2-MA'RUZA. IKKINCHI VA UCHINCHI TARTIBLI DETERMINANTLAR VA ULARNING XOSSALARI. CHIZIQLI TENGLAMALAR SISTEMASINI KRAMER USULIDA YECHISH

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Fanchi, John R. Math refresher for scientists and engineers.—3rd edition.

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Dars rejasi

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2x2- matrisaning determinanti quyidagicha hisoblanadi[1]

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

An explicit representation of the determinant of a 2×2 matrix is

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

1- misol. $\begin{vmatrix} 5 & 3 \\ -2 & 6 \end{vmatrix} = 5\cdot 6 - 3\cdot(-2) = 30 + 6 = 36.$

Shuningdek, 3x3- matrisaning determinanti quyidagicha hisoblanadi[2]

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= [a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}]$$

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[1] Fanchi J.R. - Math refreshser for scientists and engineers, 2006. Page 76

[2] Fanchi J.R. - Math refreshser for scientists and engineers, 2006. Page 76

$F = < F; +, \cdot, -, ^{-1}, 0, 1 >$ maydon va maydon ustida $F^{n\times n}$ kvadrat matritsalar to'plami berilgan bo'lsin.

1-ta'rif. Kvadrat matritsaning har bir satr va har bir ustunidan bittadan elementlar olib tuzilgan ko'paytmalarning algebraik yig'indisiga berilgan kvadrat matritsaning determinanti deyiladi.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$ matritsaning har bir satr va har bir ustunidan bittadan element olib tuzilgan n ta elementlar ko'paytmasi $a_{1\tau_1} \cdot \dots \cdot a_{n\tau_n}$ bilan n-darajali o'rniga qo'yish $\tau = \begin{pmatrix} 1 & \dots & n \\ \tau(1) & \dots & \tau(n) \end{pmatrix}$ larni birini ikkinchisiga mos qo'yuvchi o'zaro bir qiymatli moslik mavjud. Bu moslikdan n-tartibli kvadrat matritsaning determinantini aniqlashda foydalanamiz.

Uchinchi darajali o'rniga qo'yishlar to'plami $S_3 = \{\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5\}$dagi o'rniga qo'yishlar quyidagicha:

$$\varphi_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},\ \varphi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},\ \varphi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},\ \varphi_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},$$

$$\varphi_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},\ \varphi_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Uchinchi tartibli kvadrat matritsa determinanti $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ ni hisoblash uchun uchinchi darajali o'rniga qo'yishlar yordamida ko'paytmalar tuzamiz. Urniga qo'yishning ishorasi u yordamida hosil qilingan ko'paytmani qo'shish yoki ayirish kerakligini aniqlab beradi. Bundan quyidagi ifodani hosil qilamiz.

$$D=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}.$$

2-ta'rif. n-tartibli kvadrat matritsa $A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$ ning determinanti deb $|A|=\sum\limits_{\tau\in S_n}\operatorname{sgn}(\tau)a_{1\tau(1)}\cdot\ldots\cdot a_{n\tau(n)}$ ($n!$ qo'shiluvchilardan iborat) yig'indiga aytiladi.

Determinantning xossalari.

1-teorema. Nol satr yoki ustunga ega kvadrat matritsaning determinanti nolga teng.

2-teorema. Diagonal matritsaning determinanti asosiy diagonal elementlari ko'paytmasiga teng.

3-teorema. Uchburchak matritsaning determinanti asosiy diagonal elementlari ko'paytmasiga teng.

4-teorema. Kvadrat matritsa va unga transponirlangan matritsalar

determinantlari teng.

5-teorema. Kvadrat matritsaning ikkita satr (ustun)lari o'rnini almashtirish natijasida determinant ishorasi o'zgaradi.

6-teorema. Ikkita bir xil satr (ustun)ga ega kvadrat matritsa determinanti nolga teng.

7-teorema. A kvadrat matritsaning biror bir satr (ustun) elementlarini noldan farqli λ skalyarga ko'paytirilsa, u holda A matritsaning determinanti λ skalyarga ko'paytiriladi.

