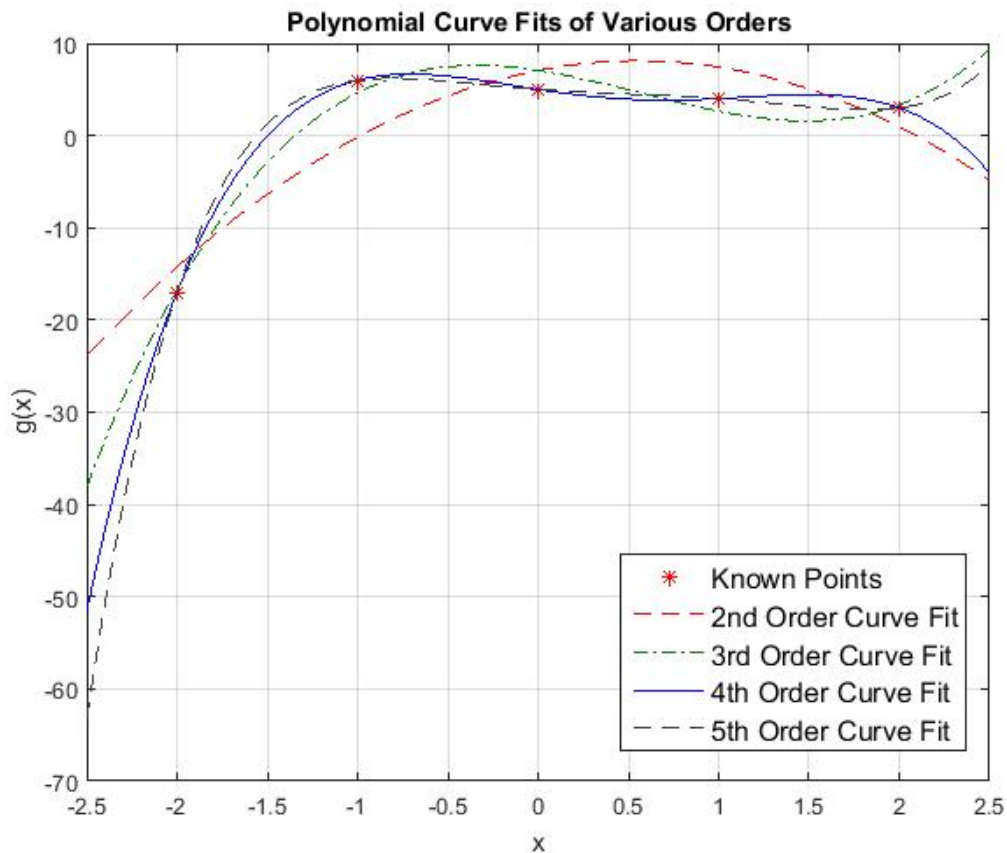




UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE & ENGINEERING
First Year Program – Core 8 and TrackOne

FIRST YEAR PROGRAM ENGINEERING PROBLEM SOLVING LABS

MAT188: Laboratory #4 *Mathematical Modelling and Curve Fitting*



CURVE FITTING AND MODELLING

In this lab, you will learn how to use polynomial curve fitting tools in MATLAB to mathematically model a system. You will learn how to use known data points to fit curves and model data, generating functions for interpolation or extrapolation.

Learning Outcomes

By the end of this lab you should...

- 1) Understand simple polynomial curve-fitting methods, and
- 2) Be able to develop a mathematical model from a curve fit based on known or measured data.

Preparation (Required doing *before* you come to the laboratory session)

1. Watch the video demonstration, *Analyzing and Visualizing Data with MATLAB* (3:26)
<http://www.mathworks.com/videos/analyzing-and-visualizing-data-with-matlab-70942.html>
2. Read through this lab document.

Review – Lab #3

At this point you should be comfortable with creating multiple plots in a figure and writing simple scripts to save and modify your code.

For example, you could use the following commands to produce plots of $\sin(2x - 1)$ and its derivative, $2 \cos(2x - 1)$, over the interval $[0, 2\pi]$ on the same figure.

```
>> x=linspace(0,2*pi,101) % x has 101 elements equally spaced from 0 to 2pi
>> y=sin(2*x-1); % calculates the value of the function
>> ydot=2*cos(2*x-1); % calculates the values of the first derivative
>> plot(x,y,'r-',x,ydot,'b--') % Plots both curves on the same figure with
different styles and colours. Alternatively plot y first then use hold and plot ydot.
```

MATLAB Skills and Knowledge:

Fitting polynomial curves to available data

The `polyval` function is useful for defining a vector of polynomial function values more easily. Instead of typing the whole function, you can provide a vector of the coefficients. `polyval(p, x)` generates a vector of polynomial values with coefficients given in vector `p` and input range given in vector `x`. For example, to define the function $f(x) = 2x^4 - 4x^3 + 7x - 5$ over the interval $[-5, 5]$:

```
>> x=linspace(-5,5,101);
>> coefs=[2 -4 0 7 -5];
>> y=polyval(coefs,x);
```

`y` now contains the values of the polynomial for each value of `x`. Notice how the x^2 coefficient must be included in the vector as a 0.

