Digital Signal Processing for Music

Part 15: Digital Filters II

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Z-Transform: Introduction

The z-transform is

- A generalization of DFT,
- Widely used in DSP as analysis,
- A useful tools to describe systems,
- The discrete-time counterpart of the Laplace transform

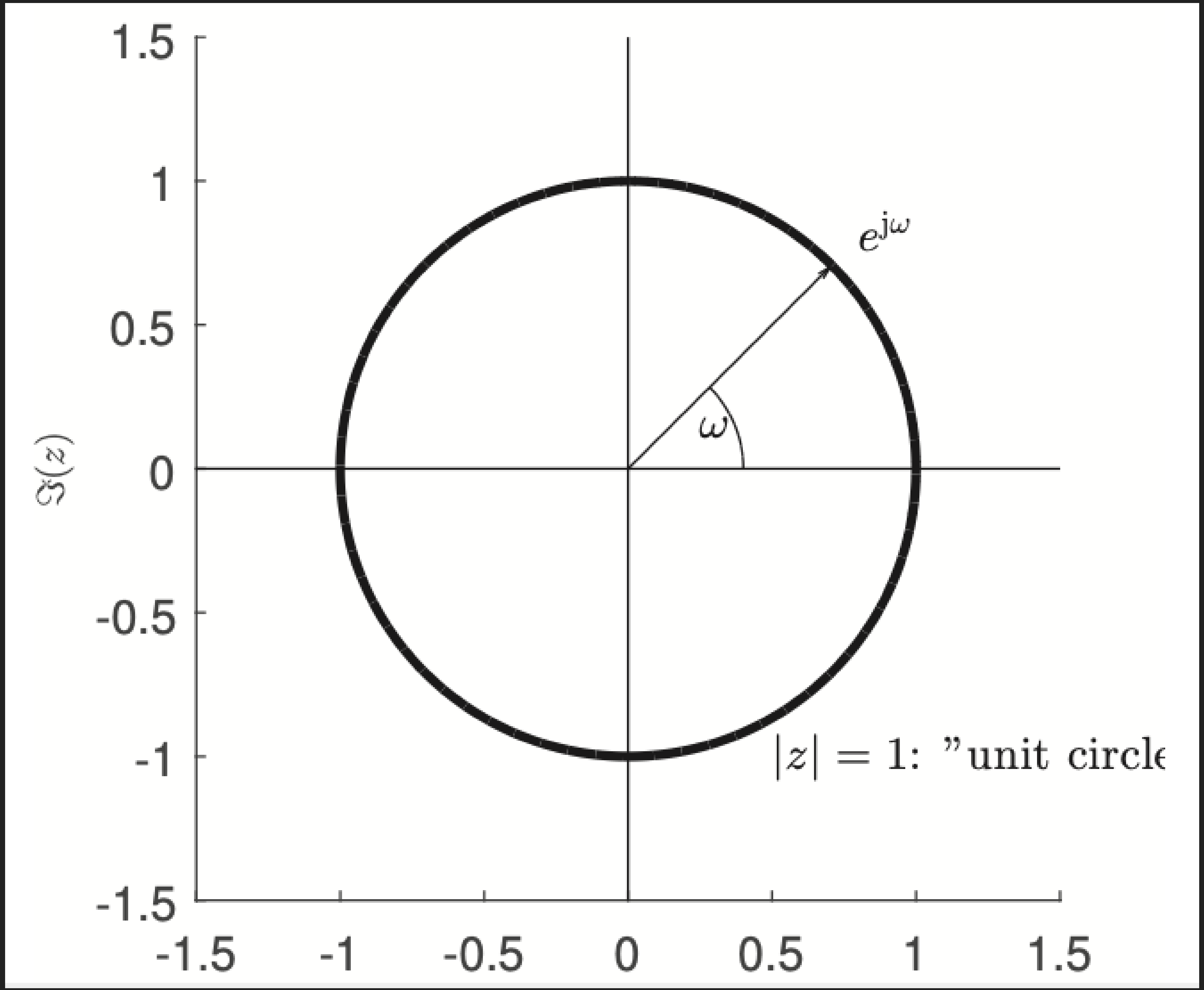
Z-Transform Definition

