CHAPTER 8

Interpolation & Approximation

(Newton Interpolation, Lagrange & Least Square Method)

METHODS

- Newton Forward-Difference
- Newton Backward-Difference
- Newton Divided-Difference
- Lagrange
- Least Square Approximation

Interpolation

- In this topic, we study methods for representing a function based on knowledge of its behavior at certain discrete points.
- From this information, we may wish to obtain estimates of function values at other points.
- Interpolation produces a function that matches the given data exactly; we seek a function that also provides a good approximation to the (unknown) data values at intermediate points.
- The data may come from measured experimental values or computed values from other numerical methods.

Interpolation (cont.)

- Suppose we have a set of points $\{x_0, x_1, \dots, x_n\}$ ordered so that $x_0<x_1<\dots<x_n$ and a set of y-values $y_0, y_1, \dots, y_n$ corresponding to the x-values (that is, (x_i, y_i) is a pair).
- The polynomial interpolation problem is to find a polynomial $p(x)$ of degree at most n that interpolates the data $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$

$$p(x_k) = y_k, \qquad k = 0,1\dots,n$$

$$y(x) \approx p_n(x)$$

Example

k	0	1	2	3	4	5
x_k	1.0	1.2	1.4	1.6	1.8	2.0
y_k	0.0000	0.1823	0.3365	0.4700	0.5878	0.6931

Given the data above, find:

y(1.1) = ??

y(1.5) = ??

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Newton Forward-Difference Formula

- $x_0, x_1, \ldots, x_n$ are arranged consecutively with **equal** spacing.

$$x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1} = h$$

- Let,

$$x = x_0 + rh$$

then the difference x-x_i can be written as

$$x - x_i = (r - i)h$$

Newton Forward-Difference Formula (cont.)

- Polynomial (Newton Forward-Difference Formula):

$$p_n(x) = y_k + r\Delta y_k + \frac{r(r-1)}{2!}\Delta^2 y_k + \ldots + \frac{r(r-1)\ldots(r-n+1)}{n!}\Delta^n y_k$$

$$r = (x - x_k)/h$$

Newton Forward-Difference Formula (cont.)

Forward-difference Notation:

Level	t Notation	Definition
0	$\Delta^0 y_k$	y_k
1	$\Delta^1 y_k$	$y_{k+1} - y_k$
2	$\Delta^2 y_k$	$\Delta y_{k+1} - \Delta y_k$
…	…	…
j	$\Delta^j y_k$	$\Delta^{j-1} y_{k+1} - \Delta^{j-1} y_k$

Newton Forward-Difference Formula (cont.)

Forward-difference Table:

k	x_k	y_k	Δy_k	$\Delta^2 y_k$	...	$\Delta^{n-1} y_k$	$\Delta^n y_k$
0	x_0	y_0	Δy_0	$\Delta^2 y_0$	...	$\Delta^{n-1} y_0$	$\Delta^n y_0$
1	x_1	y_1	Δy_1	$\Delta^2 y_1$	...	$\Delta^{n-1} y_1$	
...	...	...	...	....	...		
$n-2$	x_{n-2}	y_{n-2}	Δy_{n-2}	$\Delta^2 y_{n-2}$			
$n-1$	x_{n-1}	y_{n-1}	Δy_{n-1}				
n	x_n	y_n					

Example

- Given the following data:

k	0	1	2	3	4	5
x_k	1.0	1.2	1.4	1.6	1.8	2.0
y_k	0.5000	0.4545	0.4167	0.3846	0.3571	0.3333

- Use the Newton forward-difference formula to approximate y(1.1).

Example - Solution

i) Complete the Forward-difference Table:

k	x_k	y_k	Δy_k	$\Delta^2 y_k$	$\Delta^3 y_k$	$\Delta^4 y_k$	$\Delta^5 y_k$
0	1.0	0.5000	-0.0455	0.0077	-0.0020	0.0009	-0.0007
1	1.2	0.4545	-0.0378	0.0057	-0.0011	0.0002	
2	1.4	0.4167	-0.0321	0.0046	-0.0009		
3	1.6	0.3846	-0.0275	0.0037			
4	1.8	0.3571	-0.0238				
5	2.0	0.3333					

Example – Solution (cont.)

ii) Choose the reference point:

- $x = 1.1$ is between 1.0 and 1.2,(choose one value for the reference point which has highest forward-difference degree).
- In this case, we choose $x_0 = 1.0$ as reference point because it has 5 forward different degree, $\Delta^5 y_k$)
- Thus, **$h = 1.2 - 1.0 = 0.2$** and

$$r = (x - x_0) / h = (1.1 - 1.0) / 0.2 = 0.5$$

Example – Solution (cont.)

iii) Complete the polynomial:

$$p_5(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0$$

$$+\frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \frac{r(r-1)(r-2)(r-3)(r-4)}{5!}\Delta^5 y_0$$

$$p_5(1.1) = 0.5000 + (0.5)(-0.0455) + \frac{(0.5)(0.5-1)}{2}(0.0077)$$

$$+\frac{(0.5)(0.5-1)(0.5-2)}{6}(-0.0020) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{24}(0.0009)$$

$$+\frac{(0.5)(0.5-1)(0.5-2)(0.5-3)(0.5-4)}{120}(-0.0007)$$

$$= 0.5000 - 0.02275 - 0.0009625 - 0.000125 - 0.0000352 - 0.0000191$$

$$= 0.4761$$

Remarks

- The Newton Forward Difference formula suitable to determine the approximation of the data point lies near the beginning/centre of the table.
- This will give more accuracy of the approximation value of the data point as it involved the highest-order difference.

Exercise 1

- Given

X	0.0	1.0	2.0	3.0	4.0
f(x)	0.00	0.75	2.25	3.00	2.25

- Use the Newton forward-difference formula to approximate f(1.5).

- Compare the approximate f(x) with the actual value based on the following function, f(x)

$$f(x) = 3\sin^2(\pi x / 6)$$

Newton Backward- Difference Formula

- Apply for the consistent data points.
- For any x, $x_0 \leq x \leq x_n$ can be written as:

$$x = x_0 + rh \text{ with } 0 \leq r \leq n$$

- In general:

$$x = x_1 + (r + n - 1)h$$

Newton Backward- Difference Formula (cont.)

- Proven:

$$x - x_n = (x_n + rh) - x_n = rh$$

$$x - x_{n-1} = (x_n + rh) - (x_n - h) = (r+1)h$$

$$x - x_{n-2} = (x_n + rh) - (x_n - 2h) = (r+2)h$$

$$\vdots$$

$$x - x_1 = (x_n + rh) - (x_n - (n-1)h) = (r+n-1)h$$

$$\therefore x = x_1 + (r+n-1)h$$

Newton Backward- Difference Formula (cont.)

Backward-difference Notation:

Level	Notation	Definition
0	$\nabla^0 y_k$	y_k
1	$\nabla^1 y_k$ atau ∇y_k	$y_k - y_{k-1}$
2	$\nabla^2 y_k$	$\nabla y_k - \nabla y_{k-1}$
3	$\nabla^3 y_k$	$\nabla^2 y_k - \nabla^2 y_{k-1}$
…	…	…
$j - 1$	$\nabla^{j-1} y_k$	$\nabla^{j-2} y_k - \nabla^{j-2} y_{k-1}$
j	$\nabla^j y_k$	$\nabla^{j-1} y_k - \nabla^{j-1} y_{k-1}$

Newton Backward-Difference Formula (cont.)

Backward-difference Table:

k	x_k	y_k	∇y_k	$\nabla^2 y_k$	…	$\nabla^{n-1} y_k$	$\nabla^n y_k$
0	x_0	y_0					
1	x_1	y_1	∇y_1				
…	…	…	…	…			
n – 2	x_{n-2}	y_{n-2}	∇y_{n-2}	$\nabla^2 y_{n-2}$	…		
n – 1	x_{n-1}	y_{n-1}	∇y_{n-1}	$\nabla^2 y_{n-1}$	…	$\nabla^{n-1} y_{n-1}$	
n	x_n	y_n	∇y_n	$\nabla^2 y_n$	…	$\nabla^{n-1} y_n$	$\nabla^n y_n$

Newton Backward-Difference Formula (cont.)

