

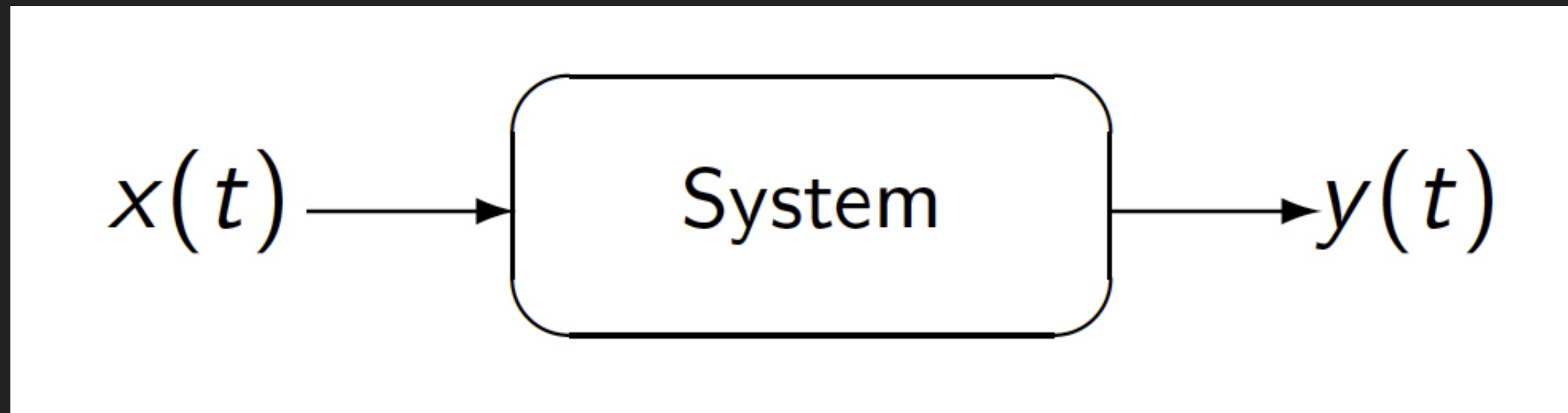
# Digital Signal Processing for Music

Part 5: LTI Systems & Convolution

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# Systems

- » Any process producing an output signal in response to an input signal



## Examples of systems in signal processing

- » Filters, Effects
- » Vocal Tract
- » Room
- » Audio cable
- » ...

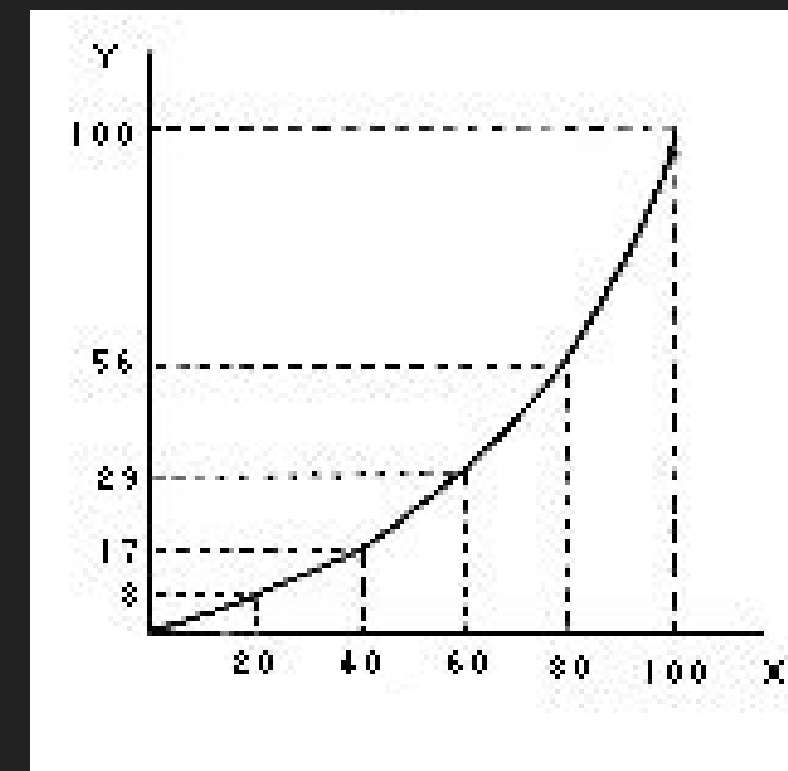
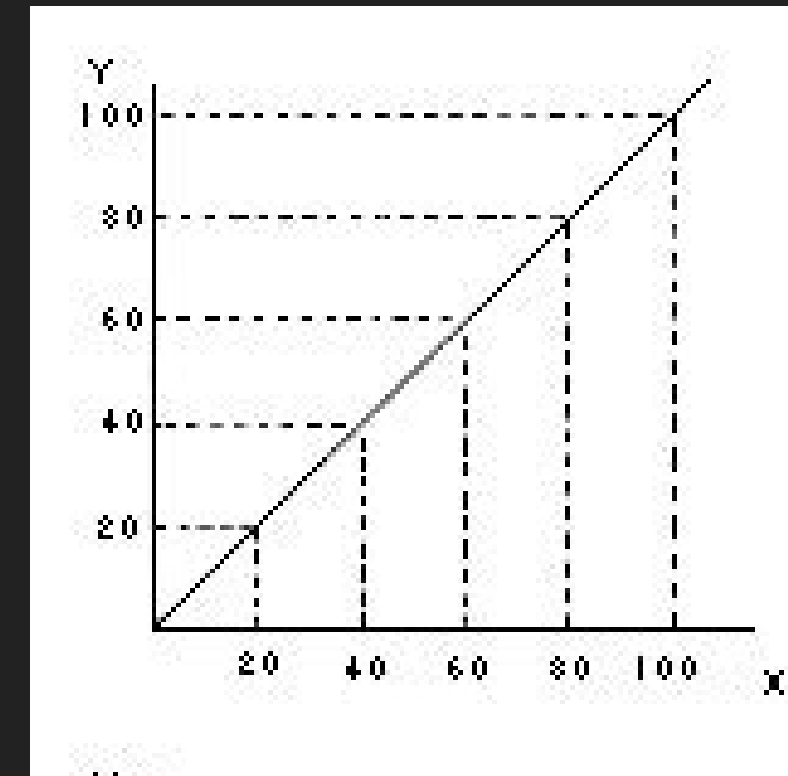
# Linearity & Non-Linearity

## Examples for (mostly) linear systems

- » Room
- » EQ
- » Echo
- » Envelope

## Examples for non-linear systems

- » Diode
- » Vacuum Tube
- » Optical Compressor
- » Distortion



# Properties of Linear Systems

1. **Homogeneity**

$$f(ax) = af(x)$$

2. **Superposition** (additivity)

$$f(x + y) = f(x) + f(y)$$

# Properties of Time Invariant Systems

Systems do not change with time

$$f(x(t - \tau)) = f(x)(t - \tau)$$

# LTI: Linear Time-Invariant Systems

Systems with these constraints are a great simplification for many real-world systems we would like to model:

- » Circuits
- » Spring-Mass-Damper
- » Reverbs
- » Resonance
- » etc...

# LTI System Example

1. Hammer gives *impulse*
2. System *responds* with velocity

## **Linearity:**

Double force, double velocity, multiple strikes add up

## **Time Invariance:**

System reacts the same whether I do it now or tomorrow

# Other LTI system characteristics

## » Causality:

Output depends only on past and present input

## » BIBO Stability:

Output is bounded for bounded input



# Convolution

*It's easy to visualize how a system reacts to an impulse, but what about a more complex input signal?'*

- » Assume that the signal is constructed from many densely packed impulses (impulse train)
- » Output is then a superposition of all individual responses
- » For discrete systems this is literal, use integration

**Convolution:**

$$y(t) = (x * h)(t) := \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$

$$y(t) = (x * h)(t) := \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$

## Steps

1. Flip one signal
2. Multiply the two signals
3. Integrate the result
4. Shift
5. Go to step 2

# Convolution Example

$$x \quad -1, \quad 0, \quad 1$$

$$h \quad 1, \quad 1, \quad 1$$

$x * h$ :

$$0 \quad x(-2) * h(0) + x(-1) * h(1) + x(0) * h(2) = -1$$

$$1 \quad x(-1) * h(0) + x(0) * h(1) + x(1) * h(2) = -1$$

$$2 \quad x(0) * h(0) + x(1) * h(1) + x(2) * h(2) = 0$$

$$3 \quad x(1) * h(0) + x(2) * h(1) + x(3) * h(2) = 1$$

$$4 \quad x(2) * h(0) + x(3) * h(1) + x(4) * h(2) = 1$$

$$x * h \quad -1, \quad -1, \quad 0, \quad 1, \quad 1$$

# Convolution Animation

[Click here for convolution animation example](#)

