# CSC2323 Discrete Structures Lecture Notes

Baffa Sani and Abubakar Sadiq Mustapha

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## Contents



Reference books:

1. Discrete Mathematics and Its Applications By Kenneth H. Rosen
2. discrete mathematics with applications By Susanna S. Epp

# 1. Propositional Logic and Predicate Logic

## Introduction

The rules of logic give precise meaning to mathematical statements. These rules are used to distinguish between valid and invalid mathematical arguments. Because a major goal of this book is to teach the reader how to understand and how to construct correct mathematical arguments, we begin our study of discrete mathematics with an introduction to logic.

Besides the importance of logic in understanding mathematical reasoning, logic has numerous applications to computer science. These rules are used in the design of computer circuits, the construction of computer programs, the verification of the correctness of programs, and in many other ways. Furthermore, software systems have been developed for constructing some, but not all, types of proofs automatically.

## 1.1 Propositions

A **proposition** (or **statement**) is a declarative sentence (that is, a sentence that declares a fact) that is either ***true*** or ***false***, but not both.

**Example 1.1.** Consider, the following sentences:.

1. Abuja is the capital of Nigeria.
2. $1 + 1 = 2$.
3. $2 + 2 = 3$.
4. China is in Europe.
5. Where are you going?
6. Do your homework.
7. $x + 1 = 2$.

The first four are propositions, the last three are not. Also, (i) and (ii) are true, but (iii) and (iv) are false.

We use letters to denote propositional variables (or sentential variables), that is, variables that represent propositions, just as letters are used to denote numerical variables. The conventional letters used for propositional variables are p, q, r, s,... . The truth value of a proposition is true, denoted by T, if it is a true proposition, and the truth value of a proposition is false, denoted by F, if it is a false proposition.

### Compound Propositions

Many propositions are *composite*, that is, composed of *subpropositions* and various connectives discussed subsequently. Such composite propositions are called **compound propositions**. A proposition is said to be **primitive** (or **atomic**) **propositions** if it cannot be broken down into simpler propositions, that is, if it is not composite.

For example, the above propositions (i) through (iv) are primitive propositions. On the other hand, the following two propositions are composite:

"It is hot and it is not sunny." and "John is smart or he studies every night."

## 1.2 Basic Logical Operations

This section discusses the three basic logical operations of conjunction, disjunction, and negation which correspond, respectively, to the English words "and," "or," and "not."

### Conjunction, $p \wedge q$

**Definition 1: Conjunction**

Let $p$ and $q$ be propositions. The ***conjunction*** of $p$ and $q$, denoted by $p \wedge q$, is the proposition "$p$ and $q$." The conjunction $p \wedge q$ is **true** when both $p$ and $q$ are true and is false otherwise.

Table 1: Truth Table for $p \wedge q$

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

**Example 1.2.** Consider the following four statements:

1. Abuja is the capital of Nigeria and $2+2=4$.
2. Abuja is the capital of Nigeria and $2+2=5$
3. China is in Europe and $2+2=4$.
4. China is in Europe and $2+2=5$.

Only the first statement is true. Each of the others is false since at least one of its substatements is false.

**Definition 2: Disjunction**

Let $p$ and $q$ be propositions. The ***disjunction*** of $p$ and $q$, denoted by $p \vee q$, is the proposition "$p$ or $q$." The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

Table 2: Truth Table for $p \vee q$

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

**Example 1.3.** Consider the following four statements:

1. Abuja is the capital of Nigeria and $2+2=4$.
2. Abuja is the capital of Nigeria and $2+2=5$
3. China is in Europe and $2+2=4$.
4. China is in Europe and $2+2=5$.

Only the last statement (4) is false. Each of the others is true since at least one of its sub-statements is true.

**Remark**: The English word "or" is commonly used in two distinct ways. Sometimes it is used in the sense of "$p$ or $q$ or both," i.e., at least one of the two alternatives occurs, as above, and sometimes it is used in the sense of "$p$ or $q$ but not both," i.e., exactly one of the two alternatives occurs. For example, the sentence "He will go to BUK or to ABU" uses "or" in the latter sense, called the ***exclusive disjunction***. Unless otherwise stated, "or" shall be used in the former sense. This discussion points out the precision we gain from our symbolic language: $p \vee q$ is defined by its truth table and always means "$p$ and/or $q$."