Polynomial (Newton Backward-difference formula):

$$p_n(x) = y_k + r\nabla y_k + \frac{r(r+1)}{2!}\nabla^2 y_k + \ldots + \frac{r(r+1)\ldots(r+n-1)}{n!}\nabla^n y_k$$

$r = (x - x_k) / h$

Example

- Given the data in the table, use the Newton backward-difference formula to approximate y(1.9).

k	0	1	2	3	4	5
x_k	1.0	1.2	1.4	1.6	1.8	2.0
y_k	0.5000	0.4545	0.4167	0.3846	0.3571	0.3333

Example - Solution

- Newton Backward Difference table

k	x_k	y_k	∇y_k	$\nabla^2 y_k$	$\nabla^3 y_k$	$\nabla^4 y_k$	$\nabla^5 y_k$
0	1.0	0.5000					
1	1.2	0.4545	-0.0455				
2	1.4	0.4167	-0.0378	0.0077			
3	1.6	0.3846	-0.0321	0.0057	-0.0020		
4	1.8	0.3571	-0.0275	0.0046	-0.0011	0.0009	
5	2.0	0.3333	-0.0238	0.0037	-0.0009	0.0002	-0.0007

Example – Solution (cont.)

- $x = 1.9$ lies between 1.8 dan 2.0,
- Choose $x_5 = 2.0$ as reference point since have highest order of backward-reference , $\nabla^5 y_k$)
- Thus, **$h = 1.2 – 1.0 = 0.2$** and

  $r = (x – x_5) / h = (1.9 – 2.0) / 0.2 = -0.5$

Example – Solution (cont.)

- Polynomial

$$p_5(x) = y_5 + r\nabla y_5 + \frac{r(r+1)}{2!}\nabla^2 y_5 + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_5$$

$$+\frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_5 + \frac{r(r+1)(r+2)(r+3)(r+4)}{5!}\nabla^5 y_5$$

$$p_5(1.9) = 0.3333 + (-0.5)(-0.0238) + \frac{(-0.5)(-0.5+1)}{2}(0.0037)$$

$$+\frac{(-0.5)(-0.5+1)(-0.5+2)}{6}(-0.0009) + \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{24}(0.0002)$$

$$+\frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)(-0.5+4)}{120}(-0.0007)$$

$$= 0.3333 + 0.0119 - 0.0004625 + 0.00005625 - 0.0000078 + 0.0000191$$

$$= 0.3448$$

Remarks

- The Newton Backward Difference formula suitable to determine the approximation of the data point lies near the end of the table.

Example

Based on the given data in the table, use the Newton backward-difference formula to approximate f(3.5).

x	0.0	1.0	2.0	3.0	4.0
f(x)	0.00	0.75	2.25	3.00	2.25

Compare the approximate f(x) with the actual value based on the following function, f(x)

$$f(x) = 3\sin^2(\pi x/6)$$

Example – Solution

- Newton Backward-Difference table

k	x	f(x)	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
0	0.0	0.00				
1	1.0	0.75	0.75			
2	2.0	2.25	1.50	0.75		
3	3.0	3.00	0.75	-0.75	-1.50	
4	4.0	2.25	-0.75	-1.50	-0.75	0.75

- Thus, $h = 1.0 - 0.0 = 1.0$ and

$r = (x - x_4) / h = (3.5 - 4.0) / 1.0 = -0.5$

Example – Solution (cont.)

- **Polynomial:**

$$p_4(x) = f(x_4) + r\nabla f(x_4) + \frac{r(r+1)}{2!}\nabla^2 f(x_4) + \frac{r(r+1)(r+2)}{3!}$$

$$\nabla^3 f(x_4) + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 f(x_4)$$

$$p_4(3.5) = 2.25 + (-0.5)(-0.75) + \frac{(-0.5)(-0.5+1)}{2}(-1.50)$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)}{6}(-0.75)$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{24}(0.75)$$

$$= 2.25 + 0.375 + 0.1875 + 0.046875 - 0.02929 = 2.83$$

Example – Solution (cont.)

- Comparison with the actual value:

$$f(x) = 3\sin^2(\pi x / 6)$$

$$f(3.5) = 3\sin^2[3.1412 * (3.5) / 6) = 2.7990$$

$$\Rightarrow p_4(3.5) = 2.83$$

$$\therefore \quad p(x) \approx f(x)$$

Exercise 2

Given

(0.0 , 0.0), (0.2 , 1.05), (0.4 , 0.85), (0.6 , 0.35),

(0.8 , 0.10), (1.0 , 1.0).

