CHAPTER 1

Part 1

System of Linear Equation and Matrices

Linear Equations

An equation of the form:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

is called a **linear equation** in

$x_1, x_2, \ldots x_n$: variables

$a_1, a_2, \ldots a_n$: real numbers, called the coefficients of the variables

b : a number called as constant term of the equation

Linear Equations

A set of numbers

$$s_1, s_2, \ldots, s_n$$

is called a **solution** to the linear equation if

$$a_1 s_1 + a_2 s_2 + \ldots a_n s_n = b$$

$$X = \left[s_1 \;\; s_2 \ldots . s_n \right]^T$$

(values $s_1, s_2, .., s_n$ are substitute for $x_1, x_2, .. x_n$)

Linear Equations

Example

Show that $X = [1, -2]^T$ is a solution to the equation

$$2x_1 - 3x_2 = 8$$

but that $Y = [1, 1]^T$ is not a solution.

Linear Equation

Example - Solution

$X = [1, -2]^T$ is a solution because $x_1 = 1$ and $x_2 = -2$ satisfy the equation:

$$2(1) - 3(-2) = 8$$

but $x_1 = 1$ and $x_2 = 1$, do not satisfy the equation:

$$2(1) - 3(1) = -1 \neq 8$$

So $Y = [1, 1]^T$ is not a solution.

Linear Equation

Example

Determine whether the points (–1, –5) and (0, –2) is a solution to the given system of equations:

$$y = 3x - 2$$

$$y = -x - 6$$

Linear Equation

Example - Solution

Do check for $(-1, -5)$:

a) $y = 3x - 2 \Rightarrow y = 3(-1) - 2 = -5$

b) $y = -x - 6 \Rightarrow y = -(-1) - 6 = -5$

Do check for $(0, -2)$:

a) $y = 3x - 2 \Rightarrow y = 3(0) - 2 = -2$

b) $y = -x - 6 \Rightarrow y = -(0) - 6 = -6 \neq -2$

Since only the point $(-1, -5)$ works in each equation, thus it is the solution to the linear equations.

System of Linear Equations (SLE)

- A finite collection of linear equations is called a system of linear equations.

- A system of m equations in n variables,

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m
\end{aligned}
$$

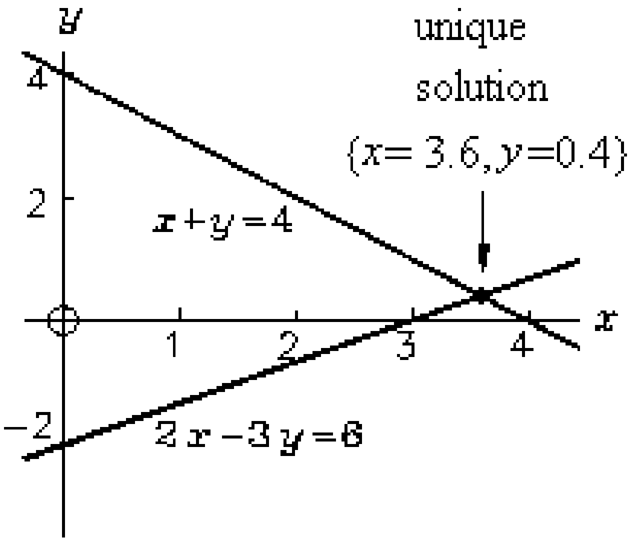
System of Linear Equations

- A system has a solution if there exist numbers $s_1, s_2, \ldots s_n$ which satisfy each of the equations simultaneously.
- The system is called consistent if it has one or more solutions. **(unique or infinite solutions)**
- The system is called inconsistent if it has no solution.

Unique Solutions for SLE

- A system,

$$\begin{cases} 1x + 1y = 4 \\ 2x - 3y = 6 \end{cases}$$



- The point where the lines cross $(x = 3.6; y = 0.4)$ is the solution that satisfies both equations simultaneously.
- The solution is unique and consistent.

Unique Solutions for SLE

- A system,

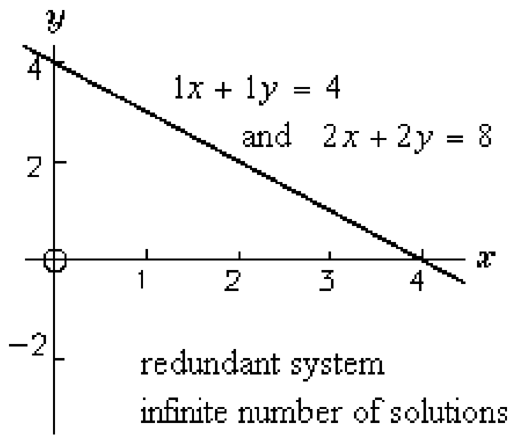
$$x_1 + x_2 = 5 \qquad (1)$$
$$3x_1 + 2x_2 = 12 \qquad (2)$$

- Has a solution, $x_1=2$ and $x_2=3$
- 2 lines are not parallel and intersect at point (2,3).
- **Solution is unique and consistent.**

Infinite (Many) Solutions for SLE

- A systems,

$$\begin{cases} 1x + 1y = 4 \\ 2x + 2y = 8 \end{cases}$$



- This system is redundant because the second equation is equivalent to the first one.
- They 'cross' at an infinite number of points, so there are an infinite number of solutions.
- The solution is not unique but infinite and consistent.

Infinite (Many) Solutions for SLE

Example

- A system,

$$x_1 + 2x_3 = 6 \qquad (1)$$

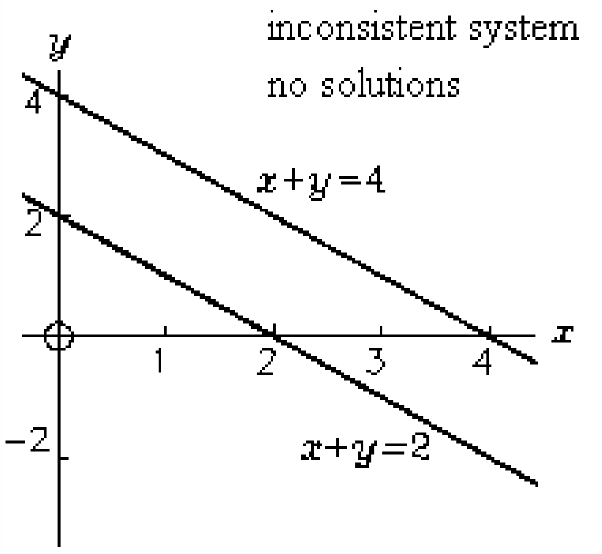
$$x_2 + x_3 = 2 \qquad (2)$$

- Has many solutions, $x_1=6-2t$, $x_2=2-t$ and $x_3=t$ where t is arbitrary. **(Consistent)**

No Solutions for SLE

- A system,

$$\begin{cases} x + y = 2 & \textbf{(1)} \\ x + y = 4 & \textbf{(2)} \end{cases}$$



- Consists of two parallel lines that never cross. Thus there is no solution.
- Solution is inconsistent.

