MTH1301: Elementary Mathematics I

Module 02 – Unit 02
Real Sequences and Series,
Quadratic Equations,
Binomial Theorem

(Draft)

Mansur Babagana
Bayero University, Kano

Update: December 15, 2021

Contents

1 Real Sequence

Sequence and series is one of the basic topics in Arithmetic. An itemized collection of elements in which repetitions of any sort are allowed is known as a **sequence**, whereas **series** is the sum of all elements.

1.1 Introduction

An arithmetic progression and geometric progression are two of the common examples of sequence and series.

Definition 1 (Sequence) A sequence is a list of items/objects which have been arranged in a sequential way. That is, as sequence $\{a_n\}$ is an infinite list of numbers

$$a_1, a_2, a_3, \ldots$$

where we have one number a_n for every positive integer n.

A sequence is a succession of terms spanned by a rule or formula.

Example 1 $1, 2, 3, 4, 5, \ldots$

Example 2 $2, 4, 6, 8, 10, \ldots$

Example 3 $a, a^3, a^5, a^7, a^9, \ldots$

A sequence may be finite or infinite. A finite sequence is one whose first and last element are known, while an infinite sequence is one whose terms are uncountable.

1.2 Defining Sequences

We can specify a sequence in various ways.

1.2.1 Pattern

We can specify it by listing some elements and implying that the pattern shown continues.

Example 4 For example

$$2, 4, 6, 8, \dots$$

would be the sequence consisting of the even positive integers.

1.2.2 Formula

We can also specify a sequence by giving a formula for the term that corresponds to the integer n.

Example 5 For example the sequence

$$2, 4, 6, 8, \dots$$

can also be specified by the explicit formula

$$a_n = 2n$$

1.2.3 Recursively

Finally, we can also provide a rule for producing the next term of a sequence from the previous ones. This is called a *recursively* defined sequence.

Example 6 For example the sequence

$$2, 4, 6, 8, \dots$$

can be specified by the rule

$$a_1 = 2 \text{ and } a_n = a_{n+1} + 2 \text{ for } n \geq 2$$

This rule says that we get the next term by taking the previous term and adding 2. Since we start at the number 2 we get all the even positive integers.

Let's discuss these ways of defining sequences in more detail, and take a look at some examples.

1.3 Arithmetic Sequences

The sequence we saw in the previous paragraph is an example of what's called an *arithmetic* sequence: each term is obtained by **adding a fixed number** to the previous term.

Alternatively, the difference between consecutive terms is always the same.

1.3.1 General Formula

If a sequence a_n is arithmetic, then there is a fixed number d so that

$$a_{n+1} - a_n = d$$

for any n. The number d is usually called the step or difference. Let's try to find a formula for the term a_n of an arithmetic sequence in terms of d and $_a1$.

Let's start with the first term a_1

$$a_1 = a_1$$

the next term is a_2 is

$$a_2 = a_1 + d$$

That is $a_2 = a_1 + d$. The next term is a_3 is

$$\begin{aligned} a_3 &= a_2 \quad\quad + d \\ a_3 &= (a_1 + d) + d \\ a_3 &= a_1 + d \quad + d \\ a_3 &= a_1 + 2d \end{aligned}$$

That is $a_3 = a_1 + 2d$. The next term is a_4 is

$$\begin{aligned} a_4 &= a_3 \quad\quad + d \\ a_4 &= (a_1 + 2d) + d \\ a_4 &= a_1 + 2d \quad + d \\ a_4 &= a_1 + 3d \end{aligned}$$

So we get

$$\begin{aligned} a_1 =& a_1 + 0d \\ a_2 =& a_1 + 1d \\ a_3 =& a_1 + 2d \\ a_4 =& a_1 + 3d \\ a_5 =& a_1 + 4d \\ \vdots =& \vdots + \vdots \\ a_n =& a_1 + (n-1)d \\ a_n =& a_1 + d(n-1) \end{aligned}$$

Therefore the n^{th}-term of an Arithmetic sequence is

$$a_n = a_1 + d(n-1) \tag{1}$$

Example 7 Consider the sequence $3, 8, 13, 18, 23, 28, \ldots$ Is it arithmetic? If so, find a formula for a_n, and use it to find a_{101}, the 101st term in the sequence.

Solution This sequence is arithmetic, since the difference between each term is 5. Let $a_1 = 3$, $a_2 = 8$, $a_3 = 13$, $a_4 = 18$, $a_5 = 23$, $a_6 = 28$,

$$\begin{aligned} a_2 - a_1 =& a_3 - a_2 = a_4 - a_3 = a_5 - a_4 = a_6 - a_5 \\ 8 - 3 = 13 - 8 =& 18 - 13 = 23 - 18 = 28 - 23 = \cdots = 5 \end{aligned}$$

So this is an arithmetic sequence with commond difference $d = 5$ and first term $a_1 = 3$. Using the formula in Equation 1

$$\begin{aligned} a_n &= a_1 + d(n-1) \\ a_{101} &= 3 + 5(101 - 1) \\ a_{101} &= 3 + 5(100) \\ a_{101} &= 3 + 500 \\ a_{101} &= 503 \end{aligned}$$

■

1.4 Geometric Sequences

Consider the sequence $2, 4, 8, 16, 32, 64, \ldots$ This sequence is not arithmetic, since the difference between terms is not always the same. If we look closely, we will see that we obtain the next term in the sequence by **multiplying the previous term by the same number**. Equivalently, the ratio of consecutive terms is always the same (namely 2).

A sequence a_n where there is a fixed r so that

$$\frac{a_n}{a_{n-1}} = r, \ \forall\, n$$

is called a geometric sequence. The number r is usually called the **ratio**.

1.4.1 General Formula

Let's try to find the formula for the term a_n of a geometric sequence in terms of r and the first term.

Let's start with the ration of the first two terms a_1, a_2,

$$\begin{aligned} \frac{a_2}{a_1} &= r \\ \implies a_2 &= a_1 \times r \end{aligned}$$

The next terms is a_2 and a_3

$$\begin{aligned} \frac{a_3}{a_2} &= r \\ \implies a_3 &= a_2 \quad \times r \\ \implies a_3 &= a_1 r \ \times r \\ \implies a_3 &= a_1 r^2 \end{aligned}$$

Next we take is a_3 and a_4

$$\begin{aligned} \frac{a_4}{a_3} &= r \\ \implies a_4 &= a_3 \quad \times r \\ \implies a_4 &= a_1 r^2 \times r \\ \implies a_4 &= a_1 r^3 \end{aligned}$$

Next we take is a_4 and a_5

$$\begin{aligned} \frac{a_5}{a_4} &= r \\ \implies a_5 &= a_4 \quad \times r \\ \implies a_5 &= a_1 r^3 \times r \\ \implies a_5 &= a_1 r^4 \end{aligned}$$

So we get

$$\begin{aligned} a_2 &= a_1 \times r \\ a_3 &= a_2 \times r = a_1 r \times r = a_1 r^2 \\ a_4 &= a_3 \times r = a_1 r^2 \times r = a_1 r^3 \\ a_5 &= a_4 \times r = a_1 r^3 \times r = a_1 r^4 \\ \vdots &= \vdots = \vdots = \vdots \\ a_n &= a_{n-1} \times r = a_1 r^{n-2} \times r = a_1 r^{n-1} \end{aligned}$$

So we get that for a geometric sequence a_n with ratio r, the formula for a_n is given by:

$$a_n = a_1 r^{n-1} \tag{2}$$

Example 8 Consider the geometric sequence $3, 6, 12, 24, 48, \ldots$. Find a formula for a_n and use it to find a_7.

Solution To find r, we should look at the ratio between successive terms:

$$r = \frac{a_1}{a_2} = \frac{6}{3} = 2$$

Then using the formula above we get

$$a_n = a_1 r^{n-1} = 3 \times 2^{n-1}$$

To find a_7 we set $n = 7$ and get

$$a_7 = 3 \times 2^{7-1} = 3 \times 2^6 = 3 \times 64 = 192$$

■

1.5 Determining the Pattern of a Sequence

We can discuss how to look for patterns and figure out given a list, how to find the sequence in question.

Example 9 When given a list, such as $1, 3, 9, 27, 81, \ldots$ we can try to look for a pattern in a few ways. Now that we have seen **arithmetic**, and **geometric** recursive sequences, one thing we can do is try to check if the given sequence is one of these types.

Arithmetic? To check if a sequence is arithmetic, we check whether or not the difference of consecutive terms is always the same. In this case, the difference changes:

$$\begin{aligned} a_2 - a_1 &= 3 - 1 = 2 \\ a_3 - a_2 &= 9 - 3 = 6 \\ a_3 - a_2 = 2 &\neq 6 = a_2 - a_1 \\ a_3 - a_2 &\neq a_2 - a_1 \end{aligned}$$

So the sequences is not arithmetic

Geometric? To check if a sequence is geometric we check whether or not the ratio of consecutive terms is always the same. In the case it is, so we conclude that the sequence is geometric:

$$\frac{a_2}{a_1} = \frac{3}{1} = 3$$
$$\frac{a_3}{a_2} = \frac{9}{3} = 3$$
$$\frac{a_4}{a_3} = \frac{27}{9} = 3$$
$$\frac{a_5}{a_4} = \frac{81}{27} = 3$$

This shows that the sequence is geometric with ratio $r = 3$, and initial term $a_1 = 1$, so we get that the sequence is given by

$$a_n = 1 \times 3^{n-1} = \times 3^{n-1}$$

Example 10 Consider the sequence $1, -3, -7, -11, -15, -19, -23, \ldots$ Determine a formula for the n^{th} term in the sequence.

Solution We quickly see that this series is not geometric, since

$$\frac{1}{-3} \neq \frac{-3}{-7} \tag{3}$$

We can now try to see if the sequence is arithmetic. If we look at the differences of consecutive terms, we get:

$$-3 - 1 = -4 = -7 - (-3) = -11 - (-7)$$

so we see that this is an arithmetic sequence with difference $d = -4$. So the general term is

$$a_n = a_1 + d(n-1) = 1 + (-4)(n-1) = -4n + 5$$

■

Exercise 1 What is the twenty-first term of the sequence given by $x_n = 4n - 3$?

Exercise 2 The fourth term of a geometric sequence is 27 and the seventh term is 1. What is the first term?

Exercise 3 The third term of an arithmetic sequence is 25 and the eighth term is 70. What is the first term?

Exercise 4 The fourth term of a geometric sequence is 135 and the seventh term is 5. What is the first term?

