is a

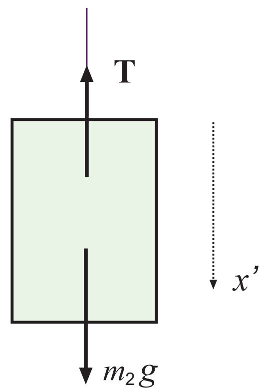

Figure 4.8: The forces acting on m_2. Coordinate x' points downward.

connection with the coordinate x used for the motion of m_1. Then the sum of forces in the x' direction gives $m_2 a_{x'}$:

$$m_2 g - T = m_2 a_{x'}$$

Now as we argued above, the accelerations are equal: $a_x = a_{x'}$. This gives us:

$$m_2 g - T = m_2 a_x \tag{4.4}$$

At this point, the physics is done and the rest of the problem is doing the math (algebra) to solve for a_x and T. We are first interested in finding a_x. We note that by adding Eqs. 4.3 and 4.4 we will eliminate T. Doing this, we get:

$$(T - m_1 g \sin\theta) + (m_2 g - T) = m_1 a_x + m_2 a_x$$

this gives:

$$m_2 g - m_1 g \sin\theta = (m_1 + m_2) a_x$$

and finally:

$$a_x = \frac{(m_2 - m_1 g \sin\theta) g}{m_1 + m_2}$$

Substituting the given values, we have:

$$\begin{aligned} a_x &= \frac{(2.30\,\mathrm{kg} - 3.70\,\mathrm{kg}\sin 30^\circ)(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})}{(3.70\,\mathrm{kg} + 2.30\,\mathrm{kg})} \\ &= +0.735\,\frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

So $a_x = +0.735\,\frac{\mathrm{m}}{\mathrm{s}^2}$. What does this mean? It means that the acceleration of m_1 *up* the slope and m_2 *downwards* has magnitude $0.735\,\frac{\mathrm{m}}{\mathrm{s}^2}$. The plus sign in our result for a_x is telling us that the acceleration *does* go in the way we (arbitrarily) set up the coordinates. If we had made the opposite ("wrong") choice for the coordinates then our acceleration would have come out with a minus sign.

(b) We've already found the answer to this part in our understanding of the result for part (a). Mass m_1 accelerates *up* the slope; mass m_2 accelerates vertically *downward*.

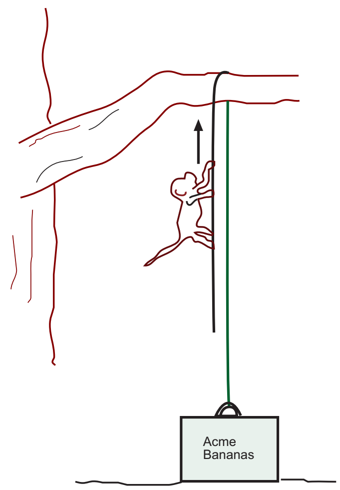

Figure 4.9: Monkey runs up the rope in Example 12.

(c) To get the tension in the string we may use either Eq. 4.3 or Eq. 4.4. Using 4.4 gives:

$$m_2 g - T = m_2 a_x \qquad \Longrightarrow \qquad T = m_2 g - m_2 a_x = m_2(g - a_x)$$

Substituting everything,

$$T = (2.30\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2} - (0.735\,\tfrac{\mathrm{m}}{\mathrm{s}^2})) = 20.8\,\mathrm{N}$$

---

12. A $10\,\mathrm{kg}$ monkey climbs up a massless rope that runs over a frictionless tree limb (!) and back down to a $15\,\mathrm{kg}$ package on the ground, as shown in Fig. 4.9. (a) What is the magnitude of the least acceleration the monkey must have if it is to lift the package off the ground? If, after the package has been lifted the monkey stops its climb and holds onto the rope, what are (b) the monkey's acceleration and (c) the tension in the rope?

(a) Before we do *anything* else, let's understand what forces are acting on the two masses in this problem. The free–body diagrams are shown in Fig. 4.10. The monkey holds onto the rope so it exerts an *upward* force of magnitude T, where T is the tension in the rope. Gravity pulls down on the monkey with a force of magnitude mg, where m is the mass of the monkey. These are all the forces. Note that they will *not* cancel since the problem talks about the monkey having an acceleration and so the net force on the monkey will *not* be zero.

The forces acting on the box are also shown. Gravity pulls downward on the box with a force of magnitude Mg, M being the mass of the box. The rope pulls upward with a force T, If the box is resting on the ground, the ground will be pushing upward with some force F_{ground}. (Here, the ground cannot pull downward.) However when the box is *not* touching the ground then F_{ground} will be zero.

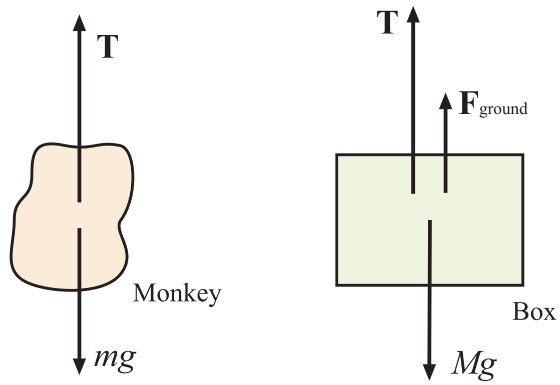

Figure 4.10: The forces acting on the two masses in Example 12.

In the first part of the problem, the monkey is moving along the rope. It is *not* stuck to any point of the rope, so there is no obvious relation between the acceleration of the monkey and the acceleration of the box. Suppose we let $a_{y,\mathrm{monkey}}$ be the vertical acceleration of the monkey and $a_{y,\mathrm{box}}$ be the vertical acceleration of the box. Then from our free–body diagrams, Newton's Second Law gives the acceleration of the monkey:

$$T - mg = ma_{y,\mathrm{monkey}}$$

When the box is on the ground its acceleration is zero and then $T + F_{\mathrm{gr}} = mg$. But when the box is off then ground then we have:

$$T - Mg = Ma_{\mathrm{box}} \qquad \text{(Box off the ground)}$$

In the first part of the problem we are solving for the condition that the monkey climbs just *barely* fast enough for the box to be lifted off the ground. If so, then the ground would exert no force but the net force on the box would be so small as to be virtually zero; the box has a *very, very* tiny acceleration upwards. From this we know:

$$T - Mg = 0 \qquad \Longrightarrow \qquad T = Mg$$

and substituting this result into the first equation gives

$$Mg - mg = ma_{\mathrm{monkey}} \qquad \Longrightarrow \qquad a_{\mathrm{monkey}} = \frac{(M-m)g}{m}$$

Substituting the given values,

$$a_{\mathrm{monkey}} = \frac{(15\,\mathrm{kg} - 10\,\mathrm{kg})(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})}{10\,\mathrm{kg}} = 4.9\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

The monkey must pull himself upwards so as to give himself an acceleration of $4.9\,\frac{\mathrm{m}}{\mathrm{s}^2}$. Anything less and the box will remain on the ground.

(b) Next, suppose that after climbing for while (during which time the box has been rising off the ground) the monkey grabs onto the rope. What new condition does this give us?

Now it is true that the distance that the monkey moves *up* is the same as the distance which the box moves *down*. The same is true of the velocities and accelerations of the monkey and box, so in this part of the problem (recalling that I defined both accelerations as being in the upward sense),

$$a_{\rm monkey} = -a_{\rm box} \ .$$

This condition is not true in general, but here it *is* because we are told that the monkey is holding fast to the rope.

If you recall the example of the Atwood machine from your textbook or lecture notes, this is the same physical situation we are dealing with here. We expect the less massive monkey to accelerate upwards and the more massive box to accelerate downwards. Let's use the symbol a for the monkey's vertical acceleration; then the box's vertical acceleration is $-a$ and our equations are:

$$T - mg = ma$$

and

$$T - Mg = M(-a) \ .$$

At this point the physics is done and the rest is math (algebra) to solve for the two unknowns, T and a. Since the first of these equations gives $T = mg + ma$, substituting this into the second equation gives:

$$mg + ma - Mg = -Ma \qquad \Longrightarrow \qquad ma + Ma = Mg - mg$$

which gives:

$$(M+m)a = (M-m)g \qquad \Longrightarrow \qquad a = \frac{(M-m)}{(M+m)}g$$

Plugging in the numbers gives

$$a = \frac{(15.0\,\mathrm{kg} - 10.0\,\mathrm{kg})}{(15.0\,\mathrm{kg} + 10.0\,\mathrm{kg})}(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 2.0\,\tfrac{\mathrm{m}}{\mathrm{s}^2} \ .$$

When the monkey is holding tight to the rope and both masses move freely, the monkey's acceleration is $2.0\,\frac{\mathrm{m}}{\mathrm{s}^2}$ upwards.

(c) Now that we have the acceleration a for this part of the problem, we can easily substitute into our results in part (b) and find the tension T. From $T - mg = ma$ we get:

$$T = mg + ma = m(g+a) = (10.0\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2} + 2.0\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 118\,\mathrm{N} \ .$$

The tension in the rope is $118\,\mathrm{N}$.

---

13. A mass M is held in place by an applied force F and a pulley system as shown in Fig. 4.11. The pulleys are massless and frictionless. Find (a) the tension in each section of rope, T_1, T_2, T_3, T_4, and T_5, and (b) the magnitude of F.

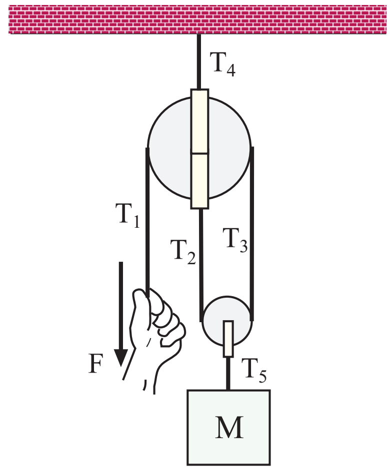

Figure 4.11: Crudely-drawn hand supports a mass M by means of a rope and pulleys

(a) We note first that the mass M (and therefore everything else) is motionless. This simplifies the problem considerably! In particular, every mass in this problem has no acceleration and so the total force on each mass is zero.

We have *five* rope tensions to find here, so we'd better start writing down some equations, fast! Actually, a few of them don't take much work at all; we know that when we have the (idealized) situation of massless rope passing around a frictionless massless pulley, the string tension is the *same* on both sides. As shown in the figure, it is a *single* piece of rope that wraps around the big upper pulley and the lower one, so the tensions T_1, T_2 and T_3 must be the *same*:

$$T_1 = T_2 = T_3$$

Not bad so far!

Next, think about the forces acting on mass M. This is pretty simple... the force of gravity Mg pulls down, and the tension T_5 pulls upward. That's all the forces but they sum to zero because M is motionless. So we must have

$$T_5 = Mg \ .$$

Next, consider the forces which act on the small pulley. These are diagrammed in Fig. 4.12(a). There is a downward pull of magnitude T_5 from the rope which is attached to M and also upward pulls of magnitude T_2 and T_3 from the long rope which is wrapped around the pulley. These forces must sum to zero, so

$$T_2 + T_3 - T_5 = 0$$

But we already know that $T_5 = Mg$ and that $T_2 = T_3$ so this tells us that

$$2T_2 - Mg = 0$$

which gives

$$T_2 = \frac{Mg}{2} \qquad \Longrightarrow \qquad T_3 = T_2 = \frac{Mg}{2} \ .$$

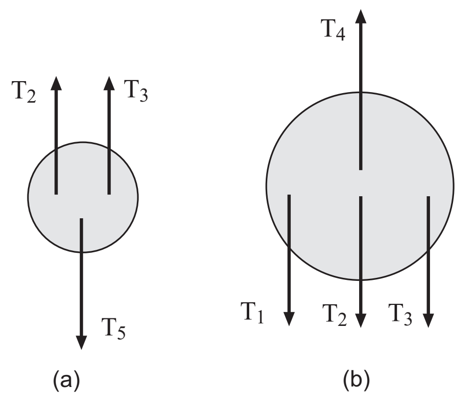

Figure 4.12: (a) Forces on the small (lower) pulley. (b) Forces on the large (upper) pulley.

We also have: $T_1 = T_2 = Mg/2$.

Next, consider the forces on the large pulley, shown in Fig. 4.12(b). Tension T_4 (in the rope attached to the ceiling) pulls upward and tensions T_1, T_2 and T_3 pull downward. These forces sum to zero, so

$$T_4 - T_1 - T_2 - T_3 = 0 \ .$$

But T_4 is the only unknown in this equation. Using our previous answers,

$$T_4 = T_1 + T_2 + T_3 = \frac{Mg}{2} + \frac{Mg}{2} + \frac{Mg}{2} = \frac{3Mg}{2}$$

and so the answers are:

$$T_1 = T_2 = T_3 = \frac{Mg}{2} \qquad T_4 = \frac{3Mg}{2} \qquad T_5 = Mg$$

(b) The force $\mathbf{F}$ is the (downward) force of the hand on the rope. It has the same magnitude as the force of the *rope on the hand*, which is T_1, and we found this to be $Mg/2$. So $F = Mg/2$.

---

14. Mass m_1 on a frictionless horizontal table is connected to mass m_2 through a massless pulley P_1 and a massless fixed pulley P_2 as shown in Fig. 4.13. (a) If a_1 and a_2 are the magnitudes of the accelerations of m_1 and m_2 respectively, what is the relationship between these accelerations? Find expressions for (b) the tensions in the strings and (c) the accelerations a_1 and a_2 in terms of m_1, m_2 and g.

(a) Clearly, as m_2 falls, m_1 will move to the right, pulled by the top string. But how do the magnitudes of the displacements, velocities and accelerations of m_2 and m_1 compare? They are not necessarily the same. Indeed, they are *not* the same.

Possibly the best way to show the relation between a_1 and a_2 is to do a little math; for a very complicated system we would have to do this anyway, and the practice won't hurt.

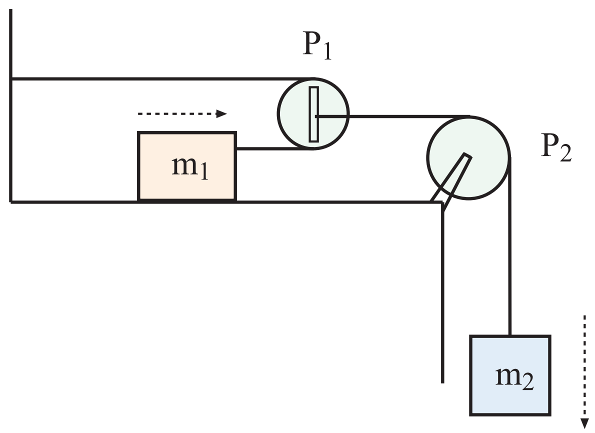

Figure 4.13: System of masses and pulleys for Example 14.

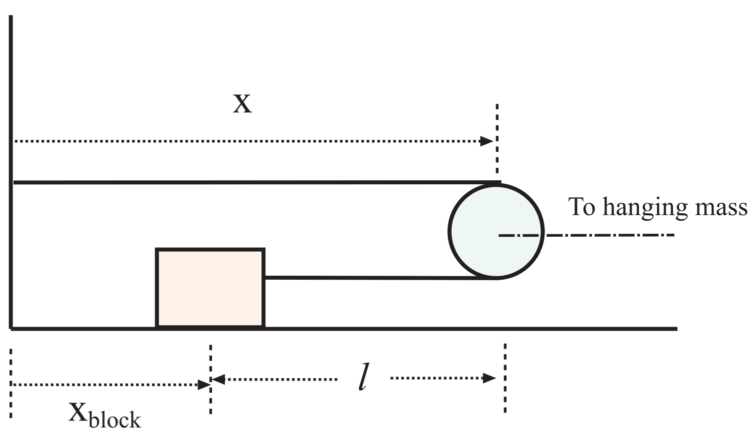

Figure 4.14: Some geometry for Example 14.

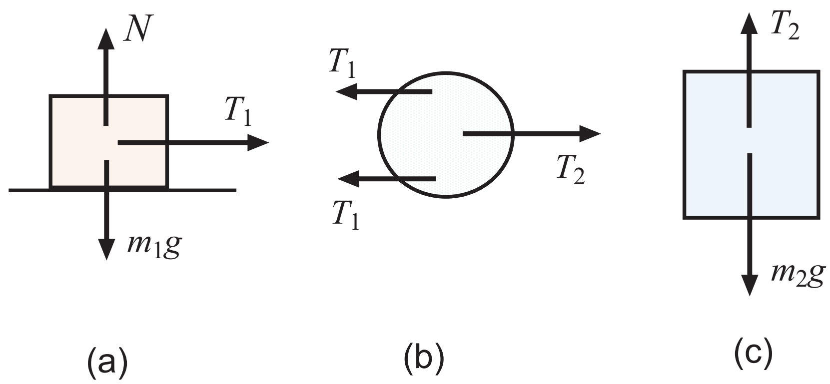

Figure 4.15: Forces on the masses (and moving pulley) in Example 14. (a) Forces on m_1. (b) Forces on the moving (massless) pulley. (c) Forces on m_2.

Focus on the upper mass (m_1) and pulley P_1, and consider the lengths labelled in Fig. 4.14. x measures the distance from the wall to the moving pulley; clearly the position of m_2 is also measured by x. ℓ is the length of string from m_1 to the pulley. $x_{\rm block}$ measures the distance from the wall to m_1. Then:

$$x_{\rm block} = x - \ell \ .$$

This really ignores the bit of string that wraps around the pulley, but we can see that it won't matter.

Now the *total* length of the string is $L = x + \ell$ and it does *not* change with time. Since we have $\ell = L - x$, we can rewrite the last equation as

$$x_{\rm block} = x - (L - x) = 2x - L$$

Take a couple time derivatives of this, keeping in mind that L is a constant. We get:

$$\frac{d^2 x_{\rm block}}{dt^2} = 2\frac{d^2x}{dt^2}$$

But the left side of this equation is the acceleration of m_1 and the right side is the (magnitude of the) acceleration of m_2. The acceleration of m_1 is twice that of m_2:

$$a_1 = 2a_2$$

We can also understand this result by realizing that when m_2 moves down by a distance x, a length $2x$ of the string must go from the "underneath" section to the "above" section in Fig. 4.14. Mass m_1 follows the end of the string so it must move forward by a distance $2x$. Its displacement is always twice that of m_2 so its acceleration is always twice that of m_2.

(b) Now we try to get some information on the forces and accelerations, and we need to draw free–body diagrams. We do this in Fig. 4.15. Mass m_1 has forces m_1g acting downward, a normal force from the table N acting upward, and the string tension T_1 pulling to the right. The vertical forces cancel since m_1 has only a horizontal acceleration, a_1. Newton's Second Law gives us:

$$T_1 = m_1 a_1 \qquad (4.5)$$

The pulley has forces acting on it, as shown in Fig. 4.15(b). The string wrapped around it exerts its pull (of magnitude T_1) both at the top and bottom so we have *two* forces of magnitude T_1 pulling to the left. The second string, which has a tension T_2, pulls to the right with a force of magnitude T_2.

Now this is a slightly subtle point, but the forces on the pulley must add to *zero* because the pulley is assumed to be *massless*. (A net force on it would give it an *infinite* acceleration!) This condition gives us:

$$T_2 - 2T_1 = 0 \tag{4.6}$$

Lastly, we come to m_2. It will accelerate downward with acceleration a_2. Summing the downward forces, Newton's Second Law gives us:

$$m_2 g - T_2 = m_2 a_2 \tag{4.7}$$

For good measure, we repeat the result found in part (a):

$$a_1 = 2a_2 \tag{4.8}$$

In these equations, the unknowns are T_1, T_2, a_1 and a_2... four of them. And we have four equations relating them, namely Eqs. 4.5 through 4.8. The *physics* is done. We just do *algebra* to finish up the problem.

There are many ways to do the algebra, but I'll grind through it in following way: Substitute Eq. 4.8 into Eq. 4.5 and get:

$$T_1 = 2m_1 a_2$$

Putting this result into Eq. 4.6 gives

$$T_2 - 2T_1 = T_2 - 4m_1 a_2 = 0 \qquad \Longrightarrow \qquad T_2 = 4m_1 a_2$$

and finally using this in Eq. 4.7 gives

$$m_2 g - 4m_1 a_2 = m_2 a_2$$

at which point we can solve for a_2 we find:

$$m_2 g = a_2(4m_1 + m_2) \qquad \Longrightarrow \qquad a_2 = \frac{m_2 g}{(4m_1 + m_2)} \tag{4.9}$$

Having solved for one of the unknowns we can quickly find the rest. Eq. 4.8 gives us a_1:

$$a_1 = 2a_2 = \frac{2m_2 g}{(4m_1 + m_2)} \tag{4.10}$$

Then Eq. 4.8 gives us T_1:

$$T_1 = m_1 a_1 = \frac{2m_1 m_2 g}{(4m_1 + m_2)} \tag{4.11}$$

Finally, since Eq. 4.6 tells us that $T_2 = 2T_1$ we get

$$T_2 = \frac{4m_1m_2g}{(4m_1 + m_2)} \tag{4.12}$$

Summarizing our results from Eqs. 4.9 through 4.12, we have:

$$T_1 = \frac{2m_1m_2g}{(4m_1 + m_2)} \qquad T_2 = \frac{4m_1m_2g}{(4m_1 + m_2)}$$

for the tensions in the two strings and:

(c)

$$a_1 = \frac{2m_2g}{(4m_1 + m_2)} \qquad a_2 = \frac{m_2g}{(4m_1 + m_2)}$$

for the accelerations of the two masses.

Chapter 5

Forces and Motion II

5.1 The Important Stuff

5.1.1 Friction Forces

Forces which are known collectively as "friction forces" are all around us in daily life. In elementary physics we discuss the friction force as it occurs between two objects whose surfaces are in contact and which slide against one another.

If in such a situation, a body is *not moving* while an applied force $\mathbf{F}$ acts on it, then **static friction** forces are opposing the applied force, resulting in zero net force. Empirically, one finds that this force can have a *maximum* value given by:

$$f_{\rm s}^{\rm max} = \mu_{\rm s} N \tag{5.1}$$

where $\mu_{\rm s}$ is the **coefficient of static friction** for the two surfaces and N is the normal (perpendicular) force between the two surfaces.

If one object is in motion relative to the other one (i.e. it is sliding on the surface) then there is a force of **kinetic friction** between the two objects. The direction of this force is such as to oppose the sliding motion and its magnitude is given by

$$f_{\rm k} = \mu_{\rm k} N \tag{5.2}$$

where again N is the normal force between the two objects and $\mu_{\rm k}$ is the **coefficient of kinetic friction** for the two surfaces.

5.1.2 Uniform Circular Motion Revisited

Recall the result given in Chapter 3: When an object is in uniform circular motion, moving in a circle of radius r with speed v, the acceleration is directed toward the center of the circle and has magnitude

$$a_{\rm cent} = \frac{v^2}{r} \ .$$

Therefore, by Newton's Second Law of Motion, the *net force* on this object must also be directed toward the center of the circle and have magnitude

$$F_{\mathrm{cent}} = \frac{mv^2}{r} \ . \tag{5.3}$$

Such a force is called a **centripetal force**, as indicated in this equation.

5.1.3 Newton's Law of Gravity (Optional for Calculus–Based)

The force of gravity is one of the fundamental forces in nature. Although in our first physics examples we only dealt with the fact that the *earth* pulls downward on all masses, in fact all masses exert an attractive gravitational force on each other, but for most objects the force is so small that we can ignore it.

Newton's Law of Gravity says that for two masses m_1 and m_2 separated by a distance r, the magnitude of the (attractive) gravitational force is

$$F = G\frac{m_1 m_2}{r^2} \qquad \text{where} \quad G = 6.67 \times 10^{-11}\ \frac{\mathrm{N{\cdot}m^2}}{\mathrm{kg^2}} \tag{5.4}$$

While the law as given really applies to *point* (i.e. small) masses, it can be used for *spherical* masses as long as we take r to be the distance between the centers of the two masses.

5.2 Worked Examples

5.2.1 Friction Forces

---

1. An ice skater moving at $12\,\frac{\mathrm{m}}{\mathrm{s}}$ coasts to a halt in $95\,\mathrm{m}$ on an ice surface. What is the coefficient of (kinetic) friction between ice and skates?

The information which we are given about the skater's (one-dimensional) motion is shown in Fig. 5.1(a). We know that the skater's notion is one of constant acceleration so we can use the results in Chapter 2. In particular we know the initial and final velocities of the skater:

$$v_0 = 12\,\tfrac{\mathrm{m}}{\mathrm{s}} \qquad v = 0$$

and we know the displacement for this interval:

$$x - x_0 = 95\,\mathrm{m}$$

we can use 2.8 to find the (constant) acceleration a. We find:

$$v_x^2 = v_{0x}^2 + 2a_x(x - x_0) \qquad \Longrightarrow \qquad a_x = \frac{(v_x^2 - v_{0x}^2)}{2(x - x_0)}$$

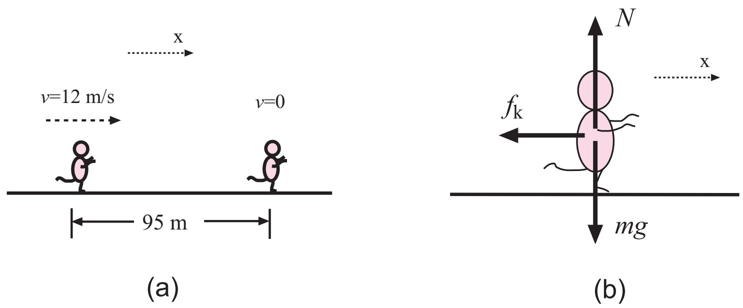

Figure 5.1: Skater slowed to a halt by friction in Example 1. Motion is shown in (a); forces acting on the skater are shown in (b).

Substituting, we get:

$$a_x = \frac{((0\,\frac{\mathrm{m}}{\mathrm{s}})^2 - (12\,\frac{\mathrm{m}}{\mathrm{s}})^2)}{2(95\,\mathrm{m})} = -0.76\,\frac{\mathrm{m}}{\mathrm{s}^2}\ .$$

Next, think about the forces acting on the skater; these are shown in Fig. 5.1(b). If the mass of the skater is m then gravity has magnitude mg and points downward; the ice exerts a normal force N upward. It also exerts a frictional force f_{k} in a direction opposing the motion. Since the skater has no motion in the vertical direction, the vertical forces must sum to zero so that $N = mg$. Also, since the *magnitude* of the force of kinetic friction is given by $f_{\mathrm{k}} = \mu_{\mathrm{k}} N$ we have:

$$f_{\mathrm{k}} = \mu_{\mathrm{k}} N = \mu_{\mathrm{k}} mg\ .$$

So the net force in the x direction is $F_x = -\mu_{\mathrm{k}} mg$.

Newton's law tells us: $F_{x,\,\mathrm{net}} = ma_x$. Using the results we have found, this gives us:

$$-\mu_{\mathrm{k}} mg = m(-0.76\,\frac{\mathrm{m}}{\mathrm{s}^2})$$

From which the m cancels to give:

$$\mu_{\mathrm{k}} = \frac{(0.76\,\frac{\mathrm{m}}{\mathrm{s}^2})}{g} = \frac{(0.76\,\frac{\mathrm{m}}{\mathrm{s}^2})}{(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})} = 7.7 \times 10^{-2}$$

The coefficient of kinetic friction between ice and skates is 7.7×10^{-2}. (Note, the coefficient of friction is *dimensionless*.)