8-teorema. Qandaydir ikkita satr (ustun)lari proporsional bo'lgan kvadrat matritsaning determinanti nolga teng.

9-teorema. Kvadrat matritsa i - qatori (ustuni)ning har bir elementi m ta qo'shiluvchilardan iborat bo'lsa, bunday kvadrat matritsaning determinanti m ta determinantlar yig'indisidan iborat bo'lib, birinchi determinant i - qatori (ustuni)da birinchi, ikkinchi determinantda ikkinchi qo’shiluvchilar va h.z. boshqa qatorlar A matritsanikidek bo’ladi.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+a & a_{22}+b & a_{23}+c \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a & b & c \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

10-teorema. Kvadrat matritsaning biror-bir satr (ustun)iga noldan farqli skalyarga ko’paytirilgan boshqa satr (ustun)ni qo’shish natijasida determinant o’zgarmaydi.

11-teorema. Kvadrat matritsaning biror-bir satr (ustun)iga qolgan satr (ustun)lar chiziqli kombinatsiyasini qo’shish natijasida determinant o’zgarmaydi.

12-teorema. Kvadrat matritsaning biror-bir satri (ustuni) qolganlarining chiziqli kombinatsiyasidan iborat bo'lsa, uning determinanti nolga teng.

13-teorema. Har qanday elementar matritsaning determinanti noldan farqli.

Properties of Determinants

Many of the properties of determinants are associated with performing elementary row (or column) operations upon the elements of the determinant. The **three basic elementary row operations being performed on determinants** are

(i) The interchange of any two rows.

(ii) The multiplication of a row by a nonzero scalar α

(iii) The replacement of the ith row by the sum of the ith row and α times the jth row, where $i \neq j$ and α is any nonzero scalar quantity.

The following are some properties of determinants stated without proof.

1. If two rows (or columns) of a determinant are equal or one row is a constant multiple of another row, then the determinant is equal to zero.
2. The interchange of any two rows (or two columns) of a determinant changes the numerical sign of the determinant.
3. If the elements of any row (or column) are all zero, then the value of the determinant is zero.
4. If the elements of any row (or column) of a determinant are multiplied by a scalar m and the resulting row vector (or column vector) is added to any other row (or column), then the value of the determinant is unchanged. As an example, take a 3×3 determinant and multiply row 3 by a nonzero constant m and add the result to row 2 to obtain

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ (d+mg) & (e+mh) & (f+mi) \\ g & h & i \end{vmatrix}.$$

5. If all the elements in a row (or column) are multiplied by the same scalar q, then the determinant is multiplied by q. This produces

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ qa_{i1} & qa_{i2} & \cdots & qa_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = q \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

6. The determinant of the product of two matrices is the product of the determinants and $|AB| = |A||B|$.
7. If each element of a row (or column) is expressible as the sum of two (or more) terms, then the determinant may also be expressed as the sum of two (or more) determinants. For example,

$$\begin{vmatrix} a_{11}+b_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1}+b_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

8. Let c_{ij} denote the cofactor of a_{ij} in the determinant of A. The value of the determinant $|A|$ is the sum of the products obtained by multiplying each element of a row (or column) of A by its corresponding cofactor and

$$|A| = a_{i1}c_{i1} + \cdots + a_{in}c_{in} = \sum_{k=1}^{n} a_{ik}c_{ik} \quad \text{row expansion}$$

$$\text{or} \quad |A| = a_{1j}c_{1j} + \cdots + a_{nj}c_{nj} = \sum_{k=1}^{n} a_{kj}c_{kj} \quad \text{column expansion}$$

If the elements of a row (or column) are multiplied by the cofactor elements from a different row (or column), then zero is obtained. These results can be used to write $AC^T = |A|I$

Deteminantning nolga teng bo'lish sharti.

Kramer formulasi

$F=<F;+,\cdot,-,^{-1},0,1>$ maydon va maydon ustida $F^{n\times n}$ matrisalar to'plami va A=
$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$ berilgan bo'lsin.

