CSCI 3104 Fall 2021
Instructors: Profs. Grochow and Waggoner

Problem Set 4

Due Date .... September 28, 2021
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Contents

1 Instructions

- The solutions **must be typed**, using proper mathematical notation. We cannot accept hand-written solutions. Here's a short intro to LaTeX.
- You should submit your work through the **class Canvas page** only. Please submit one PDF file, compiled using this LaTeX template.
- You may not need a full page for your solutions; pagebreaks are there to help Gradescope automatically find where each problem is. Even if you do not attempt every problem, please submit this document with no fewer pages than the blank template (or Gradescope has issues with it).

- You are welcome and encouraged to collaborate with your classmates, as well as consult outside resources. You must **cite your sources in this document. Copying from any source is an Honor Code violation. Furthermore, all submissions must be in your own words and reflect your understanding of the material.** If there is any confusion about this policy, it is your responsibility to clarify before the due date.
- Posting to **any** service including, but not limited to Chegg, Reddit, StackExchange, etc., for help on an assignment is a violation of the Honor Code.
- You **must** virtually sign the Honor Code (see Section 2). Failure to do so will result in your assignment not being graded.

2 Honor Code (Make Sure to Virtually Sign)

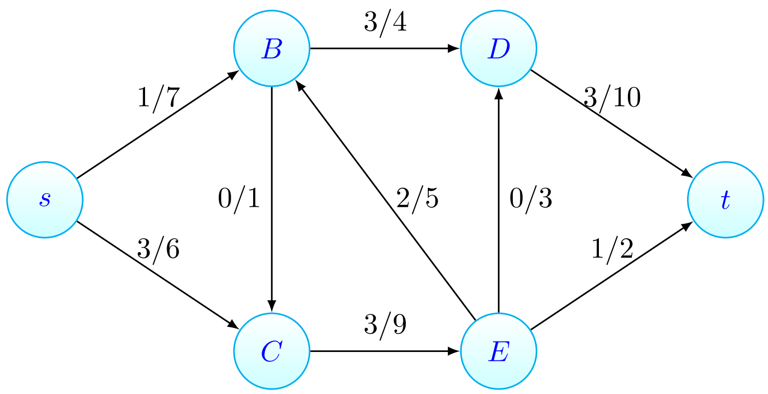
Problem 1.

- My submission is in my own words and reflects my understanding of the material.
- Any collaborations and external sources have been clearly cited in this document.
- I have not posted to external services including, but not limited to Chegg, Reddit, StackExchange, etc.
- I have neither copied nor provided others solutions they can copy.

Agreed (Abeal Sileshi). □

3 Standard 9- Network Flows: Terminology

Problem 2. Consider the following flow network, with the following flow configuration f as indicated below.



Do the following.

(a) Given the current flow configuration f, what is the maximum *additional* amount of flow that we can push across the edge (B, D) from $B \to D$? Justify using 1-2 sentences.

Answer. The maximum additional amount of flow we can push is 1 because at a capacity of 4 (from BD) and 3 flow currently being pushed, only 1 flow remains to be pushed. □

(b) Given the current flow configuration f, what is the maximum amount of flow that B can push backwards to E? Do **not** consider whether E can reroute that flow elsewhere; just whether B can push flow backwards. Justify using 1-2 sentences.

Answer. The maximum amount of flow B can push backwards to E is 2. □

(c) Given the current flow configuration f, what is the maximum amount of flow that D can push backwards to E? Do **not** consider whether D can reroute that flow elsewhere; just whether E can push flow backwards. Justify using 1-2 sentences.

Answer. D cannot push any flow backwards to E, so the maximum flow D can push backwards is zero. □

(d) How much *additional* flow can be pushed along the flow-augmenting path $s \to B \to E \to t$? Do not include the current flow along these edges. Justify using 1-2 sentences.