Exercise 5 What type of a sequence are the numbers $18, 12, 6, 0, \ldots$

Exercise 6 The second term of a geometric sequence is 405 and the seventh term is $\frac{5}{3}$. What is the first term?

Exercise 7 The fourth term of a geometric sequence is 60 and the seventh term is -3840. What is the first term?

Exercise 8 The fifth term of an arithmetic sequence is -1 and the ninth term is 31. What is the first term?

Exercise 9 The fourth term of an arithmetic sequence is 38 and the seventh term is 17. What is the first term?

Exercise 10 The third term of a geometric sequence is 64 and the eighth term is -2048. What is the first term?

2 Real Series

A series is a succession of numbers, of which each number is formed according to a definite law which is the same throughout the series.

Definition 2 (Series) A series can be highly generalized as the sum of all the terms in a sequence. However, there has to be a definite relationship between all the terms of the sequence.

If you try to add up all the terms of a sequence, you get an object called a series. In order to discuss series, it's useful to use sigma notation, so we will begin with a review of that.

2.1 Sigma Notation

When adding many terms, it's often useful to use some shorthand notation. Let a_n be a sequence of real numbers. We set

$$\sum_{i=1}^{k} a_i = a_1 + a_2 + \cdots + a_{k-1} + a_k$$

Here we add up the first terms $a_1, a_2, \ldots, a_{k-1}, a_k$ of the sequence. We can also start the sum at a different integer.

$$\sum_{i=j}^{k} a_i = a_j + a_{j+1} + \cdots + a_{k-1} + a_k$$

Example 11 Let $a_n = \dfrac{1}{2^n}$. Express the sum of the first 100 terms of the corresponding series, using sigma notation.

Solution

$$\sum_{i=1}^{100} \frac{1}{2^i}$$

■

2.1.1 Some Properties of the Sigma Notation

Given $\sum_{i=1}^{n} f(x) = f(1) + f(2) + f(3) + \cdots f(n)$ and $\sum_{i=1}^{n} g(x) = g(1) + g(2) + g(3) + \cdots g(n)$, we have for constant k

$$\begin{aligned}
\sum_{i=1}^{n} kf(x) =& kf(1) + kf(2) + kf(3) + \cdots kf(n) \\
\sum_{i=1}^{n} kf(x) =& k\Big(\underbrace{f(1) + kf(2) + kf(3) + \cdots f(n)}\Big) \\
=& k\Big(\sum_{i=1}^{n} f(x)\Big) \\
=& k\sum_{i=1}^{n} f(x)
\end{aligned}$$

We also have

$$\begin{aligned}
\sum_{i=1}^{n}\Big(f(x)+g(x)\Big) &= \underbrace{f(1)+g(1)}+\underbrace{f(2)+g(2)}+\underbrace{f(3)+g(3)}+\cdots\underbrace{f(n)+g(n)} \\
&= \underbrace{f(1)+f(2)+f(3)+\cdots f(n)}+\underbrace{g(1)+g(2)+g(3)+\cdots g(n)} \\
&= \sum_{i=1}^{n} f(x) + \sum_{i=1}^{n} g(x)
\end{aligned}$$

Exercise 11 Find the value of:

(a) $\displaystyle\sum_{i=1}^{3}\frac{i}{i+2}$ (b) $\displaystyle\sum_{k=1}^{4}5k^2$ (c) $\displaystyle\sum_{k=-2}^{2}\frac{2i}{i+3}$ (d) $\displaystyle\sum_{k=3}^{7}\frac{k+2}{k-2}$ (e) $\displaystyle\sum_{k=1}^{4}k^3$ (f) $\displaystyle\sum_{k=-1}^{3}\frac{k(k+1)}{k+2}$

(g) $\displaystyle\sum_{k=1}^{12}k^2(k-3)$ (h) $\displaystyle\sum_{j=-2}^{3}\left|j-j^2\right|$ (i) $\displaystyle\sum_{m=2}^{6}\left(m^2+m\right)$ (j) $\displaystyle\sum_{z=-2}^{2}\left(n^{n+1}\right)$

2.2 Infinite Sum

A series is an infinite sum

$$a_1 + a_2 + a_3 + \cdots$$

We typically write such an object using sigma notation

$$\sum_{i=1}^{\infty} a_i$$

If you do not use sigma notation to express a series, it's very important to include the "$+\cdots$" to make sure the person reading understands that the sum goes on forever.

An infinite series has an infinite number of terms. The sum of infinite series is denoted by S_∞, that is

$$S_\infty = \sum_{i=1}^{\infty} a_i$$

If you have an explicit expression for the term a_i you usually replacen a_i with this expression when using sigma notation.

Example 12 For example if

$$a_i = \frac{1}{2^i}$$

then the corresponding series is usually written

$$\sum_{i=1}^{\infty} \frac{1}{2^i}$$

Example 13 Consider the series

$$\frac{2}{3} + \frac{4}{9} + \frac{6}{27} + \frac{8}{81} + \cdots$$

Find an expression for the i^{th} term a_i, and use it to write the series in sigma notation.

Solution If you look at this series carefully you see that each term is given by a fraction:

$$a_i = \frac{b_i}{c_i}$$

Let's look at b_i and c_i separately. The b_i's give the list

$$2, 4, 6, 8, \ldots$$

or the even natural numbers. So

$$b_i = 2i$$

The c_i give the list

$$3, 9, 27, 81, \ldots$$

which is the geometric sequence

$$c_i = 3^i$$

■

Putting this together gives

$$a_i = \frac{2i}{3^i}$$

So the series

$$\frac{2}{3} + \frac{4}{9} + \frac{6}{27} + \frac{8}{81} + \cdots$$

can be written in sigma notation as:

$$\sum_{i=1}^{\infty} \frac{2i}{3^i}$$

2.3 Partial Sum of an Arithmetic Series

Given a series

$$a_1 + a_2 + a_3 + \cdots = \sum_{i=1}^{\infty} a_i$$

we can form the "n^{th} partial sum," usually denoted S_n. As the name might suggest, the n^{th} partial sum is obtained by taking the first n terms and adding them up. More concretely

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

To find a formula for the sum, S_n, of the first n terms of an arithmetic sequence, we can write out the terms as:

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + [a_1 + (n-1)d] \quad (4)$$

where $a_n = a_1 + (n-1)d$. Note that

$$\begin{aligned}
a_n &= a_1 + (n-1)d \\
a_{n-1} &= a_1 + (n-2)d && = a_1 + [(n-1) - 1]d && = a_1 + (n-1)d - d && = a_n - d \\
a_{n-2} &= a_1 + (n-3)d && = a_1 + [(n-1) - 2]d && = a_1 + (n-1)d - 2d && = a_n - 2d \\
a_{n-3} &= a_1 + (n-4)d && = a_1 + [(n-1) - 3]d && = a_1 + (n-1)d - 3d && = a_n - 3d \\
\vdots &= \vdots && = \vdots && = \vdots && = \vdots \\
a_{n-i} &= a_1 + (n - (i+1))d && = a_1 + [(n-1) - i)]d && = a_1 + (n-1)d - id && = a_n - id \quad (5)
\end{aligned}$$

Observe that

$$a_n = a_1 + (n-1)d \implies a_1 = a_n - (n-1)d$$

Writing Equation 4 in reverse we have,

$$S_n = a_n + a_{n-1} + a_{n-2} + \cdots + [a_n - (n-1)d] \quad (6)$$

where $a_1 = a_n - (n-1)d$. Substitute values of the a_i's from Equations 5 into Equation 6

$$S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + [a_n - (n-1)d] \quad (7)$$

Add Equation 4 and Equation 7 we get

$$\begin{array}{lccccr}
S_n = & a_1 + & (a_1 + d) + & (a_1 + 2d) + & \cdots + & [a_1 + (n-1)d] \\
S_n = & a_n + & (a_n - d) + & (a_n - 2d) + & \cdots + & [a_n - (n-1)d] \\
\hline
2S_n = & (a_1 + a_n) + & (a_1 + a_n) + & (a_1 + a_n) + & \cdots + & [a_1 + a_n]
\end{array} \quad (8)$$

The right hand side of Equation 8 contains n terms, each equal to $a_1 + a_n$, so

$$\begin{aligned}
2S_n &= n(a_1 + a_n) \\
\implies S_n &= \frac{n(a_1 + a_n)}{2} = n\left(\frac{a_1 + a_n}{2}\right)
\end{aligned}$$

Therefore, for the arithmetic sequence $a_n = a_1 + (n-1)d$, the nth partial sum is given by either

$$S_n = n\left(\frac{a_1 + a_n}{2}\right) \quad (9)$$

or by substituting $a_n = a_1 + (n-1)d$ to get

$$S_n = \frac{n}{2}[2a_1 + (n-1)d] \quad (10)$$

Example 14 A partial sum of an arithmetic sequence is given

$$1 + 8 + 15 + \cdots + 78$$

Find (a) Which ith term is 78? (b) Find S_n

Solution (a) We can use the formula $a_n = a_1 + (n-1)d$ with $a_n = 78$ to find which ith term is 78; so, we need to find a_1, d and n. $a_1 = 1$ and $d = 8 - 1 = 7$, we can find n using

$$\begin{aligned}
& a_n = a_1 + (n-1)d \\
\implies & 78 = 1 + (n-1)7 \\
\implies & 77 = (n-1)7 \\
\implies & \frac{77}{7} = (n-1) \\
\implies & 11 = n - 1 \\
\implies & 12 = n
\end{aligned}$$

There $78 = a_{12}$

(b) We can use Equation 9, and the fact that $a_1 = 1$, $n = 12$ and $a_{12} = 78$

$$S_n = n\left(\frac{a_1 + a_n}{2}\right)$$

Therefore S_{12} will be

$$\begin{aligned}
S_{12} =& 12\left(\frac{1+78}{2}\right) \\
=& 12\left(\frac{1+78}{2}\right) = \frac{12}{2}\Big(1+78\Big) = 6(1+78) = 6(79) \\
=& 474
\end{aligned}$$

■

Infinite Sum of an Arithmetic Series

If S_n tends to a limit as n tends to infinity, the limit is called the sum to infinity of the series. For an arithemic series as n tends to infinity, S_n tends to $\pm\infty$. The sum to infinity for an arithmetic series is undefined.

2.4 Geometric Series

A series of the for

$$\sum_{i=1}^{\infty} ar^{n-i} = a + ar + ar^2 + ar^3 + \cdots$$

is called a geometric series. (This is a series where the terms a_i that we are summing, form a geometric sequence.)

