MAVZU. Aniq integral, uning tatbiqlari

REJA

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1. Aniq integralning ta'rifi.

Aniq integral - matematik analizning eng muhim tushunchalaridan biridir. Egri chiziq bilan chegaralangan yuzalarni, egri chiziqli yoylar uzunliklarini, hajmlarni, bajarilgan ishlarni, yo'llarni, inersiya momentlarini va hokazolarni hisoblash masalasi shu tushuncha bilan bog'liq. $[a, b]$ kesmada $y=f(x)$ uzluksiz funksiya berilgan bo'lsin. Quyidagi amallarni bajaramiz.

1. $[a, b]$ kesmani qo'yidagi nuqtalar bilan ixtiyoriy n ta qismga bo'lamiz, va ularni qismiy intervallar deb ataymiz.

$$a=x_0<x_1<x_2<x_3<\ldots x_{i-1}<x_i\ldots<x_n=b$$

2.Qismiy intervallarning uzunliklarini bunday belgilaymiz:

$$\Delta x_1=x_1-a \qquad \Delta x_2=x_2-x_1 \ \ldots \quad \Delta x_i=x_i-x_{i-1} \ \ \ldots \quad \Delta x_n=b-x_{n-1}$$

DEFINITION OF RIEMANN SUM

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval. If c_i is *any* point in the ith subinterval $[x_{i-1}, x_i]$, then the sum

$$\sum_{i=1}^{n} f(c_i)\, \Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

is called a **Riemann sum** of f for the partition Δ.

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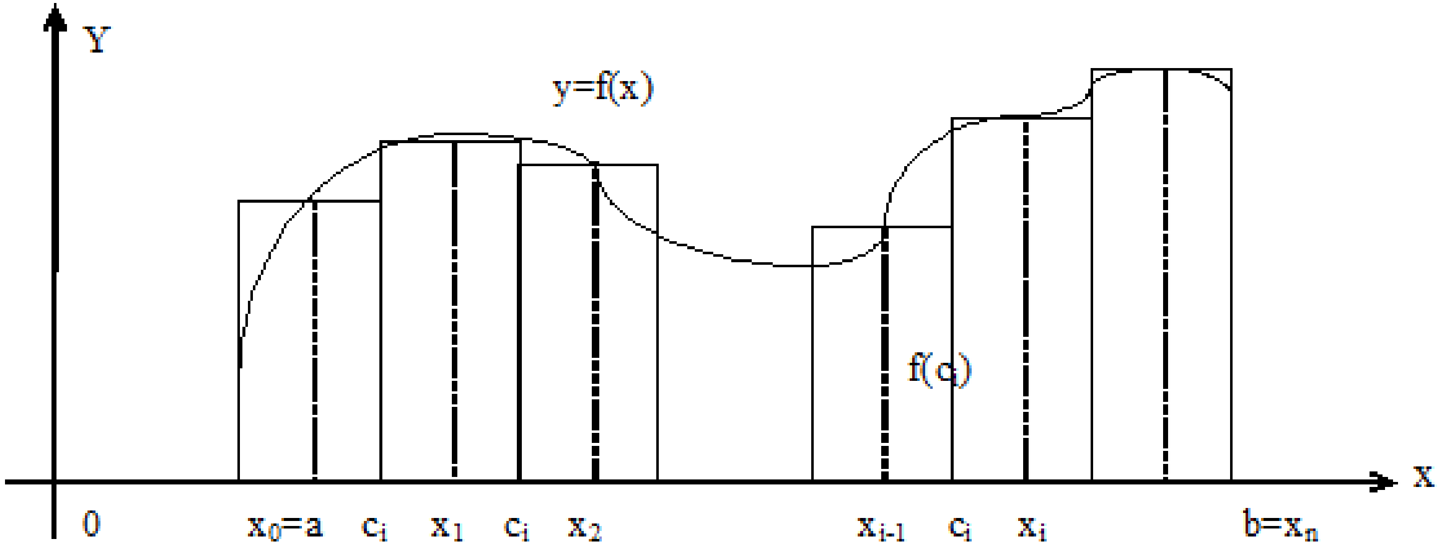
σ_n yig'indi $f(x)$ funksiya uchun $[a, b]$ kesmada tuzilgan integral yig'indi deb ataladi. σ_n integral yig'indi qisqacha bunday yoziladi:

$$\sigma_n = \sum_{i=1}^{n} f(c_i)\Delta x_i .$$



1-rasm.