No Solutions for SLE

Example

- A system,

$$3x_1 + x_2 = 5 \qquad (1)$$

$$3x_1 + x_2 = 7 \qquad (2)$$

- (2)-(1), 0=2 (no solution)
- 2 lines are parallel and do not intersect. **(Inconsistent)**

Introduction to Matrix

- A rectangular arrangement of number is called a matrix.
- Matrices are usually denoted using upper case letter A, B, C, K, P, etc.
- Example:

$$A = \begin{pmatrix} 2 & 0 & -4 \\ 3 & 1 & 7 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 2 & 8 \\ 3 & 4 & 3 \\ 3 & 9 & 6 \end{pmatrix}$$

Introduction to Matrix

- A matrix of size 1 x *n* is called a **row matrix**:

$$C = \begin{pmatrix} -1 & 5 & 2 \end{pmatrix}$$

- A matrix of size *m* x 1 is called a **column matrix:**

$$B = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

- A matrix with equal numbers of rows and columns is called a **square matrix:**

$$E = \begin{pmatrix} 7 & 3 \\ 8 & -2 \end{pmatrix}$$

Introduction to Matrix

- The size of matrix is depend on the number of rows and columns
- A matrix with *m* rows and *n* columns is called an *m*-by-*n* matrix (*m* x *n* matrix).
- Example

columns 3

rows 2

$$A = \begin{pmatrix} 2 & 0 & -4 \\ 3 & 1 & 7 \end{pmatrix}$$

2 x 3 matrix

Introduction to Matrix

- The numbers in the matrix are called its **entries**
- The entry in the i-*th row* and *j-th* column of matrix X is referred as the (i, j) – entry of the matrix X and denoted by a lower case letter with two subscripts indices, $x_{i,j}$
- If A is an $m \times n$ matrix. The (i, j) – entry of A is denoted by $a_{i,j}$:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} \cdots & a_{m,n} \end{pmatrix}$$

Introduction to Matrix

Example

A is an 2 x 3 matrix, and B is an 3 x 3 matrix

$$A = \begin{pmatrix} 2 & 0 & -4 \\ 3 & 1 & 7 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 2 & 8 \\ 7 & 4 & 3 \\ 3 & 9 & 6 \end{pmatrix}$$

The (1,2) entry of A is 0; $a_{1,2} = 0$

The (2,3) entry of A is 7; $a_{2,3} = 7$

The (3,1) entry of B is ?;

$b_{1,3} = ?$

Introduction to Matrix

- The diagonal entries in an $m \times n$ matrix $A = [a_{ij}]$ are a_{11}, a_{22}, a_{33} ... and they form the **main diagonal** of A.

$$A = \left[a_{ij}\right] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & .. & a_{1n} \\ a_{21} & a_{22} & a_{23} & .. & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & .. & a_{mn} \end{bmatrix}$$

Introduction to Matrix

- A **diagonal matrix** is a square matrix whose non-diagonal entries are zero.
- Example: *n*x*n* **identity matrix** I_n

$$I_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Introduction to Matrix

- The $m \times n$ matrix whose entries are all zero is called the **zero matrix** and will be denoted by 0.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Introduction to Matrix

Two matrices A and B are equal **(A=B)** if they have the same number of rows and columns and if corresponding entries are equal.

Example:

- Given the matrices of A, B and C

$$A = \begin{bmatrix} 2 & 0 & -4 \\ 3 & 1 & 7 \end{bmatrix} \quad ; B = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \quad ; C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- Discuss the possibility that A=B, A=C and B=C

Operation on Matrix: ADDITION

- If A and B are 2 matrices of the same size,

  A + B is defined to be the matrix of the same size formed by adding corresponding entries.

- If A=[a_{ij}] and B=[b_{ij}], [a_{ij}]+[b_{ij}]=[a_{ij}+b_{ij}]

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix} \qquad B = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \end{pmatrix}$$

$$A+B = \begin{pmatrix} a_{1,1} + b_{1,1} & a_{1,2} + b_{1,2} & a_{1,3} + b_{1,3} \\ a_{2,1} + b_{2,1} & a_{2,2} + b_{2,2} & a_{2,3} + b_{2,3} \end{pmatrix}$$

Operation on Matrix: SUBTRACTION

- A-B is defined by subtracting corresponding entries.
- If $A=[a_{ij}]$ and $B=[b_{ij}]$, $[a_{ij}]-[b_{ij}]=[a_{ij}-b_{ij}]$

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix} \qquad B = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \end{pmatrix}$$

$$A-B = \begin{pmatrix} a_{1,1}-b_{1,1} & a_{1,2}-b_{1,2} & a_{1,3}-b_{1,3} \\ a_{2,1}-b_{2,1} & a_{2,2}-b_{2,2} & a_{2,3}-b_{2,3} \end{pmatrix}$$

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Operation on Matrix

Example

$$A=\begin{bmatrix}3 & 2 & 0\\ 5 & 1 & -4\end{bmatrix} \qquad B=\begin{bmatrix}-1 & 4 & 2\\ 7 & -5 & 3\end{bmatrix}$$

$$A+B=\begin{bmatrix}3+(-1) & 2+4 & 0+2\\ 5+7 & 1+(-5) & -4+3\end{bmatrix}=\begin{bmatrix}2 & 6 & 2\\ 12 & -4 & -1\end{bmatrix}$$

$$A-B=\begin{bmatrix}3-(-1) & 2-4 & 0-2\\ 5-7 & 1-(-5) & -4-3\end{bmatrix}=\begin{bmatrix}4 & -2 & -2\\ -2 & 6 & -7\end{bmatrix}$$

Operation on Matrix - Properties

If A, B and C denote arbitrary $m \times n$ matrices, then

- $A + B = B + A$ (commutative law)
- $A + (B + C) = (A + B) + C$ (associative law)
- $0 + A = A$ (0 is the $m \times n$ zero matrix)
- $A + (-A) = 0$

Operation on Matrix: SCALAR MULTIPLICATION

- If A is a matrix and c is a number, the **scalar product** cA is the matrix formed from A by multiplying each entry of A by the number c.
- If $A = [a_{ij}]$, then $cA = c[aij]$

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix}$$

$$cA = c\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix} \Rightarrow cA = \begin{pmatrix} ca_{1,1} & ca_{1,2} & ca_{1,3} \\ ca_{2,1} & ca_{2,2} & ca_{2,3} \end{pmatrix}$$