Use an appropriate interpolation technique to determine the approximation value of :

a) y(0.1)

b) y(0.9)

Newton Divided-Difference

- Suitable for non uniform data.
- The zero divided difference of the function f with respect to x_k

$$f[x_k] = f(x_k)$$

- First divided difference of f with respect to x_k and x_{k+1}:

$$f[x_k, x_{k+1}] = \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k} = \frac{f_{k+1} - f_k}{x_{k+1} - x_k}$$

Newton's Divided-Difference (cont.)

- Second divided difference *f* with respect to x_k , x_{k+1} *and* x_{k+2}:

$$f[x_k, x_{k+1}, x_{k+2}] = \frac{f[x_{k+1}, x_{k+2}] - f[x_k, x_{k+1}]}{x_{k+2} - x_k}$$

- *k*-th divided difference f with respect to x_k , x_{k+1} , $x_{k+2} \ldots\ldots x_{k+n}$:

$$f[x_k, x_{k+1}, \ldots, x_{k+n}] = \frac{f[x_{k+1}, x_{k+2}, \ldots., x_{k+n}] - f[x_k, x_{k+1}, \ldots, x_{k+n-1}]}{x_{k+n} - x_k}$$

Newton's Divided-Difference (cont.)

Newton Divided-Difference table

k	x_k	$f[x_k]$	$f[x_k, x_{k+1}]$	$f[x_k, x_{k+1}, x_{k+2}]$	…	$f[x_k, x_{k+1},\ldots, x_{k+n}]$
0	x_0	f_0	$f[x_0,x_1]=\dfrac{f[x_1]-f[x_0]}{x_1-x_0}$	$f[x_0,x_1,x_2]=\dfrac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$	…	$f[x_0,x_1,\ldots x_n]=\dfrac{f[x_1,x_2,\ldots x_n]-f[x_0,x_1\ldots x_{n-1}]}{x_n-x_0}$
1	x_1	f_1	$f[x_1,x_2]=\dfrac{f[x_2]-f[x_1]}{x_2-x_1}$	$f[x_1,x_2,x_3]=\dfrac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}$	…	
…	…	…	…	…		
n – 1	x_{n-1}	f_{n-1}	$f[x_{n-1},x_n]=\dfrac{f[x_n]-f[x_{n-1}]}{x_n-x_{n-1}}$			
n	x_n	f_n				

Newton's Divided-Difference (cont.)

- The Newton divided-difference interpolating polynomial determine based on the first point with the highest number of divided difference:

$$p_n(x) = \sum_{k=0}^{n} f[x_0, x_1, \ldots, x_k] \prod_{j=0}^{k-1} (x - x_j)$$

$$= f[x_0] + f[x_0, x_1](x - x_0)$$

$$+ f[x_0, x_1, x_2](x - x_0)(x - x_1) + \ldots$$

$$+ f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

Example

- Construct the Newton divided-difference table based on the following data

k	0	1	2	3	4
x_k	1.0	1.6	2.5	3.0	3.2
y_k	0.5000	0.3846	0.2857	0.2500	0.2381

- Determine the approximation value of y(1.3)

Example – Solution

i) Construct the Newton divided-difference table

k	x_k	$f[x_k]$	$f^1[x_k]$	$f^2[x_k]$	$f^3[x_k]$	$f^4[x_k]$
0	1.0	0.5000	-0.1923	0.0549	-0.0137	0.0032
1	1.6	0.3846	-0.1099	0.0275	-0.0066	
2	2.5	0.2857	-0.0714	0.0170		
3	3.0	0.2500	-0.0595			
4	3.2	0.2381				

Example – Solution (cont.)

- Calculation

$$f\left[x_0,x_1\right]=\frac{y_1-y_0}{x_1-x_0}=\frac{0.3846-0.5000}{1.6-1.0}=-0.1923$$

$$f\left[x_0,x_1,x_2\right]=\frac{f\left[x_1,x_2\right]-f\left[x_0,x_1\right]}{x_2-x_0}=\frac{-0.1099-(-0.1923)}{2.5-1.0}=0.0550$$

$$f[x_0,x_1,x_2,x_3]=\frac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0}=\frac{0.0275-0.0549}{3.0-1.0}=-0.0137$$

$$f[x_0,x_1,x_2,x_3,x_4]=\frac{f[x_1,x_2,x_3,x_4]-f[x_0,x_1,x_2,x_3]}{x_4-x_0}$$

$$=\frac{-0.0066+0.0137}{3.2-1.0}=0.0032$$

Example – Solution (cont.)