Integral yig'indining geometrik ma'nosi ravshan: Agar $f(x) \geq 0$ bo'lsa, u holda σ_n - asoslari Δx_1, Δx_2, ..., Δx_i,...,Δx_n va balandliklari mos ravishda $f(c_1)$, $f(c_2)$, ..., $f(c_i)$, ..., $f(c_n)$ bo'lgan to'g'ri to'rtburchak yuzlarining yig'indisidan iborat (1-rasm).

Endi bo'lishlar soni n ni orttira boramiz ($n \to \infty$) va bunda eng katta intervalning uzunligi nolga intilishini, ya'ni $\max\Delta x_i \to 0$ deb faraz qilamiz.

Ushbu ta'rifni beramiz.

Ta'rif. Agar σ_n integral yig'indi $[a, b]$ kesmani qismiy $[x_i, x_{i\text{-}1}]$ kesmalarga ajratish usuliga va ularning har biridan c_i nuqtani tanlash usuliga bog'liq bo'lmaydigan chekli songa intilsa, u holda shu son $[a, b]$ kesmada $f(x)$ funksiyadan olingan aniq integral deyiladi va bunday belgilanadi:

$$\int_a^b f(x)dx.$$

Bu yerda $f(x)$ integral ostidagi funksiya. $[a, b]$ kesma integrallash oralig'i. a va b sonlar integrallashning qo'yi va yuqori chegarasi deyiladi.

$$\int_a^b f(x)dx = \lim_{\max\Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

DEFINITION OF DEFINITE INTEGRAL

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

exists (as described above), then f is said to be **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx.$$

The limit is called the **definite integral** of f from a to b. The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

THEOREM 4.4 CONTINUITY IMPLIES INTEGRABILITY

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is, $\int_a^b f(x)\,dx$ exists.

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Aniq integralning ta'rifidan ko'rinadiki, aniq integral hamma vaqt mavjud bo'lavermas ekan. Biz qo'yida aniq integralning mavjudlik teoremasini isbotsiz keltiramiz.

THEOREM 4.5 THE DEFINITE INTEGRAL AS THE AREA OF A REGION

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b f(x)\,dx.$$

(See Figure 4.21.)

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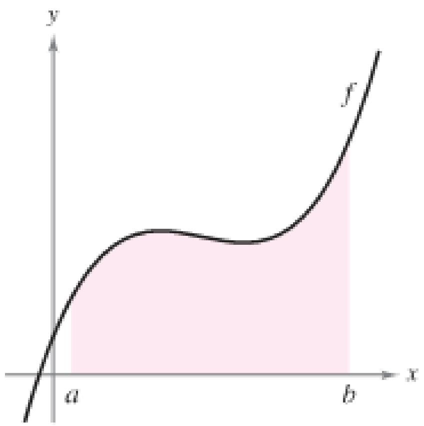

You can use a definite integral to find the area of the region bounded by the graph of f, the x-axis, $x = a$, and $x = b$.

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Teorema. Agar $u=f(x)$ funksiya $[a, b]$ kesmada uzluksiz bo'lsa, u integrallanuvchidir, ya'ni bunday funksiyaning aniq integrali mavjuddir.

Agar yuqoridan $y=f(x)\geq 0$ funksiyaning grafigi, qo'yidan OX o'qi, yon tomonlaridan esa $x=a$, $x=b$ to'g'ri chiziqlar bilan chegaralangan sohani egri chiziqli trapetsiya deb atasak, u holda

$$\int_a^b f(x)dx.$$

Aniq integralning geometrik ma'nosi ravshan bo'lib qoladi: $f(x) \geq 0$ bo'lganda u shu egri chiziqli trapetsiyaning yuziga son jihatdan teng bo'ladi.