Let's consider the nth partial sum of a geometric sequence;

$$S_n = \sum_{i=1}^{n} ar^{n-i} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \tag{11}$$

multiply both sides of 11 by r

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n \tag{12}$$

The difference between Equation 11 and Equation 12 gives

$$S_n - rS_n = a - ar^n \tag{13}$$

rearrange Equation 13 this and solve for S_n:

$$\begin{aligned}(1-r)S_n &= a(1-r^n)\\ S_n &= \frac{a(1-r^n)}{1-r}\end{aligned}$$

So for a geometric series we get the formula:

$$S_n = \frac{a(1-r^n)}{1-r} \tag{14}$$

Example 15 Consider the series

$$\sum_{i=1}^{\infty} \frac{1}{2^i}$$

Find the first 5 partial sums, (i.e. find the partial sums S_1, S_2, S_3, S_4, S_5.)

Solution

$$\begin{aligned}
&\text{For } S_1 \text{ we get : } S_1 = a_1 && && =\frac{1}{2}\\
&\text{For } S_2 \text{ we get : } S_2 = a_1 + a_2 && =\frac{1}{2}+\frac{1}{4} && =\frac{3}{4}\\
&\text{For } S_3 \text{ we get : } S_3 = a_1 + a_2 + a_3 && =\frac{1}{2}+\frac{1}{4}+\frac{1}{8} && =\frac{7}{8}\\
&\text{For } S_4 \text{ we get : } S_4 = a_1 + a_2 + a_3 + a_4 && =\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} && =\frac{15}{16}\\
&\text{For } S_5 \text{ we get : } S_5 = a_1 + a_2 + a_3 + a_4 + a_5 && =\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32} && =\frac{31}{32}
\end{aligned}$$

■

Example 16 Consider the series

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \tag{15}$$

(a) Find the 100th partial sum, using the formula.
(b) Find a formula for the nth partial sum.
(c) What happens to this expression as n gets larger and larger?

Solution (a) For the 100th partial sum, using the formula above we get

$$S_{100} = \frac{1}{2}\frac{1-\left(\frac{1}{2}\right)^{100}}{1-\frac{1}{2}} = 2\frac{1}{2}\left(1-\frac{1}{2^{100}}\right) = 1-\frac{1}{2^{100}} \tag{16}$$

(b) For the nth partial sum, we get

$$S_n = \frac{1}{2}\frac{1-\left(\frac{1}{2}\right)^{n}}{1-\frac{1}{2}} = 2\frac{1}{2}\left(1-\frac{1}{2^{n}}\right) = 1-\frac{1}{2^{n}} \tag{17}$$

(c) As n gets larger and larger, we see that the partial sum is getting closer and closer to 1.

■

Infinite Sum of a Geometric Series

For a geometric series as n tends to infinity, S_n is determined by the value of r. Note that

$$S_n = \frac{a(1-r^n)}{1-r}$$

- When $r > 1$, r^n tends to infinity as n tends to infinity, hence sum to infinity for a geometric series is undefined when $|r| > 1$
- When $r < 1$, r^n tends to zero as n tends to infinity, hence the sum to infinity for a geometric series is

$$S_\infty = \frac{a}{1-r}, \quad \text{when } |r| < 1 \tag{18}$$

Example 17 Evaluate the series

$$\sum_{i=1}^{\infty} \frac{1}{5^{i+2}}$$

Solution This series is geometric, but if we look carefully at the definition, we see that the indexing is doesn't quite match the form we're used to seeing. To fix this, let's rewrite the series:

$$\sum_{i=1}^{\infty} \frac{1}{5^{i+2}} = \sum_{i=1}^{\infty} \frac{1}{5^2}\frac{1}{5^i} = \frac{1}{5^2}\sum_{i=1}^{\infty} \frac{1}{5^i}$$

Writing the first few terms of $\sum_{i=1}^{\infty} \frac{1}{5^i}$ in order to get the first term a, and the common ratio r.

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{1}{5^i} &= \frac{1}{5^1} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \cdots \\ &= \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625} + \cdots\end{aligned}$$

We get $a_1 = \frac{1}{5}$, $a_2 = \frac{1}{25}$, $a_3 = \frac{1}{125}$, $a_4 = \frac{1}{625}$. The first term $a = a_1 = \frac{1}{5}$, the common difference r can be found as follows

$$\begin{aligned} r &= \frac{a_2}{a_1} = \frac{\frac{1}{25}}{\frac{1}{5}} = \frac{1}{25}\cdot\frac{5}{1} = \frac{1}{5} \\ &= \frac{a_3}{a_2} = \frac{\frac{1}{125}}{\frac{1}{25}} = \frac{1}{125}\cdot\frac{25}{1} = \frac{5}{25} = \frac{1}{5} \\ &= \frac{a_4}{a_3} = \frac{\frac{1}{625}}{\frac{1}{125}} = \frac{1}{625}\cdot\frac{125}{1} = \frac{1}{125}\cdot\frac{25}{1} = \frac{5}{25} = \frac{1}{5}\end{aligned}$$

Now we can use Equation 18 with $a = \frac{1}{5}$, and $r = \frac{1}{5}$ to get:

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{1}{5^{i+2}} = \frac{1}{5^2}\sum_{i=1}^{\infty} \frac{1}{5^i} &= \frac{1}{5^2}\left(\sum_{i=1}^{\infty} \frac{1}{5^i}\right) \\ &= \frac{1}{5^2}\left(\frac{a}{1-r}\right) = \frac{1}{25}\left(\frac{\frac{1}{5}}{1-\frac{1}{5}}\right) \\ &= \frac{1}{25}\left(\frac{\frac{1}{2}}{\frac{4}{5}}\right) = \frac{1}{25}\left(\frac{1}{2}\cdot\frac{5}{2}\right) \\ &= \frac{1}{25}\times\frac{1}{5}\times\frac{5}{4} = \frac{1}{100}\end{aligned}$$

■

Example 18 Find an equivalent fraction for the repeating decimal $0.\bar{3}$

Solution Notice that repeating decimal $0.\bar{3} = 0.333\ldots$ can rewritten as a sum of terms.

$$0.\bar{3} = 0.3 + 0.03 + 0.003 + \cdots$$

Looking for a pattern, rewrite the sum, noticing that the first term multiplied to 0.1 in the second term, and the second term multiplied to 0.1 in the third term.

$$0.\bar{3} = 0.3 + (0.1)\underbrace{(0.3)} + (0.1)\underbrace{(0.1)(0.3)}$$

Notice the pattern; multiply each consecutive term by a common ratio of $r = 0.1$ starting with the first term of $a_1 = 0.3$. So, substituting into our Equation 18 for an infinite geometric sum, we have

$$S_\infty = \frac{a_1}{1-r} = \frac{0.3}{1-0.1} = \frac{0.3}{0.9} = \frac{1}{3}$$

■

Exercise 12 Are the numbers $1, 1, 1, 1, 1, 1, \ldots$ an arithmetic sequence or a geometric sequence?

Exercise 13 Find the sum, if it exists, for the following:

(a) $96 + 48 + 24 + \cdots$ (b) $10 + 9 + 8 + 7 + \cdots$ (c) $248.6 + 99.44 + 39.776 + \cdots$

(d) $\displaystyle\sum_{k=1}^{\infty} 4374 \cdot \left(-\frac{1}{3}\right)^{k-1}$ (e) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{9} \cdot \left(\frac{4}{3}\right)^{k}$

2.5 Sequence and Series Formulas

List of some basic formula of arithmetic progression and geometric progression are

	Arithmetic Progression	**Geometric Progression**
Sequence	$a, a+d, a+2d, \ldots, a+(n-1)d, \ldots$	$a, ar, ar^2, \ldots, ar^{(n-1)}, \ldots$
Common Difference or Ration	$d = a_{n-1} - a_{n-2}$	$r = \dfrac{ar^{(n-1)}}{ar^{(n-2)}}$
General Term (n^{th} Term)	$a_n = a_1 + (n-1)d$	$a_n = a_1 r^{(n-1)}$
Partial Sum	$S_n = \dfrac{n(a_1 + a_n)}{2}$	$S_n = \dfrac{a(1-r^n)}{(1-r)}$ if $r < 1$ $S_n = \dfrac{a(r^n - 1)}{(r-1)}$ if $r > 1$
Infinite Sum	S_∞ is undefined	$S_\infty = \dfrac{a}{(1-r)}$ if $r < 1$ S_∞ is undefined if $r > 1$

Exercise 14 What is the sum of the first thirty terms of the arithmetic sequence $50, 45, 40, 35, \ldots$?

Exercise 15 What is the sum of the first sixteen terms of the arithmetic sequence $1, 5, 9, 13, \ldots$?

Exercise 16 What is the sum of the eleventh to twentieth terms (inclusive) of the arithmetic sequence $7, 12, 17, 22, \ldots$?

Exercise 17 What is the value of $12 + 14 + 16 + \cdots + 88$?

Exercise 18 What is the sum of the first eight terms of the geometric sequence $5, 15, 45, \ldots$?

Exercise 19 Find the sum of the numbers $11 + 12 + 13 + 14 + 15 + 16 + \cdots + 30 + 31$

Exercise 20 Determine whether or not the sequence is arithmetic. If it is arithmetic, find the common difference. (a) $2, 5, 8, 11, \ldots$ (b) $1, 2, 3, 5, \ldots$

Exercise 21 Find the nth term, the fifth term, and the 100th term, of the arithmetic sequence determined by $a = 2$ and $d = 3$

Exercise 22 Find the common difference, the fifth term, the nth term, and the 100th term of the arithmetic sequence. (a) $4, 14, 24, 34, \ldots$ (b) $t + 3, t + \frac{15}{4}, t + \frac{9}{2}, t + \frac{21}{4}, \ldots$

Exercise 23 Find the sum of the first 37 even numbers.

Exercise 24 Find the sum of the first 37 odd numbers

Exercise 25 Find the S_{50} for the sequence: $4, 9, 14, 19, 24, \ldots$

Exercise 26 Evaluate: $\sum_{i=1}^{35}(10 - 4i)$

Exercise 27 A new lecture theatre in being build for the Faculty of Computer Science and Information Technology (FCSIT), BUK. The first row of seating in the theater contains 26 seats, the second row contains 28 seats, the third row contains 30 seats, and so on. If there are 18 rows, what is the total seating capacity of the theater?

