Limitlar haqidagi asosiy teoremalar

1-teorema(yig`indining limiti haqida). Ikki funksiya algebraik yig`indisining limiti shu funksiyalar limitlarining algebraik yig`indisiga teng:

$$\lim_{x\to a}[f(x)+g(x)]=\lim_{x\to a}f(x)+\lim_{x\to a}g(x) \qquad (1)$$

Bu teorema istalgancha chekli sondagi funksiyalar algebraik yig`indisi uchun ham o`rinli.

2-teorema(ko`paytmaning limiti haqida)**.** Ikki funksiya ko`paytmasining limiti shu funksiyalar limitlarining ko`paytmasiga teng:

$$\lim_{x\to a}[f(x)\cdot g(x)]=\lim_{x\to a}f(x)\cdot\lim_{x\to a}g(x) \qquad (2)$$

Natija. O`zgarmas ko`paytuvchini (bo`luvchini) limit belgisidan tashqariga chiqarish mumkin:

$$\lim_{x\to a}[C\cdot f(x)]=C\cdot\lim_{x\to a}f(x), \quad (\text{C – o'zgarmas son}). \qquad (2')$$

3-teorema (bo`linmaning limiti haqida). Ikki funksiya nisbatining limiti, maxrajning limiti noldan farqli bo`lganda, bu funksiyalar limitlarining nisbatiga teng:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}g(x)}, \qquad (\lim_{x\to a}g(x)\neq 0) \qquad (3)$$

4-teorema (tengsizlikda limitga o`tish haqida). Agar $f(x)$ funksiya $x=a$ nuqtada limitga ega va bu nuqtaning biror atrofida $f(x)>0$ bo`lsa, u holda

$$\lim_{x\to a}f(x)\geq 0 \qquad (4)$$

bo`ladi.

5-teorema (oraliq funksiyaning limiti haqida). Agar $x=a$ nuqtaning biror atrofidagi barcha nuqtalarda

$$f_1(x)\leq\varphi(x)\leq f_2(x) \quad \text{va} \quad \lim_{x\to a}f_1(x)=\lim_{x\to a}f_2(x)=A \qquad (5)$$

bo`lsa, $\lim\limits_{x\to a}\varphi(x)=A$ bo`ladi.

Teoremalarning isbotlari

1-teoremaning isboti

$f(x)$ va $g(x)$ ning $x\to a$ dagi limitlari A va B bo`lsin:

$$\lim_{x\to a}f(x)=A, \qquad \lim_{x\to a}g(x)=B$$

*3°- xossa*ga asosan $f(x)$ ni $A+\alpha(x)$, $g(x)$ ni $B+\beta(x)$ deb yezish mumkin, bunda $\alpha(x)$ va $\beta(x)$ lar $x\to a$ da cheksiz kichik funksiyalar:

$$f(x)=A+\alpha(x), \quad g(x)=B+\beta(x), \qquad \lim_{x\to a}\alpha(x)=0, \qquad \lim_{x\to a}\beta(x)=0.$$

Unda $f(x)+g(x)=(A+B)+\alpha(x)+\beta(x)$. *2°- xossa*ga ko`ra $\alpha(x)+\beta(x)$ ham cheksiz kichik miqdor, shuning uchun $A+B$ son $f(x)+g(x)$ ning limiti bo`ladi*(3°- xossaga ko`ra):*

$$\lim_{x\to a}[f(x)+g(x)]=A+B=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)$$

Teorema isbotlandi.

2-teoremaning isboti

1-teoremaning isboti kabi bajaramiz:

$$\lim_{x\to a}f(x)=A, \ \Rightarrow\ f(x)=A+\alpha(x), \qquad \lim_{x\to a}\alpha(x)=0, \qquad (4)$$

$$\lim_{x\to a}g(x)=B \ \Rightarrow\ g(x)=B+\beta(x), \qquad \lim_{x\to a}\beta(x)=0$$

$$f(x)\cdot g(x)=[A+\alpha(x)]\cdot[B+\beta(x)]=A\cdot B+[A\cdot\beta(x)+B\cdot\alpha(x)+\alpha(x)\cdot\beta(x)].$$

O`rta qavs ichidagi ifoda *2°- xossa*ga ko`ra $x\to a$ da cheksiz kichik funksiya. Shuning uchun $A\cdot B$ *3°-xossa*ga ko`ra $x\to a$ da $f(x)\cdot g(x)$ ning limiti bo`ladi:

$$\lim_{x\to a}[f(x)\cdot g(x)]=A\cdot B=\lim_{x\to a}f(x)\cdot\lim_{x\to a}g(x)$$

Teorema isbotlandi.