1-izoh. Aniq integralning qiymati funksiyaning ko'rinishiga va integrallash chegarasiga bog'liq. Masalan:

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(z)dz.$$

2-izoh. Aniq integralning chegaralari almashtirilsa, integralning ishorasi o'zgaradi.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

3-izoh. Agar aniq integralning chegaralari teng bo'lsa, har qanday funksiya uchun ushbu tenglik o'rinli bo'ladi:

$$\int_a^b f(x)dx = 0$$

18.2. Aniq integralning asosiy xossalari.

1-xossa. O'zgarmas ko'paytuvchini aniq integral belgisidan tashqariga chiqarish mumkin:

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx \qquad (1)$$

2-xossa. Bir nechta funksiyaning algebraik yig'indisining aniq integrali qo'shiluvchilar integralining yig'indisiga teng (ikki qo'shiluvchi bo'lgan hol bilan chegaralanamiz):

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx \qquad (2)$$

THEOREM 4.7 PROPERTIES OF DEFINITE INTEGRALS

If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and

1. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

2. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$

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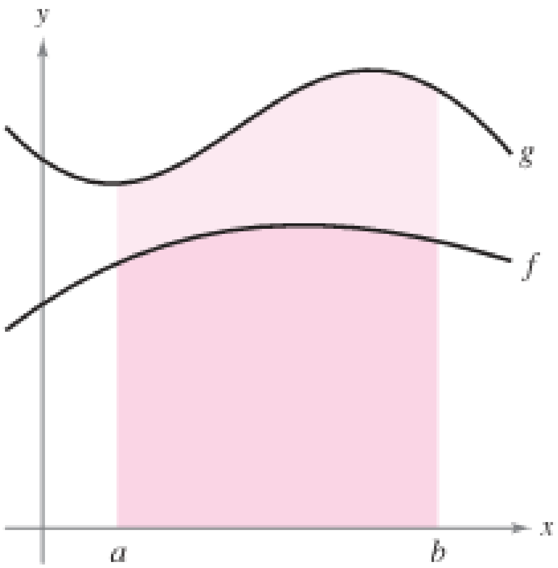
3-xossa. Agar $[a, b]$ kesmada ikki $f(x)$ va $g(x)$ funksiya $(a<b)$ $f(x) \le g(x)$ shartni qanoatlantirsa, ushbu tengsizlik o'rinli.

$$\int_a^b f(x)dx \ge \int_a^b g(x)dx. \qquad (3)$$



$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

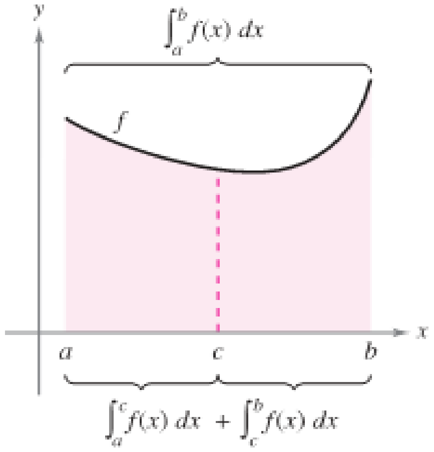
4-xossa. Agar [a, b] kesma bir necha qismga bo'linsa, u holda [a, b] kesma bo'yicha aniq integral har bir qism bo'yicha olingan aniq integrallar yig'indisiga teng. [a, b] kesma ikki qismga bo'lingan hol bilan cheklanamiz, ya'ni $a<c<b$ bo'lsa, u holda

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \qquad (4)$$

THEOREM 4.6 ADDITIVE INTERVAL PROPERTY

If f is integrable on the three closed intervals determined by a, b, and c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$



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5-xossa. Agar m va M sonlar $f(x)$ funksiyaning $[a, b]$ kesmada eng kichik va eng katta qiymati bo'lsa, ushbu tengsizlik o'rinli.