**Definition 3: Negation**

If $p$ is a proposition, the ***negation*** of $p$ is "not $p$" or "It is not the case that $p$" and is denoted $\sim p$. It has opposite truth value from $p$: if $p$ is true, $\sim p$ is false; if $p$ is false, $\sim p$ is true.

Table 3: Truth Table for $\sim p$

| $p$ | $\sim p$ |
|---|---|
| T | F |
| F | T |

**Example 1.4.** Consider the following four statements:

1. Ice floats in water.
2. It is false that ice floats in water.
3. Ice does not float in water.
4. $2+2=5$
5. It is false that $2+2=5$.
6. $2+2\neq 5$

Then (2) and (3) are each the negation of (1); and (5) and (6) are each the negation of (4).

**Remark**: The logical notation for the connectives "and," "or," and "not" is not completely standardized. For example, some texts use:

$$p \,\&\, q,\ p\,.\,q \text{ or } pq \textbf{ for } p\wedge q$$

$$p+q \textbf{ for } p\vee q$$

$$p',\ \bar{p} \text{ or } \neg p \textbf{ for } \sim p$$

## 1.3 Propositions and Truth Tables

**Definition 4: Compound Statements**

Let $P(p,q,...)$ denote an expression constructed from logical variables $p,q,...$, which take on the value TRUE (T) or FALSE (F), and the logical connectives $\wedge$, $\vee$, and $\sim$ (and others discussed subsequently). Such an expression $P(p,q,...)$ will be called a ***proposition***.

The main property of a proposition $P(p,q,...)$ is that its truth value depends exclusively upon the truth values of its variables, that is, the truth value of a proposition is known once the truth value of each of its variables is known. A simple concise way to show this relationship is through a truth table. We describe a way to obtain such a truth table below.

**Example 1.5.**

Consider, the proposition $\sim(p\wedge\sim q)$. Table 4 indicates how the truth table of $\sim(p\wedge\sim q)$ is constructed. Observe that the first columns of the table are for the variables $p,q,...$ and that there are enough rows in the table, to allow for all possible combinations of **T** and **F** for these variables. (For 2 variables, as above, 4 rows are necessary; for 3 variables, 8 rows are necessary; and, in general, for $n$ variables, $2^n$rows are required.) There is then a column for each "elementary" stage of the construction of the proposition, the truth value at each step being determined from the previous stages by the definitions of the connectives $\wedge$, $\vee$, $\sim$. Finally we obtain the truth value of the proposition, which appears in the last column.

Table 4: Truth Table for $\sim (p \wedge \sim q)$

| $p$ | $q$ | $\sim q$ | $p \wedge \sim q$ | $\sim (p \wedge \sim q)$ |
|---|---|---|---|---|
| T | T | F | F | T |
| T | F | T | T | F |
| F | T | F | F | T |
| F | F | T | F | T |

**Alternate Method for Constructing a Truth Table**

Another way to construct the truth table for $\sim (p \wedge \sim q)$ follows:

(a) First we construct the truth table shown in Table 5. That is, first we list all the variables and the combinations of their truth values. Also there is a final row labeled "step." Next the proposition is written on the top row to the right of its variables with sufficient space so there is a column under each variable and under each logical operation in the proposition. Lastly (Step 1), the truth values of the variables are entered in the table under the variables in the proposition.

(b) Now additional truth values are entered into the truth table column by column under each logical operation as shown in Table 5. We also indicate the step in which each column of truth values is entered in the table.

Table 5: Truth Table for $\sim (p \wedge \sim q)$

| $p$ | $q$ | $\sim$ | $(p$ | $\wedge$ | $\sim$ | $q)$ |
|---|---|---|---|---|---|---|
| T | T | T | T | F | F | T |
| T | F | F | T | T | T | F |
| F | T | T | F | F | F | T |
| F | F | T | F | F | T | F |
| Step | | 4 | 1 | 3 | 2 | 1 |

**Example 1.6.** Construct the truth table for the statement form $(p \vee q) \wedge \sim (p \wedge q)$.

*Solution.* Set up columns labeled $p$, $q$, $(p \vee q)$, $(p \wedge q)$, $\sim (p \wedge q)$, and $(p \vee q) \wedge \sim (p \wedge q)$. Fill in the $p$ and $q$ columns with all the logically possible combinations of T's and F's. Then use the truth tables for $\vee$ and $\wedge$ to fill in the $(p \vee q)$ and $(p \wedge q)$ columns with the appropriate truth values. Next fill in the $\sim (p \wedge q)$ column by taking the opposites of the truth values for $(p \wedge q)$.