3-teoremaning isboti

Shartga ko`ra $\lim_{x\to a}g(x)=V\neq 0.$ Oldingi teoremaning isbotiga o`xshash bajaramiz va (4) dan foydalanamiz:

$$\frac{\mathrm{f(x)}}{\mathrm{g(x)}}=\frac{A+\alpha(x)}{B+\beta(x)}=\frac{A}{B}+\frac{A+\alpha(x)}{B+\beta(x)-\frac{A}{B}}=\frac{A}{B}+\frac{B\cdot\alpha(x)-A\cdot\beta(x)}{B\cdot(B+\beta(x))}.$$

Oxirgi kasr $B\neq 0$ bo`lgani uchun $2^{\circ}$-*xossa*ga asosan cheksiz kichik. Shuning uchun $\frac{A}{B}$ son $\frac{\mathrm{f(x)}}{\mathrm{g(x)}}$ ning $x\to a$ dagi limitidir($3^{\circ}$-*xossa*ga ko`ra):

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{A}{B}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}. \qquad (\lim_{x\to a}g(x)\neq 0)$$

Teorema isbotlandi.

Keyingi ikki teoremaning isbotini ko`rsatilgan adabiyotlardan topib o`rganishni tavsiya etamiz.

Eslatma. Yuqoridagi teoremalar a cheksiz bo`lganida ham o`rinli.

1-misol

Limitni hisoblang. $\lim_{x\to\infty}\frac{4+3\mathrm{x}^2}{\mathrm{x}^2}$

Δ $\lim_{x\to\infty}\frac{4+3x^2}{x^2}=\lim_{x\to\infty}\frac{4}{x^2}=\lim_{x\to\infty}\frac{4}{x^2}+\lim_{x\to\infty}3=0+3=3.$ *Javoib*: *3.* ▲

2-misol

Limitni hisoblang. $\lim_{x\to 1}(x+1)(4x-7)$.

Δ

$$\lim_{x\to 1}(\mathrm{x}+1)(4\mathrm{x}-7)=\lim_{x\to 1}(\mathrm{x}+1)\cdot\lim_{x\to 1}(4\mathrm{x}-7)=(\lim_{x\to 1}x+1)\cdot(\lim_{x\to 1}4x-7)=$$

$=(1+1)\cdot(4\lim_{x\to 1}x-7)=2\cdot(4-7)=2\cdot(-3)=-6.$ *Javobi*: *-6* . ▲

3-misol

Limitni hisoblang. $\lim_{x\to 2}\frac{3x-2}{2x+1}$

Δ $\lim_{x\to 2}\frac{3x-2}{2x+1}=\frac{\lim_{x\to 2}(3x-2)}{\lim_{x\to 2}(2x+1)}=\frac{3\cdot 2-2}{2\cdot 2+1}=\frac{4}{5}.$ *Javob*: $\frac{4}{5}$. ▲

4-misol

Limitni hisoblang. $\lim_{x\to -1}\frac{x^2+3x+2}{x+1}.$

Δ $x\to -1$ da sura`tning ham, maxrajning ham limiti $0$ ga teng. Shuning uchun nisbatning limiti haqidagi teoremani qo`llab bo`lmaydi. Oldin shakl almashtirish bajaramiz:

$$\frac{x^2+3x+2}{x+1}=\frac{(x+1)(x+2)}{x+2}.$$

Endi limitga o`tamiz:

$$\lim_{x\to -1}\frac{x^2+3x+2}{x+1}=\lim_{x\to -1}(x+2)=-1+2=1.$$ *Javob*: *1* ▲

3. ANIQMASLIKLARNI OCHISH (YECHISH)

0/0 va ∞/∞ aniqmasliklarning ta'rifi

1-ta'rif. Agar $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x)=\infty$ bo`lsa, $\dfrac{f(x)}{g(x)}$ ifoda $x=a$ da $\dfrac{\infty}{\infty}$ *ko`rinishdagi aniqmaslik*, $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ limitni hisoblash esa $\dfrac{\infty}{\infty}$ *aniqmaslikni ochish (yechish)* deyiladi.