Operation on Matrix: SCALAR MULTIPLICATION

Example:

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 5 & 1 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 4 & 2 \\ 7 & -5 & 3 \end{bmatrix}$$

a) $$2A = 2\begin{bmatrix} 3 & 2 & 0 \\ 5 & 1 & -4 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 0 \\ 10 & 2 & -8 \end{bmatrix}$$

b) $$3B = 3\begin{bmatrix} -1 & 4 & 2 \\ 7 & -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 12 & 6 \\ 21 & -15 & 9 \end{bmatrix}$$

c) $$3B - 2A = \begin{bmatrix} -3 & 12 & 6 \\ 21 & -15 & 9 \end{bmatrix} - \begin{bmatrix} 6 & 4 & 0 \\ 10 & 2 & -8 \end{bmatrix} = \begin{bmatrix} -9 & 8 & 6 \\ 11 & -17 & 17 \end{bmatrix}$$

Operation on Matrix: SCALAR MULTIPLICATION

Theorem

- Let A and B denote matrices and let c and d denote numbers/scalar:
  - $c(A+B) = cA + cB$
  - $(c+d)A = cA + dA$
  - $c(dA) = (cd)A$
- If $cA = 0$, then either $c = 0$ or $A = 0$
  - $c(0) = 0$
  - $0A = 0$

Operation on Matrix: MULTIPLICATION

- Matrix multiplication or the matrix product is a binary operation that produces a matrix from two matrices.

- If A is an $m \times n$ matrix and B is an $n \times p$ matrix, their matrix product (AB) is the $m \times p$ matrix, in which the n-entries across a row of A are multiplied with the n-entries down a column of B and summed to produce an entry of AB.

- The number of columns of the left matrix (n) must be same as the number of rows of the right matrix (n).

Operation on Matrix: MULTIPLICATION

Matrix of size 2x3 **Matrix of size 3x1** **Matrix of size 2x1**

$$\begin{bmatrix} 3 & 2 & 0 \\ 5 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x+2y \\ 5x+y-4z \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 0 \\ 5 & 1 & -4 \end{bmatrix} \begin{bmatrix} x & a \\ y & b \\ z & c \end{bmatrix} = \begin{bmatrix} 3x+2y & 3a+2b \\ 5x+y-4z & 5a+b-4c \end{bmatrix}$$

Matrix of size 2x3 **Matrix of size 3x2** **Matrix of size 2x2**

Operation on Matrix: TRANSPOSE

Given an $m \times n$ matrix A, the transpose of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

Example:

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 5 & 1 & -4 \end{bmatrix}$$

Row

$$A^T = \begin{bmatrix} 3 & 5 \\ 2 & 1 \\ 0 & -4 \end{bmatrix}$$

Column

$$B = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

$$B^T = \begin{bmatrix} w & y \\ x & z \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Operation on Matrix: TRANSPOSE

Theorem

- Let A and B denote matrices whose sizes are appropriate for the following sums and products.
  - $(AT)^T = A$
  - $(A + B)^{\,T} = AT + BT$
  - For any scalar $r, (rA)^T = rA^T$
  - $(AB)^T = BTA^T$

Exercises

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 5 & -1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 0 & -2 \\ 7 & 11 & 8 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 1 \\ 0 & 4 \\ 1 & 3 \end{bmatrix}$$

Given the matrix above, find the answer for the following operations:

(a) $A + B$
(b) $3A - B$
(c) AC
(d) $A^T + C$
(e) $A^T + BT$
(f) $A^T A$

CHAPTER 1

Part 2

Determinants and Matrix Inverse

Determinants of Matrix

- The determinant of a matrix is a special number that can be calculated from a square matrix.

- It tells us things about the matrix that are useful in systems of linear equations, helps us find the inverse of a matrix, is useful in calculus and more.

- Symbol: $\det(A)$ or $|A|$

Determinants of Matrix: 2×2

For a 2×2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the determinant is:

$$\det(A) = |A| = ad - bc$$

Determinants of Matrix: 2×2

Example:

Compute the determinant of

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix}$$

Solution:

$$|A| = (2)(4) - (1)(5) = 3$$

Determinants of Matrix: $n \geq 2$

For $n \geq 2$, the determinant of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det(A_{1j})$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A.

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + .. + (-1)^{1+n} a_{1n} \det A$$

Determinants of Matrix: $n \geq 2$

- The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column.
  Tips: If possible, choose row or column that contains the most zeros

- The expansion across the i-th row using the cofactor

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

- The cofactor expansion down the j-th column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

Determinants of Matrix: $n \geq 2$

Sub-Matrix

- For any square matrix A, let A_{ij} denote the sub-matrix formed by deleting the i-th row and j-th column of A.

Example:

Sub-matrix

$$A = \begin{bmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ 4 & \cancel{5} & 6 \\ 7 & \cancel{8} & 9 \end{bmatrix} \Rightarrow A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$$

Determinants of Matrix: $n \geq 2$

Cofactor

- Given $A = [a_{ij}]$
- The (i, j)-cofactor of A is $C_{i,j}$ given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

- The + or – sign in the (i, j)- cofactor

depends on the position of $a_{i,j}$ in the matrix.

$(-1)^{1+1}$

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Determinants of Matrix: 3×3

The determinant of a 3×3 matrix using first row expansion:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det A = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Determinants of Matrix: 3×3

Example:

Use a cofactor expansion across the 3rd row to compute $\det(A)$.

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Determinants of Matrix: 3×3

Example- Solution:

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ \boxed{0 \quad -2 \quad 0} \end{bmatrix}$$

$$\det A = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

$$= (-1)^{3+1} a_{31} \det A_{31} + (-1)^{3+2} a_{32} \det A_{32} + (-1)^{3+3} a_{33} \det A_{33}$$

$$= 0\begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + 0\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}$$

$$= 0 + 2(-1) + 0 = -2$$

Exercises

(a)
$$A = \begin{bmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{bmatrix}$$

(b)
$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix}$$

Compute the determinants of the matrices using a cofactor expansion

i. across the 1st row

ii. down the 2nd column

Matrix Inverse

- The inverse of a square matrix A, is a matrix A^{-1}, such that

$$AA^{-1} = \mathrm{I}$$

where I is the identity matrix.

- A square matrix A has an inverse iff the determinant $|A| \neq 0$

#iff = if and only if

Matrix Inverse: 2×2

- If A= $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and ad–bc $\neq 0$,

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- (ad–bc) is the determinant of A and $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ is the adjoint of A.