- Interpolation Polynomial expression:

$$
\begin{aligned}
p_4(x) = y_0 &+ f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\
&+ f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) \\
&+ f[x_0, x_1, x_2, x_3, x_4](x - x_0)(x - x_1)(x - x_2)(x - x_3)
\end{aligned}
$$

- Assign the value into the polynomial expression:

$$
\begin{aligned}
p_4(1.3) &= 0.5 + (-0.1923)(1.3 - 1.0) + 0.0549(1.3 - 1.0)(1.3 - 1.6) \\
&\quad + (-0.0137)(1.3 - 1.0)(1.3 - 1.6)(1.3 - 2.5) \\
&\quad + 0.0032(1.3 - 1.0)(1.3 - 1.6)(1.3 - 2.5)(1.3 - 3.0) \\
&= 0.5 - 0.05769 - 0.004941 - 0.0014796 - 0.00058752 \\
&= 0.4353
\end{aligned}
$$

Remarks

- Newton Divided different formula can be used to determine the approximation of the point lies near the beginning of the table where $x_0 < x < x_1$.
- The data need to be reordered, if the point is not located at the beginning of the table
  - Determine the range of data where the point is located and label the lower bound range as, x_0 and the upper bound as x_1
  - Ordered the data starting with the value that closest with the point as x_2 followed by the other values as $x_3, x_4 .... x_n$ where x_n have the largest gap with the point.

Example

- Construct the Newton divided-difference table based on the following data:

k	0	1	2	3	4
x_k	1.0	1.6	2.5	3.0	3.2
y_k	0.5000	0.3846	0.2857	0.2500	0.2381

Determine the approximation value of y(2.8)

Example – Solution

- Position of X= 2.8

k	0	1	2	3	4
x_k	1.0	1.6	2.5	3.0	3.2
y_k	0.5000	0.3846	0.2857	0.2500	0.2381

The point lies between 2.5 and 3.0

- Assign $x_0 = 2.5$ and $x_1 = 3.0$
- X_2 = ? X_3=? X_4=?

Example – Solution (cont.)

- Newton Divided-Difference value

k	x_k	$f[x_k]$	$f^1[x_k]$	$f^2[x_k]$	$f^3[x_k]$	$f^4[x_k]$
0	2.5	0.2857	-0.0714	0.0170	-0.0066	0.0033
1	3.0	0.2500	-0.0595	0.0229	-0.0115	
2	3.2	0.2381	-0.0916	0.0458		
3	1.6	0.3846	-0.1923			
4	1.0	0.5000				

Example – Solution (cont.)

- Polynomial expression

$$
\begin{aligned}
p_4(x) = y_0 &+ f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) \\
&+ f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2) \\
&+ f[x_0,x_1,x_2,x_3,x_4](x-x_0)(x-x_1)(x-x_2)(x-x_3)
\end{aligned}
$$

- Assign the value into the expression

$$
\begin{aligned}
p_4(2.8) = 0.2857 &+ (-0.0714)(2.8-2.5) + 0.0170(2.8-2.5)(2.8-3.0) \\
&+ (-0.0066)(2.8-2.5)(2.8-3.0)(2.8-3.2) \\
&+ 0.0033(2.8-2.5)(2.8-3.0)(2.8-3.2)(2.8-1.6) \\
= 0.2857 &- 0.02142 - 0.00102 - 0.0001584 + 0.00009504 \\
= 0.2632 &
\end{aligned}
$$

Exercise 3

- Construct the **Newton divided-difference** table based on the following data:

k	**0**	**1**	**2**	**3**	**4**
x_k	2.0	3.0	6.5	8.0	12.0
$f(x_k)$	14.0	20.0	17.0	16.0	23.0

Determine the approximation value of f(2.2) and f(7.0).

Lagrange Interpolating Polynomials

- Denoted as $L_k(x)$.
- Suitable to find the approximation value of point for non uniform data.
- Interpolation is relied on the number of data that been given.
- Polynomial:

$$p_n(x) = L_0(x)y_0 + L_1(x)y_1 + \ldots\ldots L_n(x)y_n = \sum_{i=0}^{n} L_i(x)y_i$$

Lagrange Interpolating Polynomials (cont.)

