# CHAPTER 3

## General Vector Spaces

### Part 1: Definition

# Definition

A set V is called a **vector space** over the real numbers provided that there are **two operations**: **addition** and **scalar multiplication,** that satisfy the following axioms. The axioms must hold for all vectors **u**, **v** and **w** in V and all scalar *c* and *d* in R.

# Axiom

1) If **u** and **v** are objects in V, then **u** + **v** is in V
   [Closed under addition]
2) **u + v = v + u** [Addition is commutative]
3) **u + (v + w) = (u + v) + w** [Addition is associative]
4) There is an object **0** in V, called a ***zero vector*** for V, such that **0** + **u** = **u** + **0** = **u** for all **u** in V. [Additive identity]
5) For each **u** in V, there is an object –**u** in V, called a ***negative*** of **u**, such that **u** + (-**u**) = **(-u)** + **u** = **0**
   [Additive inverse]

# Axiom

Cont'd.

6) If $c$ is any scalar and $\mathbf{u}$ is any object in V, then $c\mathbf{u}$ is in V.

   [Closed under scalar multiplication]

7) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

8) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

9) $c(d\mathbf{u}) = (cd)\mathbf{u}$

10) $1\mathbf{u} = \mathbf{u}$

# Specify a Vector Space

Steps to specify a vector space V:

Step (1): Identify the set V of objects that will become vectors.

Step (2): Identify the addition and scalar multiplication operations on V.

Step (3): Verify Axioms 1 and 6; that is, adding two vectors in V produces a vector in V, and multiplying a vector in V by a scalar also produces a vector in V. **Axiom 1** is called ***closure under addition***, and **Axiom 6** is called ***closure under scalar multiplication***.

Step (4): Confirm that Axioms 2, 3, 4, 5, 7, 8, 9 and 10 hold.

# Axiom

## Example

1) The set V = $R^n$ with the standard operations of addition and scalar multiplication defined is a vector space.

2) The set V = $M_{m\times n}$ of all $m \times n$ matrices with real entries, together with the operations of matrix addition and scalar multiplication that been defined component wise is a vector space.

# Axiom

**Example**

Let V = R. Define addition and scalar multiplication by

$\mathbf{a} + \mathbf{b} = 2a + 2b$ ; $k\mathbf{a} = ka$

Show that addition is commutative but not associative.

# Axiom

## Example - Solution

Since the usual addition of real numbers (on the *rhs*) is commutative,

$$\mathbf{a} + \mathbf{b} = 2a + 2b$$

$$= 2b + 2a$$

$$= \mathbf{b} + \mathbf{a}$$

Thus, the operation is commutative.

# Axiom

## Example – Solution (cont'd)

To determine whether addition is associative, we evaluate and compare the expressions

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} \text{ and } \mathbf{a} + (\mathbf{b} + \mathbf{c})$$

In this case, we have

$$\begin{aligned}(\mathbf{a} + \mathbf{b}) + \mathbf{c} &= (2\mathbf{a} + 2\mathbf{b}) + \mathbf{c} \\ &= 2(2\mathbf{a} + 2\mathbf{b}) + 2\mathbf{c} \\ &= 4\mathbf{a} + 4\mathbf{b} + 2\mathbf{c}\end{aligned}$$

$$\begin{aligned}\mathbf{a} + (\mathbf{b} + \mathbf{c}) &= 2a+(2b+2c) \\ &= 2a + 2(2b+2c) \\ &= 2a+ 4b+4c\end{aligned}$$

Since, $(\mathbf{a} + \mathbf{b}) + \mathbf{c} \neq \mathbf{a} + (\mathbf{b} + \mathbf{c})$ => Addition is **not associative**.

# Axiom

## Example

Let, $V = \{ (a, b) \mid a, b \in R \}$. Let $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$. Define

$$(v_1, v_2) + (w_1, w_2) = (v_1 + w_1 + 1, v_2 + w_2 + 1)$$

Verify that $V$ satisfy axiom 1, 2 and 3.

# Axiom

## Example - Solution

- **Axiom 1**

  For any $v_1, v_2, w_1, w_2 \in R$, the result of sum
  $(v_1, v_2) + (w_1, w_2) = (v_1 + w_1 + 1, v_2 + w_2 + 1)$
  is in $V$.

- **Axiom 2**

  The components of the vectors are real numbers, and the addition of real numbers is commutative.

# Axiom

## Example - Solution

- **Axiom 3** $[\ (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})\ ]$

  The components of the vectors are real numbers, and the addition of real numbers is associative.

$$(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w} = [(u_1,u_2)+(v_1,v_2)]+(w_1,w_2)$$

$$= [(u_1+v_1+1)+w_1+1, (u_2+v_2+1)+w_2+1]$$

$$= [(u_1+1)+(v_1+w_1+1), (u_2+1)+(v_2+w_2+1)]$$

$$= [(u_1,u_2)+((v_1,v_2)+(w_1,w_2))]$$

$$= \boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$$

# Axiom

## Example

Let $V = R^3$ with the operations

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} \quad \text{and} \quad k\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} kx_1 \\ y_1 \\ z_1 \end{bmatrix}$$

Show that $V$ with the given operations is not a vector space.

