1 Problem 94 solution

Heron's formula states that the area of a triangle whose sides have lengths a, b, and c is:

$$A = \sqrt{p(p-a)(p-b)(p-c)} \tag{1}$$

where p is the semiperimeter ($p = \frac{a+b+c}{2}$).

As considered triangles are almost equilateral we know that $b = a$ and $c = a \pm 1$. The two cases are similar, so let's focus on $b = a + 1$. We can write (1) as:

$$\begin{aligned} A &= \sqrt{\left(\frac{3a+1}{2}\right)\left(\frac{a+1}{2}\right)^2\left(\frac{a-1}{2}\right)} \\ &= \sqrt{\frac{(3a+1)(a+1)^2(a-1)}{16}} \\ &= \frac{a+1}{4}\sqrt{(3a+1)(a-1)} \end{aligned} \tag{2}$$

From this we can see that, for the area to be integral, there are two conditions, let $x = \sqrt{(3a+1)(a-1)}$ then:

1. $x \in \mathbb{Z}$
2. $\frac{x(a+1)}{4} \in \mathbb{Z}$
   or equivalently: $x(a+1) \equiv 0 \pmod 4$

While the second condition is easy to test with a computer, the first one might be harder due to floating point precision errors. Also we want a way to enumerate all possible a without checking every time. To do this we need to solve the diophantine equation

$$(3a+1)(a-1) = x^2 \tag{3}$$

By solving the equation with a solver (like this: https://www.alpertron.com.ar/QUAD.HTM) we can get the recurrence equations for a and x:

$$a_{n+1} = -2a_n - x_n + 1 \tag{4}$$

$$x_{n+1} = -3a_n - 2x_n + 1 \tag{5}$$

Starting from the smallest solution ($a_0 = 1, x_0 = 0$) these will give all possible values for integral sides length of all almost equilateral heronian triangles in the form $(a, a, a + 1)$.
To get the ones in the form $(a, a, a - 1)$ the process is the same substituting $c = a - 1$ to (1). In the end we get the following recurrences:

$$a_{n+1} = -2a_n - x_n - 1 \tag{6}$$

$$x_{n+1} = -3a_n - 2x_n - 1 \tag{7}$$

By considering all triangles with sides obtained from positive values of (4) and (6) we can easily calculate the solution to the problem.

1.1 Merging recurrence equations

By expanding any of (4) and (6) we can see that results are the same in absolute value but have opposite sign (they alternate positive and negative a values), so we can merge the two recurrences into one that generates positive values only. To do so we need to make positive the recursive terms and flip ± 1 like this:

$$a_{n+1} = 2a_n + x_n + (-1)^{n+1} \tag{8}$$

$$x_{n+1} = 3a_n + 2x_n + (-1)^{n+1} \tag{9}$$

Note that by starting from $a_0 = 1, x_0 = 0$ the first two triangles degenerate into segments and are not considered by the problem.
Let u be the number of almost equilateral heronian triangles with perimeter that do not exceed 1 000 000 000, the final solution to the problem is:

$$\sum_{i=2}^{u} 3a_i + (-1)^i \tag{10}$$