Table 6: Truth Table for $(p \vee q) \wedge \sim (p \wedge q)$

| $p$ | $q$ | $(p \vee q)$ | $(p \wedge q)$ | $\sim (p \wedge q)$ | $(p \vee q) \wedge \sim (p \wedge q)$ |
|---|---|---|---|---|---|
| T | T | T | T | F | F |
| T | F | T | F | T | T |
| F | T | T | F | T | T |
| F | F | F | F | T | F |

■

**Example 1.7.** Construct the truth table for the statement form $(p \wedge q) \vee \sim r$.

*Solution.* Make columns headed $p$, $q$, $r$, $p \wedge q$, $\sim r$, and $(p \wedge q) \vee \sim r$. Enter the eight logically possible combinations of truth values for p, q, and r in the three left-most columns. Then fill in the truth values for p ' q and for ,r. Complete the table by considering the truth values for $(p \wedge q)$ and for $\sim r$ and the definition of an *or* statement. ■

Table 7: Truth Table for $(p \wedge q) \vee \sim r$

| $p$ | $q$ | $r$ | $p \wedge q$ | $\sim r$ | $(p \wedge q) \vee \sim r$ |
|---|---|---|---|---|---|
| T | T | T | T | F | T |
| T | T | F | T | T | T |
| T | F | T | F | F | F |
| T | F | F | F | T | T |
| F | T | T | F | F | F |
| F | T | F | F | T | T |
| F | F | T | F | F | F |
| F | F | F | F | T | T |

## 1.4 Logical Equivalence

**Definition 5: Logical Equivalence**

Two propositions $P(p, q, ...)$ and $Q(p, q, ...)$ are said to be logically equivalent, or simply equivalent or equal, denoted by

$$P(p, q, ...) \equiv Q(p, q, ...)$$

if they have identical truth tables.

**Example 1.8.** Show that $\sim (p \wedge q)$ and $(\sim p \vee \sim q)$ are logically equivalent

*Solution.* Observe that both truth tables are the same, that is, both propositions are false in the first case and true in the other three cases.

Table 8: Truth Table for $\sim (p \wedge q)$ and $(\sim p \vee \sim q)$

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \wedge q$ | $\sim (p \wedge q)$ | $\sim p \vee \sim q$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | F | F |
| T | F | F | T | F | T | T |
| F | T | T | F | F | T | T |
| F | F | T | T | F | T | T |

Symbolically,

$$\sim (p \wedge q) \equiv (\sim p \vee \sim q)$$

■

**Example 1.9.** Show that $\sim (p \vee q)$ and $(\sim p \wedge \sim q)$ are logically equivalent

*Solution.* Exercise!

■

## 1.5 Tautologies and Contradictions

**Definition 6: Tautologies**

Some propositions $P(p, q, ...)$ contain only T in the last column of their truth tables or, in other words, they are true for any truth values of their variables. Such propositions are called tautologies.

**Definition 7: Contradictions**

A proposition $P(p, q, ...)$ is called a contradiction if it contains only F in the last column of its truth table or, in other words, if it is false for any truth values of its variables.

**Example 1.10.** Show that $(p \vee \sim p)$ is tautology

*Solution.* We draw the truth table for $(p \vee \sim p)$ as follows: ■

Table 9: Truth Table for $(p \vee \sim p)$

| $p$ | $\sim p$ | $(p \vee \sim p)$ |
|---|---|---|
| T | F | T |
| F | T | T |

**Example 1.11.** Show that $(p \wedge \sim p)$ is contradiction

*Solution.* We draw the truth table for $(p \wedge \sim p)$ as follows: ■

Table 10: Truth Table for $(p \wedge \sim p)$

| $p$ | $\sim p$ | $(p \wedge \sim p)$ |
|---|---|---|
| T | F | F |
| F | T | F |

**Example 1.12.** If $\mathbf{t}$ is a tautology and $\mathbf{c}$ is a contradiction, show that $p \wedge \mathbf{t} \equiv p$ and $p \wedge \mathbf{c} \equiv \mathbf{c}$.