Recall that we were not given the mass of the skater. That didn't matter, because it cancelled out of our equations. But we did have to *consider* it in writing down our equations.

---

2. Block B in Fig. 5.2 weighs $711\,\mathrm{N}$. The coefficient of static friction between block and table is 0.25; assume that the cord between B and the knot is horizontal. Find the maximum weight of block A for which the system will be stationary.

We need to look at the forces acting at the knot (the junction of the three cables). These are shown in Fig. 5.3(a). The vertical cord must have a tension equal to the weight of block A (which we'll call W_A) because at its other end this cord is pulling *up* on A so as to support

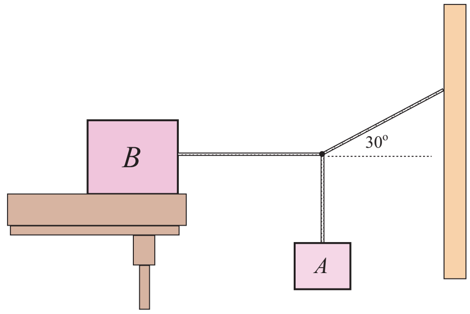

Figure 5.2: Diagram for Example 2.

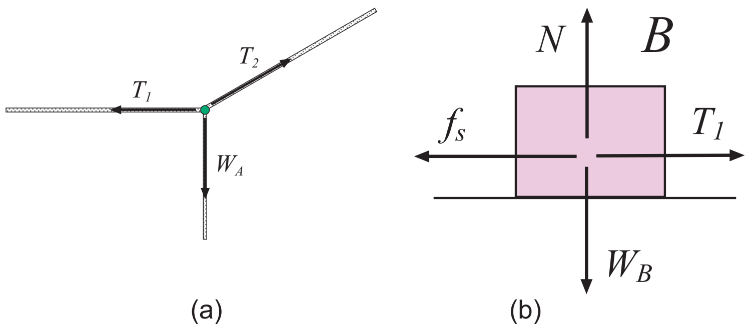

Figure 5.3: (a) Forces acting at the knot in Example 2. (b) Forces acting on block B in Example 2.

it. Let the tensions in the other cords be T_1 for the horizontal one and T_2 for the one that pulls at 30° above the horizontal. The knot is in equilibrium so the forces acting on it add to zero. In particular, the vertical components of the forces add to zero, giving:

$$T_2 \sin\theta - W_A = 0 \qquad \text{or} \qquad T_2 \sin\theta = W_A$$

(where $\theta = 30°$) and the horizontal forces add to zero, giving:

$$-T_1 + T_2 \cos\theta = 0 \qquad \text{or} \qquad T_1 = T_2 \cos\theta \ .$$

Now look at the forces acting on the block which rests on the table; these are shown in Fig. 5.3(b). There is the force of gravity pointing down, with magnitude W_B (that is, the weight of B, equal to $m_B g$). There is a normal force from the table pointing upward; there is the force from the cable pointing to the right with magnitude T_1, and there is the force of static friction pointing to the left with magnitude f_s. Since the vertical forces add to zero, we have

$$N - W_B = 0 \qquad \text{or} \qquad N = W_b$$

The horizontal forces on the block also sum to zero giving

$$T_1 - f_s = 0 \qquad \text{or} \qquad T_1 = f_s$$

Now, the problem states that the value of W_A we're finding is the *maximum* value such that the system is stationary. This means that at the value of W_A we're finding, block B is

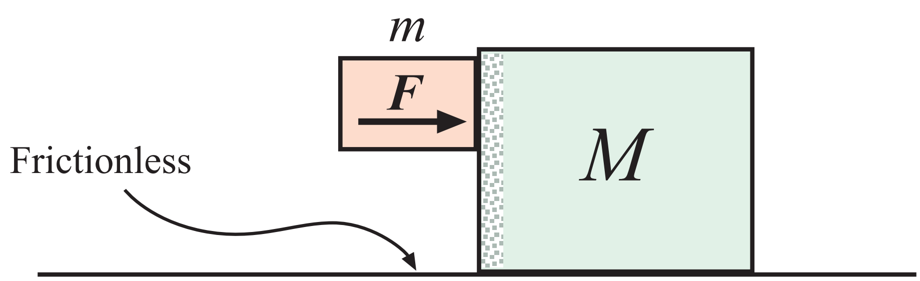

Figure 5.4: Diagram for Example 3.

just about to slip, so that the friction force f_s takes on its maximum value, $f_s = \mu_s N$. Since we also know that $N = W_B$ from the previous equation, we get:

$$T_1 = f_s = \mu_s N = \mu_s W_B$$

From before, we had $T_1 = T_2 \cos\theta$, so making this substitution we get

$$T_2 \cos\theta = \mu_s W_B$$

Almost done! Our very first equation gave $T_2 = \frac{W_A}{\sin\theta}$, so substituting for T_2 gives:

$$\left(\frac{W_A}{\sin\theta}\right)\cos\theta = \mu_s W_B \qquad \text{or} \qquad W_A \cot\theta = \mu_s W_B$$

Finally, we get:

$$W_A = \mu_s W_B \tan\theta$$

Now just plug in the numbers:

$$W_A = (0.25)(711\,\mathrm{N})\tan\,30^\circ = 103\,\mathrm{N}$$

Since we solved for W_A under the condition that block B was about to slip, this is the maximum possible value for W_A so that the system is stationary.

---

3. The two blocks (with $m = 16\,\mathrm{kg}$ and $M = 88\,\mathrm{kg}$) shown in Fig. 5.4 are not attached. The coefficient of static friction between the blocks is $\mu_s = 0.38$, but the surface beneath M is frictionless. What is the minimum magnitude of the horizontal force F required to hold m against M?

Having understood the basic set-up of the problem, we *immediately* begin thinking about the the forces acting on each mass so that we can draw free–body diagrams. The forces on mass m are: (1) The force of gravity mg which points downward. (2) The applied force F which points to the right. (3) The *normal* force with which block M pushes on m. This force necessarily points to the left. (4) The frictional force which block M exerts on m. This is to be a *static* friction force, so we have to think about its direction... in this case, it must

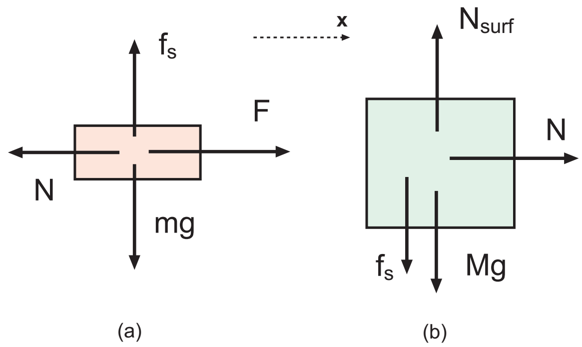

Figure 5.5: Free–body diagrams for the blocks described in Example 3.

clearly oppose the force of gravity to keep the block m from falling. So we include a force $f_{\rm s}$ pointing *up*. These forces are shown in Fig. 5.5.

Next we diagram the forces acting on M. There is the force of gravity, with magnitude Mg, pointing down; the surface beneath M exerts a normal force N pointing upward. Since this surface is frictionless, it does not exert a horizontal force on M. The mass m will exert forces on M and these will be *equal and opposite to the forces which M exerts on m*. So there is a force N on mass M pointing to the *right* and a frictional force f_s pointing *downward*.

Now that we have shown *all* the forces acting on all the masses we can start to discuss the accelerations of the masses and apply Newton's Second Law.

The problem says that mass m is not slipping downward during its motion. This must mean that the forces of friction and gravity balance:

$$f_{\rm s} = mg \ .$$

But this does us little good until we have an expression for $f_{\rm s}$. Now, in this problem we are being asked about a *critical condition* for the slippage of m. We can reasonably guess that here the force of static friction takes on its *maximum* value, namely

$$f_{\rm s} = \mu_{\rm s} N \ ,$$

N being the normal force between the two surfaces. This is an important bit of information, because combining that last two equations we get:

$$mg = \mu_{\rm s} N \ .$$

Let's consider the horizontal motion of both of the masses. Now, since the masses are always touching, their displacements, velocities and accelerations are always the same. Let the x acceleration of both masses be a. Then for mass m, Newton's Second Law gives us:

$$\sum F_x = F - N = ma$$

while for mass M, we get

$$N = Ma$$

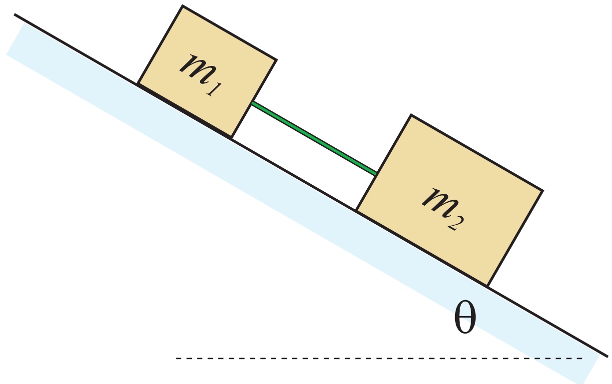

Figure 5.6: Two blocks joined by a rod slide down a slope with friction (coefficient of friction is different for the two blocks).

Combining these last two equations gives

$$F - Ma = ma \qquad \Longrightarrow \qquad F = (M+m)a \qquad \Longrightarrow \qquad a = \frac{F}{(M+m)}$$

which tells us the force N:

$$N = Ma = \frac{MF}{(M+m)}$$

Putting this result for N into our result involving the friction force gives

$$mg = \mu_{\rm s} N = \mu_{\rm s}\frac{MF}{(M+m)}$$

which lets us solve for F:

$$F = \frac{(M+m)m}{M\mu_{\rm s}}g$$

And now we can substitute the given values:

$$F = \frac{(M+m)m}{M\mu_{\rm s}}g = \frac{(16\,{\rm kg} + 88\,{\rm kg})(16\,{\rm kg})}{(88\,{\rm kg})(0.38)}(9.80\,\tfrac{\rm m}{\rm s^2}) = 488\,{\rm N}$$

---

4. In Fig. 5.6 a box of mass $m_1 = 1.65\,\mathrm{kg}$ and a box of mass $m_2 = 3.30\,\mathrm{kg}$ slide down an inclined plane while attached by a massless rod parallel to the plane. The angle of incline is $\theta = 30^\circ$. The coefficient of kinetic friction between m_1 and the incline is $\mu_1 = 0.226$; that between m_2 and the incline is $\mu_2 = 0.113$. Compute (a) the tension in the rod and (b) the common acceleration of the two boxes. c) How would the answers to (a) and (b) change if m_2 trailed m_1?

(a) We will shortly be drawing force diagrams for the two masses, but we should first pause and consider the force which comes from the rod joining the two masses. A "rod" differs

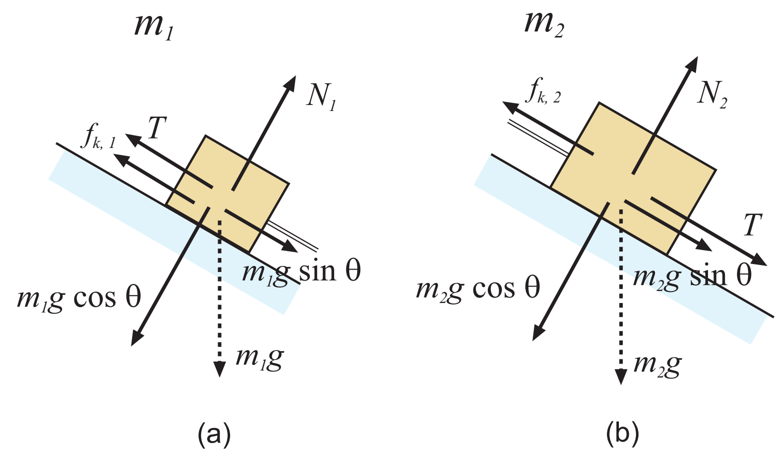

Figure 5.7: (a) Forces acting on block 1 in Example 4. We have assumed that the rod pushes outward; if that is wrong, then T will turn out to be negative. The force of gravity has be split up into components. (b) Forces acting on block 2 in Example 4.

from a "cord" in our problems in that it can pull inward on either end or else *push outward*. (Strings can only pull inward.) For the purpose of writing down our equations we need to make some assumption about what is happening and so here I will assume that the rod is pushing outward with a force of magnitude T, i.e. the rod is compressed. Should it arise in our solution that we get a negative number for T, all is not lost; we will then know that the rod is really *pulling inward* with a force of magnitude $|T|$ and so the rod is being stretched.

With that in mind, we draw a diagram for the forces acting on block 1 and there are a lot of them, as shown in Fig. 5.7(a). Rod tension T and the force of kinetic friction on block 1 (to oppose the motion) point up the slope. The "slope" component of gravity $m_1 g \sin\theta$ points down the slope. The normal component of gravity $m_1 g\cos\theta$ points into the surface and the normal force N_1 from the slope points out of the surface.

As there is no acceleration perpendicular to the slope, those force components sum to zero, giving:

$$N_1 - m_1 g\cos\theta = 0 \qquad \text{or} \qquad N_1 = m_1 g\cos\theta$$

The sum of force components in the down–the–slope direction gives $m_1 a$, where a is the down–the–slope acceleration common to both masses. So then:

$$m_1 g\sin\theta - T - f_{k,1} = m_1 a$$

We can substitute for $f_{k,1}$, since $f_{k,1} = \mu_1 N_1 = \mu_1 m_1 g\cos\theta$. That gives:

$$m_1 g\sin\theta - T - \mu_1 m_1 g\cos\theta = m_1 a \tag{5.5}$$

We have a fine equation here, but T and a are both unknown; we need another equation!

The forces acting on block 2 are shown in Fig. 5.7(b). The force of kinetic friction $f_{k,2}$ points up the slope. The rod tension T and the "slope" component of gravity $m_2 g\sin\theta$ point down the slope. The normal component of gravity $m_2 g\cos\theta$ points into the surface and the normal force of the surface on 2, N_2, points out of the slope.

Again there is no net force perpendicular to the slope, so

$$N_2 - m_2 g\cos\theta = 0 \qquad \text{or} \qquad N_2 = m_2 g\cos\theta\ .$$

The sum of the down–the–slope forces on m_2 gives $m_2 a$, so:

$$m_2 g\sin\theta + T - f_{k,2} = m_2 a$$

We can substitute for the force of kinetic friction here, with $f_{k,2} = \mu_2 N_2 = \mu_2 m_2 g\cos\theta$. Then:

$$m_2 g\sin\theta + T - \mu_2 m_2 g\cos\theta = m_2 a \tag{5.6}$$

Two equations (5.5 and 5.5) and tow unknowns (T and a). The physics is done, the rest is math!

In solving the equations I will go for an analytic (algebraic) solution, then plug in the numbers at the end. Aside from giving us some good practice with algebra, it will be useful in answering part (c).

We note that if we add Eqs. 5.5 and 5.5, T will be eliminated and we can then find a. When we do this, we get:

$$m_1 g\sin\theta + m_2 g\sin\theta - \mu_1 g\cos\theta - \mu_2 g\cos\theta = m_a + m_2 a$$

Lots of factoring to do here! Pulling out some common factors, this is:

$$g\left[(m_1+m_2)\sin\theta - \cos\theta(\mu_1 m_1 + \mu_2 m_2)\right] = (m_1+m_2)a$$

and then we get a:

$$a = \frac{g\left[(m_1+m_2)\sin\theta - \cos\theta(\mu_1 m_1 + \mu_2 m_2)\right]}{(m_1+m_2)} \tag{5.7}$$

But it's really T we want in part (a). We can eliminate a by multiplying Eq. 5.5 by m_2:

$$m_1 m_2 g\sin\theta - m_2 T - \mu_1 m_1 m_2 g\cos\theta = m_1 m_2 a$$

and Eq. 5.6 by m_2:

$$m_1 m_2 g\sin\theta + m_1 T - \mu_2 m_1 m_2 g\cos\theta = m_1 m_2 a$$

and then subtracting the second from the first. Some terms cancel, and this gives:

$$-m_2 T - m_1 T - \mu_1 m_1 m_2 g\cos\theta + \mu_2 m_1 m_2 g\cos\theta = 0$$

Factor things:

$$-T(m_1+m_2) = m_1 m_2 g\cos\theta(\mu_1 - \mu_2)$$

and finally get an expression for T:

$$T = \frac{m_1 m_2 g\cos\theta(\mu_2 - \mu_1)}{(m_1+m_2)} \tag{5.8}$$

Hey, that algebra wasn't so bad, was it? Now we have general expressions for T and a. Plugging numbers into Eq. 5.8, we get:

$$\begin{aligned} T &= \frac{(1.65\,\mathrm{kg})(3.30\,\mathrm{kg})(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})\cos 30^\circ(0.113 - 0.226)}{(1.65\,\mathrm{kg} + 3.30\,\mathrm{kg})} \\ &= -1.05\,\mathrm{N} \end{aligned}$$

Oops! T came out negative! What we find from this is that the assumption was wrong and the rod is really being *stretched* as the blocks slide down the slope, and the magnitude of the rod's tension is $1.05\,\mathrm{N}$.

(b) To find the acceleration of the blocks, plug numbers into Eq. 5.7:

$$\begin{aligned} a &= \frac{(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})\left[(1.65\,\mathrm{kg} + 3.30\,\mathrm{kg})\sin 30^\circ - \cos 30^\circ((0.226)(1.65\,\mathrm{kg}) + (0.113)(3.30\,\mathrm{kg})\right]}{(1.65\,\mathrm{kg} + 3.30\,\mathrm{kg})} \\ &= 3.62\,\frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

The (common) acceleration of the blocks is $3.62\,\frac{\mathrm{m}}{\mathrm{s}^2}$.

(c) Now we ask: What would the answers to (a) and (b) be if the blocks had slid down the slope with m_1 in the lead? Would it make any difference at all? It might, since the friction forces on the masses come from two different μ's. In any case, with our analytic results Eqs. 5.7 and 5.8 we can find the results of switching the labels "1" and "2", since that is all we would get from having the blocks switch positions.

If we switch "1" and "2" in Eq. 5.7, we can see that the result for a will not change at all because the sums within that expression are not affected by the switch. So the connected blocks will slide down the slope with the *same* acceleration, namely $3.62\,\frac{\mathrm{m}}{\mathrm{s}^2}$ for the given values.

What about T? From Eq. 5.8 we see that switching "1" and "2" gives an overall *change in sign* because of the factor $(\mu_2 - \mu_1)$. (The other factors don't change for this switch.) So we know that plugging in the numbers for the case where blocks 1 leads would give $T = +1.05\,\mathrm{N}$, and since this is a positive number, the assumption about the rod being compressed (and as a result pushing outward) would be correct. So for the case where m_1 leads, the *magnitude* of the rod's tension is the same $(1.05\,\mathrm{N})$, but now it pushing *outward*.

---

5. A $3.0-\mathrm{kg}$ **block starts from rest at the top of a** 30.0° **incline and slides** $2.0\,\mathrm{m}$ **down the incline in** $1.5\,\mathrm{s}$**. Find (a) the magnitude of the acceleration of the block, (b) the coefficient of kinetic friction between the block and the plane, (c) the frictional force acting on the block and (d) the speed of the block after it has slid** $2.0\,\mathrm{m}$**.**

(a) The basic information about the motion of the block is summarized in Fig. 5.8(a). We use a coordinate system where x points down the slope and y is perpendicular to the slope. We'll put the origin of the coordinate system at the place where the block begins its motion.

The block's motion down the slope is one of constant acceleration. (This must be so, since all of the forces acting on the block are constant.) Of course, this is an acceleration in

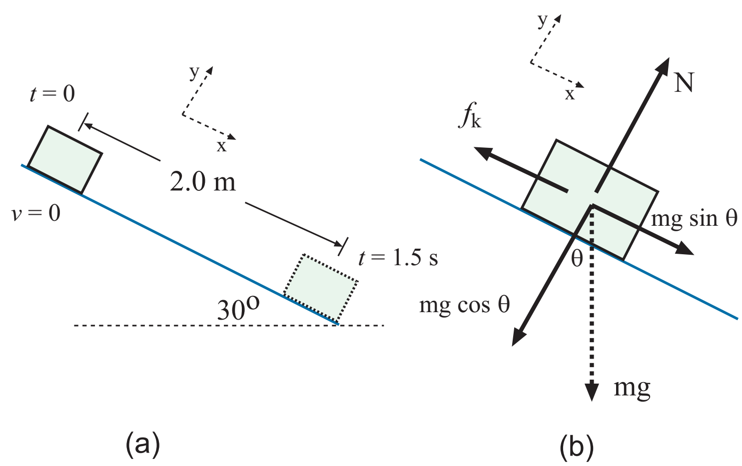

Figure 5.8: (a) Block slides down rough slope in Example 5. (b) Forces acting on the block.

the x direction, as there is no y motion. It begins its slide starting from rest ($v_{0x} = 0$) and so the block's motion is given by:

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2 = \tfrac{1}{2}a_x t^2 \ .$$

We are told that at $t = 1.5\,\mathrm{s}$, $x = 2.0\,\mathrm{m}$. Substitute and solve for a_x:

$$2.0\,\mathrm{m} = \tfrac{1}{2}a_x(1.5\,\mathrm{s})^2 \qquad \Longrightarrow \qquad a_x = \frac{2(2.0\,\mathrm{m})}{(1.5\,\mathrm{s})^2} = 1.78\,\tfrac{\mathrm{m}}{\mathrm{s}^2}$$

The magnitude of the block's acceleration is $1.78\,\frac{\mathrm{m}}{\mathrm{s}^2}$.

(b) We must now think about the forces which act on the block. They are shown in Fig. 5.8(b). Gravity pulls downward with a force mg, which we decompose into its components along the slope and perpendicular to it. The surface exerts a normal force N. There is also a force of kinetic friction from the slope. Since the block is moving down the slope, the friction force must point *up* the slope.

The block moves only along the x axis; the forces in the y direction must sum to zero. Referring to Fig. 5.8(b), we get:

$$\sum F_y = N - mg\cos\theta = 0 \qquad \Longrightarrow \qquad N = mg\cos\theta \ .$$

This gives us the normal force of the surface on the block; here, $\theta = 30.0^\circ$.

The block *does* have an acceleration in the x direction, which we've already found in part (a). The sum of the forces in the $+x$ direction gives ma_x:

$$\sum F_x = mg\sin\theta - f_\mathrm{k} = ma_x$$

Now we use the formula for the force of kinetic friction: $f_\mathrm{k} = \mu_\mathrm{k}N$. Using our expression for the normal force gives us:

$$f_\mathrm{k} = \mu_\mathrm{k}N = \mu_\mathrm{k}mg\cos\theta$$

and using this result in the last equation gives

$$mg\sin\theta - \mu_{\rm k} mg\cos\theta = ma_x \ .$$

Here, the *only* unknown is $\mu_{\rm k}$, so we find it with a little algebra: First off, we can cancel the common factor of m that appears in all terms:

$$g\sin\theta - \mu_{\rm k} g\cos\theta = a_x$$

and then solve for $\mu_{\rm k}$:

$$\begin{aligned} \mu_{\rm k} g\cos\theta &= g\sin\theta - a_x \\ &= (9.80\,\tfrac{\rm m}{\rm s^2})\sin 30.0^\circ - (1.78\,\tfrac{\rm m}{\rm s^2}) = 3.12\,\tfrac{\rm m}{\rm s^2} \end{aligned}$$

So we get:

$$\mu_{\rm k} = \frac{(3.12\,\frac{\rm m}{\rm s^2})}{(9.80\,\frac{\rm m}{\rm s^2})(\cos 30.0^\circ)} = 0.368$$

(c) As we have seen in part (b), the magnitude of the (kinetic) friction force on the mass is

$$\begin{aligned} f_{\rm k} &= \mu_{\rm k} mg\cos\theta \\ &= (0.368)(3.0\,{\rm kg})(9.80\,\tfrac{\rm m}{\rm s^2})\cos 30.0^\circ \\ &= 9.4\,{\rm N} \end{aligned}$$

The force of friction is 9.4 N.

(d) We know the acceleration of the block, its initial velocity ($v_{0x} = 0$) and the time of travel to slide 2.0 m; its final velocity is

$$v = v_{0x} + a_x t = 0 + (1.78\,\tfrac{\rm m}{\rm s^2})(1.50\,{\rm s}) = 2.67\,\tfrac{\rm m}{\rm s}$$

---

6. Three masses are connected on a table as shown in Fig. 5.9. The table has a coefficient of sliding friction of 0.35. The three masses are 4.0 kg, 1.0 kg, and 2.0 kg, respectively and the pulleys are frictionless. (a) Determine the acceleration of each block and their directions. (b) Determine the tensions in the two cords.

(a) First, a little thinking about what we expect to happen. Surely, since the larger mass is hanging on the left side we expect the 4.0 kg mass to accelerate downward, the 1.0 [kg block to accelerate to the right and the 2.0 kg block to accelerate upward. Also, since the masses are connected by strings as shown in the figure, the *magnitudes* of all three accelerations must be the same, because in any time interval the magnitudes of their *displacements* will always be the same. So each mass will have an acceleration of magnitude a with the direction appropriate for each mass.

Now we consider the forces acting on each mass. We draw free–body diagrams! If the tension in the left string is T_1 then the forces on the 4.0 kg mass are as shown in Fig. 5.10(a).

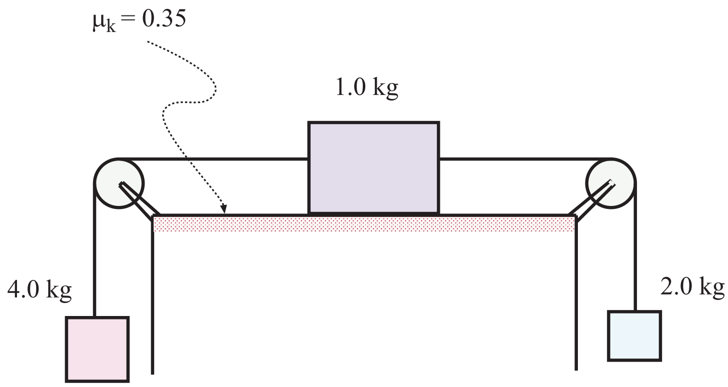

Figure 5.9: System for Example 6

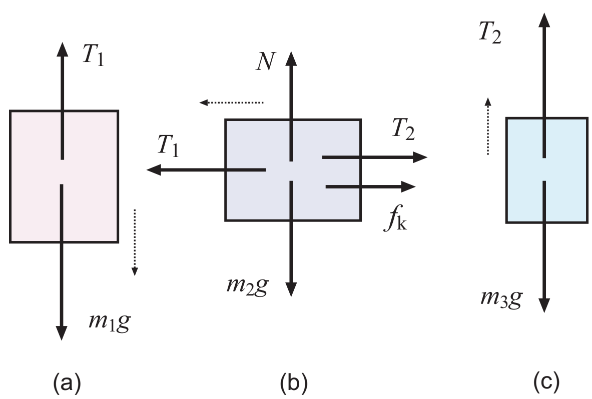

Figure 5.10: Free–body diagrams for the three masses in Example 6. (a) Forces on the mass $m_1 = 4.0\,\text{kg}$. (b) Forces on the mass $m_2 = 1.0\,\text{kg}$. (c) Forces on the mass $m_3 = 2.0\,\text{kg}$. The directions of motion assumed for each mass are also shown.

The string tension T_1 pulls upward; gravity pulls downward with a force m_1g.

The forces acting on m_2 are shown in Fig. 5.10(b). We have more of them to think about; gravity pulls with a force m_2g downward. The table pushes upward with a normal force N. It also exerts a frictional force on m_2 which opposes its motion. Since we think we know which way m_2 is going to go (left!), the friction force f_k must point to the right. There are also forces from the strings. There is a force T_1 to the left from the tension in the first string and a force T_2 pointing to the right from the tension in the other string. (Note, since these are *two different pieces of string*, they can have *different* tensions.)

The forces on m_3 are shown in Fig. 5.10(c). There is a string tension T_2 pulling up and gravity m_3g pulling down.

All right, lets write down some equations! By Newton's Second Law, the sum of the *downward* forces on m_1 should give m_1a. (We agreed that its acceleration would be downward.) This gives:

$$m_1g - T_1 = m_1a \tag{5.9}$$

Moving on to mass m_2, the vertical forces must cancel, giving

$$N = m_2g \ .$$

Newton tells us that the sum of the *left–pointing* forces must give m_2a (we decided that its acceleration would be of magnitude a, toward the left) and this gives:

$$T_1 - f_k - T_2 = m_2a$$

But since

$$f_k = \mu_k N = \mu_k m_2 g \ ,$$

this becomes

$$T_1 - \mu_k m_2 g - T_2 = m_2a \ . \tag{5.10}$$

Finally, the sum of the *upward* forces on m_3 must give m_3a. So:

$$T_2 - m_3g = m_3a \tag{5.11}$$

Having done this work in writing down these wonderful equations we stand back, admire our work and ask if we can go on to solve them. We note that there are three unknowns (a, T_1 and T_2) and we have three equations. We *can* find a solution. The *physics* is *done*...only the algebra remains.

We can do the algebra in the following way: If we just add Eqs. 5.9, 5.10 and 5.11 together (that is, add all the left–hand–sides together and the right–hand–sides together) we find that both T's cancel out. We get:

$$m_1g - T_1 + T_1 - \mu_k m_2 g - T_2 +_2 - m_3g = m_1a + m_2a + m_3a$$

which simplifies to:

$$m_1g - \mu_k m_2 g - m_3 g = (m_1 + m_2 + m_3)a$$

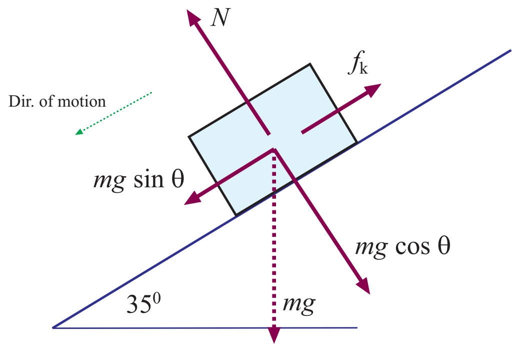

Figure 5.11: Forces on the block in Example 7.

Now we can easily find a:

$$\begin{aligned} a &= \frac{(m_1 - \mu_k m_2 - m_3)g}{(m_1 + m_2 + m_3)} \\ &= \frac{[(4.0\,\mathrm{kg}) - (0.35)(1.0\,\mathrm{kg}) - (2.0\,\mathrm{kg})](9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})}{(4.0\,\mathrm{kg} + 1.0\,\mathrm{kg} + 2.0\,\mathrm{kg})} \\ &= \frac{(1.65\,\mathrm{kg})(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})}{(7.0\,\mathrm{kg})} = 2.31\,\frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

So our *complete* answer to part (a) is: m_1 accelerates at $2.31\,\frac{\mathrm{m}}{\mathrm{s}^2}$ downward; m_2 accelerates at $2.31\,\frac{\mathrm{m}}{\mathrm{s}^2}$ to the left; m_3 accelerates at $2.31\,\frac{\mathrm{m}}{\mathrm{s}^2}$ upward.

(b) Finding the tensions in the strings is now easy; just use the equations we found in part (a).

To get T_1, we can use Eq. 5.9, which gives us:

$$T_1 = m_1 g - m_1 a = m_1(g - a) = (4.0\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2} - 2.31\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 30.0\,\mathrm{N}\ .$$

To get T_2 we can use Eq. 5.11 which gives us:

$$T_2 = m_3 g + m_3 a = m_3(g + a) = (2.0\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2} + 2.31\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 24.2\,\mathrm{N}\ .$$

The tension in the string on the left is 30.0 N. The tension in the string on the right is 24.2 N.

---

7. A block is placed on a plane inclined at 35° relative to the horizontal. If the block slides down the plane with an acceleration of magnitude $g/3$, determine the coefficient of kinetic friction between block and plane.

The forces acting on the block (which has mass m) as it slides down the inclined plane are shown in Fig. 5.11. The force of gravity has magnitude mg and points straight down; here it is split into components normal to the slope and down the slope, which have magnitudes $mg\cos\theta$ and $mg\sin\theta$, respectively, with $\theta = 35^\circ$. The surface exerts a normal force N and a force of kinetic friction, $f_{\rm k}$, which, since the block is moving *down* the slope, points *up* the slope.

The block can only accelerate along the direction of the slope, so the forces perpendicular to the slope must add to zero. This gives us:

$$N - mg\cos\theta = 0 \qquad \Longrightarrow \qquad N = mg\cos\theta$$

The acceleration of the block down the slope was given to us as $a = g/3$. Then summing the forces which point along the slope, we have

$$mg\sin\theta - f_{\rm k} = ma = mg/3$$

The force of kinetic friction is equal to $\mu_{\rm k}N$, and using our expression for N we have

$$f_{\rm k} = \mu_{\rm k}N = \mu_{\rm k}mg\cos\theta$$

and putting this into the previous equation gives:

$$mg\sin\theta - \mu_{\rm k}mg\cos\theta = mg/3 \ .$$

Fortunately, the mass m cancels from this equation; we get:

$$g\sin\theta - \mu_{\rm k}g\cos\theta = g/3$$

And now the only unknown is $\mu_{\rm k}$ which we solve for:

$$\mu_{\rm k}g\cos\theta = g\sin\theta - \frac{g}{3} = g(\sin\theta - \tfrac{1}{3})$$

Here we see that g also cancels, although we always knew the value of g! We then get:

$$\mu_{\rm k} = \frac{(\sin\theta - \frac{1}{3})}{\cos\theta} = \frac{(\sin 35^\circ - \frac{1}{3})}{\cos 35^\circ} = 0.293$$

So the coefficient of kinetic friction between block and slope is 0.293.

---

8. A $2.0\,\mathrm{kg}$ block is placed on top of a $5.0\,\mathrm{kg}$ as shown in Fig. 5.12. The coefficient of kinetic friction between the $5.0\,\mathrm{kg}$ block and the surface is 0.20. A horizontal force F is applied to the $5.0\,\mathrm{kg}$ block. (a) Draw a free–body diagram for each block. What force accelerates the $2.0\,\mathrm{kg}$ block? (b) Calculate the magnitude of the force necessary to pull both blocks to the right with an acceleration of $3.0\,\frac{\mathrm{m}}{\mathrm{s}^2}$. (c) find the minimum coefficient of static friction between the blocks such that the $2.0\,\mathrm{kg}$ block does not slip under an acceleration of $3.0\,\frac{\mathrm{m}}{\mathrm{s}^2}$.

(a) What forces act on each block?

On the big block (with mass $M = 5.0\,\mathrm{kg}$, let's say) we have the applied force **F** which pulls to the right. There is the force of gravity, Mg downward. The surface exerts a normal force N_1 upward. There is a friction force from the surface, which is directed to the left. The

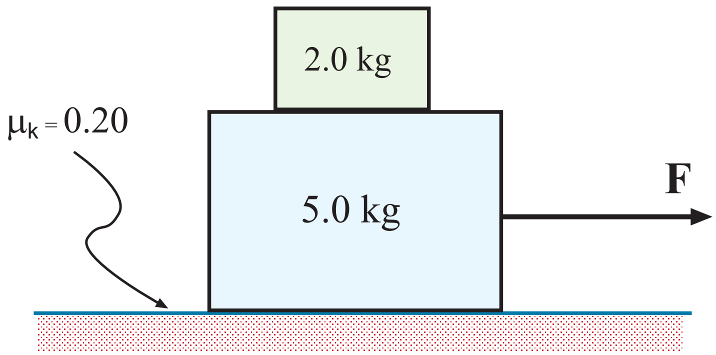

Figure 5.12: Figure for Example 8.

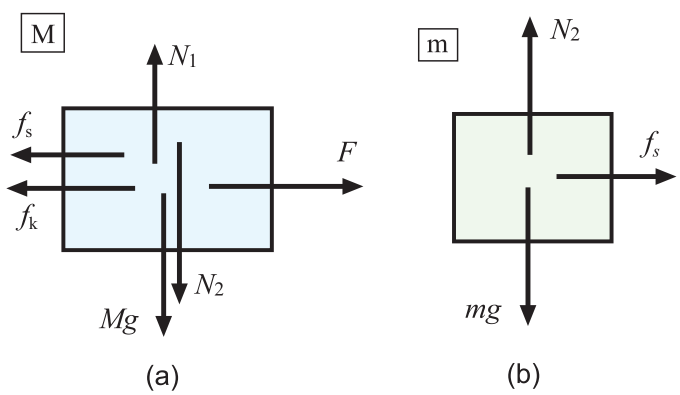

Figure 5.13: (a) Forces acting on the large block, $M = 5.0\,\mathrm{kg}$. (b) Forces acting on the small block, $m = 2.0\,\mathrm{kg}$.

small mass will also exert forces on mass M; it exerts a normal force N_2 which is directed *downward*; we know this because M is pushing upward on m. Now, M is exerting a force of static friction f_s on m which goes to the right; so m must exert a friction force f_s on M which points to the left.

These forces are diagrammed in Fig. 5.13(a).

On the small block we have the force of gravity, mg downward. Mass M exerts an upward normal force N_2 on it, and also a force of static friction f_s on it, pointing to the right. It is this force which accelerates m as it moves along with M (without slipping). These forces are diagrammed in Fig. 5.13(b).

Notice how the forces between M and m, namely N_2 (normal) and f_s, have the same magnitude but opposite directions, in accordance with Newton's Third Law. They are so–called "action–reaction pairs".

(b) The blocks will have a horizontal acceleration but no vertical motion, so that allows us to solve for some of the forces explained in part (a). The vertical forces on m must sum to zero, giving us:

$$N_2 - mg = 0 \qquad \Longrightarrow \qquad N_2 = mg = (2.0\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 19.6\,\mathrm{N}$$

and the vertical forces on M must sum to zero, giving us:

$$N_1 - N_2 - Mg = 0 \qquad \Longrightarrow \qquad N_1 = N_2 + Mg = 19.6\,\mathrm{N} + (5.0\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 68.6\,\mathrm{N}$$

We are given that the acceleration of *both* blocks is $3.0\,\frac{\mathrm{m}}{\mathrm{s}^2}$. Applying Newton's Second Law to mass m we find:

$$\sum F_x = f_s = ma_x = (2.0\,\mathrm{kg})(3.0\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 6.0\,\mathrm{N}$$

While applying it to M gives

$$\sum F_x = F - f_k - f_s = Ma_x = (5.0\,\mathrm{kg})(3.0\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 15.0\,\mathrm{N}$$

We found f_s above; we do know the force of kinetic friction (from M's sliding on the surface) because we know the coefficient of kinetic friction and the normal force N_1:

$$f_k = \mu_k N_1 = (0.20)(68.6\,\mathrm{N}) = 13.7\,\mathrm{N}$$

Now we can solve for F:

$$\begin{aligned} F &= 15.0\,\mathrm{N} + f_k + f_s \\ &= 15.0\,\mathrm{N} + 13.7\,\mathrm{N} + 6.0\,\mathrm{N} \\ &= 34.7\,\mathrm{N} \end{aligned}$$

To pull the blocks together to the right with an acceleration $3.0\,\frac{\mathrm{m}}{\mathrm{s}^2}$ we need an applied force of $34.7\,\mathrm{N}$.

(c) As we've seen, mass m accelerates because of the friction force f_s (from M's surface) which acts on it. Forces of static friction have a maximum value; here we know that we must have

$$f_s \le \mu_s N_2$$

in order for m not to slip on M. Here, we have $f_s = 6.0\,\mathrm{N}$ and $N_2 = 19.6\,\mathrm{N}$. So the critical value of μ_s for our example is

$$\mu_s = \frac{f_2}{N_2} = 0.306$$

If μ_s is less than this value, static friction cannot supply the force needed to accelerate m at $3.0\,\frac{\mathrm{m}}{\mathrm{s}^2}$. So $\mu_s = 0.306$ is the minimum value of the coefficient of static friction so that the upper block does not slip.

5.2.2 Uniform Circular Motion Revisited

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9. A toy car moving at constant speed completes one lap around a circular track (a distance of $200\,\mathrm{m}$) in $25.0\,\mathrm{s}$. (a) What is the average speed? (b) If the mass of the car is $1.50\,\mathrm{kg}$, what is the magnitude of the central force that keeps it in a circle?

(a) If a lap around the circular track is of length 200 m then the (average) speed of the car is

$$v = \frac{d}{t} = \frac{200\,\mathrm{m}}{25.0\,\mathrm{s}} = 8.00\,\frac{\mathrm{m}}{\mathrm{s}}$$

(b) The car undergoes uniform circular motion, moving in a circle of radius r with speed v. The net force on the car points toward the *center of the circle* and has magnitude

$$F_{\mathrm{cent}} = \frac{mv^2}{r}$$

Actually, we haven't found r yet. We are given the circumference of the circle, and from $C = 2\pi r$ we find

$$r = \frac{C}{2\pi} = \frac{200\,\mathrm{m}}{2\pi} = 31.8\,\mathrm{m}$$

So the net force on the car has magnitude

$$F_{\mathrm{cent}} = \frac{mv^2}{r} = \frac{(1.50\,\mathrm{kg})(8.00\,\frac{\mathrm{m}}{\mathrm{s}})^2}{(31.8\,\mathrm{m})} = 3.02\,\mathrm{N}$$

The net force on the car has magnitude 3.02 N; its direction is always inward, keeping the car on a circular path.

---

10. A mass M on a frictionless table is attached to a hanging mass M by a cord through a hole in the table, as shown in Fig. 5.14. Find the speed with which

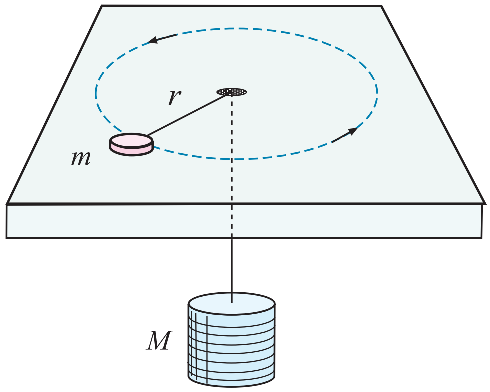

Figure 5.14: Mass m moves; mass M hangs!

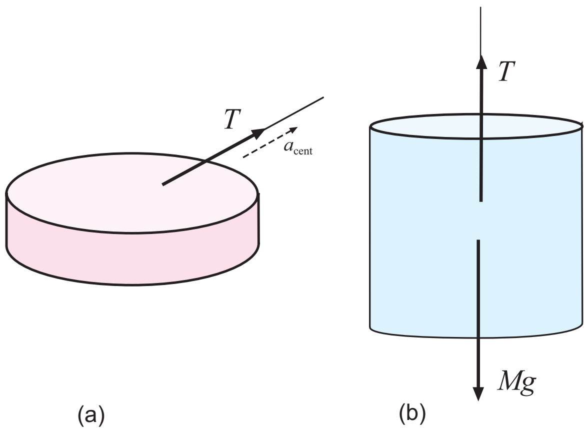

Figure 5.15: (a) Force on mass m and the direction of its acceleration. (There are also vertical forces, gravity and the table's normal force, which cancel; these are not shown.) (b) Forces acting on hanging mass M.

m must move in order for M to stay at rest.

Taking mass M to be at rest, we see that mass m must be moving in a circle of constant radius r. It is moving at (constant) speed v; so mass m undergoes uniform circular motion. So the net force on m points toward the center of the circle and has magnitude $F_{\rm cent} = mv^2/r$. The free–body diagram for m is shown in Fig. 5.15(a). The *only* force on m is the string tension (pointing toward the center of the circle). This gives us:

$$T = \frac{mv^2}{r}$$

Next consider the forces acting on M and its motion. The force diagram for M is shown in Fig. 5.15(b). Since mass M is at rest, the net force on it is zero, which gives:

$$T = Mg$$

Combining these two results, we get:

$$\frac{mv^2}{r} = Mg$$

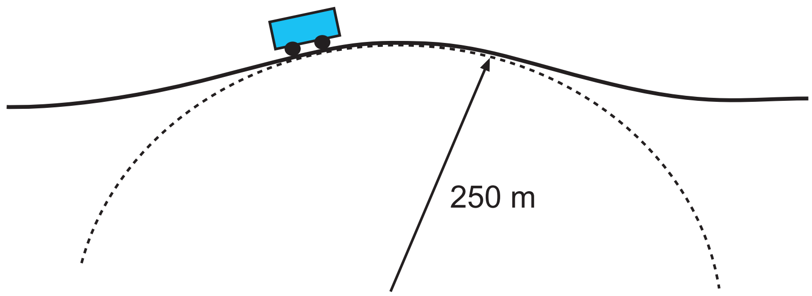

Figure 5.16: Car drives over the top of a hill in Example 11.

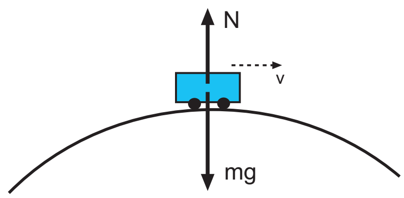

Figure 5.17: Forces acting on the car in Example 11 when it is at the top of the hill.

Solving for v, we get:

$$v^2 = \frac{Mgr}{m} \qquad \Longrightarrow \qquad v = \sqrt{\frac{Mgr}{m}}$$

---

11. A stuntman drives a car over the top of a hill, the cross section of which can be approximated by a circle of radius $250\,\mathrm{m}$, as in Fig. 5.16. What is the greatest speed at which he can drive without the car leaving the road at the top of the hill?

We begin by thinking about the forces acting on the car and its acceleration when it is at the top of the hill.

At the top of the hill, the car is moving in a circular path of radius $r = 250\,\mathrm{m}$ with some speed v. Then the car has a centripetal acceleration of magnitude v^2/r which is directed downward. (For all we know, it may also have a tangential acceleration as well, but the problem gives no information on it, and it won't be relevant for the problem.) By Newton's Second Law, the net (vertical) force on the car must have magnitude mv^2/r and must be directed *downward*.

The forces acting on the car are shown in Fig. 5.17. Then the force of gravity is mg downward. The road exerts a normal force of magnitude N upward. One may ask how we know the road's force goes upward. This is because there is no physical way in which a road can pull downward on a car driving over it. But it can push *up*.

We combine the results from the last two paragraphs. The net *downward* force must equal mv^2/r. This gives us:

$$mg - N = \frac{mv^2}{r} \ .$$

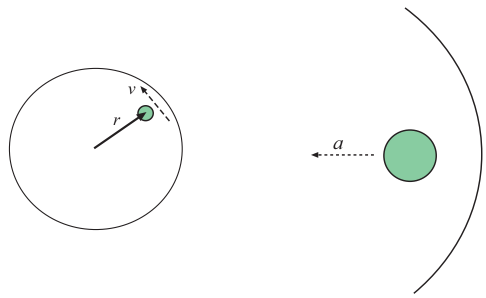

Figure 5.18: Coin moves with a rotating turntable

however without knowing anything more, we can't solve for v in this equation because we don't know N (or, for that matter, m).

We have not yet used the condition that the car is on the verge of leaving the road at the top of the hill. What does this condition give us? If we use the last equation to find the normal force:

$$N = mg - \frac{mv^2}{r}$$

we see that if we increase v there comes a point at which N must be negative in order for the car to stay on the road moving on its circular arc. But as discussed above, N *can't* be negative. But it can be zero, and it is for this speed that the car is on the verge of leaving the road at the top of the hill. The critical case has $N = 0$, and this gives us:

$$0 = mg - \frac{mv^2}{r} \qquad \Longrightarrow \qquad \frac{mv^2}{r} = mg \ .$$

Note that the mass m cancels out of this equation so we don't need to know m. We get:

$$v^2 = rg = (250\,\mathrm{m})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 2.45\,\tfrac{\mathrm{m}^2}{\mathrm{s}^2}$$

and finally

$$v = 49.5\,\tfrac{\mathrm{m}}{\mathrm{s}} \ .$$

The car may be driven as fast as $49.5\,\frac{\mathrm{m}}{\mathrm{s}}$ and it will stay on the road.

---

12. A coin placed $30.0\,\mathrm{cm}$ from the center of a rotating, horizontal turntable slips when its speed is $50.0\,\frac{\mathrm{cm}}{\mathrm{s}}$. (a) What provides the central force when the coin is stationary relative to the turntable? (b) What is the coefficient of static friction between the coin and turntable?

(a) See Fig. 5.18 for a fine illustration of the problem.

As the coin executes uniform circular motion (before it slips) it is accelerating toward the center of the turntable! So there *must* be a force (or forces) on the coin causing it to do this. This force can only come from its contact interaction with the turntable, i.e. from friction. Here, since we are dealing with the case where the coin is not sliding with respect

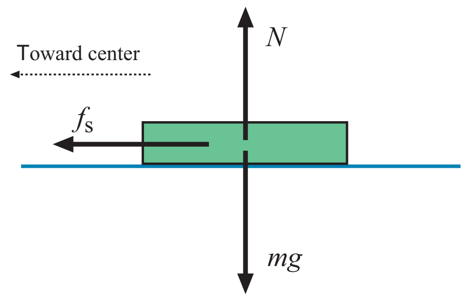

Figure 5.19: Forces acting on the coin in Example 12

to the surface, it is the force of *static* friction. Furthermore, the force of static friction is directed *toward the center of the turntable*.

(b) A view of the forces acting on the coin is given in Fig. 5.19. If the mass of the coin is m then gravity exerts a force mg downward, the turntable exerts a normal force N upward and there is a force of static friction, which as we discussed in part (a) *must* point toward the center of the turntable.

The acceleration of the coin points toward the center of the circle and has magnitude v^2/r, (r being the distance of the coin from the center). So the vertical forces must cancel, giving us $N = mg$. The net force points inward and has magnitude mv^2/r, so that $f_s = mv^2/r$.

Now for the conditions at which the coin starts to slip, the force of static friction has reached its maximum value, i.e.

$$f_s = \mu_s N$$

but here we can use our results to substitute for f_s and for N. This give us:

$$\frac{mv^2}{r} = \mu_s mg$$

which lets us solve for μ:

$$\mu_s = \frac{v^2}{rg} = \frac{(50.0\,\frac{\mathrm{cm}}{\mathrm{s}})^2}{(30.0\,\mathrm{cm})(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})} = \frac{(0.500\,\frac{\mathrm{m}}{\mathrm{s}})^2}{(0.300\,\mathrm{m})(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})} = 8.50 \times 10^{-2}$$

So the coefficient of static friction for the turntable and coin is $\mu_s = 8.50 \times 10^{-2}$.

We were never given the mass of the coin, but we did not need it because it cancelled out of our equations just before the final answer.

---

13. A Ferris wheel rotates four times each minute; it has a diameter of $18.0\,\mathrm{m}$. (a) What is the centripetal acceleration of a rider? What force does the seat exert on a $40.0-\mathrm{kg}$ rider (b) at the lowest point of the ride and (c) at the highest point of the ride? (d) What force (magnitude and direction) does the seat exert on a rider when the rider is halfway between top and bottom?

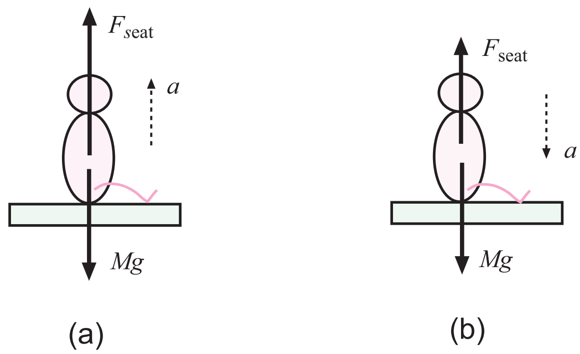

Figure 5.20: Forces acting on the Ferris wheel rider (a) at the lowest point of the ride and (b) at the highest point of the ride.

(a) First, calculate some numbers which we know are important for circular motion! The wheel turns around 4 times in one minute, so the time for *one* turn must be

$$T = \frac{1.0\,\mathrm{min}}{4} = \frac{60.0\,\mathrm{s}}{4} = 15\,\mathrm{s}\ .$$

Also, since the radius of the wheel is $R = D/2 = 18.0\,\mathrm{m}/2 = 9.0\,\mathrm{m}$, the circumference of the wheel is

$$C = 2\pi R = 2\pi(9.0\,\mathrm{m}) = 57\,\mathrm{m}$$

and then the speed of a rider is

$$v = \frac{C}{T} = \frac{57\,\mathrm{m}}{15\,\mathrm{s}} = 3.8\,\tfrac{\mathrm{m}}{\mathrm{s}}\ .$$

The rider moves at constant speed in a circular path of radius R. So the rider's acceleration is always *directed toward the center of the circle* and it has magnitude

$$a_{\mathrm{cent}} = \frac{v^2}{R} = \frac{(3.8\,\frac{\mathrm{m}}{\mathrm{s}})^2}{(9.0\,\mathrm{m})} = 1.6\,\tfrac{\mathrm{m}}{\mathrm{s}^2}$$

The centripetal acceleration of the rider is $1.6\,\frac{\mathrm{m}}{\mathrm{s}^2}$.

(b) Consider what is happening when rider is at the lowest point of the ride. His acceleration is *upward* (toward the center of the circle!) and has magnitude $1.6\,\frac{\mathrm{m}}{\mathrm{s}^2}$.

What are the forces acting on the rider (who has mass M, let's say) at this point? These are shown in Fig. 5.20(a). Gravity pulls down on the rider with a force of magnitude Mg, and the seat pushes upward on the rider with a force F_{seat}. (Usually seats *can't* pull downward; also, the force of the seat can't have any sideways component because here the net force must point *upward*). Since the net force points upward and has magnitude $F_{\mathrm{cent}} = Mv^2/R$, Newton's Second Law gives us:

$$F_{\mathrm{seat}} - Mg = \frac{Mv^2}{R}$$

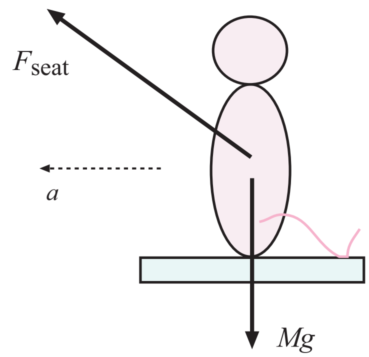

Figure 5.21: Forces on the rider when he is halfway between top and bottom.

Since $M = 40.0\,\text{kg}$, we get:

$$F_{\text{seat}} = Mg + \frac{Mv^2}{R} = (40.0\,\text{kg})(9.80\,\tfrac{\text{m}}{\text{s}^2}) + \frac{(40.0\,\text{kg})(3.8\,\frac{\text{m}}{\text{s}})^2}{(9.0\,\text{m})} = 460\,\text{N}$$

The seat pushes upward on the rider with a force of magnitude 460 N. We might say that when the rider at the lowest point, the rider has an *apparent weight* of 460 N, since that is the force of the surface on which the rider rests. Here, the apparent weight is *greater than* the true weight Mg of the rider.