Matrix Inverse: 2×2

Example:

- Find the inverse of the matrix A= $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

Solution:

$$A^{-1} = \frac{1}{4.2 - 1.3}\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{-1}{5} \\ \frac{-3}{5} & \frac{4}{5} \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} \frac{2}{5} & \frac{-1}{5} \\ \frac{-3}{5} & \frac{4}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix Inverse: 3×3

Using cofactor matrix to compute the inverse:

$$\boxed{A^{-1} = \frac{1}{|A|}(\text{adjoint } A) = \frac{1}{|A|}(\text{cofactor matrix } A)^T}$$

Example:

Find the inverse of the matrix

$$A = \begin{pmatrix} 2 & 1 & 2 \\ -3 & -1 & -1 \\ 5 & 2 & 1 \end{pmatrix}$$

Matrix Inverse: 3×3

Example- Solution:

$$A = \begin{pmatrix} 2 & 1 & 2 \\ -3 & -1 & -1 \\ 5 & 2 & 1 \end{pmatrix}$$

Step 1: Find the determinant of A (in this case, using cofactor expansion across 1^{st} row)

$$|A| = 2\begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} -3 & -1 \\ 5 & 1 \end{vmatrix} + 2\begin{vmatrix} -3 & -1 \\ 5 & 2 \end{vmatrix}$$

$$= 2(-1+2) - 1(-3+5) + 2(-6+5)$$

$$= 2 - 2 - 2$$

$$= -2$$

Matrix Inverse: 3×3

Step 2: Find the determinant of sub-matrix (matrix of minors)

Row 1:

$$A_{11} = \begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix} = (-1+2) = 1$$

$$A_{12} = \begin{vmatrix} -3 & -1 \\ 5 & 1 \end{vmatrix} = (-3+5) = 2$$

$$A_{13} = \begin{vmatrix} -3 & -1 \\ 5 & 2 \end{vmatrix} = (-6+5) = -1$$

Row 2:

$$A_{21} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = (1-4) = -3$$

$$A_{22} = \begin{vmatrix} 2 & 2 \\ 5 & 1 \end{vmatrix} = (2-10) = -8$$

$$A_{23} = \begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix} = (4-5) = -1$$

Row 3:

$$A_{31} = \begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} = (-1+2) = 1$$

$$A_{32} = \begin{vmatrix} 2 & 2 \\ -3 & -1 \end{vmatrix} = (-2+6) = 4$$

$$A_{33} = \begin{vmatrix} 2 & 1 \\ -3 & -1 \end{vmatrix} = (-2+3) = 1$$

Matrix Inverse: 3 × 3

Step 3: Find the cofactors of matrix A
(it easy ! Just need to change the sign of alternate cell)

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ -3 & -8 & -1 \\ 1 & 4 & 1 \end{bmatrix} \Rightarrow \text{matrix of cofactors} = \begin{bmatrix} 1 & -2 & -1 \\ 3 & -8 & 1 \\ 1 & -4 & 1 \end{bmatrix}$$

Step 4: Find the adjoint of matrix A

$$\boxed{\text{Adjoint } A = (\text{cofactor matrix } A)^T}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ -2 & -8 & -4 \\ -1 & 1 & 1 \end{bmatrix}$$

Matrix Inverse: 3×3

Step 5: Calculate the inverse

$$\boxed{A^{-1} = \frac{1}{|A|}(\text{adjoint } A)}$$

$$A^{-1} = -\frac{1}{2}\begin{pmatrix} 1 & 3 & 1 \\ -2 & -8 & -4 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ 1 & 4 & 2 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Exercises

1) Find the inverse of matrix A.

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

2) $$A = \begin{bmatrix} -1 & 1 & 1 \\ 3 & -3 & 3 \\ -1 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 3 & -3 \\ 0 & -1 & 2 \\ 3 & -2 & -1 \end{bmatrix}$$

A and B are matrices. Find,

i) $(AB)^T$.
ii) Determinant of A, $|A|$.
iii) Inverse of A, A^{-1}.

CHAPTER 1

Part 3

Elementary Row Operations and Gaussian Elimination

Augmented Matrix in SLE

(Recall) – What is system of linear equations?

- A finite collection of linear equation is called a system of linear equations.

- A system of m equations in n variables:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

- This is also referred to as $m \times n$ linear system.

Augmented Matrix in SLE

In order to solve a linear system using algebra (i.e., elimination), we need to represent it as matrices:

$$\begin{bmatrix} a_{11} & .. & a_{1n} \\ .. & .. & .. \\ a_{m1} & .. & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ .. \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ .. \\ b_m \end{bmatrix}$$

A is the matrix of coefficients.

X is the matrix of variables.

B is the matrix of constants.

Augmented Matrix in SLE

- When solving a linear system by the elimination method, only the **coefficients** and the **constants** are needed to find the solution.
- This matrix is called the **Augmented matrix**, [*A*|*B*], of the linear system.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Augmented Matrix in SLE

Example

- A linear system,

$$
\begin{aligned}
x_1 + x_2 + 5x_3 &= 5 \\
3x_1 + 2x_2 - x_3 &= 10 \\
2x_1 - 4x_2 + x_3 &= 8
\end{aligned}
$$

$AX=B$

$$
\begin{bmatrix} 1 & 1 & 5 \\ 3 & 2 & -1 \\ 2 & -4 & 1 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} 5 \\ 10 \\ 8 \end{bmatrix}
\Rightarrow
\left[\begin{array}{ccc|c} 1 & 1 & 5 & 5 \\ 3 & 2 & -1 & 10 \\ 2 & -4 & 1 & 8 \end{array}\right]
$$

$[A|B]$

Row Echelon Matrices

A matrix is said to be row-echelon form if the following conditions are satisfied:

- Every row with all 0 entries is below every row with nonzero entries.
- Each leading entry of a row is in a column to the right of the leading entry of the row above it(leading entry-any nonzero value).
- The second row starts with more zero than the first(left to right).
- All entries in a column below a leading entry are zeros.

$$\begin{pmatrix} \# & * & * & * & * & * \\ 0 & 0 & \# & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \# \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

#: leading entry(left most non zero entry)

***:** any number/values including 0

Row Echelon Matrices

- A row-echelon matrix is said to be in **reduced row-echelon form** if, in addition, it satisfies:
  - The leading (leftmost non zero) entry in each non zero row is **1** (called the **leading 1**)
  - Each leading 1 is the only non-zero entry in its column.

- The matrices shown below are in **row-echelon** form.

$$\begin{bmatrix} 2 & 1 & 4 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Row Echelon Matrices

- The matrices shown below are in **reduced** row-echelon form.

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Row Echelon Matrices

- These matrices are **not** in reduced row-echelon form or row-echelon form.

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 4 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Elementary Row Operations

- Elementary Row Operations (ERO) can be used to transform the matrix into its row-echelon form.

