UTM UNIVERSITI TEKNOLOGI MALAYSIA

CHAPTER 4

Linear Transformations

Part 1:
Introduction, Null Space and Range

Mapping

Define that mapping of $\boldsymbol{T}$ from V (vector space) to W (vector space) as;

$$\boldsymbol{T} : V \rightarrow W$$

This is a standard terminology, for such function;

V is called **domain** of $\boldsymbol{T}$, and

W is called **codomain** of $\boldsymbol{T}$.

if **v** is in V and **w** is in W such that;

$$\boldsymbol{T}(\mathbf{v}) = \mathbf{w}$$

Then **w** in called the **image** of **v** under **T.** The set of all images of vectors in V is called the **range** of **T**, and the set of all **v** in V such that $\boldsymbol{T}(\mathbf{v}) = \mathbf{w}$ is called the **preimage** of w.

Mapping

Example #1:

Define a mapping $T: R^2 \to R^2$ by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x - y \end{bmatrix}$$

Find the image of the coordinate vectors $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$ under the mapping T.

Mapping

Example #1 - Solution:

Since $\boldsymbol{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we have

$$T(\boldsymbol{e_1}) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1+0 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix};$$

$$T(\boldsymbol{e_2}) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0+1 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Mapping

Example #2:

For any vector $\mathbf{v} = (\mathrm{v}_1, \mathrm{v}_2)$ in R^2, let $T: R^2 \rightarrow R^2$ defined by

$$T(\mathrm{v}_1, \mathrm{v}_2) = (\mathrm{v}_1 - \mathrm{v}_2, \mathrm{v}_1 + 2\mathrm{v}_2).$$

a) Find the image of **v** = (-1,2).

b) Find the preimage of **w** =(-1,11).

Mapping

Example #2 - Solution:

a) Find the image of **v** = (-1,2).

$$T(-1,2) = \left(-1 - 2, -1 + 2(2)\right) = (-3,3)$$

b) Find the preimage of **w** =(-1,11).

$.: T(\mathrm{v}_1, \mathrm{v}_2) = (\mathrm{v}_1 - \mathrm{v}_2, \mathrm{v}_1 + 2\mathrm{v}_2)$= (-1, 11)

$\mathrm{v}_1 - \mathrm{v}_2$ = -1; $\mathrm{v}_1 + 2\mathrm{v}_2$ = 11

=> solve the SLE, give: v_1 = 3; v_2 = 4

Linear Transformation

- Let **V** and **W** be vector spaces.
- The function of $\boldsymbol{T}: \mathbf{V} \rightarrow$ **W** is called a **Linear Transformation** of **V** into **W** if the following two properties are true for all $\boldsymbol{u}, \boldsymbol{v} \in$ **V** and for any scalar *c*.

(i) $T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$

[$T(\boldsymbol{u} + \boldsymbol{v})$: addition in **V**]

[$T(\boldsymbol{u}) + T(\boldsymbol{v})$:addition in **W**]

(ii) $T(c\boldsymbol{u}) = cT(\boldsymbol{u})$

[$T(c\boldsymbol{u})$: scalar multiplication in **V**]

[$cT(\boldsymbol{u})$: scalar multiplication in **W**]

(iii) $T(c\boldsymbol{u} + \boldsymbol{v}) = cT(\boldsymbol{u}) + T(\boldsymbol{v})$

In the case which **V** = **W**, then *T* is also called a **Linear Operator.**

Linear Transformation

Example:

Show that

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2),$$

is a Linear Transformation from R^2 into R^2

Linear Transformation

Example:

Let $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{u} = (u_1, u_2)$ be vectors in R^2 and let $\boldsymbol{c}$ be any real number. Then construct the following;

1. Because $\boldsymbol{u} + \boldsymbol{v} = (u_1, u_2) + (v_1, v_2)$
$= (u_1 + v_1, u_2 + v_2)$, we have

$$\begin{aligned}
T(\boldsymbol{u} + \boldsymbol{v}) &= T(u_1 + v_1, u_2 + v_2) \\
&= ((u_1 + v_1) - (u_2 + v_2), (u_1 + v_1) + 2(u_2 + v_2)) \\
&= \big((u_1 - u_2) + (v_1 - v_2), (u_1 + 2u_2) + (v_1 + 2v_2)\big) \\
&= (u_1 - u_2, u_1 + 2u_2) + (v_1 - v_2, v_1 + 2v_2) \\
&= T(\boldsymbol{u}) + T(\boldsymbol{v})
\end{aligned}$$