Exercise 28 Evaluate $\sum_{x=1}^{51}\left(\frac{3}{8} + \frac{1}{4}\right)$

Exercise 29 Discuss methods for calculating sums where the index does not start at 1. For example,

$$\sum_{k=15}^{35}(3k + 4) = 1659$$

3 Quadratic Equations

Quadratic equations are second-degree algebraic expressions and are of the form

$$ax^2 + bx + c = 0$$

The word "***Quadratic***" is derived from the word "***Quad***" which means square. In other words, a quadratic equation is an "***equation of degree 2***". There are many scenarios where a quadratic equation is used. Did you know that when a rocket is launched, its path is described by a quadratic equation? Further, a quadratic equation has numerous applications in physics, engineering and astronomy.

Quadratic equations are second-degree equations in x that have two answers for x. These two answers for x are also called the roots of the quadratic equations and are designated as (α, β). We shall learn more about the roots of a quadratic equation in the below content.

3.1 What is Quadratic Equation?

A quadratic equation is an algebraic expression of the second degree in x. The quadratic equation in its standard form is $ax^2 + bx + c = 0$, where a, b are the coefficients, x is the variable, and c is the constant term. The first condition for an equation to be a quadratic equation is the coefficient of x^2 is a non-zero term ($a \neq 0$). For writing a quadratic equation in standard form, the x^2 term is written first, followed by the x term, and finally, the constant term is written. The numeric values of a, b, c are generally not written as fractions or decimals but are written as integral values.

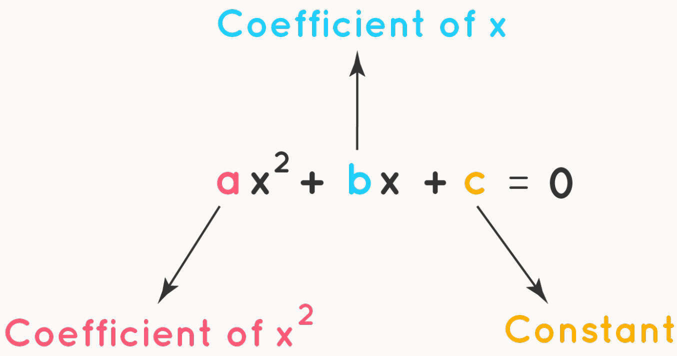


Figure 1: Standard Form of a Quadratic Equation [3]

Further in real math problems the quadratic equations are presented in different forms:

$$(x-1)(x+2) = 0$$
$$-x2 = -3x + 1$$
$$5x(x+3) = 12x$$
$$x3 = x(x2 + x - 3)$$

All of the above equations need to be transformed into standard form of the quadratic equation before performing further operations.

3.2 Quadratic Equation Formula

Quadratic Formula is the simplest way to find the roots of a quadratic equation. There are certain quadratic equations that cannot be easily factorized, and here we can conveniently use this quadratic formula to find the roots in the quickest possible way. The roots of the quadratic equation further help to find the sum of the roots and the product of the roots of the quadratic equation. The two roots in the quadratic formula are presented as a single expression. The positive sign and the negative sign can be alternatively used to obtain the two distinct roots of the equation.

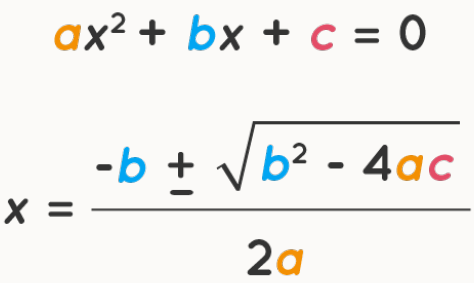


Figure 2: Quadratic Formula [3]

3.3 Important Formulas for Solving Quadratic Equations

The following list of important formulas is helpful to solve quadratic equations.

1. The quadratic equation in its standard form is $ax^2 + bx + c = 0$
2. The formula to find the roots of the quadratic equation is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
3. The discriminant of the quadratic equation is $d = b^2 - 4ac$
4. For $d > 0$ the roots are real and distinct and are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$..
5. For $d = 0$ the roots are real and equal and are given by $x_1 = -\dfrac{b}{2a}$.
6. For $d < 0$ the roots do not exist, or the roots are imaginary.
7. The sum of the roots α, β of a quadratic equation is $\alpha + \beta = \dfrac{-b}{a}$, where a is the coefficient of x^2 and b is the coefficient of x.
8. The product of the roots α, β of a quadratic equation is $\alpha\beta = \dfrac{c}{a}$, where a is the coefficient of x^2 and c is the constant term.
9. The quadratic equation having roots α, β, is $x^2 - (\alpha + \beta)x + \alpha\beta = 0$.
10. The condition for the quadratic equations

$$a_1x^2 + b_1x + c1 = 0, \text{ and}$$
$$a_2x^2 + b_2x + c2 = 0$$

having the same roots is

$$\frac{a_1b_2 - a_2b_1}{b_1c_2 - b_2c_1} = (a_2c_1 - a_1c_2)^2$$

11. For positive values of a, that is $(a > 0)$, the quadratic expression $f(x) = ax^2 + bx + c$ has a minimum value at $x = \dfrac{-b}{2a}$
12. For negative values of a, that is $(a < 0)$, the quadratic expression $f(x) = ax^2 + bx + c$ has a maximum value at $x = \dfrac{-b}{2a}$
13. For $a > 0$, the range of the quadratic equation $ax^2 + bx + c = 0$ is $\left[\dfrac{b^2 - 4ac}{4a}, \infty\right)$.
14. For $a < 0$, the range of the quadratic equation $ax^2 + bx + c = 0$ is $\left(-\infty, \dfrac{b^2 - 4ac}{4a}\right]$.

3.4 Quadratic Formula Proof

Consider an arbitrary quadratic equation: $ax^2 + bx + c = 0, ane0$. To determine the roots of this equation, we proceed as follows:

$$
\begin{aligned}
ax^2+bx+c &= 0 && \text{subtract } c \text{ from both sides}\\
ax^2+bx &= -c && \text{divide both sides by } a \neq 0\\
x^2+\frac{bx}{a} &= -\frac{c}{a} && \text{add } \left(\frac{b}{2a}\right)^2 \text{ to both sides}\\
x^2+\frac{bx}{a}+\left(\frac{b}{2a}\right)^2 &= -\frac{c}{a}+\left(\frac{b}{2a}\right)^2 && \text{multiply the } x \text{ coefficient with } \frac{2}{2}=1\\
x^2+\frac{bx}{a}\cdot\frac{2}{2}+\left(\frac{b}{2a}\right)^2 &= -\frac{c}{a}+\frac{b^2}{4a^2} && \text{simplify and collect like terms}\\
x^2+2x\left(\frac{b}{2a}\right)+\left(\frac{b}{2a}\right)^2 &= \frac{-4ac+b^2}{4a^2}\\
x^2+2x\left(\frac{b}{2a}\right)+\left(\frac{b}{2a}\right)^2 &= \frac{b^2-4ac}{4a^2}\\
\left(x+\frac{b}{2a}\right)^2 &= \frac{b^2-4ac}{4a^2} && \text{take square root of both sides}\\
x+\frac{b}{2a} &= \pm\sqrt{\frac{b^2-4ac}{4a^2}} && \text{valid only when } b^2-4ac \geq 0\\
x+\frac{b}{2a} &= \frac{\pm\sqrt{b^2-4ac}}{2a} && \text{subtract } \frac{b}{2a} \text{ from both sides}\\
x+\frac{b}{2a}-\frac{b}{2a} &= \frac{\pm\sqrt{b^2-4ac}}{2a}-\frac{b}{2a}\\
x &= \frac{\pm\sqrt{b^2-4ac}-b}{2a}\\
x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}
\end{aligned}
$$

One should be careful when taking square roots on both sides of an equation — which is why there is "$\pm$"

3.5 Methods to Solve Quadratic Equations

A quadratic equation can be solved to obtain two values of x or the two roots of the equation. There are four different methods to find the roots of the quadratic equation. The four methods of solving the quadratic equations are as follows.

- Factorizing of Quadratic Equation
- Formula Method of Finding Roots
- Graphing Method to Find the Roots

Let us look in detail at each of the above methods to understand how to use these methods, their applications, and their uses.

3.6 Factorization of Quadratic Equation

Factorization of quadratic equation follows a sequence of steps. For a general form of the quadratic equation $ax^2 + bx + c = 0$, we need to first split the middle term into two terms, such that the product of the terms is equal to the constant term. Further, we can take the common terms from the available term, to finally obtain the required factors. For understanding factorization, the general form of the quadratic equation can be presented as follows.

$$\begin{aligned} x^2 + (m+n)x + mn &= 0 \\ x^2 + mx + nx + mn &= 0 \\ x(x+m) + n(x+m) &= 0 \\ (x+m)(x+n) &= 0 \end{aligned}$$

this means that $(x+m) = 0$ or $(x+n) = 0$, which implies that $x = -m$ or $x = -n$

Let us understand factorization through the below example.

$$\begin{aligned} x^2 + 7x + 12 &= 0 \\ x^2 + (4x + 3x) + 12 &= 0 \\ x^2 + 4x + 3x + 12 &= 0 \\ x(x+4) + 3(x+4) &= 0 \\ (x+3)(x+4) &= 0 \end{aligned}$$

this means that $(x+3) = 0$ or $(x+4) = 0$, which implies that $x = -3$ or $x = -4$

3.7 Quadratic Formula to Find Roots

The quadratic equations which cannot be solved through the method of factorization can be solved with the help of a formula. The formula to solve the quadratic equation uses the terms from the standard form of a quadratic equation. Through the below formula we can obtain the two roots of x by first using the positive sign in the formula and then using the negative sign. Any quadratic equation can be solved using this formula. Further to the above-mentioned

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Figure 3: Quadratic Formula [3]

two methods of solving quadratic equations, there is another important method of solving a quadratic equation.

3.8 Graphing a Quadratic Equation

The graph of the quadratic equation $ax^2 + bx + c = 0$ can be obtained by representing the quadratic equation as a function $y = ax^2 + bx + c$. Further on solving and substituting values for x, we can obtain values of y, we can obtain numerous points. These points can be presented in the coordinate axis to obtain a parabola-shaped graph for the quadratic equation. The point

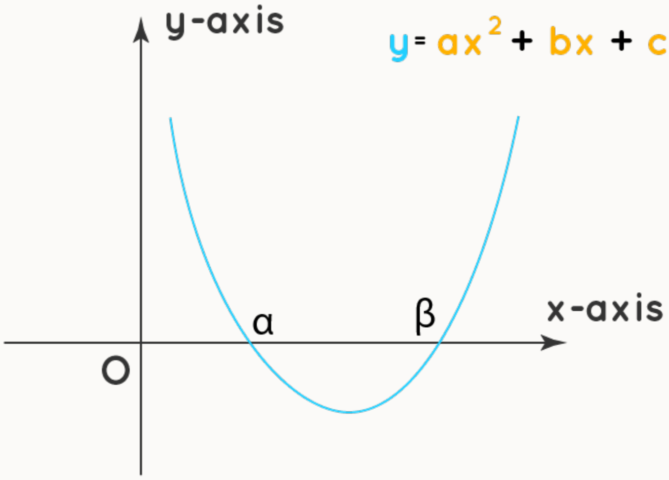


Figure 4: Graph of a Quadratic Equation [3]

where the graph cuts the horizontal x-axis is the solution of the quadratic equation. These points can also be algebraically obtained by equalizing the y value to 0 in the function $y = ax^2 + bx + c$ and solving for x.