- If we denote row 1 by R1, row 2 by R2, etc., and a is a scalar, the three elementary row operations are as follows:

  1) Swap two rows (equations), denoted R1↔R2 (this would swap rows 1 and 2).
  2) Multiply a row by a nonzero scalar, denoted aR1 (this would multiply row 1 by a).
  3) Add a multiple of a row to another row, denoted R1 + aR2 (this would add a multiply row 2 to row 1, and replace the previous row 1).

Gaussian Elimination

- Assume that a system of linear equations has one solution.
  - Carry the augmented matrix of the system to row-echelon form.
  - Assign the non-leading variables as parameters.
  - Solve the leading variables using back-substitution.

- Every matrix can be carried to row-echelon form by a sequence of elementary row operations:
  - Interchanging two rows.
  - Multiplying a row by a non-zero scalar.
  - Adding a multiple of one row to another row.

Gaussian Elimination

Example

Solve the following system of linear equations.

$$
\begin{aligned}
2x_1 + x_2 + x_3 &= 7 \\
3x_1 + 2x_2 - x_3 &= 4 \\
x_1 - 4x_2 + 2x_2 &= -1
\end{aligned}
$$

Gaussian Elimination

Example - Solution

Step 1: Convert to augmented form:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 7 \\ 3x_1 + 2x_2 - x_3 &= 4 \\ x_1 - 4x_2 + 2x_3 &= -1 \end{aligned} \rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 3 & 2 & -1 & 4 \\ 1 & -4 & 2 & -1 \end{array}\right]$$

Step 2: Apply ERO

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 3 & 2 & -1 & 4 \\ 1 & -4 & 2 & -1 \end{array}\right) \quad R_1 \longleftrightarrow R_3 \quad \left(\begin{array}{ccc|c} 1 & -4 & 2 & -1 \\ 3 & 2 & -1 & 4 \\ 2 & 1 & 1 & 7 \end{array}\right)$$

Gaussian Elimination

Example – Solution (cont'd)

Calculate the scalar, $m_{21} = -\frac{a_{21}}{a_{11}} = -\frac{3}{1} = -3,$ then update row R2:

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & -1 \\ 3 & 2 & -1 & 4 \\ 2 & 1 & 1 & 7 \end{array}\right] \Rightarrow R2 \to R2 + m_{21}R1$$

3+(-3)(1)=0
2+(-3)(-4)=14
-1+(-3)(2)=-7
4+(-3)(-1)=7

Calculate the scalar, $m_{31} = -\frac{a_{31}}{a_{11}} = -\frac{2}{1} = -2$, then update row R3:

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & -1 \\ 0 & 14 & -7 & 7 \\ 0 & 9 & -3 & 9 \end{array}\right] \Rightarrow R3 \to R3 + m_{31}R1$$

2+(-2)(1)=0
1+(-2)(-4)=9
1+(-2)(2)=-3
7+(-2)(-1)=9

Gaussian Elimination

Example – Solution (cont'd)

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & -1 \\ 0 & 14 & -7 & 7 \\ 0 & 9 & -3 & 9 \end{array}\right] \quad R2 \rightarrow (\tfrac{1}{7})R2 \quad \left[\begin{array}{ccc|c} 1 & -4 & 2 & -1 \\ 0 & 2 & -1 & 1 \\ 0 & 9 & -3 & 9 \end{array}\right]$$

Calculate the scalar, $m_{32} = -\dfrac{a_{32}}{a_{22}} = -\dfrac{9}{2}$; then update row R3:

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & -1 \\ 0 & 2 & -1 & 1 \\ 0 & 9 & -3 & 9 \end{array}\right] \Rightarrow \quad R3 \rightarrow R3 + m_{32}R2$$

0+(-9/2)(0)=0
9+(-9/2)(2)=0
-3+(-9/2)(-1)=3/2
9+(-9/2)(1)=9/2

Gaussian Elimination

Example – Solution (cont'd)

At the last step, we had obtained in the row echelon form.

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & -1 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & \frac{3}{2} & \frac{9}{2} \end{array}\right] \rightarrow \begin{aligned} x_1 - 4x_2 + 2x_3 &= -1 \\ 2x_2 - x_3 &= 1 \\ \tfrac{3}{2}x_3 &= \tfrac{9}{2} \end{aligned}$$

Use **back substitution** to get the solutions.

$$x_3 = 3 \;;\qquad x_2 = \frac{1+3}{2} = 2 \;;\qquad x_1 = -1 + 4(2) - 2(3) = 1$$

Gaussian Elimination

Example

Solve the system of linear equations

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 10$$
$$x_1 + 2x_2 + 4x_3 + 5x_4 = 8$$
$$2x_1 + 4x_2 + 6x_3 + 8x_4 = 20$$

Gaussian Elimination

Example - Solution

1) The augmented matrix for this system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 10 \\ x_1 + 2x_2 + 4x_3 + 5x_4 &= 8 \\ 2x_1 + 4x_2 + 6x_3 + 8x_4 &= 20 \end{aligned} \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 1 & 2 & 4 & 5 & 8 \\ 2 & 4 & 6 & 8 & 20 \end{array}\right]$$

Gaussian Elimination

Example – Solution (cont'd)

$$R2 \rightarrow R2 + m_{21}R1$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 1 & 2 & 4 & 5 & 8 \\ 2 & 4 & 6 & 8 & 20 \end{array}\right]$$

$$m_{21} = -\frac{a_{21}}{a_{11}} = -\frac{1}{1} = -1$$

1+(-1)(1)=0
2+(-1)(2)=0
4+(-1)(3)=1
5+(-1)(4)=1
8+(-1)(10)=-2

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 0 & 0 & 1 & 1 & -2 \\ 2 & 4 & 6 & 8 & 20 \end{array}\right]$$

Gaussian Elimination

Example – Solution (cont'd)

$$R3 \rightarrow R3 + m_{31}R1$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 0 & 0 & 1 & 1 & -2 \\ 2 & 4 & 6 & 8 & 20 \end{array}\right]$$

$$m_{31} = -\frac{a_{31}}{a_{11}} = -\frac{2}{1} = -2$$

2+(-2)(1)=0
4+(-2)(2)=0
6+(-2)(3)=0
8+(-2)(4)=0
20+(-2)(10)=0

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Gaussian Elimination

Example – Solution (cont'd)

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let non-leading variables

$x_2 = s$ and $x_4 = t$

(assign non-leading variables as parameter)

leading variables,

(solve leading variables using back-substitutions)

$$x_3 = -2 - t$$

$$x_1 = 10 - 2s - 3(-2 - t) - 4t = 16 - 2s - t$$

Exercises #1

Consider the linear system

$$
\begin{aligned}
x + 2y - 3z &= 1 \\
2x + 5y - 8z &= 4 \\
-2x - 4y + 6z &= -2
\end{aligned}
$$

i) Define the coefficient matrix A for the linear system.
ii) Solve the linear system using Gaussian elimination.

