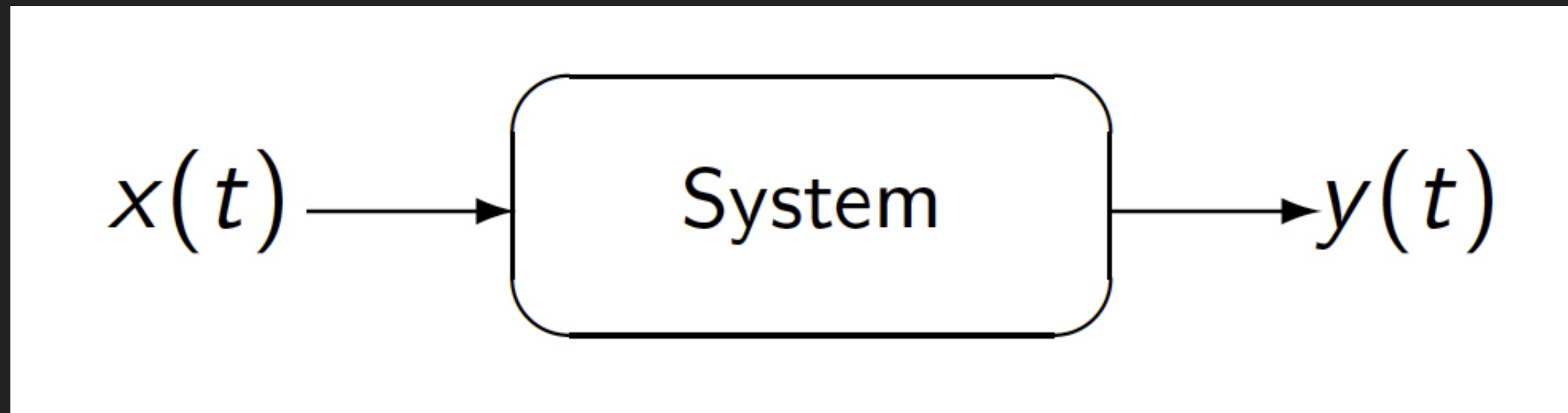# Digital Signal Processing for Music

Part 5: LTI Systems & Convolution

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Systems

>> Any process producing an output signal in response to an input signal

Examples of systems in signal processing

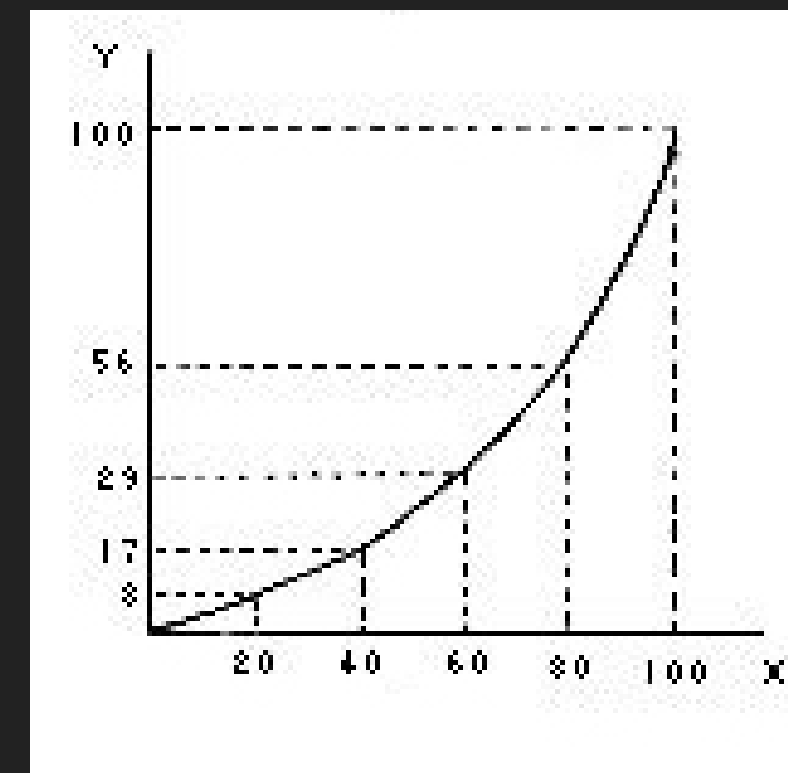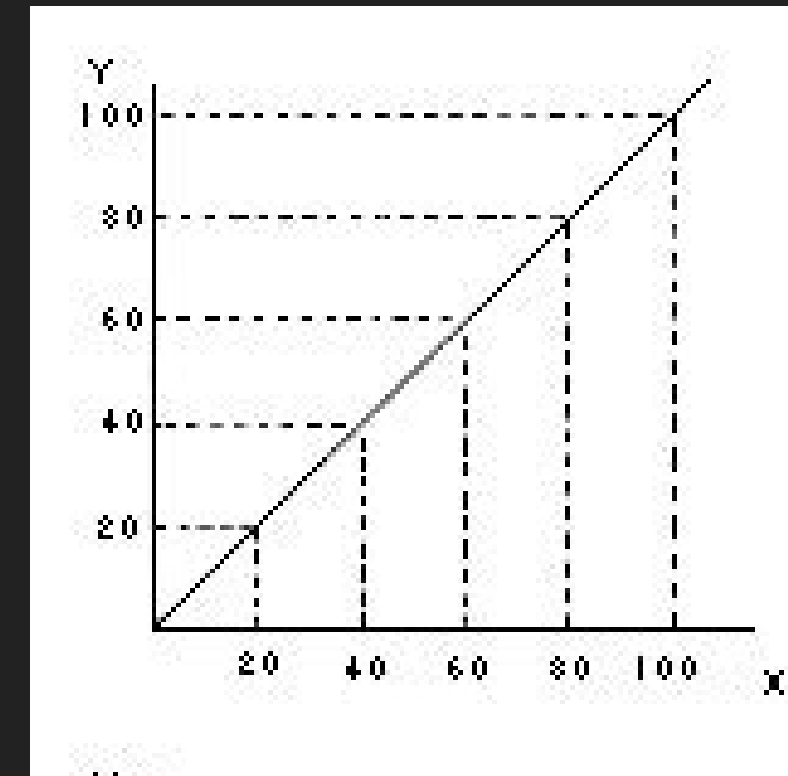>> Filters, Effects

>> Vocal Tract

>> Room

>> Audio cable

>> ...

Linearity & Non-Linearity

Examples for (mostly) linear systems

» Room

» EQ

» Echo

» Envelope

Examples for non-linear systems

» Diode

» Vacuum Tube

» Optical Compressor

» Distortion

Properties of Linear Systems

1.
   Homogeneity

   $$f(ax) = af(x)$$

2.
   Superposition (additivity)

   $$f(x + y) = f(x) + f(y)$$

Properties of Time Invariant Systems

Systems do not change with time

$$f(x(t - \tau)) = f(x)(t - \tau)$$

LTI: Linear Time-Invariant Systems

Systems with these constraints are a great simplification for many real-world systems we would like to model:

» Circuits

» Spring-Mass-Damper

» Reverbs

» Resonance

» etc…

LTI System Example

1. Hammer gives *impulse*

2. System *responds* with velocity

Linearity:

Double force, double velocity, multiple strikes add up

Time Invariance:

System reacts the same whether I do it now or tomorrow

Other LTI system characteristics

