Generalizations of Rohlfs' Model of Demand

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1 Overview

In 1974, as a member of the technical staff at Bell Laboratories, Jeffrey Rohlfs investigated how network externalities - in which the demand for a service depends not only on its price, but also on the number of people subscribing - affect the demand for a good or service. His paper attempted to explain the public's recent failure to adopt a video communication network created by Bell Labs earlier that year [1]. In this paper, he critically assumed a uniform distribution of maximum willingness to pay (WtP) for all members of a group looking to use a service. In other words, he assumed no single price is more favorable to a group than any other price. While this does decrease the complexity of the problem, it is not incredibly realistic, as, in practice, WtP might follow a Gaussian-like distribution - in which the rate of subscribing to a product begins to decrease past some set price [2-4].

Here, I present an extension of Rohlfs' Model to arbitrary Beta distributions, Beta(α,β), which can be shown to encapsulate his original $U\{0,1\}$ WtP assumption. Furthermore, I will look at how varying distribution parameters α, β affect the *maximum charging price*, the *unsteady state fraction* of the population before which the service will always fail, and the *steady state fraction* of the population that will ultimately be reached once the adoption fraction has passed the unsteady state equilibrium.

2 Prior Work

Rohlfs was not the first researcher to look into market behavior in the presence of network externalities. He was, however, one of the first to derive a model that could explain how network externalities affect equilibrium adoption rates of a service. Rohlfs cited earlier papers by Artle & Averous [5] and by Squire [6] that assumed network externalities; and models such as the Bass model (1969) [7] and Logistic Model (1935) [8] already provided analyses of how markets evolve *over time* taking into account network externalities. Previous work, however, appeared to care more for proper pricing in the presence of network externalities, while Rohlfs looked for a deeper understanding behind market evolution.

Rohlfs' model includes three key assumptions (taken from Prof. Mitra's slides):

1. Additive utility. For individual i, the Utility, or WtP is given by:

$$U_i = \sum_{j \in F} w_{ij} \tag{1}$$

where w_{ij} is the incremental utility to individual i of a communication link to j, and F is the subscriber set.

2. Uniform calling, i.e. homogeneity of individual's calling, with no preferences such as friendships or communities of interest. In this case, the utility equation above can be modified as:

$$\begin{aligned} U_i &= \textstyle\sum_{j \in F} w_{ij} \\ &= \textstyle\sum_{j \in F} w_i \\ &= (N * f) w_i \end{aligned} \tag{2}$$

since $w_{ij} = w_i$ for all j - where N is the population, F is the set of subscribers, and f is the subscribing fraction of population.

3. Uniform distribution of Willingness-to-Pay, such that w_i is ordered between 0 and 1. Here, individuals are indexed in increasing order of their WtP such that if $w_i > w_{\hat{i}}$, and $\hat{i}$ is a subscriber, i must also be a subscriber.

To summarize Rohlfs' model, let p be the price of a service, and $\hat{i}$ be the marginal customer such that:

$$U_{\hat{i}} = p = (N * f) w_{\hat{i}} \tag{3}$$

$$w_{\hat{i}} = 1 - f \tag{4}$$

from the assumption that WtP is $U\{0,1\}$. Thus, by combining (3) and (4), we obtain the canonical demand function in the presence of network externality,

$$p = N * f(1 - f) \tag{5}$$

which can be seen in Figure 1, where C represents the steady state equilibrium position, and B represents the unsteady state equilibrium. The following section, *Assertion and Validation* will modify the last assumption, and look at how non-uniform distributions of WtP affect properties of the demand function.

