Digital Signal Processing for Music

Part 7: Fourier Transform, Part 1

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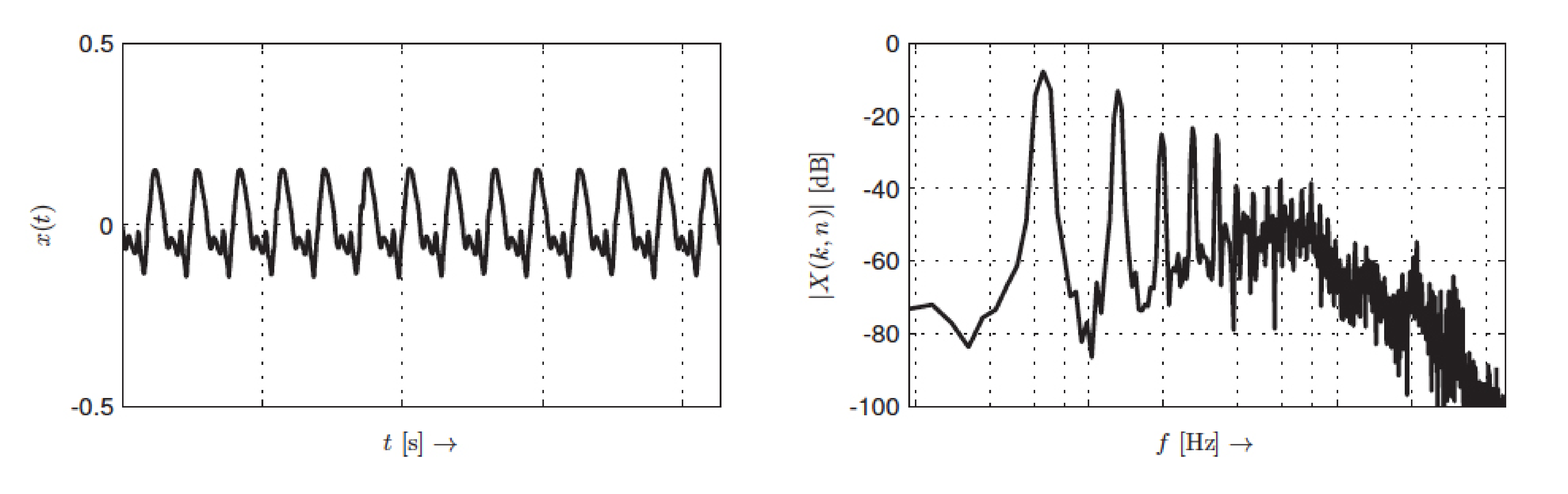
Fourier Transform: Overview

- Fourier series to Fourier Transform
- Properties of the Fourier Transform
- Windowed Fourier Transform (STFT)
- Transform of sampled time signals
- Discrete Fourier Transform

Fourier Transform: Introduction



Fourier series is a brilliant insight, but:

- Works only for periodic signals
- Difficult to use for real-world analysis as it requires knowledge of fundamental frequency

→ **Fourier Transform**

Fourier series, revisited

$$c_k = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\mathrm{e}^{-\mathrm{j}\omega_0 k t}\, dt$$

- Fourier series coefficient can be interpreted as **correlation coefficient** between signal and sinusoidals of different frequencies
- Only frequencies $k\omega_0$ are used (ω_0 *has to be known*)
- Fourier series produces a **'line spectrum'**

Fourier series, revisited

$$c_k = \frac{1}{T_0} \int\limits_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\mathrm{e}^{-\mathrm{j}\omega_0 k t}\, dt$$

- Distance betwen frequency components decreases as T_0 increases
- Aperiodic functions could be analyzed by increasing $T_0 \to \infty$

Fourier series, revisited

$$c_k = \frac{1}{T_0} \int\limits_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\mathrm{e}^{-\mathrm{j}\omega_0 k t}\, dt$$