$$X(z) = \sum_{i=-\infty}^{\infty} x(i) z^{-i}, \quad z \in \mathfrak{C}$$

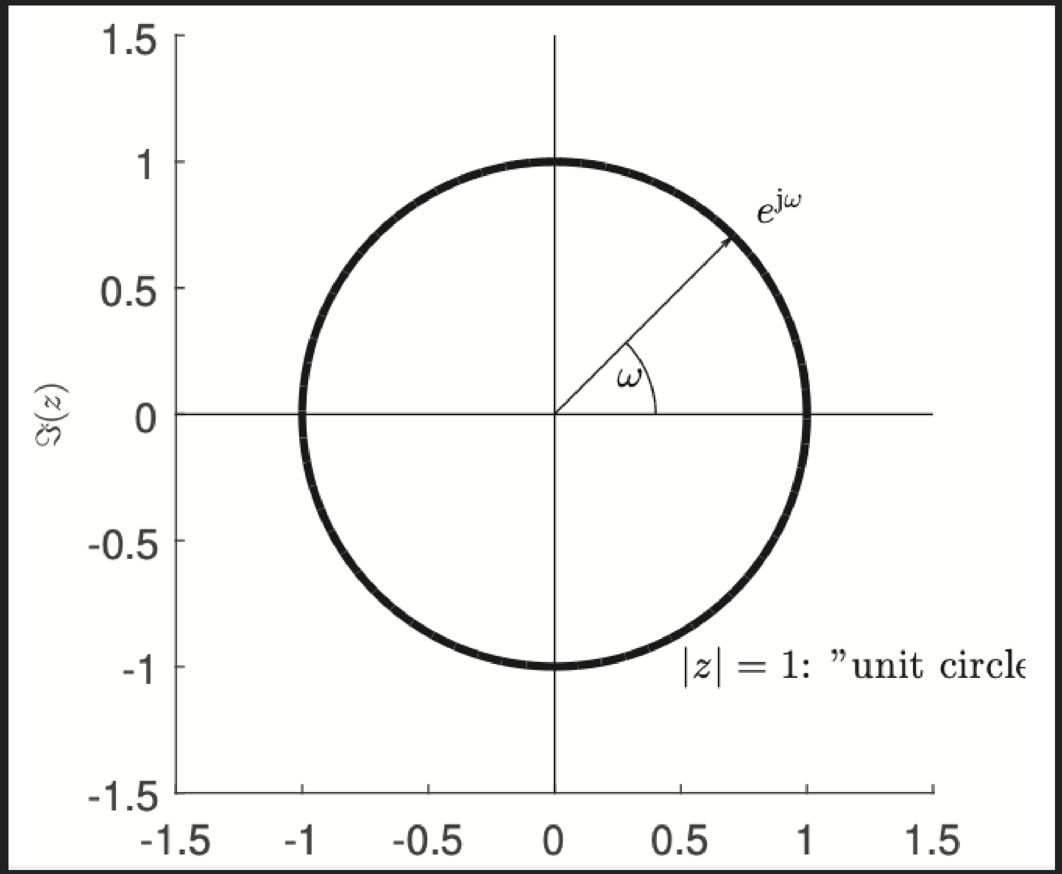
- $X(z)$: complex function of a complex number
- Compare Fourier transform $X(\mathrm{j}\omega)$: complex function of real-valued ω

$$X(\mathrm{j}\omega) = \sum_{i=-\infty}^{\infty} x(i) e^{-\mathrm{j}\omega i} \Rightarrow X(\mathrm{j}\omega) = X(z) \text{ at } z = e^{\mathrm{j}\omega}$$