*Solution.* We draw the truth table for $p \wedge \mathbf{t} \equiv p$ and $p \wedge \mathbf{c} \equiv \mathbf{c}$ as follows: ■

Table 11: Truth Table for $p \wedge \mathbf{t} \equiv p$ and $p \wedge \mathbf{c} \equiv \mathbf{c}$

| $p$ | $\mathbf{t}$ | $p \wedge \mathbf{t}$ | $\mathbf{c}$ | $p \wedge \mathbf{c}$ |
|---|---|---|---|---|
| T | T | T | F | F |
| F | T | F | F | F |

**Theorem 1.1.** *Logical Equivalences*

Given any statement variables $p$, $q$, and $r$, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

| | | |
|---|---|---|
| 1. Commutative laws: | $p \wedge q \equiv q \wedge p$ | $p \vee q \equiv q \vee p$ |
| 2. Associative laws: | $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ | $(p \vee q) \vee r \equiv p \vee (q \vee r)$ |
| 3. Distributive laws: | $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ | $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ |
| 4. Identity laws: | $p \wedge \mathbf{t} \equiv p$ | $p \vee \mathbf{c} \equiv p$ |
| 5. Negation laws: | $p \vee \sim p \equiv \mathbf{t}$ | $p \wedge \sim p \equiv \mathbf{c}$ |
| 6. Double negative law: | $\sim(\sim p) \equiv p$ | |
| 7. Idempotent laws: | $p \wedge p \equiv p$ | $p \vee p \equiv p$ |
| 8. Universal bound laws: | $p \vee \mathbf{t} \equiv \mathbf{t}$ | $p \wedge \mathbf{c} \equiv \mathbf{c}$ |
| 9. De Morgan's laws: | $\sim(p \wedge q) \equiv \sim p \vee \sim q$ | $\sim(p \vee q) \equiv \sim p \wedge \sim q$ |
| 10. Absorption laws: | $p \vee (p \wedge q) \equiv p$ | $p \wedge (p \vee q) \equiv p$ |
| 11. Negations of $\mathbf{t}$ and $\mathbf{c}$: | $\sim \mathbf{t} \equiv \mathbf{c}$ | $\sim \mathbf{c} \equiv \mathbf{t}$ |

**Example 1.13.** Prove theorem 1.1

*Solution.* Exercise! ■

**Example 1.14.** Use Theorem 1.1 to verify the logical equivalence $\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$

*Solution.* Use the laws of Theorem 1.1 to replace sections of the statement form on the left by logically equivalent expressions. Each time you do this, you obtain a logically equivalent statement form. Continue making replacements until you obtain the statement form on the right.

$$\begin{aligned}
\sim(\sim p \wedge q) \wedge (p \vee q) &\equiv (\sim(\sim p) \vee \sim q) \wedge (p \vee q) && \text{by De Morgan's laws}\\
&\equiv (p \vee \sim q) \wedge (p \vee q) && \text{by the double negative law}\\
&\equiv p \vee (\sim q \wedge q) && \text{by the distributive law}\\
&\equiv p \vee (q \wedge \sim q) && \text{by the commutative law for } \wedge\\
&\equiv p \vee \mathbf{c} && \text{by the negation law}\\
&\equiv p && \text{by the identity law}
\end{aligned}$$

■

**Example 1.15.** Show that $\sim(p \vee (\sim p \wedge q))$ and $\sim p \wedge \sim q$ are logically equivalent by developing a series of logical equivalences.

*Solution.* Use the laws of Theorem 1.1 to replace sections of the statement form on the left by logically equivalent expressions. Each time you do this, you obtain a logically equivalent statement form. Continue making replacements until you obtain the statement form on the right.

$$\begin{aligned}
\sim(p \vee (\sim p \wedge q)) &\equiv \sim p \wedge \sim(\sim p \wedge q) && \text{by the second De Morgan law}\\
&\equiv \sim p \wedge (\sim(\sim p) \vee \sim q) && \text{by the first De Morgan law}\\
&\equiv \sim p \wedge (p \vee \sim q) && \text{by the double negation law}\\
&\equiv (\sim p \wedge p) \vee (\sim p \wedge \sim q) && \text{by the second distributive law}\\
&\equiv \mathbf{c} \vee (\sim p \wedge \sim q) && \text{because } \sim p \wedge p \equiv \mathbf{c}\\
&\equiv (\sim p \wedge \sim q) \vee \mathbf{c} && \text{by the commutative law for disjunction}\\
&\equiv (\sim p \wedge \sim q) && \text{by the identity law for } \mathbf{c}
\end{aligned}$$