(c) When the rider is at the highest point of the wheel, his acceleration is *downward*. The forces acting on the rider are shown in Fig. 5.20(b); these are the same forces as in part (a) but now the net force points *downward* and has magnitude $F_{\text{cent}} = Mv^2/R$. Adding up the *downward* forces, Newton's Second Law now gives us:

$$Mg - F_{\text{seat}} = \frac{Mv^2}{R}$$

which now gives us

$$F_{\text{seat}} = Mg - \frac{Mv^2}{R} = (40.0\,\text{kg})(9.80\,\tfrac{\text{m}}{\text{s}^2}) - \frac{(40.0\,\text{kg})(3.8\,\frac{\text{m}}{\text{s}})^2}{(9.0\,\text{m})} = 330\,\text{N}\ .$$

The seat pushes upward on the rider with a force of magnitude 330 N. We would say that at the top of the ride, the apparent weight of the rider is 330 N. This time the apparent weight is *less than* the true weight of the rider.

(d) When the rider is halfway between top and bottom, the net force on him still points toward the center of the circle (and has magnitude Mv^2/R), but in this case the direction is *horizontal*, as indicated in Fig. 5.21. (In this picture the rider is on the *right* side of the Ferris wheel, as we look at it face–on.) The forces acting on the rider are also shown in this picture. The force of gravity, Mg can only pull downward. The only other force on the rider, namely that of the seat does *not* push straight upward in this case. We know that it can't, because the sum of the two forces must point horizontally (to the right). The force of the seat must also have a horizontal component; it must point as shown in Fig. 5.21.

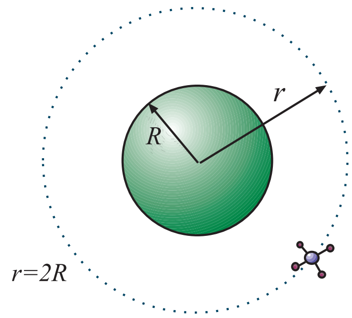

Figure 5.22: Satellite orbits the Earth in Example 14.

Without being overly formal about the mathematics we can see that the vertical component of $\mathbf{F}_{\rm seat}$ must be Mg so as to cancel the force of gravity. The vertical component of $\mathbf{F}_{\rm seat}$ must have magnitude

$$F_{\rm seat,\ vert} = Mg = (40.0\,{\rm kg})(9.80\,\tfrac{\rm m}{\rm s^2}) = 392\,{\rm N}$$

The horizontal component of $\mathbf{F}_{\rm seat}$ must equal Mv^2/R since as we've seen, that is the net force on the rider. So:

$$F_{\rm seat,\ horiz} = \frac{Mv^2}{R} = \frac{(40.0\,{\rm kg})(3.8\,\frac{\rm m}{\rm s})^2}{(9.0\,{\rm m})} = 64\,{\rm N}$$

Now we can find the magnitude of the force of the seat:

$$\begin{aligned} F_{\rm seat} &= \sqrt{{F_{\rm seat,\ vert}}^2 + {F_{\rm seat,\ horiz}}^2} \\ &= \sqrt{(392\,{\rm N})^2 + (64\,{\rm N})^2} \\ &= 397\,{\rm N} \end{aligned}$$

and this force is directed at an angle θ above the horizontal, where θ is given by

$$\theta = \tan^{-1}\left(\frac{F_{\rm seat,\ vert}}{F_{\rm seat,\ horiz}}\right) = \tan^{-1}\left(\frac{392\,{\rm N}}{64\,{\rm N}}\right) = 81^\circ$$

The force of the seat has magnitude 397 N and is directed at 81° above the horizontal.

5.2.3 Newton's Law of Gravity (Optional for Calculus–Based)

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14. A satellite of mass $300\,{\rm kg}$ is in a circular orbit around the Earth at an altitude equal to the Earth's mean radius (See Fig. 5.22.) Find (a) the satellite's orbital speed, (b) the period of its revolution, and (c) the gravitational force acting on it. Use: $R_{\rm Earth} = 6.37\times10^6\,{\rm m}$ and $M_{\rm Earth} = 5.98\times10^{24}\,{\rm kg}$.

(a) (Comment: This was the way the problem was originally stated. I will find the answers in a different order!)

We are told that the height of satellite *above the surface of the Earth* is equal to the Earth's radius. This means that the radius of the satellite's orbit is equal to *twice* the radius of the Earth. Since the mean radius of the Earth is $R = 6.37 \times 10^6$ m, then the orbit radius is

$$r = 2R = 2(6.37 \times 10^6\,\mathrm{m}) = 1.27 \times 10^7\,\mathrm{m}\ .$$

The satellite is always at this distance from the center of the Earth; Newton's law of gravitation tells us the force which the Earth exerts on the satellite:

$$\begin{aligned} F_{\rm grav} &= G\frac{m_{\rm sat}M_{\rm Earth}}{r^2} \\ &= \left(6.67 \times 10^{-11}\,\tfrac{\mathrm{N\cdot m^2}}{\mathrm{kg^2}}\right)\frac{(300\,\mathrm{kg})(5.98 \times 10^{24}\,\mathrm{kg})}{(1.27 \times 10^7\,\mathrm{m})^2} \\ &= 7.42 \times 10^2\,\mathrm{N} \end{aligned}$$

This force is always directed toward the center of the Earth. Since this is the only force which acts on the satellite, it is also the (net) centripetal force on it:

$$F_{\rm cent} = \frac{mv^2}{r} = 7.42 \times 10^2\,\mathrm{N}$$

We can now find the speed of the satellite. It is:

$$v^2 = \frac{rF_{\rm cent}}{m} = \frac{(1.27 \times 10^7\,\mathrm{m})(7.42 \times 10^2\,\mathrm{N})}{(300\,\mathrm{kg})} = 3.14 \times 10^7\,\tfrac{\mathrm{m^2}}{\mathrm{s^2}}$$

which gives

$$v = 5.60 \times 10^3\,\tfrac{\mathrm{m}}{\mathrm{s}}\ .$$

So the satellite's orbital speed is $5.60 \times 10^3\,\frac{\mathrm{m}}{\mathrm{s}}$.

(b) Recall that the speed of an object in uniform circular motion is related to the period and radius by:

$$v = \frac{2\pi r}{T}$$

From this we get the period of the satellite's orbit:

$$\begin{aligned} T &= \frac{2\pi r}{v} \\ &= \frac{2\pi(1.27 \times 10^7\,\mathrm{m})}{(5.60 \times 10^3\,\frac{\mathrm{m}}{\mathrm{s}})} \\ &= 1.42 \times 10^4\,\mathrm{s} = 3.96\,\mathrm{hr} \end{aligned}$$

The period of the satellite is 3.96 hr.

(c) The answer to this part has been found already! The gravitational force acting on the satellite is 7.42×10^2 N.

Chapter 6

Work, Kinetic Energy and Potential Energy

6.1 The Important Stuff

6.1.1 Kinetic Energy

For an object with mass m and speed v, the **kinetic energy** is defined as

$$K = \tfrac{1}{2}mv^2 \tag{6.1}$$

Kinetic energy is a scalar (it has magnitude but no direction); it is always a positive number; and it has SI units of $\mathrm{kg \cdot m^2/s^2}$. This new combination of the basic SI units is known as the **joule**:

$$1\,\mathrm{joule} = 1\,\mathrm{J} = 1\,\tfrac{\mathrm{kg \cdot m^2}}{\mathrm{s^2}} \tag{6.2}$$

As we will see, the joule is also the unit of work W and potential energy U. Other energy units often seen are:

$$1\,\mathrm{erg} = 1\,\tfrac{\mathrm{g \cdot cm^2}}{\mathrm{s^2}} = 10^{-7}\,\mathrm{J} \qquad 1\,\mathrm{eV} = 1.60 \times 10^{-19}\,\mathrm{J}$$

6.1.2 Work

When an object moves while a force is being exerted on it, then **work** is being done on the object by the force.

If an object moves through a displacement $\mathbf{d}$ while a *constant* force $\mathbf{F}$ is acting on it, the force does an amount of work equal to

$$W = \mathbf{F} \cdot \mathbf{d} = Fd\cos\phi \tag{6.3}$$

where ϕ is the angle between $\mathbf{d}$ and $\mathbf{F}$.

Work is also a scalar and has units of $1\,\mathrm{N \cdot m}$. But we can see that this is the same as the joule, defined in Eq. 6.2.

Work can be negative; this happens when the angle between force and displacement is larger than 90°. It can also be *zero*; this happens if $\phi = 90°$. To do work, the force must have a component along (or opposite to) the direction of the motion.

If several different (constant) forces act on a mass while it moves though a displacement **d**, then we can talk about the **net work** done by the forces,

$$\begin{aligned} W_{\rm net} &= \mathbf{F}_1 \cdot \mathbf{d} + \mathbf{F}_1 \cdot \mathbf{d} + \mathbf{F}_1 \cdot \mathbf{d} + \ldots && (6.4) \\ &= \left(\sum \mathbf{F}\right) \cdot \mathbf{d} && (6.5) \\ &= \mathbf{F}_{\rm net} \cdot \mathbf{d} && (6.6) \end{aligned}$$

If the force which acts on the object is *not* constant while the object moves then we must perform an integral (a sum) to find the work done.

Suppose the object moves along a straight line (say, along the x axis, from x_i to x_f) while a force whose x component is $F_x(x)$ acts on it. (That is, we know the force F_x as a function of x.) Then the work done is

$$W = \int_{x_i}^{x_f} F_x(x)\, dx \qquad (6.7)$$

Finally, we can give the most general expression for the work done by a force. If an object moves from $\mathbf{r}_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}$ to $\mathbf{r}_f = x_f\mathbf{i} + y_f\mathbf{j} + z_f\mathbf{k}$ while a force $\mathbf{F}(\mathbf{r})$ acts on it the work done is:

$$W = \int_{x_i}^{x_f} F_x(\mathbf{r})\, dx + \int_{y_i}^{y_f} F_y(\mathbf{r})\, dy + \int_{z_i}^{z_f} F_z(\mathbf{r})\, dz \qquad (6.8)$$

where the integrals are calculated along the path of the object's motion. This expression can be abbreviated as

$$W = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F} \cdot d\mathbf{r}\ . \qquad (6.9)$$

This is rather abstract! But most of the problems where we need to calculate the work done by a force will just involve Eqs. 6.3 or 6.7

We're familiar with the force of gravity; gravity does work on objects which move vertically. One can show that if the height of an object has changed by an amount Δy then gravity has done an amount of work equal to

$$W_{\rm grav} = -mg\Delta y \qquad (6.10)$$

regardless of the horizontal displacement. Note the minus sign here; if the object increases in height it has moved *oppositely* to the force of gravity.

6.1.3 Spring Force

The most famous example of a force whose value depends on position is the **spring force**, which describes the force exerted on an object by the end of an **ideal spring**. An ideal spring will pull inward on the object attached to its end with a force proportional to the amount by which it is stretched; it will push outward on the object attached to its with a force proportional to amount by which it is compressed.

If we describe the motion of the end of the spring with the coordinate x and put the origin of the x axis at the place where the spring exerts no force (the equilibrium position) then the spring force is given by

$$F_x = -kx \tag{6.11}$$

Here k is **force constant**, a number which is different for each ideal spring and is a measure of its "stiffness". It has units of $\mathrm{N/m} = \mathrm{kg/s^2}$. This equation is usually referred to as **Hooke's law**. It gives a decent description of the behavior of real springs, just as long as they can oscillate about their equilibrium positions and they are not stretched by *too* much!

When we calculate the work done by a spring on the object attached to its end as the object moves from x_i to x_f we get:

$$W_{\mathrm{spring}} = \tfrac{1}{2}kx_i^2 - \tfrac{1}{2}kx_f^2 \tag{6.12}$$

6.1.4 The Work–Kinetic Energy Theorem

One can show that as a particle moves from point $\mathbf{r}_i$ to $\mathbf{r}_f$, the change in kinetic energy of the object is equal to the net work done on it:

$$\Delta K = K_f - K_i = W_{\mathrm{net}} \tag{6.13}$$

6.1.5 Power

In certain applications we are interested in the *rate* at which work is done by a force. If an amount of work W is done in a time Δt, then we say that the **average power** $\overline{P}$ due to the force is

$$\overline{P} = \frac{W}{\Delta t} \tag{6.14}$$

In the limit in which both W and Δt are very small then we have the instantaneous power P, written as:

$$P = \frac{dW}{dt} \tag{6.15}$$

The unit of power is the **watt**, defined by:

$$1\,\mathrm{watt} = 1\,\mathrm{W} = 1\,\tfrac{\mathrm{J}}{\mathrm{s}} = 1\,\tfrac{\mathrm{kg\cdot m^2}}{\mathrm{s^3}} \tag{6.16}$$

The watt is related to a quaint old unit of power called the horsepower:

$$1\,\mathrm{horsepower} = 1\,\mathrm{hp} = 550\,\tfrac{\mathrm{ft\cdot lb}}{\mathrm{s}} = 746\,\mathrm{W}$$

One can show that if a force $\mathbf{F}$ acts on a particle moving with velocity $\mathbf{v}$ then the instantaneous rate at which work is being done on the particle is

$$P = \mathbf{F}\cdot\mathbf{v} = Fv\cos\phi \tag{6.17}$$

where ϕ is the angle between the directions of $\mathbf{F}$ and $\mathbf{v}$.

6.1.6 Conservative Forces

The work done on an object by the force of gravity does not depend on the path taken to get from one position to another. The same is true for the spring force. In both cases we just need to know the initial and final coordinates to be able to find W, the work done by that force.

This situation also occurs with the general law for the force of gravity (Eq. 5.4.) as well as with the electrical force which we learn about in the second semester!

This is a different situation from the friction forces studied in Chapter 5. Friction forces do work on moving masses, but to figure out how much work, we need to know *how* the mass got from one place to another.

If the net work done by a force does not depend on the path taken between two points, we say that the force is a **conservative force**. For such forces it is also true that the net work done on a particle moving on around any closed path is zero.

6.1.7 Potential Energy

For a conservative force it is possible to find a function of position called the **potential energy**, which we will write as $U(\mathbf{r})$, from which we can find the work done by the force.

Suppose a particle moves from $\mathbf{r}_i$ to $\mathbf{r}_f$. Then the work done on the particle by a conservative force is related to the corresponding potential energy function by:

$$W_{\mathbf{r}_i \to \mathbf{r}_f} = -\Delta U = U(\mathbf{r}_i) - U(\mathbf{r}_f) \tag{6.18}$$

The potential energy $U(\mathbf{r})$ also has units of joules in the SI system.

When our physics problems involve forces for which we *can* have a potential energy function, we usually think about the *change in potential energy* of the objects rather than the work done by these forces. However for non–conservative forces, we *must* directly calculate their work (or else deduce it from the data given in our problems).

We have encountered two conservative forces so far in our study. The simplest is the force of gravity near the surface of the earth, namely $-mg\mathbf{j}$ for a mass m, where the y axis points upward. For this force we can show that the potential energy function is

$$U_{\mathrm{grav}} = mgy \tag{6.19}$$

In using this equation, it is *arbitrary* where we put the origin of the y axis (i.e. what we call "zero height"). But once we make the choice for the origin we must *stick* with it.

The other conservative force is the spring force. A spring of force constant k which is extended from its equilibrium position by an amount x has a potential energy given by

$$U_{\mathrm{spring}} = \tfrac{1}{2}kx^2 \tag{6.20}$$

6.1.8 Conservation of Mechanical Energy

If we separate the forces in the world into conservative and non-conservative forces, then the work–kinetic energy theorem says

$$W = W_{\rm cons} + W_{\rm non-cons} = \Delta K$$

But from Eq. 6.18, the work done by *conservative* forces can be written as a change in potential energy as:

$$W_{\rm cons} = -\Delta U$$

where U is the sum of *all* types of potential energy. With this replacement, we find:

$$-\Delta U + W_{\rm non-cons} = \Delta K$$

Rearranging this gives the general theorem of the **Conservation of Mechanical Energy**:

$$\Delta K + \Delta U = W_{\rm non-cons} \tag{6.21}$$

We define the **total energy** E of the system as the sum of the kinetic and potential energies of all the objects:

$$E = K + U \tag{6.22}$$

Then Eq. 6.21 can be written

$$\Delta E = \Delta K + \Delta U = W_{\rm non-cons} \tag{6.23}$$

In words, this equation says that the total mechanical energy changes by the amount of work done by the non–conservative forces.

Many of our physics problems are about situations where all the forces acting on the moving objects are conservative; loosely speaking, this means that there is no friction, or else there is negligible friction.

If so, then the work done by non–conservative forces is zero, and Eq. 6.23 takes on a simpler form:

$$\Delta E = \Delta K + \Delta U = 0 \tag{6.24}$$

We can write this equation as:

$$K_i + U_i = K_f + U_f \qquad \text{or} \qquad E_i = E_f$$

In other words, for those cases where we can ignore friction–type forces, if we add up all the kinds of energy for the particle's *initial* position, it is equal to the sum of all the kinds of energy for the particle's *final* position. In such a case, the amount of mechanical energy stays the same. . . it is conserved.

Energy conservation is useful in problems where we only need to know about positions or speeds but not *time* for the motion.

6.1.9 Work Done by Non–Conservative Forces

When the system does have friction forces then we must go back to Eq. 6.23. The change in total mechanical energy equals the work done by the non–conservative forces:

$$\Delta E = E_f - E_i = W_{\rm non-cons}$$

(In the case of sliding friction with the rule $f_k = \mu_k N$ it *is* possible to compute the work done by the non–conservative force.)

6.1.10 Relationship Between Conservative Forces and Potential Energy (Optional?)

Eqs. 6.9 (the general expression for work W) and 6.18 give us a relation between the force $\mathbf{F}$ on a particle (as a function of position, $\mathbf{r}$) and the change in potential energy as the particle moves from $\mathbf{r}_i$ to $\mathbf{r}_f$:

$$\int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F} \cdot d\mathbf{r} = -\Delta U \qquad (6.25)$$

Very loosely speaking, potential energy is the (negative) of the integral of $\mathbf{F}(\mathbf{r})$. Eq. 6.25 can be rewritten to show that (loosely speaking!) the force $\mathbf{F}(\mathbf{r})$ is the (minus) derivative of $U(\mathbf{r})$. More precisely, the components of $\mathbf{F}$ can be gotten by taking *partial derivatives* of U with respect to the Cartesian coordinates:

$$F_x = -\frac{\partial U}{\partial x} \qquad F_y = -\frac{\partial U}{\partial y} \qquad F_z = -\frac{\partial U}{\partial z} \qquad (6.26)$$

In case you haven't come across partial derivatives in your mathematics education yet: They come up when we have functions of several variables (like a function of x, y and z); if we are taking a partial derivative with respect to x, we treat y and z as constants.

As you may have already learned, the three parts of Eq. 6.26 can be compactly written as

$$\mathbf{F} = -\nabla U$$

which can be expressed in words as "$\mathbf{F}$ is the negative gradient of U".

6.1.11 Other Kinds of Energy

This chapter covers the mechanical energy of particles; later, we consider extended objects which can rotate, and they will also have *rotational kinetic energy*. Real objects also have temperature so that they have *thermal energy*. When we take into account *all* types of energy we find that total energy is *completely* conserved...we never lose any! But here we are counting only the *mechanical* energy and if (in real objects!) friction is present some of it can be lost to become thermal energy.

6.2 Worked Examples

6.2.1 Kinetic Energy

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1. If a Saturn V rocket with an Apollo spacecraft attached has a combined mass of $2.9 \times 10^5\,\mathrm{kg}$ and is to reach a speed of $11.2\,\frac{\mathrm{km}}{\mathrm{s}}$, how much kinetic energy will it then have?

(Convert some units first.) The speed of the rocket will be

$$v = (11.2\,\tfrac{\mathrm{km}}{\mathrm{s}})\left(\frac{10^3\,\mathrm{m}}{1\,\mathrm{km}}\right) = 1.12 \times 10^4\,\tfrac{\mathrm{m}}{\mathrm{s}}\ .$$

We know its mass: $m = 2.9 \times 10^5\,\mathrm{kg}$. Using the definition of kinetic energy, we have

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(2.9 \times 10^5\,\mathrm{kg})(1.12 \times 10^4\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 = 1.8 \times 10^{13}\,\mathrm{J}$$

The rocket will have $1.8 \times 10^{13}\,\mathrm{J}$ of kinetic energy.

---

2. If an electron (mass $m = 9.11 \times 10^{-31}\,\mathrm{kg}$) in copper near the lowest possible temperature has a kinetic energy of $6.7\times10^{-19}\,\mathrm{J}$, what is the speed of the electron?

Use the definition of kinetic energy, $K = \frac{1}{2}mv^2$ and the given values of K and m, and solve for v. We find:

$$v^2 = \frac{2K}{m} = \frac{2(6.7 \times 10^{-19}\,\mathrm{J})}{(9.11 \times 10^{-31}\,\mathrm{kg})} = 1.47 \times 10^{12}\,\tfrac{\mathrm{m}^2}{\mathrm{s}^2}$$

which gives:

$$v = 1.21 \times 10^6\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

The speed of the electron is $1.21 \times 10^6\,\frac{\mathrm{m}}{\mathrm{s}}$.

6.2.2 Work

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3. A floating ice block is pushed through a displacement of $\mathbf{d} = (15\,\mathrm{m})\mathbf{i} - (12\,\mathrm{m})\mathbf{j}$ along a straight embankment by rushing water, which exerts a force $\mathbf{F} = (210\,\mathrm{N})\mathbf{i} - (150\,\mathrm{N})\mathbf{j}$ on the block. How much work does the force do on the block during the displacement?

Here we have the simple case of a straight–line displacement $\mathbf{d}$ and a *constant* force $\mathbf{F}$. Then the work done by the force is $W = \mathbf{F} \cdot \mathbf{d}$. We are given all the components, so we can compute the dot product using the components of $\mathbf{F}$ and $\mathbf{d}$:

$$W = \mathbf{F} \cdot \mathbf{d} = F_x d_x + F_y d_y = (210\,\mathrm{N})((15\,\mathrm{m}) + (-150\,\mathrm{N})(-12\,\mathrm{m}) = 4950\,\mathrm{J}$$

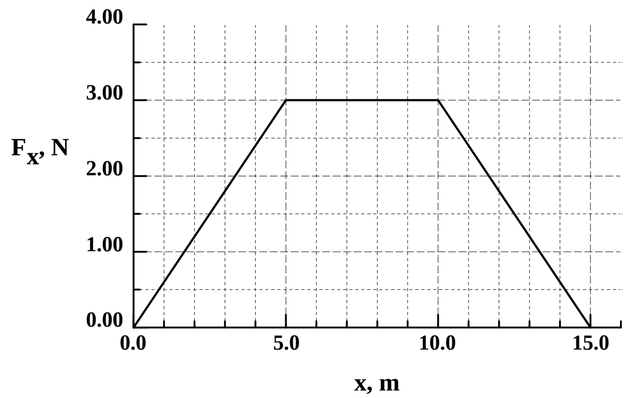

Figure 6.1: Force F_x, which depends on position x; see Example 4.

The force does 4950 J of work.

---

4. A particle is subject to a force F_x that varies with position as in Fig. 6.1. Find the work done by the force on the body as it moves (a) from $x = 0$ to $x = 5.0\,\mathrm{m}$, (b) from $x = 5.0\,\mathrm{m}$ to $x = 10\,\mathrm{m}$ and (c) from $x = 10\,\mathrm{m}$ to $x = 15\,\mathrm{m}$. (d) What is the total work done by the force over the distance $x = 0$ to $x = 15\,\mathrm{m}$?

(a) Here the force is *not* the same all through the object's motion, so we can't use the simple formula $W = F_x x$. We must use the more general expression for the work done when a particle moves along a straight line,

$$W = \int_{x_i}^{x_f} F_x\,dx \ .$$

Of course, this is just the "area under the curve" of F_x vs. x from x_i to x_f.

In part (a) we want this "area" evaluated from $x = 0$ to $x = 5.0\,\mathrm{m}$. From the figure, we see that this is just *half* of a rectangle of base 5.0 m and height 3.0 N. So the work done is

$$W = \tfrac{1}{2}(3.0\,\mathrm{N})(5.0\,\mathrm{m}) = 7.5\,\mathrm{J} \ .$$

(Of course, when we evaluate the "area", we just keep the *units* which go along with the base and the height; here they were meters and newtons, the product of which is a *joule*.)

So the work done by the force for this displacement is 7.5 J.

(b) The region under the curve from $x = 5.0\,\mathrm{m}$ to $x = 10.0\,\mathrm{m}$ is a *full* rectangle of base 5.0 m and height 3.0 N. The work done for this movement of the particle is

$$W = (3.0\,\mathrm{N})(5.0\,\mathrm{m}) = 15.\,\mathrm{J}$$

(c) For the movement from $x = 10.0\,\mathrm{m}$ to $x = 15.0\,\mathrm{m}$ the region under the curve is a *half* rectangle of base 5.0 m and height 3.0 N. The work done is

$$W = \tfrac{1}{2}(3.0\,\mathrm{N})(5.0\,\mathrm{m}) = 7.5\,\mathrm{J} \ .$$

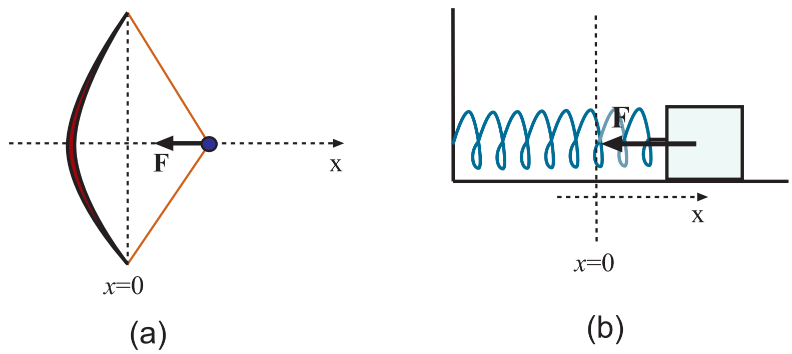

Figure 6.2: The force of a bow string (a) on the object pulling it back can be modelled as as ideal spring (b) exerting a restoring force on the mass attached to its end.

(d) The total work done over the distance $x = 0$ to $x = 15.0\,\mathrm{m}$ is the sum of the three separate "areas",

$$W_{\mathrm{total}} = 7.5\,\mathrm{J} + 15.\,\mathrm{J} + 7.5\,\mathrm{J} = 30.\,\mathrm{J}$$

---

5. What work is done by a force $\mathbf{F} = (2x\,\mathrm{N})\mathbf{i} + (3\,\mathrm{N})\mathbf{j}$, with x in meters, that moves a particle from a position $\mathbf{r}_i = (2\,\mathrm{m})\mathbf{i} + (3\,\mathrm{m})\mathbf{j}$ to a position $\mathbf{r}_f = -(4\,\mathrm{m})\mathbf{i} - (3\,\mathrm{m})\mathbf{j}$?

