UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE & ENGINEERING
First Year Program – Core 8 and TrackOne

FIRST YEAR PROGRAM
ENGINEERING PROBLEM SOLVING LABS

MAT188: Laboratory #7
Solving Systems of Linear Equations

$$\boldsymbol{f(a,b,c) = \begin{pmatrix} a & b & c \\ b & a & b \\ c & b & a \end{pmatrix}}$$

SOLVING SYSTEMS OF LINEAR EQUATIONS

In many practical scenarios, engineers must solve systems of linear equations to find a specific solution or inform their work towards finding an optimal solution to a design challenge. A system of linear equations represents a problem with multiple variables that are governed by rules or fundamental principles (e.g., physical laws.) In this lab, you will learn how to create systems of linear equations to model engineering problems and use matrices to solve those linear systems using MATLAB.

Learning Outcomes

By the end of this lab students will…

1) Be able to solve systems of linear equations using MATLAB, and
2) Apply these techniques to solve engineering related problems.

Preparation *(Required to do **before** you come to the laboratory session)*

1. Read through this lab document and make sure you understand the basic process you have been learning in Linear Algebra for solving systems of linear equations.
2. Develop the required system of linear equations that you will need to solve to address the problem(s), then solve it/them using MATLAB.

Course Connection (MAT188: Linear Algebra)

In MAT188: Linear Algebra you have recently been learning about solving a system of linear equations, which can be expressed in matrix form as $Ax = b$, by using the following basic steps:

1) Create the ***augmented matrix*** from the system of linear equations of the form $[A \quad | b]$

$$\begin{aligned} 5x_1 - 15x_2 &= 0 \\ -10x_1 + 14x_2 &= 32 \end{aligned} \rightarrow \left[\begin{array}{cc|c} 5 & -15 & 0 \\ -10 & 14 & 32 \end{array}\right]$$

2) Through the process of ***Gauss-Jordan Elimination*** and through the use of ***elementary row operations*** you put the augmented matrix into its ***reduced row-echelon form***:

$$\left[\begin{array}{cc|c} 5 & -15 & 0 \\ -10 & 14 & 32 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 5 & -15 & 0 \\ 0 & -16 & 32 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 1 & -2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & -6 \\ 0 & 1 & -2 \end{array}\right]$$

In MATLAB there are a number of techniques that you can use to solve systems like $Ax = b$, where A is an $m \times n$ matrix and b is a $m \times 1$ vector:

1) *Gauss-Jordan Elimination*: The `[x]=rref(R)` command in MATLAB will calculate the reduced row echelon form of the augmented matrix $R = [A \quad | b]$. Note that you <u>don't type the '|' symbol</u> in MATLAB when you create your R matrix. This approach is useful when the system is *underdetermined*, meaning there are more unknowns than equations ($m < n$), or when there are *infinitely many solutions* (i.e., $rank(A) = rank(R) < n$).
2) *Backslash Operator*: One can use the *backslash* operator, '\' in MATLAB to solve the system $Ax = b$ through the calculation `x=A\b`.
3) *Inverse*: One can also write the solution to these systems as $x = A^{-1}b$ where A^{-1} represents the *inverse of* A. In MATLAB one can calculate this as `x=inv(A)*b`. This is of course only usable when the inverse of A exists, meaning the matrix A must be square (i.e., $m = n$) and $rank(A) = n$. In addition, the backslash calculation in MATLAB is generally quicker and less prone to errors, and thus is preferred over the use of the `inv(A)` command for solving systems.

4) *Symbolic*: If appropriate, one can also use the symbolic computation command `linsolve` so solve a system of linear equations.

Do the following exercise:
Write a short script to solve the following system of linear equations:

$$
\begin{gathered}
16I_1 - 2I_2 - 4I_3 = 6 \\
-2I_1 + 6I_2 - 2I_4 = 0 \\
-4I_1 - 2I_2 + 4I_3 + 6I_4 = 0 \\
-I_3 + I_4 = 3
\end{gathered}
$$

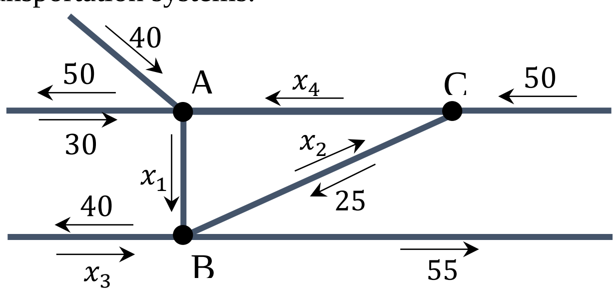
You should first check the rank of the coefficient matrix so you know what kind of solution to expect (use `rank(A)`). Use two different methods to confirm your solution.