$$m(b-a) \le \int_a^b f(x)dx. \le M(b-a) \qquad (5)$$

Isboti Shartga ko'ra $m \le f(x) \le M$ ekani kelib chiqadi. 3-xossaga asosan qo'yidagiga ega bo'lamiz:

$$\int_a^b m dx \le \int_a^b f(x)dx \le \int_a^b M dx \qquad (5^*)$$

Biroq

$$\int_a^b m dx = m\int_a^b dx = m \lim_{\max \Delta x_i \to 0} \sum_{i=1}^n \Delta x_i = m(b-a)$$

$$\int_a^b M dx = M\int_a^b dx = M \lim_{\max \Delta x_i \to 0} \sum_{i=1}^n \Delta x_i = M(b-a)$$

bo'lgani uchun (5*) tengsizlik

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

bo'ladi.

6-xossa (o'rta qiymat haqidagi teorema).

Agar $f(x)$ funksiya $[a, b]$ kesmada uzluksiz bo'lsa, bu kesmaning ichida shunday $x=s$ nuqta topiladiki, bu nuqtada funksiyaning qiymati uning shu kesmadagi o'rtacha qiymatiga teng bo'ladi, ya'ni

$$f(s)=\frac{1}{(b-a)}\int_a^b f(x)dx.$$

Isboti. Faraz qilaylik, m va M sonlar $f(x)$ uzluksiz funksiyaning $[a, b]$ kesmadagi eng kichik va eng katta qiymati bo'lsin. Aniq integralni baholash haqidagi xossaga ko'ra qo'yidagi qo'sh tengsizlik to'g'ri:

$$m(b-a)\leq\int_a^b f(x)dx.\leq M(b-a)$$

tengsizlikning hamma qismlarini b-a>0 ga bo'lamiz. Natijada

$$m \leq \frac{1}{(b-a)}\int_a^b f(x)dx \leq M$$

ni hosil qilamiz. Ushbu $\mu = \frac{1}{(b-a)}\int_a^b f(x)dx.$ belgilashni kiritib, qo'sh tengsizlikni qayta yozamiz. $m \leq \mu \leq M$

$f(x)$ funksiya $[a, b]$ kesmada uzluksiz bo'lgani uchun u m va M orasidagi hamma oraliq qiymatlarni qabul qiladi.

Demak, biror $x=s$ qiymatda $\mu=f(s)$ bo'ladi, ya'ni

$$f(s) = \frac{1}{(b-a)}\int_a^b f(x)dx.$$

Teorema isbotlandi.

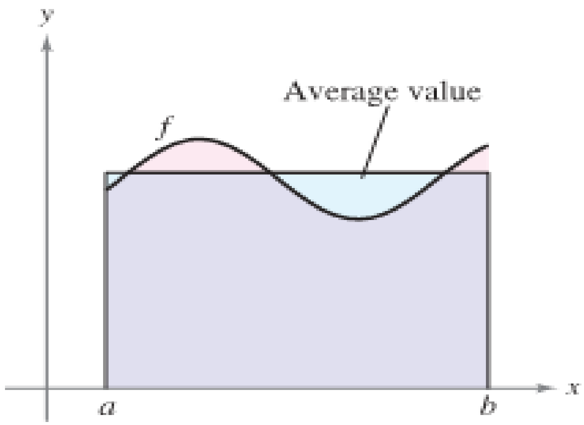
Average Value of a Function

The value of $f(c)$ given in the Mean Value Theorem for Integrals is called the **average value** of f on the interval $[a, b]$.

DEFINITION OF THE AVERAGE VALUE OF A FUNCTION ON AN INTERVAL

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$



Average value $= \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

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1. Nyuton-Leybnits formulasi.

Aniq integrallarni integral yig'indining limiti sifatida bevosita hisoblash ko'p hollarda juda qiyin, uzoq hisoblashlarni talab qiladi va amalda juda kam qo'llaniladi. Integrallarni topish formulasi Nyuton-Leybnits teoremasi bilan beriladi.