Exercise 30 Solve the following quadratic equations by the method of factorisation:

(a) $x^2 - x - 6 = 0$
(b) $x^2 - 16 = 0$
(c) $x^2 - 2x = 0$
(d) $x^2 - 6x + 9 = 0$
(e) $6x^2 + 18x + 12 = 0$
(f) $6p^2 - 31p + 35 = 0$
(g) $6x^2 - 11x - 7 = 0$
(h) $-3r^2 - 14r + 5 = 0$
(i) $14x^2 = 29x - 12$

Exercise 31 Solve the following quadratic equations using the formula:

(a) $3x^2 - 8x + 2 = 0$
(b) $-2x^2 + 3x + 7 = 0$
(c) $4x^2 - 3x - 2 = 0$
(d) $7r^2 + 8r - 2 = 0$
(e) $x^2 + x + \frac{1}{4} = \frac{1}{9}$
(f) $5x^2 - 4x - 1 = 0$
(g) $2a^2 - 5.3a + 1.25 = 0$
(h) $x(x + 4) + 2x(x + 3) = 5$
(i) $\frac{3}{2x-3} - \frac{2}{x+1} = 5$
(j) $\frac{2}{x+2} - \frac{3}{x+1} = 5$

Exercise 32 Draw the graphs of the quadratic functions given below and indicate clearly where the curves intersect the x and y axes

(a) $y = (x+1)^2 - 1$
(b) $y = -(x+1)^2 - 1$
(c) $y = (x+2)^2 - 3$
(d) $y = (x+2)^2 + 3$
(e) $y = -(x+2)^2 + 3$

Exercise 33 The sum of a number and its reciprocal is $\frac{53}{14}$. What is the number?

Exercise 34 The sum of the first n natural numbers is given by the formula:

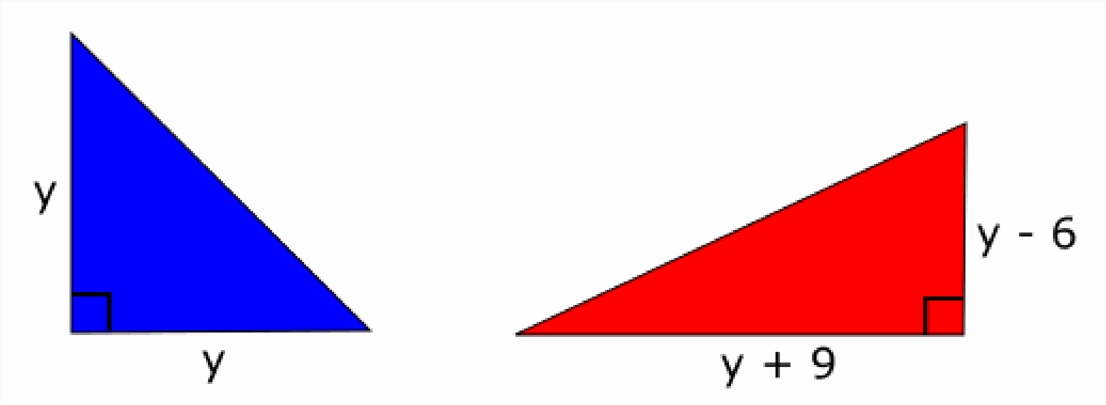
$$S_n = \frac{1}{2}n(n+1)$$

How many natural numbers added together give the sum 91?

Exercise 35 Auwal is two years older than Babatunde, and Chima is three years younger than Babatunde. The product of Auwal's age and Babatunde's age is 66. How old is Babatunde?

Exercise 36 The difference of a number and its reciprocal is $\frac{15}{4}$. What is the number?

Exercise 37 The areas of the triangles below are equal. Find the value of y.



(**Hint:** Use Area of triangle $= \frac{1}{2} \times$ base $\times$ height)

Exercise 38 The sum of the cubes of the first n natural numbers is given by the formula:

$$S_n^3 = \frac{1}{4}n^2(n+1)^2$$

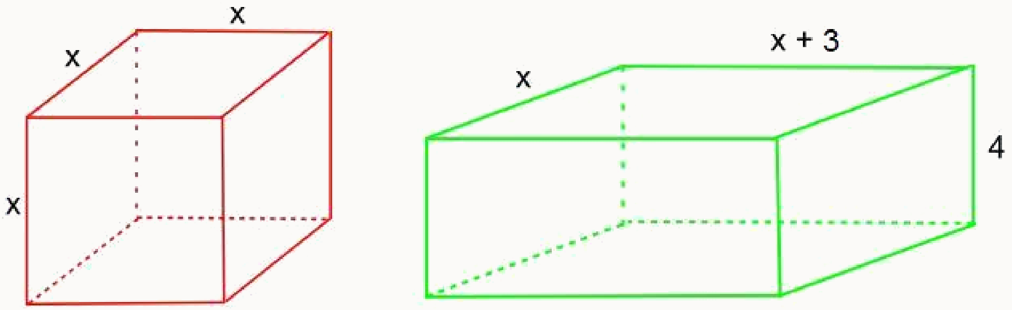
For what value of n is S_n^3 equal to 18496

Exercise 39 A stone is projected vertically upwards. Its height h meters after time t seconds is given by the equation:

$$h = 60t - 5t^2$$

After how many seconds is its height equal to 135 meters?

Exercise 40 The volumes of the cube and the rectangular prism are equal. What is the value of x?



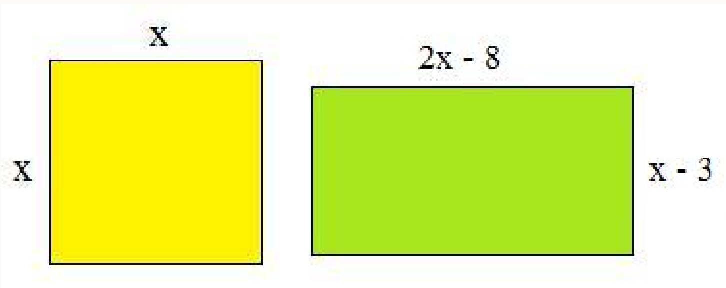
(**Hint:** Use Volume of a Rectangle $=$ length $\times$ height $\times$ width)

Exercise 41 A particle moves so that its distance d meters from the starting point (origin) after time t seconds is given by the equation

$$s = 50t - 150t^2$$

After how many seconds is the distance from the origin equal to 4 meters?

Exercise 42 The areas of the square and the rectangle are equal. What is the value of x?



4 Binomial Theorem

4.1 Polynomials

Polynomials are algebraic expressions that consist of variables and coefficients as shown in Figure!5 Polynomial is made up of two terms, namely **Poly** (meaning "many") and **Nominal**

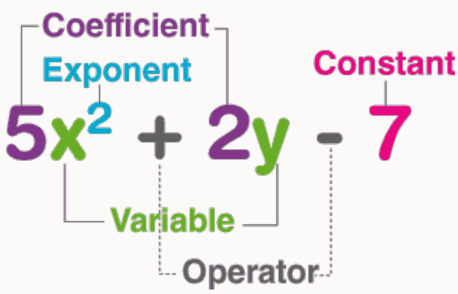


Figure 5: Polynomial [1]

(meaning ""terms"). A polynomial is defined as an expression which is composed of **variables**, **constants** and **exponents**, that are combined using the mathematical operations such as addition, subtraction, multiplication and division (No division operation by a variable). Based on the numbers of terms present in the expression, it is classified as **monomial**, **binomial**, and **trinomial**. Examples of constants, variables and exponents are as follows:

Constants: $1, 2, 3, 5, 6$ etc.
Variables. f, c, s, i, t etc.
Exponents: 5 in x^5 etc.

4.1.1 Notation for Polynomial

The polynomial function is denoted by $P(x)$ where x represents the variable. For example,

$$P(x) = x^2 - 5x + 11$$

If the variable is denoted by a, then the function will be $P(a)$.

4.1.2 Degree of a Polynomial

The degree of a polynomial is defined as the highest degree of a monomial within a polynomial. Thus, a polynomial equation having one variable which has the largest exponent is called a degree of the polynomial.

Polynomial	Degree	Example
Constant or Zero Polynomial	0	6
Linear Polynomial	1	$2x + 5$
Quadratic Polynomial	2	$8x^2 - 3x + 4$
Cubic Polynomial	3	$2x^3 + 3x^2 + 5x + 7$
Quartic Polynomial	4	$5x^4 + 3x^3 + 11x^2 - 13x + 5$

Example 19 Find the degree of the polynomial $6s^4 + 3x^2 + 5x + 19$

Solution The degree of the polynomial is 4. ■

4.1.3 Terms of a Polynomial

The terms of polynomials are the parts of the equation which are generally separated by "+" or "−" signs. So, each part of a polynomial in an equation is a term. For example, in a polynomial, say, $2x^2 + 5 + 4$, the number of terms will be 3. The classification of a polynomial is done based on the number of terms in it.

Polynomial	Terms	Degree
$P(x) = x^3 - 2x^2 + 3x + 4$	x^3, $-2x^2$, $3x$ and 4	3

4.1.4 Types of Polynomials

Polynomials are of 3 different types and are classified based on the number of terms in it. The three types of polynomials are:

Monomial: A monomial is an expression which contains only one term. For an expression to be a monomial, the single term should be a non-zero term. A few examples of monomials are: $5x$, 3, $6y^7$, $-6mn$

Binomial: A binomial is a polynomial expression which contains exactly two terms. A binomial can be considered as a sum or difference between two or more monomials. A few examples of binomials are: $-5x+3$, $7k^5+10j$, m^2n+nm

Trinomial: A trinomial is an expression which is composed of exactly three terms. A few examples of trinomial expressions are: $6k^5+3r-5$, x^8+23k^6-87

	Monomial	**Binomial**	**Trinomial**
Number of Terms	One Term	Two Terms	Three Terms
Example	x, $x^3y^6z^2$, $\frac{p}{4}$	$a+a^3$, $fc^2s^3+i^4t^5$	$cs^4+it^3+se^2$, y^3+y-12

These polynomials can be combined using addition, subtraction, multiplication, and division but is never division by a variable. A few examples of Non Polynomials are: $\dfrac{1}{x+2}$, x^{-3}.