Curve Fitting (Guided Exercise)

In MAT188, you will soon solve problems on systems of linear equations, which can be applied to polynomial curve fitting¹.

Consider the following problem that we will solve:

Find a polynomial of the form $g(x) = ax^4 + bx^3 + cx^2 + dx + e$, such that it goes through the points $g(-2) = -17$, $g(-1) = 6$, $g(0) = 5$, $g(1) = 4$, $g(2) = 3$.

If you were to solve this problem on paper, you would obtain a system of five linear equations and solve it to find the five unknown coefficients²:

$$a = -1, b = 2, c = 1, d = -3, e = 5$$

Using these coefficients and `polyval`, generate a plot of this 4th-order polynomial, similar to Figure 1. Include the five known points as a separate function on your plot with a different line style (refer to the MATLAB Summary document for Lab 3).

The `polyfit` function calculates the coefficients of the *optimal curve fit*, in which the differences between the polynomial curve and the known data points are minimized.

This enables us to predict the behavior of a system based on only a few known or measured pieces of data. This process is known as *interpolation* or *extrapolation*, and the curve fit is a *mathematical model* of that system.

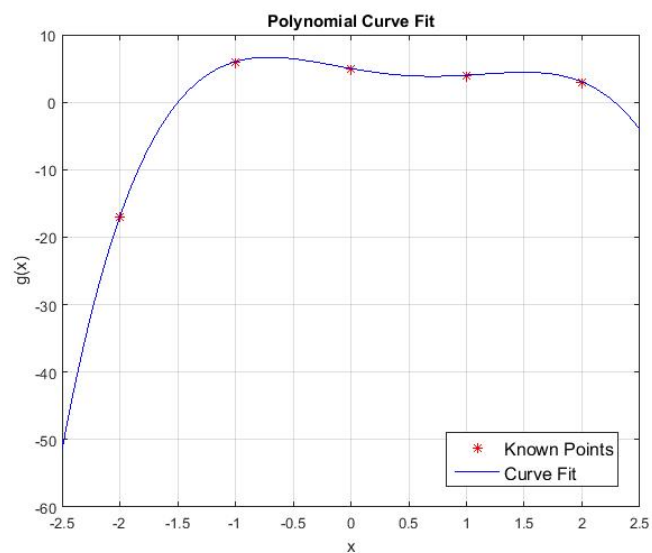


Figure 1. Fourth-order polynomial curve fit

Consider the same generic polynomial as above, $g(x)$, with the same set of known data points. Use the `polyfit` function in MATLAB to determine the coefficients for a fourth-order polynomial fit:

```
>> xpts = [-2, -1, 0, 1, 2];
>> ypts = [-17, 6, 5, 4, 3];
>> coefs = polyfit(xpts, ypts, 4)
```

How do the elements of the `coefs` vector compare to the solution given above?

¹ If you are interested a simple example using MATLAB can be found at: <https://www.youtube.com/watch?v=y2EF-d-15Oo>
We are extending these ideas here to deal with polynomial curve fits as opposed to linear (or straight line) fit in the video.

² Following the same process as shown in the video above, can you come up with these five linear equations?

A good engineer is able to identify the simplest possible mathematical model of a system that still predicts its behaviour accurately. We will now explore different polynomial curve fits for this system.

Use `polyfit` to determine the coefficients of a 2nd-order, a 3rd-order, and a 5th-order polynomial curve fit for this set of known data points. *Did you get any errors in these calculations?*

Practice by creating a script that obtains the coefficients of various polynomial curve fits (2nd, 3rd, 4th, and 5th order), then plots the known data points and the four curve fits on the same figure, similar to Figure 2. After getting the coefficients using `polyfit`, use `polyval` to create the y-vectors. Note that for the modelled functions based on the curve fits, you should create a proper input vector `x` using `linspace` and not use `xpts` as your input vector.