<u>Engineering Applications</u>: Solving Systems of Linear Equations

Application #1: Traffic Flow

Adapted from Jeffery Holt, *Linear Algebra with Applications*, W. H. Freeman Co., New York, NY, 2013, pgs. 37 – 43.

One of the main areas of specialization in Civil Engineering is *Transportation Engineering and Planning*. This has been identified by the National Academy of Engineering as one of the 14 Grand Challenges for engineers of our time[1]. Systems of linear equations play an important role in the design of new transportation systems.



Consider this traffic scenario where five roads meet at three intersections, A, B, and C (Q3 from Section 1.4 of the Jeffry Holt textbook, pg. 43). Some sections are one-way roads, while others allow traffic flow in both directions. From road measurements over a few months, the average number of cars travelling in a certain direction at a particular time of day is known for certain sections of these roads. However, for four of the directions, the average numbers are not known (labeled as x_1, x_2, x_3, x_4).

Find the minimum average traffic flow at this time of day from C to A. To do this:

1) Identify the fundamental rule that applies at each intersection.
2) Find the three linear equations that govern these four unknowns.
3) Chose the appropriate method and use MATLAB to solve this system of linear equations. Note that you will need to carefully interpret the resulting RREF matrix, as your solution will have a free parameter.
4) Make a conclusion based on your parameterized solution. Observe that the four unknowns cannot be negative as they represent flow of traffic in a specific (positive) direction.

[1] The list of these Grand Challenges can be found at: http://www.engineeringchallenges.org/

Application #2: Chemical Reactions

Adapted from Jeffery Holt, *Linear Algebra with Applications*, W. H. Freeman Co., New York, NY, 2013, pgs. 37 – 43.

Understanding the laws that govern chemical reactions are a critical part of the scientific foundation for engineers. Balancing chemical reactions based on stoichiometry considerations can be assisted by using systems of linear equations.

Consider the two reactions presented below that are described by *unbalanced* equations with unknown coefficients *a* to *h*:

$$aC_3H_8 + bO_2 \rightarrow cCO_2 + dH_2O$$

$$eCO_2 + fH_2O \rightarrow gC_6H_{12}O_6 + hO_2$$

Balance these equations by finding the unknown coefficients. To do this:

1) Identify the fundamental rule that applies to these situations.
2) Determine the sets of three linear equations that govern the four unknowns for each case.
3) Chose the appropriate method and use MATLAB to solve this system of linear equations. Note that you will need to carefully interpret the resulting RREF matrix, as your solution will have a free parameter.
4) Make a conclusion on how to balance these equations.
5) Can you recognize what events these equations represent?

<u>Engineering Problem Solving</u>: Solving Systems of Linear Equations

Engineering Problem Solving Process

Consider once again George Pólya's Problem Solving Process that has been discussed in Lab #2 and APS100: Orientation to Engineering:

Understanding the Problem

What is the unknown? (GOAL)
What relevant information is provided? (KNOWNS)
What is the condition? What fundamental principles are related? (FOUNDATION)
Draw a diagram. (VISUALIZATION)

Devising a Plan

Consider the unknown. What are the possible connections between the information provided and the unknown.
Can the problem be restated in a more useful manner?
Do you know a related problem? Can you use its answer or its method?

Carrying Out the Plan

Check each step as you carry out the plan. Is it leading you in the right direction?

Looking Back

Can you check the result? How about the solution?
Is the answer complete? Have you solved the problem asked?

