CSC2323 Discrete Structures Lecture Notes

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Contents

Reference books:

1. Discrete Mathematics and Its Applications By Kenneth H. Rosen

2. discrete mathematics with applications By Susanna S. Epp

1.   Propositional Logic and Predicate Logic

Introduction

The rules of logic give precise meaning to mathematical statements. These rules are used to distinguish between valid and invalid mathematical arguments. Because a major goal of this book is to teach the reader how to understand and how to construct correct mathematical arguments, we begin our study of discrete mathematics with an introduction to logic.

Besides the importance of logic in understanding mathematical reasoning, logic has numerous applications to computer science. These rules are used in the design of computer circuits, the construction of computer programs, the verification of the correctness of programs, and in many other ways. Furthermore, software systems have been developed for constructing some, but not all, types of proofs automatically.

1.1   Propositions

A **proposition** (or **statement**) is a declarative sentence (that is, a sentence that declares a fact) that is either *true* or *false*, but not both.

Example 1.1. Consider, the following sentences:.

1. Abuja is the capital of Nigeria.

2. $1 + 1 = 2$.

3. $2 + 2 = 3$.

4. China is in Europe.

5. Where are you going?

6. Do your homework.

7. $x + 1 = 2$.

The first four are propositions, the last three are not. Also, (i) and (ii) are true, but (iii) and (iv) are false.

We use letters to denote propositional variables (or sentential variables), that is, variables that represent propositions, just as letters are used to denote numerical variables. The conventional letters used for propositional variables are $p, q, r, s; \dots$. The truth value of a proposition is true, denoted by T, if it is a true proposition, and the truth value of a proposition is false, denoted by F, if it is a false proposition.

Compound Propositions

Many propositions are *composite*, that is, composed of *subpropositions* and various connectives discussed subsequently. Such composite propositions are called **compound propositions**. A proposition is said to be **primitive** (or **atomic**) **propositions** if it cannot be broken down into simpler propositions, that is, is not composite.

For example, the above propositions (i) through (iv) are primitive propositions. On the other hand following two propositions are composite:

"It is hot and it is not sunny." and "John is smart or he studies every night."

1.2   Basic Logical Operations

This section discusses the three basic logical operations of conjunction, disjunction, and negation, correspond, respectively, to the English words "and," "or," and "not."

Conjunction, $p \wedge q$

> **Definition 1: Conjunction**
>
> Let p and q be propositions. The **conjunction** of p and q, denoted by $p \wedge q$, is the proposition "p and q." The conjunction $p \wedge q$ is **true** when both p and q are true and is false otherwise.

Table 1: Truth Table for $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 1.2. Consider the following four statements:

1. Abuja is the capital of Nigeria and $2 + 2 = 4$.

2. Abuja is the capital of Nigeria and $2 + 2 = 5$

3. China is in Europe and $2 + 2 = 4$.

4. China is in Europe and $2 + 2 = 5$.

Only the first statement is true. Each of the others is false since at least one of its substatements is false.

> **Definition 2: Disjunction**
>
> Let p and q be propositions. The **disjunction** of p and q, denoted by $p \vee q$, is the proposition "p or q." The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

Table 2: Truth Table for $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 1.3. Consider the following four statements:

1. Abuja is the capital of Nigeria and $2 + 2 = 4$.

2. Abuja is the capital of Nigeria and $2 + 2 = 5$

3. China is in Europe and $2 + 2 = 4$.

4. China is in Europe and $2 + 2 = 5$.

Only the last statement (4) is false. Each of the others is true since at least one of its sub-statements is true.

Remark: The English-word "or" is commonly used in two distinct ways. Sometimes it is used in the sense of "p or q or both," i.e., at least one of the two alternatives occurs, as above, and sometimes it is used in the sense of "p or q but not both," i.e., exactly one of the two alternatives occurs. For example, the sentence "He will go to BUK or to ABU" uses "or" in the latter sense, called the **exclusive disjunction**. Unless otherwise stated, "or" shall be used in the former sense. This discussion points out the precision we gain from our symbolic language: $p \vee q$ is defined by its truth table and always means "p and/or q."

Definition 3: Negation

If p is a proposition, the **negation** of p is "not p" or "It is not the case that p" and is denoted $\sim p$. It has opposite truth value from p: if p is true, $\sim p$ is false; if p is false, $\sim p$ is true.

