# Mathematics Extension 2

## HSC study notes

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Written by Vincent Liu (2018)

### De Moivre's Theorem and other identities

$|zw| = |z||w|$ $\qquad$ $\arg zw = \arg z + \arg w$

$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ $\qquad$ $\arg \frac{z}{w} = \arg z - \arg w$

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n\left(\cos n\theta + i\sin n\theta\right)$$

### Complex Loci

*Circle*: $|z - z_1| = k$

*Perpendicular Bisector*: $|z - z_1| = |z - z_2|$

*Ray*: $\arg(z - z_1) = \theta$

### Complex Roots of Unity

If $z^n = 1$ and $z = \cos\theta + i\sin\theta$,
$\cos n\theta + i\sin n\theta = \cos 2k\pi + i\sin 2k\pi$

### Ellipse

*Eccentricity*: $e = \sqrt{1 - \frac{b^2}{a^2}}$

*Equations*: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $x = a\cos\theta$, $y = b\sin\theta$

*Foci*: $(\pm ae, 0)$ $\qquad$ *Directrices*: $x = \pm\frac{a}{e}$

*Tangent*:

Cartesian: $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

Parametric: $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

*Normal*:

Cartesian: $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

Parametric: $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

*Chord of Contact*: $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$

### Hyperbola

*Eccentricity*: $e = \sqrt{1 + \frac{b^2}{a^2}}$

*Equations*: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $x = a\sec\theta$, $y = b\tan\theta$

*Foci*: $(\pm ae, 0)$ $\qquad$ *Directrices*: $x = \pm\frac{a}{e}$

*Asymptotes*: $y = \pm\frac{b}{a}x$

*Tangent*:

Cartesian: $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

Parametric: $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

*Normal*:

Cartesian: $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$

Parametric: $\frac{ax}{\cos\theta} + \frac{by}{\sin\theta} = a^2 + b^2$

*Chord of Contact*: $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$

### Rectangular Hyperbola $\quad (e = \sqrt{2})$

*Equations*: $xy = c^2$ and $x = cp$, $y = \frac{c}{p}$

*Foci*: $(\pm c\sqrt{2}, \pm c\sqrt{2})$ $\qquad$ *Directrices*: $x + y = \pm c\sqrt{2}$

*Tangent*:

Cartesian: $xy_1 + x_1y = 2c^2$

Parametric: $x + p^2y = 2cp$

*Normal*:

Cartesian: $xx_1 - yy_1 = (x_1)^2 - (y_1)^2$

Parametric: $p^3x - py = c(p^4 - 1)$

*Chord of Contact*: $xy_0 + x_0y = 2c^2$

**Integration by $t$-substitution**

If $t = \tan\left(\frac{x}{2}\right)$, and $t$ is defined, then:

$$dx = \frac{2}{1+t^2}\,dt \text{ and}$$

$$\sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad \tan x = \frac{2t}{1-t^2}$$

**Integration by Parts**

$$\int u\,dv = uv - \int v\,du$$

**Properties of Definite Integrals**

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

**Volumes of Solids with Similar Cross-sections**

$V = \int_a^b A\,dh$ ($A$ is the cross-sectional area)

**Volumes by Subtraction**

If the cross section of the volume is a ring, then:

$$V = \pi\int_a^b \left(r_{\text{outer}}\right)^2 - \left(r_{\text{inner}}\right)^2\,dh$$

**Volumes by Cylindrical Shells**

$$V = 2\pi\int_a^b rh\,dr$$

**Different Forms of Acceleration**

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = v\frac{dv}{dx} = \frac{dv}{dt}$$

**Circular Motion**

*Period*: $T = \frac{2\pi}{\omega}$

*Displacement*: $\ell = r\theta$

*Velocity:*

*Tangential*: $v = r\omega$

*Angular:* $v_A = \omega$

*Acceleration:*

*Centripetal:* $a_N = r\omega^2 = \frac{v^2}{r}$

*Tangential:* $a_T = r\dot{\omega}$

Where $\omega = \frac{d\theta}{dt}$, expressed in radians/second.

**The AM-GM Inequality**

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}$$

# Topic 1: Graphs

## Symmetry

| $\times$ | **odd** | **even** |
|---|---|---|
| **odd** | even | odd |
| **even** | odd | even |

A function $f(x)$ is even if and only if $f(x) = f(-x)$.
A function $f(x)$ is odd if and only if $f(x) = -f(-x)$.

Even functions have symmetry about the $y$-axis. Odd functions have symmetry about the origin.

The product or quotient of two odd or even functions is an even function.
The product or quotient of an odd and even function is an odd function.

If $f(x)$ is even and differentiable everywhere then $f'(x)$ is odd.
If $f(x)$ is odd and differentiable everywhere then $f'(x)$ is even.

## Finding the Asymptotes of a Rational Function

A rational function is a function with a polynomial numerator and denominator, i.e. $f(x) = \dfrac{P(x)}{Q(x)}$.

<u>*Vertical*</u>

A vertical asymptote exists wherever the denominator is zero.

<u>*Horizontal*</u>

The equation of the horizontal asymptote(s), if they exist, are given by $y = \lim\limits_{x \to \pm\infty} f(x)$.

The limit can be evaluated by dividing both the numerator and denominator by the highest degree of $x$ and using the result $\lim\limits_{x \to \pm\infty} \dfrac{1}{x^n} = 0$.

<u>*Oblique*</u>

An oblique asymptote exists if and only if the degree of numerator is one greater than the degree of denominator.

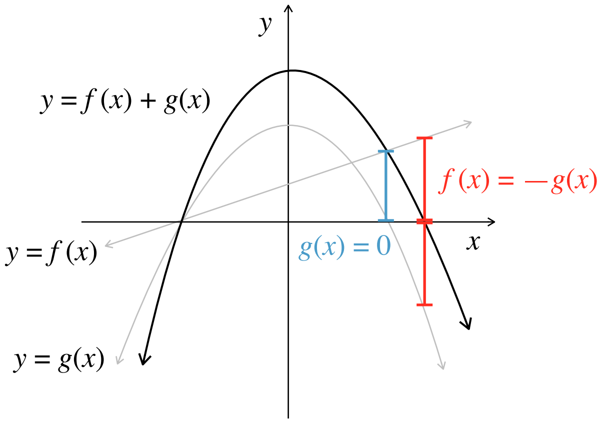
If $f(x) = \dfrac{P(x)}{Q(x)}$ then a long division may be performed to obtain:

$$\frac{P(x)}{Q(x)} = g(x) + \frac{R(x)}{Q(x)}$$

Where the equation of the oblique asymptote is given by $y = g(x)$.

<u>*Note:*</u> The degree of the asymptote is the difference in the degree between the numerator and denominator. If the numerator is more than one degree higher than the denominator, then the asymptote will be quadratic/cubic/quartic etc.

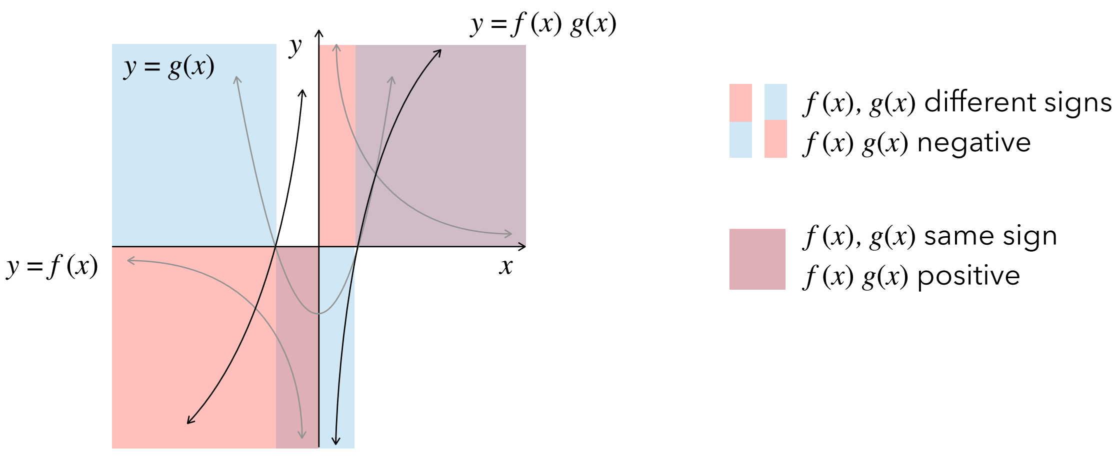
## Addition/Subtraction of Ordinates




The graph of $f(x) + g(x)$ can be sketched by taking the graphs of $f(x)$ and $g(x)$ and adding their $y$ values.
The graph of $f(x) - g(x)$ can be sketched by using the graphs of $f(x)$ and $-g(x)$.

- When one of the functions is zero, the sum is the value of the other function.
- When the functions are equal in magnitude and opposite in direction, their sum is zero.
- When the functions are equal, their sum is double the function value.

## Multiplication of Ordinates



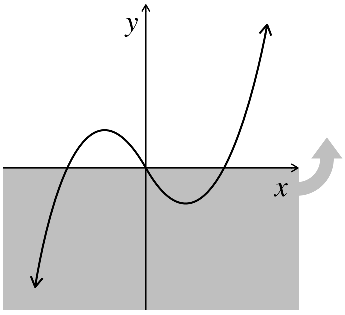

The graph of $f(x)g(x)$ can be sketched by taking the graphs of $f(x)$ and $g(x)$ and multiplying their $y$ values.
The graph of $f(x) \div g(x)$ can be sketched by using the graphs of $f(x)$ and $g(x)^{-1}$.

- If both functions are the same sign, their product is positive.
- If both functions are different signs, their product is negative.
- If either function is zero, their product is zero.

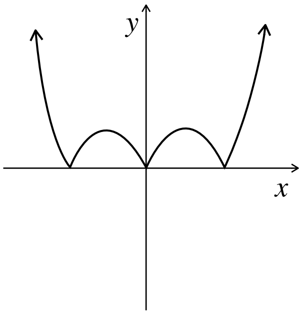
**Absolute Values**

$y = |f(x)|$ *(reflection in the x-axis)*

$y = f(x)$



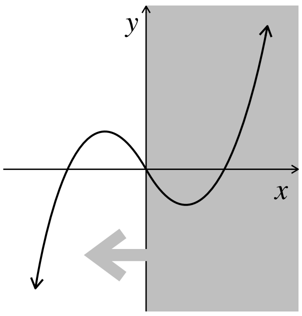
$y = |f(x)|$



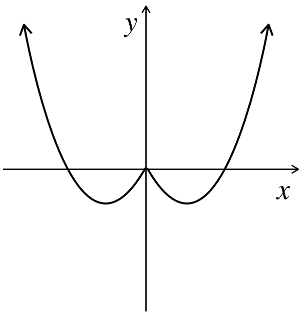
Take the sections of the graph where $y < 0$ and **reflect** them across the $x$-axis.

$y = f(|x|)$ *(symmetry in the y-axis)*

$y = f(x)$



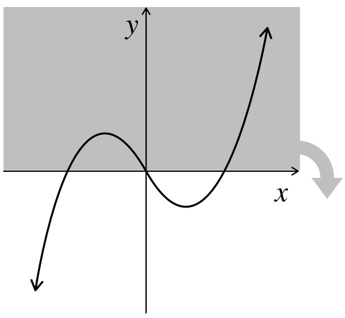
$y = f(|x|)$



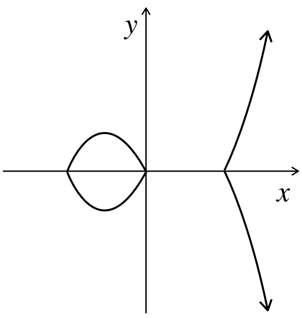
**Discard** the part of the graph where $x < 0$ and **mirror** the rest of the graph across the $y$-axis.

$|y| = f(x)$ *(symmetry in the x-axis)*

$y = f(x)$



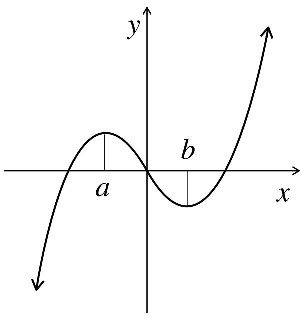
$|y| = f(x)$



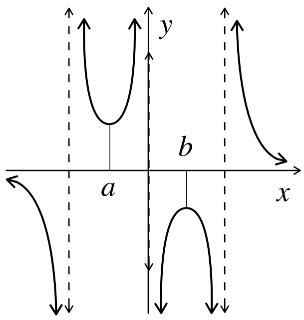
**Discard** the part of the graph where $y < 0$ and **mirror** the rest of the graph across the $x$-axis.

**Taking the Reciprocal**

$y = f(x)$




$y = \dfrac{1}{f(x)}$

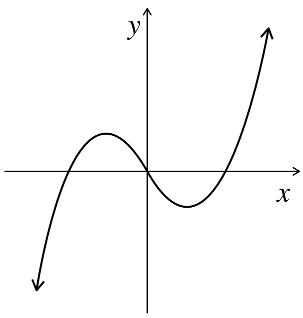



- The sign of the function is preserved.
- Any points where $f(x) = 1$ are preserved.
- The $x$-coordinates of stationary points are preserved. Maxima become minima, and vice versa.
- Any $x$-intercepts become vertical asymptotes, and vice versa.

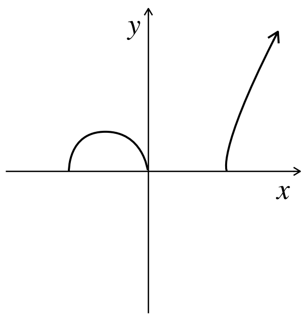
As $f(x) \to \pm\infty$, $\dfrac{1}{f(x)} \to 0^{\pm}$.  As $f(x) \to 0^{\pm}$, $\dfrac{1}{f(x)} \to \pm\infty$.

**Taking the Square Root**

$y = f(x)$




$y = \sqrt{f(x)}$



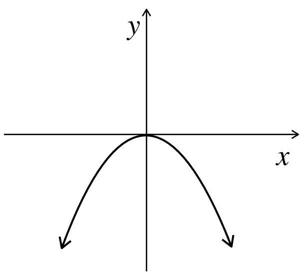

- $\sqrt{f(x)}$ does not exist for $f(x) < 0$.
- Any points where $f(x) = 1$ are preserved.
- The $x$-coordinates of stationary points are preserved. Their nature is also preserved.

$$\text{If } \begin{cases} f(x) > 1, \ \sqrt{f(x)} < f(x) \\ f(x) < 1, \ \sqrt{f(x)} > f(x) \end{cases}$$

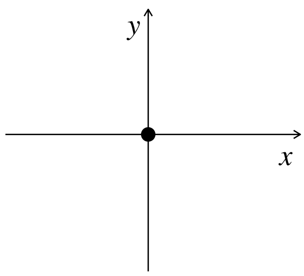
(If $f(x) = 1$, $\sqrt{f(x)} = f(x)$)

- $\sqrt{f(x)}$ will have isolated points if $f(x)$ is zero at one point and negative around it.

$y = f(x)$

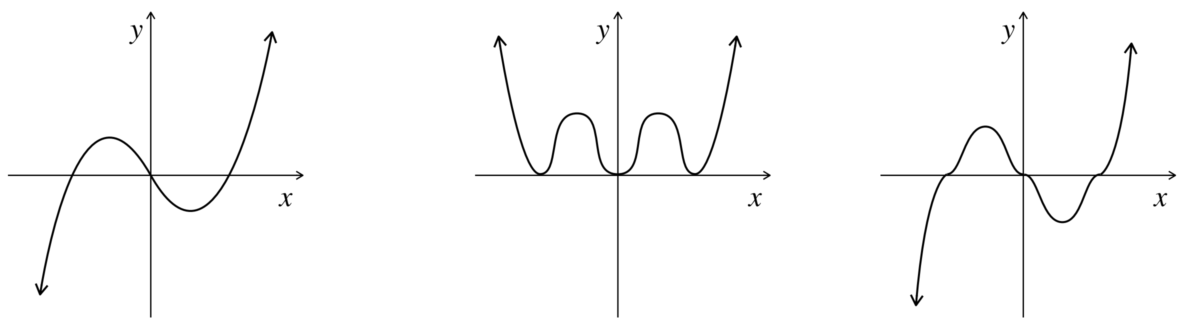



$y = \sqrt{f(x)}$



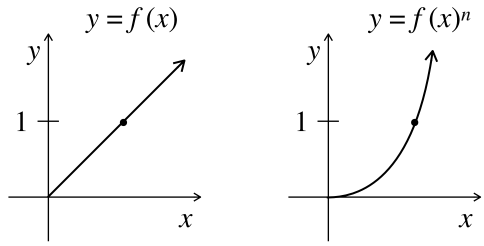
## Integer Powers

$y = f(x)$ $\qquad$ $y = f(x)^{2n}$ $\qquad$ $y = f(x)^{2n+1}$




*If $n$ is negative:*
The graph of $f(x)^{-n}$ can be taking by graphing $f(x)^n$ and taking its reciprocal.

*If $n$ is positive:*




- If $|f(x)| < 1$ then $|f(x)^n| < |f(x)| < 1$.
- If $|f(x)| > 1$ then $|f(x)^n| > |f(x)|$.

- If $n$ is even, $f(x)^n$ is an even function that is always positive.
- If $n$ is odd, the sign of $f(x)^n$ is the same as $f(x)$.

- The $x$-coordinates of stationary points are preserved.
- The $x$-intercepts of $f(x)$ become stationary points of $f(x)^n$.

## Implicit Differentiation

With some equations, it is very tedious to make $y$ the subject when differentiating, e.g. $x^2 + xy + y^2$.

We may treat $y$ as an unknown function of $x$ and then differentiate normally using the chain rule.

In this way, $y^2$ becomes $2y \cdot \dfrac{dy}{dx}$ when differentiated.

$xy$ can be differentiated with the product rule to get $x \cdot \dfrac{dy}{dx} + 1 \cdot y$.

# Topic 2: Complex Numbers

Complex numbers (set notation $\mathbb{C}$) are a superset of the real number system. They consist of a real and imaginary part. The imaginary unit $i$ is introduced, where $i^2 = -1$.

Any complex number can always be written in the Cartesian form $a + ib$, where $a, b \in \mathbb{R}$.

**Real and Imaginary Parts**

For $z = a + ib$: The real part of $z$, $\mathrm{Re}(z)$, is $a$. The imaginary part of $z$, $\mathrm{Im}(z)$, is $b$. (**not** $ib$)

A real number $z$ has $\mathrm{Im}(z) = 0$, and a purely imaginary number $z$ has $\mathrm{Re}(z) = 0$.

Two complex numbers $a + ib$ and $c + id$ are equal if and only if $a = c$ and $b = d$.

Conversely, if two complex numbers are equal, then we can equate their real and imaginary parts.

**The Conjugate**

The conjugate of a complex number $z$ is denoted $\bar{z}$ and is obtained by reversing the sign of the imaginary part. If $z = a + ib$, then $\bar{z} = a - ib$.

*Some important identities*: $z + \bar{z} = 2\,\mathrm{Re}(z)$ $\quad z\bar{z} = |z|^2 \implies \dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2} \ (z \neq 0)$

**Operations on Complex Numbers**

We can perform operations on complex numbers by treating $i$ as a constant, remembering that $i^2 = -1$.

*Addition*: We can add two complex numbers by adding their real and imaginary parts.
$(a + ib) + (c + id) = (a + c) + i(b + d)$

*Subtraction*: We can subtract two complex numbers in a similar fashion.
$(a + ib) - (c + id) = (a - c) + i(b - d)$

*Multiplication*: We can multiply two complex numbers by treating them as binomials.

$$\begin{aligned}(a + ib)(c + id) &= ac + iad + ibc + i^2bd \\ &= (ac - bd) + i(ad + bc)\end{aligned}$$

*Division*: When dividing two complex numbers, we can "realise the denominator" by multiplying by its conjugate (similar to rationalising the denominator).

$$\begin{aligned}\frac{a + ib}{c + id} &= \frac{a + ib}{c + id} \cdot \frac{c - id}{c - id} \\ &= \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}\end{aligned}$$

*Note*: $i(-i) = -(i^2) = 1$.

*Absolute Value:* The absolute value, or modulus of a complex number $z = a + ib$ is given by $|z| = \sqrt{a^2 + b^2}$.

**The Commutativity of Taking the Conjugate**

Taking the conjugate is commutative with addition, subtraction, division and multiplication, i.e.

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \qquad \overline{z_1 - z_2} = \overline{z_1} - \overline{z_2} \qquad \overline{z_1 \times z_2} = \overline{z_1} \times \overline{z_2} \qquad \overline{z_1 \div z_2} = \overline{z_1} \div \overline{z_2} \qquad (\overline{z})^n = \overline{z^n}$$

**Finding the Square Root**

Suppose that the complex number $x + iy$ is a square root of $a + ib$. Then,

$$\begin{aligned} (x+iy)^2 &= a + ib \\ (x^2 - y^2) + i(2xy) &= a + ib \end{aligned}$$

Equating the imaginary parts and squaring gives $\begin{cases} x^2 - y^2 &= a \\ 2xy &= b \end{cases} \implies \begin{cases} (x^2 - y^2)^2 &= a^2 \\ (2xy)^2 &= b^2 \end{cases}$

Using the identity $(A + B)^2 = (A - B)^2 + 4AB$ with $A = x^2,\ B = y^2$, we obtain $(x^2 + y^2)^2 = a^2 + b^2$.

Since both $x^2 + y^2$ and $a^2 + b^2$ are positive, we may take the positive square root of both sides to get:

$$\begin{cases} x^2 - y^2 &= a \\ x^2 + y^2 &= \sqrt{a^2 + b^2} \end{cases}$$

Adding these equations and solving gives $\quad x = \pm\sqrt{\dfrac{a + \sqrt{a^2 + b^2}}{2}} \quad$ and $\quad y = \dfrac{b}{2x}$

**Solving Quadratics**

*Difference of two squares:* $\quad z^2 - a^2$

Factorising into $(z + a)(z - a)$, the roots are $z = \pm a$.

*Sum of two squares:* $\quad z^2 + a^2$

This can be rewritten as a difference of two squares, $z^2 - \left(-a^2\right)$.
Factorising into $(z + ia)(z - ia)$, the roots are $z = \pm ia$.

*General form:* $\quad az^2 + bz + c$

The quadratic formula $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ can be used, even if the coefficients are not real.

(It is necessary to find the square roots of the discriminant.)

**The Argand Diagram** (or complex plane)

All complex numbers can be represented on a two dimensional plane with a vertical and horizontal axis. This plane is called the Argand diagram, and has axes labelled either $x$ and $y$, or $\mathrm{Re}$ and $\mathrm{Im}$.

**Mod-Arg Form** (or polar form)

The mod-arg form is an alternate way to represent complex numbers.

$$z = r(\cos\theta + i\sin\theta) \qquad \text{or} \qquad z = r\,\mathrm{cis}\,\theta \qquad \text{(short for } \underline{\mathrm{c}}\mathrm{os} + \underline{\mathrm{i}}\ \underline{\mathrm{s}}\mathrm{in})$$

The *modulus* $r$ is the distance between a complex number and the origin on the Argand diagram.

The *argument* $\theta$ is the angle (in radians) between the complex number and the positive real axis. Conventionally, the argument $\theta$ that satisfies $-\pi < \theta \leq \pi$ is used – this is the *principal argument*.

<u>*Note:*</u> $\arg 0$ is undefined.

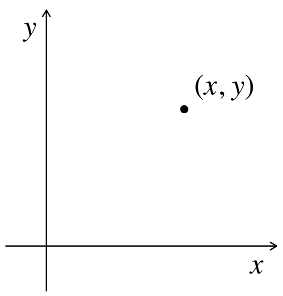
<u>*Converting From the Cartesian Form $z = x + iy$*</u>

The modulus $r$ is given by $r = |z| = \sqrt{x^2 + y^2}$.

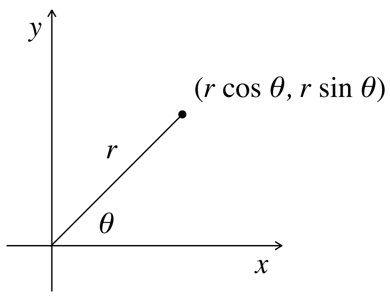
The argument $\theta$ is given by $\tan\theta = \dfrac{y}{x}$ (when finding $\theta$, keep in mind which quadrant $z$ should be in).

**The Representation of Complex Numbers on the Argand Diagram**

$z = x + iy$




$z = r(\cos\theta + i\sin\theta)$




<u>*Note*</u>: $\bar{z}$ is $z$ reflected across the $x$ axis, and $iz$ is the rotation of $z$ by $90^{\mathrm{o}}$ anti-clockwise about the origin.

**De Moivre's Theorem**

When multiplying two complex numbers together, we can add their arguments and multiply their moduli.

$$|zw| = |z||w| \qquad \arg zw = \arg z + \arg w \qquad \text{and} \qquad \left|\frac{z}{w}\right| = \frac{|z|}{|w|} \qquad \arg\frac{z}{w} = \arg z - \arg w$$

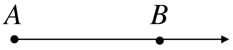
<u>*Proof:*</u> Let $z = a\,\mathrm{cis}\,\alpha,\ w = b\,\mathrm{cis}\,\beta.$

$$\begin{aligned} zw &= a(\cos\alpha + i\sin\alpha)\cdot b(\cos\beta + i\sin\beta) \\ &= ab(\cos\alpha\cos\beta - \sin\alpha\sin\beta + i[\sin\alpha\cos\beta + \sin\beta\cos\alpha)] \\ &= ab[\cos(\alpha+\beta) + i\sin(\alpha+\beta)] \end{aligned}$$

From this, we obtain De Moivre's theorem: $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$.

**Vectors**

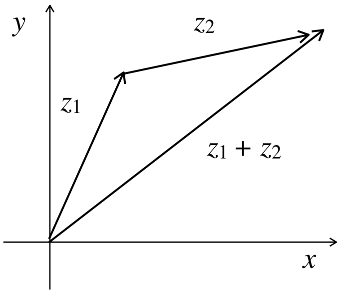
Complex numbers can be represented as vectors, which have a magnitude and direction.

The vector  can be written as $\overrightarrow{AB}$.



Any two vectors with the same magnitude and direction are equal, *regardless of where they are*.

<u>*Vector addition*</u>

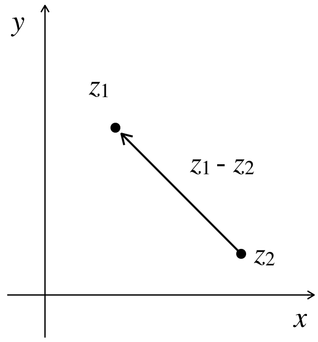




Vectors can be added by arranging them from tip to tail.

The resulting vector from the *tail of the first vector* to the *tip of the last* is the sum.

<u>*Vector subtraction*</u>



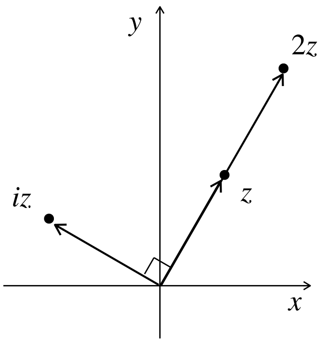


Vector subtraction can be performed by reversing the subtracted vector and adding (since $-\overrightarrow{AB} = \overrightarrow{BA}$).

For two vectors, the result is the vector drawn from the *vector that is subtracted,* to the *vector it is subtracted from*.

From the diagram, $|z_1 - z_2|$ represents the distance between $z_1$ and $z_2$.

<u>*Vector multiplication*</u>





A vector can be enlarged or rotated by multiplying it.

Multiplication by $\operatorname{cis}\theta$ rotates the vector by $\theta$ anti-clockwise.

Multiplication by a real number $k$ scales the vector by a factor of $k$.

For example, $i = \operatorname{cis}(\pi/2)$, so multiplying by $i$ is an anti-clockwise rotation of 90°.

**The Triangle Inequality**

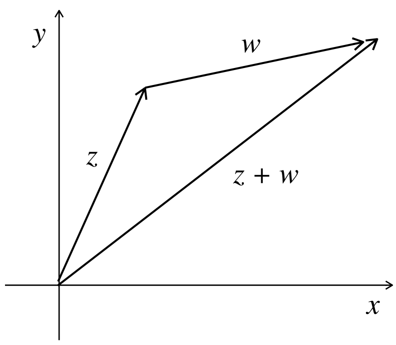
By considering the three sides of a triangle, it is obvious that:

- the sum of the lengths of any two sides is greater than or equal to the length of the third.
- the length of any one side is greater than or equal to the difference in lengths of the two other sides.

From this, we get the triangle inequality:

$$\big|\,|z| - |w|\,\big| \leq |z + w| \leq |z| + |w|$$

Equality occurs only in the degenerate case, when the three points are collinear and the triangle becomes a line.

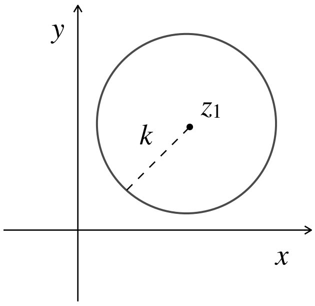


## Curves and Regions of Complex Numbers

Remember that $\arg 0$ is undefined, so points should be excluded using an open circle, where appropriate.

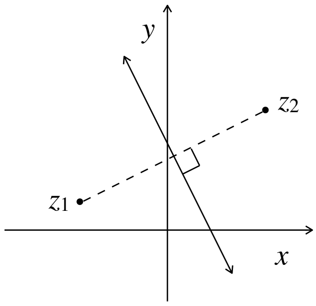
*Circle*

$|z - z_1| = k$



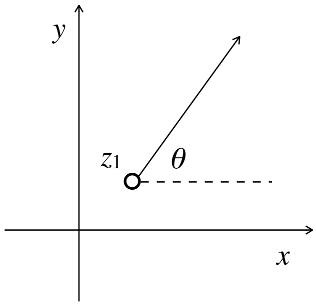

*Perpendicular Bisector*

$|z - z_1| = |z - z_2|$



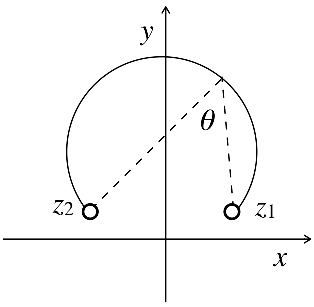

*Ray*

$\arg(z - z_1) = \theta$

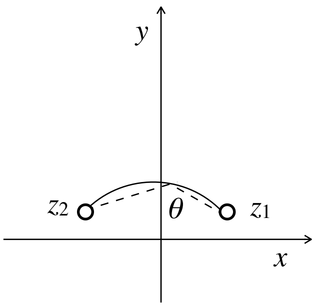



*Arc (or interval/rays)* $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \theta$ *Note*: the arc moves counterclockwise from $z_1$ to $z_2$.

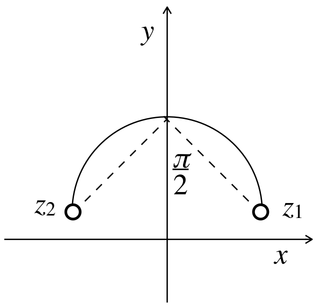
$\theta < \dfrac{\pi}{2}$ (acute)



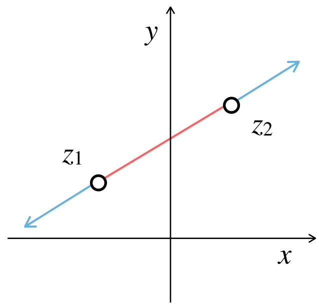

$\theta > \dfrac{\pi}{2}$ (obtuse)




$\theta = \dfrac{\pi}{2}$




$\theta = \pi$ $\theta = 0$




*Negative values of $\theta$*: If $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = -\theta$, then $\arg\left(\dfrac{z - z_2}{z - z_1}\right) = \theta$.

*Proof*: $\arg(z - z_1) - \arg(z - z_2) = -\theta$ so $\theta = \arg(z - z_2) - \arg(z - z_1)$.

## Complex Roots of Unity

We can solve equations of the form $z^n = 1$ by using De Moivre's theorem.

Let $z = \cos\theta + i\sin\theta$, so $z^n = \cos n\theta + i\sin n\theta$ ($|z| = 1$ since $z^n = 1$)

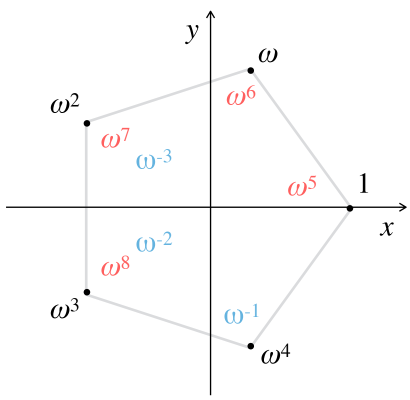
Since $z^n = 1$, and $1 = \cos 2k\pi + i\sin 2k\pi$ for some $k \in \mathbb{Z}$,

$\cos n\theta + i\sin n\theta = \cos 2k\pi + i\sin 2k\pi$

Equating real and imaginary parts,

$$\begin{cases} \cos n\theta &= \cos 2k\pi \\ \sin n\theta &= \sin 2k\pi \end{cases} \implies \theta = \frac{2k\pi}{n}$$

The values of $k$ that give unique solutions are $k = 0, 1, 2 \ldots (n-1)$.

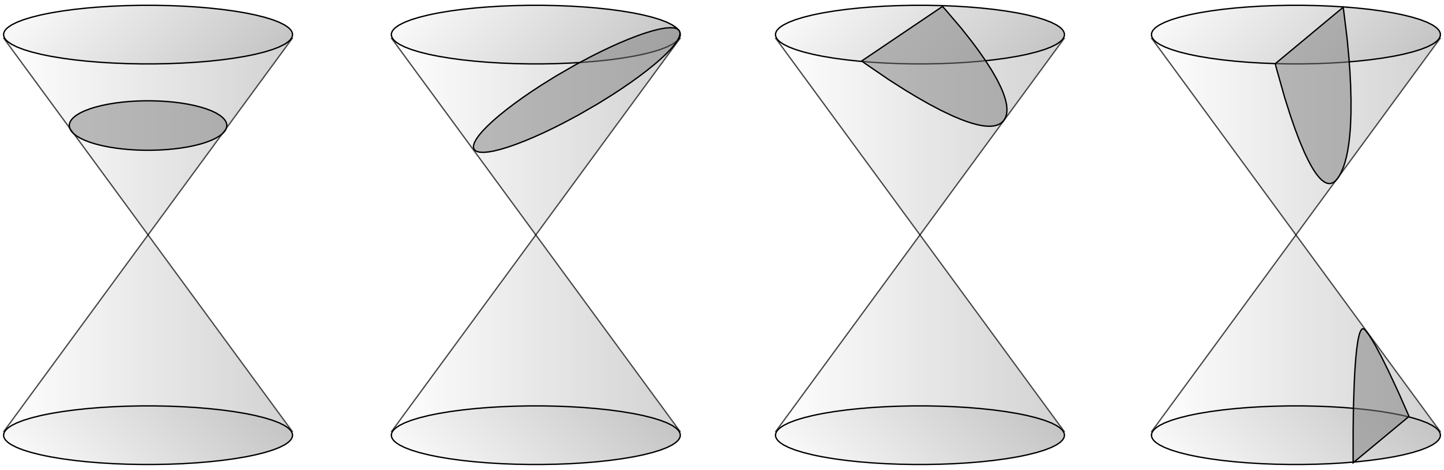



The roots of unity are equally spread out across the Argand diagram.

$1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0$

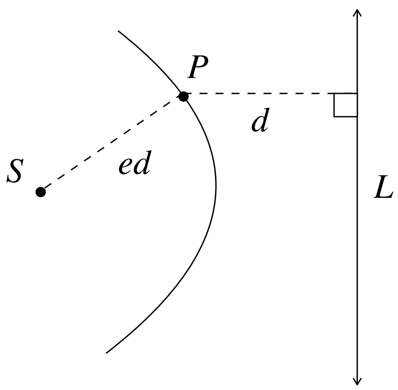
# Topic 3: Conics

The conic sections are the curves obtained when a "double cone" is sliced.



*Circle* *Ellipse* *Parabola* *Hyperbola*

Most conics can alternatively be defined as the locus of a point such that its distance from a fixed point and a fixed line is in the ratio $e : 1$, where $e$ is the eccentricity of the curve (not to be confused with Euler's number $e = 2.718...$).



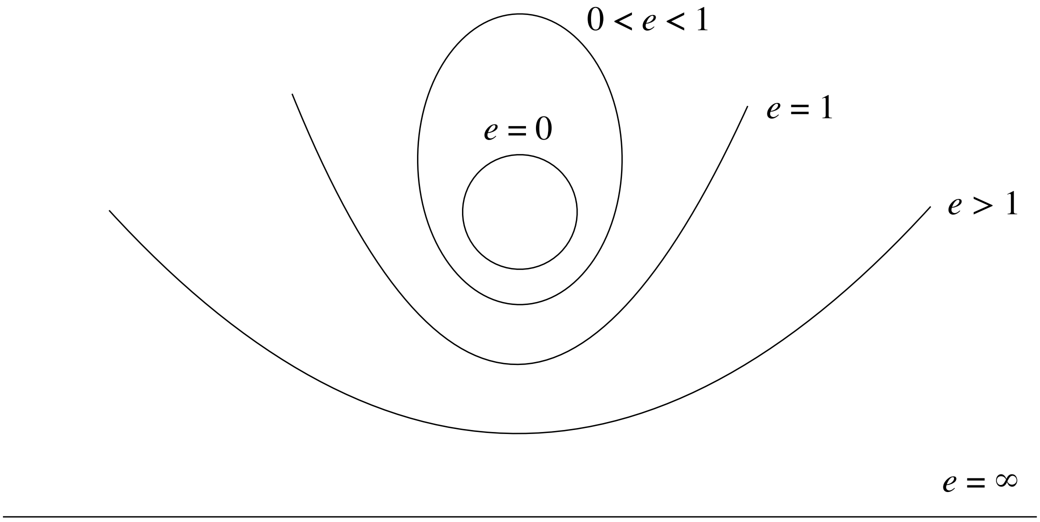

If $e = 0$, the conic is a **circle**.

If $0 < e < 1$, the conic is an **ellipse** (*that is not also a circle*).

If $e = 1$, the conic is a **parabola**.

If $e > 1$, the conic is a **hyperbola**.

Note: The circle cannot be defined using the focus and directrix definition. However, it can be thought of as the limiting case as $e$ approaches zero.

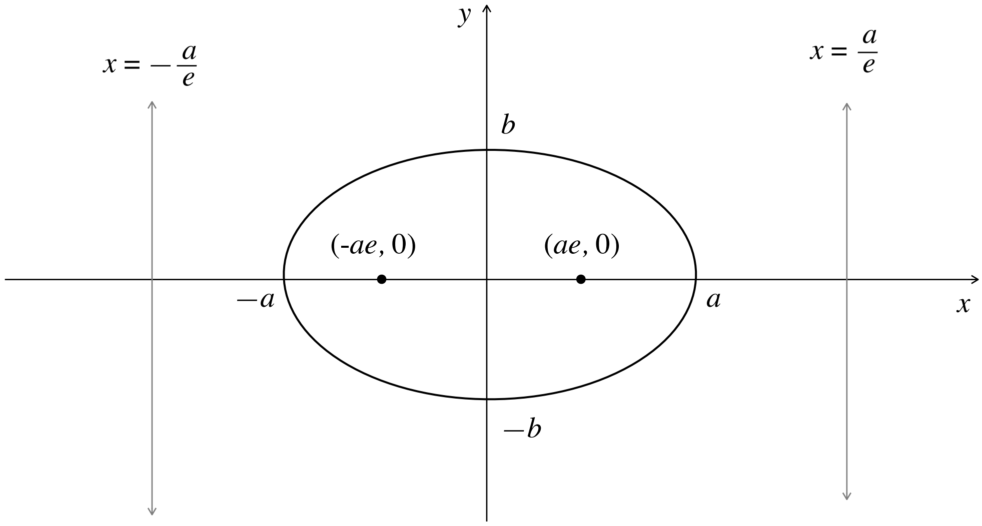



How the shape of a conic varies with the eccentricity $e$.

**Ellipse**

*Cartesian:* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

*Parametric:* $\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases}$




*Eccentricity*: $e = \sqrt{1 - \dfrac{b^2}{a^2}}$

*Foci*: $(\pm ae, 0)$

*Directrices:* $x = \pm \dfrac{a}{e}$

*Note:* We require $b < a$ for the eccentricity to be defined. If $b > a$, rearrange to get $\dfrac{y^2}{b^2} + \dfrac{x^2}{a^2} = 1$.

This is the graph of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, but with the values of $a$ and $b$, $x$ and $y$ swapped.

---

| | *Cartesian* | *Parametric* |
|---|---|---|
| *Gradient:* | $\dfrac{dy}{dx} = -\dfrac{b^2x}{a^2y}$ | $\dfrac{dy}{dx} = -\dfrac{b\cos\theta}{a\sin\theta}$ |
| *Tangent:* | $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$ | $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$ |
| *Normal:* | $\dfrac{a^2x}{x_1} - \dfrac{b^2y}{y_1} = a^2 - b^2$ | $\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2$ |

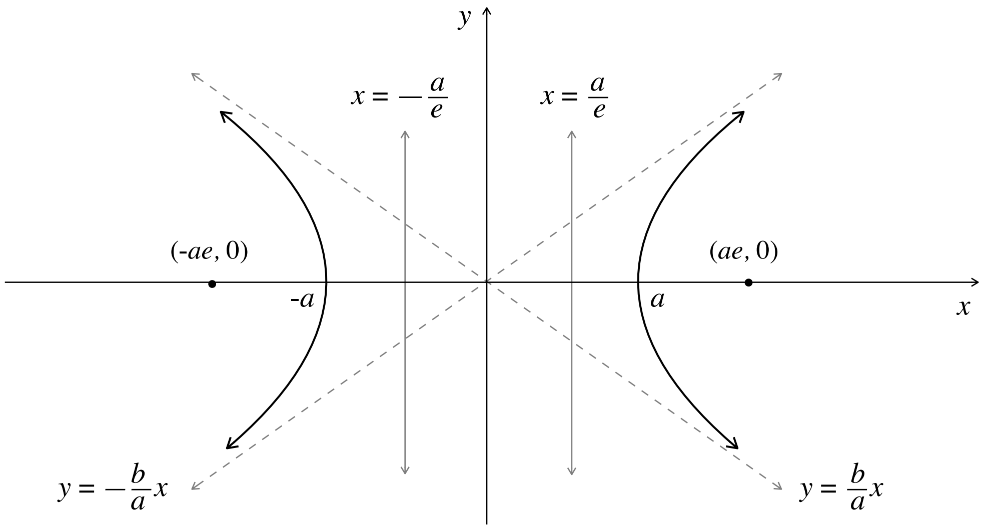
*Chord of Contact*: $\dfrac{xx_0}{a^2} + \dfrac{yy_0}{b^2} = 1$

*Note:* Use implicit differentiation to find the gradient function in Cartesian form.

**Hyperbola**

*Cartesian:* $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

*Parametric:* $\begin{cases} x = a\sec\theta \\ y = b\tan\theta \end{cases}$




*Eccentricity:* $e = \sqrt{1 + \dfrac{b^2}{a^2}}$

*Foci:* $(\pm ae, 0)$

*Directrices:* $x = \pm\dfrac{a}{e}$

*Asymptotes:* $y = \pm\dfrac{b}{a}x$

*Note:* If given $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$, rearrange to get $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$.

This is the graph of $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, but reflected about the origin and with the values of $a$ and $b$ swapped.

---

| | *Cartesian* | *Parametric* |
|---|---|---|
| *Gradient:* | $\dfrac{dy}{dx} = \dfrac{b^2x}{a^2y}$ | $\dfrac{dy}{dx} = \dfrac{b\sec\theta}{a\tan\theta}$ |
| *Tangent:* | $\dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} = 1$ | $\dfrac{x\sec\theta}{a} - \dfrac{y\tan\theta}{b} = 1$ |
| *Normal:* | $\dfrac{a^2x}{x_1} + \dfrac{b^2y}{y_1} = a^2 + b^2$ | $\dfrac{ax}{\cos\theta} + \dfrac{by}{\sin\theta} = a^2 + b^2$ |

*Chord of Contact:* $\dfrac{xx_0}{a^2} - \dfrac{yy_0}{b^2} = 1$

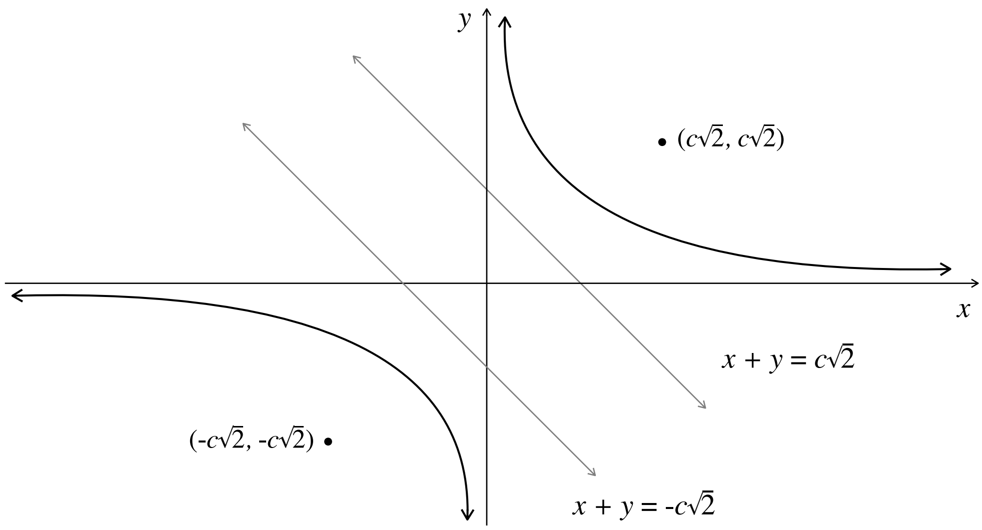
*Note:* Use implicit differentiation to find the gradient function in Cartesian form.

## **Rectangular Hyperbola**

A hyperbola is rectangular if its asymptotes are perpendicular.

*Cartesian:* $xy = c^2$

*Parametric:* $\begin{cases} x = cp \\ y = \frac{c}{p} \end{cases}$




*Eccentricity:* $e = \sqrt{2}$

*Foci:* $(\pm c\sqrt{2}, \pm c\sqrt{2})$

*Directrices:* $x + y = \pm c\sqrt{2}$

*Asymptotes:* $x = 0, \quad y = 0$

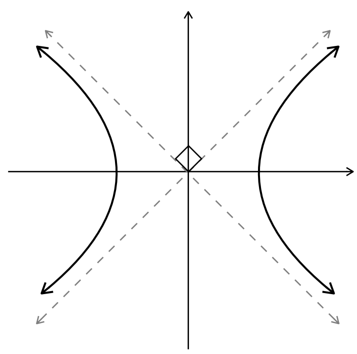
---

| | *Cartesian* | *Parametric* |
|---|---|---|
| *Gradient:* | $\frac{dy}{dx} = -\frac{y}{x}$ | $\frac{dy}{dx} = -\frac{1}{p^2}$ |
| *Tangent:* | $xy_1 + x_1y = 2c^2$ | $x + p^2y = 2cp$ |
| *Normal:* | $xx_1 - yy_1 = (x_1)^2 - (y_1)^2$ | $p^3x - py = c(p^4 - 1)$ |

*Chord of Contact:* $xy_0 + x_0y = 2c^2$

The curve $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$ or $x^2 - y^2 = a^2$ is also a rectangular hyperbola.

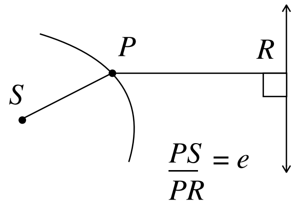
This hyperbola is oriented differently to $xy = c^2$. Here, $a = c\sqrt{2}$.



*Note:* Use implicit differentiation to find the gradient function in Cartesian form.

## Geometric Properties of Conic Sections

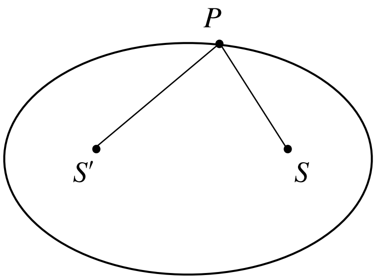
The ratio of the distance from the curve to the focus, and the distance from the curve to the directrix is $e : 1$, where $e$ is the eccentricity of the curve.



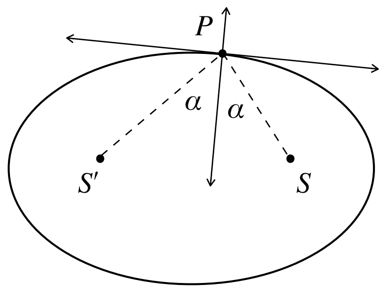

*Ellipse*

The sum of focal lengths is constant.

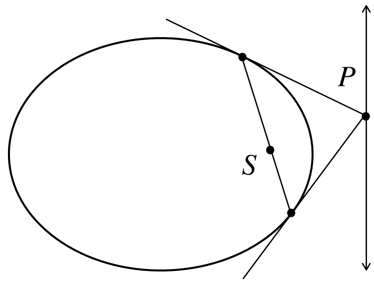
$PS + PS' = 2a$




(*Reflection Property*) The normal at any point bisects the angle subtended by the foci at that point.



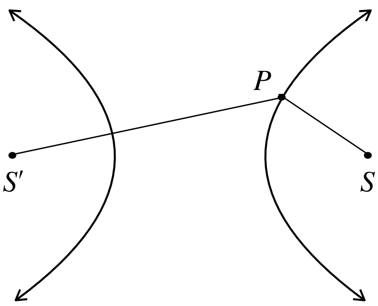

The chord of contact from a point on the directrix is a focal chord.



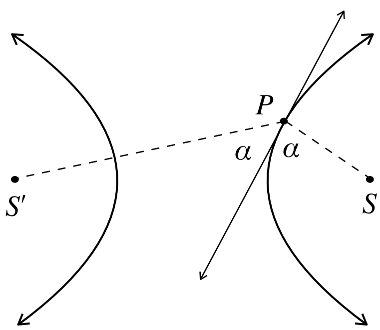

*Hyperbola*

The difference between focal lengths is constant.

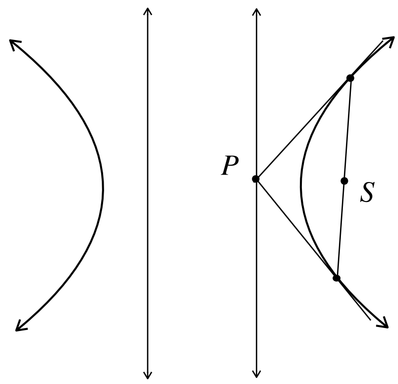
$|PS - PS'| = 2a$




(*Reflection Property*) The tangent at any point bisects the angle subtended by the foci at that point.



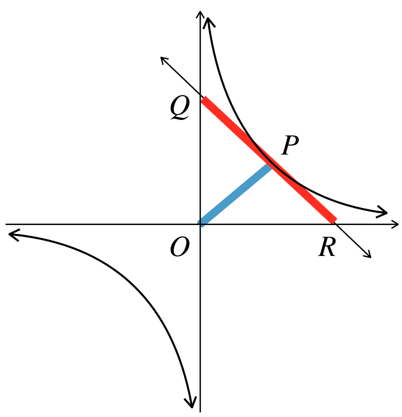

The chord of contact from a point on the directrix is a focal chord.



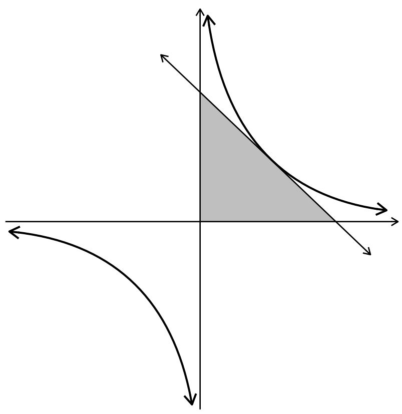

*Rectangular Hyperbola*

The length of the tangent between the asymptotes is twice the length from the point of contact to the intersection of the asymptotes.

$$QR = 2OP$$




The area of the triangle bounded by the tangent and asymptotes is constant.



## Topic 4: Integration

### Substitutions

*Trigonometric substitutions*

We can integrate some algebraic expressions by using trigonometric identities.

| Expression | Substitution | | |
|---|---|---|---|
| $\sqrt{a^2 - x^2}$ | $x = a \sin\theta$ | $dx = a\cos\theta \, d\theta$ | *Note*: the expression becomes $\lvert \cos x \rvert$, not $\cos x$. |
| $x^2 + a^2$ | $x = a\tan\theta$ | $dx = a\sec^2\theta \, d\theta$ | |
| $x^2 - a^2$ | $x = a\sec\theta$ | $dx = a\sec\theta\tan\theta \, d\theta$ | *Note*: this requires $x \leq \lvert a \rvert$, or else no value of $\theta$ exists. |

*$t$-substitutions*

If $t = \tan\frac{x}{2}$, then $dx = \dfrac{2}{1+t^2}\,dt$ and:

$$\sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad \tan x = \frac{2t}{1-t^2}$$

*Note*: this method can only be used when $\tan\frac{x}{2}$ is defined, i.e. $x$ cannot be an odd multiple of $\pi$.

### Partial Fractions

For integrals of the form $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$ where the degree of the numerator is *less than* the denominator,

We can decompose polynomial fractions of the form $\dfrac{A(x)}{B(x)\,C(x)\,D(x)\ldots}$ by setting:

$$\frac{A(x)}{B(x)\,C(x)\,D(x)\ldots} \equiv \frac{R_1(x)}{B(x)} + \frac{R_2(x)}{C(x)} + \frac{R_3(x)}{D(x)} + \ldots$$ (which is an identity, i.e. valid for all values of $x$)

For the fractions on the RHS, the degree of the numerator is one less than the degree of the denominator.

$R_1(x), R_2(x), R_3(x)$ … can be found by equating coefficients or substituting for $x$ to eliminate terms.

*Repeated Factors*

If the denominator has a repeated factor, e.g. $\dfrac{A(x)}{B(x)\,C(x)^3}$,

then the decomposition must contain the repeated factor several times, each with reducing multiplicity, e.g.

$$\frac{A(x)}{B(x)\,C(x)^2} \equiv \frac{R_1(x)}{B(x)} + \frac{R_2(x)}{C(x)} + \frac{R_3(x)}{C(x)^2} + \frac{R_4(x)}{C(x)^3}$$

**Long Division**

For integrals of the form $\int \frac{P(x)}{Q(x)}\,dx$ where the degree of the numerator is *greater than* the denominator,

We may perform a polynomial long division and then integrate:

$$\frac{P(x)}{Q(x)} = D(x) + \frac{R(x)}{Q(x)}$$

$$\int \frac{P(x)}{Q(x)}\,dx = \int D(x)\,dx + \int \frac{R(x)}{Q(x)}\,dx$$ (the degree of $R(x)$ must be less than the degree of $Q(x)$.)

**Integration by Parts**

For integrals of the form $\int f(x)\,g(x)\,dx$,

We can reverse the product rule $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$ by integrating it with respect to $x$, which results in

$$\int u\,dv = uv - \int v\,du$$ after rearranging.

We can then choose $u = f(x)$ and $dv = g(x)\,dx$.

When choosing $u$ and $dv$, note that $u$ should usually (but not always) become simpler when differentiated and $dv$ should not become more complicated after integration.

<u>Setting $dv = dx$</u>

When the integrand is not a product of two other functions e.g. of the form $\int f(x)\,dx$, we can note that $f(x) = f(x) \cdot 1$ and then set $dv = 1\,dx \implies v = x$. This is useful for integrals like $\int \sin^{-1} x\,dx$.

**Reduction Formulae** (recurrence relations)

Reduction formulae allow us to easily evaluate integrals $I_n$ involving an index $n$ (e.g. $I_n = \int \sin^n x\,dx$) by recursively defining the integral in terms of one with a reduced index, for example $I_{n-1}$ or $I_{n-2}$.

e.g. if $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$ then $I_n = \frac{n-1}{n} \cdot I_{n-2}$.

Integration by parts is frequently used to prove reduction formulae.

**Properties of Definite Integrals**

*Symmetry*

It can be shown that $\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x) + f(-x)\,dx.$

*Proof:* $\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{-a}^{0} f(x)\,dx$

Let $u = -x \implies du = -dx$

$$= \int_{0}^{a} f(x)\,dx - \int_{a}^{0} f(-u)\,du$$
$$= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(-x)\,dx \text{ (using dummy variables)}$$
$$= \int_{0}^{a} f(x) + f(-x)\,dx$$

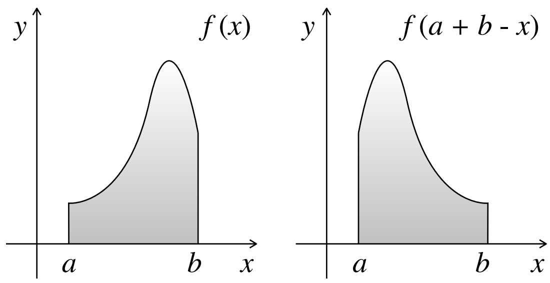
*Corollaries*: If $f(x)$ is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$

If $f(x)$ is odd, then $\int_{-a}^{a} f(x)\,dx = 0.$

*Reflection*

By reflecting the graph in such a way that the area remains the same, we get:

$$\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx$$




Note that the area remains the same even when flipped.

This can be generalised to:

$$\int_{a}^{b} f(x)\,dx = \int_{a}^{b} f(a+b-x)\,dx$$

*Proof:* Let $x = a + b - u \implies dx = -du$

$$\int_{a}^{b} f(x)\,dx = -\int_{b}^{a} f(a+b-u)\,du$$
$$= \int_{a}^{b} f(a+b-u)\,du$$
$$= \int_{a}^{b} f(a+b-x)\,dx \quad \text{(dummy variables)}$$

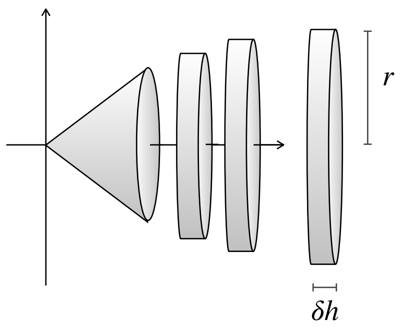
# Topic 5: Volumes

### Volumes of Solids of Revolution

The volume of a solid of revolution can be estimated by breaking up the solid into individual cylinders and finding a formula for the area of each cross-section. Then the volumes can be calculated and summed.

The volume of an individual cylinder $\delta V$ is given by:

$$\begin{aligned}\delta V &= A\ \delta h \\ &= \pi r^2\ \delta h\end{aligned}$$




In this diagram, $r = y$ and $\delta h = \delta x$.
The area is given by $A = \pi y^2$.

The volume of the whole solid is given by summing up the volumes of each individual cylinder and taking the limit as $\delta h \to 0$.

$$V = \lim_{\delta h \to 0} \sum_{h=a}^{b} \pi r^2\ \delta h$$

This is the definition of an integral, so we may rewrite the formula as:

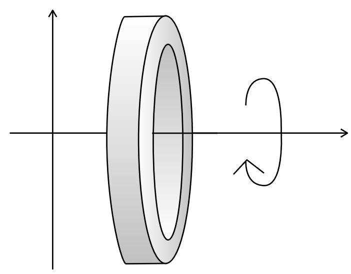
$$V = \pi \int_a^b r^2\ dh$$ (depending on which axis is used, $dh$ will be replaced with either $dx$ or $dy$).

### Volumes by Subtraction

If the cross-section of a solid is an annulus (ring), we can calculate its area by subtracting the inner circle from the outer one.

$$\begin{aligned}\delta V &= A\ \delta h \\ &= \pi\left(r_{\text{outer}}\right)^2 - \pi\left(r_{\text{inner}}\right)^2\ \delta h \\ &= \pi\left[\left(r_{\text{outer}}\right)^2 - \left(r_{\text{inner}}\right)^2\right]\ \delta h\end{aligned}$$

$$\therefore V = \pi \int_a^b \left(r_{\text{outer}}\right)^2 - \left(r_{\text{inner}}\right)^2\ dh$$



An annulus formed by revolution.

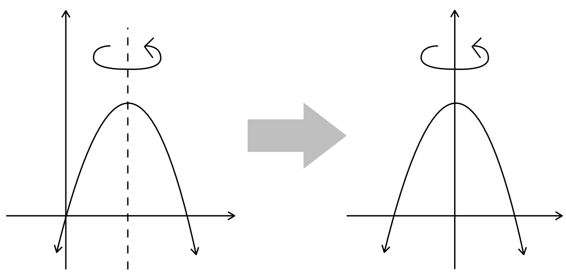
### Shifts and Reflections

If the axis of rotation is not the the $x$ or $y$-axis but some line parallel to them, a shift or reflection can simplify the problem.

<u>Shift:</u> To shift right $a$ units, replace $x$ with $x - a$.
To shift left $a$ units, replace $x$ with $x + a$.

To shift up $a$ units, replace $y$ with $y - a$.
To shift down $a$ units, replace $y$ with $y + a$.

<u>Reflect:</u> To reflect along the line $x = a$, replace $x$ with $2a - x$.
To reflect along the line $y = a$, replace $x$ with $2a - y$.



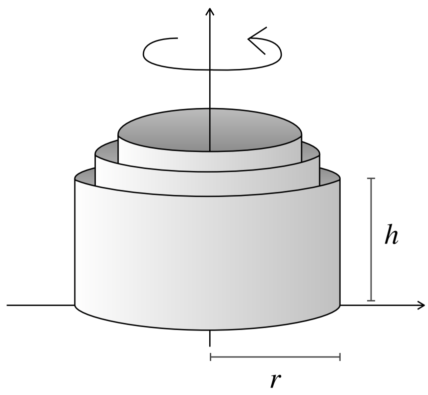
## Volumes by Cylindrical Shells

In this method, we divide the solid into many hollow cylindrical shells. Then we calculate the volume of each shell and sum.

$$\delta V = 2\pi r h \,\delta r$$

$$V = 2\pi \int_a^b r h \, dr$$

Integration occurs in the direction perpendicular to the axis of rotation.




The volume of each shell is the circumference times the height.

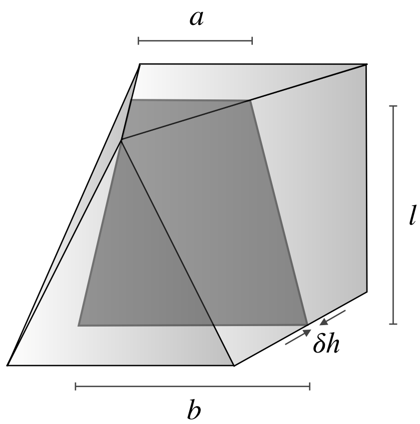
## Volumes of Solids with Known Cross-Sections

If the area of a cross section can be expressed in terms of the height then the volume can be determined by slicing using the general formula:

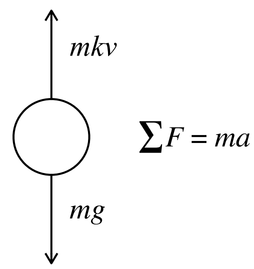
$$\delta V = A \,\delta h$$

$$V = \int_a^b A \, dh$$

and determining an expression for $A$ in terms of $h$.




$A = ½\, l\,(a + b)$




Start by drawing a free body diagram, showing all the forces acting on an object.

- Conventionally, the initial direction of motion is taken to be positive.
- The gravitational force is given by $mg$.
- By Newton's second law, the sum of all forces $\sum F$ is equal to $ma$ (or $m\ddot{x}$).
- By Newton's third law, if two objects are touching, then they both exert equal forces on each other.
- If a force is not purely horizontal or purely vertical, resolve it into its horizontal and vertical components.
- If two objects are connected by a string then both undergo a tension force in the direction of the other.
  - Tension is a non-negative quantity. If it is zero then the string is not taut.
  - The tension forces on both objects connected by the same string are equal.

**Resisted Motion**

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = v\frac{dv}{dx} = \frac{dv}{dt}$$ (the last two are the most useful with resisted motion)

Form an equation for acceleration: $m\ddot{x} = \dots$ and integrate using one of the above forms.
Find the constant of integration using the initial conditions.

*Terminal Velocity:* The terminal velocity is given by $\lim_{t\to\infty} v$ or $v$ when $\ddot{x} = 0$.

**Circular Motion**

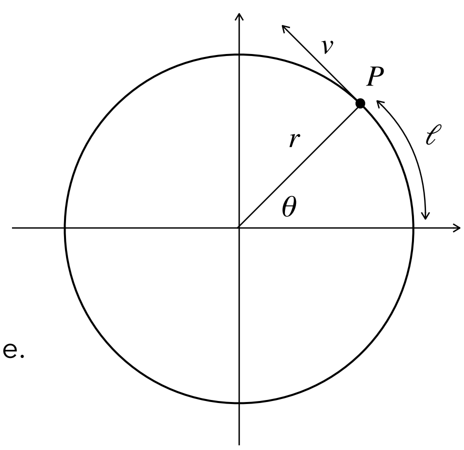
*Displacement*

The displacement of a point on a circle is the arc length, which is given by $\ell = r\theta$, where $\theta$ is the angle swept out by the point. Note that $\theta$ itself is also a function of time.

*Angular Velocity*

Angular velocity is the rate of change of the angle $\theta$ swept out by an object (around a point) over time.
It is typically measured in radians per second (this is required for the period formula to work).
Angular velocity is usually denoted by $\omega$, where $\omega = \dot{\theta} = \dfrac{d\theta}{dt}$.

*Period:* The period of motion is given by $T = \dfrac{2\pi}{\omega}$.

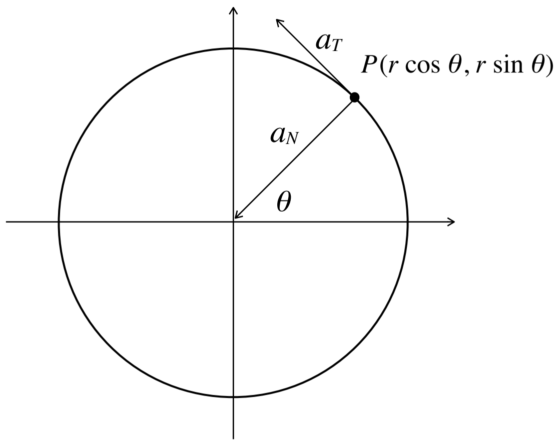



*Tangential Velocity*

Tangential velocity is how fast the object is moving around the circle.
It can be obtained by differentiating displacement with respect to time.

$$v = \frac{d}{dt}(r\theta) = r\dot{\theta} = r\omega$$

*Acceleration*




Given a point $P(r\cos\theta,\ r\sin\theta)$, we can find acceleration by differentiating the vertical and horizontal components separately.

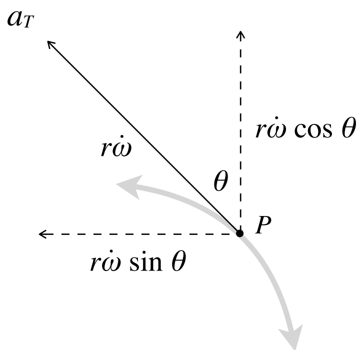
Keeping in mind that $\theta$ is a function of $t$ and that $\omega = \dot{\theta}$, applying the chain rule gives:

| | |
|---|---|
| $x = r\cos\theta$ | $y = r\sin\theta$ |
| $\dot{x} = -r\omega\sin\theta$ | $\dot{y} = r\omega\cos\theta$ |
| $\dot{x} = -r\omega^2\cos\theta - r\dot{\omega}\sin\theta$ | $\dot{y} = -r\omega^2\sin\theta + r\dot{\omega}\cos\theta$ |

The expressions for $\ddot{x}$ and $\ddot{y}$ can be split into two separate vectors:

**Normal** acceleration $\begin{cases}\ddot{x} = -r\omega^2\cos\theta \\ \ddot{y} = -r\omega^2\sin\theta\end{cases}$ and **tangential** acceleration $\begin{cases}\ddot{x} = -r\dot{\omega}\sin\theta \\ \ddot{y} = \quad r\dot{\omega}\cos\theta\end{cases}$.

*Tangential Acceleration*

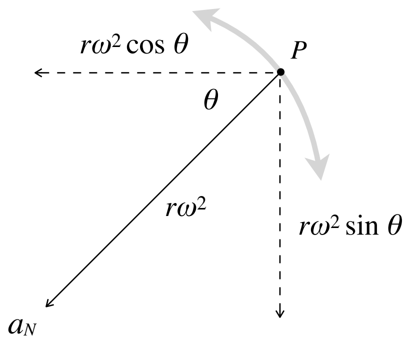



Tangential acceleration changes the tangential velocity.
It can also be obtained by differentiating tangential velocity.

$$a_T = \frac{dv}{dt} = r\dot{\omega}$$

In the case of uniform circular motion ($\omega$ is constant), $\dot{\omega} = 0$ and $a_T = 0$.