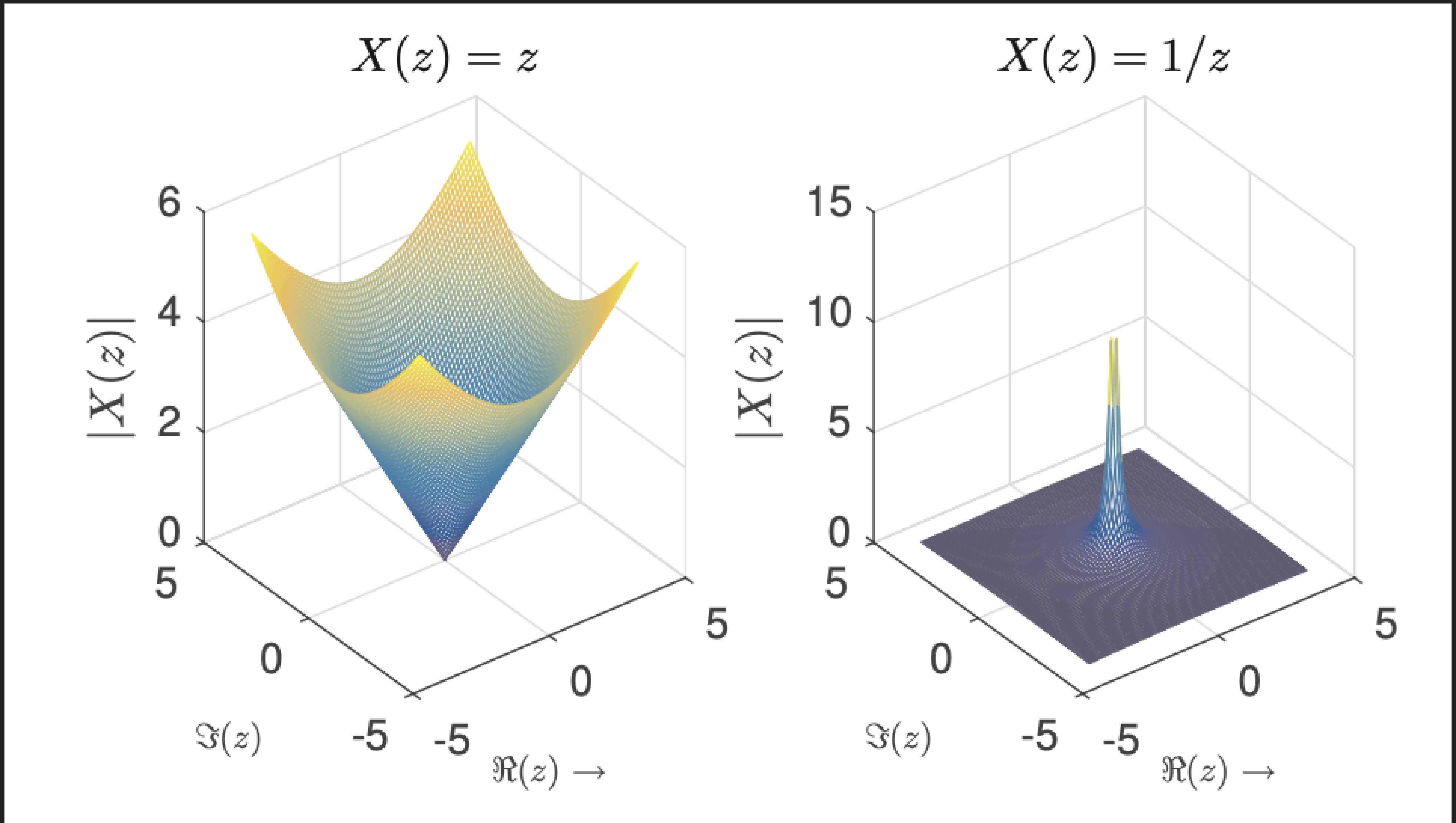
Z-plane



- $X(z)$ defined on complex plane
- $X(\mathrm{j}\omega)$ defined on unit circle
- Observation: $X(\mathrm{j}\omega)$ is periodic with 2π



Trivial Examples



What is the magnitude for $X(z) = \frac{1}{(z-0.5)}$

Same as $\frac{1}{z}$ but shifted

System Description

Fourier transform and z-transform have largely similar properties, most importantly