We use the general definition of work (for a two–dimensional problem),

$$W = \int_{x_i}^{x_f} F_x(\mathbf{r})\,dx + \int_{y_i}^{y_f} F_y(\mathbf{r})\,dy$$

With $F_x = 2x$ and $F_y = 3$ [we mean that F in newtons when x is in meters; work W will come out with units of *joules*!], we find:

$$\begin{aligned} W &= \int_{2\,\mathrm{m}}^{-4\,\mathrm{m}} 2x\,dx + \int_{3\,\mathrm{m}}^{-3\,\mathrm{m}} 3\,dy \\ &= x^2\Big|_{2\,\mathrm{m}}^{-4\,\mathrm{m}} + 3x\Big|_{3\,\mathrm{m}}^{-3\,\mathrm{m}} \\ &= [(16) - (4)]\,\mathrm{J} + [(-9) - (9)]\,\mathrm{J} \\ &= -6\,\mathrm{J} \end{aligned}$$

---

6. An archer pulls her bow string back $0.400\,\mathrm{m}$ by exerting a force that increases from zero to $230\,\mathrm{N}$. (a) What is the equivalent spring constant of the bow? (b) How much work is done in pulling the bow?

(a) While a bow string is not literally *spring*, it may behave like one in that it exerts a force on the thing attached to it (like a hand!) that is *proportional to the distance of pull* from the equilibrium position. The correspondence is illustrated in Fig. 6.2.

We are told that when the string has been pulled back by $0.400\,\mathrm{m}$, the string exerts a restoring force of $230\,\mathrm{N}$. The magnitude of the string's force is equal to the force constant k times the magnitude of the displacement; this gives us:

$$|F_{\mathrm{string}}| = 230\,\mathrm{N} = k(0.400\,\mathrm{m})$$

Solving for k,

$$k = \frac{(230\,\mathrm{N})}{(0.400\,\mathrm{m})} = 575\,\frac{\mathrm{N}}{\mathrm{m}}$$

The (equivalent) spring constant of the bow is $575\,\frac{\mathrm{N}}{\mathrm{m}}$.

(b) Still treating the bow string as if it were an ideal spring, we note that in pulling the string from a displacement of $x = 0$ to $x = 0.400\,\mathrm{m}$ the *string* does as amount of work *on the hand* given by Eq. 6.12:

$$\begin{aligned} W_{\mathrm{string}} &= \tfrac{1}{2}kx_i^2 - \tfrac{1}{2}kx_f^2 \\ &= 0 - \tfrac{1}{2}(575\,\tfrac{\mathrm{N}}{\mathrm{m}})(0.400\,\mathrm{m})^2 \\ &= -46.0\,\mathrm{J} \end{aligned}$$

Is this answer to the question? Not quite... we were really asked for the work done *by the hand on the bow string*. But at all times during the pulling, the hand exerted an equal and opposite force on the string. The force had the opposite direction, so the work that it did has the opposite sign. The work done (by the hand) in pulling the bow is $+46.0\,\mathrm{J}$.

6.2.3 The Work–Kinetic Energy Theorem

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7. A $40\,\mathrm{kg}$ box initially at rest is pushed $5.0\,\mathrm{m}$ along a rough horizontal floor with a constant applied horizontal force of $130\,\mathrm{N}$. If the coefficient of friction between the box and floor is 0.30, find (a) the work done by the applied force, (b) the energy lost due to friction, (c) the change in kinetic energy of the box, and (d) the final speed of the box.

(a) The motion of the box and the forces which do work on it are shown in Fig. 6.3(a). The (constant) applied force points in the same direction as the displacement. Our formula for the work done by a constant force gives

$$W_{\mathrm{app}} = Fd\cos\phi = (130\,\mathrm{N})(5.0\,\mathrm{m})\cos 0^\circ = 6.5\times 10^2\,\mathrm{J}$$

The applied force does $6.5\times 10^2\,\mathrm{J}$ of work.

(b) Fig. 6.3(b) shows *all* the forces acting on the box.

The vertical forces acting on the box are gravity (mg, downward) and the floor's normal force (N, upward). It follows that $N = mg$ and so the magnitude of the friction force is

$$f_{\mathrm{fric}} = \mu N = \mu mg = (0.30)(40\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 1.2\times 10^2\,\mathrm{N}$$

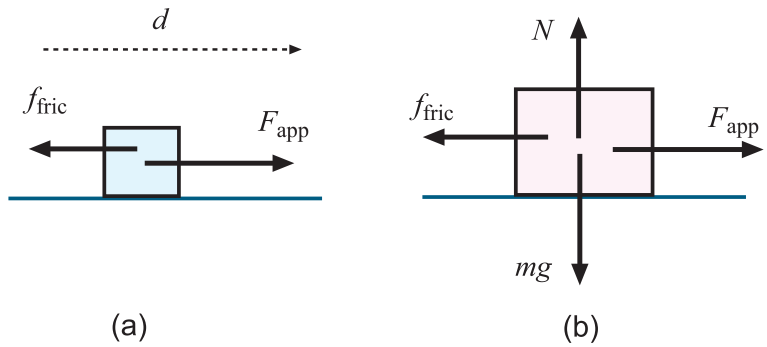

Figure 6.3: (a) Applied force and friction force both do work on the box. (b) Diagram showing *all* the forces acting on the box.

The friction force is directed opposite the direction of motion ($\phi = 180^\circ$) and so the work that it does is

$$\begin{aligned} W_{\rm fric} &= Fd\cos\phi \\ &= f_{\rm fric}\,d\cos 180^\circ = (1.2\times10^2\,\mathrm{N})(5.0\,\mathrm{m})(-1) = -5.9\times10^2\,\mathrm{J} \end{aligned}$$

or we might say that 5.9×10^2 J is *lost* to friction.

(c) Since the normal force and gravity do no work on the box as it moves, the net work done is

$$W_{\rm net} = W_{\rm app} + W_{\rm fric} = 6.5\times10^2\,\mathrm{J} - 5.9\times10^2\,\mathrm{J} = 62\,\mathrm{J}\ .$$

By the work–Kinetic Energy Theorem, this is equal to the change in kinetic energy of the box:

$$\Delta K = K_f - K_i = W_{\rm net} = 62\,\mathrm{J}\ .$$

(d) Here, the initial kinetic energy K_i was *zero* because the box was initially at rest. So we have $K_f = 62\,\mathrm{J}$. From the definition of kinetic energy, $K = \frac{1}{2}mv^2$, we get the final speed of the box:

$$v_f^2 = \frac{2K_f}{m} = \frac{2(62\,\mathrm{J})}{(40\,\mathrm{kg})} = 3.1\,\frac{\mathrm{m}^2}{\mathrm{s}^2}$$

so that

$$v_f = 1.8\,\frac{\mathrm{m}}{\mathrm{s}}$$

---

8. A crate of mass $10.0\,\mathrm{kg}$ is pulled up a rough incline with an initial speed of $1.50\,\frac{\mathrm{m}}{\mathrm{s}}$. The pulling force is $100\,\mathrm{N}$ parallel to the incline, which makes an angle of 20.0° with the horizontal. The coefficient of kinetic friction is 0.400, and the crate is pulled $5.00\,\mathrm{m}$. (a) How much work is done by gravity? (b) How much energy is lost due to friction? (c) How much work is done by the $100\,\mathrm{N}$ force? (d) What is the change in kinetic energy of the crate? (e) What is the speed of the crate after being pulled $5.00\,\mathrm{m}$?

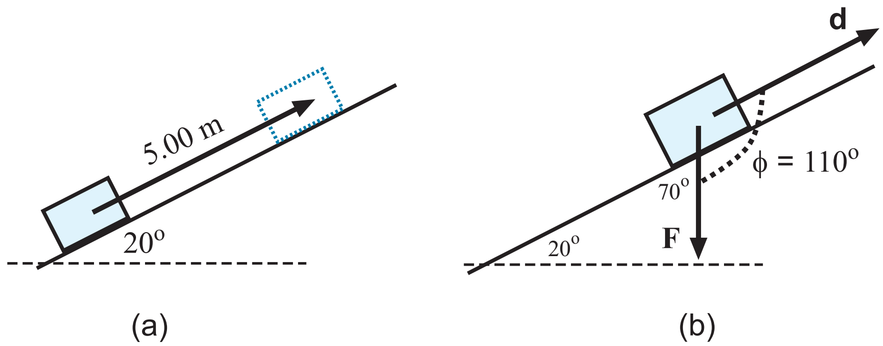

Figure 6.4: (a) Block moves 5.00 m up plane while acted upon by gravity, friction and an applied force. (b) Directions of the displacement and the force of gravity.

(a) We can calculate the work done by gravity in two ways. First, we can use the definition: $W = \mathbf{F}\cdot\mathbf{d}$. The magnitude of the gravity force is

$$F_{\rm grav} = mg = (10.0\,\mathrm{kg}(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 98.0\,\mathrm{N}$$

and the displacement has magnitude 5.00 m. We see from geometry (see Fig. 6.4(b)) that the angle between the force and displacement vectors is 110°. Then the work done by gravity is

$$W_{\rm grav} = Fd\cos\phi = (98.0\,\mathrm{N})(5.00\,\mathrm{m})\cos\,110^\circ = \boxed{-168\,\mathrm{J}}\ .$$

Another way to work the problem is to plug the right values into Eq. 6.10. From simple geometry we see that the change in height of the crate was

$$\Delta y = (5.00\,\mathrm{m})\sin\,20^\circ = +1.71\,\mathrm{m}$$

Then the work done by gravity was

$$W_{\rm grav} = -mg\Delta y = -(10.0\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(1.71\,\mathrm{m}) = -168\,\mathrm{J}$$

(b) To find the work done by friction, we need to know the force of friction. The forces on the block are shown in Fig. 6.5(a). As we have seen before, the normal force between the slope and the block is $mg\cos\theta$ (with $\theta = 20^\circ$) so as to cancel the normal component of the force of gravity. Then the force of kinetic friction on the block points down the slope (opposite the motion) and has magnitude

$$\begin{aligned} f_{\rm k} &= \mu_{\rm k}N = \mu mg\cos\theta \\ &= (0.400)(10.0\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})\cos\,20^\circ = 36.8\,\mathrm{N} \end{aligned}$$

This force points exactly opposite the direction of the displacement **d**, so the work done by friction is

$$W_{\rm fric} = f_{\rm k}d\cos\,180^\circ = (36.8\,\mathrm{N})(5.00\,\mathrm{m})(-1) = -184\,\mathrm{J}$$

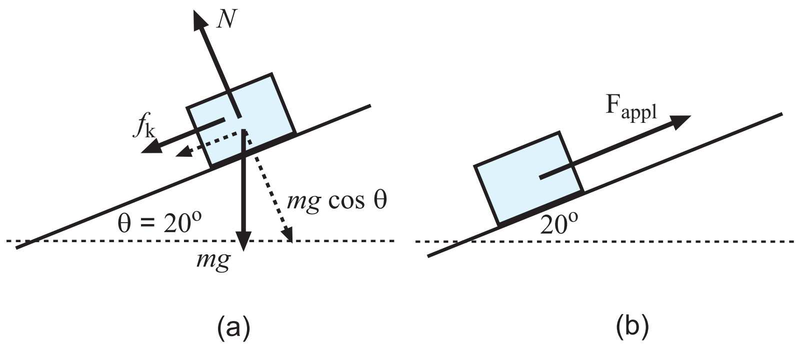

Figure 6.5: (a) Gravity and friction forces which act on the block. (b) The applied force of 100 N is along the direction of the motion.

(c) The 100 N applied force pulls in the direction up the slope, which is *along* the direction of the displacement **d**. So the work that is does is

$$W_{\rm appl} = Fd\cos 0^\circ = (100\,{\rm N})(5.00\,{\rm m})(1) = 500.\,{\rm J}$$

(d) We have now found the work done by each of the forces acting on the crate as it moved: Gravity, friction and the applied force. (We should note the the normal force of the surface *also* acted on the crate, but being perpendicular to the motion, it did no work.) The net work done was:

$$\begin{aligned} W_{\rm net} &= W_{\rm grav} + W_{\rm fric} + W_{\rm appl} \\ &= -168\,{\rm J} - 184\,{\rm J} + 500.\,{\rm J} = 148\,{\rm J} \end{aligned}$$

From the work–energy theorem, this is equal to the change in kinetic energy of the box: $\Delta K = W_{\rm net} = 148\,{\rm J}$.

(e) The initial kinetic energy of the crate was

$$K_i \;=\; \tfrac{1}{2}(10.0\,{\rm kg})(1.50\,\tfrac{\rm m}{\rm s})^2 \;=\; 11.2\,{\rm J}$$

If the final speed of the crate is v, then the change in kinetic energy was:

$$\Delta K = K_f - K_i = \tfrac{1}{2}mv^2 - 11.2\,{\rm J}\ .$$

Using our answer from part (d), we get:

$$\Delta K \;=\; \tfrac{1}{2}mv^2 - 11.2\,{\rm J} = 148\,{\rm J} \qquad\Longrightarrow\qquad v^2 = \frac{2(159\,{\rm J})}{m}$$

So then:

$$v^2 = \frac{2(159\,{\rm J})}{(10.0\,{\rm kg})} = 31.8\,\tfrac{\rm m^2}{\rm s^2} \qquad\Longrightarrow\qquad v = 5.64\,\tfrac{\rm m}{\rm s}\ .$$

The final speed of the crate is $5.64\,\frac{\rm m}{\rm s}$.

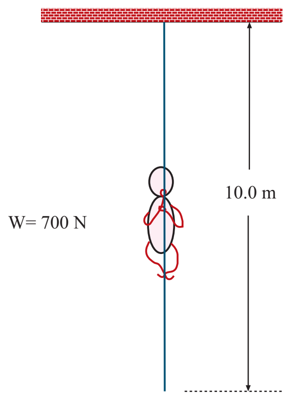

Figure 6.6: Marine climbs rope in Example 9. You don't like my drawing? Tell it to the Marines!

6.2.4 Power

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9. A $700\,\mathrm{N}$ **marine in basic training climbs a** $10.0\,\mathrm{m}$ **vertical rope at a constant speed of** $8.00\,\mathrm{s}$**. What is his power output?**

Marine is shown in Fig. 6.6. The speed of the marine up the rope is

$$v = \frac{d}{t} = \frac{10.0\,\mathrm{m}}{8.00\,\mathrm{s}} = 1.25\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

The forces acting on the marine are gravity (700 N, downward) and the force of the rope which must be 700 N upward since he moves at constant velocity. Since he moves in the same direction as the rope's force, the rope does work on the marine at a rate equal to

$$P = \frac{dW}{dt} = \mathbf{F}\cdot\mathbf{v} = Fv = (700\,\mathrm{N})(1.25\,\tfrac{\mathrm{m}}{\mathrm{s}}) = 875\,\mathrm{W}\ .$$

(It may be hard to think of a stationary rope doing work on anybody, but that is what is happening here.)

This number represents a rate of change in the potential energy of the marine; the energy comes from someplace. *He* is losing (chemical) energy at a rate of 875 W.

---

10. Water flows over a section of Niagara Falls at a rate of $1.2\times10^{6}\,\mathrm{kg/s}$ **and falls** $50\,\mathrm{m}$**. How many** $60\,\mathrm{W}$ **bulbs can be lit with this power?**

Whoa! Waterfalls? Bulbs? What's going on here??

If a certain mass m of water *drops* by a height h (that is, $\Delta y = -h$), then from Eq. 6.10, gravity does an amount of work equal to mgh. If this change in height occurs over a time interval Δt then the rate at which gravity does work is $mgh/\Delta t$.

For Niagara Falls, if we consider the amount of water that falls in one second, then a mass $m = 1.2\times10^{6}\,\mathrm{kg}$ falls through 50 m and the work done by gravity is

$$W_{\mathrm{grav}} = mgh = (1.2\times10^{6}\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(50\,\mathrm{m}) = 5.88\times10^{8}\,\mathrm{J}\ .$$

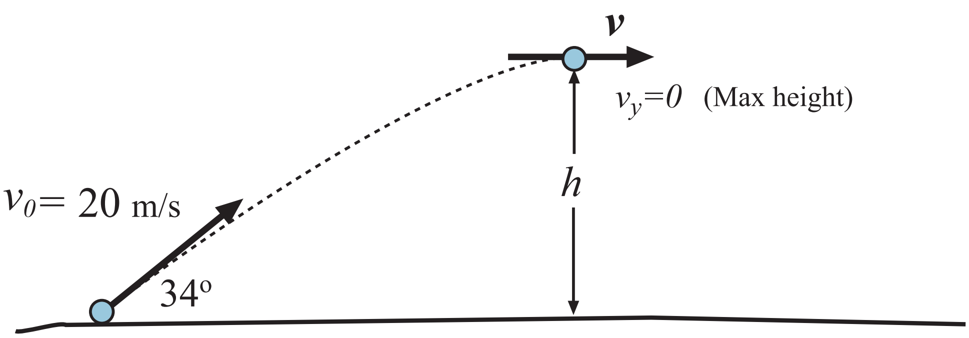

Figure 6.7: Snowball is launched at angle of 34° in Example 11.

This occurs every second, so gravity does work at a rate of

$$P_{\text{grav}} = \frac{mgh}{\Delta t} = \frac{5.88 \times 10^8\,\text{J}}{1\,\text{s}} = 5.88 \times 10^8\,\text{W}$$

As we see later, this is also the rate at which the water *loses potential energy*. This energy can be converted to other forms, such as the electrical energy to make a light bulb function. In this highly idealistic example, all of the energy is converted to electrical energy.

A 60 W light bulb uses energy at a rate of $60\,\frac{\text{J}}{\text{s}} = 60\,\text{W}$. We see that Niagara Falls puts out energy at a rate much bigger than this! Assuming *all* of it goes to the bulbs, then dividing the *total* energy consumption rate by the rate for *one* bulb tells us that

$$N = \frac{5.88 \times 10^8\,\text{W}}{60\,\text{W}} = 9.8 \times 10^6$$

bulbs can be lit.

6.2.5 Conservation of Mechanical Energy

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11. A $1.50\,\text{kg}$ snowball is shot upward at an angle of 34.0° to the horizontal with an initial speed of $20.0\,\frac{\text{m}}{\text{s}}$. (a) What is its initial kinetic energy? (b) By how much does the gravitational potential energy of the snowball–Earth system change as the snowball moves from the launch point to the point of maximum height? (c) What is that maximum height?

(a) Since the initial speed of the snowball is $20.0\,\frac{\text{m}}{\text{s}}$, we have its initial kinetic energy:

$$K_i = \tfrac{1}{2}mv_0^2 = \tfrac{1}{2}(1.50\,\text{kg})(20.0\,\tfrac{\text{m}}{\text{s}})^2 = 300.\,\text{J}$$

(b) We need to remember that since this projectile was not fired straight up, it will still have *some* kinetic energy when it gets to maximum height! That means we have to think a little harder before applying energy principles to answer this question.

At maximum height, we know that the y component of the snowball's velocity is zero. The x component is *not* zero.

But we do know that since a projectile has no horizontal acceleration, the x component will remain *constant*; it will keep its initial value of

$$v_{0x} = v_0 \cos\theta_0 = (20.0\,\tfrac{\mathrm{m}}{\mathrm{s}})\cos 34^\circ = 16.6\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

so the speed of the snowball at maximum height is $16.6\,\frac{\mathrm{m}}{\mathrm{s}}$. At maximum height, (the final position) the kinetic energy is

$$K_f = \tfrac{1}{2}mv_f^2 = \tfrac{1}{2}(1.50\,\mathrm{kg})(16.6\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 = 206.\,\mathrm{J}$$

In this problem there are only conservative forces (namely, gravity). The mechanical energy is conserved:

$$K_i + U_i = K_f + U_f$$

We already found the initial kinetic energy of the snowball: $K_i = 300.\,\mathrm{J}$. Using $U_{\mathrm{grav}} = mgy$ (with $y = 0$ at ground level), the initial potential energy is $U_i = 0$. Then we can find the final potential energy of the snowball:

$$\begin{aligned} U_f &= K_i + U_i - K_f \\ &= 300.\,\mathrm{J} + 0 - 206.\,\mathrm{J} \\ &= 94.\,\mathrm{J} \end{aligned}$$

The final gravitational potential energy of the snowball–earth system (a long–winded way of saying what U is!) is then $94.\,\mathrm{J}$. (Since its original value was zero, this is the answer to part (b).)

(c) If we call the maximum height of the snowball h, then we have

$$U_f = mgh$$

Solve for h:

$$h = \frac{U_f}{mg} = \frac{(94.\,\mathrm{J})}{(1.5\,\mathrm{kg})(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})} = 6.38\,\mathrm{m}$$

The maximum height of the snowball is $6.38\,\mathrm{m}$.

---

12. A pendulum consists of a $2.0\,\mathrm{kg}$ stone on a $4.0\,\mathrm{m}$ string of negligible mass. The stone has a speed of $8.0\,\frac{\mathrm{m}}{\mathrm{s}}$ when it passes its lowest point. (a) What is the speed when the string is at 60° to the vertical? (b) What is the greatest angle with the vertical that the string will reach during the stone's motion? (c) If the potential energy of the pendulum–Earth system is taken to be zero at the stone's lowest point, what is the total mechanical energy of the system?

(a) The condition of the pendulum when the stone passes the lowest point is shown in Fig. 6.8(a). Throughout the problem we will measure the height y of the stone from the

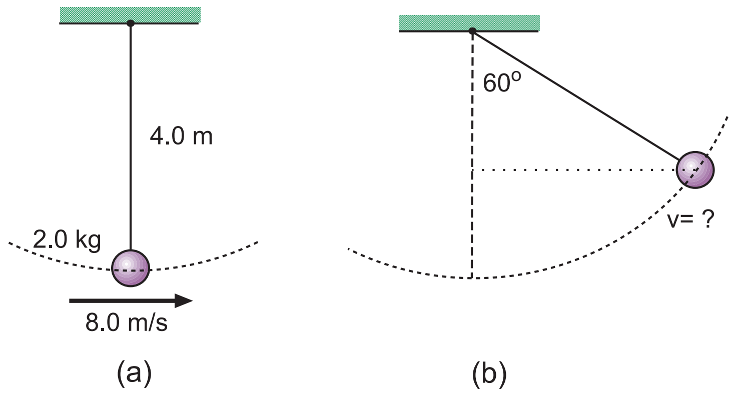

Figure 6.8: (a) Pendulum in Example 12 swings through lowest point. (b) Pendulum has swung 60° past lowest point.

bottom of its swing. Then at the bottom of the swing the stone has zero potential energy, while its kinetic energy is

$$K_i = \tfrac{1}{2}mv_0^2 = \tfrac{1}{2}(2.0\,\mathrm{kg})(8.0\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 = 64\,\mathrm{J}$$

When the stone has swung up by 60° (as in Fig. 6.8(b)) it has some potential energy. To figure out how much, we need to calculate the height of the stone *above the lowest point of the swing*. By simple geometry, the stone's position is

$$(4.0\,\mathrm{m})\cos 60^\circ = 2.0\,\mathrm{m}$$

down from the top of the string, so it must be

$$4.0\,\mathrm{m} - 2.0\,\mathrm{m} = 2.0\,\mathrm{m}$$

up from the lowest point. So its potential energy at this point is

$$U_f = mgy = (2.0\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(2.0\,\mathrm{m}) = 39.2\,\mathrm{J}$$

It will also have a kinetic energy $K_f = \frac{1}{2}mv_f^2$, where v_f is the final speed.

Now in this system there are only a *conservative force* acting on the particle of interest, i.e. the stone. (We should note that the string tension also acts on the stone, but since it always pulls perpendicularly to the motion of the stone, it does no work.) So the total mechanical energy of the stone is conserved:

$$K_i + U_i = K_f + U_f$$

We can substitute the values found above to get:

$$64.0\,\mathrm{J} + 0 = \tfrac{1}{2}(2.0\,\mathrm{kg})v_f^2 + 39.2\,\mathrm{J}$$

which we can solve for v_f:

$$(1.0\,\mathrm{kg})v_f^2 = 64.0\,\mathrm{J} - 39.2\,\mathrm{J} = 24.8\,\mathrm{J} \qquad \Longrightarrow \qquad v_f^2 = 24.8\,\tfrac{\mathrm{m}^2}{\mathrm{s}^2}$$

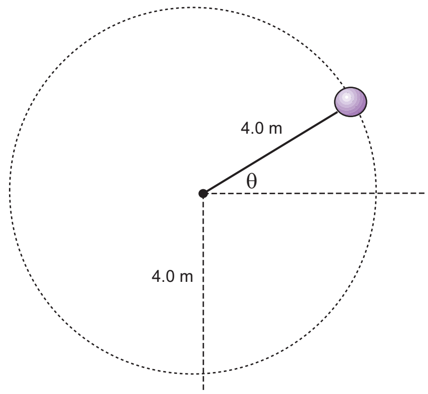

Figure 6.9: Stone reaches its highest position in the swing, which we specify by some angle θ measured above the horizontal.

and then:

$$v_f = 5.0\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

The speed of the stone at the 60° position will be $5.0\,\frac{\mathrm{m}}{\mathrm{s}}$.

(b) Clearly, since the stone is still in motion at an angle of 60°, it will keep moving to greater angles and larger heights above the bottom position. For all we know, it may keep rising until it gets to some angle θ above the position where the string is horizontal, as shown in Fig. 6.9. We *do* assume that the string will stay straight until this point, but that is a reasonable assumption.

Now at this point of maximum height, the speed of the mass is instantaneously zero. So in *this* final position, the kinetic energy is $K_f = 0$. Its height above the starting position is

$$y = 4.0\,\mathrm{m} + (4.0\,\mathrm{m})\sin\theta = (4.0\,\mathrm{m})(1 + \sin\theta) \tag{6.27}$$

so that its potential energy there is

$$U_f = mgy_f = (2.0\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(4.0\,\mathrm{m})(1+\sin\theta) = (78.4\,\mathrm{J})(1+\sin\theta)$$

We use the conservation of mechanical energy (from the position at the bottom of the swing) to find θ: $K_i + U_i = K_f + U_f$, so:

$$U_f = K_i + U_i - K_f \qquad \Longrightarrow \qquad (78.4\,\mathrm{J})(1+\sin\theta) = 64\,\mathrm{J} + 0 - 0$$

This gives us:

$$1 + \sin\theta = \frac{78.4\,\mathrm{J}}{64\,\mathrm{J}} = 1.225 \qquad \Longrightarrow \qquad \sin\theta = 0.225$$

and finally

$$\theta = 13^\circ$$

We do get a sensible answer of θ so we were right in writing down Eq. 6.27. Actually this equation would also have been correct if θ were negative and the pendulum reached its highest point with the string below the horizontal.

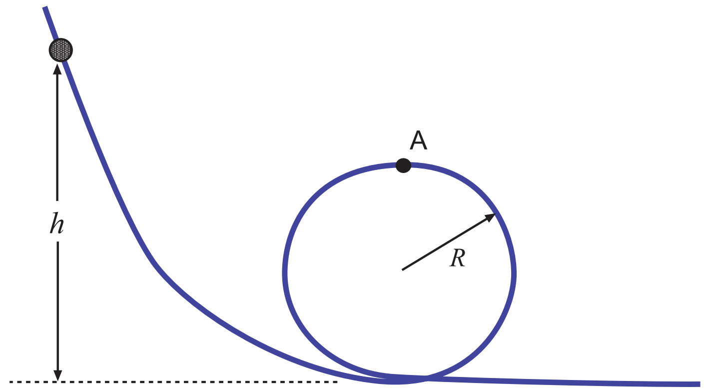

Figure 6.10: Bead slides on track in Example 13.

---

13. A bead slides without friction on a loop–the–loop track (see Fig. 6.10). If the bead is released from a height $h = 3.50R$, what is its speed at point A? How large is the normal force on it if its mass is $5.00\,\mathrm{g}$?

In this problem, there are no friction forces acting on the particle (the bead). Gravity acts on it and gravity is a conservative force. The track will exert a *normal* forces on the bead, but this force does no work. So the total energy of the bead —kinetic plus (gravitational) potential energy— will be conserved.