Exercise #2

Use Gaussian Elimination to solve the following systems of linear equations.

(a)
$$
\begin{aligned}
x_1 + x_2 - x_3 &= 1 \\
2x_1 - x_2 + x_3 &= -1 \\
-x_1 - x_2 + 3x_3 &= 2
\end{aligned}
$$

(b)
$$
\begin{aligned}
-2x - 2y + 2z &= 1 \\
x \qquad\quad + 5z &= -1 \\
3x + 2y + 3z &= -2
\end{aligned}
$$

CHAPTER 1

Part 4

Gauss-Jordan Elimination and Matrix Factorization

Gauss-Jordan Elimination

- In Gauss-Jordan elimination, elements above the diagonal are eliminated in the same manner as are elements below the diagonal, thus avoiding the back substitution phase of the solution process.

- The elimination step results in an **identity matrix** rather than a triangular matrix.

Gauss-Jordan Elimination

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} u_{11} & u_{12} & u_{13} & d_1 \\ 0 & u_{22} & u_{23} & d_2 \\ 0 & 0 & u_{33} & d_3 \end{array}\right]$$

Gaussian Elimination

$$\left[\begin{array}{ccc|c} u_{11} & u_{12} & u_{13} & d_1 \\ 0 & u_{22} & u_{23} & d_2 \\ 0 & 0 & u_{33} & d_3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} d_{11} & 0 & 0 & e_1 \\ 0 & d_{22} & 0 & e_2 \\ 0 & 0 & d_{33} & e_3 \end{array}\right]$$

Gauss-Jordan Elimination

Gauss-Jordan Elimination

$$\left[\begin{array}{ccc|c} d_{11} & 0 & 0 & e_1 \\ 0 & d_{22} & 0 & e_2 \\ 0 & 0 & d_{33} & e_3 \end{array}\right] \rightarrow \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

$$d_{11}x_1 = e_1$$
$$d_{22}x_2 = e_2$$
$$d_{33}x_3 = e_3$$

$$x_1 = \frac{e_1}{d_{11}}, \quad x_2 = \frac{e_2}{d_{22}}, \quad x_3 = \frac{e_3}{d_{33}}$$

Gauss-Jordan Elimination

Example

Given the following matrix which is a result from Gaussian elimination process to solve a particular linear system. Proceed to Gauss-Jordan elimination until you obtain the solution.

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & -2 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & -31 & 13 \end{array}\right]$$

Gauss-Jordan Elimination

Example - Solution

- Column 2, row 1:

$$m_{12} = -\frac{a_{12}}{a_{22}} = -\frac{-2}{1} = 2$$

$$\left[\begin{array}{ccc|c} 1 & \boxed{-2} & 4 & -2 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & -31 & 13 \end{array}\right]$$

R1+ m_{12}R2 = R1

1+(2)(0)= 1
-2+(2)(1)= 0
4+(2)(5)=14
-2+(2)(-1)=-4

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 14 & -4 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & -31 & 13 \end{array}\right]$$

Gauss-Jordan Elimination

Example – Solution (cont'd)

- Column 3, row 1:

$$m_{13} = -\frac{a_{13}}{a_{33}} = -\frac{14}{-31} = \frac{14}{31}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \boxed{14} & -4 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & -31 & 13 \end{array}\right]$$

R1+ m_{13}R3 = R1

1+(14/31)(0)= 1
0+(14/31)(0)= 0
14+(14/31)(-31)=0
-4+(14/31)(13)=58/31

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 58/31 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & -31 & 13 \end{array}\right]$$

Gauss-Jordan Elimination

Example – Solution (cont'd)

- Column 3, row 3:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 58/31 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & \boxed{-31} & 13 \end{array}\right]$$

R3 ÷ (-31) = R3

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 58/31 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 1 & 13/-31 \end{array}\right]$$

Gauss-Jordan Elimination

Example – Solution (cont'd)

- Column 3, row 2:

$$m_{23} = -\frac{a_{23}}{a_{33}} = -\frac{5}{1} = -5$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 58/31 \\ 0 & 1 & \boxed{5} & -1 \\ 0 & 0 & 1 & 13/-31 \end{array}\right]$$

R2+ m_{23}R3 = R2

0+(-5)(0)= 0
1+(-5)(0)= 1
5+(-5)(1)=0
-1+(-5)(13/-31)=34/31

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 58/31 \\ 0 & 1 & 0 & 34/31 \\ 0 & 0 & 1 & 13/-31 \end{array}\right]$$

Gauss-Jordan Elimination

Example – Solution (cont'd)

- Solution:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{58}{31} \\ 0 & 1 & 0 & \frac{34}{31} \\ 0 & 0 & 1 & -\frac{13}{31} \end{array}\right] \Rightarrow x = \left[\frac{58}{31}, \frac{34}{31}, -\frac{13}{31}\right]^T$$

Gauss-Jordan Elimination

- Although the Gauss-Jordan technique and Gauss elimination might appear almost identical, the former requires more work.
- Gauss-Jordan involves approximately 50% more operations than Gauss elimination.
- Therefore, Gauss elimination is the simple elimination method of preference for obtaining solutions of linear equations.

Exercise

Solve the following linear system using Gauss-Jordan elimination.

$$2x_1 + 4x_2 - 6x_3 = -4$$
$$x_1 + 5x_2 + 3x_3 = 10$$
$$x_1 + 3x_2 + 2x_3 = 5$$

Use fraction in your calculations. Check your answers by substituting them into the original equations.

Matrix Inverse using Linear System

Suppose A is a square matrix and there exists a sequence of elementary row operations that carry $A \rightarrow I$.

Then A is invertible and this same sequence carries $I \rightarrow A^{-1}$.

$$[\,AI\,] \rightarrow [\,IA^{-1}]$$

where the row operations on A and I are carried out simultaneously.