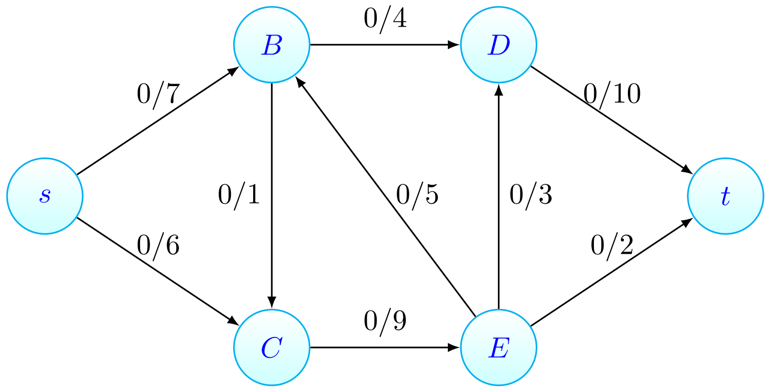
Answer. Only 1 additional flow can be pushed from $s \to B \to E \to t$. You can push 6 additional flow from $S \to B$, and then 2 backwards from $B \to E$, but there only remains space for 1 additional unit flow from $E \to t$ □

(e) Find a second flow-augmenting path and indicate the maximum amount of additional flow that can be pushed along the path. Assume that the flow-augmenting path from part (d) has **not** been applied. Justify using 1-2 sentences.

Answer. Another flow-augmenting path is $s \to B \to D \to t$. The maximum amount of additional flow that can be pushed is 1 because while you can push 6 additional flow from $S \to B$, from $B \to D$ 1 additional flow can be pushed and therefore it can only push one more additional flow from $D \to t$. □

4 Standard 10- Network Flows: Ford-Fulkerson

Problem 3. Consider the following flow network, with no initial flow along the graph.

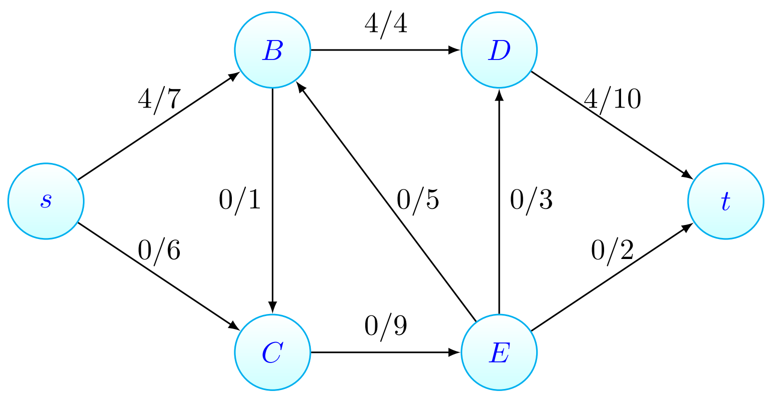


Do the following.

4.1 Problem 3(a)

(a) Consider the flow-augmenting path $s \to B \to D \to t$. Push as much flow through the flow-augmenting path and draw the updated flow network below.

Answer.

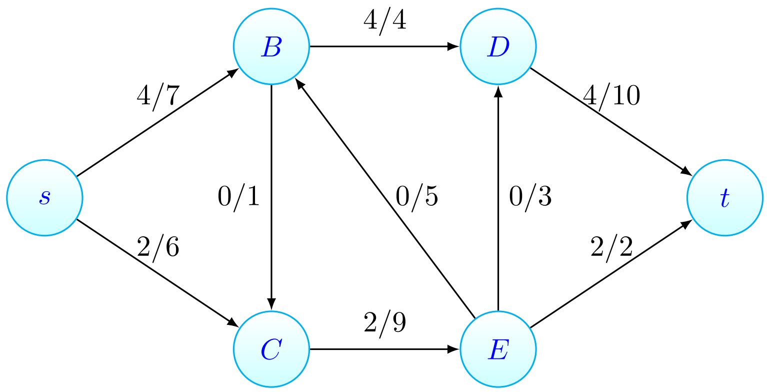


□

4.2 Problem 3(b)

(b) Find a flow-augmenting path using the updated flow configuration from part (a). Then do the following: (i) clearly identify both the flow-augmenting path and the maximum amount of flow that can be pushed through said path; and then (ii) push as much flow through the flow-augmenting path and draw the updated flow network below.

Answer. The next flow-augmented path chosen is $S \to C \to E \to t$

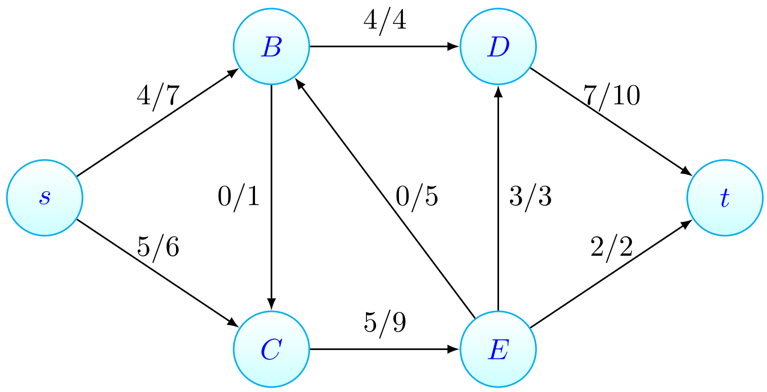


□

4.3 Problem 3(c)

(c) Find a flow-augmenting path using the updated flow configuration from part (b). Then do the following: (i) clearly identify both the flow-augmenting path and the maximum amount of flow that can be pushed through said path; and then (ii) push as much flow through the flow-augmenting path and draw the updated flow network below.

Answer. The next flow-augmented path chosen is $S \to C \to E \to D \to t$



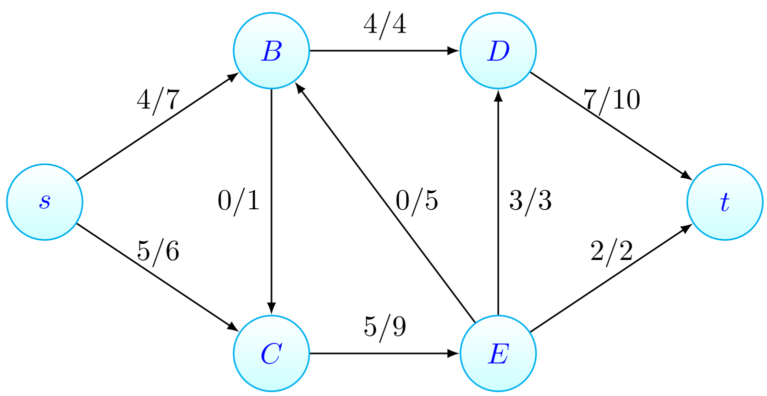
□

4.4 Problem 3(d)

(d) Using the flow configuration from part ((c)), finish executing the Ford–Fulkerson algorithm. Include the following here: (i) your flow network, reflecting the maximum-valued flow configuration you found, and (ii) the corresponding minimum capacity cut. There may be multiple minimum capacity cuts, but you should identify the one corresponding to your maximum-valued flow configuration. Then (iii) finally, compare the value of your flow to the capacity of the cut.

Note: You do **not** need to include the remaining steps of the Ford–Fulkerson algorithm. We will not check these steps when grading.

Answer. The max flow of the network is 9.
The corresponding minimum capacity cut is 9
We find that the value of the flow is equal to the minimum capacity cut.



□

5 Standard 11- Network Flows: Reductions and Applications

5.1 Problem 4

Problem 4. In this problem, we reduce the Maximum Flow problem where multiple sources and sinks are allowed to the One-Source, One-Sink Maximum Flow problem. The reduction is as follows. Let $\mathcal{N}(G, c, S, T)$ be our flow network with multiple sources and multiple sinks. We construct a new flow network $\mathcal{N}'(H, c', S', T')$, as follows.