Table 3: Truth Table for $\sim p$

p	$\sim p$
T	F
F	T

Example 1.4. Consider the following four statements:

1. Ice floats in water.

2. It is false that ice floats in water.

3. Ice does not float in water.

4. $2 + 2 = 5$

5. It is false that $2 + 2 = 5$.

6. $2 + 2 \neq 5$

Then (2) and (3) are each the negation of (1); and (5) and (6) are each the negation of (4).

Remark: The logical notation for the connectives "and," "or," and "not" is not completely standardized. For example, some texts use:

$$p \text{ \& } q, \, p \cdot q \text{ or } pq \text{ for } p \wedge q$$

$$p + q \text{ for } p \vee q$$

$$p', \bar{p} \text{ or } \neg p \text{ for } \sim p$$

1.3 Propositions and Truth Tables

Definition 4: Compound Statements

Let $P(p, q, ...)$ denote an expression constructed from logical variables $p, q, ...$, which take on the value TRUE (T) or FALSE (F), and the logical connectives $\wedge$, $\vee$, and $\sim$ (and others discussed subsequently). Such an expression $P(p, q, ...)$ will be called a **proposition**.

The main property of a proposition $P(p, q, ...)$ is that its truth value depends exclusively upon the truth values of its variables, that is, the truth value of a proposition is known once the truth value of each of its variables is known. A simple concise way to show this relationship is through a truth table. We describe a way to obtain such a truth table below.

Example 1.5.

Consider, the proposition $\sim (p \wedge \sim q)$. Table 4 indicates how the truth table of $\sim (p \wedge \sim q)$ is constructed. Observe that the first columns of the table are for the variables $p, q, ...$ and that there are enough rows in the table, to allow for all possible combinations of T and F for these variables. (For 2 variables, as above, 4 rows are necessary; for 3 variables, 8 rows are necessary; and, in general, for n variables, 2^n rows are required. There is then a column for each "elementary" stage of the construction of the proposition, the truth value of each step being determined from the previous stages by the definitions of the connectives $\wedge$, $\vee$, $\sim$. Finally we obtain the truth value of the proposition, which appears in the last column.

Table 4: Truth Table for $\sim (p\wedge \sim q)$

p	q	$\sim q$	$p\wedge \sim q$	$\sim (p\wedge \sim q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

Alternate Method for Constructing a Truth Table

Another way to construct the truth table for $\sim (p\wedge \sim q)$ follows:

(a) First we construct the truth table shown in Table 5. That is, first we list all the variables and the combinations of their truth values. Also there is a final row labeled "step." Next the proposition is written on the top row to the right of its variables with sufficient space so there is a column under each variable and under each logical operation in the proposition. Lastly (Step 1), the truth values of the variables are entered in the table under the variables in the proposition.

(b) Now additional truth values are entered into the truth table column by column under each logical operation as shown in Table 5. We also indicate the step in which each column of truth values is entered in the table.

Table 5: Truth Table for $\sim (p\wedge \sim q)$

p	q	$\sim$	$(p$	$\wedge$	$\sim$	$q)$
T	T	T	T	F	F	T
T	F	F	T	T	T	F
F	T	T	F	F	F	T
F	F	T	F	F	T	F
Step	4	1	3	2	1	

Example 1.6. Construct the truth table for the statement form $(p\vee q)\wedge \sim (p\wedge q)$.

Solution. Set up columns labeled p, q, $(p\vee q)$, $(p\wedge q)$, $\sim (p\wedge q)$, and $(p\vee q)\wedge \sim (p\wedge q)$. Fill in the p and q columns with all the logically possible combinations of T's and F's. Then use the truth tables for $\vee$ and $\wedge$ to fill in the $(p\vee q)$ and $(p\wedge q)$ columns with the appropriate truth values. Next fill in the $\sim (p\wedge q)$ column by taking the opposites of the truth values for $(p\wedge q)$.

Table 6: Truth Table for $(p\vee q)\wedge \sim (p\wedge q)$

p	q	$(p\vee q)$	$(p\wedge q)$	$\sim (p\wedge q)$	$(p\vee q)\wedge \sim (p\wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Example 1.7. Construct the truth table for the statement form $(p\wedge q)\vee \sim r$.