Matrix Inverse using Linear System

Example

Find the inverse of the matrix,

$$A = \begin{bmatrix} 1 & 4 & -1 \\ 2 & 7 & 1 \\ 1 & 3 & 0 \end{bmatrix}$$

Matrix Inverse using Linear System

Example - Solution

$$[\,A \mid I\,]$$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 2 & 7 & 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

Target $\Rightarrow$ $[\,I \mid A^{-1}\,]$

Matrix Inverse using Linear System

Example – Solution (cont'd)

$$\left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 2 & 7 & 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$R2 - 2R1 = R2$$

2 -2(1) =0
7 -2(4) =-1
1 -2(-1)=3

0 -2(1) =-2
1 -2(0) =1
0 -2(0) =0

$$\left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

Matrix Inverse using Linear System

Example – Solution (cont'd)

$$\left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$R3 - R1 = R3$$

1 - 1 =0
3 - 4 =-1
0 - (-1) =1

0 - 1 = -1
0 - 0 = 0
1 - 0 = 1

$$\left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right]$$

Matrix Inverse using Linear System

Example – Solution (cont'd)

$$\left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$R3 - R2 = R3$$

0 – (0) = 0
-1- (-1) = 0
1 – (3) = -2

-1 – (-2) = 1
0 – (1) = -1
1 – (0) = 1

$$\left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 & -1 & 1 \end{array}\right]$$

Matrix Inverse using Linear System

Example – Solution (cont'd)

$$\left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 & -1 & 1 \end{array}\right]$$

$$R1 + 4R2 = R1$$

1 + 4(0) = 1
4 + 4(-1)= 0
-1+ 4(3) =11

1 +4(-2)= -7
0 +4(1) = 4
0 +4(0) = 0

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 11 & -7 & 4 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 & -1 & 1 \end{array}\right]$$

Matrix Inverse using Linear System

Example – Solution (cont'd)

$$R2 \times (-1) = R2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 11 & -7 & 4 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 11 & -7 & 4 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & -2 & 1 & -1 & 1 \end{array}\right]$$

$$R3 \div (-2) = R3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 11 & -7 & 4 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & -2 & 1 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 11 & -7 & 4 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \end{array}\right]$$

Matrix Inverse using Linear System

Example – Solution (cont'd)

$$\left[\begin{array}{ccc|ccc} \boxed{1 & 0 & 11 & -7 & 4 & 0} \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \end{array}\right]$$

$$R1 - 11R3 = R1$$

1 - 11(0) = 1
0 - 11(0) = 0
11 - 11(1) = 0

-7 -11(-1/2) = -3/2
4 -11 (1/2) = -3/2
0 -11(-1/2) = 11/2

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-3}{2} & \frac{-3}{2} & \frac{11}{2} \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \end{array}\right]$$

Matrix Inverse using Linear System

Example – Solution (cont'd)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-3}{2} & \frac{-3}{2} & \frac{11}{2} \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \end{array}\right] \rightarrow R2+3R3=R2$$

0 + 3(0) = 0
1 + 3(0) = 1
-3+ 3(1) = 0

2 +3(-1/2) = 1/2
-1 +3(1/2) = 1/2
0 +3(-1/2) =-3/2

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-3}{2} & \frac{-3}{2} & \frac{11}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{-3}{2} \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \end{array}\right]$$

Matrix Inverse using Linear System

Example – Solution (cont'd)

$$[\,A \mid I\,] \qquad\qquad [\,I \mid A^{-1}\,]$$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 2 & 7 & 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Target}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-3}{2} & \frac{-3}{2} & \frac{11}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{-3}{2} \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \end{array}\right]$$

Exercise

Find the inverse of the matrix,

$$A= \begin{pmatrix} 2 & 1 & 2 \\ -3 & -1 & -1 \\ 5 & 2 & 1 \end{pmatrix}$$

using the linear system.

Matrix Factorization

- The solution of a system of linear equations $AX = B$ can be computed much more quickly if the matrix A can be factored in the form

$$A = LU$$

where

L is a lower triangular matrix

U is an upper triangular matrix

Matrix Factorization

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow (Ax = B)$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \Rightarrow (A = LU)$$

Matrix Factorization

$$L \times U = A$$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} l_{11}u_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + l_{22}u_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + l_{33}u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Matrix Factorization

$AX = B$ can be solved in 2 stages:

1) First solve $LY = B$ for Y by forward substitution.

2) Then solve $UX = Y$ for X by back substitution.

Matrix Factorization

Stage 1:

$$L \times Y = B$$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Forward substitution:

$$y_1 = \frac{b_1}{l_{11}},$$

$$y_2 = \frac{b_2 - l_{21} y_1}{l_{22}},$$

$$y_3 = \frac{b_3 - l_{31} y_1 - l_{32} y_2}{l_{33}}$$

Matrix Factorization

Stage 2:

$$U \times X = Y$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Back substitution:

$$x_3 = \frac{y_3}{u_{33}},$$

$$x_2 = \frac{y_2 - u_{23}x_3}{u_{22}},$$

$$x_1 = \frac{y_1 - u_{12}x_2 - u_{13}x_3}{u_{11}}$$

Matrix Factorization - Doolittle

The diagonal elements of matrix L are 1's.

$$\begin{array}{ccc} A & & L & U \end{array}$$

$$\begin{bmatrix} a_{11} & a_{12} & .. & a_{1n} \\ a_{21} & a_{22} & .. & a_{2n} \\ : & : & .. & : \\ a_{n1} & a_{n2} & .. & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & .. & 0 \\ l_{21} & 1 & .. & 0 \\ : & : & .. & : \\ l_{n1} & l_{n2} & .. & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & .. & u_{1n} \\ 0 & u_{22} & .. & u_{2n} \\ : & : & .. & : \\ 0 & 0 & .. & u_{nn} \end{bmatrix}$$

Matrix Factorization - Doolittle

$$L \times U = A$$

$$\begin{bmatrix} 1 & 0 & .. & 0 \\ l_{21} & 1 & .. & 0 \\ \vdots & \vdots & .. & \vdots \\ l_{n1} & l_{n2} & .. & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & .. & u_{1n} \\ 0 & u_{22} & .. & u_{2n} \\ \vdots & \vdots & .. & \vdots \\ 0 & 0 & .. & u_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & .. & a_{1n} \\ a_{21} & a_{22} & .. & a_{2n} \\ \vdots & \vdots & .. & \vdots \\ a_{n1} & a_{n2} & .. & a_{nn} \end{bmatrix}$$

Matrix Factorization - Doolittle

Example

Find the LU factorization of the matrix using Doolittle form and use it to solve the linear system.

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 14 \end{bmatrix}$$

Matrix Factorization - Doolittle

Example -Solution

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 4 & 2 \end{bmatrix}$$

$$u_{11} = 1 \qquad u_{12} = 2 \qquad u_{13} = 1$$

Matrix Factorization - Doolittle

Example –Solution (cont'd)

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 4 & 2 \end{bmatrix}$$

$$l_{21} = \frac{3}{1} = 3 \qquad l_{31} = \frac{1}{1} = 1$$

Matrix Factorization - Doolittle

Example –Solution (cont'd)

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & l_{32} & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 4 & 2 \end{bmatrix}$$

$$u_{22} = 1 - (3)(2) = -5 \qquad u_{23} = 1 - (3)(1) = -2$$

Matrix Factorization - Doolittle

Example –Solution (cont'd)

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & l_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -2 \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 4 & 2 \end{bmatrix}$$

$$l_{32} = \frac{4-(1)(2)}{-5} = -\frac{2}{5}$$

Matrix Factorization - Doolittle

Example –Solution (cont'd)

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & \frac{-2}{5} & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -2 \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 4 & 2 \end{bmatrix}$$

$$u_{33} = 2 - (1)(1) - \left(-\frac{2}{5}\right)(-2) = \frac{1}{5}$$

Matrix Factorization - Doolittle

Example –Solution (cont'd)

Answer for Doolittle factorization:

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & \frac{-2}{5} & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -2 \\ 0 & 0 & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 4 & 2 \end{bmatrix}$$

Matrix Factorization - Doolittle

Example –Solution (cont'd)

Proceed to solve the linear equation system.