- $S' = \{s'\}$ is the set containing our one source.
- $T' = \{t'\}$ is the set containing our one source.
- We now construct H by starting with G (including precisely the vertices and edges of G) and adding s' and t'. For each source $s \in S$ of G, we add a directed edge (s', s) (that is, $s' \to s$) in H. For each sink $t \in T$ of G, we add a directed edge (t, t') (that is, $t \to t'$) in H.
- We construct the capacity function c' of H as follows.
  - If (u, v) corresponds to an edge of G, then $c'(u, v) = c(u, v)$.
  - If $s \in S$ is a source of G, then $c'(s', s)$ is the amount of flow that we can push from s in G. That is:

$$c(s', s) = \sum_{(s,v) \in E(G)} c(s, v).$$

  - If $t \in T$ is a sink of G, then $c'(t, t')$ is the maximum amount of flow that t can receive along its incoming edges in G. That is:

$$c'(t, t') = \sum_{(v,t) \in E(G)} c(v, t).$$

Do the following. [**Hint:** Before attempting either part (a) or part (b), we highly recommend doing the following scratch work first. Construct your own flow network with multiple sources and multiple sinks. Then go through the above construction carefully to obtain a new flow network with one source and one sink. Trying to construct your own examples is extremely beneficial when working to understand a new construction.]

5.1.1 Problem 5(a)

(a) Show that, for every feasible flow f on $\mathcal{N}$, there exists a (corresponding) feasible flow f' on $\mathcal{N}'$ such that $\text{val}(f) = \text{val}(f')$.

Proof. First we have to establish that both f and f' are flows that adhere to flow rules
We know that val (f', s) = $\sum_{(s,v) \in E(G)} f(s, v) \leq \sum_{(s,v) \in E(G)} c(s, v)$ = c(s',v)
This establishes f' as a flow
This follows from $c'(u, v) = c(u, v)$
val (f) = $\sum \sum_{(s,v) \in E(G)} f(s, v)$
the first summation in the double summation is 'for all S'
val (f') = $\sum_{s' \in E(G)} f(s', s)$
Following from the above asserts that val(f) = val(f')

□

5.1.2 Problem 5(b)

(b) For every feasible flow g' on $\mathcal{N}'$, show how to recover a feasible flow g on $\mathcal{N}$ such that $\mathrm{val}(g) = \mathrm{val}(g')$.

Proof. The only thing that changes between N and N' is that there is one sink and one source added. But seeing as the sink's flow cannot exceed what comes before it, and the source can't pump more flow that exceed the following nodes, the flow stays the same between N and N'. □

5.2 Problem 6

Problem 5. Suppose a restaurant has 3 customers, each of whom can order at most one entree. The restaurant is running low on inventory. It has the following in stock:

- Two burgers,
- Three servings of crab cakes, and
- One salmon filet.

Each customer submits their order, indicating only whether they will eat a given meal. **Of the meals a given customer is willing to eat, that customers does NOT indicate whether they prefer one meal to another.** The restaurant wishes to assign entrees in such a way that respects whether the customers will eat their entrees and maximizes the number of entrees sold.

Do the following.

5.2.1 Problem 6(a)

(a) Carefully describe how to construct a flow network corresponding to the customers' order preferences. That is, we take as input the customers' order preferences, and you need to describe how to construct the corresponding flow network.

Note that you just have to give a construction. You are **not** being asked to prove anything about your construction.

Answer. □

5.2.2 Problem 6(b)

(b) Suppose that the three customers provide their orders:

- Customer 1: Crab cakes, Burger. [Note: Customer 1 will not eat Salmon]
- Customer 2: Salmon, Burger, Crab cakes.
- Customer 3: Salmon, Crab cakes, Burger.

Using your construction from part (a), find a maximum-valued flow on your flow network and identify the corresponding allocation of entrees to customers. You may hand-draw your flow network, but the explanation must be typed.

Answer. □