# Axiom

## Example - Solution

- **Axiom 1:**

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$$

is in $V$, (closure of addition)

- **Axiom 2:**

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} = \begin{bmatrix} x_2 + x_1 \\ y_2 + y_1 \\ z_2 + z_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

(Addition is commutative)

# Axiom

## Example – Solution (cont'd)

- **Axiom 3:**

$$\left(\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}+\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}\right)+\begin{bmatrix}x_3\\y_3\\z_3\end{bmatrix}=\begin{bmatrix}(x_1+x_2)+x_3\\(y_1+y_2)+y_3\\(z_1+z_2)+z_3\end{bmatrix}$$

$$=\begin{bmatrix}x_1+(x_2+x_3)\\y_1+(y_2+y_3)\\z_1+(z_2+z_3)\end{bmatrix}=\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}+\left(\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}+\begin{bmatrix}x_3\\y_3\\z_3\end{bmatrix}\right)$$

(Addition is associative)

# Axiom

## Example – Solution (cont'd)

- Axiom 4

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

(Additive identity)

- Axiom 5

$$\begin{bmatrix} -x_1 \\ -y_1 \\ -z_1 \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(Additive inverse)

# Axiom

## Example – Solution (cont'd)

- Axiom 6:

$$k\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} kx_1 \\ y_1 \\ z_1 \end{bmatrix}$$

is in $V$
(Closure under scalar multiplication)

- Axiom 7:

$$k\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = \begin{bmatrix} k(x_1 + x_2) \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} = \begin{bmatrix} (kx_1 + kx_2) \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} = k\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + k\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right)$$

# Axiom

## Example – Solution (cont'd)

- Axiom 8:

$$(k+m)\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) = \begin{bmatrix} (k+m)x_1 \\ y_1 \\ z_1 \end{bmatrix} \neq k\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + m\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) = \begin{bmatrix} (k+m)x_1 \\ 2y_1 \\ 2z_1 \end{bmatrix}$$

Thus, $V$ is not a vector space

# Axiom

## Example

Let $V = R$. The addition and scalar multiplication is defined as follows:

$$a + b = ab \,; \quad ka = ka$$

Show that $V$ is not a vector space.

# Axiom

## Example - Solution

$$a + b = a^b \neq b + a = b^a$$

.: Axiom 2 is not satisfied.

# Exercise #1

Let $V = R^2$ and define addition as the standard component wise addition and define scalar multiplication by

$$c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+c \\ y \end{bmatrix}$$

Determine whether $V$ is a vector space.

# Vector Subspace

- A **subspace** of $W$ of a vector space $V$ is nonempty subset that itself a vector space with respect to the inherited operations of vector addition and scalar multiplication on $V$.
  - **Closed under addition**

    If **u** and **v** are vectors in $W$ , then **u** + **v** is in $W$

  - **Closed scalar multiplication**

    If $k$ is any scalar and **u** is any vector in $W$, then $k\boldsymbol{u}$ is in $W$

# Vector Subspace

## Example

- Let $V$ be the vector space $R^2$ with the standard definitions of addition and scalar multiplication.

- Let $W$ $R^2$ be the subset defined by

$$W = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} \middle| a \in R \right\}$$

# Vector Subspace

## Example

- The sum of two vectors is in $W$

$$\begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} a+b \\ 0 \end{bmatrix}$$

(The subset $W$ is closed under addition)

- Let $c$ be any real number,

$$c\begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} ca \\ 0 \end{bmatrix} \quad \text{is in } W$$

(The subset $W$ is closed under scalar multiplication)

Thus, $W$ is a subspace of $V$

# Vector Subspace

## Example

Let, $W = \left\{ \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \middle| a, b \in R \right\}$

be a subset of $V$, the vector space $R^3$ for all vectors,

$$X = (a_1, b_1, 1); Y = (a_2, b_2, 1)$$

Determine whether $W$ is a subspace of $V$.

# Vector Subspace

## Example - Solution

The sum of two vectors,

$$X + Y = \begin{bmatrix} a_1 \\ b_1 \\ 1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ 2 \end{bmatrix}$$ is not in $W$.

$\therefore$ The subset $W$ is not closed under addition. Thus $W$ is not a subspace of $V$.

# Vector Subspace

## Example

Let $M_{2\times 2}$ be the vector space of $2 \times 2$ matrices with the standard operations for addition and scalar multiplication and let $W$ be the subset of all $2 \times 2$ matrices with **trace** 0, that is,

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a + d = 0 \right\}$$

Show that $W$ is a subspace of $M_{2\times 2}$

**Note:** Trace of a square matrix is the sum of the entries on the diagonal.

# Vector Subspace

## Example - Solution

Let, $w_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $w_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

be matrices in $W$, so that $a_1 + d_1 = 0$ and $a_2 + d_2 = 0$.

The sum of two matrices in $W$,

$$w_1 + w_2 = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

Since the trace of $w_1 + w_2 = (a_1 + a_2) + (d_1 + d_2)$

$$= (a_1 + d_1) + (a_2 + d_2) = 0$$

**($W$ is closed under addition)**

# Vector Subspace

## Example – Solution (cont'd)

- Let $c$ be any real number

$$cw_1 = c\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} ca_1 & cb_1 \\ cc_1 & cd_1 \end{bmatrix}$$

Since the trace of $cw_1 = ca_1 + cd_1 = c(a_1 + d_1) = 0$

**($W$ is closed under scalar multiplication)**

Thus, $W$ is a subspace of $M_{2\times 2}$

# Exercise #2

Let,

$$W = \left\{ \begin{bmatrix} a \\ a+1 \end{bmatrix} \middle| a \in R \right\}$$

be a subset of the vector space $V = R^2$ with the standard definitions of addition and scalar multiplication. Determine whether $W$ is a subspace of $V$.

# Exercise #3

Determine whether the subset $S$ of $R^2$ is a subspace.

If $S$ is not a subspace, find vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $S$ such that $\boldsymbol{u} + \boldsymbol{v}$ is not in $S$; or a vector $\boldsymbol{u}$ and a scalar $c$ such that $c\boldsymbol{u}$ is not in $S$.