■

## 1.6 Conditional and Biconditional Statements

**Definition 8: Conditional**

Let $p$ and $q$ be propositions. The conditional statement $p \rightarrow q$ is the proposition "if $p$, then $q$." The conditional statement $p \rightarrow q$ is false when $p$ is true and $q$ is false, and true otherwise. In the conditional statement $p \rightarrow q$, $p$ is called the **hypothesis** (or **antecedent** or **premise**) and $q$ is called the **conclusion** (or **consequence**).

The conditional $p \rightarrow q$ is frequently read "$p$ implies $q$" or "$p$ only if $q$." In expressions that include $\rightarrow$ as well as other logical operators such as $\wedge$, $\vee$, and $\sim$, the **order of operations** is that $\rightarrow$ is performed last. Thus, according to the specification of order of operations, $\sim$ is performed first, then $\wedge$ and $\vee$, and finally $\rightarrow$.

**Example 1.16.** Construct a truth table for the statement form $p \vee \sim q \rightarrow \sim p$.

*Solution.* By the order of operations given above: ■

Table 12: Truth Table for $p \to q$

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

Table 13: Truth Table for $p\vee \sim q \to \sim p$

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p\vee \sim q$ | $p\vee \sim q \to \sim p$ |
|---|---|---|---|---|---|
| T | T | F | F | T | F |
| T | F | F | T | T | F |
| F | T | T | F | F | T |
| F | F | T | T | T | T |

**Example 1.17.** Show That $p \vee q \to r \equiv (p \to r) \wedge (q \to r)$ ■

*Solution.* Exercise.

**Representation of If-Then as Or**

The truth table of $\sim p \vee q$ and $p \to q$ are identical, that is, they are both *false* only in the second case. Accordingly, $p \to q$ is logically equivalent to $\sim p \vee q$; that is,

$$p \to q \equiv \sim p \vee q$$

**Example 1.18.** Show that $\sim (p \to q)$ and $p\wedge \sim q$ are logically equivalent.

*Solution.* We could use a truth table to show that these compound propositions are equivalent. Indeed, it would not be hard to do so. However, we want to illustrate how to use logical identities that we already know to establish new logical identities, something that is of practical importance for establishing equivalences of compound propositions with a large number of variables. So, we will establish this equivalence by developing a series of logical equivalences, using one of the equivalences in Theorem 1.1

| | | |
|---|---|---|
| $\sim (p \to q)$ | $\equiv \sim (\sim p \vee q)$ | by the conditional-disjunction equivalence |
| | $\equiv \sim (\sim p)\wedge \sim q$ | by the second De Morgan law |
| | $\equiv p\wedge \sim q$ | by the double negation law |

■

**Example 1.19.** Show that $(p \wedge q) \to (p \wedge q)$ is a tautology.

*Solution.* To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to **t**. (Note: This could also be done using a truth table.)

| | | |
|---|---|---|
| $(p \wedge q) \to (p \vee q)$ | $\equiv \sim (p \wedge q) \vee (p \vee q)$ | by the conditional-disjunction equivalence |
| | $\equiv (\sim p\vee \sim q) \vee (p \vee q)$ | by the first De Morgan law |
| | $\equiv (\sim p \vee p) \vee (\sim q \vee q)$ | by the associative and commutative laws for disjunctic |
| | $\equiv \mathbf{t} \vee \mathbf{t}$ | by Example 1 and the commutative law for disjunctior |
| | $\equiv \mathbf{t}$ | by the domination law |

■

### The Contrapositive of a Conditional Statement

**Definition 9: Contrapositive**

The **contrapositive** of a conditional statement of the form "If **p** then **q**" is

$$\text{If} \sim q \text{ then} \sim q.$$

Symbolically,

$$\text{The contrapositive of } p \rightarrow q \text{ is } \sim q \rightarrow \sim p.$$

**Example 1.20.** Write each of the following statements in its equivalent contrapositive form:

a If John can swim across the lake, then John can swim to the island.

b If today is Sunday, then tomorrow is Monday.

