CHAPTER 8

Analysis of Variance (ANOVA)

Outline

- Introduction
- One-Way ANOVA with Equal Sample Sizes.
- One-Way ANOVA with Unequal Sample Sizes.
- Two-Way ANOVA.

Not covered in this syllabus

Comparisons of inference test

Test Type	Used For	Dependent Variable	Independent Variable	No. of Groups / Type	Common Inferential Test / Method / Algorithm	Example
T-test	Compare means between two groups	Numeric (e.g. order amount)	Categorical (2 groups)	2 groups	Independent Samples T-test	Order amounts between men and women
ANOVA	Compare means across two or more groups	Numeric (e.g. weight)	Categorical (≥2 groups)	2 or more groups	One-way ANOVA	Weight differences among marital statuses
Chi-squared Test	Test association between two categorical variables	Categorical (e.g. pet choice)	Categorical (e.g. gender)	2 or more categories	Pearson's Chi-squared Test	Whether men prefer dogs over cats
Correlation	Measure strength & direction of relationship between numeric variables	Numeric (e.g. blood pressure)	Numeric (e.g. age)	Continuous variables	Pearson / Spearman Correlation Coefficient	Relationship between age and blood pressure
Regression	Predict numeric outcome from one or more numeric predictors	Numeric (e.g. sales)	Numeric (e.g. ad spending)	Continuous variables	Simple Linear Regression	Predicting sales from ad spending, population, conversion rate

formula

Introduction

- ANOVA is a method of testing the equality of three or more population means by analyzing sample variances.
- The purpose of ANOVA is to test for significant differences between Means.
- Elementary concepts provide a brief introduction to the basics of statistical significance testing.

One-Way ANOVA with Equal Sample Sizes

- We assume that the populations have normal distribution and same variance (or standard deviation), and the samples are random and independent of each other.

ANOVA Notation:

n = size of each sample

k = number of populations or treatments being compared

$s_{\bar{X}}^2$ = variance sample means

s_P^2 = pooled variance obtained by calculating the mean of the sample variances

One-Way ANOVA with Equal Sample Sizes

- Test statistic: $$F = \frac{\text{variance between sample}}{\text{variance within sample}} = \frac{nS_{\overline{X}}^2}{S_P^2}$$

- Variance between sample:
  - Also called variation due to treatment
  - An estimate of the common population variance σ^2 that is based on the variability among the sample **means.**
- Variance within sample:
  - Also called variation due to error.
  - An estimate of the common population variance σ^2 based on the sample **variances.**

One-Way ANOVA with Equal Sample Sizes

- The critical value of F:
  - numerator degrees of freedom = $k - 1$
  - denominator degrees of freedom = $k(n - 1)$
  - k = number of population or treatments being compared.
  - n = sample size

Example 1

Table below lists the head injury to car crash test dummies for four different types of cars. Use a 0.05 significance level to test the null hypothesis that the different types of car have the same mean.

No. of head injury			
Subcompact Cars	Compact Cars	Midsize Cars	Full-Size Cars
681	643	469	384
428	655	727	656
917	442	525	602
898	514	454	687
420	525	259	360

Example 1

- Step 1: Define hypothesis.

$H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$

H_1 : at least one mean is different.

- Step 2: For each category, find $n, \bar{x}$ and s

Example 1

- Category 1: (Subcompact Cars)

$$n = 5$$

$$\bar{x} = \frac{681 + 428 + 917 + 898 + 420}{5} = 668.8$$

$$s = \sqrt{\frac{(681 - 668.8)^2 + (428 - 668.8)^2 + (917 - 668.8)^2 + (898 - 668.8)^2 + (420 - 668.8)^2}{5 - 1}}$$

$$= 242.0$$

- Category 2: (Compact Cars)

$$n = 5$$

$$\bar{x} = \frac{643 + 655 + 442 + 514 + 525}{5} = 555.8$$

$$s = \sqrt{\frac{(643 - 555.8)^2 + (655 - 555.8)^2 + (442 - 555.8)^2 + (514 - 555.8)^2 + (525 - 555.8)^2}{5 - 1}}$$