# Convolution as Echo

## Steps

1. Scale
2. Delay
3. Sum
4. Repeat

# Convolution as Echo Example

$x$   $-1, 0, 1$

$h$   $1, 1, 1$

$x * h$ :

$x(0)$   $-1, -1, -1$

$x(1)$   $0, 0, 0$

$x(2)$   $1, 1, 1$

$x * h$   $-1, -1, 0, 1, 1$

# Identity and Impulse Response

$$x(t) = \delta(t) * x(t)$$

$$h(t) = \delta(t) * h(t)$$

- » Describes the response of a system to an impulse as a function of time
- » As an impulse includes all frequency, the resulting IR defines the response for all frequencies
- » The convolution of  $\delta(t)$  with a signal/impulse response results in that impulse response

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau) \cdot x(t - \tau) d\tau$$

## Convolution - Properties

### » Commutativity

$$h(t) * x(t) = x(t) * h(t)$$

### » Associativity

$$(g(t) * h(t)) * x(t) = g(t) * (h(t) * x(t))$$

### » Distributivity

$$g(t) * (h(t) + x(t)) = (g(t) * h(t)) + (g(t) * x(t))$$



# Derivation: Commutativity

$$h(t) * x(t) = x(t) * h(t)$$

Substituting  $\tau' = t - \tau$

$$\begin{aligned} x(t) * h(t) &= \int_{-\infty}^{\infty} h(\tau) \cdot x(t - \tau) d\tau \\ &= \int_{-\infty}^{\infty} h(t - \tau') \cdot x(\tau') d(t - \tau') \\ &= \int_{-\infty}^{\infty} x(\tau') \cdot h(t - \tau') d\tau' \\ &= h(t) * x(t) \end{aligned}$$

# Derivation: Associativity

$$(g(t) * h(t)) * x(t) = g(t) * (h(t) * x(t))$$

Changing the order  
of sums and  
shifting the  
operands as shown

$$\begin{aligned}(g(t) * h(t)) * x(t) &= \int_{\tau=-\infty}^{\infty} (g(\tau) * h(\tau)) \cdot x(t - \tau) d\tau \\&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\xi) \cdot h(\tau - \xi) \cdot x(t - \tau) d\tau d\xi \\&= \int_{-\infty}^{\infty} g(\xi) \cdot \int_{-\infty}^{\infty} h(\tau - \xi) \cdot x(t - \tau) d\tau d\xi \\&= \int_{-\infty}^{\infty} g(\xi) \cdot \int_{-\infty}^{\infty} h(\tau') \cdot x(t - \xi - \tau') d\tau' d\xi \\&= \int_{-\infty}^{\infty} g(\xi) \cdot (h(t - \xi) * x(t - \xi)) d\xi \\&= g(t) * (h(t) * x(t))\end{aligned}$$

# Derivation: Distributivity

$$g(t) * (h(t) + x(t)) = g(t) * h(t) + g(t) * x(t)$$

$$g(t) * (h(t) + x(t)) = \int_{-\infty}^{\infty} g(\tau) \cdot (h(t - \tau) + x(t - \tau)) d\tau$$

$$= \int_{-\infty}^{\infty} g(\tau) \cdot h(t - \tau) + g(\tau) \cdot x(t - \tau) d\tau$$

$$= \int_{-\infty}^{\infty} g(\tau) \cdot h(t - \tau) d\tau + \int_{-\infty}^{\infty} g(\tau) \cdot x(t - \tau) d\tau$$

$$= g(t) * h(t) + g(t) * x(t)$$

# Importance of convolution for audio DSP

- »» Ability to model LTI systems
- »» Used both as a runtime technique and mathematical tool

## Uses

- »» FIR Filters
- »» Reverbs
- »» Windowing effects
- »» Modeling analog systems

# Summary - LTI

- » Many real-world systems can be approximated by an **LTI system**
- » Properties of an LTI system:
  - Linearity 1: Homogeneity (Scaling)
  - Linearity 2: Superposition (additivity)
  - Time Invariance (system doesn't change')
- » Additional Properties
  - Causality (no future input)
  - BIBO - Bounded input bounded output
- » Impulse response is a **complete** description of an LTI system

# Summary - Convolution

Convolution:

- » Describes the process of generating the output of an LTI system from the input
- » Is commutative
- » Is associative
- » Is distributive