*Solution.* a If John cannot swim to the island, then John cannot swim across the lake.

b If tomorrow is not Monday, then today is not Sunday.

∎

When you are trying to solve certain problems, you may find that the contrapositive form of a conditional statement is easier to work with than the original statement. Replacing a statement by its contrapositive may give the extra push that helps you over the top in your search for a solution. This logical equivalence is also the basis for the contrapositive method of proof

### The Converse and Inverse of a Conditional Statement

**Definition 10: Converse and Inverse**

Suppose a conditional statement of the form "If $p$ then $q$" is given.

1. The **converse** is "If $q$ then $p$."
2. The **inverse** is "If $\sim p$ then $\sim q$."

Symbolically,

$$\text{The converse of } p \rightarrow q \text{ is } q \rightarrow p,$$

and

$$\text{The inverse of } p \rightarrow q \text{ is } \sim p \rightarrow \sim q.$$

**Example 1.21.** Write the converse and inverse of each of the following statements:

a If John can swim across the lake, then John can swim to the island.

b If today is Sunday, then tomorrow is Monday.

*Solution.* a *Converse*: If Howard can swim to the island, then Howard can swim across the lake.
*Inverse*: If John cannot swim across the lake, then John cannot swim to the island.

b *Converse*: If tomorrow is Monday, then today is Sunday.
*Inverse*: If today is not Sunday, then tomorrow is not Monday.

∎

1. A conditional statement and its converse are not logically equivalent.
2. A conditional statement and its inverse are not logically equivalent.
3. The converse and the inverse of a conditional statement are logically equivalent to each other.

**Biconditional**

**Definition 11: Biconditional**

Given statement variables $p$ and $q$, the biconditional of p and q is "$p$ if, and only if, $q$" and is denoted $p \leftrightarrow q$. It is true if both $p$ and $q$ have the same truth values and is false if $p$ and $q$ have opposite truth values. The words if and only if are sometimes abbreviated *iff*.

The biconditional has the following truth table:

Table 14: Truth Table for $p \leftrightarrow q$

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

**Order of Operations for Logical Operators**

1. $\sim$ Evaluate negations first.
2. $\wedge, \vee$ Evaluate $\wedge$ and $\vee$ second. When both are present.
3. $\rightarrow, \leftrightarrow$ Evaluate $\rightarrow$ and $\leftrightarrow$ third. When both are present.

According to the separate definitions of *if* and only *if*, saying "$p$ if, and only if, $q$" should mean the same as saying both "$p$ if $q$" and "$p$ only if $q$." The following annotated truth table shows that this is the case:

**Example 1.22.** Rewrite the following statement as a conjunction of two if-then statements:

> This computer program is correct if, and only if, it produces correct answers for all possible sets of input data.

*Solution.* If this program is correct, then it produces the correct answers for all possible sets of input data; and if this program produces the correct answers for all possible sets of input data, then it is correct. ■

## 1.7 Arguments

**Definition 12: Arguments**

An *argument* is an assertion that a given set of propositions $P_1, P_2, ..., P_n$, called *premises*, yields (has a consequence) another proposition $Q$, called the *conclusion*. Such an argument is denoted by

$$P_1, P_2, ..., P_n \vdash Q$$

An argument $P_1, P_2, ..., P_n \vdash Q$ is said to be valid if $Q$ is true whenever all the premises $P_1, P_2, ..., P_n$ are true.
An argument which is not valid is called *fallacy*.

**Example 1.23.** Determine the validity of the following argument:

If Socrates is a man, then Socrates is mortal.
Socrates is a man.
∴ Socrates is mortal.

*Solution.* The argument has the abstract form

if $p$ the $q$
$p$
$\therefore q$

Table 15: Truth Table for the argument

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

■

**Example 1.24.** Determine the validity of the following argument:

$p \to q \vee \sim r$
$q \to p \wedge r$
$\therefore p \to r$

*Solution.* The truth table shows that even though there are several situations in which the premises and the conclusion are all true (rows 1, 7, and 8), there is one situation (row 4) where the premises are true and the conclusion is false. ■

| $p$ | $q$ | $r$ | $\sim r$ | $q \vee \sim r$ | $p \wedge r$ | $p \to q \vee \sim r$ | $q \to p \wedge r$ | $p \to r$ |
|---|---|---|---|---|---|---|---|---|
| T | T | T | F | T | T | T | T | T |
| T | T | F | T | T | F | T | F | |
| T | F | T | F | F | T | F | T | |
| T | F | F | T | T | F | T | T | T |
| F | T | T | F | T | F | T | F | |
| F | T | F | T | T | F | T | F | |
| F | F | T | F | F | F | T | T | T |
| F | F | F | T | T | F | T | T | T |

**Example 1.25.** Determine the validity of the following argument:

If Zeus is human, then Zeus is mortal.
Zeus is not mortal.
∴ Zeus is not human.