- Where

$$L_k(x) = \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{(x-x_j)}{(x_k-x_j)} = \frac{(x-x_0)\ldots(x-x_{k-1})(x-x_{k+1})(x-x_n)}{(x_k-x_0)\ldots(x_k-x_{k-1})(x_k-x_{k+1})(x_k-x_n)}$$

- Subject to

$$L_k(x_j) = \begin{cases} 1, & \text{jika}, k = j \\ 0, & \text{jika}, k \neq j \end{cases}$$

- It shows that ⇨ $\sum_{k=0}^{n} L_k(x) = 1$
    - It is true for any $x \in [x_0, x_n]$
    - And can be used to check the calculation

Example

- Given

k	0	1	2	3	4
x_k	1.0	1.6	2.5	3.0	3.2
y_k	0.5000	0.3846	0.2857	0.2500	0.2381

Determine the approximation value of y(1.3).

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Example – Solution

- Based on the data given,

$$p_4(x) = \sum_{i=0}^{4} L_i(x) y_i$$

$$p_4(x) = L_0(x)y_0 + L_1(x)y_1 + L_2(x)y_2 + L_3(x)y_3 + L_4(x)y_4$$

where $L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{4} \frac{(x - x_j)}{(x_i - x_j)}$

Therefore,

$$p_4(1.3) = \sum_{i=0}^{4} L_i(1.3) y_i \text{ where } L_i(1.3) = \prod_{\substack{j=0 \\ j\neq i}}^{4} \frac{(1.3 - x_j)}{(x_i - x_j)}$$

Example – Solution (cont.)

- Calculate $L_0(1.3)$

$$L_0(1.3) = \frac{(1.3 - x_1)(1.3 - x_2)(1.3 - x_3)(1.3 - x_4)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_0 - x_4)}$$

$$= \frac{(1.3-1.6)(1.3-2.5)(1.3-3.0)(1.3-3.2)}{(1.0-1.6)(1.0-2.5)(1.0-3.0)(1.0-3.2)} = 0.2936$$

- Calculate $L_1(1.3)$

$$L_1(1.3) = \frac{(1.3 - x_0)(1.3 - x_2)(1.3 - x_3)(1.3 - x_4)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)}$$

$$= \frac{(1.3-1.0)(1.3-2.5)(1.3-3.0)(1.3-3.2)}{(1.6-1.0)(1.6-2.5)(1.6-3.0)(1.6-3.2)} = 0.9613$$

Example – Solution (cont.)

- Calculate $L_2(1.3)$

$$L_2(1.3)=\frac{(1.3-x_0)(1.3-x_1)(1.3-x_3)(1.3-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}$$

$$=\frac{(1.3-1.0)(1.3-1.6)(1.3-3.0)(1.3-3.2)}{(2.5-1.0)(2.5-1.6)(2.5-3.0)(2.5-3.2)}=-0.6152$$

- Calculate $L_3(1.3)$

$$L_3(1.3)=\frac{(1.3-x_0)(1.3-x_1)(1.3-x_2)(1.3-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}$$

$$=\frac{(1.3-1.0)(1.3-1.6)(1.3-2.5)(1.3-3.2)}{(3.0-1.0)(3.0-1.6)(3.0-2.5)(3.0-3.2)}=0.7329$$

Example – Solution (cont.)

- Calculate $L_4(1.3)$

$$L_4(1.3) = \frac{(1.3-x_0)(1.3-x_1)(1.3-x_2)(1.3-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}$$
$$= \frac{(1.3-1.0)(1.3-1.6)(1.3-2.5)(1.3-3.0)}{(3.2-1.0)(3.2-1.6)(3.2-2.5)(3.2-3.0)} = -0.3726$$

- Check:

$$\sum_{i=0}^{4} L_i(1.3) = L_0(1.3) + L_1(1.3) + L_2(1.3) + L_3(1.3) + L_4(1.3)$$
$$= 0.2936 + 0.9613 - 0.6152 + 0.7329 - 0.3726 = 1$$

Example – Solution (cont.)