At the initial position, when the bead is released, the bead has no speed; $K_i = 0$. But it is at a height h above the bottom of the track. If we agree to measure height from the bottom of the track, then the initial potential energy of the bead is

$$U_i = mgh$$

where $m = 5.00\,\mathrm{g}$ is the mass of the bead.

At the final position (A), the bead has *both* kinetic and potential energy. If the bead's speed at A is v, then its final kinetic energy is $K_f = \frac{1}{2}mv^2$. At position A its height is $2R$ (it is a full diameter above the "ground level" of the track) so its potential energy is

$$U_f = mg(2R) = 2mgR \ .$$

The total energy of the bead is conserved: $K_i + U_i = K_f + U_f$. This gives us:

$$0 + mgh = \tfrac{1}{2}mv^2 + 2mgR \ ,$$

where we want to solve for v (the speed at A). The mass m cancels out, giving:

$$gh = \tfrac{1}{2}v^2 = 2gR \qquad \Longrightarrow \qquad \tfrac{1}{2}v^2 = gh - 2gR = g(h - 2R)$$

and then

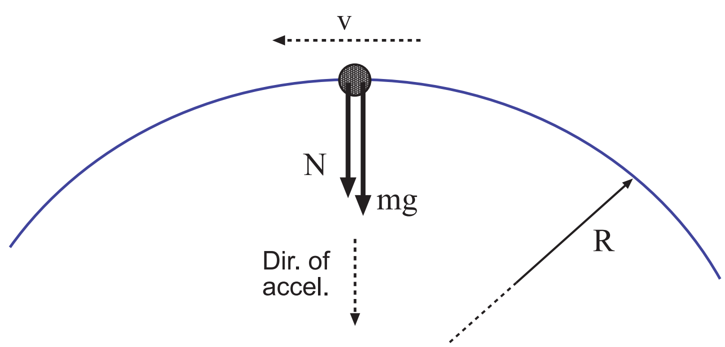

Figure 6.11: Forces acting on the bead when it is at point A (the top of the loop).

$$v^2 = 2g(h - 2R) = 2g(3.50R - 2R) = 2g(1.5R) = 3.0\,gR \tag{6.28}$$

and finally

$$v = \sqrt{3.0\,gR}\ .$$

Since we don't have a numerical value for R, that's as far as we can go.

In the next part of the problem, we think about the *forces* acting on the bead at point A. These are diagrammed in Fig. 6.11. Gravity pulls down on the bead with a force mg. There is also a normal force from the track which I have *drawn* as having a downward component N. But it is possible for the track to be pushing *upward* on the bead; if we get a negative value for N we'll know that the track was pushing *up*.

At the top of the track the bead is moving on a circular path of radius R, with speed v. So it is accelerating *toward the center of the circle*, namely downward. We know that the downward forces must add up to give the centripetal force mv^2/R:

$$mg + N = \frac{mv^2}{R} \qquad \Longrightarrow \qquad n = \frac{mv^2}{R} - mg = m\left(\frac{v^2}{R} - g\right)\ .$$

But we can use our result from Eq. 6.28 to substitute for v^2. This gives:

$$N = m\left(\frac{3.0\,gR}{R} - g\right) = m(2g) = 2mg$$

Plug in the numbers:

$$N = 2(5.00 \times 10^{-3}\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 9.80 \times 10^{-2}\,\mathrm{N}$$

At point A the track is pushing *downward* with a force of $9.80 \times 10^{-2}\,\mathrm{N}$.

---

14. Two children are playing a game in which they try to hit a small box on the floor with a marble fired from a spring –loaded gun that is mounted on the table. The target box is $2.20\,\mathrm{m}$ horizontally from the edge of the table; see Fig. 6.12.

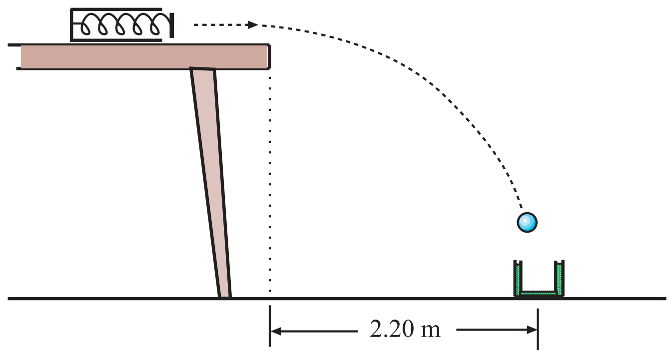

Figure 6.12: Spring propels marble off table and hits (or misses) box on the floor.

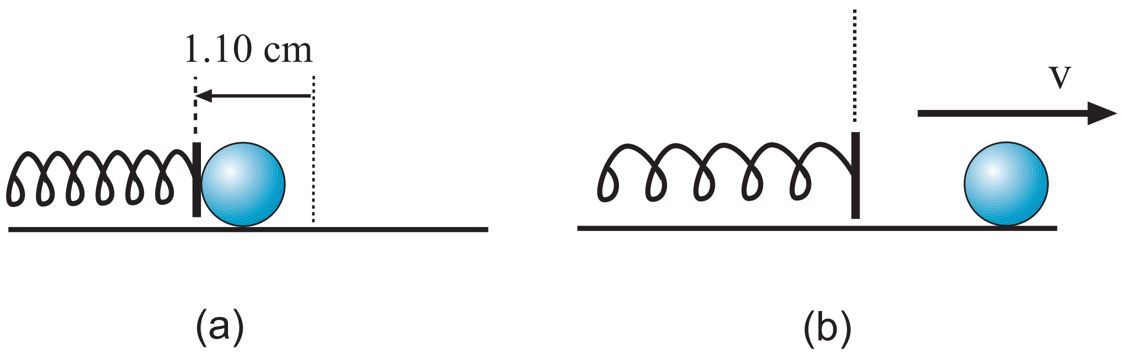

Figure 6.13: Marble propelled by the spring–gun: (a) Spring is compressed, and system has potential energy. (b) Spring is released and system has kinetic energy of the marble.

Bobby compresses the spring $1.10\,\mathrm{cm}$**, but the center of the marble falls** $27.0\,\mathrm{cm}$ **short of the center of the box. How far should Rhoda compress the spring to score a direct hit?**

Let's put the origin of our coordinate system (for the motion of the marble) at the edge of the table. With this choice of coordinates, the object of the game is to insure that the x coordinate of the marble is $2.20\,\mathrm{m}$ when it reaches the level of the floor.

There are many things we are not told in this problem! We don't know the spring constant for the gun, or the mass of the marble. We don't know the height of the table above the floor, either!

When the gun propels the marble, the spring is initially compressed and the marble is motionless (see Fig. 6.13(a).) The energy of the system here is the energy stored in the spring, $E_i = \frac{1}{2}kx^2$, where k is the force constant of the spring and x is the amount of compression of the spring.) When the spring has returned to its natural length and has given the marble a speed v, then the energy of the system is $E_f = \frac{1}{2}mv^2$. If we can neglect friction then mechanical energy is conserved during the firing, so that $E_f = E_i$, which gives us:

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}kx^2 \qquad \Longrightarrow \qquad v = \sqrt{\frac{k}{m}x^2} = x\sqrt{\frac{k}{m}}$$

We will let x and v be the compression and initial marble speed for Bobby's attempt. Then we have:

$$v = (1.10 \times 10^{-2}\,\mathrm{m})\sqrt{\frac{k}{m}} \tag{6.29}$$

The marble's trip from the edge of the table to the floor is (by now!) a fairly simple kinematics problem. If the time the marble spends in the air is t and the height of the table is h then the equation for vertical motion tells us:

$$h = \tfrac{1}{2}gt^2 \ .$$

(This is true because the marble's initial velocity is all *horizontal*. We do know that on Bobby's try, the marble's x coordinate at impact was

$$x = 2.20\,\mathrm{m} - 0.27\,\mathrm{m} = 1.93\,\mathrm{m}$$

and since the horizontal velocity of the marble is v, we have:

$$vt = 1.93\,\mathrm{m} \ . \tag{6.30}$$

There are too many unknowns to solve for k, v, h and t... but let's go on.

Let's suppose that Rhoda compresses the spring by an amount x' so that the marble is given a speed v'. As before, we have

$$\tfrac{1}{2}mv'^2 = \tfrac{1}{2}kx'^2$$

(it's the same spring and marble so that k and m are the same) and this gives:

$$v' = x'\sqrt{\frac{k}{m}} \ . \tag{6.31}$$

Now when Rhoda's shot goes off the table and through the air, then if its time of flight is t' then the equation for vertical motion gives us:

$$h = \tfrac{1}{2}gt'^2 \ .$$

This is the same equation as for t, so that the times of flight for both shots is the same: $t' = t$. Since the x coordinate of the marble for Rhoda's shot will be $x = 2.20\,\mathrm{m}$, the equation for horizontal motion gives us

$$v't = 2.20\,\mathrm{m} \tag{6.32}$$

What can we do with these equations? If we divide Eq. 6.32 by Eq. 6.30 we get:

$$\frac{v't}{vt} = \frac{v'}{v} = \frac{2.20}{1.93} = 1.14$$

If we divide Eq. 6.31 by Eq. 6.29 we get:

$$\frac{v'}{v} = \frac{x'\sqrt{\frac{k}{m}}}{(1.10 \times 10^{-2}\,\mathrm{m})\sqrt{\frac{k}{m}}} = \frac{x'}{(1.10 \times 10^{-2}\,\mathrm{m})} \ .$$

With these last two results, we can solve for x'. Combining these equations gives:

$$1.14 = \frac{x'}{(1.10 \times 10^{-2}\,\mathrm{m})} \qquad \Longrightarrow \qquad x' = 1.14(1.10 \times 10^{-2}\,\mathrm{m}) = 1.25\,\mathrm{cm}$$

Rhoda should compress the spring by 1.25 cm in order to score a direct hit.

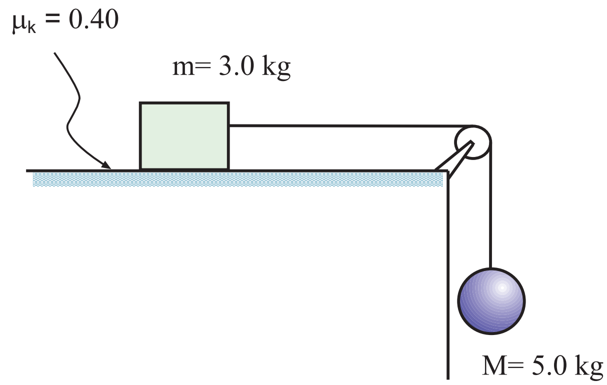

Figure 6.14: Moving masses in Example 15. There is friction between the surface and the 3.0 kg mass.

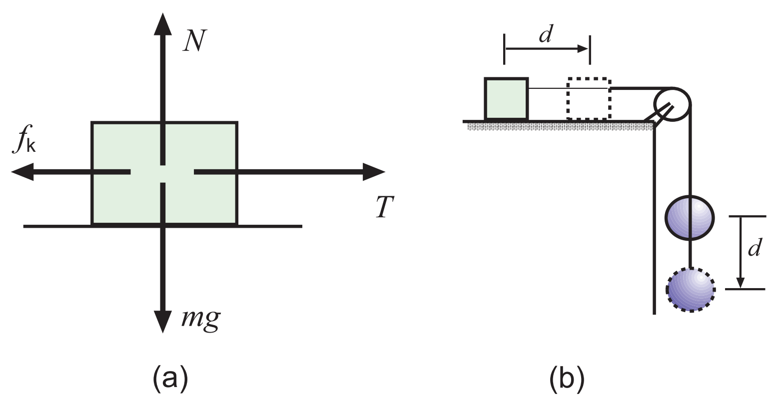

Figure 6.15: (a) Forces acting on m. (b) Masses m and M travel a distance $d = 1.5\,\mathrm{m}$ as they increase in speed from 0 to v.

6.2.6 Work Done by Non–Conservative Forces

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15. The coefficient of friction between the $3.0\,\mathrm{kg}$ **mass and surface in Fig. 6.14 is** 0.40**. The system starts from rest. What is the speed of the** $5.0\,\mathrm{kg}$ **mass when it has fallen** $1.5\,\mathrm{m}$**?**

When the system starts to move, both masses accelerate; because the masses are connected by a string, *they always have the same speed*. The block (m) slides on the rough surface, and friction does work on it. Since its height does not change, its potential energy does not change, but its kinetic energy increases. The hanging mass (M) drops freely; its potential energy decreases but its kinetic energy increases.

We want to use energy principles to work this problem; since there *is* friction present, we need to calculate the work done by friction.

The forces acting on m are shown in Fig. 6.15(a). The normal force N must be equal to mg, so the force of kinetic friction on m has magnitude $\mu_k N = \mu_k mg$. This force *opposes* the motion as m moves a distance $d = 1.5\,\mathrm{m}$, so the work done by friction is

$$W_{\mathrm{fric}} = f_k d \cos\phi = (\mu_k m g)(d)(-1) = -(0.40)(3.0\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(1.5\,\mathrm{m}) = -17.6\,\mathrm{J}\ .$$

Mass m's initial speed is zero, and its final speed is v. So its change in kinetic energy is

$$\Delta K = \tfrac{1}{2}(3.0\,\mathrm{kg})v^2 - 0 = (1.5\,\mathrm{kg})v^2$$

As we noted, m has no change in potential energy during the motion.

Mass M's change in kinetic energy is

$$\Delta K = \tfrac{1}{2}(5.0\,\mathrm{kg})v^2 - 0 = (2.5\,\mathrm{kg})v^2$$

and since it has a *change in height* given by $-d$, its change in (gravitational) potential energy is

$$\Delta U = Mg\Delta y = (5.0\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(-1.5\,\mathrm{m}) = -73.5\,\mathrm{J}$$

Adding up the changes for both masses, the total change in mechanical energy of this system is

$$\begin{aligned} \Delta E &= (1.5\,\mathrm{kg})v^2 + (2.5\,\mathrm{kg})v^2 - 73.5\,\mathrm{J} \\ &= (4.0\,\mathrm{kg})v^2 - 73.5\,\mathrm{J} \end{aligned}$$

Now use $\Delta E = W_{\mathrm{fric}}$ and get:

$$(4.0\,\mathrm{kg})v^2 - 73.5\,\mathrm{J} = -17.6\,\mathrm{J}$$

Solve for v:

$$(4.0\,\mathrm{kg})v^2 = 55.9\,\mathrm{J} \qquad \Longrightarrow \qquad v^2 = \frac{55.9\,\mathrm{J}}{4.0\,\mathrm{kg}} = 14.0\,\tfrac{\mathrm{m}^2}{\mathrm{s}^2}$$

which gives

$$v = 3.74\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

The final speed of the $5.0\,\mathrm{kg}$ mass (in fact of both masses) is $3.74\,\frac{\mathrm{m}}{\mathrm{s}}$.

---

16. A $10.0\,\mathrm{kg}$ block is released from point A in Fig. 6.16. The track is frictionless except for the portion BC, of length $6.00\,\mathrm{m}$. The block travels down the track, hits a spring of force constant $k = 2250\,\mathrm{N/m}$, and compresses it $0.300\,\mathrm{m}$ from its equilibrium position before coming to rest momentarily. Determine the coefficient of kinetic friction between surface BC and block.

We know that we *must* use energy methods to solve this problem, since the path of the sliding mass is curvy.

The forces which act on the mass as it descends and goes on to squish the spring are: gravity, the spring force and the force of kinetic friction as it slides over the rough part. Gravity and the spring force are conservative forces, so we will keep track of them with the potential energy associated with these forces. Friction is a non-conservative force, but in

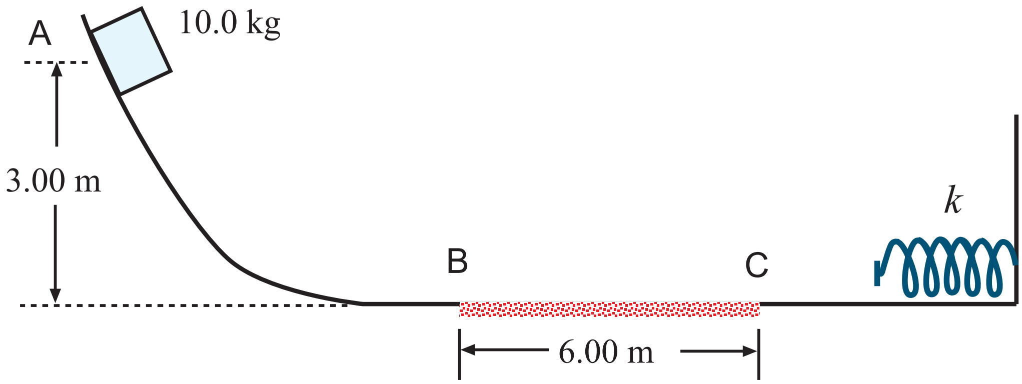

Figure 6.16: System for Example 16

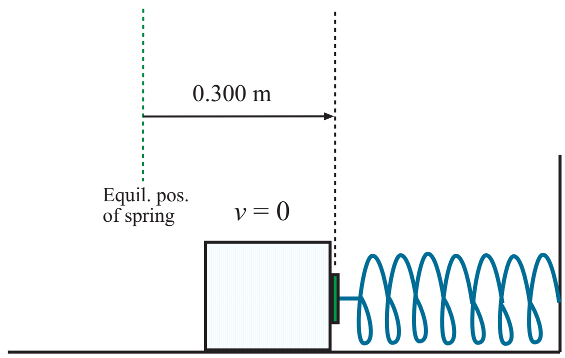

Figure 6.17: After sliding down the slope and going over the rough part, the mass has maximally squished the spring by an amount $x = 0.300\,\mathrm{m}$.

this case we can calculate the work that it does. Then, we can use the energy conservation principle,

$$\Delta K + \Delta U = W_{\mathrm{non-cons}} \tag{6.33}$$

to find the unknown quantity in this problem, namely μ_{k} for the rough surface. We *can* get the answer from this equation because we have numbers for all the quantities except for $W_{\mathrm{non-cons}} = W_{\mathrm{friction}}$ which depends on the coefficient of friction.

The block is *released* at point A so its initial speed (and hence, kinetic energy) is zero: $K_i = 0$. If we measure height upwards from the level part of the track, then the initial potential energy for the mass (all of it *gravitational*) is

$$U_i = mgh = (10.0\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(3.00\,\mathrm{m}) = 2.94 \times 10^2\ \mathrm{J}$$

Next, for the "final" position of the mass, consider the time at which it has maximally compressed the spring and it is (instantaneously) at rest. (This is shown in Fig. 6.17.) We don't need to think about what the mass was doing in between these two points; we don't care about the speed of the mass during its slide.

At this final point, the mass is again at rest, so its kinetic energy is zero: $K_f = 0$. Being at zero height, it has no gravitational potential energy but now since there is a compressed spring, there is stored (potential) energy in the *spring*. This energy is given by:

$$U_{\rm spring} = \tfrac{1}{2}kx^2 = \tfrac{1}{2}(2250\,\tfrac{\rm N}{\rm m})(0.300\,{\rm m})^2 = 1.01 \times 10^2\,{\rm J}$$

so the final potential energy of the system is $U_f = 1.01 \times 10^2$ J.

The total mechanical energy of the system changes because there is a non–conservative force (friction) which does work. As the mass (m) slides over the rough part, the vertical forces are gravity (mg, downward) and the upward normal force of the surface, N. As there is no vertical motion, $N = mg$. The magnitude of the force of kinetic friction is

$$f_{\rm k} = \mu_{\rm k}N = \mu_{\rm k}mg = \mu_{\rm k}(10.0\,{\rm kg})(9.80\,\tfrac{\rm m}{\rm s^2}) = \mu_{\rm k}(98.0\,{\rm N})$$

As the block moves 6.00 m this force points *opposite* (180° from) the direction of motion. So the work done by friction is

$$W_{\rm fric} = f_{\rm k}d\cos\phi = \mu_{\rm k}(98.0\,{\rm N})(6.00\,{\rm m})\cos 180^\circ = -\mu_{\rm k}(5.88 \times 10^2\,{\rm J})$$

We now have everything we need to substitute into the energy balance condition, Eq. 6.33. We get:

$$(0-0) + (1.01 \times 10^2\,{\rm J} - 2.94 \times 10^2\,{\rm J}) = -\mu_{\rm k}(5.88 \times 10^2\,{\rm J})\ .$$

The physics is done. We do algebra to solve for $\mu_{\rm k}$:

$$-1.93 \times 10^2\,{\rm J} = -\mu_{\rm k}(5.88 \times 10^2\,{\rm J}) \qquad \Longrightarrow \qquad \mu_{\rm k} = 0.328$$

The coefficient of kinetic friction for the rough surface and block is 0.328.

6.2.7 Relationship Between Conservative Forces and Potential Energy (Optional?)

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17. A potential energy function for a two–dimensional force is of the form $U = 3x^3y - 7x$. Find the force that acts at the point (x, y).

(We presume that the expression for U will give us U in *joules* when x and y are in meters!)

We use Eq. 6.26 to get F_x and F_y:

$$\begin{aligned} F_x &= -\frac{\partial U}{\partial x} \\ &= -\frac{\partial}{\partial x}(3x^3y - 7x) \\ &= -(9x^2y - 7) = -9x^2y + 7 \end{aligned}$$

and:

$$\begin{aligned} F_y &= -\frac{\partial U}{\partial y} \\ &= -\frac{\partial}{\partial y}(3x^3y - 7x) \\ &= -(3x^3) = -3x^3 \end{aligned}$$

Then in unit vector form, **F** is:

$$\mathbf{F} = (-9x^2y + 7)\mathbf{i} + (-3x^3)\mathbf{j}$$

where, if x and y are in meters then **F** is in newtons. Got to watch those units!

Chapter 7

Linear Momentum and Collisions

7.1 The Important Stuff

7.1.1 Linear Momentum

The **linear momentum** of a particle with mass m moving with velocity $\mathbf{v}$ is defined as

$$\mathbf{p} = m\mathbf{v} \tag{7.1}$$

Linear momentum is a *vector*. When giving the linear momentum of a particle you *must* specify its *magnitude* and *direction*. We can see from the definition that its units must be $\frac{\mathrm{kg \cdot m}}{\mathrm{s}}$. Oddly enough, this combination of SI units does not have a commonly–used named so we leave it as $\frac{\mathrm{kg \cdot m}}{\mathrm{s}}$!

The momentum of a particle is related to the net force on that particle in a simple way; since the mass of a particle remains constant, if we take the time derivative of a particle's momentum we find

$$\frac{d\mathbf{p}}{dt} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a} = \mathbf{F}_{\mathrm{net}}$$

so that

$$\mathbf{F}_{\mathrm{net}} = \frac{d\mathbf{p}}{dt} \tag{7.2}$$

7.1.2 Impulse, Average Force

When a particle moves freely then interacts with another system for a (brief) period and then moves freely again, it has a definite change in momentum; we define this change as the **impulse I** of the interaction forces:

$$\mathbf{I} = \mathbf{p}_f - \mathbf{p}_i = \Delta\mathbf{p}$$

Impulse is a vector and has the same units as momentum.

When we integrate Eq. 7.2 we can show:

$$\mathbf{I} = \int_{t_i}^{t_f} \mathbf{F}\, dt = \Delta\mathbf{p}$$

We can now define the **average force** which acts on a particle during a time interval Δt. It is:

$$\overline{\mathbf{F}} = \frac{\Delta p}{\Delta t} = \frac{\mathbf{I}}{\Delta t}$$

The value of the average force depends on the time interval chosen.

7.1.3 Conservation of Linear Momentum

Linear momentum is a useful quantity for cases where we have a few particles (objects) which interact with *each other* but not with the rest of the world. Such a system is called an **isolated system**.

We often have reason to study systems where a few particles interact with each other very briefly, with forces that are strong compared to the other forces in the world that they may experience. In those situations, and for that brief period of time, we can treat the particles as if they *were* isolated.

We can show that when two particles interact *only* with each other (i.e. they are isolated) then their total momentum remains constant:

$$\mathbf{p}_{1i} + \mathbf{p}_{2i} = \mathbf{p}_{1f} + \mathbf{p}_{2f} \tag{7.3}$$

or, in terms of the masses and velocities,

$$m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = m_1\mathbf{v}_{1f} + m_2\mathbf{v}_{2f} \tag{7.4}$$

Or, abbreviating $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{P}$ (total momentum), this is: $\mathbf{P}_i = \mathbf{P}_f$.

It is important to understand that Eq. 7.3 is a *vector* equation; it tells us that the total x component of the momentum is conserved, *and* the total y component of the momentum is conserved.

7.1.4 Collisions

When we talk about a **collision** in physics (between two particles, say) we mean that two particles are moving freely through space until they get close to one another; then, for a short period of time they exert strong forces on each other until they move apart and are again moving freely.

For such an event, the two particles have well-defined momenta $\mathbf{p}_{1i}$ and $\mathbf{p}_{2i}$ before the collision event and $\mathbf{p}_{1f}$ and $\mathbf{p}_{2f}$ afterwards. But the sum of the momenta before and after the collision is conserved, as written in Eq. 7.3.

While the *total momentum* is conserved for a system of isolated colliding particles, the *mechanical energy* may or may not be conserved. If the mechanical energy (usually meaning the total kinetic energy) is the same before and after a collision, we say that the collision is **elastic**. Otherwise we say the collision is **inelastic**.

If two objects collide, stick together, and move off as a combined mass, we call this a **perfectly inelastic** collision. One can show that in such a collision more kinetic energy is lost than if the objects were to bounce off one another and move off separately.

When two particles undergo an *elastic* collision then we also know that

$$\tfrac{1}{2}m_1v_{1i}^2 + \tfrac{1}{2}m_2v_{2i}^2 = \tfrac{1}{2}m_1v_{1f}^2 + \tfrac{1}{2}m_2v_{2f}^2 \ .$$

In the special case of a one-dimensional elastic collision between masses m_1 and m_2 we can relate the final velocities to the initial velocities. The result is

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right) v_{2i} \qquad (7.5)$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_{2i} \qquad (7.6)$$

This result can be useful in solving a problem where such a collision occurs, but it is *not* a fundamental equation. So don't memorize it.

7.1.5 The Center of Mass

For a system of particles (that is, lots of 'em) there is a special point in space known as the **center of mass** which is of great importance in describing the overall motion of the system. This point is a weighted average of the positions of all the mass points.