15-teorema. Kvadrat matritsaning determinanti nolga teng bo'lishi uchun uning satr (ustun)lari chiziqli bog'langan bo'lishi zarur va yetarli.

Isbot. 1. Matritsaning satrlari chiziqli erkli bo'lsa, $|A| \neq 0$ ekanligini isbotlaymiz.

Agar berilgan kvadrat matritsaning satrlari chiziqli erkli bo'lsa, u holda uni elementar matritsalar ko'paytmasi ko'rinishida ifodalash mumkin, ya'ni $A = E_1 \cdot E_2 \cdot \ldots \cdot E_k$. U holda determinant xossalariga ko'ra

$|A| = |E_1| \cdot |E_2| \cdot \ldots \cdot |E_k|$ va $|E_i| \neq 0 (i = \{1, \ldots, k\}$. Bundan $|A| \neq 0$.

To'g'ri teorema bilan teskari teoremaga qarama-qarshi teoremalar teng kuchli bo'lganligidan, $|A| = 0$ ekanligidan A matritsa chiziqli erkliligi kelib chiqadi.

2. A matritsaning satrlari chiziqli bog’liq bo’lsa, $|A| = 0$ ekanligini isbotlaymiz.

Satrlari chiziqli bog’liq matritsaning kamida bitta satri qolganlari orqali chiziqli ifodalanadi. Determinantlar xossalariga ko’ra $|A| = 0$.

1-misol. $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 2 & 4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 7 \\ 0 & 0 & 0 \end{vmatrix} = 0.$

16-teorema. Har qanday kvadrat matritsa uchun quyidagi shartlar teng kuchli:

1. $|A| \neq 0$.

2. Matritsaning satr (ustun)lari chiziqli erkli.

3. A matritsa teskarilanuvchi.

4. A matritsa elementar matritsalar yordamida ifodalanadi.

17-teorema. A matritsaning rangi uning noldan farqli minorlarining eng yuqori tartibiga teng.

Isboti. Noldan farqli A= $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ matritsa berilgan

bo'lsin. U holda uning rangi $r = r(A) > 0$. Matritsaning kamida bitta noldan farqli r tartibli minori mavjudligini isbotlaymiz.

$r = r(A) > 0$ bo'lganligi uchun, A matritsaning r ta chiziqli erkli satrlari bor. Shu satrlardan tuzilgan A matritsaning $B \in F^{r \times n}$ matritsaostisini tuzamiz B=

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{r1} & a_{r2} & \dots & a_{rn} \end{pmatrix}$$, bu matritsaning rangi $r(B)=r$. Matritsaning satr va ustun ranglari tengligidan $\rho(B)=r$. Demak, B matritsaning r ta chiziqli erkli ustunlari mavjud. B matritsaning r ta chiziqli erkli ustunlaridan tashkil topgan matritsaostisini C bilan belgilaymiz. U holda $C \in F^{r \times r}$ va $r(C)=r$. Yuqoridagi 18.2-teorema shartlariga ko'ra, C matritsaning ustunlari chiziqli erkli bo'lganligi uchun $|C| \neq 0$.

Demak, C matritsa A matritsaning tartibi r ga teng bo'lgan noldan farqli minori bo'ladi.

Agar $k > r(A)$ bo'lsa, A matritsaning k tartibli har qanday minori nolga teng bo'ladi.

Haqiqatdan ham, $k > r(A)$ bo'lsa, A matritsaning har qanday k ta satri chiziqli bog'langan bo'ladi. Bundan A matritsaning har qanday $(k \times k)$ tartibli qismmatritsasida satrlari chiziqli bog'langan bo'ladi va 18.1-teoremaga ko'ra bunday qismmatritsalar determinanti, ya'ni A matritsaning k tartibli har qanday minori nolga teng.