Solution. Make columns headed p, q, r, $p\wedge q$, $\sim r$, and $(p\wedge q)\vee \sim r$. Enter the eight logically possible combinations of truth values for p, q, and r in the three left-most columns. Then fill in the truth values for $p\wedge q$ and for r. Complete the table by considering the truth values for $(p\wedge q)$ and for $\sim r$ and the definition of an *or* statement.

5A

Table 7: Truth Table for $(p \wedge q) \vee \sim r$

p	q	r	$p \wedge q$	$\sim r$	$(p \wedge q) \vee \sim r$
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	T	T
F	F	T	F	F	F
F	F	F	F	T	T

1.4 Logical Equivalence

Definition 5: Logical Equivalence

Two propositions $P(p, q, ...)$ and $Q(p, q, ...)$ are said to be logically equivalent, or simply equivalent or equal, denoted by

$$P(p, q, ...) \equiv Q(p, q, ...)$$

if they have identical truth tables.

Example 1.8. Show that $\sim (p \wedge q)$ and $(\sim p \vee \sim q)$ are logically equivalent

Solution. Observe that both truth tables are the same, that is, both propositions are false in the first case and true in the other three cases.

Table 8: Truth Table for $\sim (p \wedge q)$ and $(\sim p \vee \sim q)$

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim (p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Symbolically,

$$\sim (p \wedge q) \equiv (\sim p \vee \sim q)$$

Example 1.9. Show that $\sim (p \vee q)$ and $(\sim p \wedge \sim q)$ are logically equivalent

Solution. Exercise!

1.5 Tautologies and Contradictions

Definition 6: Tautologies

Some propositions $P(p, q, ...)$ contain only T in the last column of their truth tables or, in other words, they are true for any truth values of their variables. Such propositions are called tautologies.

Definition 7: Contradictions

A proposition $P(p, q, ...)$ is called a contradiction if it contains only F in the last column of its truth table or, in other words, if it is false for any truth values of its variables.

Example 1.10. Show that $(p \lor \sim p)$ is tautology

Solution. We draw the truth table for $(p \lor \sim p)$ as follows:

Table 9: Truth Table for $(p \lor \sim p)$

p	$\sim p$	$(p \lor \sim p)$
T	F	T
F	T	T

Example 1.11. Show that $(p \land \sim p)$ is contradiction

Solution. We draw the truth table for $(p \land \sim p)$ as follows:

Table 10: Truth Table for $(p \land \sim p)$

p	$\sim p$	$(p \land \sim p)$
T	F	F
F	T	F

Example 1.12. If t is a tautology and c is a contradiction, show that $p \land t \equiv p$ and $p \land c \equiv c$.

Solution. We draw the truth table for $p \land t \equiv p$ and $p \land c \equiv c$ as follows:

Table 11: Truth Table for $p \land t \equiv p$ and $p \land c \equiv c$

p	t	$p \land t$	c	$p \land c$
T	T	T	F	F
F	T	F	F	F

Theorem 1.1. *Logical Equivalences*

Given any statement variables p, q, and r, a tautology t and a contradiction c, the following logical equivalences hold.

1. Commutative laws: $\quad p \land q \equiv q \land p \qquad\qquad\qquad p \lor q \equiv q \lor p$
2. Associative laws: $\quad (p \land q) \land r \equiv p \land (q \land r) \qquad (p \lor q) \lor r \equiv p \lor (q \lor r)$
3. Distributive laws: $\quad p \land (q \lor r) \equiv (p \land q) \lor (p \land r) \qquad p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$
4. Identity laws: $\quad p \land t \equiv p \qquad\qquad\qquad p \lor c \equiv p$
5. Negation laws: $\quad p \lor \sim p \equiv t \qquad\qquad\qquad p \land \sim p \equiv c$
6. Double negative law: $\quad \sim (\sim p) \equiv p$
7. Idempotent laws: $\quad p \land p \equiv p \qquad\qquad\qquad p \lor p \equiv p$
8. Universal bound laws: $\quad p \lor t \equiv t \qquad\qquad\qquad p \land c \equiv c$
9. De Morgan's laws: $\quad \sim (p \land q) \equiv \sim p \lor \sim q \qquad \sim (p \lor q) \equiv \sim p \land \sim q$
10. Absorption laws: $\quad p \lor (p \land q) \equiv p \qquad\qquad p \land (p \lor q) \equiv p$
11. Negations of t and c: $\quad \sim t \equiv c \qquad\qquad\qquad \sim c \equiv t$