*i.* $S = \left\{ \begin{matrix} x \\ 2x - 1 \end{matrix} \middle| x \in R \right\}$

*ii.* $S = \left\{ \begin{matrix} x \\ y \end{matrix} \middle| xy \leq 0 \right\}$

# CHAPTER 3

## General Vector Spaces

Part 2:
Span, Basis and Dimension

# Span of Sets of Vector

- Let $V$ be a vector space and let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a (finite) set of vectors in $V$.
- The **span** of $S$, denoted by $\boldsymbol{span}(S)$ or span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, is the set:

$$\boldsymbol{span}(S) = \{c_1\mathbf{v}_1, c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n | c_1, c_2, \ldots, c_n \in R\}$$

- In other word, determine if every vector in $V$ can be expressed as **linear combination** of the vectors in $S$ or not.
- $span(S)$ is a subspace of $V$.

# Span of Single Vector

The span of a single nonzero vector in $R^n$ is a **line** through the origin as shown in figure 1(a).

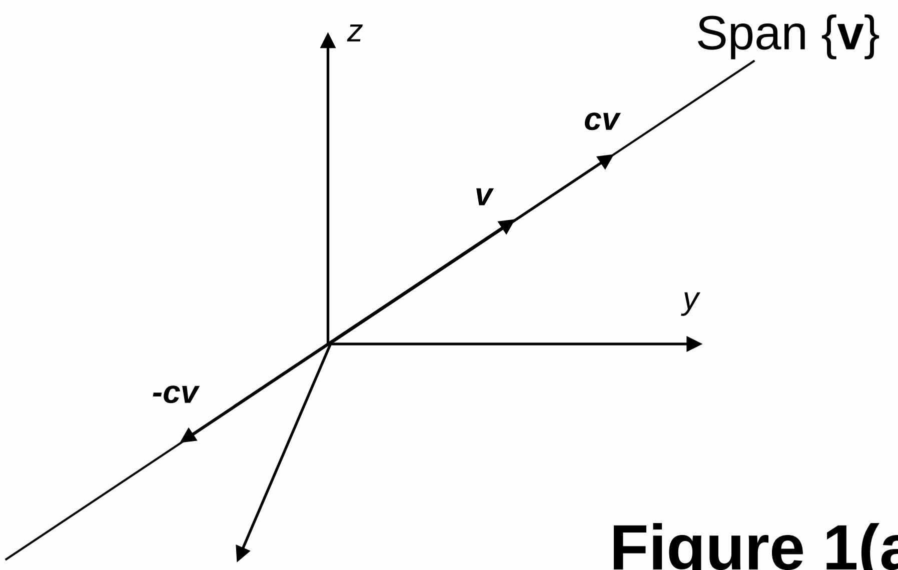


**Figure 1(a)**

# Span of Two Vectors

- The span of two linearly independent vectors is a **plane** through the origin as shown in figure 1(b).

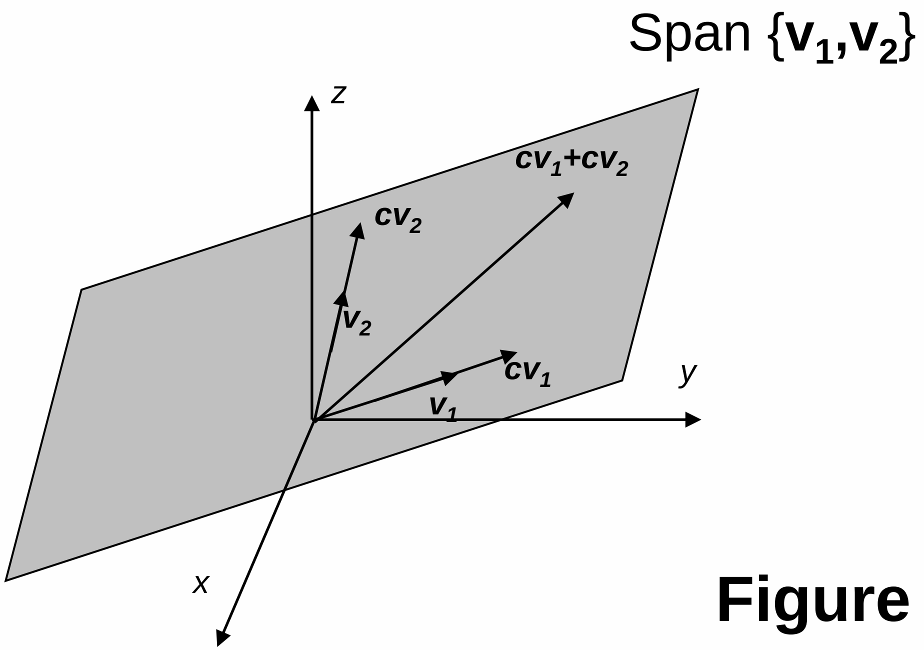


**Figure 1(b)**

# Span of Sets of Vector

## Example #1:

Let $S$ be the subset of the vector space $R^3$ defined by

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$$

Show that the vector, **v** is in **span**($S$).

$$\mathbf{v} = \begin{bmatrix} -4 \\ 4 \\ -6 \end{bmatrix}$$

# Span of Sets of Vector

## Example #1 - Solution:

- The vector, **v** is in **span**($S$) provided that there are scalars $c_1, c_2$ and $c_3$ such that

$$c_1\begin{bmatrix}2\\-1\\0\end{bmatrix}+c_2\begin{bmatrix}1\\3\\-2\end{bmatrix}+c_3\begin{bmatrix}1\\1\\4\end{bmatrix}=\begin{bmatrix}-4\\4\\-6\end{bmatrix}$$

- This linear system in matrix form is given by

$$\left[\begin{array}{ccc|c}2 & 1 & 1 & -4\\-1 & 3 & 1 & 4\\0 & -2 & 4 & -6\end{array}\right]$$

# Span of Sets of Vector

## Example #1 – Solution (cont'd):

- Solve the linear system:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & -4 \\ -1 & 3 & 1 & 4 \\ 0 & -2 & 4 & -6 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -3 & -1 & -4 \\ 2 & 1 & 1 & -4 \\ 0 & -2 & 4 & -6 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -3 & -1 & -4 \\ 0 & 7 & 3 & 4 \\ 0 & -2 & 4 & -6 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -3 & -1 & -4 \\ 0 & 7 & 3 & 4 \\ 0 & 1 & -2 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -1 & -4 \\ 0 & 1 & \frac{3}{7} & \frac{4}{7} \\ 0 & 1 & -2 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{7} & \frac{-16}{7} \\ 0 & 1 & \frac{3}{7} & \frac{4}{7} \\ 0 & 1 & 2 & -3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{7} & \frac{-16}{7} \\ 0 & 1 & \frac{3}{7} & \frac{4}{7} \\ 0 & 0 & \frac{-17}{7} & \frac{-17}{7} \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{7} & \frac{-16}{7} \\ 0 & 1 & \frac{3}{7} & \frac{4}{7} \\ 0 & 0 & 1 & -1 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & \frac{3}{7} & \frac{4}{7} \\ 0 & 0 & 1 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]; \quad c_3 = -1; c_2 = 1; c_3 = -2$$

Since every vector in $R^3$ can be written as a linear combination of the three given vectors, thus the vector **v** is in span($S$).

# Span of Sets of Vector

## Example #2:

Show that

$$span\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}-2\\3\\1\end{bmatrix},\begin{bmatrix}1\\2\\4\end{bmatrix}\right\}=R^3$$

# Span of Sets of Vector

## Example #2 - Solution:

Let, $v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an arbitrary element of $R^3$. The vector **v** is in **span**(*S*) provided that there are scalars $c_1, c_2$ and $c_3$ such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = c_1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

# Span of Sets of Vector

## Example #2 - Solution:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & x_1 \\ -1 & 3 & 2 & x_2 \\ 0 & 1 & 4 & x_3 \end{array}\right] \cdots \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & x_1 \\ 0 & 1 & 3 & x_2 + x_1 \\ 0 & 0 & 1 & x_3 - (x_2+x_1) \end{array}\right]$$

This shows that a solution exists for all choice of $x_1, x_2$ and $x_3$ . Thus every vector in $R^3$ can be written as a linear combination of the three given vectors. Hence, the span of the three vectors is all of $R^3$ .

# Exercise #4

1) Determine whether

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$$ span the vector space $R^3$.

2) Show that

$$span\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\} = R^3$$

# Basis For Vector Space

- A subset B of a vector space $V$ is a **basis** for $V$ provided that
  - i) $B$ is linearly independent set of vectors in $V$
  - ii) $span(B) = V$

- For any $R^n$, $\boldsymbol{e}_n = \left\{ \begin{bmatrix} 1 \\ 0 \\ \cdot\cdot \\ \cdot\cdot \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \cdot\cdot \\ \cdot\cdot \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \cdot\cdot \\ \cdot\cdot \\ 1 \end{bmatrix} \right\}$

is the standard basis and denote by $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots \ldots . \boldsymbol{e}_n$

# Basis For Vector Space

## Example #1

Show that the set,

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

is the basis for $R^3$.

# Basis For Vector Space

## Example #1 – Solution:

1) Show that **span**($B$) = $V$

$$c_1\begin{bmatrix}1\\1\\0\end{bmatrix}+c_2\begin{bmatrix}1\\1\\1\end{bmatrix}+c_3\begin{bmatrix}0\\1\\-1\end{bmatrix}=\begin{bmatrix}a\\b\\c\end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c}1&1&0&a\\1&1&1&b\\0&1&-1&c\end{array}\right]\ldots\Rightarrow\left[\begin{array}{ccc|c}1&0&0&2a-b-c\\0&1&0&-a+b+c\\0&0&1&-a+b\end{array}\right]$$

This show that a solution exists for all choices of $a, b$ and $c$.
Thus, $span(B) = R^3$

# Basis For Vector Space

## Example #1 – Solution (cont'd):

2) Show that *B* is linearly independent set of vectors in *V*

$$c_1\begin{bmatrix}1\\1\\0\end{bmatrix}+c_2\begin{bmatrix}1\\1\\1\end{bmatrix}+c_3\begin{bmatrix}0\\1\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c}1&1&0&0\\1&1&1&0\\0&1&-1&0\end{array}\right]\ldots.\Rightarrow\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]$$

This implies that $c_1=0,\ c_2=0$ and $c_3=\ 0$. (trivial solution)

Thus set *B* is linearly independent.

# Basis For Vector Space

## Example #1 – Solution (cont'd):

- Alternatively, compute the determinant of the coefficients matrix and set of vectors is linearly independent if and only if the determinant is nonzero.

Use 1st row:

$$1\begin{vmatrix}1 & 1\\1 & -1\end{vmatrix} - 1\begin{vmatrix}1 & 1\\0 & -1\end{vmatrix} + 0\begin{vmatrix}1 & 1\\0 & 1\end{vmatrix} = (-2) - (-1) + (0) = -1$$

Hence the set *B* is a basis for $R^3$

# Basis For Vector Space

## Example #2:

Determine whether

$$B = \left\{ \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix} \right\}$$

is the basis for $M_{2\times 2}$.

# Basis For Vector Space

## Example #2 – Solution:

$Let\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, show that $A$ is in $span(B)$.

$$c_1 \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & a \\ 3 & 2 & 1 & b \\ 2 & 1 & 0 & c \\ 1 & 0 & -4 & d \end{array}\right]$$

# Basis For Vector Space

## Example #2 – Solution (cont'd):

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & a \\ 3 & 2 & 1 & b \\ 2 & 1 & 0 & c \\ 1 & 0 & -4 & d \end{array}\right] .. \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & a \\ 0 & 5 & 1 & -3a+b \\ 0 & 0 & -3 & -a-3b+5c \\ 0 & 0 & 0 & a+4b-7c+d \end{array}\right]$$

- Observe that the solution of linear system is inconsistent.
- Hence, $B$ does not span $M_{2x2}$ . Therefore, the set $B$ is not a basis for $M_{2x2}$

# Dimension of Vector Space

- The dimension of a finite dimensional vector space, denoted by **dim(V)**, is defined to be the number of vectors in a basis for V.

- In additional, the *zero vector space* is defined to have dimensional *zero*.

# Dimension of Vector Space

- $\dim(R^n) = n$
The standard basis has $n$ vectors $(\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n)$.
- $\dim(M_{2\times 2}) = 4$
The standard basis of $M_{2\times 2}$ has 4 vectors.
- $\dim(M_{m\times n}) = m * n$
The standard basis has $m * n$ vectors.
- $\dim(P_n) = n + 1$
The standard basis has $n + 1$ vectors.

# Dimension of Vector Space

**Example:**

Determine a basis and the dimension of the solution space of the homogeneous system.

$$2x_1 + 2x_2 - 3x_3 + x_5 = 0$$
$$-x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0$$
$$x_1 + x_2 - 2x_3 - x_5 = 0$$
$$x_3 + x_4 + x_5 = 0$$

# Dimension of Vector Space

## Example #2 – Solution:

Find the solution space of the homogeneous system by using Gaussian elimination method.

$$\left(\begin{array}{ccccc|c} 2 & 2 & -3 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right) \dots \Rightarrow \left(\begin{array}{ccccc|c} 2 & 2 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0.5 & -3 & 1.5 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \end{array}\right)$$

It shows that $x_1, x_3, x_4$ and $x_5$ are leading variables in which $x_3 = x_4 = x_5 = 0$ and $x_2$ is a free variable. Let $x_2 = a$, then we obtain $x_1 = -x_2 = -a$.

# Dimension of Vector Space

## Example #2 – Solution (cont'd):

Write the solution as follows:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -a \\ a \\ 0 \\ 0 \\ 0 \end{pmatrix} = a\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow A = \left\{ x \mid x = a\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, a \in R \right\}$$

# Dimension of Vector Space

## Example #2 – Solution (cont'd):

This is an evident that the vector **v** is spans the solution space *A* and linearly independent which implies that **v** is a basis of *A.*

$$\mathbf{v} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, the dimension of the solution space *A* is equal to 1 since *A* contains only one vector basis.

# Exercise #5

Find the basis for the subspace *S* of the vector space V and specify the dimension of *S*.

$$S = \left\{ \begin{pmatrix} s+2t \\ -s+t \\ t \end{pmatrix} \middle| \, s, t \in R \right\}, V = R^3$$

# CHAPTER 3

## General Vector Spaces

## Part 3:
## Coordinate & Change of Basis

# Coordinate

Let $\boldsymbol{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be an ordered basis for the vector space $V$. Let **v** be a vector in $V$, and let $c_1, c_2, \dots . c_n$ be a unique scalars such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots . + c_n\mathbf{v}_n$$

Then $c_1, c_2, \dots . c_n$ are called the **coordinates of v relative to** B, and written as

$$[\mathrm{v}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

# Coordinate

## Example #1:

Let $V = R^2$ and **B** be the ordered basis,

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

Find the coordinates of the vector, $\mathbf{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ relative to *B.*

# Coordinate

## Example #1 – Solution:

The coordinates $c_1$ and $c_2$ are found by writing **v** as a linear combination of the two vectors in $B$.

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

and we obtain $c_1 = 3$ and $c_2 = 2$. Therefore the coordinate vector of $v = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ relative to $B$ is $[\mathbf{v}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

# Coordinate

**Example #2:**

Let W be the subspace of all symmetric matrices in the vector space $\mathrm{M}_{2\times 2}$. Let,

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Show that $B$ is a basis for $W$ and find the coordinates of vector, $v = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ relative to $B$.

# Coordinate

## Example #2 – Solution:

Since the matrix is 2x2 <u>symmetric</u> matrices, thus $B$ **spans** $W$. The matrices in $B$ are also linearly independent and hence are a basis for $W$. Observe that **v** can be written as,

$$2\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}+3\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}+5\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}=\begin{bmatrix}2 & 3\\3 & 5\end{bmatrix}$$

Then, the coordinate vector of **v** relative to the ordered basis $B$ is

$$[\mathbf{v}]_B=\begin{bmatrix}2\\3\\5\end{bmatrix}$$

# Exercise #6

Find the coordinates of the vector **v** relative to the ordered basis $B$.

$$a)\ B = \left\{\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}-2\\2\end{bmatrix}\right\}, \mathbf{v} = \begin{bmatrix}8\\0\end{bmatrix}$$

$$b)\ B = \left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}3\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}\right\}, \mathbf{v} = \begin{bmatrix}2\\-1\\9\end{bmatrix}$$

# Change of Basis

**Definition:**

Change from coordinates relative to one basis for $V$ to another basis for $V$.

## Example #1:

Let $V$ be a vector space of dimension 2 and let $B = \{\mathbf{v_1}, \mathbf{v}_2\}$ and $B' = \{\mathbf{v}_1', \mathbf{v}_2'\}$ be ordered bases for $V$. Now let **v** be a vector in $V$, and suppose that the coordinates of **v** relative to $B$ are given by $[\mathbf{v}]_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ that is $v = x_1\mathbf{v}_1 + x_2\mathbf{v}_2$.