$$= 91.0$$

Example 1

- Category 3: (Midsize Cars)

$$n = 5$$

$$\bar{x} = \frac{469 + 727 + 525 + 454 + 259}{5} = 486.8$$

$$s = \sqrt{\frac{(469 - 486.8)^2 + (727 - 486.8)^2 + (525 - 486.8)^2 + (454 - 486.8)^2 + (259 - 486.8)^2}{5 - 1}}$$

$$= 167.7$$

- Category 4: (Fullsize Cars)

$$n = 5$$

$$\bar{x} = \frac{384 + 656 + 602 + 687 + 360}{5} = 537.8$$

$$s = \sqrt{\frac{(384 - 537.8)^2 + (656 - 537.8)^2 + (602 - 537.8)^2 + (687 - 537.8)^2 + (360 - 537.8)^2}{5 - 1}}$$

$$= 154.6$$

Example 1

- Step 3: Find variance between samples

Step 3a: Find mean between samples

$$\bar{\bar{x}} = \frac{668.8 + 555.8 + 486.8 + 537.8}{k = 4} = 562.3$$

Step 3b: Find standard deviation between samples

$$s_{\bar{x}} = \sqrt{\frac{(668.8 - 562.3)^2 + (555.8 - 562.3)^2 + (486.8 - 562.3)^2 + (537.8 - 562.3)^2}{4 - 1}}$$

$$= 76.779$$

Step 3c: Find variance between samples

$$ns_{\bar{x}}^2 = 5(76.779)^2 = 29475.1$$

Example 1

- Step 4: Find variance within samples

$$s_p^2 = \frac{(242.0)^2 + (91.0)^2 + (167.7)^2 + (154.6)^2}{k = 4}$$

$$= 29717.4$$

- Step 5: Calculate test statistic, *F*

$$F = \frac{\text{variance between samples}}{\text{variance within samples}} = \frac{ns_{\bar{x}}^2}{s_p^2}$$

$$= \frac{29475.1}{29717.4}$$

$$= 0.992$$

Example 1

- Step 6: Calculate numerator and denominator degree of freedom
  - Numerator = $k - 1 = 4 - 1 = 3$
  - Denominator = $k(n - 1) = 4(5 - 1) = 16$

- Step 7: Find critical value of F with $\alpha = 0.05$ from F-distribution table
  - F critical value = 3.24

Example 1

- Step 8: Test the claim and state the conclusion.

  - Since $F_{\textit{test statistic}} < F_{\textit{critical value}}$ (0.992 < 3.24), we fail to reject the null hypothesis.

  - There is sufficient evidence to claim that the different types of cars have the same mean for head injury.

Example 2

Table below lists the chest deceleration to car crash test dummies for four different types of cars. Use a 0.05 significance level to test the null hypothesis that the different types of car have the same mean.

Subcompact Cars	Compact Cars	Midsize Cars	Full-Size Cars
n = 5	n = 5	n = 5	n = 5
$\bar{x} = 50.4$	$\bar{x} = 53.0$	$\bar{x} = 48.8$	$\bar{x} = 46.0$
s = 6.69	s = 4.64	s = 3.35	s = 7.11

Example 2

- Step 1: State the hypothesis.

  $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$

  H_1 : at least one mean is different.

- Step 2: (Done – refer table).
- Step 3: Find variance between samples

  Step 3a: Find mean between samples

$$\bar{\bar{x}} = \frac{50.4 + 53.0 + 48.8 + 46.0}{4} = 49.55$$

  Step 3b: Find variance between samples

$$s_{\bar{x}} = \sqrt{\frac{(50.4 - 49.55)^2 + (53.0 - 49.55)^2 + (48.8 - 49.55)^2 + (46.0 - 49.55)^2}{4 - 1}}$$

$$= 2.93$$

Example 2

- Step 4: Find variance within samples

$$s_p^2 = \frac{(6.69)^2 + (4.64)^2 + (3.35)^2 + (7.11)^2}{4}$$
$$= 32.02$$

- Step 5: Calculate test statistic, *F*

$$F = \frac{\text{variance between samples}}{\text{variance within samples}} = \frac{ns_{\bar{x}}^2}{s_p^2}$$
$$= \frac{42.92}{32.02} \longleftarrow \quad ns_{\bar{x}}^2 = 5(2.93)^2 = 42.92$$
$$= 1.34$$