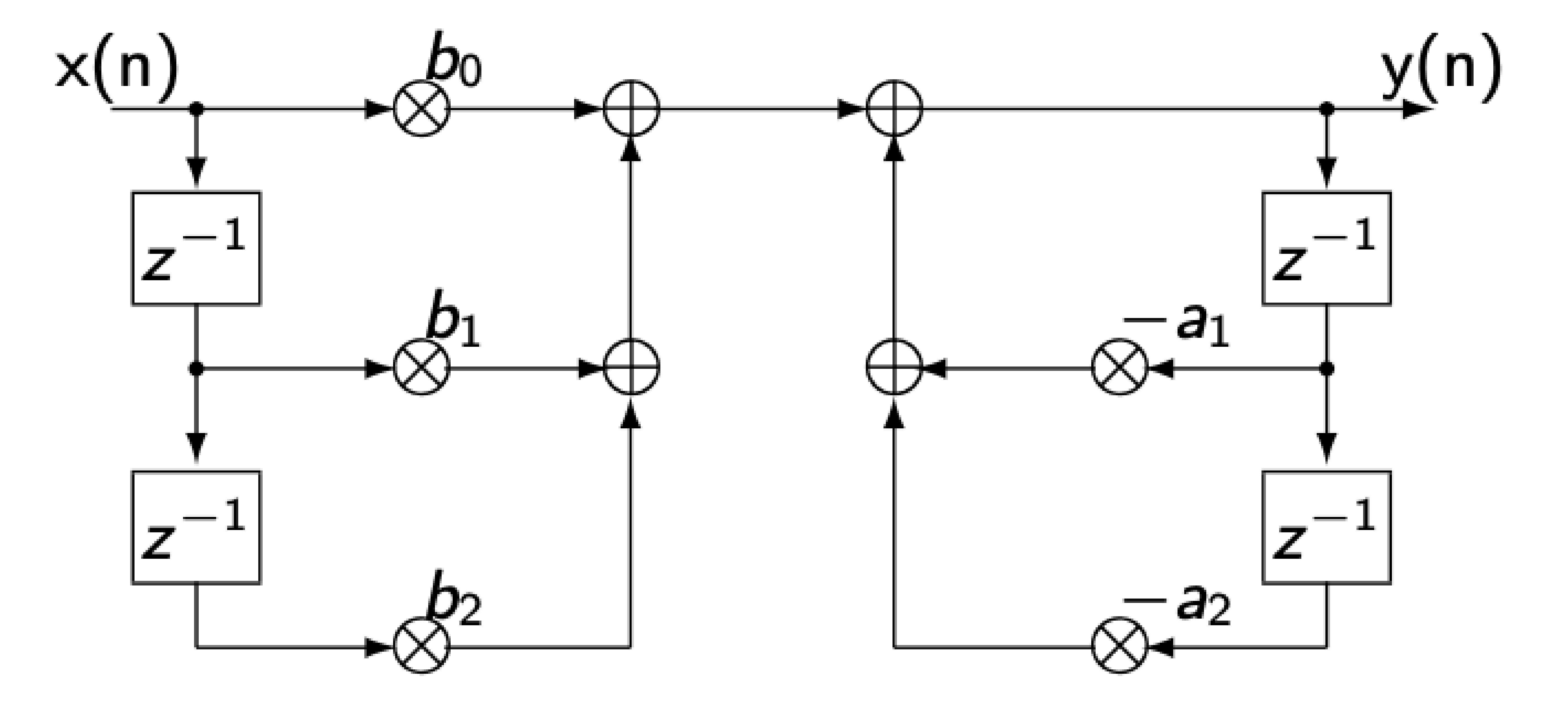
- **Linearity**

$$y(i) = c_1 x_1(i) + c_2 x_2(i) \Rightarrow Y(\mathrm{j}\omega) = c_1 X_1(\mathrm{j}\omega) + c_2 X_2(\mathrm{j}\omega)$$
$$\Rightarrow Y(z) = c_1 X_1(z) + c_2 X_2(z)$$

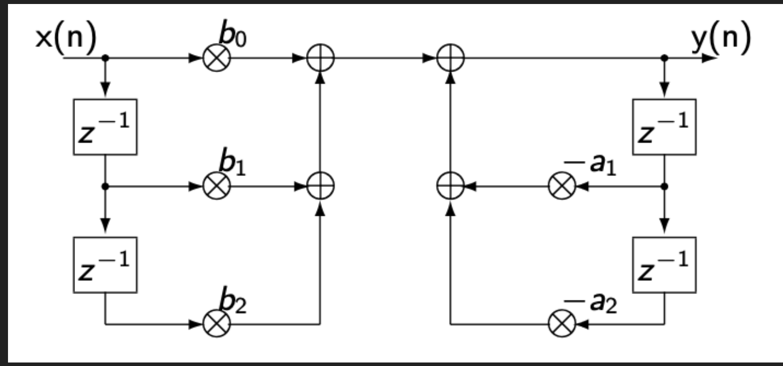
- **Time Shift**

$$y(i) = x(i-n) \Rightarrow Y(\mathrm{j}\omega) = e^{-\mathrm{j}\omega n} X(\mathrm{j}\omega)$$
$$\Rightarrow Y(z) = z^{-n} X(z)$$

Biquad: Difference Equation



Biquad: Difference Equation



$$y(i) = \sum_{j=0}^{2} b_j x(i-j) - \sum_{k=1}^{2} a_j y(i-j)$$

$$Y(z) = \sum_{j=0}^{2} b_j X(z) z^{-j} - \sum_{k=1}^{2} a_j Y(z) z^{-j}$$

$$Y(z)\left(1 + \sum_{j=1}^{2} a_j z^{-j}\right) = X(z) \sum_{j=0}^{2} b_j z^{-j}$$