Teorema. Agar *F(x)* funksiya *f(x)* funksiyaning [*a*, *b*] kesmadagi boshlang'ich funksiyasi bo'lsa, u holda aniq integral boshlang'ich funksiyaning integrallash oralig'idagi orttirmasiga teng, ya'ni

$$\int_a^b f(x)dx = F(b) - F(a) \qquad (1)$$

(1)tenglik Nyuton-Leybnits formulasi deyiladi.

1. *Provided you can find* an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.
2. When applying the Fundamental Theorem of Calculus, the following notation is convenient.

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b$$
$$= F(b) - F(a)$$

For instance, to evaluate $\int_1^3 x^3\,dx$, you can write

$$\int_1^3 x^3\,dx = \frac{x^4}{4}\Big]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$

3. It is not necessary to include a constant of integration C in the antiderivative because

$$\int_a^b f(x)\,dx = \Big[F(x) + C\Big]_a^b$$
$$= [F(b) + C] - [F(a) + C]$$
$$= F(b) - F(a).$$

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Isboti. $F(x)$ funksiya $f(x)$ funksiyaning biror boshlang'ich funksiyasi bo'lsin, u holda 1-teoremaga ko'ra $\int_a^x f(t)dt$ funksiya ham $f(x)$ funksiyaning boshlang'ich funksiyasi bo'ladi. Berilgan funksiyaning ikkita istalgan boshlang'ich funksiyalari bir-biridan o'zgarmas C qo'shiluvchiga farq qiladi, ya'ni $F(x)=F(x)+C$.

Shuning uchun:

$$\int_a^x f(t)dt = F(x) + C$$

C-o'zgarmas miqdorni aniqlash uchun bu tenglikda x=a deb olamiz:

$$\int_a^a f(t)dt = F(a) + C \ , \ \int_a^a f(t)dt = 0$$

bo'lgani uchun $F(a)+C=0$. Bundan, $S=-F(a)$. Demak, $\int_a^x f(t)dt = F(x) - F(a)$

Endi $x=b$ deb Nyuton-Leybnits formulasini hosil qilamiz:

$$\int_a^b f(t)dt = F(b)\text{-}F(a)$$

yoki integrallash o'zgaruvchisini x bilan almashtirsak:

$$\int_a^b f(x)dx = F(b)\text{-}F(a)$$

$F(b)\text{-}F(a)=F(x)\Big|_a^b$ belgilash kiritib, oxirgi formulani qo'yidagicha qayta yozish mumkin:

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b)\text{-}F(a)$$

Teorema isbotlandi.

Integral ostidagi funksiyaning boshlang'ich funksiyasi ma'lum bo'lsa, u golda Nyuton-Leybnits formulasi aniq integrallarni hisoblash uchun amalda qulay usulni beradi. Faqat shu formulaning kashf etilishi aniq integralni hozirgi zamonda matematik analizda tutgan o'rnini olishga imkon bergan. Nyuton-Leybnits formulasi aniq integralning tatbiqi sohasini ancha kengaytirdi, chunki matematika bu formula yordamida xususiy kurinishdagi turli masalalarni yechish uchun umumiy usulga ega bo'ldi.

Misollar.

1) $$\int_0^1 \frac{dx}{1+x^2} = arctgx|_0^1 = arctg1 - arctg0 = \frac{\pi}{4}$$

2) $$\int_{\sqrt{3}}^{\sqrt{8}} \frac{xdx}{\sqrt{1+x^2}} = \frac{1}{2}\int_{\sqrt{3}}^{\sqrt{8}} \frac{d(1+x^2)}{\sqrt{1+x^2}} = \frac{1}{2}\int_{\sqrt{3}}^{\sqrt{8}} (1+x^{2})^{-\frac{1}{2}} d(1+x^2) = (1+x^2)^{\frac{1}{2}}|_{\sqrt{3}}^{\sqrt{8}} =$$

$$= \sqrt{9} - \sqrt{4} = 3 - 2 = 1.$$

3) $$\int_0^{\frac{\pi}{2}} \sin^2 xdx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2x)dx = \frac{1}{2}(x - \frac{1}{2}\sin 2x)|_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

2. O'zgaruvchini almashtirish.

Bizga $\int_a^b f(x)dx$ aniq integral berilgan bo'lsin, bunda $f(x)$ funksiya $[a, b]$ kesmada uzluksizdir.