4.2 Factorial, Permutation and Combination

4.2.1 Factorial

Definition 3 (Factorial) [14] In mathematics, the factorial of a non-negative integer n, denoted by $n!$, is the product of all positive integers less than or equal to n:

$$n! = n\times(n-1)\times(n-2)\times(n-3)\times\cdots\times 3\times 2\times 1 \text{ for } n\geq 1 \tag{19}$$

with $0! = 1$.

Example 20

$$7! = 7\times 6\times 5\times 4\times 3\times 2\times 1 = 5040$$

Note that Equation 19 can be written as:

$$n! = \prod_{i=1}^{n} i$$

This leads to the recurrence relation

$$n! = n\times(n-1)! \tag{20}$$

Example 21

$$\begin{aligned} 4! &= 4\times 3! \\ 12! &= 12\times 11! \\ 123! &= 123\times 122! \end{aligned}$$

Exercise 43 Evaluate (a) $2!$ (b) $5!$ (c) $0!$

4.2.2 Permutation

Definition 4 (Permutation) [15] A permutation is a mathematical calculation of the number of ways a particular set can be arranged, where the order of the arrangement matters. The formula for a permutation is:

$$P(n,r) = {}^nP_r = \frac{n!}{(n-r)!} \tag{21}$$

where

$$\begin{aligned} n =& \text{total items in the set} \\ r =& \text{items taken for the permutation} \\ ! =& \text{denotes factorial} \end{aligned}$$

The generalized expression of the formula is, "How many ways can you arrange r from a set of n if the order matters?".

A permutation of a set of objects is an arrangement of those objects into a particular order. For example, there are six permutations of the set $1, 2, 3$: $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, and $(3, 2, 1)$.

Example 22 Consider a race in which 3 different prizes are awarded to the top 3 fastest competitors. If 25 competitors participate in the race, in how many distinct orders could the 3 prizes be awarded?

Solution To solve this problem, we want to evaluate the number of possible permutations of 3 elements from the set of 25 elements; in other words, $r = 3$ and $n = 25$. Plugging these values into the Equation 21, we have:

$$\begin{aligned} P(25,3) = {}^{25}P_3 =& \frac{25!}{(25-3)!} \\ =& \frac{25!}{(22)!} \end{aligned}$$

by Equation 20: $n! = n(n-1)!$

$$\begin{aligned} \implies \frac{25!}{(22)!} =& \frac{25 \times 24 \times 23 \times 22!}{(22)!} \\ =& 25 \times 24 \times 23 \\ =& 13800 \end{aligned}$$

■

4.2.3 Combination

Definition 5 (Combination) A combination is a mathematical calculation of the number of ways a particular set can be arranged, where the order of the arrangement **does not** matter. The formula for a combination is:

$$C(n,r) = {}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} \tag{22}$$

where

$$\begin{aligned} n =& \text{total items in the set} \\ r =& \text{items taken for the combination} \\ ! =& \text{denotes factorial} \end{aligned}$$

The generalized expression of the formula is, "n choose r".

Example 23 Faculty of Computer Science and Information Technology at Bayero University Kano offers 12 course for fresh level 100 students. Each student must choose 6 of them to fulfil his/her minimum requirement. How many possible combinations can be made?

Solution To solve this problem, we want to evaluate the number of possible combinations of 6 elements from the set of 12 elements; in other words, $r = 6$ and $n = 12$. Plugging these values into the Equation 22, we have:

$$\begin{aligned} C(12,6) = {}^{12}C_6 = \binom{12}{6} &= \frac{12!}{6!(12-6)!} \\ &= \frac{12!}{6!(6!)} \\ &= \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6!}{6!(6!)} \\ &= \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7}{6!} \\ &= \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7}{6 \times 5 \times 4 \times 3 \times 2 \times 1} \\ &= 11 \times 2 \times 3 \times 2 \times 7 \\ &= 924 \end{aligned}$$

■

4.3 Pascal's Triangle

One of the most interesting Number Patterns is Pascal's Triangle.

Definition 6 (Pascal Triangle) Pascal's triangle is a triangular array constructed by summing adjacent elements in preceding rows. It is named after the 17th century French mathematician, *Blaise Pascal* (1623 – 1662).

To build the triangle, start with "1" at the top, then continue placing numbers below it in a triangular pattern.

Each number is the numbers directly above it added together.

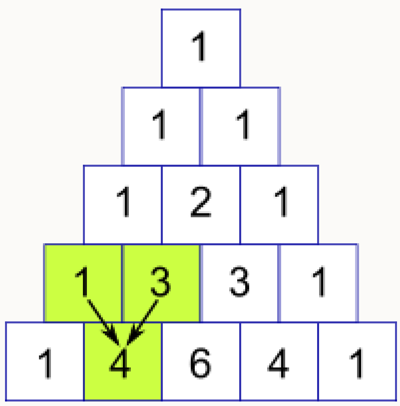


Figure 6: Pascal Triangle with $1 + 3 = 4$ highlighted [7]

4.3.1 Patterns Within the Triangle

Diagonals

Note that from Figure [**?**]

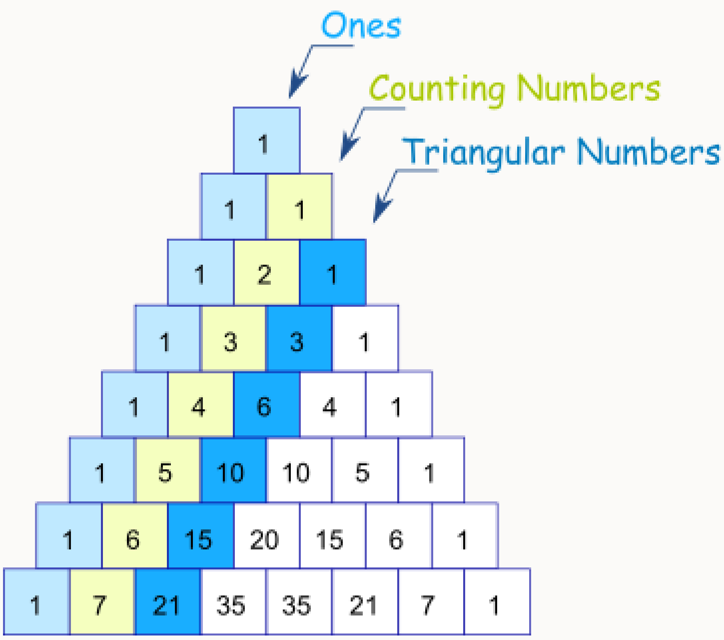


Figure 7: Pascal Triangle Pattern: Diagonals [7]

- The first diagonal is, of course, just "1"s
- The next diagonal has the Counting Numbers $\mathbb{N}$ (1,2,3, etc).
- The third diagonal has the triangular numbers. A triangular number is a number that can be represented by a pattern of dots arranged in an equilateral triangle with the same number of dots on each side as shown in Figure 8.

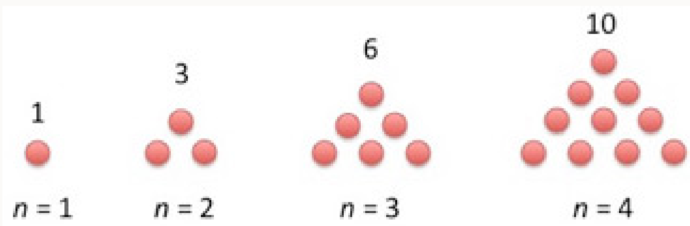


Figure 8: Triangular Numbers [11]

Symmetry

The triangle is also symmetrical. The numbers on the left side have identical matching numbers on the right side, like a mirror image as shown in Figure 9.

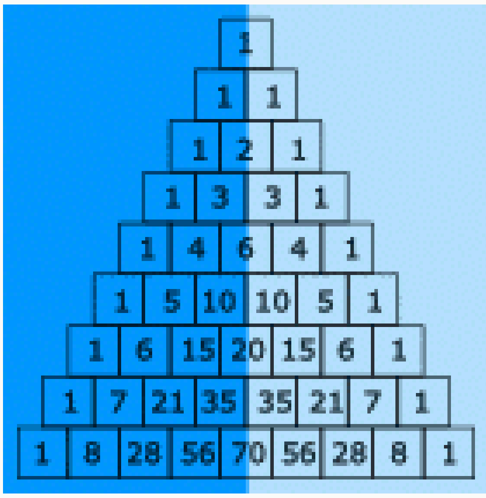


Figure 9: Pascal Triangle Pattern: Symmetry [7]

Horizontal Sums

The triangle is also symmetrical. The numbers on the left side have identical matching numbers on the right side, like a mirror image as shown in Figure 10.

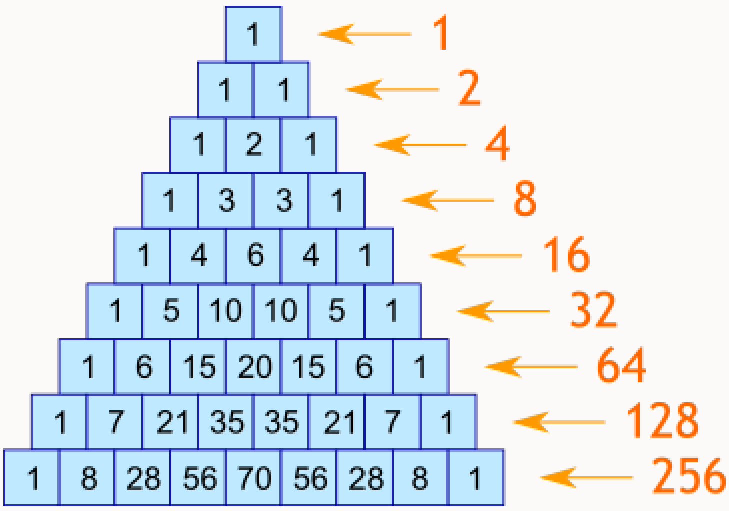


Figure 10: Pascal Triangle Pattern: Horizontal Sums [7]

Exponents of 11

Each line is also the powers (exponents) of 11 as shown in Figure 11:

$11^0 = 1$ (the first line is just a "1")
$11^1 = 11$ (the first line is just a "1" and "1")
$11^2 = 121$ (the first line is just a "1", "2" and "1"))

and so on. But what happens with 11^5? Simple! The digits just overlap, like this: The same thing happens with 11^6 etc.