■

*Solution.* Exercise

**Example 1.26. Generalization**: The following argument forms are valid:

(a)

$$\begin{array}{l} p \\ \therefore p \vee q \end{array}$$

(b)

$$\begin{array}{l} q \\ \therefore p \vee q \end{array}$$

**Example 1.27. Specialization**: The following argument forms are valid:

(a)

$$\begin{array}{l} p \wedge q \\ \therefore p \end{array}$$

(b)

$$\begin{array}{l} p \wedge q \\ \therefore q \end{array}$$

**Example 1.28. Elimination**: The following argument forms are valid:

(a)

$$\begin{array}{l} p \vee q \\ \sim q \\ \therefore p \end{array}$$

(b)

$$\begin{array}{l} p \vee q \\ \sim p \\ \therefore q \end{array}$$

**Example 1.29. Transitivity**: The following argument forms are valid:

$$\begin{array}{l} p \to q \\ q \to r \\ \therefore p \to r \end{array}$$

If $n$ is divisible by 18, then $n$ is divisible by 9.
If $n$ is divisible by 9, then the sum of the digits of $n$ is divisible by 9.
$\therefore$ If $n$ is divisible by 18, then the sum of the digits of $n$ is divisible by 9.

**Example 1.30. Proof by Division into Cases**: The following argument forms are valid:

$$\begin{array}{l} p \vee q \\ p \to r \\ q \to r \\ \therefore r \end{array}$$

$x$ is positive or $x$ is negative.
If $x$ is positive, then $x^2 > 0$.
If $x$ is negative, then $x^2 > 0$.
$\therefore x^2 > 0$.

**Example 1.31.** Show that the following argument is invalid:

$p \to q$
$q$
$\therefore p$

If Zeke is a cheater, then Zeke sits in the back row.
Zeke sits in the back row.
$\therefore$ Zeke is a cheater.

**Theorem 1.2.** *The argument $P_1, P_2, ..., P_n \vdash Q$ is valid if and only if the proposition $(P_1 \wedge P_2 ... \wedge P_n) \to Q$ is a tautology.*

**Example 1.32.** A fundamental principle of logical reasoning states:

"If $p$ implies $q$ and $q$ implies $r$, then $p$ implies $r$"

Show that the above argument is valid ■

*Solution.* Construct the truth table for "If $p$ implies $q$ and $q$ implies $r$, then $p$ implies $r$"

| $p$ | $q$ | $r$ | $p \to q$ | $q \to r$ | $p \to r$ | $(p \to q) \wedge (q \to r)$ | $[(p \to q) \wedge (q \to r)] \to (p \to r)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | T | F | F | F | T |
| T | F | T | F | T | T | F | T |
| T | F | F | F | T | F | F | T |
| F | T | T | T | T | T | T | T |
| F | T | F | T | F | T | F | T |
| F | F | T | T | T | T | T | T |
| F | F | F | T | T | T | T | T |

**Example 1.33.** Show that the following argument is a fallacy: $p \to q, \sim p \vdash \sim q$.

*Solution.* Construct the truth table for $[(p \to q) \wedge \sim p] \to \sim q$ as in Fig. below Since the proposition $[(p \to q) \wedge \sim p] \to \sim q$ is not a tautology, the argument is a fallacy. Equivalently, the argument is a fallacy since in the third line of the truth table $p \to q$ and $\sim p$ are true but $\sim q$ is false. ■

| $p$ | $q$ | $p \to q$ | $\sim p$ | $(p \to q) \wedge \sim p$ | $\sim q$ | $[(p \to q) \wedge \sim p] \to \sim q$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | F | T |
| T | F | F | F | F | T | T |
| F | T | T | T | T | F | F |
| F | F | T | T | T | T | T |

**Example 1.34.** Determine the validity of the following argument: