UTM
UNIVERSITI TEKNOLOGI MALAYSIA

CHAPTER 7

Eigen Value and Eigen Vector

Introduction

- Let $\mathbf{A}$ be a square matrix of dimension $n \times n$ and let $\mathbf{v}$ be a vector of dimension n.
- The product, $\mathbf{Y} = \mathbf{Av}$ can be viewed as a linear transformation from n-dimensional into itself.
- We want to find scalars (λ) for which there exists a nonzero vector $\mathbf{v}$ such that

$$\mathbf{Av} = \lambda\mathbf{v}$$

  that is, the linear transformation $T(\mathbf{v}) = \mathbf{Av}$ maps $\mathbf{v}$ onto the multiple $\lambda\mathbf{v}$.
- When this occurs, we call $\mathbf{v}$ an eigen vector that corresponds to the eigen value (λ), and together they form the eigen pair (λ), $\mathbf{v}$ for $\mathbf{A}$.

Introduction

- The identity matrix **[I]** can be used to express equation

$$\mathbf{Av} = \lambda\mathbf{v}$$

as

$$\mathbf{Av} = \lambda\mathbf{Iv}$$

which is then rewritten in the standard form for a linear system as,

$$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{v} = \mathbf{0}$$

Introduction

- This linear system,

$$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{v} = \mathbf{0}$$

has nontrivial solutions if and only if the matrix $\mathbf{A} - \lambda\mathbf{I}$ is singular, that is

$$|\mathbf{A} - \lambda\mathbf{I}| = 0$$

- When the determinant is expanded, it becomes a polynomial of degree *n*, which is called the *characteristic polynomial*,

$$p(\lambda)=|\mathbf{A} - \lambda\mathbf{I}|$$

Eigen Value and Eigen Vector

- If **A** is an $n \times n$ matrix, then its n eigen values λ_1, λ_2,…, λ_n are the roots of the characteristic polynomial

$$p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$$

- If λ is an **eigen value** of **A** and the nonzero vector **v** has the property that

$$\mathbf{Av} = \lambda\mathbf{v}$$

then **v** is called an **eigen vector** of **A** corresponding to the eigen value λ.

Eigen Value and Eigen Vector

Example:

- Let, $A = \begin{bmatrix} 3 & 6 \\ 1 & 4 \end{bmatrix}$
- The characteristic polynomial of the matrix *A:*

$$\boldsymbol{A} - \lambda \boldsymbol{I} = \begin{bmatrix} 3 & 6 \\ 1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & 6 \\ 1 & 4-\lambda \end{bmatrix}$$

$$P_1(\lambda) = det|A - \lambda I| = \lambda^2 - 7\lambda + 6 = (\lambda - 6)(\lambda - 1)$$

- Therefore, the eigen values of **A** are λ= 6 and λ= 1.

Eigen Value and Eigen Vector

Example:

- Let, $B = \begin{bmatrix} -1 & 6 & -12 \\ 0 & -13 & 30 \\ 0 & -9 & 20 \end{bmatrix}$
- Then,

$$\boldsymbol{B} - \lambda \boldsymbol{I} = \begin{bmatrix} -1-\lambda & 6 & -12 \\ 0 & -13-\lambda & 30 \\ 0 & -9 & 20-\lambda \end{bmatrix}$$

- The characteristic polynomial of the matrix *B*

$$P_1(\lambda) = det|B - \lambda I| = \lambda^3 - 6\lambda^2 + 3\lambda + 10$$
$$= (\lambda + 1)(\lambda - 5)(\lambda - 2)$$

- *The eigen values are* λ= -1; λ= 5 and λ= 2.

Eigen Value

- To find eigen values when the dimension n is small:
  - Find the coefficient of the characteristic polynomial
  - Finds its root
  - Find the nonzero solutions of the homogeneous linear system $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{v} = \mathbf{0}$
- Finding the determinant of an $n \times n$ matrix is computationally expensive and finding good approximations to the roots of $p(\lambda)$ is also difficult.

Theorem

- Assume that **A** is $n \times n$ matrix and let

$$\lambda_i, i = 1, 2, \ldots, n$$

are eigen values of **A**

(1) $$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii}$$

2) $$\prod_{i=1}^{n} \lambda_i = |A|$$

Gerschgorin's Circle Theorem

- Assume that **A** is $n \times n$ matrix, and let B_i denote the disk in the complex plane with center a_{ii} and radius r_i

$$r_i = \sum_{\substack{j=1 \\ i \neq j}}^{n} \left| a_{ij} \right|$$

$$B_i = \left| \lambda - a_{ii} \right| \leq r_i$$

- If $S = \bigcup_{i=1}^{n} B_i$ then all the eigen values of **A** lie in the set S.

Gerschgorin's Circle Theorem

Example:

Let, $A = \begin{bmatrix} 3 & -2 & 1 \\ -1 & 3 & 1 \\ 1 & -2 & -4 \end{bmatrix}.$

Estimate the eigen values of matrix A using the Gerschgorin's circle theorem.

Gerschgorin's Circle Theorem

Example - Solution:

Row 1:

$$A = \begin{bmatrix} \boxed{3 \quad -2 \quad 1} \\ -1 \quad 3 \quad 1 \\ 1 \quad -2 \quad -4 \end{bmatrix}$$

center

radius

$$|\lambda - 3| < \underbrace{|-2| + |1|}_{\text{radius}}$$

$$|\lambda - 3| < 3 \quad \{\lambda - 3 = \pm 3 \Rightarrow 0,6\}$$

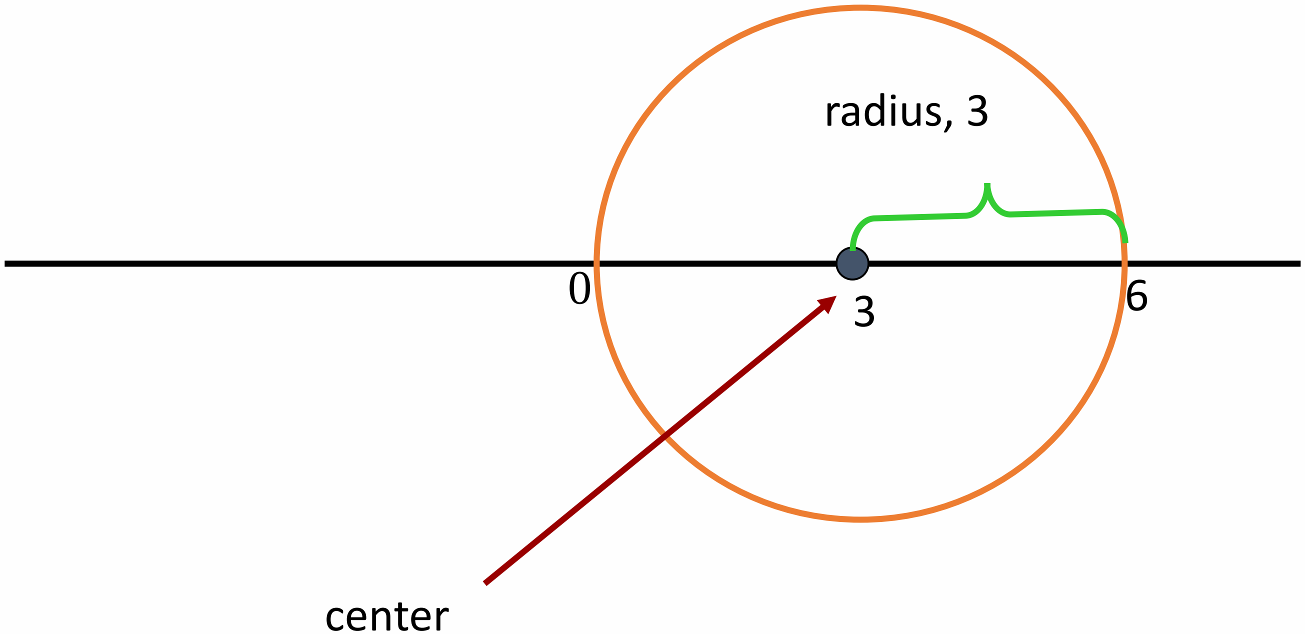
$$0 < \lambda < 6$$

$$\lambda \in (0,6)$$

Gerschgorin’s Circle Theorem

Example – Solution (cont’d):

- Circle



Gerschgorin's Circle Theorem

Example – Solution (cont'd):

Row 2:

center

radius

$$A = \begin{bmatrix} 3 & -2 & 1 \\ \boxed{-1 \quad 3 \quad 1} \\ 1 & -2 & -4 \end{bmatrix} \longrightarrow$$

$$|\lambda - 3| < \underbrace{|-1| + |1|}_{\text{radius}}$$

$$|\lambda - 3| < 2$$

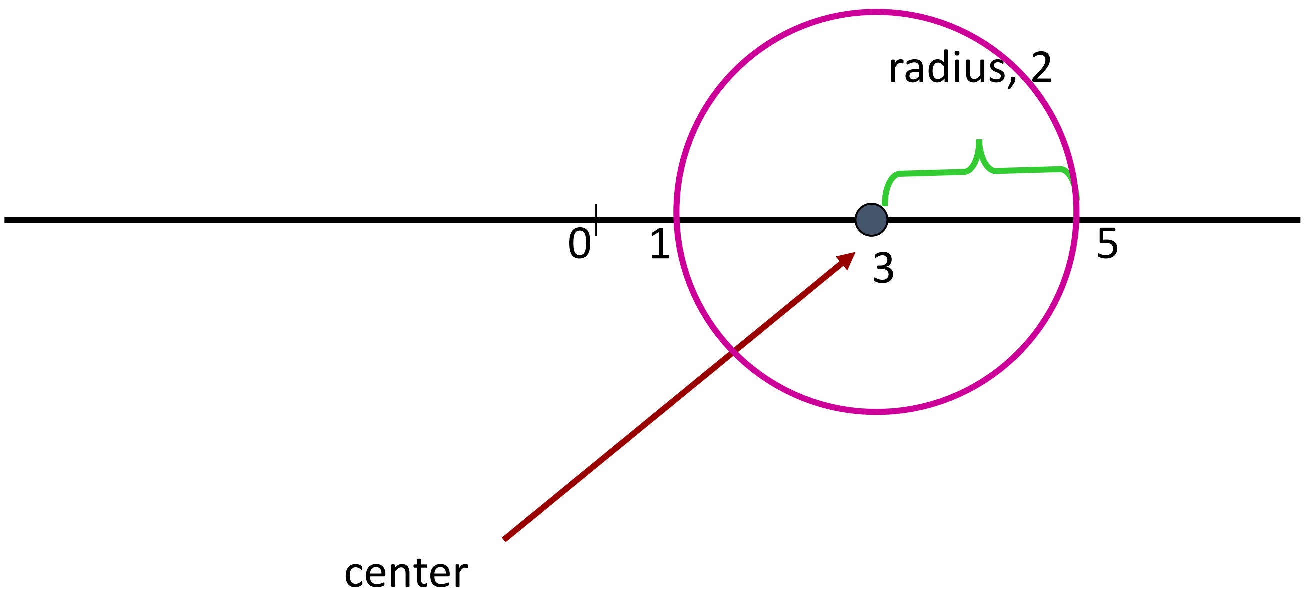
$$1 < \lambda < 5$$

$$\lambda \in (1,5)$$

Gerschgorin's Circle Theorem

Example – Solution (cont'd):

- Circle



Gerschgorin's Circle Theorem

Example – Solution (cont'd):

Row 3:

$$A = \begin{bmatrix} 3 & -2 & 1 \\ -1 & 3 & 1 \\ 1 & -2 & -4 \end{bmatrix}$$

center

$$\left|\lambda - (-4)\right| < \underbrace{|1| + |-2|}_{\text{radius}}$$

$$|\lambda + 4| < 3$$

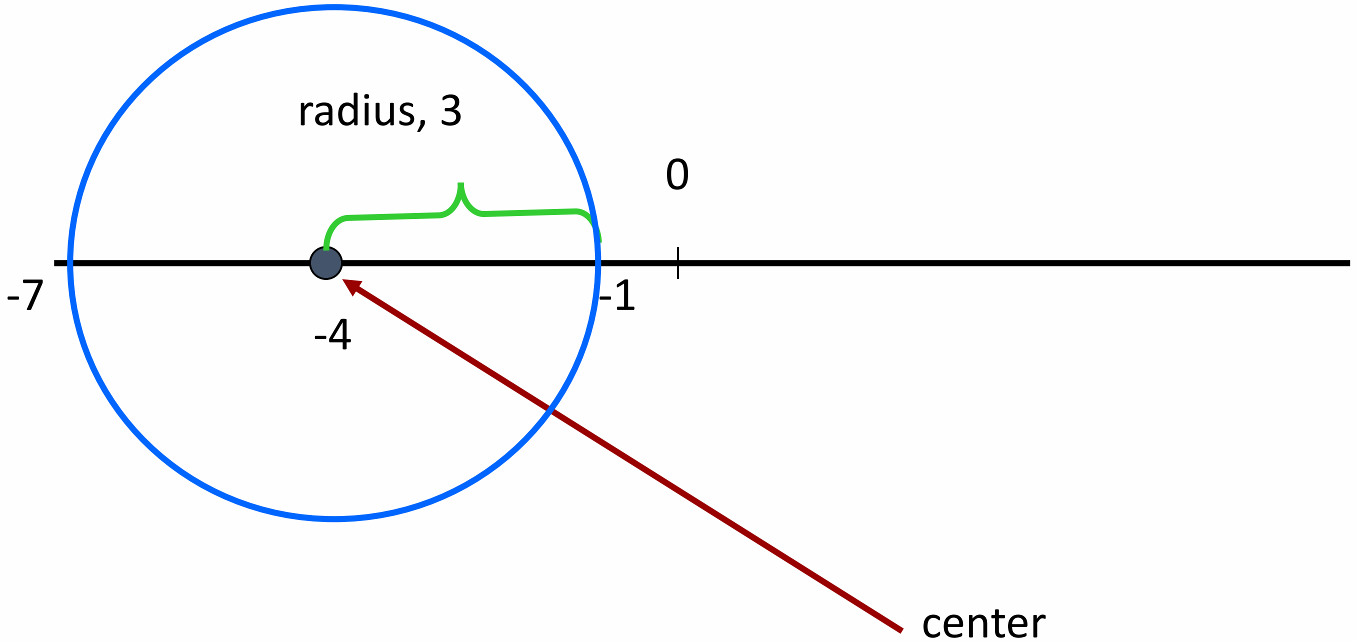
$$-7 < \lambda < -1$$

$$\lambda \in (-7, -1)$$

Gerschgorin's Circle Theorem

Example – Solution (cont'd):

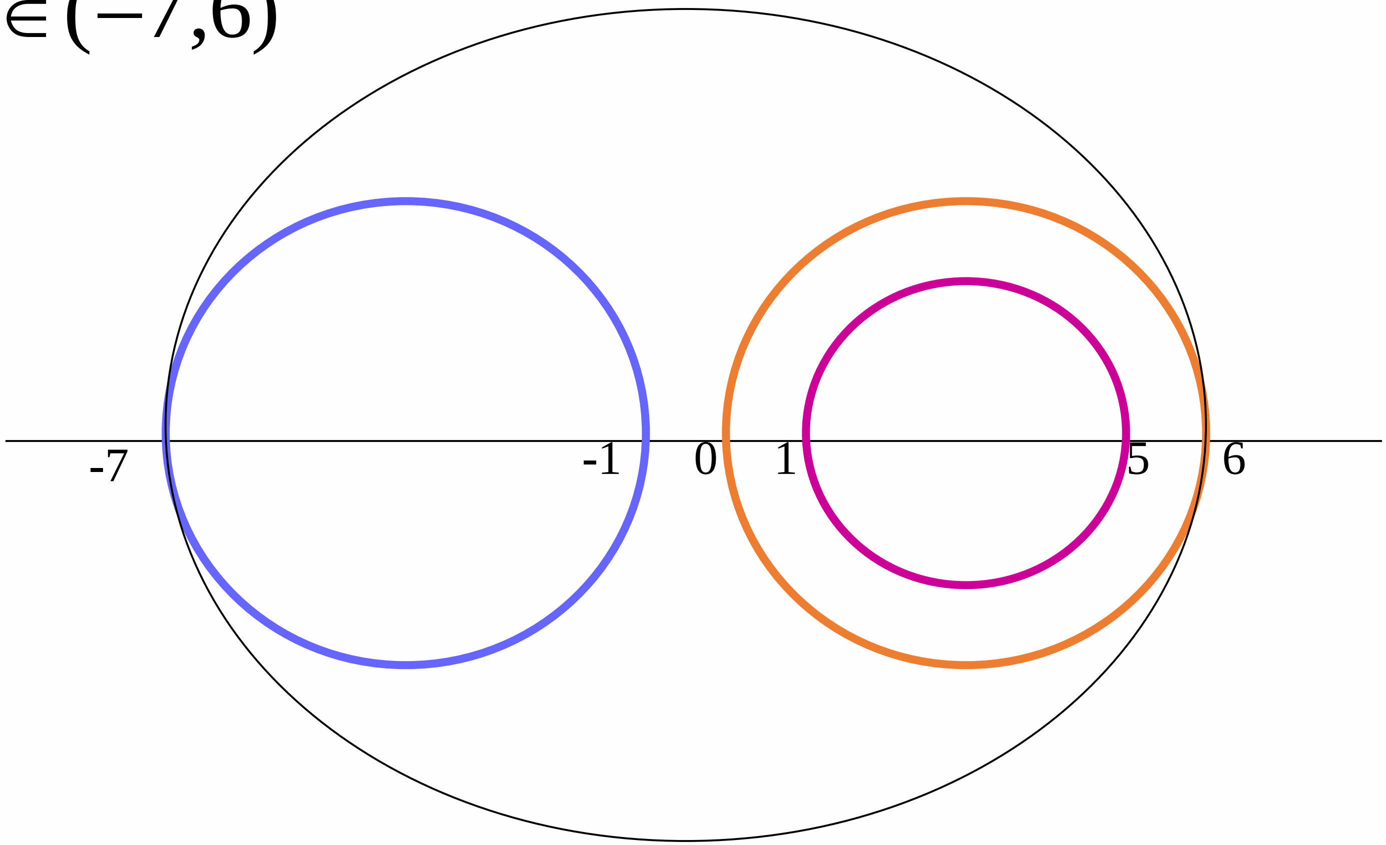
- Circle



Gerschgorin's Circle Theorem

Example – Solution (cont'd):

$$\lambda \in (-7,6)$$



Power Method

- The **power method** is an iterative technique used to determine the **dominant eigen value** of a matrix – that is, the eigen value with the largest magnitude.
- By modifying the method slightly, it can also be used to determine other eigen values.
- One useful feature of the power method is that it produces not only an eigen value, but also the associated eigen vector.

Power Method

- If λ_1 is an eigen value of $\boldsymbol{A}$ that is larger in absolute value than any other eigen value, it is called the **dominant eigen value.**

- An eigen vector $\boldsymbol{v}_1$ corresponding to λ_1 is called a **dominant eigen vector**.

Power Method

- To apply the power method, we assume that the $n \times n$ matrix **A** has n eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ with an associated collection of linearly independent eigenvectors $\{\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(2)}, \ldots \boldsymbol{v}^{(n)}\}$

- We assume that **A** has precisely one eigen value, λ_1 that is largest in magnitude, so that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n|$$

Power Method Algorithm

$$A\mathbf{v} = \lambda\mathbf{v}$$

$$\mathbf{v} = \frac{1}{\lambda} A\mathbf{v}$$

- Start with vector, $\boldsymbol{v}^{(0)}$

$$\mathbf{v}^{(1)} = \frac{1}{m_1} A\mathbf{v}^{(0)}$$

Power Method Algorithm

- Generate the sequence $\{v^{(k)}\}$ recursively, using

$$\mathbf{v}^{(k+1)} = \frac{1}{m_{k+1}} A\mathbf{v}^{(k)} \qquad k = 0,1,2\ldots.$$

- Where m_{k+1} is the coordinate of $\boldsymbol{Av}^{(k)}$ of largest magnitude (in the case of a tie, choose the coordinate that comes first).

Power Method Algorithm

- The sequences $\{v^{(k)}\}$ and $\{m_k\}$ will converge to $\boldsymbol{v}$ and λ, respectively.

$$\lim_{k\to\infty} v_k = v \qquad \lim_{k\to\infty} m_k = \lambda$$

- **Termination criteria:** Select a tolerance (error), $\varepsilon>0$ and generate $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{k+1}$ until

$$\left\|\mathbf{v}^{(k+1)} - \mathbf{v}^{(k)}\right\| < \varepsilon$$

Power Method Algorithm

Example:

Let,

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$$

Use the Power method to approximate the most dominant eigen value of the matrix. Let $v^{(0)}=(0,0,1)^T$ and iterate until $\varepsilon=0.001$.

Power Method Algorithm

Example – Solution:

$$\mathbf{v} = A\mathbf{v}^{(0)} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ \boxed{5} \end{bmatrix}$$

$$\boxed{m_1 = 5}$$

(largest magnitude)

$$\mathrm{v}^{(1)} = \frac{1}{m_1} * A\mathrm{v}^{(0)} = \mathrm{v}/m_1 = 1/5 \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -0.2 \\ 0.2 \\ 1.0 \end{bmatrix}$$

$$\left\| \mathbf{v}^{(1)} - \mathbf{v}^{(0)} \right\| > \varepsilon$$

(next iteration)

Power Method Algorithm

Example – Solution (cont'd):

$$\mathbf{v} = A\mathbf{v}^{(1)} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} \begin{bmatrix} -0.2 \\ 0.2 \\ 1.0 \end{bmatrix} = \begin{bmatrix} -0.8 \\ 0.8 \\ \boxed{3.4} \end{bmatrix}$$

$$\boxed{m_2 = 3.4}$$

$$\mathbf{v}^{(2)} = \mathbf{v}/m_2 = 1/3.4 \begin{bmatrix} -0.8 \\ 0.8 \\ 3.4 \end{bmatrix} = \begin{bmatrix} -0.235 \\ 0.235 \\ 1.0 \end{bmatrix}$$

$$\left\| \mathbf{v}^{(2)} - \mathbf{v}^{(1)} \right\| > \varepsilon$$

(next iteration)

Power Method Algorithm

Example – Solution (cont'd):

k	$(v^{(k)})^T$			$(Av^{(k)})^T$			m_{k+1}
0	0	0	1	-1	1	5	5
1	-0.2	0.2	1	-0.8	0.8	3.4	3.4
2	-0.235	0.235	1	-0.765	0.765	3.12	3.12
3	-0.245	0.245	1	-0.755	0.755	3.04	3.04
4	-0.248	0.248	1	-0.752	0.752	3.016	3.016
5	-0.249	0.249	1	-0.751	0.751	3.008	3.008
6	-0.250	0.250	1	-0.750	0.750	3.000	3.000
7	-0.250	0.250	1				

Power Method Algorithm

Example – Solution (cont'd):

$$\left\| \mathbf{v}^{(7)} - \mathbf{v}^{(6)} \right\| < \varepsilon$$

$$\lambda_1 \approx m_7 = 3.000$$

$$\mathbf{v}_1 \approx \mathbf{v}^{(7)} = \begin{bmatrix} -0.250 \\ 0.250 \\ 1.000 \end{bmatrix}$$

Power Method Algorithm

Example – Solution (cont'd):

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}\begin{bmatrix} -0.25 \\ 0.25 \\ 1.0 \end{bmatrix} = 3.000 \begin{bmatrix} -0.25 \\ 0.25 \\ 1.0 \end{bmatrix}$$

$$\begin{bmatrix} -0.75 \\ 0.75 \\ 3.00 \end{bmatrix} = \begin{bmatrix} -0.75 \\ 0.75 \\ 3.00 \end{bmatrix} \qquad \checkmark$$

Exercise

Given a matrix A as follows.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

a) Use the **Gerschgorin's Circle Theorem** to determine a region containing all the eigenvalues of $\boldsymbol{A}$.

b) Find the dominant eigenvalue (λ_1) and the corresponding eigenvector of matrix A using **Power method.** Use $v^{(0)} = [0, 1, 0]^T$. Do calculation in 4 decimal points and take $\varepsilon = 0.005$

c) Suppose that the smallest eigenvalue (λ_3) of matrix **A** is -1. Find the intermediate eigenvalue.

Shifted Power Method

- Consider the standard eigen value problem

$$\boldsymbol{Av}=\lambda\boldsymbol{v}$$

- By subtracting a scalar $\boldsymbol{s}$ from both sides of the standard eigen value problem, the eigen values of the matrix are shifted:

$$\boldsymbol{Av} - s\mathbf{I}\boldsymbol{v} = \lambda\boldsymbol{v} - s\boldsymbol{v}$$

which yields $(\boldsymbol{A} - s\mathbf{I})\,\boldsymbol{v} = (\lambda - s)\boldsymbol{v}$

and can be written as $\boldsymbol{A}_{\text{shifted}}\boldsymbol{v} = \lambda_{\text{shifted}}\boldsymbol{v}$

Shifted Power Method

- $\boldsymbol{A}_{\text{shifted}}$ is the shifted matrix,

$$\boldsymbol{A}_{\text{shifted}} = (\boldsymbol{A} - s\mathbf{I})$$

- $\lambda_{\text{shifted}} = \lambda - \boldsymbol{s}$ is the eigen value of the shifted matrix.
- Shifting matrix ***A*** by a scalar ***s*** shifts the eigen value by ***s***.
- Shifting a matrix by a scalar does not affect the eigen vectors.

Shifted Power Method

- Shifting of eigen values of a matrix can be used to find the opposite extreme eigen value, which is either
  - the smallest magnitude eigen value, or
  - the largest magnitude eigen value of opposite sign.

Shifted Power Method

- Consider a matrix whose eigen values are all the same sign; λ=1, 2, 4, and 8.
- λ= 8 is the eigen value of largest magnitude.
- λ=1 is the opposite extreme eigen value.
- The eigen value of largest magnitude λ_1=8 can be found by the power method.
- Shifting the eigen values by ***s*** = 8, yields the shifted eigen values λ = -7, -6, -4, 0.

Shifted Power Method

- The largest magnitude eigen value of the shifted matrix can be found by the power method; $\lambda_{\text{shifted, largest}} = -7$.

- The smallest eigen value of the original matrix may be found by

$$\lambda_{\text{smallest}} = \lambda_{\text{shifted, largest}} + 8 = -7 + 8 \quad = 1$$

Shifted Power Method

- Consider a matrix whose eigen values are both positive and negative; λ=-1, -2, 4, and 8.
- λ=8 is the eigen value of largest magnitude.
- λ=-2 is the opposite extreme eigen value.
- The eigen value of largest magnitude λ_1=8 can be found by the power method.
- Shifting the eigen values by ***s*** = 8, yields the shifted eigen values λ = -9, -10, -4, 0.

Shifted Power Method

- The largest magnitude eigen value of the shifted matrix can be found by the power method; $\lambda_{shifted, largest} = -10$.
- The largest magnitude eigen value of opposite sign of the original matrix may be found by

$$\lambda_{largest,negative} = \lambda_{shifted,\ largest} + 8 = -10 + 8 = -2$$

Shifted Power Method

The shifted power method can be summarized as follows:

1. Solve for the eigen value of largest magnitude λ_1 using the power method.
2. Shift the matrix $\boldsymbol{A}$ by $\boldsymbol{s} = \lambda_1$ to obtain the shifted matrix $\boldsymbol{A}_{\text{shifted}}$.
3. Solve for the eigen value λ_{shifted} by the power method.
4. Compute the opposite extreme eigen value of matrix $\boldsymbol{A}$ by

$$\lambda = \lambda_{\text{shifted}} + \boldsymbol{s}$$

Shifted Power Method

Example:

Let, $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$

The dominant eigen value of ***A*** is λ_1= 3.0.

Use the shifted power method to find the remaining eigen values of ***A***.