Linear Transformation

Example (cont'd):

2. Because $c\boldsymbol{u} = c(u_1, u_2) = (cu_1, cu_2)$, we have

$$\begin{aligned} T(c\boldsymbol{u}) &= T(cu_1\ , cu_2) \\ &= (cu_1 - cu_2\ , cu_1 + 2cu_2) \\ &= c(u_1 - u_2\ ,\ u_1 + 2u_2) \\ &= cT(\boldsymbol{u}). \end{aligned}$$

So, T is a linear transformation.

Linear Transformation

Example:

Let A be an $m \times n$ matrix. Define a mapping $T: R^n \rightarrow R^m$ by

$$T(\mathbf{x}) = A\mathbf{x}$$

a) Show that T is a linear transformation.

b) Let A be the 2×3 matrix,

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}$$

Find the image of

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix}$$

under the mapping $T: R^3 \rightarrow R^2$ $with$ $T(\mathbf{x}) = A\mathbf{x}$

Linear Transformation

Example - Solution:

a) Show that T is a linear transformation.

$$A(c\mathbf{u} + \mathbf{v}) = cA\mathbf{u} + A\mathbf{v}$$

Therefore, $T(c\mathbf{u} + \mathbf{v}) = cT(\mathbf{u}) + T(\mathbf{v})$

b) Since T is defined by matrix multiplication, we have

$$\mathrm{T}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}1 & 2 & -1\\-1 & 3 & 2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix};$$ and

$$\mathrm{T}\left(\begin{bmatrix}7\\-1\\5\end{bmatrix}\right) = \begin{bmatrix}1 & 2 & -1\\-1 & 3 & 2\end{bmatrix}\begin{bmatrix}7\\-1\\5\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Linear Transformation

Most of the common functions studied in calculus are not linear transformation.

a) $f(x) = \sin x$ is not a linear transformation from R into R because in general:

$$\sin(x_1 + x_2) \neq \sin(x_1) + \sin(x_2)$$

b) $f(x) = x^2$ not a linear transformation from R into R because in general:

$$(x_1 + x_2)^2 \neq (x_1^2 + x_2^2)$$

c) $f(x) = x + 1$ is not linear transformation from R into R because

$$f(x_1 + x_2) = x_1 + x_2 + 1$$

whereas $f(x_1) + f(x_2) = x_1 + x_2 + 2$,

which is $f(x_1 + x_2) \neq f(x_1) + f(x_2)$.

Linear Transformation

Two simple linear transformations are **the zero transformations** and **the identity transformation**, which are defined as follows:

1. $T(\mathbf{v}) = \mathbf{0}$ for all **v** (Zero transformation ($T: V \rightarrow W$))

   (In general, zero transformation is the transformation T: $R^n \rightarrow R^m$ that maps every vector **v** in R^n to zero vector in R^m)

2. $T(\mathbf{v}) = \mathbf{v}$ for all **v** (Identity transformation ($T: V \rightarrow V$))

   (In general, identity transformation is the transformation T: $R^n \rightarrow R^n$ that maps every vector **v** into itself)

Properties of Linear Transformation

Let T be a linear transformation from V into W, where $\mathbf{u}$ and $\mathbf{v}$ are in V. Then the following proposition/properties are true:

1. $T(\mathbf{0}) = \mathbf{0}$

2. $T(-\mathbf{v}) = -T(\mathbf{v})$

3. $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$

4. If $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$, then

$$T(\mathbf{v}) = T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n)$$
$$= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \ldots + c_nT(\mathbf{v}_n).$$

Properties of Linear Transformation

Example:

Define a mapping $T: R^2 \to R^2$ by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \left(\begin{bmatrix} e^x \\ e^y \end{bmatrix}\right)$$

Determine whether T is a linear transformation.

Properties of Linear Transformation

Example - Solution:

Since

$$\mathrm{T}(\mathbf{0}) = T\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = \left(\begin{bmatrix}e^0\\e^0\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix}$$

By the contrapositive of Proposition 1, we know that T is not a linear transformation.