Determine the coordinates of **v** relative to $B$'.

# Change of Basis

## Example #1 – Solution:

To determine the coordinates of $\mathbf{v}$ relative to $B'$:

1) Write $\mathbf{v}_1$ and $\mathbf{v}_2$ in terms of the vectors $\mathbf{v}'_1$ and $\mathbf{v}'_2$. Since $B'$ is a basis, there are scalars $a_1, a_2, b_1$ and $b_2$ such that

$$\mathbf{v}_1 = a_1\mathbf{v}'_1 + a_2\mathbf{v}'_2$$
$$\mathbf{v}_2 = b_1\mathbf{v}'_1 + b_2\mathbf{v}'_2$$

2) Then $\mathbf{v}$ can be write as

$$\mathbf{v} = x_1(a_1\mathbf{v}'_1 + a_2\mathbf{v}'_2) + x_2(b_1\mathbf{v}'_1 + b_2\mathbf{v}'_2)$$

3) Collecting the coefficient of $\mathbf{v}'_1$ and $\mathbf{v}'_2$, gives

$$\mathbf{v} = (x_1a_1 + x_2b_1)\mathbf{v}'_1 + (x_1a_2 + x_2b_2)\mathbf{v}'_2$$

# Change of Basis

## Example #1 – Solution (cont'd):

4) Coordinates of **v** relative to the basis $B'$ are given by

$$[\mathbf{v}]_B = \begin{bmatrix} x_1 a_1 + x_2 b_1 \\ x_1 a_2 + x_2 b_2 \end{bmatrix}$$

5) Rewriting the vector on the right-hand side as a matrix product, we have

$$[\mathbf{v}]_{\boldsymbol{B}'} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} [\mathbf{v}]_{\boldsymbol{B}}$$

# Change of Basis

## Example #1 – Solution (cont'd):

6) Notice that the column vectors of the matrix are the coordinate vectors $[\mathbf{v}]_{\boldsymbol{B}}$ and $[\mathbf{v}]_{\boldsymbol{B'}}$. The matrix,

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

is called the **transition matrix** from $B$ to $B$' and is denote by $[I]_B^{B'}$. So that, $[\mathbf{v}]_{\boldsymbol{B'}} = [I]_B^{B'}[\mathbf{v}]_B$

# Change of Basis

## Example #2

Let, $V = R^2$ with bases

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \text{ and } B' = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

a) Find the transition matrix from $B$ to $B'$.

b) Let, $[\mathbf{v}]_B = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, find $[\mathbf{v}]_{\boldsymbol{B'}}$

# Change of Basis

## Example #2 – Solution:

**a.** By denoting the vectors in $B$ by $\mathbf{v}_1$ and $\mathbf{v}_2$ and those in $B'$ by $\mathbf{v}'_1$ and $\mathbf{v}'_2$, the column vectors of the transition matrix are $[\mathbf{v}_1]_{B'}$ and $[\mathbf{v}_2]_{B'}$. These coordinate vectors are found from the equations

$$c_1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad d_1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + d_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Solving these equations gives $c_1 = 2$ and $c_2 = 3$, and $d_1 = 0$ and $d_2 = -1$, so that

$$[\mathbf{v}_1]_{B'} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{and} \quad [\mathbf{v}_2]_{B'} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Therefore, the transition matrix is

$$[I]_B^{B'} = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}$$

# Change of Basis

## Example #2 – Solution (cont'd):

**b.** Since

$$[\mathbf{v}]_{B'} = [I]_B^{B'}[\mathbf{v}]_B$$

then

$$[\mathbf{v}]_{B'} = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \end{bmatrix}$$

Observe that the same vector, relative to the different bases, is obtained from the coordinates $[\mathbf{v}]_B$ and $[\mathbf{v}]_{B'}$. That is,

$$3\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} = 6\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 11\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

# Change of Basis

## Example #3

Let, $B = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ be a standard ordered basis for $R^2$.

Let $B$’ be the ordered basis given by,

$$B' = \{\mathbf{v}_1', \mathbf{v}_2'\} = \left\{\begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$$

a. Find the transition matrix from *B* to *B’*.

b. Let $\mathbf{v} = \begin{bmatrix}3\\4\end{bmatrix}$, find $[\mathbf{v}]_{\boldsymbol{B'}}$

# Change of Basis

## Example #3 – Solution:

**a.** The transition matrix from $B$ to $B'$ is computed by solving the equations

$$c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad d_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + d_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

That is, we must solve the linear systems

$$\begin{cases} -c_1 + c_2 = 1 \\ \phantom{-}c_1 + c_2 = 0 \end{cases} \quad \text{and} \quad \begin{cases} -d_1 + d_2 = 0 \\ \phantom{-}d_1 + d_2 = 1 \end{cases}$$

The unique solutions are given by $c_1 = -\frac{1}{2}, c_2 = \frac{1}{2}$ and $d_1 = \frac{1}{2}, d_2 = \frac{1}{2}$. The transition matrix is then given by

$$[I]_B^{B'} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

# Change of Basis

## Example #3 – Solution (cont'd):

b.

$$[\mathbf{v}]_{B'} = [I]_B^{B'}[\mathbf{v}]_B$$

$$[\mathbf{v}]_{B'} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{7}{2} \end{bmatrix}$$

# Exercise #7

i) Find the **transition matrix** between the ordered bases $B_1 = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$ and $B_2 = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$

ii) Given, $\mathbf{v} = \begin{bmatrix}2\\3\end{bmatrix}$, find $[\mathbf{v}]_{\boldsymbol{B_2}}$

# CHAPTER 3

## General Vector Spaces

Part 4:
Row & Column Space, Rank & Nullity

# Definition - Row Vector

For an $m \times n$ matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

the vectors

$$\begin{aligned} \mathrm{r}_1 &= (a_{11} \quad a_{12} \quad \cdots \quad a_{1n}) \\ \mathrm{r}_2 &= (a_{21} \quad a_{22} \quad \cdots \quad a_{2n}) \\ &\quad\ \vdots \qquad \vdots \qquad\qquad \vdots \\ \mathrm{r}_m &= (a_{m1} \quad a_{m2} \quad \cdots \quad a_{mn}) \end{aligned}$$

in $R^n$ formed from the rows of $A$ are called the *row vectors of $A$*.