- $T_0 \to \infty$
- $k\omega_0 \to \omega$
- $\frac{1}{T_0} \to 0$

To avoid zero result, multiply with T_0

Definition of Fourier Transform (Continuous)

$$X(\mathrm{j}\omega) = \mathfrak{F}[x(t)] = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathrm{j}\omega t}\, dt$$

Example 1: Rect Window

$$w_{\mathrm{R}}(t) = \begin{cases} 1, & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$W_{\mathrm{R}}(t) = \int_{-\infty}^{\infty} w_{\mathrm{R}}(t) \mathrm{e}^{-\mathrm{j}\omega t}\, dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{-\mathrm{j}\omega t}\, dt$$

$$= \frac{1}{-\mathrm{j}\omega} \underbrace{\left(\mathrm{e}^{-\mathrm{j}\frac{\omega}{2}} - \mathrm{e}^{\mathrm{j}\frac{\omega}{2}}\right)}_{=-2\mathrm{j}\sin\left(\frac{\omega}{2}\right)}$$

$$= \frac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}} = \operatorname{sinc}\left(\frac{\omega}{2}\right)$$

Example 2: Dirac (Impulse)

$$\delta(t) = \begin{cases} 1, & t = 0 \\ 0, & t \neq 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

$$\Delta(\mathrm{j}\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-\mathrm{j}\omega t} dt = e^{-\mathrm{j}\omega \cdot 0} = 1$$

Shifted Dirac $\delta(t - \tau_0)$

Properties of Fourier Transform

Property 1: Invertibility

$$x(t) = \mathfrak{F}^{-1}[X(\mathrm{j}\omega)]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega t}\, d\omega$$

Reminder: Signal reconstruction with Fourier series coefficients

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\mathrm{j}\omega_0 k t}$$

Comments:

Property 2: Superposition

$$y(t) = c_1 \cdot x_1(t) + c_2 \cdot x_2(t)$$
$$\mapsto Y(\mathrm{j}\omega) = c_1 \cdot X_1(\mathrm{j}\omega) + c_2 \cdot X_2(\mathrm{j}\omega)$$

$$Y(\mathrm{j}\omega) = \int_{-\infty}^{\infty} \big(c_1 \cdot x_1(t) + c_2 \cdot x_2(t)\big) \cdot \mathrm{e}^{-\mathrm{j}\omega t}\, dt$$

$$= c_1 \cdot \int_{-\infty}^{\infty} x_1(t)\mathrm{e}^{-\mathrm{j}\omega t}\, dt + c_2 \cdot \int_{-\infty}^{\infty} x_2(t)\mathrm{e}^{-\mathrm{j}\omega t}\, dt$$

$$= c_1 \cdot X_1(\mathrm{j}\omega) + c_2 \cdot X_2(\mathrm{j}\omega)$$

Property 3: Convolution and Multiplication

$$y(t) = \int_{-\infty}^{\infty} h(\tau) \cdot x(t-\tau)\, d\tau$$

$$\mapsto Y(\mathrm{j}\omega) = H(\mathrm{j}\omega) \cdot X(\mathrm{j}\omega)$$

$$\begin{aligned}
Y(\mathrm{j}\omega) &= \int_{-\infty}^{\infty} y(t)\mathrm{e}^{-\mathrm{j}\omega t}\, dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(\tau) \cdot x(t-\tau)\, d\tau \right) \mathrm{e}^{-\mathrm{j}\omega t}\, dt \\
&= \int_{-\infty}^{\infty} h(\tau) \int_{-\infty}^{\infty} x(t-\tau)\mathrm{e}^{-\mathrm{j}\omega t}\, dt\, d\tau
\end{aligned}$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau) \cdot x(t-\tau)\, d\tau$$