Stage 1: $LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & -\frac{2}{5} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 14 \end{bmatrix}$$

$$y_1 = 8$$

$$y_2 = 11 - (3)(8) = -13$$

$$y_3 = 14 - (1)(8) - (-\tfrac{2}{5})(-13) = \tfrac{4}{5}$$

$$Y = \begin{bmatrix} 8 & -13 & \frac{4}{5} \end{bmatrix}^T$$

Matrix Factorization - Doolittle

Example –Solution (cont'd)

Stage 2: $UX = Y$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -2 \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -13 \\ \frac{4}{5} \end{bmatrix}$$

$$x_3 = \frac{\frac{4}{5}}{\frac{1}{5}} = 4$$

$$x_2 = \frac{-13-(-2)(4)}{-5} = 1$$

$$x_1 = \frac{8-(2)(1)-(1)(4)}{1} = 2$$

$$X = \begin{bmatrix} 2 & 1 & 4 \end{bmatrix}^T$$

Exercise

Find the LU factorization of the matrix A using Doolittle form and use it to solve the linear system $AX = B$.

$$A = \begin{bmatrix} 1 & 1 & 6 \\ -1 & 2 & 9 \\ 1 & -2 & 3 \end{bmatrix} \quad ; B = [7 \;\; 2 \;\; 10]^T$$

Matrix Factorization - Crout

The diagonal elements of matrix U are 1's.

$$L \times U = A$$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Matrix Factorization - Crout

Example

Find the LU factorization of the matrix using Crout form and use it to solve the linear system.

$$\begin{bmatrix} 7 & 2 & -3 \\ 2 & 5 & -3 \\ 1 & -1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -12 \\ -20 \\ -26 \end{bmatrix}$$

Matrix Factorization - Crout

Example –Solution

$$L \times U = A$$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -3 \\ 2 & 5 & -3 \\ 1 & -1 & -6 \end{bmatrix}$$

Matrix Factorization - Crout

Example –Solution (cont'd)

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -3 \\ 2 & 5 & -3 \\ 1 & -1 & -6 \end{bmatrix}$$

$l_{11} = 7$ $\qquad$ $l_{21} = 2$ $\qquad$ $l_{31} = 1$

Matrix Factorization - Crout

Example –Solution (cont'd)

$$\begin{bmatrix} 7 & 0 & 0 \\ 2 & l_{22} & 0 \\ 1 & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -3 \\ 2 & 5 & -3 \\ 1 & -1 & -6 \end{bmatrix}$$

$$u_{12} = \frac{2}{7} \qquad u_{13} = \frac{-3}{7}$$

Matrix Factorization - Crout

Example –Solution (cont'd)

$$\begin{bmatrix} 7 & 0 & 0 \\ 2 & l_{22} & 0 \\ 1 & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} 1 & 2/7 & -3/7 \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -3 \\ 2 & 5 & -3 \\ 1 & -1 & -6 \end{bmatrix}$$

$$l_{22} = 5 - (2)\frac{2}{7} = \frac{31}{7}$$

$$\frac{31}{7}u_{23} = -3 - \frac{3}{7}(2)$$

$$u_{23} = -\frac{15}{31}$$

Matrix Factorization - Crout

Example –Solution (cont'd)

$$\begin{bmatrix} 7 & 0 & 0 \\ 2 & 31/7 & 0 \\ 1 & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & 2/7 & -3/7 \\ 0 & 1 & -15/31 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -3 \\ 2 & 5 & -3 \\ 1 & -1 & -6 \end{bmatrix}$$

$$l_{32} = -1 - \frac{2}{7} = -\frac{9}{7}$$

$$l_{33} = -6 + \frac{3}{7} - \frac{(-9)}{7}\frac{(-15)}{31} = \frac{-192}{31}$$

Matrix Factorization - Crout

Example –Solution (cont'd)

Answer for Crout factorization:

$$\begin{bmatrix} 7 & 0 & 0 \\ 2 & 31/7 & 0 \\ 1 & -9/7 & -192/31 \end{bmatrix} \begin{bmatrix} 1 & 2/7 & -3/7 \\ 0 & 1 & -15/31 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -3 \\ 2 & 5 & -3 \\ 1 & -1 & -6 \end{bmatrix}$$

Matrix Factorization - Crout

Example –Solution (cont'd)

Proceed to solve the linear equation system.

Stage 1: $LY = B$

$$\begin{bmatrix} 7 & 0 & 0 \\ 2 & 31/7 & 0 \\ 1 & -9/7 & -192/31 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -12 \\ -20 \\ -26 \end{bmatrix}$$

$$y_1 = -\frac{12}{7},$$

$$y_2 = -\frac{116}{31},$$

$$y_3 = \frac{902}{192}$$

Matrix Factorization - Crout

Example –Solution (cont'd)

Stage 2: $UX = Y$

$$\begin{bmatrix} 1 & 2/7 & -3/7 \\ 0 & 1 & -15/31 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -12/7 \\ -116/31 \\ 902/192 \end{bmatrix}$$

$$x_3 = \frac{902}{192}$$

$$x_2 = \frac{-116}{31} + \frac{15}{31}\left(\frac{902}{192}\right) = \frac{-282}{192}$$

$$x_1 = \frac{-12}{7} - \frac{2}{7}\left(\frac{-282}{192}\right) + \frac{3}{7}\left(\frac{902}{192}\right) = \frac{138}{192}$$

Exercise #1

Find the LU factorization of the matrix A using Crout form and use it to solve the linear system $AX = B$.

$$A = \begin{bmatrix} 2 & 4 & -6 \\ 1 & 5 & 3 \\ 1 & 3 & 2 \end{bmatrix}; \quad B = \begin{bmatrix} -4 \\ 10 \\ 5 \end{bmatrix}$$

Exercise #2

Given a matrix, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{bmatrix}$; **and**

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & l_{22} & 0 \\ 3 & l_{32} & l_{33} \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Complete the LU decomposition for A.