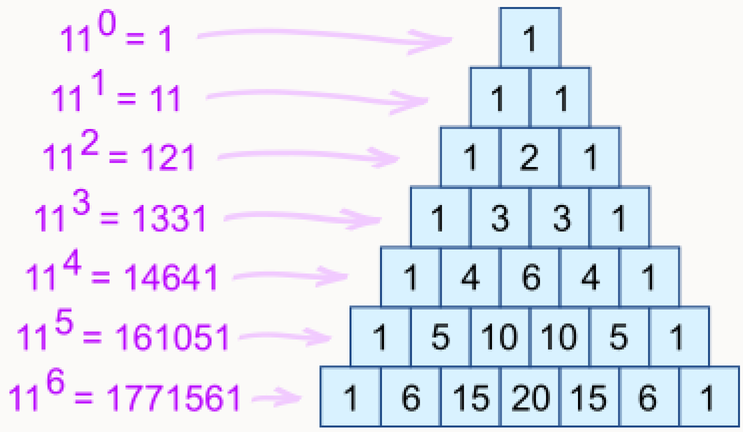
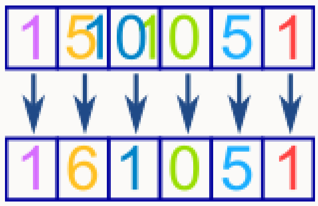


Figure 11: Pascal Triangle Pattern: Exponents of 11 [7]



The Pascal's triangle will be a valuable tool in our discussion on the Binomial Theorem.

4.4 Binomial Theorem

Remember: A binomial is a polynomial with two terms.

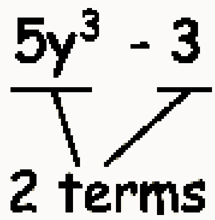


Figure 12: A Binomial [6]

What happens when we multiply a binomial by itself ... many times?

Example 24 $a + b$ is a binomial (the two terms are a and b). Let us multiply a+b by itself

$$(a + b)(a + b) = a^2 + 2ab + b^2$$

Now take that result and multiply by $a + b$ again:

$$(a^2 + 2ab + b^2)(a + b) = a^3 + 3a^2b + 3ab^2 + b^3$$

And again:

$$(a^3 + 3a^2b + 3ab^2 + b^3)(a + b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

The calculations get longer and longer as we go, but there is some kind of pattern developing. That pattern is summed up by the Binomial Theorem:

$$(a+b)^n = \sum_{k=0}^{n} = \binom{n}{k} a^{n-k} b^k \qquad \text{(\textbf{Binomial Theorem})}$$

We will now take a look at the component of the **Binomial Theorem**

4.4.1 Exponents

First, a quick summary of Exponents. An exponent says **how many times** to use something in a multiplication.

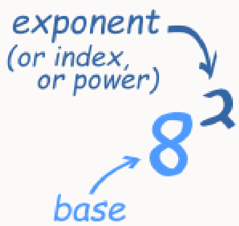


Figure 13: Exponent [6]

Example 25 $8^2 = 8 \times 8 = 64$

An exponent of 1 means just to have it appear once, so we get the original value:

Example 26 $8^1 = 8$

An exponent of 0 means not to use it at all, and we have only 1:

Example 27 $8^0 = 0$

4.4.2 Exponents of (a+b)

Now on to the binomial. We will use the simple binomial $a+b$, but it could be any binomial.

Let us start with an exponent of 0 and build upwards.

Exponent of 0 When an exponent is 0, we get 1:

$$(a+b)^0 = 1$$

Exponent of 1 When the exponent is 1, we get the original value, unchanged:

$$(a+b)^1 = (a+b)$$

Exponent of 2 An exponent of 2 means to multiply by itself

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$

Exponent of 3 For an exponent of 3 just multiply again:

$$(a+b)^3 = (a^2+2ab+b^2)(a+b) = a^3 + 3a^2b + 3ab^2 + b^3 \qquad (23)$$

We have enough now to start talking about the pattern.

4.4.3 The Pattern

In the last result, that is Equation 23, we got

$$a^3 + 3a^2b + 3ab^2 + b^3$$

Now, notice the exponents of a. They start at 3 and go down: 3, 2, 1, 0:

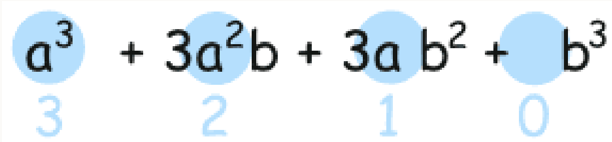


Figure 14: Exponent of a [6]

Likewise the exponents of b go upwards: 0, 1, 2, 3:

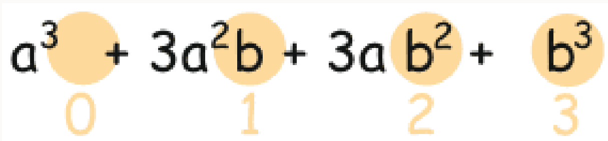


Figure 15: Exponent of b [6]

If we number the terms 0 to n, we get this:

$\mathbf{k=0}$	$\mathbf{k=1}$	$\mathbf{k=2}$	$\mathbf{k=3}$
a^3	a^2	$a^1 = a$	$a^0 = 1$
$b^0 = 1$	$b^1 = b$	b^2	b^3

Which can be brought together into this:

$$a^{n-k}b^k \tag{24}$$

Here is an example to see Equation 24 works:

$k=0$	$k=1$	$k=2$	$k=3$
$a^{n-k}b^k$ $= a^{3-0}b^0$ a^3	$a^{n-k}b^k$ $= a^{3-1}b^1$ a^2b	$a^{n-k}b^k$ $= a^{3-2}b^2$ ab^2	$a^{n-k}b^k$ $= a^{3-3}b^3$ b^3

4.4.4 Coefficients

So far we have: $a^3 + a^2b + ab^2 + b^3$

But we **really** need: $a^3 + 3a^2b + 3ab^2 + b^3$

We are missing the numbers (which are called **coefficients**).

Let's look at all the results we got before, from $(a+b)^0$ up to $(a+b)^3$:

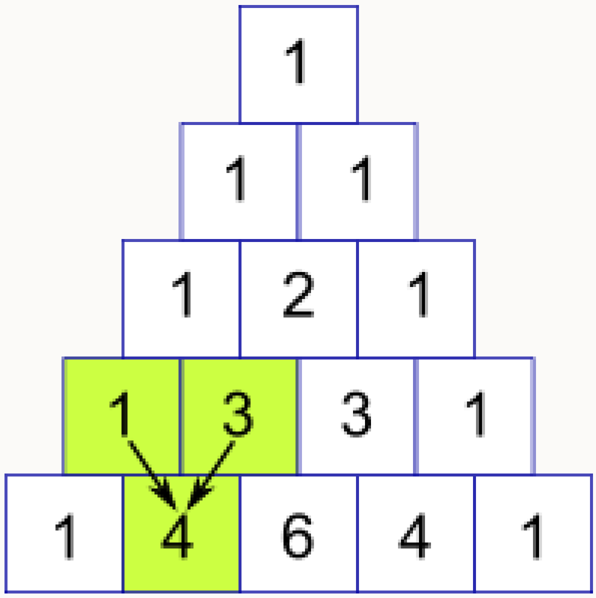
$$\begin{gathered} 1 \\ a+b \\ a^2+2ab+b^2 \\ a^3+3a^2b+3ab^2+b^3 \end{gathered}$$

And now look at just the coefficients (with a "1" where a coefficient wasn't shown):

$$\begin{gathered} \mathbf{1} \\ \mathbf{1}a+\mathbf{1}b \\ \mathbf{1}a^2+\mathbf{2}ab+\mathbf{1}b^2 \\ \mathbf{1}a^3+\mathbf{3}a^2b+\mathbf{3}ab^2+\mathbf{1}b^3 \end{gathered}$$

This is actually a **Pascal's Triangle!**

Remember: "Each number is just the two numbers above it added together (except for the edges, which are all "1")"



Equipped with this information let us try something new . . . an **exponent of 4**:

$$\begin{array}{rccccccccccc}
a \text{ exponents are } 4, 3.\ 2, 1, 0: & a^4 & + & a^3 & + & a^2 & + & a & + & 1 & \\
b \text{ exponents are } 0, 1, 2, 3, 4: & a^4 & + & a^3b & + & a^2b^2 & + & ab^3 & + & b^4 & \\
\textit{coefficients} \text{ are } 1, 4, 6, 4, 1: & a^4 & + & 4a^3b & + & 6a^2b^2 & + & 4ab^3 & + & b^4 & \checkmark
\end{array}$$

We can now use that pattern for exponents of 5, 6, 7, . . ., 50, . . ., 112, . . . you name it!

Exercise 44 Expand $(a+b)^5$

Solution $a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$ ■

4.5 Using a Formula

Our next task is to write it all as a formula. We already have the exponents figured out:

$$a^{n-k}b^k \qquad \text{(exponents)}$$

But how do we write a formula for "find the coefficient from Pascal's Triangle" ...? We can use the combination formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

And it matches to Pascal's Triangle as show in Figure 16 (Note how the top row is row zero and also the leftmost column is zero!) Pascals Triangle Combinations:

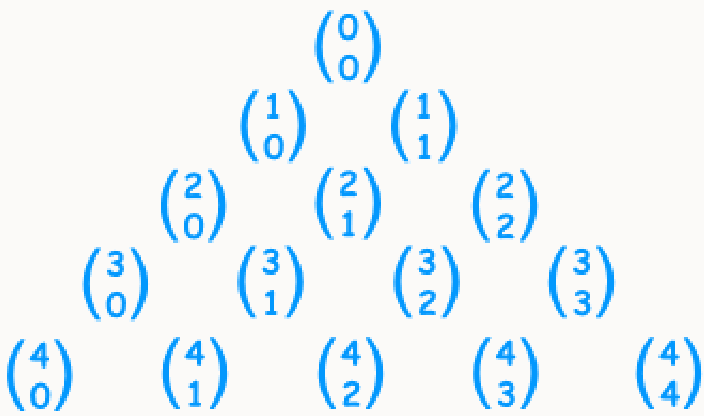


Figure 16: Pascal's Triangle and Combination a [6]

Example 28 Show that the value row 4, term 2 in Pascal's Triangle is 6.