Example 2

- Step 6: Calculate numerator and denominator degree of freedom
  - Numerator = $k - 1 = 4 - 1 = 3$
  - Denominator = $k(n - 1) = 4(5 - 1) = 16$

- Step 7: Find critical value of F with $\alpha = 0.05$ from F-distribution table
  - F critical value = 3.24

Example 2

- Step 8: Test the claim and state the conclusion.

    - Since $F_{test\ statistic} < F_{critical\ value}$ (1.34 < 3.24), we fail to reject the null hypothesis.

    - There is sufficient evidence to claim that the different types of cars have the same mean for chest deceleration.

Exercise #1

Table 1 lists the body temperatures of 5 randomly selected subjects from each of 3 different age groups. Informal examination of the 3 sample means (97.940, 98.580, 97.800) seems to suggest that the 3 samples come from populations with means that are not significantly different. Test the claim that the 3 age-group populations have the same mean body temperature. Use $\alpha = 0.05$.

Exercise #1

Table 1: Body Temperature (°F) Categorized by Age

18-20	21-29	30 and older
98.0	99.6	98.6
98.4	98.2	98.6
97.7	99.0	97.0
98.5	98.2	97.5
97.1	97.9	97.3
$n_1=5$	$n_2=5$	$n_3=5$
Mean: $x_1=97.940$	$x_2=98.580$	$x_3=97.800$
Std. Dev: $s_1=0.568$	$s_2=0.701$	$s_3=0.752$

Exercise #2

Data in the following Table 1 represent the number of hours of relief provided by five different brands of headache tablets given to 25 patients experiencing fevers of 38° Celsius or more. Perform the analysis of variance (ANOVA) and test the hypothesis at 0.05 level of significance that the mean number of hours of relief provided by the tablets is the same for all five brands. Provide your evidence.

Table 1

Tablet brand	A	B	C
Number of hoursof relief	5.2	9.1	2.4
	4.7	7.1	3.4
	8.1	8.2	4.1
	6.2	6.0	1.0
	3.0	9.1	1.0

Exercise #2

a) Define the hypothesis statement.

H_0: $\mu_A = \mu_B = \mu_C$
H_1: At least one tablet brand gives different average relief hours.

b) Calculate mean and variance.

Tablet Brand	Mean	Variance
A	5.44	3.553
B	7.9	1.805
C	2.38	1.952

Exercise #2

c) Calculate the test statistics. $F = \frac{\text{variance between sample}}{\text{variance within sample}} = \frac{nS_{\bar{X}}^{2}}{S_{P}^{2}}$

Mean between samples:

$$\bar{\bar{x}} = \frac{5.44 + 7.9 + 2.38}{3} = 5.24$$

Variance between samples (numerator):

$$s_{\bar{x}}^{2} = \frac{(5.44-5.24)^2 + (7.9-5.24)^2 + (2.38-5.24)^2}{3-1} = 7.6476$$

Variance within samples (denominator):

$$s_{p}^{2} = \frac{3.553+1.805+1.952}{3} = 2.436$$

$nS_x^2 = 5 \times 7.6476 = 38.2372$

$$F = \frac{38.2372}{2.436} = 15.697$$

Exercise #2

d) Calculate numerator and denominator degree of freedom. Use α = 0.05

df_1 (numerator) = k - 1 = 3 - 1 = 2,
df_2 (denominator) = k(n - 1) = 3(5-1) = 12

e) State the critical value.

F (0.05, 2, 12) = 3.89

f) Test the claim and state the conclusion.

Since 15.697 > 3.89, reject H_0.
Conclusion: There is a significant difference among 3 tablet brands