Shifted Power Method

Example – Solution:

Let ***B*** = ***A***shifted

$$B = A - \lambda_1 I = A - 3I$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 & -1 \\ 1 & -3 & 1 \\ 4 & -4 & 2 \end{bmatrix}$$

Shifted Power Method

Example – Solution (cont'd):

- Let, v $^{(0)}$ =[0, 1, 0]^T and ε=0.001

$$\mathbf{v} = B\mathbf{v}^{(0)} = \begin{bmatrix} -2 & 2 & -1 \\ 1 & -3 & 1 \\ 4 & -4 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ \boxed{-4} \end{bmatrix}$$

$$\boxed{m_1 = -4}$$

Shifted Power Method

Example – Solution (cont'd):

$$\mathbf{v}^{(1)} = \mathbf{v}/m_1 = \begin{bmatrix} -0.5 \\ 0.75 \\ 1.0 \end{bmatrix} \qquad \left|\mathbf{v}^{(1)} - \mathbf{v}^{(0)}\right| > \varepsilon$$

(next iteration)

Shifted Power Method

Example – Solution (cont'd):

k	$(v^{(k)})^T$			$(Bv^{(k)})^T$			m_{k+1}
0	0	1	0	2	-3	-4	-4
1	-0.5	0.75	1	1.5	-1.75	-3	-3
2	-0.5	0.583	1	1.166	-1.249	-2.332	-2.332
3	-0.5	0.536	1	1.072	-1.108	-2.144	-2.144
4	-0.5	0.517	1	1.034	-1.051	-2.068	-2.068
5	-0.5	0.508	1	1.016	-1.024	-2.032	-2.032
6	-0.5	0.504	1	1.008	-1.012	-2.016	-2.016
7	-0.5	0.502	1	1.004	-1.006	-2.008	-2.008
8	-0.5	0.501	1	1.002	-1.003	-2.004	-2.004
9	-0.5	0.5	1	1.000	-1.000	-2.000	-2.000
10	-0.5	0.5	1				

Shifted Power Method

Example – Solution (cont'd):

- The largest magnitude eigen value of the shifted matrix and its eigenvector,

$$\lambda_{shifted} = m_{10} = -2.0 \qquad \mathbf{v} = [-0.5, 0.5, 1.0]^T$$

- The opposite extreme eigen value of matrix ***A*** and its eigenvector,

$$\lambda = \lambda_{shifted} + \lambda_1 = -2.0 + 3.0 = 1.0$$

$$\mathbf{v} = [-0.5, 0.5, 1.0]^T$$

Shifted Power Method

Example – Solution (cont'd):

$$A\mathbf{v} = \lambda \mathbf{v}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} \begin{bmatrix} -0.5 \\ 0.5 \\ 1.0 \end{bmatrix} = 1.0 \begin{bmatrix} -0.5 \\ 0.5 \\ 1.0 \end{bmatrix}$$

$$\begin{bmatrix} -0.5 \\ 0.5 \\ 1.0 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.5 \\ 1.0 \end{bmatrix} \quad \checkmark$$

Shifted Power Method

Example – Solution (cont'd):

Since the opposite extreme eigenvalue $\lambda=1$, has the same sign as the largest magnitude eigen value of the original matrix $\lambda_1=3$, all eigen values of ***A*** are positive and $\lambda=1$ is the smallest eigen value of matrix ***A***.

Shifted Power Method

Example – Solution (cont'd):

- Matrix ***A*** has 3 eigen values, λ_1,λ_2, and λ_3.
- Let,

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3|$$

- The largest eigen value of matrix ***A***, λ_1=3
- The smallest eigen value of matrix ***A***, λ_3=1

Shifted Power Method

Example – Solution (cont'd):

- Find the intermediate eigen value of matrix ***A***, λ_2

$$\sum_{i=1}^{3} \lambda_i = \sum_{i=1}^{3} a_{ii}$$

$$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33} = 1 + 0 + 5 = 6$$

$$3 + \lambda_2 + 1 = a_{11} + a_{22} + a_{33} = 6$$

$$\lambda_2 = 6 - 1.0 - 3.0 = 2.0$$

Exercise

Given a matrix A as follows.

$$A = \begin{bmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{bmatrix}$$

a) Find the characteristics of polynomial of the matrix A.

b) Find the smallest eigenvalue, λ_3 and the corresponding eigenvector of matrix A using **Shifted Power method**. Use $v^{(0)} = [1.0, \quad 0.25, \quad 1.0]$ and dominant eigenvalue, $\lambda_1 = 4$. Do calculation in 3 decimal points and take $\varepsilon = 0.005$. Round your smallest eigenvalue, λ_3 to the nearest integer.