Solution $n = 4$ (row) and $r = 2$ (term), therefore the value of row 4, term 2 is

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4!}{2!2!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1} = 6$$

■

Exercise 45 Evaluate the values of the terms in Figure 16 and compare them those of Figure 6

4.6 Putting ut All Together

The last step is to put all the terms together into one formula. But we are adding lots of terms together – can that be done using one formula?

Yes! The handy Sigma Notation allows us to sum up as many terms as we want: Now it can all go into one formula:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k \qquad \text{(\textbf{The Binomial Theorem})}$$

Example 29 Find $(a+b)^3$

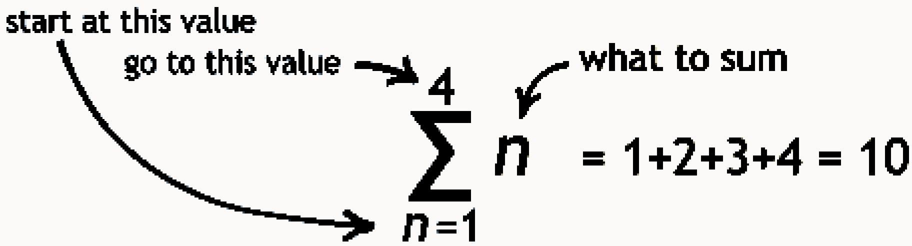


Figure 17: Sigma Notation a [6]

Solution

$$\begin{aligned}(a+b)^3 &= \sum_{k=0}^{3}\binom{3}{k}a^{3-k}b^k \\ &= \binom{3}{0}a^{3-0}b^0 + \binom{3}{1}a^{3-1}b^1 + \binom{3}{2}a^{3-2}b^2 + \binom{3}{3}a^{3-3}b^3 \\ &= 1\cdot a^3b^0 + 3\cdot a^2b^1 + 3\cdot a^1b^2 + 1\cdot a^0b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

■

BUT, it is usually much easier just to remember the patterns:

- The first term's exponents start at n and go down.
- The second term's exponents start at 0 and go up.
- Coefficients are from Pascal's Triangle, or by calculation using

$$\frac{n!}{k!(n-k)!}$$

Example 30 Expand $(y+5)^4$

Solution

Start with the exponents:	$y^4 5^0$	$y^3 5^1$	$y^2 5^2$	$y^1 5^3$	$y^0 5^4$
Include Coefficients:	$\mathbf{1} y^4 5^0$	$\mathbf{4} y^3 5^1$	$\mathbf{6} y^2 5^2$	$\mathbf{4} y^1 5^3$	$\mathbf{1} y^0 5^4$

Then write down the answer (including all calculations, such as 4×5, 6×52, etc):

$$(y+5)^4 = y^4 + 20y^3 + 150y^2 + 500y + 625$$

■

We may also want to calculate just one term:

Example 31 What is the coefficient for x^3 in $(2x+4)^8$.

Solution $n = 8$, so the exponent for x^3 is 3 and the exponent of the 4 is $8-3=5$.Why? Because: But we don't need to calculate all the other values if we only want one term. And let's

2x:	8	7	6	5	4	3	2	1	0
4:	0	1	2	3	4	5	6	7	8
	$(2x)^8 4^0$	$(2x)^7 4^1$	$(2x)^6 4^2$	$(2x)^5 4^3$	$(2x)^4 4^4$	$(2x)^3 4^5$	$(2x)^2 4^6$	$(2x)^1 4^7$	$(2x)^0 4^8$

not forget

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

"8 choose 5" – we can use Pascal's Triangle, or calculate directly:

$$\binom{8}{5} = \frac{8!}{5!(8-5)!} = \frac{8!}{5!(3)!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$$

And we get:

$$56(2x)^3 4^5$$

Which simplifies to:

$$458752x^3$$

■

4.6.1 Binomial Theorem Geometrically

The Binomial Theorem can be shown using Geometry:

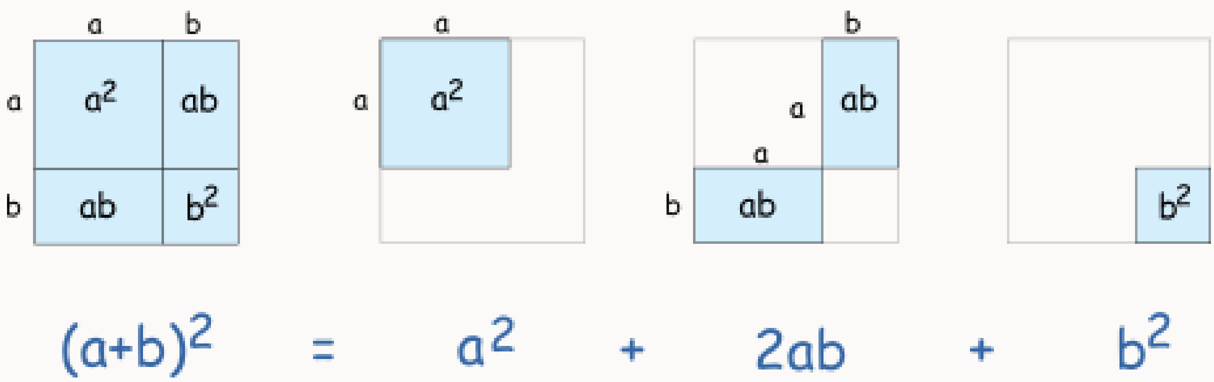


Figure 18: In 2 dimensions, $(a+b)^2 = a^2 + 2ab + b^2$ [6]

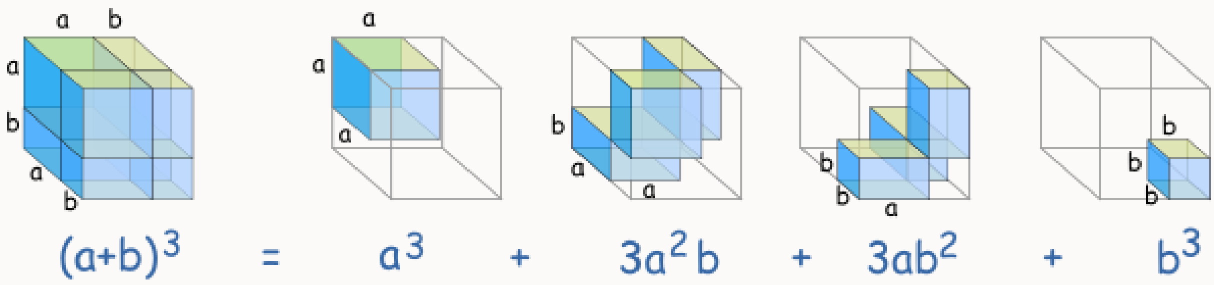


Figure 19: In 3 dimensions, $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ [6]

Exercise 46 Expand $(x+1)^6$

Solution $(x+1)^6 = x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$

■

Exercise 47 Expand $(2+x)^7$

Solution $(2+x)^7 = 128 + 448x + 672x^2 + 560x^3 + 280x^4 + 84x^5 + 14x^6 + x^7$

■

Exercise 48 Expand $(2x-3)^5$

Solution $(2x-3)^5 = 32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243$ ■

Exercise 49 Expand $(3x-5)^4$

Solution $(3x-5)^4 = 81x^4 - 540x^3 + 1350x^2 - 1500x + 625$ ■

Exercise 50 What is the binomial expansion of $(x+1)^8$?

Solution $(x+1)^8 = x^8 + 8x^7 + 28x^6 + 56x^5 + 70x^4 + 56x^3 + 28x^2 + 8x + 1$ ■

Exercise 51 What is the 5th term in the expansion of $(x+2)^7$ (in decreasing powers of x)?

Solution $580x^3$ ■

Exercise 52 What is the 3rd term in the expansion of $(y-3)^6$ (in decreasing powers of y)?

Solution $135y^4$ ■

Exercise 53 What is the 7th term in the expansion of $(5x+2)^8$ (in decreasing powers of x)?

Solution $44800x^2$ ■

Exercise 54 What is the 2nd term in the expansion of $(3z-4)^9$ (in decreasing powers of z)?

Solution $-236196z^8$ ■

Exercise 55 What is the 3rd term in the expansion of $(7y-5z)^4$ (in decreasing powers of y)?

Solution $7350y^2z^2$ ■

References

[1] BYJU'S (2021a). Polynomials (definition, types and examples). https://byjus.com/maths/polynomial/.

[2] BYJU'S (2021b). Sequence and series-definition, types, formulas and examples. https://byjus.com/maths/sequence-and-series/.

[3] Cue Learn Private Limited (2021). Quadratic equation - formula, examples quadratic formula. https://www.cuemath.com/algebra/quadratic-equations/.

[4] Lumen Learning (2021a). The binomial theorem boundless algebra. https://courses.lumenlearning.com/boundless-algebra/chapter/the-binomial-theorem/.

[5] Lumen Learning (2021b). Combinatorics boundless algebra. https://courses.lumenlearning.com/boundless-algebra/chapter/combinatorics.

[6] MathisFun (2021a). Binomial theorem.html. https://www.mathsisfun.com/algebra/binomial-theorem.html.

[7] MathisFun (2021b). Pascal's triangle. https://www.mathsisfun.com/pascals-triangle.html.

[8] mathsmutt.co.uk (2021). Infinite series. http://www.mathsmutt.co.uk/files/inser.html.

[9] plus.maths.org (2021). Maths in a minute triangular numbers. https://plus.maths.org/content/maths-minute-triangular-numbers.

[10] Saylor Academy (2012). Arithmetic sequences and series. https://saylordotorg.github.io/text_intermediate-algebra/s12-02-arithmetic-sequences-and-serie.html.

[11] Study.com (2021). What are triangular numbers - definition, formula & examples. https://study.com/academy/lesson/what-are-triangular-numbers-definition-formula-examples.html.

[12] Department of Mathematics, University of Toronto (2021a). Sequences and series - worked examples. https://www.math.toronto.edu/preparing-for-calculus/9_sequences/we_1_sequences.html.

[13] Department of Mathematics, University of Toronto (2021b). Sequences and series - worked examples. https://www.math.toronto.edu/preparing-for-calculus/9_sequences/we_2_series.html.

[14] Encyclopedia of Mathematics (2021). Factorial. http://encyclopediaofmath.org/index.php?title=Factorial&oldid=44632.

[15] Investpedia (2021). Permutation definition. https://www.investopedia.com/terms/p/permutation.asp.