Masalan: $\lim\limits_{x\to\infty}\dfrac{x^2+5}{7-x+4x^2}$, $\lim\limits_{n\to\infty}\dfrac{\sqrt{2+3n^4}}{1+n-6n^2}$ lar $\dfrac{\infty}{\infty}$ aniqmasliklardir.

2-ta'rif. Agar $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$ bo`lsa, $\dfrac{f(x)}{g(x)}$ ifoda $x=a$ da $\dfrac{0}{0}$ *ko`rinishdagi aniqmaslik*, $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ limitni hisoblash esa $\dfrac{0}{0}$ *aniqmaslikni ochish(yechish)* deyiladi.

Masalan: $\lim\limits_{x\to 1}\dfrac{x^2-1}{x^2-3x+1}$, $\lim\limits_{n\to 2}\dfrac{\sqrt{16-n^4}}{3n-6}$, $\lim\limits_{x\to 0}\dfrac{x^2+x}{\sin 2x}$ lar $\dfrac{0}{0}$ aniqmasliklardir.

3-ta'rif. Agar $\lim\limits_{x\to a} f(x) = 0$, $\lim\limits_{x\to a} g(x)=\infty$ bo`lsa, $f(x)\cdot g(x)$ ifoda $x=a$ da $0\cdot\infty$ *ko`rinishdagi aniqmaslik*, $\lim\limits_{x\to a} f(x)\cdot g(x)$ limitni hisoblash esa $0\cdot\infty$ *aniqmaslikni ochish (yechish)* deyiladi.

Masalan: $\lim\limits_{x\to 0} x\,ctgx$, $\lim\limits_{n\to\infty} n(\sqrt{n^2+1}-\sqrt{n^2-1})$ lar $0\cdot\infty$ aniqmasliklardir.

Shu kabi: $\infty-\infty$, 1^∞, ∞^0, 0^0 aniqmasliklar ham ta'riflanadi.

1-eslatma

∞ /∞ aniqmaslikni ochishda, $x\to\infty$, $f(x)$ va $g(x)$ - ko`phadlar, $n$- ular darajalarng eng kattasi bo`lsa, avval kasrning surat va maxrajini x^n ga bo`lib, keyin limitga o'tiladi. Masalan:

$$\lim_{x\to\infty}\frac{3x^4-5x+2}{4x^4+7x+1}$$

∞ /∞ ko`rinishdagi aniqmaslik. Uni ochish uchun, avval kasrning surat va maxrajini $x^4$ ga bo`lib, so'ng limitga o`tamiz:

$$\lim_{x\to\infty}\frac{3x^4-5x^2+2}{4x^4+7x^3+1}=\lim_{x\to\infty}\frac{3-\frac{5}{x^2}+\frac{2}{x^4}}{4+\frac{7}{x}+\frac{1}{x^4}}=\frac{3-0+0}{4+0+0}=\frac{3}{4}.\ \ Javob:\ \frac{3}{4}$$

1-eslatmada aytilgan usulni surat yoki maxrajdagi ko'phadlar ildiz ostida kelganida ham qo'llash mumkin.

1-misol

Limitni hisoblang. $\lim\limits_{x\to\infty}\dfrac{4-15x^2}{\sqrt{7+9x^4}}$.

$$\Delta\quad \lim_{x\to\infty}\frac{4-15x^2}{\sqrt{7+9x^4}}.=\lim_{x\to\infty}\frac{\frac{4-15x^2}{x^2}}{\frac{\sqrt{7+9x^4}}{x^2}}=\lim_{x\to\infty}\frac{\frac{4}{x^2}-15}{\sqrt{\frac{7}{x^4}+9}}=\frac{0-15}{\sqrt{0+9}}=-\frac{15}{3}=-3.\ \ Javob:\ -3\quad\blacktriangle$$

2-eslatma

0/0 aniqmaslikni ochishda, $x\to a$, $f(x)$ va $g(x)$ – ko`phadlar bo'lsa, avval, kasrning surat va maxrajini 0 ga aylantiruvchi (bu ko'pincha $x-a$ yoki uning darajasi) ifodaga qisqartirib, keyin limitga o'tiladi. Masalan:

$$\lim_{x\to 2}\frac{x^2-3x+2}{2x-x^2}$$

limit ostidagi kasr $x=2$ da 0/0 aniqmaslikdir. Uni ochish uchun, avval kasrning surat va maxrajini $x-2$ ga qisqartirib, so'ng limitga o`tamiz:

$$\lim_{x\to 2}\frac{x^2-3x+2}{2x-x^2}=\lim_{x\to 2}\frac{(x-x_1)(x-x_2)}{x(2-x)}=\lim_{x\to 2}\frac{(x-2)(x-1)}{x(2-x)}=\lim_{x\to 2}\frac{1-x}{x}=\frac{1-2}{2}=-\frac{1}{2}.$$

Bu yerda $x_1=2,\ x_2=1$ $\quad x^2-3x+2=0$ kvadrat tenglamaning ildizlari.