$$\mapsto Y(\mathrm{j}\omega) = H(\mathrm{j}\omega) \cdot X(\mathrm{j}\omega)$$

$$\begin{aligned} Y(\mathrm{j}\omega) &= \int_{-\infty}^{\infty} y(t)\mathrm{e}^{-\mathrm{j}\omega t}\, dt \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(\tau) \cdot x(t-\tau)\, d\tau \right) \mathrm{e}^{-\mathrm{j}\omega t}\, dt \\ &= \int_{-\infty}^{\infty} h(\tau) \int_{-\infty}^{\infty} x(t-\tau)\mathrm{e}^{-\mathrm{j}\omega t}\, dt\, d\tau \end{aligned}$$

$$= \int_{-\infty}^{\infty} h(\tau)\mathrm{e}^{-\mathrm{j}\omega\tau} \underbrace{\int_{-\infty}^{\infty} x(t-\tau)\mathrm{e}^{-\mathrm{j}\omega(t-\tau)}\, d(t-\tau)}_{X(\mathrm{j}\omega)}\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\mathrm{e}^{-\mathrm{j}\omega\tau}\, d\tau \cdot X(\mathrm{j}\omega)$$

$$= H(\mathrm{j}\omega) \cdot X(\mathrm{j}\omega)$$

Property 4: Parseval`s Theorem

$$\int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\mathrm{j}\omega)|^2\,d\omega$$

$$\int_{-\infty}^{\infty} h(\tau)\cdot x(t-\tau)\,d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\mathrm{j}\omega)\cdot X(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega t} d\omega$$

$$H(\mathrm{j}\omega) \longrightarrow X^*(\mathrm{j}\omega)/h(\tau) \longrightarrow x(-\tau),\ t=0$$

$$\int_{-\infty}^{\infty} x(-\tau)\cdot x(-\tau)\,d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\mathrm{j}\omega)\cdot X(\mathrm{j}\omega)\,d\omega$$

$$\int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\mathrm{j}\omega)|^2\,d\omega$$

Property 5: Time & Frequency Shift

$$y(t) = x(t - t_0) \rightarrow Y(\mathrm{j}\omega) = X(\mathrm{j}\omega)\mathrm{e}^{-\mathrm{j}\omega t_0}$$

$$\int_{-\infty}^{\infty} x(t - t_0)\mathrm{e}^{-\mathrm{j}\omega t}\,dt = \int_{-\infty}^{\infty} x(\tau)\mathrm{e}^{-\mathrm{j}\omega(\tau + t_0)}\,d\tau$$

$$= \mathrm{e}^{-\mathrm{j}\omega t_0} \int_{-\infty}^{\infty} x(\tau)\mathrm{e}^{-\mathrm{j}\omega\tau}\,d\tau$$

$$= \mathrm{e}^{-\mathrm{j}\omega t_0} \cdot X(\mathrm{j}\omega)$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\mathrm{j}(\omega - \omega_0))\mathrm{e}^{\mathrm{j}\omega t}\,d\omega = \mathrm{e}^{\mathrm{j}\omega_0 t} \cdot x(t)$$

Property 6: Symmetry 1/2

$$|X(\mathrm{j}\omega)| = |X(-\mathrm{j}\omega)|$$

$$\Phi_{\mathrm{X}}(\omega) = -\Phi_{\mathrm{X}}(-\omega)$$

Time signal sum of even and odd component $x_e(t),\ x_o(t)$

$$x(t) = \underbrace{\frac{1}{2}(x(t) + x(-t))}_{x_e(t)} + \underbrace{\frac{1}{2}(x(t) - x(-t))}_{x_o(t)}$$

$$X_e(\mathrm{j}\omega) = \int_{-\infty}^{\infty} x_e(t)\cos(\omega t)\,dt - \mathrm{j}\underbrace{\int_{-\infty}^{\infty} x_e(t)\sin(\omega t)\,dt}_{=0}$$