If the particles in the system have masses m_1, m_2, $\ldots m_N$, with total mass

$$\sum_i^N m_i = m_1 + m_2 + \cdots + m_N \equiv M$$

and respective positions $\mathbf{r_1}$, $\mathbf{r_2}$, $\ldots$,$\mathbf{r_N}$, then the center of mass $\mathbf{r}_{\rm CM}$ is:

$$\mathbf{r}_{\rm CM} = \frac{1}{M}\sum_i^N m_i\mathbf{r}_i \qquad (7.7)$$

which means that the x, y and z coordinates of the center of mass are

$$x_{\rm CM} = \frac{1}{M}\sum_i^N m_ix_i \qquad y_{\rm CM} = \frac{1}{M}\sum_i^N m_iy_i \qquad z_{\rm CM} = \frac{1}{M}\sum_i^N m_iz_i \qquad (7.8)$$

For an extended object (i.e. a continuous distribution of mass) the definition of $\mathbf{r}_{\rm CM}$ is given by an *integral* over the mass elements of the object:

$$\mathbf{r}_{\rm CM} = \frac{1}{M}\int \mathbf{r}\, dm \qquad (7.9)$$

which means that the x, y and z coordinates of the center of mass are now:

$$x_{\rm CM} = \frac{1}{M}\int x\, dm \qquad y_{\rm CM} = \frac{1}{M}\int y\, dm \qquad z_{\rm CM} = \frac{1}{M}\int z\, dm \qquad (7.10)$$

When the particles of a system are in motion then in general their center of mass is also in motion. The velocity of the center of mass is a similar weighted average of the individual velocities:

$$\mathbf{v}_{\rm CM} = \frac{d\mathbf{r}_{\rm CM}}{dt} = \frac{1}{M}\sum_i^N m_i \mathbf{v}_i \tag{7.11}$$

In general the center of mass will accelerate; its acceleration is given by

$$\mathbf{a}_{\rm CM} = \frac{d\mathbf{v}_{\rm CM}}{dt} = \frac{1}{M}\sum_i^N m_i \mathbf{a}_i \tag{7.12}$$

If $\mathbf{P}$ is the total momentum of the system and M is the total mass of the system, then the motion of the center of mass is related to $\mathbf{P}$ by:

$$\mathbf{v}_{\rm CM} = \frac{\mathbf{P}}{M} \qquad \text{and} \qquad \mathbf{a}_{\rm CM} = \frac{1}{M}\frac{d\mathbf{P}}{dt}$$

7.1.6 The Motion of a System of Particles

A system of *many* particles (or an extended object) in general has a motion for which the description is very complicated, but it is possible to make a simple statement about the motion of its center of mass. Each of the particles in the system may feel forces from the other particles in the system, but it may also experience a net force from the (external) environment; we will denote this force by $\mathbf{F}_{\rm ext}$. We find that when we add up all the *external* forces acting on all the particles in a system, it gives the acceleration of the *center of mass* according to:

$$\sum_i^N \mathbf{F}_{{\rm ext},\,i} = M\mathbf{a}_{\rm CM} = \frac{d\mathbf{P}}{dt} \tag{7.13}$$

Here, M is the total mass of the system; $\mathbf{F}_{{\rm ext},\,i}$ is the external force acting on particle i.

In words, we can express this result in the following way: For a system of particles, the center of mass moves as if it were a *single* particle of mass M moving under the influence of the sum of the external forces.

7.2 Worked Examples

7.2.1 Linear Momentum

1. A $3.00\,\mathrm{kg}$ particle has a velocity of $(3.0\mathbf{i} - 4.0\mathbf{j})\,\frac{\mathrm{m}}{\mathrm{s}}$. Find its x and y components of momentum and the magnitude of its total momentum.

Using the definition of momentum and the given values of m and $\mathbf{v}$ we have:

$$\mathbf{p} = m\mathbf{v} = (3.00\,\mathrm{kg})(3.0\mathbf{i} - 4.0\mathbf{j})\,\tfrac{\mathrm{m}}{\mathrm{s}} = (9.0\mathbf{i} - 12.\mathbf{j})\,\tfrac{\mathrm{kg\cdot m}}{\mathrm{s}}$$

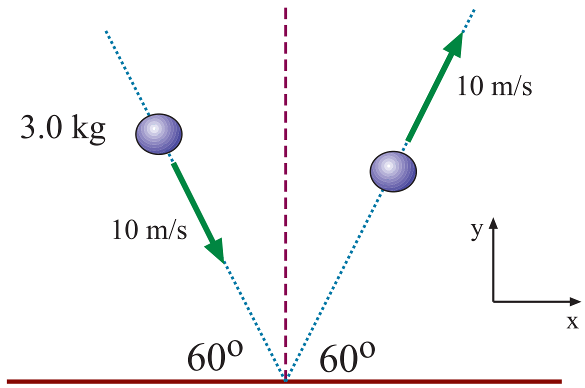

Figure 7.1: Ball bounces off wall in Example 3.

So the particle has momentum components

$$p_x = +9.0\,\frac{\text{kg}\cdot\text{m}}{\text{s}} \qquad \text{and} \qquad p_y = -12.\,\frac{\text{kg}\cdot\text{m}}{\text{s}}\ .$$

The magnitude of its momentum is

$$p = \sqrt{p_x^2 + p_y^2} = \sqrt{(9.0)^2 + (-12.)^2}\,\frac{\text{kg}\cdot\text{m}}{\text{s}} = 15.\,\frac{\text{kg}\cdot\text{m}}{\text{s}}$$

7.2.2 Impulse, Average Force

---

2. A child bounces a superball on the sidewalk. The linear impulse delivered by the sidewalk is $2.00\,\text{N}\cdot\text{s}$ during the $\frac{1}{800}\,\text{s}$ of contact. What is the magnitude of the average force exerted on the ball by the sidewalk.

The magnitude of the change in momentum of (impulse delivered to) the ball is $|\Delta\mathbf{p}| = |\mathbf{I}| = 2.00\,\text{N}\cdot\text{s}$. (The *direction* of the impulse is upward, since the initial momentum of the ball was downward and the final momentum is upward.)

Since the time over which the force was acting was

$$\Delta t = \frac{1}{800}\,\text{s} = 1.25 \times 10^{-3}\,\text{s}$$

then from the definition of average force we get:

$$|\overline{\mathbf{F}}| = \frac{|\mathbf{I}|}{\Delta t} = \frac{2.00\,\text{N}\cdot\text{s}}{1.25 \times 10^{-3}\,\text{s}} = 1.60 \times 10^{3}\,\text{N}$$

---

3. A $3.0\,\text{kg}$ steel ball strikes a wall with a speed of $10\,\frac{\text{m}}{\text{s}}$ at an angle of 60° with the surface. It bounces off with the same speed and angle, as shown in Fig. 7.1. If the ball is in contact with the wall for $0.20\,\text{s}$, what is the average force exerted on the wall by the ball?

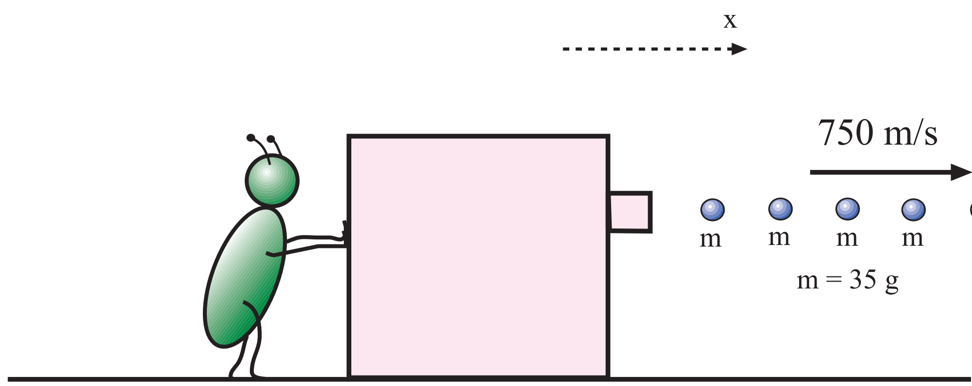

Figure 7.2: *Simplified* picture of a machine gun spewing out bullets. An external force is necessary to hold the gun in place!

The average force is defined as $\overline{\mathbf{F}} = \Delta\mathbf{p}/\Delta t$, so first find the change in momentum of the ball. Since the ball has the same speed before and after bouncing from the wall, it is clear that its x velocity (see the coordinate system in Fig. 7.1) stays the same and so the x momentum stays the same. But the y momentum *does* change. The initial y velocity is

$$v_{iy} = -(10\,\tfrac{\mathrm{m}}{\mathrm{s}})\sin 60^\circ = -8.7\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

and the final y velocity is

$$v_{fy} = +(10\,\tfrac{\mathrm{m}}{\mathrm{s}})\sin 60^\circ = +8.7\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

so the change in y momentum is

$$\Delta p_y = mv_{fy} - mv_{iy} = m(v_{fy} - v_{iy}) = (3.0\,\mathrm{kg})(8.7\,\tfrac{\mathrm{m}}{\mathrm{s}} - (-8.7\,\tfrac{\mathrm{m}}{\mathrm{s}})) = 52\,\tfrac{\mathrm{kg\cdot m}}{\mathrm{s}}$$

The average y force on the ball is

$$\overline{F}_y = \frac{\Delta p_y}{\Delta t} = \frac{I_y}{\Delta t} = \frac{(52\,\frac{\mathrm{kg\cdot m}}{\mathrm{s}})}{(0.20\,\mathrm{s})} = 2.6 \times 10^2\,\mathrm{N}$$

Since $\overline{\mathbf{F}}$ has no x component, the average force has magnitude $2.6 \times 10^2\,\mathrm{N}$ and points in the y direction (away from the wall).

---

4. A machine gun fires $35.0\,\mathrm{g}$ bullets at a speed of $750.0\,\frac{\mathrm{m}}{\mathrm{s}}$. If the gun can fire $200\,\mathrm{bullets/\,min}$, what is the average force the shooter must exert to keep the gun from moving?

Whoa! Lots of things happening here. Let's draw a diagram and try to sort things out. Such a picture is given in Fig. 7.2.

The gun interacts with the bullets; it exerts a brief, strong force on each of the bullets which in turn exerts an "equal and opposite" force on the gun. The gun's force changes the bullet's momentum from *zero* (as they are initially at rest) to the final value of

$$p_f = mv = (0.0350\,\mathrm{kg})(750\,\tfrac{\mathrm{m}}{\mathrm{s}}) = 26.2\,\tfrac{\mathrm{kg\cdot m}}{\mathrm{s}}\ .$$

so this is also the *change* in momentum for each bullet.

Now, since 200 bullets are fired every minute (60 s), we should count the interaction time as the time to fire *one* bullet,

$$\Delta t = \frac{60\,\mathrm{s}}{200} = 0.30\,\mathrm{s}$$

because every 0.30 s, a firing occurs again, and the *average* force that we compute will be valid for a length of time for which many bullets are fired. So the average force of the gun on the bullets is

$$\overline{F}_x = \frac{\Delta p_x}{\Delta t} = \frac{26.2\,\frac{\mathrm{kg\cdot m}}{\mathrm{s}}}{0.30\,\mathrm{s}} = 87.5\,\mathrm{N}$$

From Newton's Third Law, there must an average backwards force *of the bullets on the gun* of magnitude 87.5 N. If there were no other forces acting on the gun, it would accelerate backward! To keep the gun in place, the shooter (or the gun's mechanical support) must exert a force of 87.5 N in the forward direction.

We can also work with the numbers as follows: In one minute, 200 bullets were fired, and a *total* momentum of

$$P = (200)(26.2\,\tfrac{\mathrm{kg\cdot m}}{\mathrm{s}}) = 5.24 \times 10^3\,\tfrac{\mathrm{kg\cdot m}}{\mathrm{s}}$$

was imparted to them. So during this time period (60 seconds!) the average force on the whole set of bullets was

$$\overline{F}_x = \frac{\Delta P}{\Delta t} = \frac{5.24 \times 10^3\,\frac{\mathrm{kg\cdot m}}{\mathrm{s}}}{60.0\,\mathrm{s}} = 87.5\,\mathrm{N}\ .$$

As before, this is also the average backwards force of the *bullets on the gun* and the force required to keep the gun in place.

7.2.3 Collisions

---

5. A $10.0\,\mathrm{g}$ **bullet is stopped in a block of wood (**$m = 5.00\,\mathrm{kg}$**). The speed of the bullet–plus–wood combination immediately after the collision is** $0.600\,\frac{\mathrm{m}}{\mathrm{s}}$**. What was the original speed of the bullet?**

A picture of the collision just before and after the bullet (quickly) embeds itself in the wood is given in Fig. 7.3. The bullet has some initial speed v_0 (we don't know what it is.)

The collision (and embedding of the bullet) takes place very rapidly; for that brief time the bullet and block essentially form an isolated system because any external forces (say, from friction from the surface) will be of no importance compared to the *enormous* forces between the bullet and the block. So the total momentum of the system will be conserved; it is the same before and after the collision.

In this problem there is only motion along the x axis, so we only need the condition that the total x momentum (P_x) is conserved.

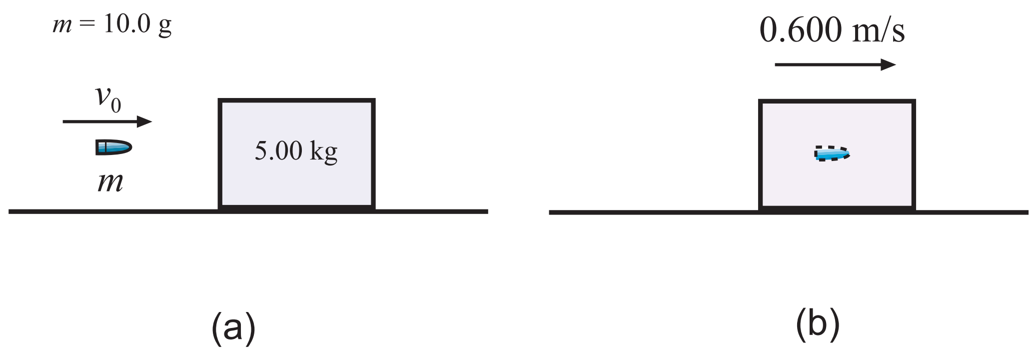

Figure 7.3: Collision in Example 5. (a) Just before the collision. (b) Just after.

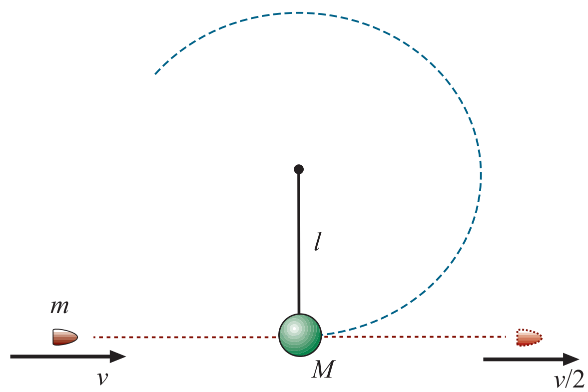

Figure 7.4: Bullet passes through a pendulum bob and emerges with half its original speed; the bob barely swings through a complete circle!

Just before the collision, only the bullet (with mass m) is in motion and its x velocity is v_0. So the initial momentum is

$$P_{i,x} = mv_0 = (10.0 \times 10^{-3}\,\mathrm{kg})v_0$$

Just after the collision, the bullet–block combination, with its mass of $M + m$ has an x velocity of $0.600\,\frac{\mathrm{m}}{\mathrm{s}}$. So the final momentum is

$$P_{f,x} = (M + m)v = (5.00\,\mathrm{kg} + 10.0 \times 10^{-3}\,\mathrm{kg})(0.600\,\tfrac{\mathrm{m}}{\mathrm{s}}) = 3.01\,\tfrac{\mathrm{kg\cdot m}}{\mathrm{s}}$$

Since $P_{i,x} = P_{f,x}$, we get:

$$(10.0 \times 10^{-3}\,\mathrm{kg})v_0 = 3.01\,\tfrac{\mathrm{kg\cdot m}}{\mathrm{s}} \qquad \Longrightarrow \qquad v_0 = 301\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

The initial speed of the bullet was $301\,\frac{\mathrm{m}}{\mathrm{s}}$.

---

6. As shown in Fig. 7.4, a bullet of mass m and speed v passes completely through a pendulum bob of mass M. The bullet emerges with a speed $v/2$. The pendulum bob is suspended by a stiff rod of length ℓ and negligible mass. What

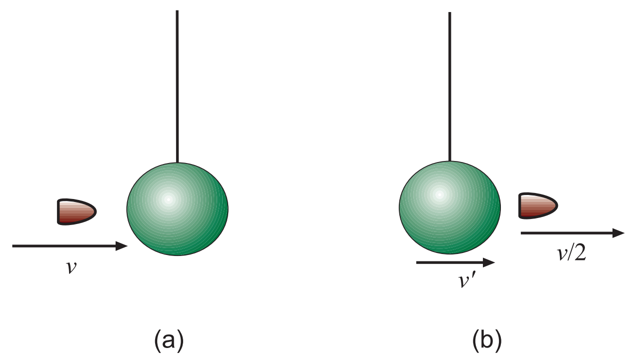

Figure 7.5: Collision of the bullet with the pendulum bob. (a) Just before the collision. (b) Just after. The bullet has gone through the bob, which has acquired a velocity v'.

is the minimum value of v such that the pendulum bob will barely swing through a complete vertical circle?

Whoa! There's a hell of a lot of things going on in this problem. Let's try to sort them out.

We break things down into a sequence of events: First, the bullet has a very rapid, very strong interaction with the pendulum bob, where is quickly passes through, imparting a velocity to the bob which at first will have a horizontal motion. Secondly, the bob swings upward and, as we are told, will get up to the top of the vertical circle.

We show the collision in Fig. 7.5. In this rapid interaction there are no net external forces acting on the system that we need to worry about. So its total momentum will be conserved. The total horizontal momentum before the collision is

$$P_{i,x} = mv + 0 = mv$$

If after the collision the bob has velocity v', then the total momentum is

$$P_{f,x} = m\left(\frac{v}{2}\right) + Mv'$$

Conservation of momentum, $P_{i,x} = P_{f,x}$ gives

$$mv = m\left(\frac{v}{2}\right) + Mv' \qquad \Longrightarrow \qquad Mv' = m\left(\frac{v}{2}\right)$$

and so:

$$v' = \frac{1}{M}\frac{mv}{2} = \frac{mv}{2M} \tag{7.14}$$

Now consider the trip of the pendulum bob up to the top of the circle (it *must* get to the top, by assumption). There are no friction–type forces acting on the system as M moves, so *mechanical energy is conserved*.

If we measure height from the bottom of the swing, then the initial potential energy is zero while the initial kinetic energy is

$$K_i = \tfrac{1}{2}M\,(v')^2 \ .$$

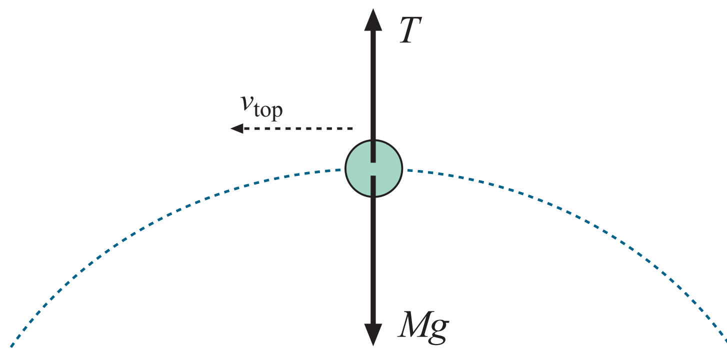

Figure 7.6: Forces acting on pendulum bob M at the top of the swing.

Now suppose at the top of the swing mass M has speed $v_{\rm top}$. Its height is 2ℓ and its potential energy is $Mg(2\ell)$ so that its final energy is

$$E_f = \tfrac{1}{2}Mv_{\rm top}^2 + 2Mg\ell$$

so that conservation of energy gives:

$$\tfrac{1}{2}M\,(v')^2 = \tfrac{1}{2}Mv_{\rm top}^2 + 2Mg\ell \tag{7.15}$$

What do we know about $v_{\rm top}$? A drawing of the forces acting on M at the top of the swing is shown in Fig. 7.6. Gravity pulls down with a force Mg. There may be a force from the suspending rod; here, I've happened to draw it pointing upward. *Can* this force point upward? Yes it can... we need to read the problem carefully. It said the bob was suspended by a *stiff rod* and such an object can exert a force (still called the tension T) *inward or outward* along its length. (A string can only pull inward.) The bob is moving on a circular path with (instantaneous) speed $v_{\rm top}$ so the net force on it points *downward* and has magnitude $Mv_{\rm top}^2/\ell$:

$$Mg - T = \frac{Mv_{\rm top}^2}{\ell} \ .$$

Since T can be positive or negative, $v_{\rm top}$ can take on any value. It could be zero. What condition are we looking for which corresponds to the smallest value of the bullet speed v?

We note that as v gets bigger, so does v' (the bob's initial speed). As v' increases, so does $v_{\rm top}$, as we see from conservation of energy. But it is entirely possible for $v_{\rm top}$ to be zero, and *that* will give the smallest possible value of v. That would correspond to the case where M picked up enough speed to *just barely* make it to the top of the swing. (And when the bob goes past the top point then gravity moves it along through the full swing.)

So with $v_{\rm top} = 0$ then Eq. 7.15 gives us

$$\tfrac{1}{2}M\,(v')^2 = 2Mg\ell \qquad \Longrightarrow \qquad v'^2 = 4g\ell$$

and:

$$v' = \sqrt{4g\ell}$$

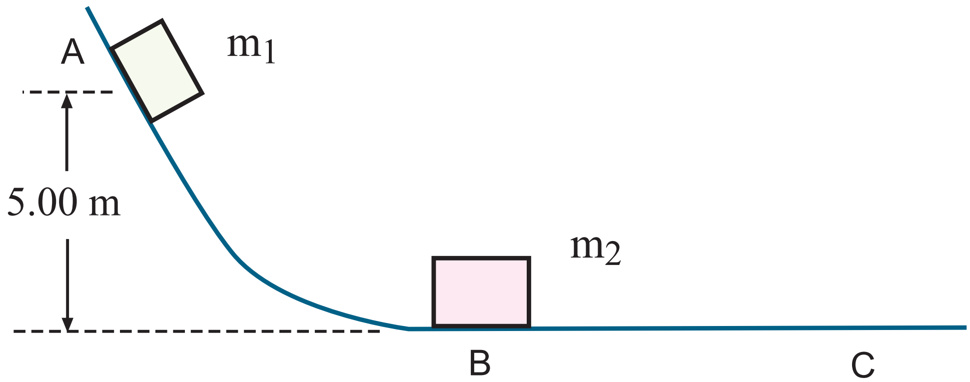

Figure 7.7: Frictionless track, for Example 7. Mass m_1 is released and collides elastically with mass m_2.

and putting this result back into Eq. 7.15, we have

$$\sqrt{4g\ell} = \frac{1}{M}\frac{mv}{2} = \frac{mv}{2M} \ .$$

Finally, solve for v:

$$v = \frac{2M}{m}\sqrt{4g\ell} = \frac{4M\sqrt{g\ell}}{m}$$

The minimum value of v required to do the job is $v = 4M\sqrt{g\ell}/m$.

---

7. Consider a frictionless track ABC as shown in Fig. 7.7. A block of mass $m_1 = 5.001\,\mathrm{kg}$ is released from A. It makes a head–on elastic collision with a block of mass $m_2 = 10.0\,\mathrm{kg}$ at B, initially at rest. Calculate the maximum height to which m_1 rises after the collision.

Whoa! What is this problem talking about??

We release mass m_1; it slides down to the slope, picking up speed, until it reaches B. At B it makes a collision with mass m_2, and we are told it is an *elastic* collison. The last sentence in the problem implies that in this collision m_1 will reverse its direction of motion and head *back up* the slope to some maximum height. We would also guess that m_2 will be given a forward velocity.

This sequence is shown in Fig. 7.8. First we think about the instant of time just before the collision. Mass m_1 has velocity v_{1i} and mass m_2 is still stationary. How can we find v_{1i}? We can use the fact that *energy is conserved* as m_1 slides down the smooth (frictionless) slope. At the top of the slope m_1 had some potential energy, $U = m_1gh$ (with $h = 5.00\,\mathrm{m}$) which is changed to kinetic energy, $K = \frac{1}{2}m_1v_{1i}^2$ when it reaches the bottom. Conservation of energy gives us:

$$m_1gh = \tfrac{1}{2}mv_{1i}^2 \qquad \Longrightarrow \qquad v_{1i}^2 = 2gh = 2(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(5.00\,\mathrm{m}) = 98.0\,\tfrac{\mathrm{m}^2}{\mathrm{s}^2}$$

so that

$$v_{1i} = +9.90\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

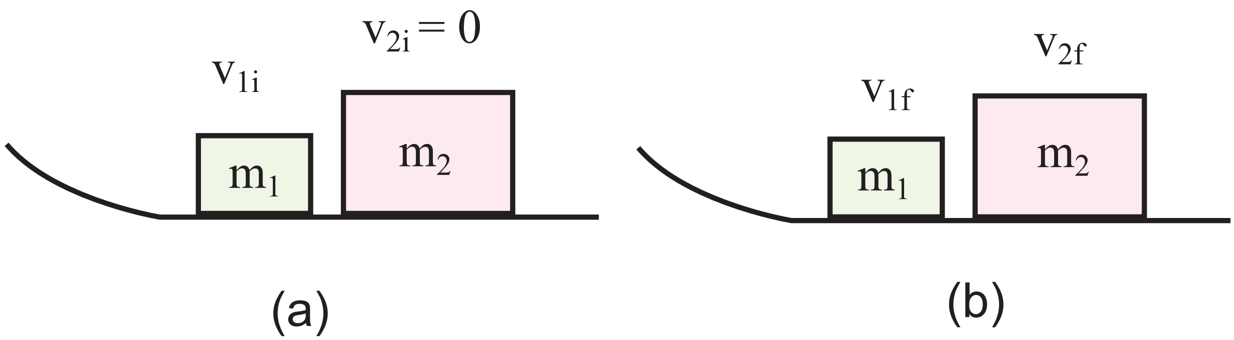

Figure 7.8: (a) Just before the collision; m_1 has acquired a velocity of v_{1i} from sliding down the slope. (b) Just after the collision; mass m_1 has *velocity* v_{1f} and mass m_2 has *velocity* v_{2f}.

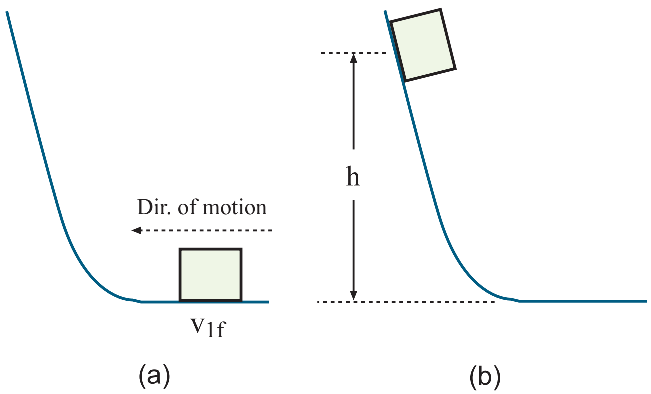

Figure 7.9: (a) After the collision, m_1 goes to the left and will move back up the slope. (b) After moving back up the slope, m_1 reaches some maximum height h.

We chose the *positive* value here since m_1 is obviously moving *forward* at the bottom of the slope. So m_1's velocity just before striking m_2 is $+9.90\,\frac{\mathrm{m}}{\mathrm{s}}$.

Now m_1 makes an elastic (one–dimensional) collision with m_2. What are the final velocities of the masses? For this we can use the result given in Eqs. 7.5 and 7.6, using $v_{2i} = 0$. We get:

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} = \left(\frac{5.001\,\mathrm{kg} - 10.0\,\mathrm{kg}}{5.001\,\mathrm{kg} + 10.0\,\mathrm{kg}}\right)(+9.90\,\tfrac{\mathrm{m}}{\mathrm{s}}) = -3.30\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} = \left(\frac{2(5.001\,\mathrm{kg})}{5.001\,\mathrm{kg} + 10.0\,\mathrm{kg}}\right)(+9.90\,\tfrac{\mathrm{m}}{\mathrm{s}}) = +6.60\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

So after the collision, m_1 has a *velocity* of $-3.30\,\frac{\mathrm{m}}{\mathrm{s}}$; that is, it has *speed* $3.30\,\frac{\mathrm{m}}{\mathrm{s}}$ and it is now moving *to the left*. After the collision, m_2 has velocity $+6.60\,\frac{\mathrm{m}}{\mathrm{s}}$, so that it is moving to the right with speed $6.60\,\frac{\mathrm{m}}{\mathrm{s}}$.