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Aniq integral (2) formula bo'yicha hisoblaganda yangi o'zgaruvchidan eski o'zgaruvchiga qaytish kerak emas, balki eski o'zgaruvchining chegaralarini keyingi boshlang'ich funksiyaga qo'yish kerak.

THEOREM 4.15 CHANGE OF VARIABLES FOR DEFINITE INTEGRALS

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

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Misollar.

1) $\int\limits_{3}^{8} \frac{xdx}{\sqrt{x+1}}$ integralni hisoblang.

Yechish. x+1=u^2 deb almashtirsak, x=u^2-1, dx=2udu bo'ladi. Integrallashning yangi chegaralari: x=3 bo'lganda t=2.

x=8 bo'lganda u=3 u holda:

$$\int\limits_{3}^{8} \frac{xdx}{\sqrt{x+1}} = \int\limits_{2}^{3} \frac{(t^2-1)2udu}{t} = 2\int\limits_{2}^{3} (u^2-1)du = 2(\frac{u^3}{3} - u)\Big|_2^3 = 2(6-\frac{2}{3}) = \frac{32}{3};$$

2) $\int\limits_{0}^{1} \sqrt{1-x^2}\, dx$ integralni hisoblang.

Yechish. x=sin$u$ deb almashtirsak, dx=cosudu, 1-x^2=cos^2u bo'ladi. Integrallashning yangi chegaralarini aniqlaymiz: x=0 bo'lganda u=0

x=1 bo'lganda u=π/2

U holda:

$$\int\limits_{0}^{1} \sqrt{1-x^2}\, dx = \int\limits_{0}^{\pi/2} \cos^2 udu = \frac{1}{2}\int\limits_{0}^{\pi/2}(1+\cos 2udu = \frac{1}{2}(u+\frac{1}{2}\sin 2u)\Big|_0^{\pi/2} = \frac{\pi}{4}$$

3. Aniq integralni bo'laklab integrallash.

Faraz qilaylik, $u(x)$ va $v(x)$ funksiyalar $[a, b]$ kesmada differensiallanuvchi funksiyalar bo'lsin. U holda: $(uv)' = u'v + uv'$

Bu tenglikni ikkala tomonini a dan b gacha bo'lgan oraliqda integrallaymiz.

$$\int_a^b (uv)'dx = \int_a^b u'vdx + \int_a^b uv'dx \qquad (3)$$

Lekin $\int (uv)'dx = uv + C$ bo'lgani sababli

$$\int (uv)'dx = uv\Big|_a^b$$

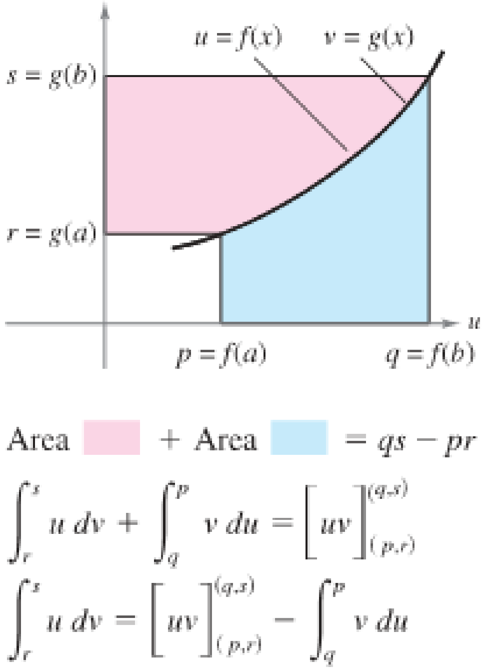
Demak, (3) tenglikni qo'yidagi ko'rinishda yozish mumkin:

$$uv\Big|_a^b = \int_a^b vdu + \int_a^b udv$$

Bundan

$$\int_a^b udv = uv\,\Big|_a^b - \int_a^b vdu \qquad (4)$$

Bu formula aniq integralni bo'laklab integrallash formulasi deyiladi.