$X_e(\mathrm{j}\omega)$ is real

$X_e(\mathrm{j}\omega) = X_e(-\mathrm{j}\omega)$ (substitute $x(t)$ with $x(-t)$)

Property 6: Symmetry 2/2

$$|X(\mathrm{j}\omega)| = |X(-\mathrm{j}\omega)|$$

$$\Phi_\mathrm{X}(\omega) = -\Phi_\mathrm{X}(-\omega)$$

Time signal sum of even and odd component $x_e(t),\ x_o(t)$

$$x(t) = \underbrace{\frac{1}{2}(x(t) + x(-t))}_{x_e(t)} + \underbrace{\frac{1}{2}(x(t) - x(-t))}_{x_o(t)}$$

$$X_o(\mathrm{j}\omega) = \underbrace{\int_{-\infty}^{\infty} x_o(t)\cos(\omega t)\,dt}_{=0} - \mathrm{j}\int_{-\infty}^{\infty} x_o(t)\sin(\omega t)\,dt$$

$X_o(\mathrm{j}\omega)$ is imaginary

$X_o(\mathrm{j}\omega) = X_o(-\mathrm{j}\omega)$ (substitute $x(t)$ with $-x(-t)$)

Property 7: Time & Frequency Scaling

$$y(t) = x(c \cdot t)$$

$$\mapsto Y(\mathrm{j}\omega) = \frac{1}{|c|} X\left(\mathrm{j}\frac{\omega}{c}\right)$$

$$Y(\mathrm{j}\omega) = \int_{-\infty}^{\infty} x(c \cdot t) \mathrm{e}^{-\mathrm{j}\omega t} \, dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \mathrm{e}^{-\mathrm{j}\omega \frac{\tau}{c}} \, d\frac{\tau}{c}$$

$$= \frac{1}{c} \int_{-\infty}^{\infty} x(\tau) \mathrm{e}^{-\mathrm{j}\frac{\omega}{c}\tau} \, d\tau$$

$$= \frac{1}{c} X\left(\mathrm{j}\frac{\omega}{c}\right)$$

Verifying Fourier Transform Implementation

- *Property 1*: **Invertibility**: Running IFT returns the EXACT original signal

$$x(t) = \mathfrak{F}^{-1}[X(\mathrm{j}\omega)]$$

- *Property 2*: **Superposition**: Scaled addition in time domain maps to linear scale in magnitudes in frequency domain

$$y(t) = c_1 \cdot x_1(t) + c_2 \cdot x_2(t)$$
$$\mapsto Y(\mathrm{j}\omega) = c_1 \cdot X_1(\mathrm{j}\omega) + c_2 \cdot X_2(\mathrm{j}\omega)$$

Verifying Fourier Transform Implementation

- *Property 4*: **Parseval`s Theorem**: Energy conservation between time domain and frequency domain

$$\int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\mathrm{j}\omega)|^2\,d\omega$$

- *Property 6*: **Symmetry**: Frequency domain is symmetric across zero frequency (and for DFT across windowSize)

$$|X(\mathrm{j}\omega)| = |X(-\mathrm{j}\omega)|$$
$$\Phi_{\mathrm{X}}(\omega) = -\Phi_{\mathrm{X}}(-\omega)$$

Key Properties for Future Topics

- *Property 3*: **Convolution and Multiplication**: Convolution in the time domain is multiplication in the frequency domain

$$y(t) = \int_{-\infty}^{\infty} h(\tau) \cdot x(t-\tau)\, d\tau$$

$$\mapsto Y(\mathrm{j}\omega) = H(\mathrm{j}\omega) \cdot X(\mathrm{j}\omega)$$

- *Property 5*: **Time & Frequency Shift**: Time shift in time domain is phase shift in frequency domain

$$y(t) = x(t - t_0) \rightarrow Y(\mathrm{j}\omega) = X(\mathrm{j}\omega)\mathrm{e}^{-\mathrm{j}\omega t_0}$$