A plethora of academic literature came after the publication of this paper, and its impact was widespread. Papers by Joseph Farrell and Garth Saloner [9] and by Michael Katz and Carl Shapiro [10] in the mid-

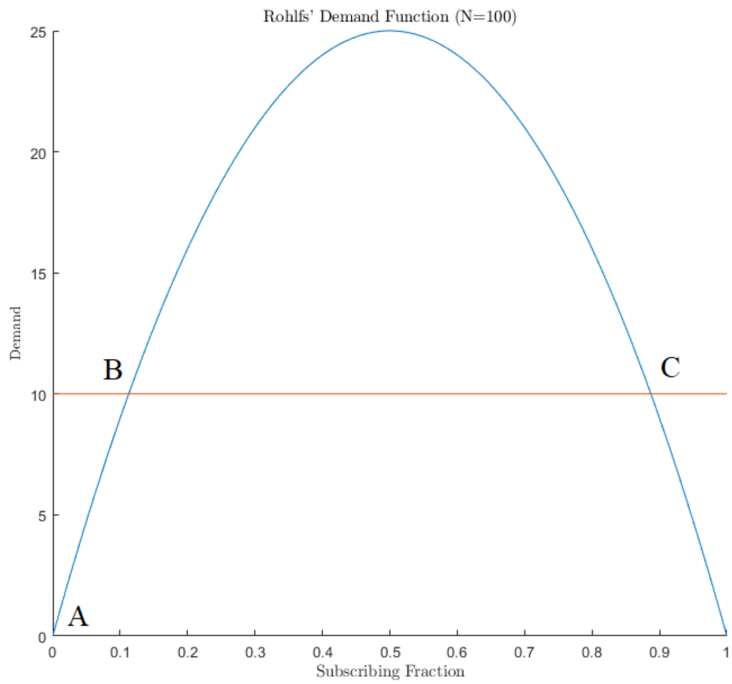

Figure 1: Canonical Demand Function with Network Externality

1980's and 1990's served to popularize Rohlfs' model. Shapiro et. al indicated that regulators should pay increasingly close attention to markets which have network effects to ensure that the this, in combination of short-term market power does not spiral into anti-competitive monopolies. Katz et. al, furthered Rohlfs' ideas and applied them to decisions of purchasing fax machines, and in Beta versus VHS decisions. In addition, the paper was mentioned during a government case against Microsoft to explain markets with non-negligible network externalities.

In recent years however, Rohlfs' model has decreased in applicability, as it does not take into account the ease at which consumers can now switch between competing products. In social media for example, one can choose from a variety of websites and be active on all of them at the same time. Thus, researchers now tend to focus on critical periods of market development that can foster anti-competitive tendencies, instead of looking at the broader picture.

3 Assertion and Validation

Rohlfs' model assumed a uniform distribution of willingness to pay of teh form below:

$$f(w) = \begin{cases} 0 & w < 0,\ w > 1 \\ 1 & 0 \le w \le 1 \end{cases}$$

This work extends his uniform willingness to pay assumption to the following equation:

$$f(w, \alpha, \beta) = \begin{cases} 0 & w < 0,\ w > 1 \\ \frac{1}{B(\alpha,\beta)} w^{\alpha-1}(1-w)^{\beta-1} & 0 \le w \le 1 \end{cases}$$

where:

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{6}$$

Several examples of this function can be seen in the first column of Figure 2. As is shown, setting $\alpha = \beta = 1$ results in Rohlfs' original uniform distribution assumption. Setting $\alpha = 1, \beta = 2$, or $\alpha = 2, \beta = 1$ results in a strictly decreasing/increasing $f(w)$ with a slope of $-\frac{1}{2}$ and $\frac{1}{2}$ respectively. Beyond this, we obtain Gaussian-like densities with varying degrees of skew.

In the case of Rohlfs' model, solving for the WtP, $w_{\hat{i}} \in [0, 1]$ of the marginal customer is trivial, due to the uniform WtP function. More formally however, solving for $w_{\hat{i}}$ can be done as:

$$1 - f = \int_0^{w_{\hat{i}}} f(w)dw \tag{7}$$

where $1 - f$ is the fraction of the population not subscribed, and $f(w)$ is the PDF in question. In Rohlfs' case:

$$\begin{aligned} 1 - f &= \textstyle\int_0^{w_{\hat{i}}} f(w)dw \\ &= \textstyle\int_0^{w_{\hat{i}}} dw \\ &= w_{\hat{i}} \end{aligned} \tag{8}$$

However, for the beta distribution, this becomes more complex.

$$\begin{aligned} 1 - f &= \textstyle\int_0^{w_{\hat{i}}} f(w)dw \\ &= \tfrac{1}{B(\alpha,\beta)} \textstyle\int_0^{w_{\hat{i}}} w^{\alpha-1}(1-w)^{\beta-1}dw \end{aligned} \tag{9}$$

This is what's known as the incomplete beta distribution [11], where the integral must be written as an

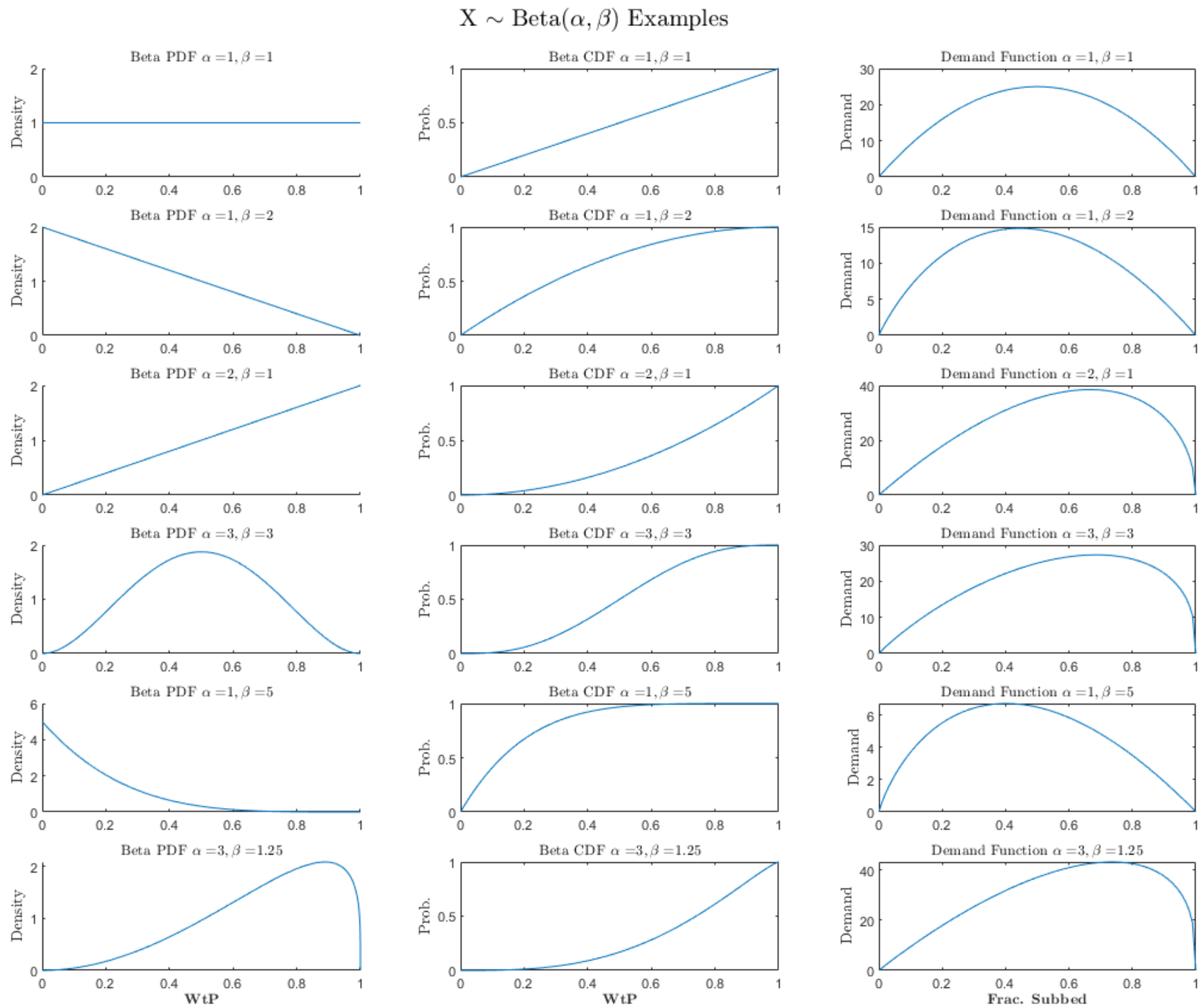

Figure 2: $X \sim Beta(\alpha, \beta)$, CDFs, and Demand Function Examples

infinite sum of the following form:

$$Beta(w_{\hat{i}}, \alpha, \beta) = w_{\hat{i}}^{\alpha} \sum_{n=0}^{\infty} \frac{(1-b)_n}{n!(a+n)} w_{\hat{i}}^{n} \tag{10}$$

where $(1-b)_n$ is a Pochhammer symbol [12]:

$$(1-b)_n = \frac{\Gamma(1-b+n)}{\Gamma(1-b)} \tag{11}$$

So, while there is no closed form solution in the general case, this work will look at individual values of α, β that produce closed form solutions. Eq. (7) is simply just the CDF of the density function, evaluated at $w_{\hat{i}}$ as a function of f, the fraction of the population subscribed to the service. In this work, CDFs and values of $w_{\hat{i}}$ were obtained analytically by varying f in discrete intervals $\in [0,1]$. After finding $w_{\hat{i}}$, Eq. (5) was used to develop the final demand function. Graphs of several CDFs and demand functions can be seen in Figure 2.

Of interest, however, is how these α and β values affect the *maximum charging price*, the *unsteady state fraction* of the population before which the service will always fail, and the *steady state fraction* of the population that will ultimately be reached once the adoption fraction has passed the unsteady state equilibrium.

As can be seen in the last column of Figure 2, the maximum charging price varies significantly as a function of α and β. Precisely how this value changes was also modeled analytically, and results can be seen in Figure 3. Several important results can be derived from this image.

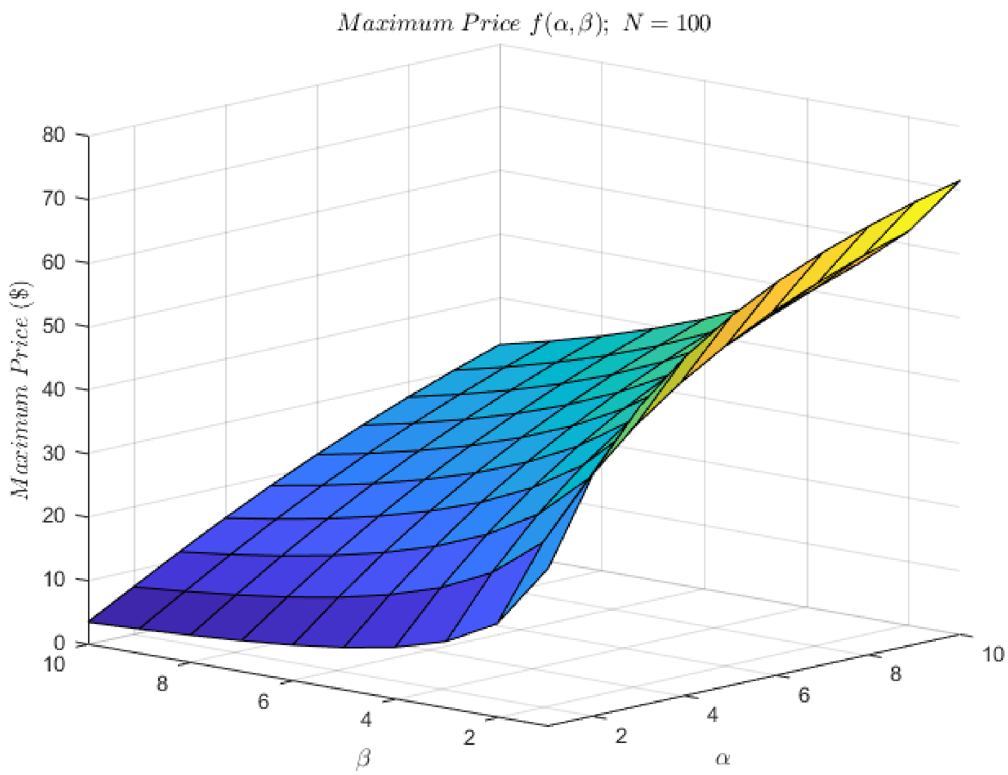

Figure 3: Maximum Chargeable Price $f(\alpha,\beta), N = 100$

1. The maximum charging price with $\alpha = \beta = 1$ and $N = 100$ is 25. This is in line with the $\frac{N}{4}$ derivation previously found in Rohlfs' model.