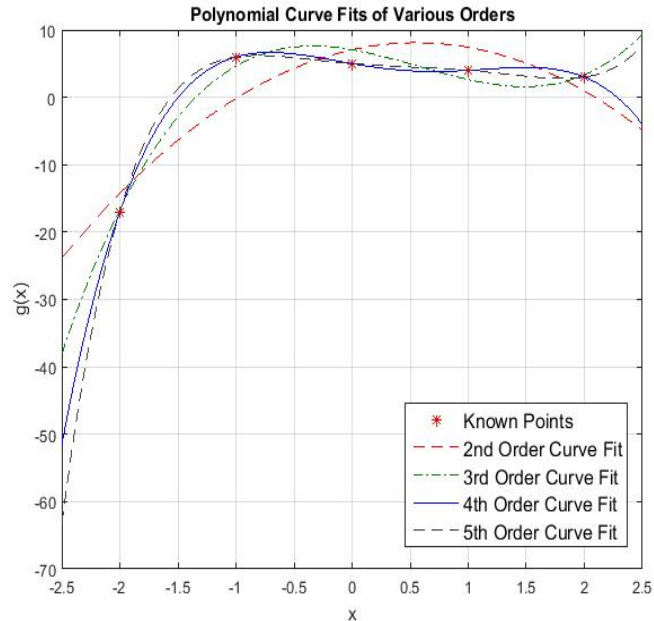


Figure 2. Comparison of various polynomial curve fits

What do you notice about the 2nd and 3rd order curve fits that is different from the 4th and 5th order curve fits?

If you were to use these models to predict or extrapolate the behavior of the system for values of $x > 2.5$, which curve fit appears to be the best?

(Do not submit responses to these questions above, they are to help you practice).

Engineering Applications

The Engineer's Toolbox - Mathematical Modelling

One of the most fundamental components of an “*engineer's toolbox*” is the ability to apply the principles of math and science to a problem to identify an optimal solution. Using *mathematical modelling*, an engineer creates an approximate description of a system using equations, graphs, or other tools, often based on fundamental principles combined with measured data. With such a model, the engineer can predict the behaviour of the system beyond what is known from direct measurements, and can virtually experiment with a wide range of design approaches or ideas without the need for physical prototyping or experimentation.

In this part of the lab, you will use curve fitting for a specific application. For a situation, you will plot the original data, determine and compare polynomial and exponential models for the data, choose which one better models the situation, and use the better model to predict future behaviour.

Try to complete both exercises below, but submit your response to ONLY ONE of them. (Either Penicillin Clearance OR US Population).

It is suggested that you take a screenshot of all relevant figures and code, and paste them into a neatly formatted document, and submitted as a single PDF file. Ideally you can complete this before the end of day on your Lab day (it is not designed to take you a long time to do). However, if you'd like, you can submit anytime up until Saturday October 6th, 2018, by 11:59pm.

Penicillin Clearance:

You will examine what happens when a patient is given an injection of penicillin and model its concentration in the body over time as the kidneys remove it. Write a script to do the following:

- Plot the five experimental data points in the table at http://mathinsight.org/penicillin_clearance_model.
- On the same figure, develop and plot a polynomial fit for this data. Experiment with different order curve fits and choose the best one.
- On the same figure, develop and plot an exponential fit for this data ($y = ae^{bt}$, where t is the time in minutes). You can determine a and b from two of the data points; do this by manual calculation on paper or in the MATLAB command window.
- Which type of fit seems to be more realistic as the better model of the real-life situation?
- Using the better model:
 - Predict when the penicillin concentration would drop below 40 $\mu\text{g/ml}$.
 - Suppose it is known that for penicillin to remain effective at killing bacteria, a level of $>20 \mu\text{g/ml}$ must be maintained within a person's bloodstream. Identify the longest time one should wait between penicillin injections.
- Please include a brief 75-150 word paragraph discussing the use of MatLab as a computational tool to help solve any of the problems above.**

US Population:

You will examine the growth of the US population over time, as recorded by census data. Write a script to do the following:

- Load the US Census data of US population in millions from 1790 to 1990 by typing `load census` in the MATLAB command window. This will create the vectors `cdate` and `pop` in the workspace. Plot this data.
- On the same figure, develop and plot a polynomial fit for this data. Experiment with different order curve fits and choose the best one.
- On the same figure, develop and plot an exponential fit for this data ($y = ae^{bt}$, where t is the time in years). You can determine a and b from two of the data points; do this by manual calculation on paper or in the MATLAB command window.
- Which type of fit seems to be more realistic as the better model of the real-life situation?
- Using the better model:

- i. Estimate the US population in 2017. Does this match what is expected (see <http://www.worldometers.info/world-population/us-population/>)?
 - ii. What would you predict the US population to be when you turn 40 years old?
 - iii. If it is expected that the current US agricultural system and economy will not be able to sustain a population over 750 million, estimate how many years the country's engineers, scientists, and government officials have to identify a long-term solution to these critical issues.
- f) Please include a brief 75-150 word paragraph discussing the use of MatLab as a computational tool to help solve any of the problems above.**