# Definition - Column Vector

and the vectors $\mathrm{c}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \quad \mathrm{c}_2 = \begin{pmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2} \end{pmatrix}, \cdots, \ \mathrm{c}_m = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$

in $R^m$ formed from the columns of $A$ are called the *column vectors* of $A$.

# Example

Let, $A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$

The row vectors of $A$ are:

$$r_1 = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}; r_2 = \begin{bmatrix} 3 & -1 & 4 \end{bmatrix}$$

The column vectors of $A$ are:

$$c_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}; c_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}; c_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

# Row, Column & Null Space

Let $A$ be an $m \times n$ matrix.

| | |
|---|---|
| Row space of $A$ | Subspace of $R^n$ spanned by the **row vectors** of $A$. |
| Column space of $A$ | Subspace of $R^m$ spanned by the **column vectors** of $A$. |
| Nullspace of $A$ | The **solution space** of the homogeneous system of equation $Ax = 0$, which is a subspace of $R^n$ |

# Row, Column & Null Space

## Example #3:

Given, $A = \begin{pmatrix} 1 & 5 & 3 \\ 2 & 6 & 2 \end{pmatrix}$

The **row space** of $A$:

$\{x^T | x^T = c_1(1,5,3) + c_2(2,6,2); c_1, c_2 \in R\}$
$= \{x^T | x^T = (c_1 + 2c_2, 5c_1 + 6c_2, 3c_1 + 2c_2); c_1, c_2 \in R\}$

The **column space** of $A$:

$\{x | x = c_1(1,2)^T + c_2(5,6)^T + c_3(3,2)^T; c_1, c_2, c_3 \in R\}$
$= \{x | x = (c_1 + 5c_2 + 3c_3, 2c_1 + 6c_2 + 2c_3; c_1, c_2, c_3 \in R\}$

# Row, Column & Null Space

## Example #3 (cont'd):

The **nullspace** of $A$: (we need to solve it using the linear system)

$$A = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 6 & 2 \end{bmatrix}$$

$$\Rightarrow \tilde{A} = \left[\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 2 & 6 & 2 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$.: \; x_2 = -x_3; \; x_1 = 2x_3$$

$$\Rightarrow X = \begin{pmatrix} 2x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \Rightarrow \{x | x = c(2, -1, 1)^T; c \in R\}$$

# Basis for Row Space

Let $A$ is a $m \times n$ matrix and matrix E is in row echelon form of $A$. Then the row vectors with the leading 1's (the nonzero row vector) of matrix E will form a basis for the row space of $A$.

**Example:**

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \Rightarrow E = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The basis vectors are:

$r_1 = (1 \quad -3 \quad 4 \quad -2 \quad 5 \quad 4)$
$r_2 = (0 \quad 0 \quad 1 \quad 3 \quad -2 \quad -6)$
$r_3 = (0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 5)$

Refer: Module - Example 3.28 (page 108)

# Basis for Column Space

Let $A$ is a $m \times n$ matrix and E is in row echelon form of $A$. Then the column vectors with the leading 1's of the row vectors form a basis for the column space of E. Then, the *corresponding* column vector of $A$ will form a basis for the column space of $A$.

**Example:**

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \Rightarrow E = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The column space are:

$$c_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \end{pmatrix}, c_3 = \begin{pmatrix} 4 \\ 9 \\ 9 \\ -4 \end{pmatrix}, c_4 = \begin{pmatrix} 5 \\ 8 \\ 9 \\ -5 \end{pmatrix}$$

Refer: Module - Example 3.29 (page 108)

# Dimension and Rank

- Dimension of the row space of $A$:

The number of non-zero rows in the echelon form of $A$ (number of basis for row space).

- Dimension of the column space of $A$:

Number of leading 1's in the echelon form of $A$ (number of basis for column space)

Also called the rank of $A$ (rank($A$))

# Dimension and Rank

## Example #1:

Determine the rank and basis of the row space of the matrix:

$$A = \begin{bmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{bmatrix}$$

# Dimension and Rank

## Example #1 - Solution:

The reduced row-echelon form of $A$ is

$$A = \begin{bmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Than rank of $A$ equal to 2. The row space of $A$ are spanned by,

$$\mathbf{v}_1 = [1 \quad 6 \quad 9], \mathbf{v}_2 = [0 \quad 1 \quad 2]$$

# Dimension and Rank

## Example #2:

If $A = \begin{bmatrix} 1 & -1 & 3 & -2 \\ 2 & -2 & 2 & -1 \\ -1 & 1 & 5 & -4 \end{bmatrix}$, find

i) Basis for row space of matrix

ii) Basis for column space of matrix

iii) rank ($A$)

Solution – Refer Module - Example 3.31 (page 110)