Properties of Linear Transformation

Example:

Define a mapping $T: M_{m\times n} \rightarrow M_{n\times m}$ by

$$T(A) = A'$$

Show that the mapping is a linear transformation.

Properties of Linear Transformation

Example - Solution:

$$T(A+B) = (A+B)' = A' + B' = T(A) + T(B)$$

Also by this same theorem,

$$T(cA) = (cA)' = cA' = cT(A)$$

Thus, T is a linear transformation.

Properties of Linear Transformation

Example:

Let $T: R^3 \rightarrow R^2$ be a linear transformation and let B be the standard basis for R^3. If

$$T(e_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; T(e_2) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \text{ and } T(e_3) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Find $T(\mathbf{v})$, where $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

Properties of Linear Transformation

Example - Solution:

To find the image of vector **v**, we need to write the vector as a linear combination of the basis vector.

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

$$\begin{bmatrix}1\\3\\2\end{bmatrix} = c_1\begin{bmatrix}1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\\0\end{bmatrix}+c_3\begin{bmatrix}0\\0\\1\end{bmatrix} \Rightarrow c_1{=}1;\, c_2 = 3;\, c_3 = 2$$

Applying T to this linear combination and using the linearity properties of T, we have

$$T(\mathbf{v}) = T(e_1 + 3e_2 + 2e_3) = T(e_1) + 3T(e_2) + 2T(e_3)$$

$$= \begin{bmatrix}1\\1\end{bmatrix} + 3\begin{bmatrix}-1\\2\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-2\\9\end{bmatrix}$$

Exercises #1

Given the following vectors **v** and **w**, find:

a) image of **v**

b) preimage of **w**

1) $T(v_1, v_2) = (v_1 + v_2, v_1 - v_2),$
$\mathbf{v} = (3,4); \mathbf{w} = (3,19)$

2) $T(v_1, v_2, v_3) = (v_1 - v_2, v_1 + v_2, 2v_2)$
$\mathbf{v} = (2,3,0); \mathbf{w} = (-11,-1,10)$

The Null Space (Kernel) and Range

Definition

- If $T: V \rightarrow W$ is a linear transformation, then the set of vectors in V that T maps into **0**, is called **kernel** (or **null space**) of T denoted by $\ker(T)$, or

  $\ker(T) = \{\mathbf{v} \in V | T(\mathbf{v}) = \mathbf{0}\}$ or $N(T) = \{\mathbf{v} \in V | T(\mathbf{v}) = \mathbf{0}\}$

- The set of all vectors in W that are images under T of at least one vector in V is call the **range** of T; denoted by $R(T)$, or

$$R(T) = \{T(\mathbf{v}) \mid \mathbf{v} \in V\}$$

The Null Space (Kernel) and Range

The null space of a linear transformation is then the set of all vectors in V that are mapped to the zero vector, with the range being the set of all images of the mapping, as shown in Fig. 1.

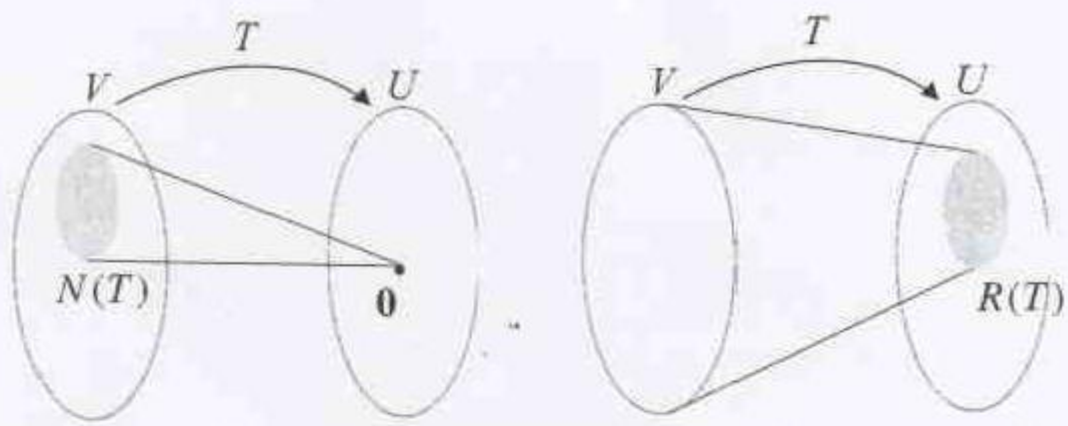

Figure 1

The Null Space (Kernel) and Range

Example:

Find the kernel of linear transformation $T: R^2 \rightarrow R^3$ represented by $T(x_1, x_2) = (x_1 - 2x_2, 0, -x_1)$

Solution:

To find $\ker(T)$ we need to find all $x = (x_1, x_2)$ in R^2 such that,

$T(x_1, x_2) = (x_1 - 2x_2, 0, -x_1) = (0,0,0)$

this lead to the homogenous system

$$x_1 - 2x_2 = 0; \quad 0 = 0 \quad and \quad -x_1 = 0$$

which lead the trivial solution $(x_1, x_2) = (0,0)$, so we have

$$\ker(T) = \{(0,0)\} = \{\mathbf{0}\}.$$

The Null Space (Kernel) and Range

Example:

Define the linear transformation $T: R^4 \to R^3$ by

$$T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a+b \\ b-c \\ a+d \end{bmatrix}$$

Find a basis for the null space of T and its dimension.

The Null Space (Kernel) and Range

Example - Solution:

The null space of T is found by setting each component of the image vector equal to 0. This yields the linear system:

$$\begin{cases} a+b=0 \\ b-c=0 \\ a+d=0 \end{cases}$$

This linear system has infinitely many solutions, given by

$$S=\left\{\left.\begin{bmatrix}-t\\t\\t\\t\end{bmatrix}\right| t\in R\right\}, \text{ hence } N(T)=span\left\{\begin{bmatrix}-1\\1\\1\\1\end{bmatrix}\right\}.$$

A basis for $N(T)$ consists of the one vector, $\begin{bmatrix}-1\\1\\1\\1\end{bmatrix}$. Consequently, dim $(N(T)) = 1$

Theorem

Let V and W be finite dimensional vector spaces and

$B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis of V.

If $T: V \rightarrow W$ is a linear transformation, then

$$R(T) = span\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)\}$$

The Null Space (Kernel) and Range

Example:

Let $T: R^3 \rightarrow R^3$ be a linear operator and $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a basis for R^3. Suppose that

$$T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; T(\mathbf{v}_2) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}; T(\mathbf{v}_3) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

a) Is $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ in $R(T)$?

b) Find a basis for $R(T)$.

The Null Space (Kernel) and Range

Example - Solution:

a) Based on the theorem, the vector $\begin{bmatrix}1\\2\\1\end{bmatrix}$ is in $R(T)$ if there are scalars

c_1, c_2 and c_3 such that $c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3) = \begin{bmatrix}1\\2\\1\end{bmatrix}$. That is,

$$c_1\begin{bmatrix}1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}1\\0\\-1\end{bmatrix} + c_3\begin{bmatrix}2\\1\\-1\end{bmatrix} = \begin{bmatrix}1\\2\\1\end{bmatrix}$$

The set of solutions to the linear system is given by $S = \{(2-t, -1-t, t | t \in R\}$.
If $t = 0, S = \{c_1$=2; $c_2 = -1; c_3 = 0\}$.

Thus, vector $\begin{bmatrix}1\\2\\1\end{bmatrix}$ is in $R(T)$.

The Null Space (Kernel) and Range

Example - Solution:

b. To find a basis for $R(T)$, we row-reduce the matrix

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} \quad \text{to obtain} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the leading 1s are in columns 1 and 2, a basis for $R(T)$ is given by

$$R(T) = \textbf{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

#basis for column vectors

Observe that since the range is spanned by two linearly independent vectors, $R(T)$ is a plane in $\mathbb{R}^3$

Exercises # 2

i) The function $T : R^3 \rightarrow R^2$ is defined as

$$T(v) = A(v) = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

a) Find the image of $v = \begin{bmatrix} -5 \\ 7 \\ -3 \end{bmatrix}$

b) Find the kernel of the linear transformation, ker(T)

ii) Let $T : R^3 \rightarrow R^3$ be a linear transformation such that

$$T(1,0,0) = (-2,-1,4)$$
$$T(0,1,0) = (1,-5,3)$$
$$T(0,0,1) = (6,0,-5)$$

Find $T(7, -1, 4)$