2. For increasingly skewed Beta distributions $\beta >> \alpha$ (for which can be seen in the 4th row of Figure

2), the maximum charging price tends toward 0. This makes intuitive sense, as the Beta distribution approaches the delta function, $\delta(w)$ (at $w = 0$), in which no one is willing to pay for the product.

3. For $\alpha >> \beta$ (seen in the 5th row of Figure 2), the Beta distribution approaches $\delta(w-1)$ and the maximum charging price tends toward N.

Secondly, it is important to see how the unsteady state equilibrium point (point B in Figure 1) varies with α and β as well. For this, I have chosen an arbitrary price, p, such that there are two solutions for *every* value of α, β in the range tested. The value of p is largely irrelevant, as the unsteady state point will vary in similar ways regardless of p. The results can be seen in Figure 4. Again, several key insights can be found in this image (bare in mind the difference in α, β ranges).

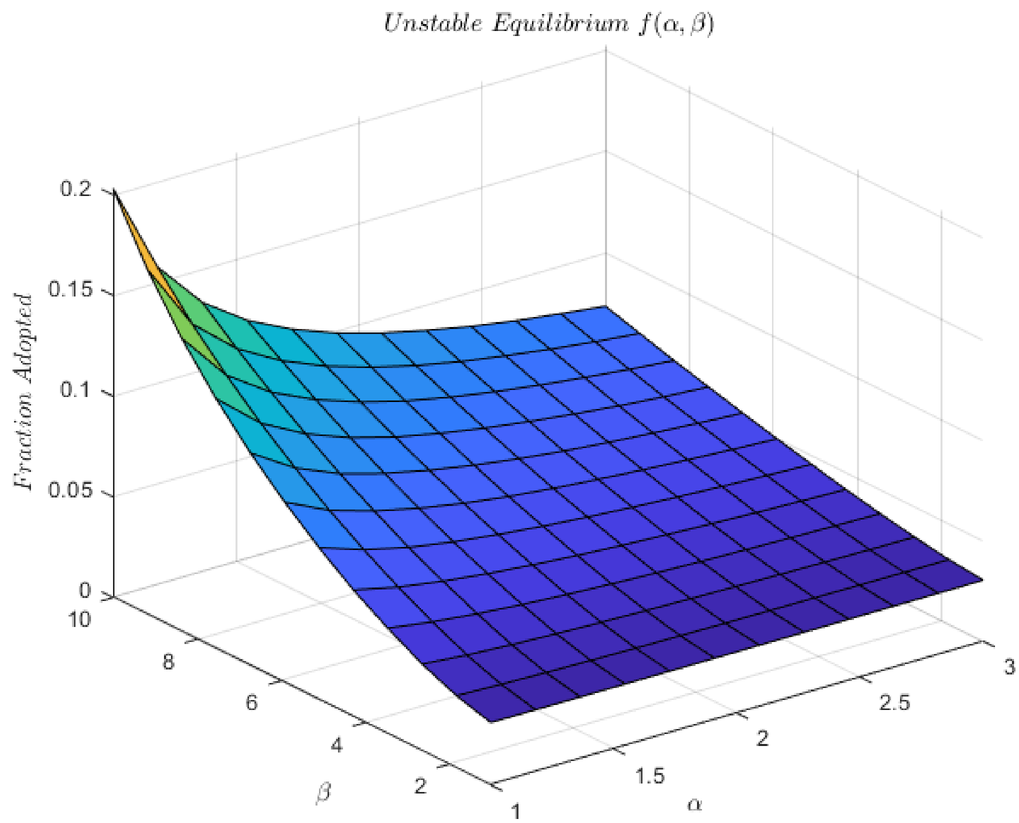

Figure 4: Unstable equilibrium point $f(\alpha, \beta)$

1. For very skewed beta distributions $\beta >> \alpha$, the fraction of the population required to adopt the product before success increases dramatically. In this figure, we see $(\alpha, \beta) = (1, 10)$ results in an unsteady

equilibrium adoption value of around 20% of the population. This makes sense, as a set price is less appealing to the group, when they, on average, aren't willing to pay much at all for it.

2. For $\alpha >> \beta$, this value more gradually approaches 0. Again, this makes intuitive sense, as the higher people are willing to pay for the service, the more easily it will be successful.

Finally, it is important to see how the steady state equilibrium point (C in Figure 1) changes with α and β. Here, the same price p is used as with the unsteady state point for consistency. Similar insights as with the unsteady state point can be made.

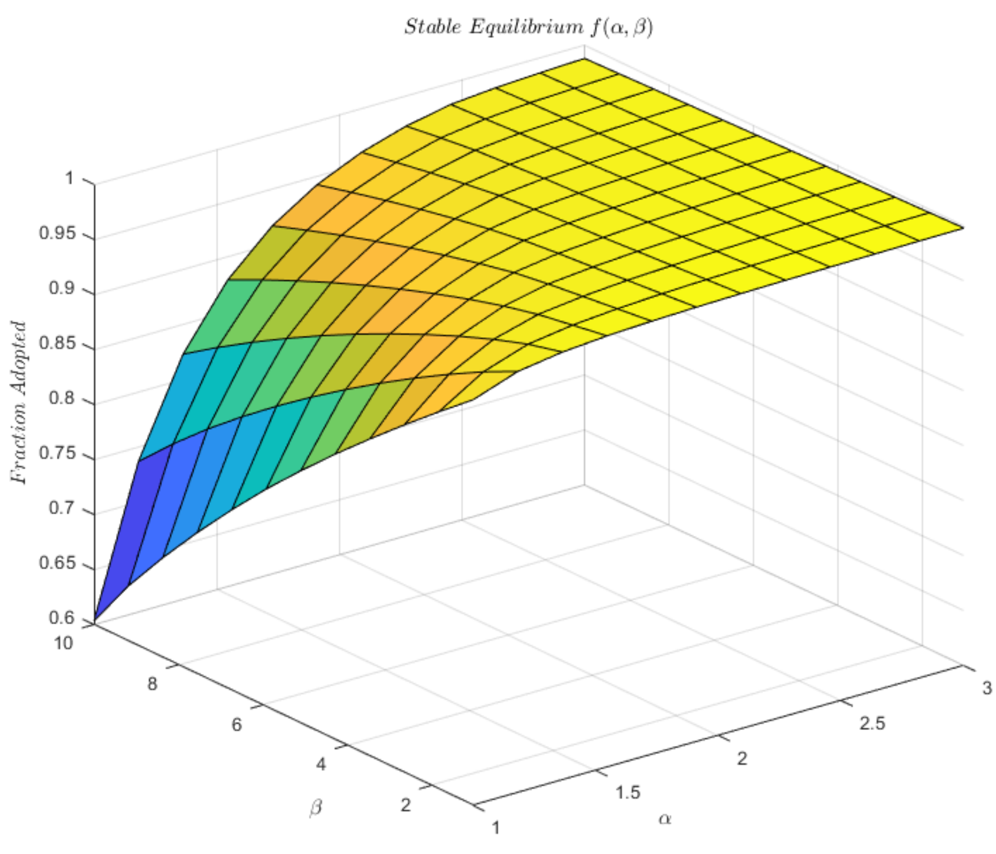

Figure 5: Stable equilibrium point $f(\alpha, \beta)$

1. As $\beta >> \alpha$, the steady state adoption rate decreases sharply. For the same (α, β) = (1,10), only around 60% of the total population will adopt the product (assuming it passes the unsteady state point).

2. If $\alpha >> \beta$, the stable adoption rate of the service tends toward 100%.

Again, it is important to note, the implicit assumption of a Gaussian-like willingness to pay PDF is not unfounded. Much research has been done estimating these distributions in the real world - many of which have this form. Just a few examples of this can be found in [2-4].

4 Conclusion

This work set out to understand how willingness to pay distributions outside of the uniform distribution affect the maximum charging price, the unsteady state fraction of the population before which the service will always fail, and the steady state fraction of the population that will ultimately be reached once the adoption fraction has passed the unsteady state equilibrium. By using the Beta distribution, this work explores many such PDFs with varying degrees of skew by modifying the α and β parameters. From this, we gain key insights into dynamics of market evolution not present in Rohlfs' original work.

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