Since m_1 is now moving to the left, it will head back up the slope. (See Fig. 7.9.) How high will it go? Once again, we can use energy conservation to give us the answer. For the trip back up the slope, the initial energy (all kinetic) is

$$E_i = K_i = \tfrac{1}{2}m(3.30\,\tfrac{\mathrm{m}}{\mathrm{s}})^2$$

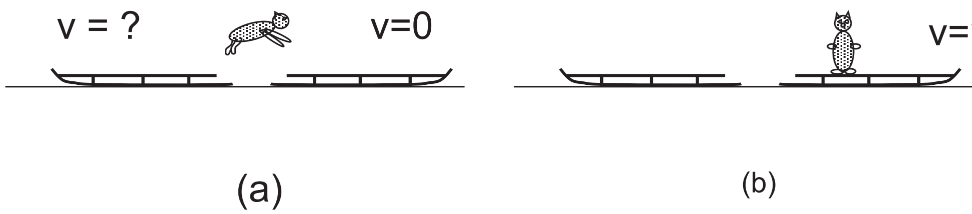

Figure 7.10: (a) Cat leaps from left sled to the right sled. What is new velocity of left sled? (b) Cat has landed on the right sled. What is its velocity now?

and when it reaches maximum height (h) its speed is zero, so its energy is the potential energy,

$$E_f = U_f = mgh$$

Conservation of energy, $E_i = E_f$ gives us:

$$\tfrac{1}{2}m(3.30\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 = mgh \qquad \Longrightarrow \qquad h = \frac{(3.30\,\frac{\mathrm{m}}{\mathrm{s}})^2}{2g} = 0.556\,\mathrm{m}$$

Mass m_1 will travel back up the slope to a height of 0.556 m.

---

8. Two $22.7\,\mathrm{kg}$ ice sleds are placed a short distance apart, one directly behind the other, as shown in Fig. 7.10 (a). A $3.63\,\mathrm{kg}$ cat, standing on one sled, jumps across to the other and immediately back to the first. Both jumps are made at a speed on $3.05\,\frac{\mathrm{m}}{\mathrm{s}}$ relative to the ice. Find the final speeds of the two sleds.

We will let the x axis point to the right. In the initial picture (not shown) the cat is sitting on the left sled and both are motionless. Taking our system of interacting "particles" to be the cat and the left sled, the initial momentum of the system is $P = 0$.

After the cat has made its first jump, the velocity of this sled will be $v_{L,x}$, and the (final) total momentum of the system will be

$$P_f = (22.7\,\mathrm{kg})v_{L,x} + (3.63\,\mathrm{kg})(+3.05\,\tfrac{\mathrm{m}}{\mathrm{s}})$$

Note, we are using velocities with respect to the ice, and that is how we were given the velocity of the cat. Now as there are no net external forces, the momentum of this system is conserved. This gives us:

$$0 = (22.7\,\mathrm{kg})v_{L,x} + (3.63\,\mathrm{kg})(+3.05\,\tfrac{\mathrm{m}}{\mathrm{s}})$$

with which we easily solve for $v_{L,x}$:

$$v_{L,x} = -\frac{(3.63\,\mathrm{kg})(+3.05\,\frac{\mathrm{m}}{\mathrm{s}})}{(22.7\,\mathrm{kg})} = -0.488\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

so that the left sled moves at a speed of $0.488\,\frac{\mathrm{m}}{\mathrm{s}}$ *to the left* after the cat's first jump.

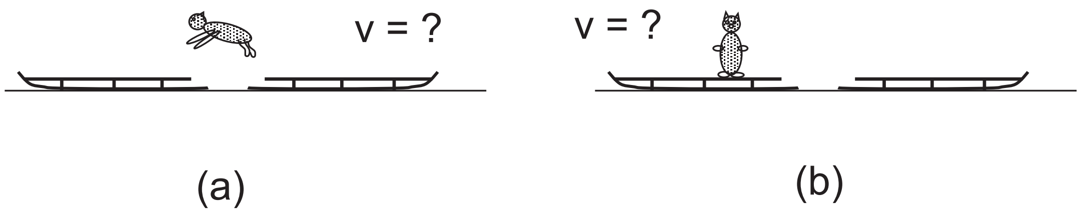

Figure 7.11: (a) Cat leaps from right sled back to left sled. What is new velocity of right sled? (b) Cat has landed back on left sled. What is *its* velocity now? Note that this is basically the mirror image of the previous figure, so I only had to draw it once! Hah!

The cat lands on the right sled and after landing it moves with the same velocity as that sled; the collision here is completely inelastic. For this part of the problem, the system of "interacting particles" we consider is *the cat and the right sled*. (The left sled does not interact with *this* system.) The initial momentum of this system is just that of the cat,

$$P_i = (3.63\,\mathrm{kg})(+3.05\,\tfrac{\mathrm{m}}{\mathrm{s}}) = 11.1\,\tfrac{\mathrm{kg\cdot m}}{\mathrm{s}}$$

If the final velocity of both cat and sled is $v_{R,x}$ then the final momentum is

$$P_f = (22.7\,\mathrm{kg} + 3.63\,\mathrm{kg})v_{R,x} = (26.3\,\mathrm{kg})v_{R,x}$$

(The cat and sled move as one mass, so we can just add their individual masses.) Conservation of momentum of this system, $P_i = P_f$ gives

$$11.1\,\tfrac{\mathrm{kg\cdot m}}{\mathrm{s}} = (26.3\,\mathrm{kg})v_{R,x}$$

so

$$v_{R,x} = \frac{\left(11.1\,\frac{\mathrm{kg\cdot m}}{\mathrm{s}}\right)}{(26.3\,\mathrm{kg})} = 0.422\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

Now we have the velocities of both sleds as they are pictured in Fig. 7.10 (b).

And now the cat makes a jump back to the left sled, as shown in Fig. 7.11 (a). Again, we take the system to be the cat and the right sled. Its initial momentum is

$$P_i = (22.7\,\mathrm{kg} + 3.63\,\mathrm{kg})(0.422\,\tfrac{\mathrm{m}}{\mathrm{s}}) = 11.1\,\tfrac{\mathrm{kg\cdot m}}{\mathrm{s}}$$

Now after the cat leaps, the velocity of the cat (with respect to the ice) is $-3.05\,\frac{\mathrm{m}}{\mathrm{s}}$, as specified in the problem. If the velocity of the right sled after the leap is $v'_{R,x}$ then the final momentum of the system is

$$P_f = (3.63\,\mathrm{kg})(-3.05\,\tfrac{\mathrm{m}}{\mathrm{s}}) + (22.7\,\mathrm{kg})v'_{R,x}$$

Conservation of momentum for the system, $P_i = P_f$, gives

$$11.1\,\tfrac{\mathrm{kg\cdot m}}{\mathrm{s}} = (3.63\,\mathrm{kg})(-3.05\,\tfrac{\mathrm{m}}{\mathrm{s}}) + (22.7\,\mathrm{kg})v'_{R,x}$$

so that we can solve for $v'_{R,x}$:

$$v'_{R,x} = \frac{(11.1\,\frac{\mathrm{kg\cdot m}}{\mathrm{s}} + 11.1\,\frac{\mathrm{kg\cdot m}}{\mathrm{s}})}{(22.7\,\mathrm{kg})} = 0.975\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

so during its second leap the cat makes the right sled go faster!

Finally, for the cat's landing on the left sled we consider the (isolated) system of the cat and the left sled. We already have the velocities of the cat and sled at this time; its initial momentum is

$$P_i = (22.7\,\mathrm{kg})(-0.488\,\tfrac{\mathrm{m}}{\mathrm{s}}) + (3.63\,\mathrm{kg})(-3.05\,\tfrac{\mathrm{m}}{\mathrm{s}}) = -22.1\,\tfrac{\mathrm{kg\cdot m}}{\mathrm{s}} \quad .$$

After the cat has landed on the sled, it is moving with the same velocity as the sled, which we will call $v'_{L,x}$. Then the final momentum of the system is

$$P_f = (22.7\,\mathrm{kg} + 3.63\,\mathrm{kg})v'_{L,x} = (26.3\,\mathrm{kg})v'_{L,x}$$

And momentum conservation for *this* collision gives

$$-22.1\,\tfrac{\mathrm{kg\cdot m}}{\mathrm{s}} = (26.3\,\mathrm{kg})v'_{L,x}$$

and then

$$v'_{L,x} = \frac{(-22.1\,\frac{\mathrm{kg\cdot m}}{\mathrm{s}})}{(26.3\,\mathrm{kg})} = -0.842\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

Summing up, the final velocities of the sleds (after the cat is done jumping) are:

$$\text{Left Sled:} \qquad v'_{L,x} = -0.842\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

$$\text{Right Sled:} \qquad v'_{r,x} = +0.975\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

7.2.4 Two–Dimensional Collisions

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9. An unstable nucleus of mass $17 \times 10^{-27}\,\mathrm{kg}$ initially at rest disintegrates into three particles. One of the particles, of mass $5.0 \times 10^{-27}\,\mathrm{kg}$, moves along the y axis with a speed of $6.0 \times 10^{6}\,\frac{\mathrm{m}}{\mathrm{s}}$. Another particle of mass $8.4 \times 10^{-27}\,\mathrm{kg}$, moves along the x axis with a speed of $4.0 \times 10^{6}\,\frac{\mathrm{m}}{\mathrm{s}}$. Find (a) the velocity of the third particle and (b) the total energy given off in the process.

(a) First, draw a picture of what is happening! Such a picture is given in Fig. 7.12. In the most general sense of the word, this is indeed a "collision", since it involves the rapid interaction of a few isolated particles.

The are no external forces acting on the particles involved in the disintegration; the total momentum of the system is conserved. The parent nucleus is *at rest*, so that the total momentum was (and remains) zero: $\mathbf{P}_i = \mathbf{0}$.

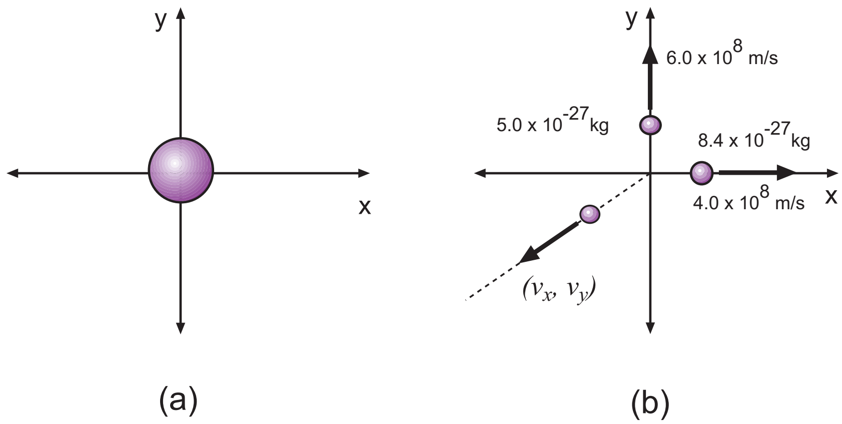

Figure 7.12: Nucleus disintegrates in Example 9. (a) Before the split; nucleus is at rest. (b) Afterwards; three pieces fly off in different directions.

Afterwards, the system consists of three particles; for two of these particles, we are given the masses and velocities. We are not given the mass of the third piece, but since we were given the mass of the parent nucleus, we might think that we can use the fact that the masses must sum up to the same value before and after the reaction to find it. In fact relativity tells us that masses don't really add in this way and when nuclei break up there *is* a measurable mass difference, but it is small enough that we can safely ignore it in this problem. So we would say that *mass* is conserved, and if m is the mass of the unknown fragment, we get:

$$17 \times 10^{-27}\,\mathrm{kg} = 5.0 \times 10^{-27}\,\mathrm{kg} + 8.4 \times 10^{-27}\,\mathrm{kg} + m$$

so that

$$m = 3.6 \times 10^{-27}\,\mathrm{kg}\ .$$

We will let the velocity components of the third fragment be v_x and v_y. Then the total x momentum after the collision is

$$P_{f,x} = (8.4 \times 10^{-27}\,\mathrm{kg})(4.0 \times 10^{6}\,\tfrac{\mathrm{m}}{\mathrm{s}}) + (3.6 \times 10^{-27}\,\mathrm{kg})v_x$$

Using $P_{i,x} = 0 = P_{f,x}$, we find:

$$(8.4 \times 10^{-27}\,\mathrm{kg})(4.0 \times 10^{6}\,\tfrac{\mathrm{m}}{\mathrm{s}}) + (3.6 \times 10^{-27}\,\mathrm{kg})v_x = 0$$

which easily gives:

$$v_x = -9.33 \times 10^{6}\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

Similarly, the total y momentum after the collision is:

$$P_{f,y} = (5.0 \times 10^{-27}\,\mathrm{kg})(6.0 \times 10^{6}\,\tfrac{\mathrm{m}}{\mathrm{s}}) + (3.6 \times 10^{-27}\,\mathrm{kg})v_y$$

and using $P_{i,y} = 0 = P_{f,y}$, we have:

$$(5.0 \times 10^{-27}\,\mathrm{kg})(6.0 \times 10^{6}\,\tfrac{\mathrm{m}}{\mathrm{s}}) + (3.6 \times 10^{-27}\,\mathrm{kg})v_y = 0$$

which gives

$$v_y = -8.33 \times 10^6 \, \frac{\mathrm{m}}{\mathrm{s}}$$

This really does specify the velocity of the third fragment (as requested), but it is also useful to express it as a magnitude and direction. The *speed* of the third fragment is

$$v = \sqrt{(-9.33 \times 10^6 \, \tfrac{\mathrm{m}}{\mathrm{s}})^2 + (-8.33 \times 10^6 \, \tfrac{\mathrm{m}}{\mathrm{s}})^2} = 1.25 \times 10^7 \, \tfrac{\mathrm{m}}{\mathrm{s}}$$

and its direction θ (measured counterclockwise from the x axis) is given by

$$\tan\theta = \left(\frac{-8.33}{-9.33}\right) = 0.893$$

Realizing that θ must lie in the third quadrant, we find:

$$\theta = \tan^{-1}(0.893) - 180^\circ = -138^\circ \; .$$

(b) What is the gain in energy by the system for this disintegration? By this we mean the gain in kinetic energy. Initially, the system has *no* kinetic energy. After the breakup, the kinetic energy is the sum of $\frac{1}{2}mv^2$ for all the particles, namely

$$\begin{aligned} K_f &= \tfrac{1}{2}(5.0 \times 10^{-27}\,\mathrm{kg})(6.0 \times 10^6 \, \tfrac{\mathrm{m}}{\mathrm{s}})^2 + \tfrac{1}{2}(8.4 \times 10^{-27}\,\mathrm{kg})(4.0 \times 10^6 \, \tfrac{\mathrm{m}}{\mathrm{s}})^2 \\ &\qquad + \tfrac{1}{2}(3.6 \times 10^{-27}\,\mathrm{kg})(1.25 \times 10^7 \, \tfrac{\mathrm{m}}{\mathrm{s}})^2 \\ &= 4.38 \times 10^{-13}\,\mathrm{J} \end{aligned}$$

We might say that the process gives off 4.38×10^{-13} J of energy.

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10. A billiard ball moving at $5.00 \, \frac{\mathrm{m}}{\mathrm{s}}$ strikes a stationary ball of the same mass. After the collision, the first ball moves at $4.33 \, \frac{\mathrm{m}}{\mathrm{s}}$ at an angle of 30.0° with respect to the original line of motion. Assuming an elastic collision (and ignoring friction and rotational motion), find the struck ball's velocity.

The collision is diagrammed in Fig. 7.13. We don't know the final speed of the struck ball; we will call it v, as in Fig. 7.13(b). We don't know the final direction of motion of the struck ball; we will let it be some angle θ, measured below the x axis, also as shown in Fig. 7.13(b).

Since we are dealing with a "collision" between the two objects, we know that the total momentum of the system is conserved. So the x and y components of the total momentum is the same before and after the collision.

Suppose we let the x and y components of the struck ball's final velocity be v_x and v_y, respectively. Then the condition that the total x momentum be conserved gives us:

$$m(5.00 \, \tfrac{\mathrm{m}}{\mathrm{s}}) + 0 = m(4.33 \, \tfrac{\mathrm{m}}{\mathrm{s}}) \cos 30.0^\circ + m v_x$$

(The struck ball has *no* momentum initially; after the collision, the incident ball has an x velocity of $(4.33 \, \frac{\mathrm{m}}{\mathrm{s}}) \cos 30.0^\circ$.)

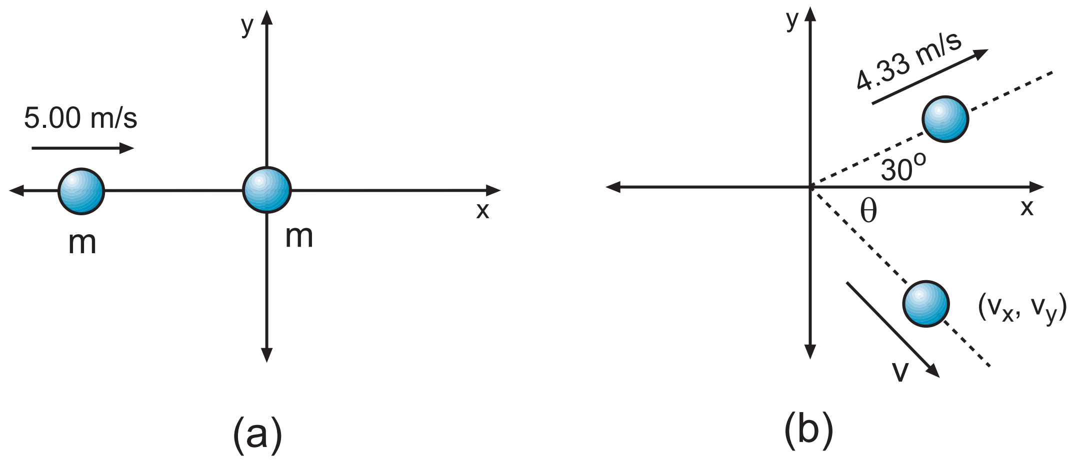

Figure 7.13: Collision in Example 10. (a) Before the collision. (b) After the collision.

Luckily, the m's cancel out of this equation and we can solve for v_x:

$$(5.00\,\tfrac{\mathrm{m}}{\mathrm{s}}) = (4.33\,\tfrac{\mathrm{m}}{\mathrm{s}})\cos 30.0^\circ + v_x$$

and then:

$$v_x = (5.00\,\tfrac{\mathrm{m}}{\mathrm{s}}) - (4.33\,\tfrac{\mathrm{m}}{\mathrm{s}})\cos 30.0^\circ = 1.25\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

We can similarly find v_y by using the condition that the total y momentum be conserved in the collision. This gives us:

$$0 + 0 = m(4.33\,\tfrac{\mathrm{m}}{\mathrm{s}})\sin 30.0^\circ + mv_y$$

which gives

$$v_y = -(4.33\,\tfrac{\mathrm{m}}{\mathrm{s}})\sin 30.0^\circ = -2.16\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

Now that we have the components of the final velocity we can find the speed and direction of motion. The speed is:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(1.25\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 + (-2.16\,\tfrac{\mathrm{m}}{\mathrm{s}})^2} = 2.50\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

and the direction is found from

$$\tan\theta = \frac{v_y}{v_x} = \frac{-2.16\,\frac{\mathrm{m}}{\mathrm{s}}}{1.25\,\frac{\mathrm{m}}{\mathrm{s}}} = -1.73 \qquad \Longrightarrow \qquad \theta = \tan^{-1}(-1.73) = -60^\circ$$

So the struck ball moves off with a speed of $2.50\,\frac{\mathrm{m}}{\mathrm{s}}$ at an angle of 60° downward from the x axis.

This really completes the problem but we notice something strange here: We were given *more* information about the collision than we used. We were also told that the collision was *elastic*, meaning that the total kinetic energy of the system was the same before and after the collision. Since we now have all of the speeds, we can *check* this.

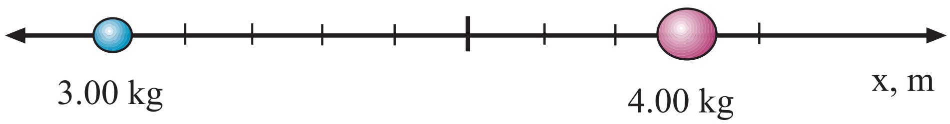

Figure 7.14: Masses and positions for Example 11.

The total kinetic energy before the collision was

$$K_i = \tfrac{1}{2}m(5.00\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 = m(12.5\,\tfrac{\mathrm{m}^2}{\mathrm{s}^2})\ .$$

The total kinetic energy after the collision was

$$K_f = \tfrac{1}{2}m(4.33\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 + \tfrac{1}{2}m(2.50\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 = m(12.5\,\tfrac{\mathrm{m}^2}{\mathrm{s}^2})$$

so that K_i is the same as K_f; the statement that the collision is elastic is *consistent* with the other data; but in this case we were given *too much* information in the problem!

7.2.5 The Center of Mass

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11. A $3.00\,\mathrm{kg}$ **particle is located on the** x **axis at** $x = -5.00\,\mathrm{m}$ **and a** $4.00\,\mathrm{kg}$ **particle is on the** x **axis at** $x = 3.00\,\mathrm{m}$**. Find the center of mass of this two–particle system.**

The masses are shown in Fig. 7.14. There is only one coordinate (x) and two mass points to consider here; using the definition of x_{CM}, we find:

$$\begin{aligned} x_{\mathrm{CM}} &= \frac{m_1x_1 + m_2x_2}{m_1+m_2} \\ &= \frac{(3.00\,\mathrm{kg})(-5.00\,\mathrm{m}) + (4.00\,\mathrm{kg})(3.00\,\mathrm{m})}{(3.00\,\mathrm{kg} + 4.00\,\mathrm{kg})} \\ &= -0.429\,\mathrm{m} \end{aligned}$$

The center of mass is located at $x = -0.429\,\mathrm{m}$.

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12. An old Chrysler with mass $2400\,\mathrm{kg}$ **is moving along a straight stretch of road at** $80\,\mathrm{km/h}$**. It is followed by a Ford with mass** $1600\,\mathrm{kg}$ **moving at** $60\,\mathrm{km/h}$**. How fast is the center of mass of the two cars moving?**

The cars here have 1–dimensional motion along (say) the x axis.
From Eq. 7.11 we see that the velocity of their center of mass is given by:

$$v_{\mathrm{CM},x} = \frac{1}{M}\sum_i^N m_i v_{i,x} = \frac{m_1v_{1x} + m_2v_{2x}}{m_1+m_2}$$

Plugging in the masses and velocities of the two cars, we find

$$v_{\mathrm{CM},\,x} = \frac{(2400\,\mathrm{kg})\left(80\,\frac{\mathrm{km}}{\mathrm{h}}\right) + (1600\,\mathrm{kg})\left(60\,\frac{\mathrm{km}}{\mathrm{h}}\right)}{(2400\,\mathrm{kg} + 1600\,\mathrm{kg})} = 72\,\frac{\mathrm{km}}{\mathrm{h}}$$

In units of $\frac{\mathrm{m}}{\mathrm{s}}$ this is

$$72\,\frac{\mathrm{km}}{\mathrm{h}}\left(\frac{10^3\,\mathrm{m}}{1\,\mathrm{km}}\right)\left(\frac{1\,\mathrm{h}}{3600\,\mathrm{s}}\right) = 20\,\frac{\mathrm{m}}{\mathrm{s}}$$

7.2.6 The Motion of a System of Particles

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13. A $2.0\,\mathrm{kg}$ particle has a velocity of $\mathbf{v}_1 = (2.0\mathbf{i} - 3.0\mathbf{j})\,\frac{\mathrm{m}}{\mathrm{s}}$, and a $3.0\,\mathrm{kg}$ particle has a velocity $(1.0\mathbf{i} + 6.0\mathbf{j})\,\frac{\mathrm{m}}{\mathrm{s}}$. Find (a) the velocity of the center of mass and (b) the total momentum of the system.

(a) We are given the masses and velocity components for the two particles. Then writing out the x and y components of Eq. 7.11 we find:

$$\begin{aligned} v_{\mathrm{CM},x} &= \frac{(m_1 v_{1x} + m_2 v_{2x})}{(m_1 + m_2)} \\ &= \frac{(2.0\,\mathrm{kg})(2.0\,\frac{\mathrm{m}}{\mathrm{s}}) + (3.0\,\mathrm{kg})(1.0\,\frac{\mathrm{m}}{\mathrm{s}})}{(2.0\,\mathrm{kg} + 3.0\,\mathrm{kg})} \\ &= 1.4\,\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

$$\begin{aligned} v_{\mathrm{CM},y} &= \frac{(m_1 v_{1y} + m_2 v_{2y})}{(m_1 + m_2)} \\ &= \frac{(2.0\,\mathrm{kg})(-3.0\,\frac{\mathrm{m}}{\mathrm{s}}) + (3.0\,\mathrm{kg})(6.0\,\frac{\mathrm{m}}{\mathrm{s}})}{(2.0\,\mathrm{kg} + 3.0\,\mathrm{kg})} \\ &= 2.4\,\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

The velocity of the center of mass of the two–particle system is

$$\mathbf{v}_{\mathrm{CM}} = (1.4\mathbf{i} + 2.4\mathbf{j})\,\frac{\mathrm{m}}{\mathrm{s}}$$

(b) The total momentum of a system of particles is related to the velocity of the center of mass by $\mathbf{P} = M\mathbf{v}_{\mathrm{CM}}$ so we can use the answer from part (a) to get:

$$\begin{aligned} \mathbf{P} &= M\mathbf{v}_{\mathrm{CM}} = (2.0\,\mathrm{kg} + 3.0\,\mathrm{kg})((1.4\mathbf{i} + 2.4\mathbf{j})\,\frac{\mathrm{m}}{\mathrm{s}}) \\ &= (7.00\mathbf{i} + 12.0\mathbf{j})\,\frac{\mathrm{kg\cdot m}}{\mathrm{s}} \end{aligned}$$

Appendix A: Conversion Factors

Length	**cm**	**meter**	**km**	**in**	**ft**	**mi**
1 cm =	1	10^{-2}	10^{-5}	0.3937	3.281×10^{-2}	6.214×10^{-6}
1 m =	100	1	10^{-3}	39.37	3.281	6.214×10^{-4}
1 km =	10^{5}	1000	1	3.937×10^{4}	3281	06214
1 in =	2.540	2.540×10^{-2}	2.540×10^{-5}	1	8.333×10^{-2}	1.578×10^{-5}
1 ft =	30.48	0.3048	3.048×10^{-4}	12	1	1.894×10^{-4}
1 mi =	1.609×10^{5}	1609	1.609	6.336×10^{4}	5280	1

Mass	**g**	**kg**	**slug**	**u**
1 g =	1	0.001	6.852×10^{-2}	6.022×10^{26}
1 kg =	1000	1	6.852×10^{-5}	6.022×10^{23}
1 slug =	1.459×10^{4}	14.59	1	8.786×10^{27}
1 u =	1.661×10^{-24}	1.661×10^{-27}	1.138×10^{-28}	1

An object with a *weight* of 1 lb has a *mass* of 0.4536 kg.