CHAPTER 6

Non-Linear Equations

Outline

Introduction

- Finding a solution to an equation $f(x)$, is finding the value of x when $f(x) = 0$.

$$f(x) = 6x - 12$$
$$0 = 6x - 12$$
$$6x = 12$$
$$x = 2$$

For a linear equation, we can bring x to the left of equation to solve it.

For a quadratic equation (polynomial degree=2), we can factorise it to solve the equation.

$$f(x) = x^2 - 5x + 6$$
$$0 = x^2 - 5x + 6$$
$$x^2 - 5x + 6 = 0$$
$$(x-2)(x-3) = 0$$
$$\Longrightarrow x_0 = 2 \quad , x_0 = 3$$

Introduction

How about if the equation is a polynomial with degree >2?

$$f(x) = 3x^4 + 6x^3 + 4x^2 - 9x - 12$$

$$f(x) = x^3 - 6x^2 + 3x + 10$$

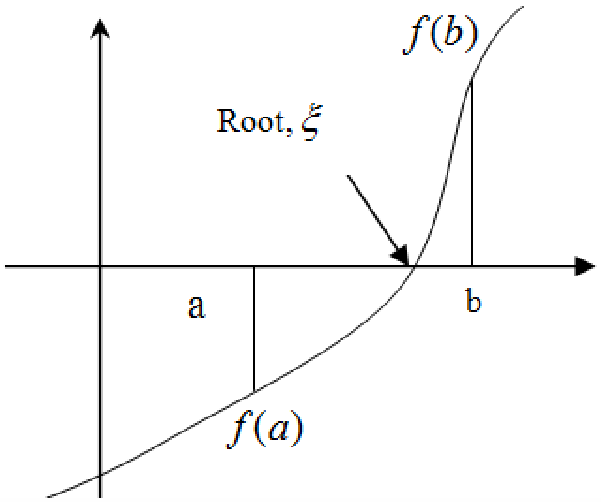
NLE Solution Using Numerical Method

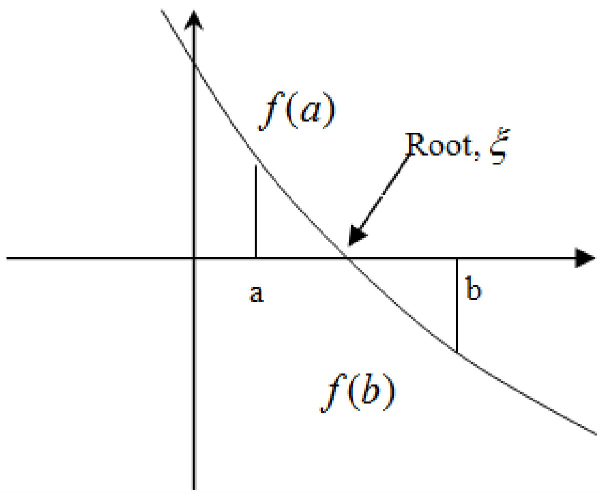
- Two kinds of numerical method to solve Non-Linear Equation. (NLE):
  - Bracketing methods:
    - Need to solve intermediate value theorem first
    - **Bisection Method**
    - **False Positive Method**
  - Fixed-Point iteration method
    - **Secant Method**
    - **Newton's Method**

Intermediate Value Theorem

Definition:

If $f(x)$ is a continuous function at interval $[a, b]$ and $f(a)$ and $f(b)$ is on opposite direction or sign, $f(a).f(b) < 0$ therefore a root, ξ is exist between the interval $[a. b]$.





Intermediate Value Theorem

Example:

Using the intermediate value theorem, check whether $f(x) = x^2 + 2x - 1$ contains real root in interval:

i. [-1,0]
ii. [0,1]
iii. [1,2]

Intermediate Value Theorem

Example – Solution (i):

$[-1,0]$

Given $f(x) = x^2 + 2x - 1$

Assume $a = -1$ and $b = 0$

So

$$f(a) = f(-1) = (-1)^2 + 2(-1) - 1$$
$$= 1 - 2 - 1$$
$$= -2$$

so $f(-1) = -2$

$$f(b) = f(0) = (0)^2 + 2(0) - 1$$
$$= 0 - 0 - 1$$
$$= -1$$

so, $f(0) = -1$

Since both value of $f(a)$ and $f(b)$ is negative and does not meet intermediate value theorem, so there is no real root in interval $[a, b]$ that is [-1,0].

Intermediate Value Theorem

Example – Solution (i):

Prove can also be performed by:

$$
\begin{aligned}
f(a).f(b) &= f(-1).f(0) \\
&= (-2)(-1) \\
&= 2 > 0
\end{aligned}
$$

Since $f(a).\ f(b) > 0$, does not meet intermediate value theorem, we conclude that no real root in interval [-1,0].

Intermediate Value Theorem

Example – Solution (ii):

$[0,1]$

Given $f(x) = x^2 + 2x - 1$

Assume $a = 0\ \ 1$ and $b = 1$

So

$$f(a) = f(0) = (0)^2 + 2(0) - 1$$
$$= 0 - 0 - 1$$
$$= -1$$

so $f(0) = -1$

$$f(b) = f(1) = (1)^2 + 2(1) - 1$$
$$= 1 + 2 - 1$$
$$= 2$$

so, $f(0) = 2$

Both value of $f(a)$ and $f(b)$ do not in same sign or direction, that is both in opposite direction. Since intermediate value theorem is meet, so there at least one real root for the continuous function $f(x)$ in interval [0,1].

Intermediate Value Theorem

Example – Solution (ii):

Prove can also be performed by:

$$
\begin{aligned}
f(a).f(b) &= f(0).f(1) \\
&= (-1)(2) \\
&= -2 < 0
\end{aligned}
$$

Since $f(a).\ f(b) < 0$, meet the intermediate value theorem, we conclude that there is at least one real root in interval [0,1].

Intermediate Value Theorem

Example – Solution (iii):

[1,2]

Given $f(x) = x^2 + 2x - 1$

Assume $a = 1$ 1 and $b = 2$

So

$$f(a) = f(1) = (1)^2 + 2(1) - 1$$
$$= 1 + 2 - 1$$
$$= 2$$

so $f(1) = 2$

$$f(b) = f(2) = (2)^2 + 2(2) - 1$$
$$= 4 + 4 - 1$$
$$= 7$$

so, $f(2) = 7$

Since both value of $f(a)$ and $f(b)$ is positive and does not meet intermediate value theorem, so there is no real root in interval $[a, b]$ that is [1,2].

Intermediate Value Theorem

Example – Solution (iii):

Prove can also be performed by:

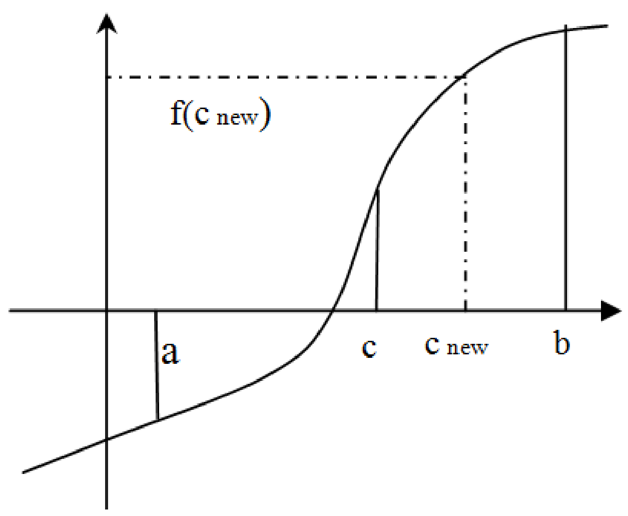
$$f(a).f(b) = f(1).f(2)$$
$$= (2)(7)$$
$$= 14 > 0$$

Since $f(a).\ f(b) > 0$, does not meet intermediate value theorem, we conclude that no real root in interval [1,2].

Bisection Method

- Have to check intermediate value theorem first.
- The root can be obtained through **midpoint** for $[a, b]$ which is

$$c = \frac{a+b}{2}$$



Bisection Method

$f(a) = -ve$ and $f(b) = +ve$, there are real root that lies in the interval [a, b]. Bisection method divides the interval $[a,b]$ into two equal halves of $[a,c]$ and $[c,b]$.To determine whether the point x_c is a point that we want to find, we need to substitute the value of c into the $f(x)$. There are three possibilities of $f(c)$ that might arise:

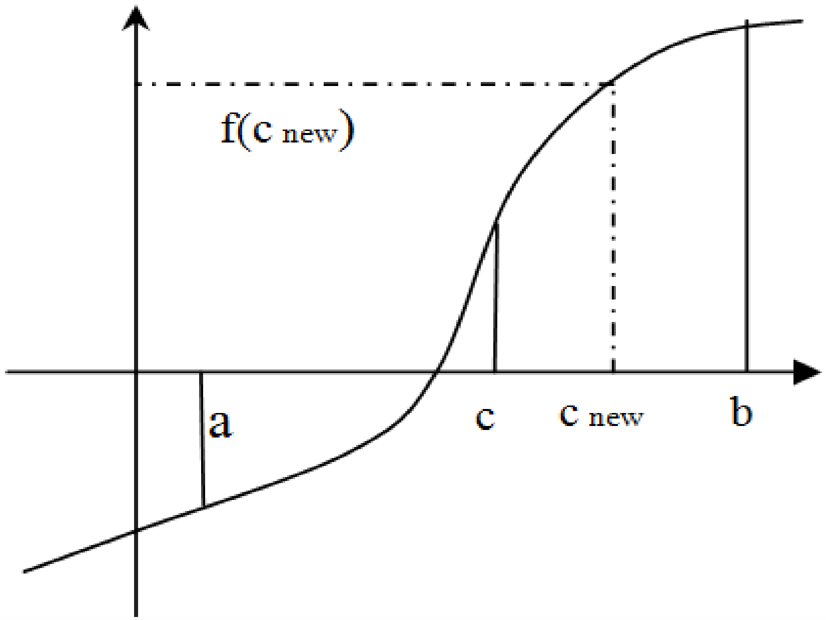
i. $f(c) = 0$

ii. $f(c) < 0$, or

iii. $f(c) > 0$

If $f(c) = 0$, then $\boldsymbol{c}$ = root.
Otherwise, compute new $\boldsymbol{c}$:

$$\boldsymbol{c = \frac{a+b}{2}}$$



Bisection Method

If $f(a) < 0$ and $f(b) > 0$ then there might be three possibilities arise:

i. $f(c) = 0$, then $\xi = c$,

ii. $f(c) < 0$, then $\xi \in [c, b]$, and

iii. $f(c) > 0$, then $\xi \in [a, c]$

Bisection Method

But otherwise if $f(a) > 0$ and $f(b) < 0$, then there are three possibilities:

i. $f(c) = 0$, then $\xi = c$,

ii. $f(c) < 0$, then $\xi \in [a, c]$, and

iii. $f(c) > 0$, then $\xi \in [c, b]$

Convergence Criteria

- When the root is determined by **indirect methods**, the real root that obtained through **iteration process** is repeated until the root was discovered in a specified accuracy limits.
- This limit is called the **convergence criteria** which is usually obtained by the following procedure:

i) Define $\varepsilon > 0$. Stops when the absolute value of the function at a point which is less than ε: $|f(x_k)| < \varepsilon$

ii) Specify a number > 0. Stops when the absolute value of difference between two successive approximation x_k and x_{k+1} is less than ε: $|x_k - x_{k+1}| < \varepsilon$.

Convergence Criteria

- In addition to the above criteria, we can also use the numerator criteria for determining the number of iteration. The formula is as follows:

$$\frac{b-a}{2^n} \leq \varepsilon$$

- This criteria can also be used to stop the equations that have no solution.

Convergence Criteria

Example:

Suppose we have $f(x) = x^3 - 3x^2 + 8x - 5$, the root is obtained using bisection method that lies in the interval $[0,1]$ and approximates to two decimal places of accuracy. Check whether $f(x)$ satisfy the intermediate value theorem on the interval $[0,1]$

Convergence Criteria

Example - Solution:

Substitute $a = 0$ and $b = 1$ into $f(x) = x^3 - 3x^2 + 8x - 5$

$f(0) = (0)^3 - 3(0)^2 + 8(0) - 5 = -5$

$f(1) = (1)^3 - 3(1)^2 + 8(1) - 5 = 1$

.: $f(a).f(b) = (-5)(1) = -5 < 0$, then the intermediate value theorem is satisfied. **Bisection method** can be used to find the root of $f(x)$ lies in the interval $[0,1]$.

Since it approximates to two d.p, the value $\varepsilon = \frac{1}{2} \times 10^2 = 0.005$

Convergence Criteria

Example - Solution:

$$c = \frac{a+b}{2}$$

then $c = \frac{0+1}{2} = 0.5$ and substitute $c = 0.5$ into $f(x)$ and identify the interval for new root

$$\begin{aligned} f(0.5) &= 0.5^3 - 3(0.5)^2 + 8(0.5) - 5 \\ &= 0.125 - 0.75 + 4 - 5 \\ &= -1.625 \end{aligned}$$

Since $f(0.5) = -1.625 > \varepsilon$, then the next roots need to be identified. The new interval for the root is [0.5, 1]. This new interval has been chosen so that the root must in between $f(a)$ and $f(b)$ have opposite sign. The calculation for the next root is shown in the Table 6.1.

Convergence Criteria

Example - Solution:

Table 6.1

i	a	b	$f(a)$	$f(b)$	c	$f(c)$
0	0	1	-5	1	0.5	-1.625
1	0.5	1	-1.625	1	0.75	-0.266
2	0.75	1	-0.266	1	0.875	0.373
3	0.75	0.875	-0.266	0.373	0.8125	0.056
4	0.75	0.813	-0.266	0.056	0.7815	-0.103
5	0.782	0.813	-0.103	0.056	0.7975	-0.021
6	0.798	0.813	-0.021	0.056	0.8055	0.020
7	0.798	0.806	-0.021	0.02	0.802	0.002

In the seventh iteration, the value $f(c) = 0.002 < 0.005$. Therefore the calculation to find the root, ξ can be stopped. Hence, the root of $f(x)$ in the interval $[0,1]$ is $0.798 \approx 0.80$ (approximate to 2 decimal places).

Exercises

Given a non-linear function, $f(x) = x^3 - \sin(x)$

a) Using intermediate value theorem, check whether there is a root in the following interval:

   i) [1, 1.25]

   ii) [0.75, 1]

b) Based on the interval in (a), find the root at 3 d.p using Bisection method.

Secant Method

- A graphical representation on how the secant method working is shown in Figure (a).

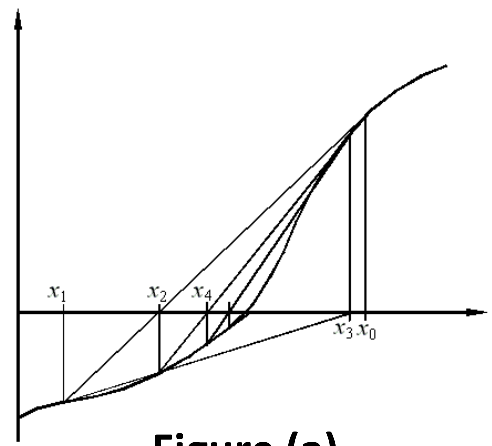


Figure (a)

- To find x_2, initially we use x_0 and x_1. Then to obtain x_3, we discard the oldest value, in this case x_0 and use $x_1$and x_2. To obtain x_4, we only use the two latest value, i.e., x_2 and x_3.

Secant Method

- Sometimes, the method can diverge instead of converge (see example – Fig.(b)).
- Swapping the two initial guesses x_0 and x_1 may change the behaviour of the method from divergent to convergent.

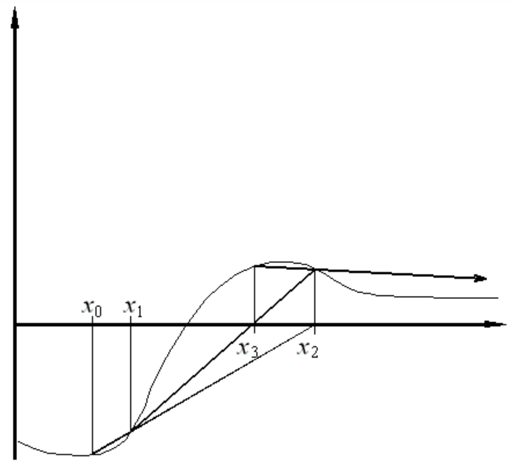


Figure (b): Divergence using the secant method

Secant Method

Therefore, in secant method, the convergence is highly depends on $f(x)$ function and selection of initial approximation of x_1 and x_2. If the incorrect initial values are chose, the probability of getting the divergence solution is higher. The secant method formula can be represented as

$$x_{i+2} = \frac{x_i f(x_{i+1}) - x_{i+1} f(x_i)}{f(x_{i+1}) - f(x_i)}, \text{ where } i = 0,1,2..n$$

Continue the calculation until we found the root with the specified accuracy.

Secant Method

Example:

Solve this function using the secant method.

$$f(x) = \sin(x) + 3x - e^{x}$$

If the initial guess are $x_0 = 1$ and $x_1 = 0$.

Do calculation in 3 decimal points.

Secant Method

Example - Solution:

Since the solution is correct to decimal points,

$$\varepsilon = \frac{1}{2} \times 10^{-3}$$
$$= 0.0005$$

Given

$x_0 = 1$ and $x_1 = 0$,

Then $f(x_0) = \sin(1) + 3(1) - e^1 = 1.1232$, $\quad f(x_1) = \sin(0) + 3(0) - e^0 = -1$

Secant Method

Example – Solution (cont'd):

The first root, x_2

$$
\begin{aligned}
x_2 &= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \\
&= \frac{1(-1) - 0(1.1232)}{-1 - 1.1232} \\
&= 0.47098 \\
&\approx 0.4710
\end{aligned}
$$

Replace $x_2 = 0.4710$ to $f(x) = \sin(x) + 3x - e^x$

Secant Method

Example – Solution (cont'd):

Then

$$f(0.4710) = \sin(0.4710) + 3(0.4710) - e^{0.4710}$$
$$= 0.2652$$

Since $f(x_2) = 0.2652 > \varepsilon$, then we need to find the next root using both points, x_1 and x_2.

The next calculations are shown in Table 6.3

Secant Method

Example – Solution (cont'd):

Table 6.3

Iteration, i	x_i	x_{i+1}	x_{i+2}	$f(x_{i+2})$
0	1	0	0.4710	0.2652
1	0	0.4710	0.3723	0.0295
2	0.4710	0.3723	0.3599	−0.0012
3	0.3723	0.3599	0.3604	0.0000

The calculation can be stopped at third iteration since $|f(x_5)| = 0.0000 < 0.0005$.

Then we can conclude that the root for $f(x)$ which is correct to decimal points is $x_5 = 0.3604 \approx 0.360$.

Secant Method

Example:

Suppose we wish to find a root of the function $f(x) = \cos(x) + 2\sin(x) + x^2$. A closed form solution for x does not exist so we must use a numerical technique. We will use $x_0 = 0$ and $x_1 = -0.1$ as our initial approximations and $\varepsilon = 0.0005$.

Secant Method

Example - Solution:

We will use 4 d.p to find a solution and the resulting iteration is shown in Table 6.4

Table 6.4. The secant method applied to $f(x) = \cos(x) + 2\sin(x) + x^2$.

n	x_{n-1}	x_n	x_{n+1}	$\lvert f(x_{n+1})\rvert$	$\lvert x_{n+1} - x_n\rvert$
1	0.0	-0.1	-0.5136	0.1522	0.4136
2	-0.1	-0.5136	-0.6100	0.0457	0.0964
3	-0.5136	-0.6100	-0.6514	0.0065	0.0414
4	-0.6100	-0.6514	-0.6582	0.0013	0.0068
5	-0.6514	-0.6582	-0.6598	0.0006	0.0016
6	-0.6582	-0.6598	-0.6595	0.0002	0.0003

Thus, with the last step, both halting conditions are met, $f(x_{n+1})$ and $\lvert x_{n+1} - x_n\rvert < 0.0005$ and therefore, after six iterations, our approximation to the root is -0.6595.

Newton Method

- Assume that the initial estimate of the zero, x_0 is close to the true root, draw the tangent line to the $f(x)$ at point x_0 and find the next approximation to the zero value, x_1 where the tangent crosses the x-axis.

- Then, draw the new tangent line to the curve at the point $(x_1, f(x_1))$ and as shown in Figure (c), the tangent line is now intersects the x-axis at point x_2. Notice that the point where this tangent crosses the x axis usually represents an improved estimate of the root.

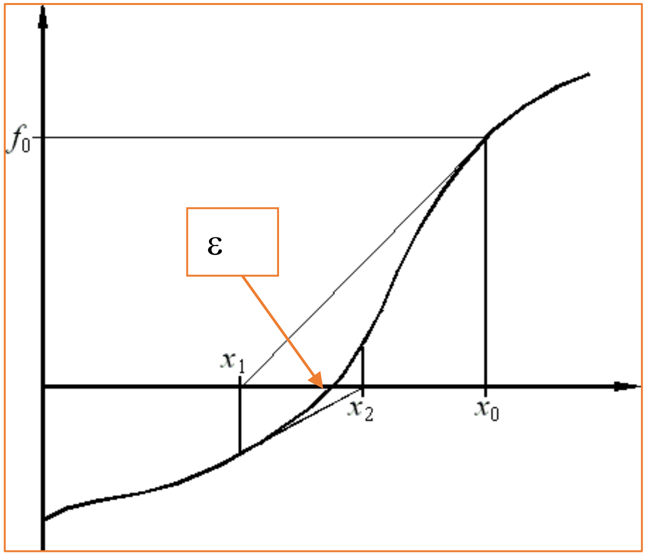


Figure (c): Graphical depiction of Newton's method

Newton Method

In general, the Newton Method formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \text{where } n = 0, 1, 2, \dots m$$

Although the Newton method is often very efficient compared with the secant method, there are situations where it performs poorly.

- Its convergence depends on the nature of the function and on the accuracy of the initial guess.
- The difficulty to get the derivative function can occur when dealing with the complex function.
- The Newton approach will often diverge if the initial guesses are not sufficiently close to the true roots or with the poor initial estimate.

Newton Method

Example:

Use the Newton's Method to estimate the root of $f(x)=x^3-\sin x$ employing an initial guess $x_0=1$. Do calculations in 5 decimal points and obtain a solution accurate to 3 decimal places.

$$f(x)=x^3-\sin x$$

For calculation in 5 decimals points, $\varepsilon=0.00005$.

Newton Method

Example - Solution:

Given $f(x) = x^3 - \sin x$, then $f'(x) = 3x^2 - (\cos x)$

Substitute $x_0 = 1$ into $f(x)$ and $f'(x)$.

$$f(x) = x^3 - \sin x; \qquad \text{and} \qquad f'(x) = 3x^2 - \cos x$$

$$f(1) = 1^3 - \sin 1 = 0.15853 \qquad \qquad f'(1) = 3(1)^2 - \cos 1 = 2.45970$$

Obtain x_1 value using Newton Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \text{where} \quad n = 0,1,2,..m$$

$$= 1 - \frac{0.15853}{2.45970}$$

$$= 0.93555$$

Newton Method

Example - Solution:

Thus,

$$f(x_1) = x_1^3 - \sin x_1;$$

$$f(0.93555) = 0.93555^3 - \sin 0.93555 = 0.01392$$

Convergence check:

$$\left|x_{n+1} - x_n\right| < 0.00005$$

$$\left|x_1 - x_0\right| = \left|1 - 0.93555\right| = 0.06445 > 0.00005$$

Since the convergence condition is not met, then further iterations to determine the root for $f(x)$ are required. The result is summarized in Table 6.5.

Newton Method

Example - Solution:

Table 6.5

n	x_n	$f(x_n)$	$f'(x_n)$
0	1	0.15853	2.45970
1	0.93555	0.01392	2.03239
2	0.92870	0.00015	1.98858
3	0.92862	-0.00001	1.98807
4	0.92862	-0.00001	

The calculation can now stop at $n = 4$ as the convergence condition is been satisfied where $|x_4 - x_3| = 0 < 0.00005$,

Therefore the root of $f(x) = x^3 - \sin x$ is $x_4 = 0.92862$ and for solution correct to 3 decimal places the answer is round up to $x_4 = 0.929$.

Newton Method – Special Function, $x^n - c = 0$

- The Newton Method also can be used for finding the square root of a number.
- For example, if one wishes to find the square root of 612, this is equivalent to finding the solution to:

$$x^2 = 612$$

- The function to use in Newton's method is,

$$f(x) = x^2 - 612$$

Newton Method – Special Function, $x^n - c = 0$

Consider

$$f(x) = x^k - c,$$

then

$$f'(x) = kx^{k-1}.$$

It gives $x_{n+1} = x_n - \dfrac{x_n^k - c}{kx_n^{k-1}}$ or $x_{n+1} = \dfrac{1}{k}\left((k-1)x_n + \dfrac{c}{x_n^{k-1}}\right)$, $n = 0,1,2,\ldots$

Newton Method – Special Function, $x^n - c = 0$

Example:

Use the Newton's method to locate the root for

$$f(x) = x^2 - 2$$

Solution:

Use the initial guess of $x_0 = 1$ and iterate until $\varepsilon = 0.005$.

Given

$$f(x) = x^2 - 2$$

Thus,

$$f'(x) = 2x$$

Newton Method – Special Function, $x^n - c = 0$

Example – Solution (cont'd):

Then by substituting value $f(x)$ and $f'(x)$ to the standard formula, it gives

$$x_{n+1} = x_n - \frac{x^2 - 2}{2x}$$ and can be summarized as

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

Substitute value $x_0 = 1$ into $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$ to obtain the value x_1

$$\begin{aligned} x_{n+1} &= \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \\ x_1 &= \frac{1}{2}\left(x_0 + \frac{2}{x_0}\right) \\ &= \frac{1}{2}\left(1 + \frac{2}{1}\right) \\ &= 1.5 \end{aligned}$$

Newton Method – Special Function, $x^n - c = 0$

Example – Solution (cont'd):

Substitute value $x_1 = 1.5$ into $f(x) = x^2 - 2$

$$f(1.5) = (1.5)^2 - 2$$
$$= 2.25 - 2 = 0.25$$

Check if the convergence situation is satisfied: $|x_{n+1} - x_n| < \varepsilon$

$$|x_{n+1} - x_n| < \varepsilon \Rightarrow |x_1 - x_0| = |1.5 - 1 = 0.5 > \varepsilon|$$

Thus, further iterations to determine the root for $f(x)$ are required. The result is shown in Table 6.6.

Table 6.6

n	x_n	x_{n+1}	$f(x_{n+1})$
0	1	1.5	0.25
1	1.5	1.4167	0.007
2	1.4167	1.4142	0.00006

It is found that , $|x_3 - x_2| = 0.0025$, which is less than the ε value. Therefore, the root for $f(x)$ is $x_3 = 1.4142$.

Exercises

1) Find root for the function $f(x) = e^x - x^2$ using Secant Method. Use $x_0 = -1, x_1 = 0$ and error, $\varepsilon = 0.0005$

   Hint: Root of the function is -0.704

2) Given an equation $f(x) = x^3 - 2x^2 - 5$. Find the root using Newton method. Take initial value, $x_0 = 2$ and error, $\varepsilon = 0.0005$

   Hint: Root of the function is 2.691