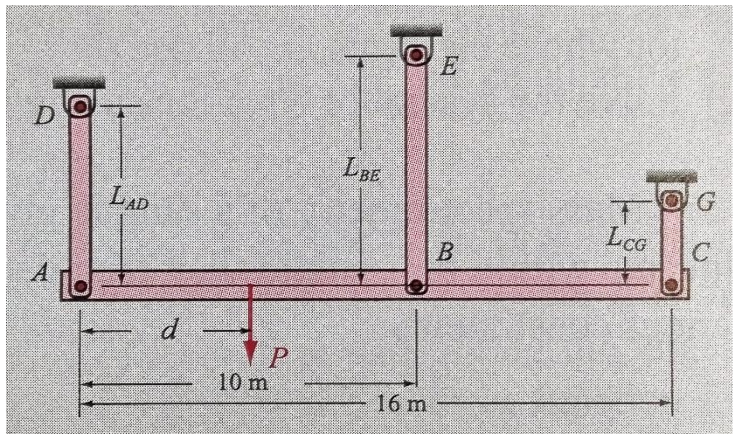
Make use of this process to solve the engineering problem described below.

***The following problem is for submission by Saturday, October 27th 11:59pm via MAT188 Labs website*:** While not required, you are encouraged to work with a partner on this lab. Both students in each pair must submit their work via the course website. In other words, two students can produce one document, but each of those students must upload that (identical) document. Make sure you write your name and student number and your partner's name and student number on the submission! Failure to follow this submission procedure will result in an incomplete submission.

Include your group's script, your two (or more if needed) plots, and a brief statement describing which material you would recommend and where, if possible, the weight should be placed to keep the large bar horizontal. Make sure to briefly explain your reasons for your conclusions.

Problem: *Bar Material Selection*[2]

As shown in the figure to the right, a rigid bar ABC is suspended by three vertical bars that have square cross-sections. As part of your summer research position you have been asked to answer the following using your MATLAB skills:



1) Determine which of the following materials can be used for all of the vertical bars in order to ensure that for $P = 90$ kN and any value of d (i.e., $0 \leq d \leq 16$ m) none of the bars elongate by more than 0.1% of their original length. If more than one material qualifies, you need to pick the lightest one.

- Aluminum Alloy 1100 (E=75GPa, density=2740kg/m^3)
- Nickel 200 (E=209GPa, density=8890kg/m^3)
- Steel Alloy 4340 (E=197GPa, density=7850kg/m^3)

2) For your choice of material, identify where, if possible, to properly locate the weight on the horizontal bar such that the bar remains horizontal.

Your colleagues have already determined that the forces in these bars are governed by the following set of equations:

$$F_{AD} + F_{BE} + F_{CG} = P$$
$$10F_{BE} + 16F_{CG} = dP$$
$$6\Delta L_{AD} - 16\Delta L_{BE} + 10\Delta L_{CG} = 0$$

Where E is the elastic (or Young's) modulus of the material used for the three vertical bars and A is the cross-sectional area each vertical bar, which in this case is known to be $A = 0.0004$ m^2. The original lengths of the bars are $L_{AD} = 4$m, $L_{BE} = 5$m, and $L_{CG} = 2$m. You have also discovered that the elongation of the bars can be expressed in terms of the force in the bar as:

$$\Delta L = \frac{FL}{EA}$$

[2] Problem adapted and figure from A. Gilat, V. Subramaniam, *Numerical Methods for Engineers and Scientists: An Introduction with Applications Using MATLAB*, John Wiley & Sons, Inc., 2008, pg. 176.

Work with the person next to you to set up the solution to these problems by:

1) *Clearly understanding these problems*: What exactly are your being asked for? What will a good answer look like? How do you expect the elongation of these bars to change as d increases?
2) *Devise plans to solve these problems*: How do you plan to use MATLAB to solve these two problems? What basic steps might your solution script have? What types of plots might be helpful to support your answers?

Now, with your group of two, create a script in MATLAB that solves these problems. Your script should generate at least two plots to support your conclusions.

3) *Carry out your plans*: Turn your basic steps into MATLAB code. Test and verify your code for a simple situation (i.e., Does it give you the expected results when d is large (16 m) or small (zero)? When d is 10 m are the results correct? Does the sum of the forces in the vertical bars equal to what you would expect for all values of d?)
4) *Look back*: Did you solve the problems that were asked? Have your adequately supported your conclusions? Do the results of the numeric computation make sense? What other considerations did you have to consider?