2-misol

Limitni toping. $\quad \lim_{x\to 3}\frac{x^2-5x+6}{x^2-9}$

Δ

$$\lim_{x\to 3}\frac{x^2-5x+6}{x^2-9}=\lim_{x\to 3}\frac{(x-3)(x-2)}{(x-3)(x+3)}=\lim_{x\to 3}\frac{x-2}{x+3}=\frac{3-2}{3+3}=\frac{1}{6}.\ \textit{Javob}:\ \frac{1}{6}.\quad ▲$$

2-eslatmada aytilgan usulni surat yoki maxrajdagi ko'phadlar ildiz ostida kelganida ham qo'llash mumkin.

3-misol

Limitni toping. $\quad \lim_{x\to 0}\frac{x^2+x}{\sqrt{1+x}-\sqrt{1-x}}.$

Δ

$$\lim_{x\to 0}\frac{x^2+x}{\sqrt{1+x}-\sqrt{1-x}}=\lim_{x\to 0}\frac{x(x+1)(\sqrt{1+x}+\sqrt{1-x})}{(\sqrt{1+x}-\sqrt{1-x})(\sqrt{1+x}+\sqrt{1-x})}=$$

$$=\lim_{x\to 0}\frac{x(x+1)(\sqrt{1+x}+\sqrt{1-x})}{(1+x)-(1-x)}=\lim_{x\to 0}\frac{x(x+1)(\sqrt{1+x}+\sqrt{1-x})}{2x}=$$

$$=\lim_{x\to 0}\frac{x(x+1)(\sqrt{1+x}+\sqrt{1-x})}{2}=\frac{1\cdot(1+1)}{2}=1.\quad \text{Javob}:\ 1.$$

▲

∞-∞, 0·∞, 0^0, 1^∞, ∞^0 aniqmasliklar, ko`p hollarda *0/0* yoki *∞ /∞* aniqmasliklarga keltirilib yechiladi (ochiladi).

4-misol

Limitni toping. $\quad \lim_{x\to\infty} x(\sqrt{x^2+4}-\sqrt{x^2+1})$

Δ Bu $x\to\infty$ da *∞-∞* aniqmaslik. Limitga o`tishdan oldin limit ostidagi kasrning surat va maxrajini $\sqrt{x^2+4}+\sqrt{x^2+1}$ ga ko`paytiramiz:

$$\lim_{x\to\infty} x(\sqrt{x^2+4}-\sqrt{x^2+1})=\lim_{x\to\infty}\frac{x(\sqrt{x^2+4}-\sqrt{x^2+1})(\sqrt{x^2+4}+\sqrt{x^2+1})}{\sqrt{x^2+4}-\sqrt{x^2+1}}=$$

$$\lim_{x\to\infty}\frac{x((x^2+4)-(x^2+1))}{\sqrt{x^2+4}+\sqrt{x^2+1}}=\lim_{x\to\infty}\frac{3x}{\sqrt{x^2+4}+\sqrt{x^2+1}}.$$

Hosil bo`lgan *∞/∞* aniqmaslikni ochish uchun kasrning surat va maxrajini $x$ ga bo`lamiz, keyin limitga o`tamiz:

$$\lim_{x\to\infty}\frac{3}{\dfrac{\sqrt{x^2+4}+\sqrt{x^2+1}}{x}}=\lim_{x\to\infty}\frac{3}{\sqrt{x+\dfrac{4}{x^2}}+\sqrt{x+\dfrac{1}{x^2}}}=\frac{3}{1+1}=1{,}5. \qquad \blacktriangle$$

1-misoldagi limitning **Maple KMS** da hisoblablanishi quyidagicha:

```
> restart:
Limit((4-15*x^2)/sqrt(7+9*x^4),x=infinity)=
limit((4-15*x^2)/sqrt(7+9*x^4), x=infinity);
```

$$\lim_{x\to\infty}\frac{4-15\,x^2}{\sqrt{7+9\,x^4}}=-5$$