- Thus,

$$p_4(1.3) = \sum_{i=0}^{4} L_i(1.3) y_i$$

$$= 0.2936(0.5000) + 0.9613(0.3846) - 0.6152(0.2857)$$

$$+ 0.7329(0.2500) - 0.3726(0.2381)$$

$$= 0.4353$$

Example

- Given

k	0	1	2	3	4
x_k	2.0	3.0	6.5	8.0	12.0
$f(x_k)$	14.0	20.0	17.0	16.0	23.0

Determine the approximation value of $f(2.2)$ based on Lagrange interpolating polynomials method.

Example – Solution

- Location of x = 2.2:

k	0	1	2	3	4
x_k	2.0	3.0	6.5	8.0	12.0
$f(x_k)$	14.0	20.0	17.0	16.0	23.0

Lies between 2.0 dan 3.0

Example – Solution (cont.)

- Polynomial:

$$p_4(2.2) = \sum_{i=0}^{4} L_i(2.2) y_i \text{ dengan } L_i(2.2) = \prod_{\substack{j=0 \\ j \neq i}}^{4} \frac{(2.2 - x_j)}{(x_i - x_j)}$$

- Calculate:

$$L_0(2.2) = \frac{(2.2 - x_1)(2.2 - x_2)(2.2 - x_3)(2.2 - x_4)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_0 - x_4)}$$

$$= \frac{(2.2 - 3.0)(2.2 - 6.5)(2.2 - 8.0)(2.2 - 12.0)}{(2.0 - 3.0)(2.0 - 6.5)(2.0 - 8.0)(2.0 - 12.0)} = 0.7242$$

Example – Solution (cont.)

- Calculate

$$L_1(2.2) = \frac{(2.2 - x_0)(2.2 - x_2)(2.2 - x_3)(2.2 - x_4)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)}$$

$$= \frac{(2.2 - 2.0)(2.2 - 6.5)(2.2 - 8.0)(2.2 - 12.0)}{(3.0 - 2.0)(3.0 - 6.5)(3.0 - 8.0)(3.0 - 12.0)} = 0.3104$$

- Calculate

$$L_2(2.2) = \frac{(2.2 - x_0)(2.2 - x_1)(2.2 - x_3)(2.2 - x_4)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)}$$

$$= \frac{(2.2 - 2.0)(2.2 - 3.0)(2.2 - 8.0)(2.2 - 12.0)}{(6.5 - 2.0)(6.5 - 3.0)(6.5 - 8.0)(6.5 - 12.0)} = -0.069991$$

Example – Solution (cont.)

- Calculate

$$L_3(2.2) = \frac{(2.2 - x_0)(2.2 - x_1)(2.2 - x_2)(2.2 - x_4)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)(x_2 - x_4)}$$

$$= \frac{(2.2 - 2.0)(2.2 - 3.0)(2.2 - 6.5)(2.2 - 12.0)}{(8.0 - 2.0)(8.0 - 3.0)(8.0 - 6.5)(8.0 - 12.0)} = 0.037458$$

- Calculate

$$L_4(2.2) = \frac{(2.2 - x_0)(2.2 - x_1)(2.2 - x_2)(2.2 - x_3)}{(x_4 - x_0)(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)}$$

$$= \frac{(2.2 - 2.0)(2.2 - 3.0)(2.2 - 6.5)(2.2 - 8.0)}{(12.0 - 2.0)(12.0 - 3.0)(12.0 - 6.5)(12.0 - 8.0)} = -0.002015$$

Example – Solution (cont.)

- Check

$$\sum_{i=0}^{4} L_i(2.2) = L_0(2.2) + L_1(2.2) + L_2(2.2) + L_3(2.2) + L_4(2.2)$$
$$= 0.7242 + 0.3104 - 0.069991 + 0.037458 - 0.002015$$
$$= 1.000052 \approx 1.0$$

- Thus: $p_4(2.2) = \sum_{i=0}^{4} L_i(2.2) y_i$

$$= 0.7242(14) + 0.3104(20) - 0.069991(17)$$
$$+ 0.037458(16) - 0.002015(23)$$
$$= 15.709936$$

Exercise 4

- Given

k	0	1	2	3	4
x_k	2.0	3.0	6.5	8.0	12.0
$f(x_k)$	14.0	20.0	17.0	16.0	23.0

Determine the approximation value of f(7.0) based on Lagrange interpolating polynomials method.