*Centripetal Acceleration (Normal Acceleration)*




Centripetal (or normal) acceleration controls the object's turning.
It is always directed towards the centre of motion (e.g. centre of the circle).

$$a_N = r\omega^2 \qquad \text{or} \qquad a_N = \frac{v^2}{r}, \qquad \text{since } v = r\omega$$

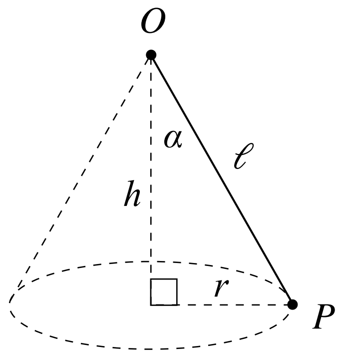
---

**Circular Motion Equations**

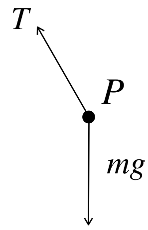
*Note*: For uniform circular motion, $\omega$ is constant and so $\dot{\omega} = 0$.

| | | | | |
|---|---|---|---|---|
| *Period:* | $T = \dfrac{2\pi}{\omega}$ | | | |
| *Displacement:* | $\ell = r\theta$ | | | |
| *Velocity:* | *Tangential:* $v = r\omega$ | | *Angular:* | $v_A = \omega$ |
| *Acceleration:* | *Centripetal:* $a_N = r\omega^2 = \dfrac{v^2}{r}$ | | *Tangential:* | $a_T = r\dot{\omega}$ |

## Conical Pendulum




In a simple conical pendulum, the only forces acting on the particle $P$ are the tension $T$ and gravity.

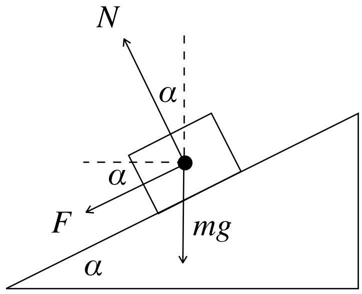



If the particle has no vertical acceleration, the sum of vertical forces must be zero, i.e. $T\cos\alpha - mg = 0$.

Because this motion is circular, the net horizontal force is the centripetal force. In this case, $T\sin\alpha = mr\omega^2$.

Also, $\sin\alpha = \dfrac{r}{\ell}$ and $\cos\alpha = \dfrac{h}{\ell}$.

## Banked Track

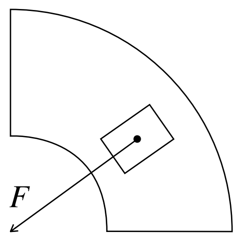



A banked track is a slanted track that enables a higher turning speed.

Any objects on it experience a normal force (perpendicular to the slant) and the gravitational force.

If the object is not moving at a certain speed, it will tend to slip off the track.
If this happens, a lateral force due to friction is introduced, which helps the object stay on the track.

This lateral force can be in one of two directions:
- down the slope (if the object tries to slip up the track by moving too fast), or
- up the slope (if the object tries to slip down the track by moving too slow).




A top-down view.

If the object has no vertical acceleration (i.e. its friction prevents slipping), then net vertical force is zero. i.e. $N\cos\alpha - F\sin\alpha - mg = 0$.

Because this motion is circular, the net horizontal force is the centripetal force. In this case, $N\sin\alpha + F\cos\alpha = mr\omega^2$.

## Topic 7: Polynomials

A polynomial is an algebraic expression of the form $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0$ for $n \in \mathbb{Z}^+$.

**The Integer Root Theorem**

An integer root of any polynomial with integer coefficients is a factor of the constant term.

<u>Proof:</u> Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0$.
Suppose $\beta$ is an integer root of $P(x)$. Then, $P(\beta) = 0$.

$$\begin{aligned} a_n\beta^n + a_{n-1}\beta^{n-1} + \ldots + a_1\beta + a_0 &= 0 \\ a_n\beta^n + a_{n-1}\beta^{n-1} + \ldots\ldots\ldots + a_1\beta &= -a_0 \\ \beta(a_n\beta^{n-1} + a_{n-1}\beta^{n-2} + \ldots\ldots\ldots + a_1) &= -a_0 \end{aligned}$$

Since $\beta$ is an integer, it is therefore a divisor of the constant term $a_0$.

<u>Note</u>: this proof can be generalised to rational roots: if $p/q$ is a rational root of a polynomial, then $p$ is a factor of the constant term, and $q$ is a factor of the leading coefficient.

**Multiple root theorem**

If a polynomial $P(x)$ has a root $\alpha$ of multiplicity $r$ (i.e. $P(x)$ has $(x-\alpha)^r$ as a factor), then $\alpha$ is also a root of all derivatives of $P(x)$ up to the $(r-1)^{\text{th}}$ derivative.

Conversely, if $\alpha$ is a root of $P(x)$ and all its derivatives up to the $(r-1)^{\text{th}}$ derivative, then it is a root of multiplicity $r$ (that is, $P(x)$ has $(x-\alpha)^r$ as a factor).

<u>Proof:</u> Let $P(x)$ be a polynomial with a root $\alpha$ of multiplicity $r$ where $r > 1$.

$$\begin{aligned} P(x) &= (x-\alpha)^r Q(x) \\ P'(x) &= (x-\alpha)^r \cdot Q'(x) + Q(x) \cdot r(x-\alpha)^{r-1} \quad \text{(by the product rule)} \\ &= (x-\alpha)^{r-1}\big(Q'(x)(x-\alpha) + rQ(x)\big) \end{aligned}$$

Therefore $\alpha$ is also a root of $P'(x)$.

**Complex conjugate root theorem**

If $P(z)$ is a polynomial *with real coefficients* then $P(\overline{\alpha}) = \overline{P(\alpha)}$.

<u>Proof</u>: Suppose $P(\alpha) = a_n\alpha^n + a_{n-1}\alpha^{n-1} + \ldots + a_0$.
Then, $P(\overline{\alpha}) = a_n\overline{\alpha}^n + a_{n-1}\overline{\alpha}^{n-1} + \ldots + a_0$
Since the coefficients are real, taking the conjugate is commutative ($\overline{\alpha}^n = \overline{\alpha^n}$ and $\overline{\alpha^n} + \overline{\alpha^m} = \overline{\alpha^n + \alpha^m}$).
$\therefore P(\overline{\alpha}) = \overline{a_n\alpha^n + a_{n-1}\alpha^{n-1} + \ldots + a_0} = \overline{P(\alpha)}$

<u>Corollary</u>: If $\alpha$ is a root of a polynomial, then $\overline{\alpha}$ is a root too. (If $P(\alpha) = 0$, then $P(\overline{\alpha}) = \overline{P(\alpha)} = \overline{0} = 0$.)

**Fundamental Theorem of Algebra**

The fundamental theorem of algebra states that every polynomial with a degree of at least one has at least one zero (which may or may not be real).

**Polynomial Root Transformations**

To transform the roots of a polynomial $P(x)$ by a function $f$, replace $x$ with $f^{-1}(x)$ in the polynomial.

> *Proof*: Consider the polynomial $P(x)$. If $x$ is a root, then $P(x) = 0$.
> To transform the roots by a function $f$, we want to satisfy the equation $P(f(x)) = 0$.
> Replacing $x$ with $f^{-1}(x)$ gives $P(f(f^{-1}(x))) = P(x) = 0$.

*Common Transformations:*

| Desired Transformation | | Replacement |
|---|---|---|
| Addition | $\alpha \to \alpha + k$ | $x \to x - k$ |
| Multiplication | $\alpha \to k\alpha$ | $x \to \dfrac{x}{k}$ |
| Squaring | $\alpha \to \alpha^2$ | $x \to \sqrt{x}$ |
| Reciprocation | $\alpha \to \dfrac{1}{\alpha}$ | $x \to \dfrac{1}{x}$ |

*Note:* $\alpha$ is any root of the polynomial $P(x)$.

# Topic 8: Harder 3 Unit Topics

**Inequalities**

Inequalities can be added together or subtracted from each other. They can also be multiplied or divided if the quantities do not change sign.

To prove $a > b$, it is sufficient to show that $a - b > 0$.

*The AM-GM Inequality*

The arithmetic mean is always greater than the geometric mean.

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}$$

Proof (*for $n = 2$):*

$$\begin{aligned} \left(\sqrt{x} - \sqrt{y}\right)^2 &> 0 \\ x + y - 2\sqrt{xy} &> 0 \\ x + y &> 2\sqrt{xy} \\ \frac{x + y}{2} &> \sqrt{xy} \end{aligned}$$

This inequality can be used to prove other inequalities.

*Other Useful Inequalities*

Noting that the square of a real number is always non-negative,

$$(a - b)^2 \geq 0 \qquad \Longrightarrow \qquad a^2 + b^2 \geq 2ab$$

$$(a + b)^2 = (a - b)^2 + 4ab \qquad \Longrightarrow \qquad (a + b)^2 \geq 4ab$$

*Using Calculus*

An inequality such as $a > b$ can be proved by showing that $a - b > 0$.

This can be done using calculus to analyse the function e.g. showing that the absolute minimum of the function is zero (or greater).

**Strong Induction**

With normal ("weak") induction, you prove a statement $S(n)$ to be true by:
- proving a base case,
- assuming $S(k)$ is true for some $k$, and
- proving that $S(k+1)$ must then be true.

With strong induction, you prove a statement $S(n)$ to be true by:
- proving a base case,
- assuming $S(k)$ **and all preceding statements** $S(k-1),\ S(k-2)\ldots$, down to the base case are true for some $k$, and
- proving that $S(k+1)$ must therefore be true.

These additional assumptions provide more information to work with, which may be helpful when attempting to proves the inductive step.

_Note:_ For many scenarios, it may not be necessary to use ALL preceding statements to prove $S(k+1)$.