Misol.

1) $\int_0^1 arctgx dx$ integral hisoblansin.

$$\int_0^1 arctgx dx = \begin{vmatrix} u = arctgx & du = \dfrac{dx}{1+x^2} \\ dv = dx & v = x \end{vmatrix} = x arctgx \Big|_0^1 - \int_0^1 \frac{xdx}{1+x^2} = arctg1 -$$

$$\frac{1}{2}\ln\left(1+x^2\right)\Big|_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

2) $\int_0^1 xe^{-x} dx$ integral hisoblansin.

$$\int_0^1 xe^{-x} dx = \begin{vmatrix} u = x & du = dx \\ dv = e^{-x} dx & v = -e^{-x} \end{vmatrix} = -xe^{-x}\Big|_0^1 + \int_0^1 e^{-x} dx = -e^{-1} - e^{-x}\Big|_0^1 =$$

$$= -e^{-1} - e^{-1} + 1 = 1 - \frac{2}{e};$$

Izoh: Ba'zi integrallarni hisoblashda bo'laklab integrallash formulasini bir necha marta qo'llash mumkin.

3) $\int_0^1 \arcsin x dx$ integral hisoblansin.

Evaluate $\int_0^1 \arcsin x \, dx$.

Solution Let $dv = dx$.

$$dv = dx \quad \Rightarrow \quad v = \int dx = x$$

$$u = \arcsin x \quad \Rightarrow \quad du = \frac{1}{\sqrt{1 - x^2}} \, dx$$

Integration by parts now produces

$$\int u \, dv = uv - \int v \, du$$

Integration by parts formula

$$\int \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1 - x^2}} \, dx$$

Substitute.

$$= x \arcsin x + \frac{1}{2} \int (1 - x^2)^{-1/2} (-2x) \, dx$$

Rewrite.

$$= x \arcsin x + \sqrt{1 - x^2} + C.$$

Integrate.

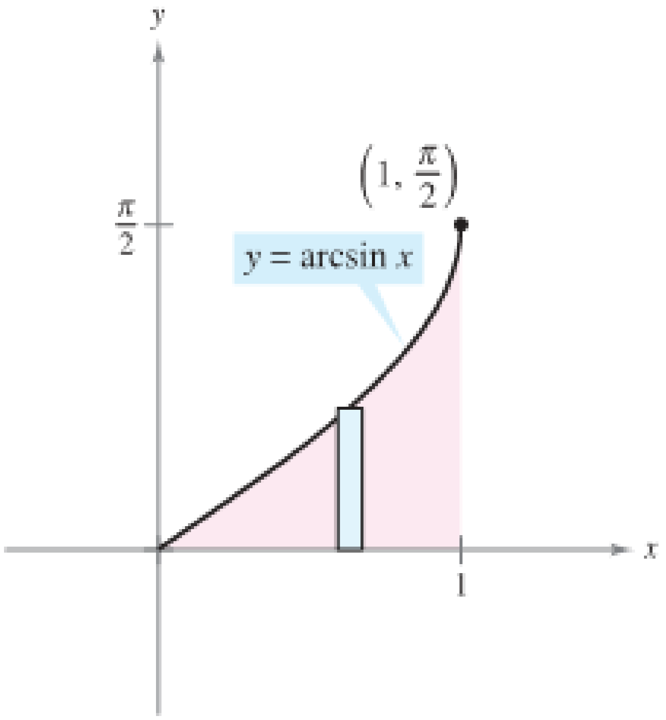
Using this antiderivative, you can evaluate the definite integral as follows.

$$\int_0^1 \arcsin x \, dx = \left[x \arcsin x + \sqrt{1 - x^2} \right]_0^1$$

$$= \frac{\pi}{2} - 1$$

$$\approx 0.571$$

The area represented by this definite integral is shown in Figure 8.2.



The area of the region is approximately 0.571.
Figure 8.2

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