Least Square Approximation

- Determine the coefficient values of polynomials, (a_0, a_1, …, a_m) to minimize the sum of error square difference between $p(x_k)$ and f_k at any points based on

$$S = \sum_{k=0}^{n} \varepsilon_k^2 = \sum_{k=0}^{n} [p(x_k) - f_k]^2$$

  where ε_k is the error at point x_k which is the difference between the approximation value based on the polynomial , $p(x_k)$ and the actual value at point x_k.
- S is the summation of error square at all points $k=0,1,\ldots n$.

Least Square Approximation (cont.)

- Thus, we require that

$$S = \sum_{k=0}^{n} \left[a_0 + a_1 x + \ldots + a_m x^m - f_k \right]^2$$, be minimum.

- The minimum value of S can be obtained when the partial differential is equal 0 for j = 0, 1, …, m

$$\frac{\partial S}{\partial a_j} = 2 \sum_{k=0}^{n} \left[a_0 + a_1 x + \ldots + a_m x^m - f_k \right] x_k^j = 0$$

Least Square Approximation (cont.)

- It gives

$$a_0 \sum_{k=0}^{n} x_k^j + a_1 \sum_{k=0}^{n} x_k^{j+1} + \ldots + a_m \sum_{k=0}^{n} x_k^{j+m} = \sum_{k=0}^{n} x_k^j f_k$$

- In linear system equation form :

$$\begin{aligned}
&a_0 s_0 + a_1 s_1 + \ldots + a_m s_m = v_0 \\
&a_0 s_1 + a_1 s_2 + \ldots + a_m s_{m+1} = v_1 \\
&\ldots \\
&\ldots \\
&a_0 s_m + a_1 s_{m+1} + \ldots + a_m s_{2m} = v_m
\end{aligned}$$

Least Square Approximation (cont.)

- where

$$s_j = \sum_{k=0}^{n} x_k^j, j = 0, 1, 2, \ldots, m$$

$$v_l = \sum_{k=0}^{n} x_k^l f_k, l = 0, 1, 2, \ldots, m$$

- In matrix form:

$$\begin{bmatrix} s_0 & s_1 & \cdots & s_m \\ s_1 & s_2 & \cdots & s_{m+1} \\ \cdots & \cdots & \cdots & \cdots \\ s_m & s_{m+1} & \cdots & s_{2m} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \cdots \\ a_m \end{bmatrix} = \begin{bmatrix} v_0 \\ v_1 \\ \cdots \\ v_m \end{bmatrix}$$

Least Square Approximation (cont.)

- The coefficient values of a_0, a_1, …, a_m can be determine using the system of linear equations method that has been discussed in Chapter 1.

Example

- Determine the appropriate linear polynomial expression, $p(x) = a_0 + a_1x$ based on the following data:

k	**0**	**1**	**2**	**3**	**4**
$\mathbf{x_k}$	1	3	4	5	8
$\mathbf{f_k}$	5	9	11	13	19

- Then, determine f(4.5).

Example – Solution

- System of linear equations

$$\begin{bmatrix} s_0 & s_1 \\ s_1 & s_2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}$$

where

$$s_j = \sum_{k=0}^{4} x_k^j \ , \ \ j = 0,1,2$$

and $v_l = \sum_{k=0}^{4} x_k^l f_k \ , \ l = 0,1$

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Example – Solution (cont.)

- Calculation table:

	x_k^0	X_k^1	x_k^2	f_k	$x_k^0 f_k$	$x_k^1 f_k$
	1	1	1	5	5	5
	1	3	9	9	9	27
	1	4	16	11	11	44
	1	5	25	13	13	65
	1	8	64	19	19	152
$\sum_{k=0}^{4}$	**5**	**21**	**115**	**-**	**57**	**293**

Example – Solution (cont.)

- In matrix form : $$\begin{bmatrix} 5 & 21 \\ 21 & 115 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 57 \\ 293 \end{bmatrix}$$
- Solution, $a_0 = 3.0$ and $a_1 = 2.0$
- Therefore, the polynomial expression is $p(x) = 2.0x + 3.0$
- To determine $f(4.5)$:

$$p(4.5) = 2.0(4.5) + 3.0 = 12$$
$$f(4.5) \approx p(4.5) = 12$$

Exercise 5

- Determine the appropriate linear polynomial expression, $p(x) = a_0 + a_1x$ based on the following data:

x	1	2	3	4	5
f(x)	0.50	1.40	2.00	2.50	3.10

- Determine f(2.3)