# Dimension and Rank

## Example #2 - Solution:

The matrix $E = \begin{bmatrix} 1 & -1 & 3 & -2 \\ 0 & 0 & 1 & -3/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

i) Basis for row space of matrix $A$:

$$r_1 = (1 \;\; -1 \;\; 3 \;\; -2);\, r_2 = (0 \;\; 0 \;\; 1 \;\; -3/4)$$

ii) Basis for column space of matrix $A$:

$$c_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix};\; c_3 = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$$

iii) rank $(A) \; = 2$

Solution – Refer Module - Example 3.31 (page 110)

# Basis for Vector Space - Row

Suppose given a vector space $V$ is spanned by a set of row vectors,

$$\mathbf{v}_1 = (v_{11}, v_{12}, \dots, v_{1n})$$
$$\mathbf{v}_2 = (v_{21}, v_{22}, \dots, v_{2n})$$
$$\dots.$$
$$\mathbf{v}_m = (v_{m1}, v_{m2}, \dots, v_{mn})$$

Then, we can form a matrix:

$$A = \begin{pmatrix} v_{11} & v_{12} & ,\cdots v_{1n} \\ v_{21} & v_{22} & ,\cdots v_{2n} \\ \dots & \dots & ,\cdots \quad \dots \\ v_{m1} & v_{m2} & ,\cdots v_{mn} \end{pmatrix}$$

# Basis for Vector Space - Row

## Example:

Find the dimension and a set of basis for row space vectors for the vector space $V$ spanned by

$$\mathbf{v}_1 = (1, 2, 0, 2, 5), \mathbf{v}_2 = (-2, 5, 1, -1, 8),$$
$$\mathbf{v}_3 = (0, -3, 3, 4, 1), \mathbf{v}_4 = (3, 6, 0, -7, 2)$$

Refer: Module - Example 3.32 (page 111 )

# Basis for Vector Space - Row

## Example - Solution:

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix} \ldots \Rightarrow E = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

.: dimension of $V$ = 3, the basis for the row space are:

$$r_1 = (1, 2, 0, 2, 5), r_2 = (0, 1, -1, -3, -2), r_3 = (0, 0, 0, 1, 1)$$

**Note:** $r_1 = (1, 2, 0, 2, 5), r_2 = (0, -1, 1, 3, 2), r_3 = (0, 0, 0, -5, -5)$ also a basis.

Refer: Module - Example 3.32 (page 111 )

# Basis for Vector Space - Row

Then we obtain the important rules as follows:

- The vector space $V$ is the row space of $A$.
- Dimension of $V$ equals to the dimension of row space of $A$ and also equals to rank $(A)$.
- Basis for row space vectors of $V$ equals to the nonzero row vector in the echelon form of $A$.

# Basis for Vector Space - Column

Suppose given a vector space $V$ is spanned by a set of column vectors

$$\mathbf{v}_1 = (v_{11}, v_{21}, \dots, v_{m1})^T$$
$$\mathbf{v}_2 = (v_{12}, v_{22}, \dots, v_{m2})^T$$
$$\dots.$$
$$\mathbf{v}_n = (v_{1n}, v_{2n}, \dots, v_{mn})^T$$

which are not necessarily linear independent.

Form a matrix $A$ having vector $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ as a column vector:

$$A = \begin{pmatrix} v_{11} & v_{21} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{pmatrix}$$

# Basis for Vector Space - Column

## Example:

Find the dimension and a set of basis vectors for the vector space V spanned by

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{v}_5 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

Refer: Module - Example 3.33 (page 112 )

# Basis for Vector Space- Column

## Example - Solution:

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 2 \end{bmatrix} \ldots \Rightarrow E = \begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

.: dimension of $V$ = 3, the basis for the column space are:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Refer: Module - Example 3.33 (page 112 )

# Basis for Vector Space - Column

The important rules obtained as follows.

- Dimension of $V$ equals to the dimension of column space of $A$ and also equals to rank($A$).

- Basis for vectors span by $v_1, v_2, \dots . v_n$ equals to the corresponding column vector of $A$.

# Nullity

Nullity of matrix $A$ defined as the dimension of the nullspace of $A$.

## Example:

Find a basis for the nullspace of the homogeneous system as follows.

$$2x_1 + 2x_2 - 3x_3 + x_5 = 0$$
$$-x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0$$
$$x_1 + x_2 - 2x_3 - x_5 = 0$$
$$x_3 + x_4 + x_5 = 0$$

# Nullity

## Example - Solution:

The nullspace of $A$ is the solution space of the homogeneous system, $Ax = 0$

$$A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} .. \Rightarrow E = \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow\ x_2 = s; x_5 = t; x_3 = -t; x_4 = 0; x_1 = -s - t$$

$$.: X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \left( s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

# Nullity

## Example – Solution (cont'd):

The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form the basis for nullspace of $A$:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Thus, the nullity of $A$ is 2.

# Dimension Theorem of Matrices

If $A$ is a matrix with $n$ columns, then

$$\text{rank}\,(A) + \text{nullity}(A) = n$$

$$[\textbf{no. of leading variables}] + [\textbf{no. of free variables}] = \boldsymbol{n}$$

# Exercise #8

Find the rank and nullity of the following matrix.

a) $A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 2 & 4 & 2 & 0 \\ -1 & -3 & 0 & 5 \end{bmatrix}$

# Exercise #9

Given,

$$A = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4 \quad \mathbf{v}_5] = \begin{bmatrix} 1 & -1 & 5 & 1 & 0 \\ 2 & 1 & 4 & 2 & 1 \\ 3 & 0 & 9 & 3 & 1 \\ -1 & -1 & -1 & -1 & 2 \end{bmatrix}$$

a) What is the rank of $A$

b) Find the basis for the row space $A$

c) Find the basis of the column space of $A$

d) Find the basis of null space of $A$