» **Causality:**

Output depends only on past and present input

» **BIBO Stability:**

Output is bounded for bounded input

Convolution

It's easy to visualize how a system reacts to an impulse, but what about a more complex input signal?'

» Assume that the signal is constructed from many densely packed impulses (impulse train)

» Output is then a superposition of all individual responses

» For discrete systems this is literal, use integration

Convolution:

$$y(t) = (x * h)(t) := \int\limits_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$y(t) = (x * h)(t) := \int\limits_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$

Steps

1. Flip one signal

2. Multiply the two signals

3. Integrate the result

4. Shift

5. Go to step 2

Convolution Example

x -1,   0,   1 h 1,   1,   1

$x * h$:

0      $x(-2) * h(0) + x(-1) * h(1) + x(0) * h(2) = -1$

1      $x(-1) * h(0) + x(0) * h(1) + x(1) * h(2) = -1$

2      $x(0) * h(0) + x(1) * h(1) + x(2) * h(2) = 0$

3      $x(1) * h(0) + x(2) * h(1) + x(3) * h(2) = 1$

4      $x(2) * h(0) + x(3) * h(1) + x(4) * h(2) = 1$

       $x * h$ -1, -1, 0,   1,   1

Convolution Animation

Click here for convolution animation example

Convolution as Echo

Steps

1. Scale

2. Delay

3. Sum

4. Repeat

Convolution as Echo Example

x -1, 0, 1 h 1, 1, 1

$x * h$:

$x(0)$ -1, -1, -1
$x(1)$ 0, 0, 0
$x(2)$ 1, 1, 1

$x * h$ -1, -1, 0, 1, 1

Identity and Impulse Response

$$x(t) = \delta(t) * x(t)$$

$$h(t) = \delta(t) * h(t)$$

» Describes the response of a system to an impulse as a function of time

» As an impulse includes all frequency, the resulting IR defines the response for all frequencies

» The convolution of $\delta(t)$ with a signal/impulse response results in that impulse response

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau) \cdot x(t - \tau) d\tau$$

Convolution - Properties

>> **Commutativity**

$$h(t) * x(t) = x(t) * h(t)$$

>> **Associativity**

$$\big(g(t) * h(t)\big) * x(t) = g(t) * \big(h(t) * x(t)\big)$$

>> **Distributivity**

$$g(t) * \big(h(t) + x(t)\big) = \big(g(t) * h(t)\big) + \big(g(t) * x(t)\big)$$

Derivation: Commutativity

$$h(t) * x(t) = x(t) * h(t)$$

Substituting $\tau' = t - \tau$

$$x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau) \cdot x(t - \tau) d\tau$$

$$= \int_{-\infty}^{\infty} h(t - \tau') \cdot x(\tau') d(t - \tau')$$

$$= \int_{-\infty}^{\infty} x(\tau') \cdot h(t - \tau') d\tau'$$

$$= h(t) * x(t)$$

Derivation: Associativity

$$\big(g(t) * h(t)\big) * x(t) = g(t) * \big(h(t) * x(t)\big)$$

Changing the order of sums and shifting the operands as shown

$$\big(g(t) * h(t)\big) * x(t) = \int_{\tau=-\infty}^{\infty} \big(g(\tau) * h(\tau)\big) \cdot x(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\xi) \cdot h(\tau-\xi) \cdot x(t-\tau)\,d\tau\,d\xi$$

$$= \int_{-\infty}^{\infty} g(\xi) \cdot \int_{-\infty}^{\infty} h(\tau-\xi) \cdot x(t-\tau)\,d\tau\,d\xi$$

$$= \int_{-\infty}^{\infty} g(\xi) \cdot \int_{-\infty}^{\infty} h(\tau') \cdot x(t-\xi-\tau')\,d\tau'\,d\xi$$

$$= \int_{-\infty}^{\infty} g(\xi) \cdot \big(h(t-\xi) * x(t-\xi)\big)\,d\xi$$

$$= g(t) * \big(h(t) * x(t)\big)$$

Derivation: Distributivity

$$g(t) * \big(h(t) + x(t)\big) = g(t) * h(t) + g(t) * x(t)$$

$$g(t) * \big(h(t) + x(t)\big) = \int\limits_{-\infty}^{\infty} g(\tau) \cdot \big(h(t - \tau) + x(t - \tau)\big) d\tau$$

$$= \int\limits_{-\infty}^{\infty} g(\tau) \cdot h(t - \tau) + g(\tau) \cdot x(t - \tau) d\tau$$

$$= \int\limits_{-\infty}^{\infty} g(\tau) \cdot h(t - \tau) d\tau + \int\limits_{-\infty}^{\infty} g(\tau) \cdot x(t - \tau) d\tau$$

$$= g(t) * h(t) + g(t) * x(t)$$

Importance of convolution for audio DSP

» Ability to model LTI systems

» Used both as a runtime technique and mathematical tool

Uses

» FIR Filters

» Reverbs

» Windowing effects

» Modeling analog systems

Summary – LTI

» Many real-world systems can be approximated by an **LTI system**

» Properties of an LTI system:

       Linearity 1: Homogeneity (Scaling)

       Linearity 2: Superposition (additivity)

       Time Invariance (system doesn't change')

» Additional Properties

       Causality (no future input)

       BIBO - Bounded input bounded output

» Impulse response is a **complete** description of an LTI system

Summary – Convolution

Convolution:

» Describes the process of generating the output of an LTI system from the input

» Is commutative

» Is associative

» Is distributive