2-misol. $A = \begin{pmatrix} 1 & -1 & 0 & 2 \\ 2 & 1 & 1 & -2 \\ -1 & 0 & 1 & 3 \\ 2 & -2 & 0 & 4 \end{pmatrix}$ matritsa rangini minorlar yordamida aniqlang.

Yechish. Matritsa rangi haqidagi teoremaga ko'ra matritsaning noldan farqli minorlarini aniqlaymiz.

Matritsaning berilishidan, unda kamida bitta noldan farqli birinchi tartibli minor mavjud, masalan, $A_1 = (1)$ matritsaostining determinanti 1ga teng, ya'ni $M_1 = |1| = 1 \neq 0$.

Matritsaning $A_2 = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$ matritsaostining determinanti

$$M_2 = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 1 - (-2) = 3 \neq 0 .$$

Matritsaning $A_3 = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ matritsaostining determinanti

Matritsaning 4-tartibli minori berilgan matritsaning determinantidan iborat, uni hisoblaymiz:

$$|A| = \begin{vmatrix} 1 & -1 & 0 & 2 \\ 2 & 1 & 1 & -2 \\ -1 & 0 & 1 & 3 \\ 2 & -2 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 2 \\ 0 & 3 & 1 & -6 \\ 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0 .$$

Demak, berilgan matritsaning noldan farqli minorlari 1-tartibli, 2-tartibli va 3-tartibli. Ulardan yuqori tartibligi 3-tartibli minor bo'lganligi uchun, berilgan matritsaning rangi 3 ga teng.

2-ta'rif. A= $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$ matritsaning a_{ij} elementining

A_{ij} $(i, j \in \{1,\dots,n\}$ algebraik to'ldiruvchilaridan iborat

$$A^* = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ . & . & \dots & . \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$ matritsaga A matritsaga biriktirilgan

matritsa deyiladi.

18-teorema. Agar $|A| \neq 0$ bo'lsa, u holda A matritsa teskarilanuvchi va $A^{-1} = |A|^{-1} \cdot A^*$.

Isbot. 17.3-Laplas teoremasi va 17.4-teoremalarga ko'ra

$$A_i (A^*)^j = (a_{i1},\dots,a_{in}) \cdot \begin{pmatrix} A_{j1} \\ \vdots \\ A_{jn} \end{pmatrix} = a_{i1}A_{j1} + \dots + a_{in}A_{jn} = \begin{cases} |A, агар, i = j; | \\ 0, агар, i \neq j. \end{cases}$$

$$|A| = \begin{vmatrix} 5 & -1 & -1 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{vmatrix} = 5(4-9)+1(2-12)-1(3-8)=-25-10+5=-30.$$

Determinant noldan farqli, demak, matritsaning teskarisi mavjud. Matritsaning har bir elementi algebraik to'ldiruvchisini topamiz:

$A_{11} = (-1)^{1+1}$ ™ $M_{11} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -5;$

$A_{12} = (-1)^{1+2}$ ™ $M_{12} = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = 10;$

$A_{13} = (-1)^{1+3}$ ™ $M_{13} = \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = -5;$

$A_{21} = (-1)^{2+1}$ ™ $M_{21} = \begin{vmatrix} -1 & -1 \\ 3 & 2 \end{vmatrix} = -1;$

$A_{22} = (-1)^{2+2}$ ™ $M_{22} = \begin{vmatrix} 5 & -1 \\ 4 & 2 \end{vmatrix} = 14;$

$A_{23} = (-1)^{2+3}$ ™ $M_{23} = \begin{vmatrix} 5 & -1 \\ 4 & 3 \end{vmatrix} = -19;$

$A_{31} = (-1)^{3+1}$ ™$M_{31} = \begin{vmatrix} -1 & -1 \\ 2 & 3 \end{vmatrix} = -1;$

$A_{32} = (-1)^{3+2}$ ™$M_{32} = \begin{vmatrix} 5 & -1 \\ 1 & 3 \end{vmatrix} = -16;$

$A_{33} = (-1)^{3+3}$ ™ $M_{33} = \begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 11;$