Biquad: Transfer Function

$$
\begin{aligned}
H(z) &= \frac{Y(z)}{X(z)} \\
&= \frac{\sum_{j=0}^{2} b_j z^{-j}}{1 + \sum_{j=1}^{2} a_j z^{-j}} \\
&= \frac{b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2}}{1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}} \\
&= \frac{\text{numerator polynomial}}{\text{denominator polynomial}}
\end{aligned}
$$

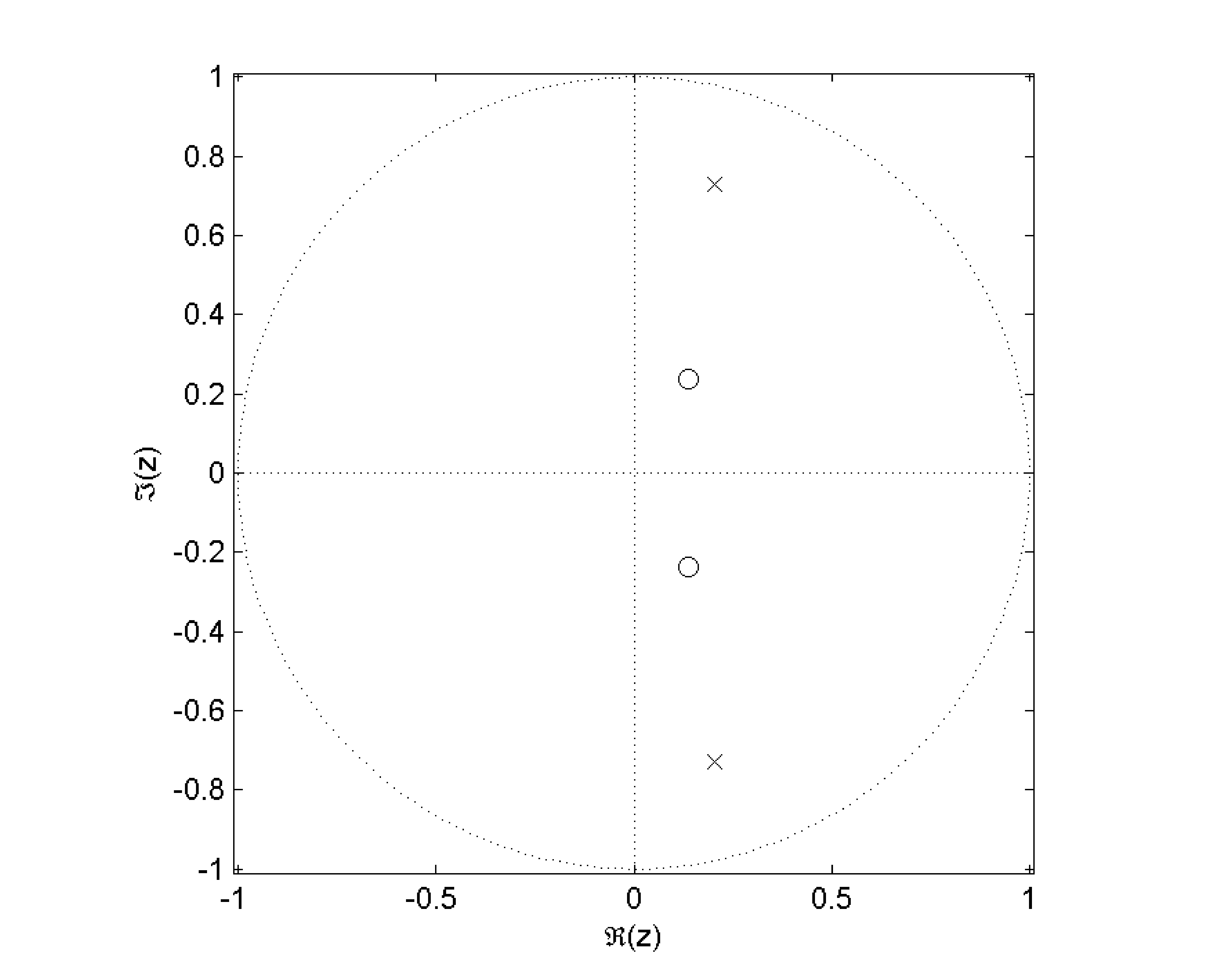
Biquad: Poles and Zeroes

- Numerator ➞ 0: Zero
- Denominator ➞ 0: Pole

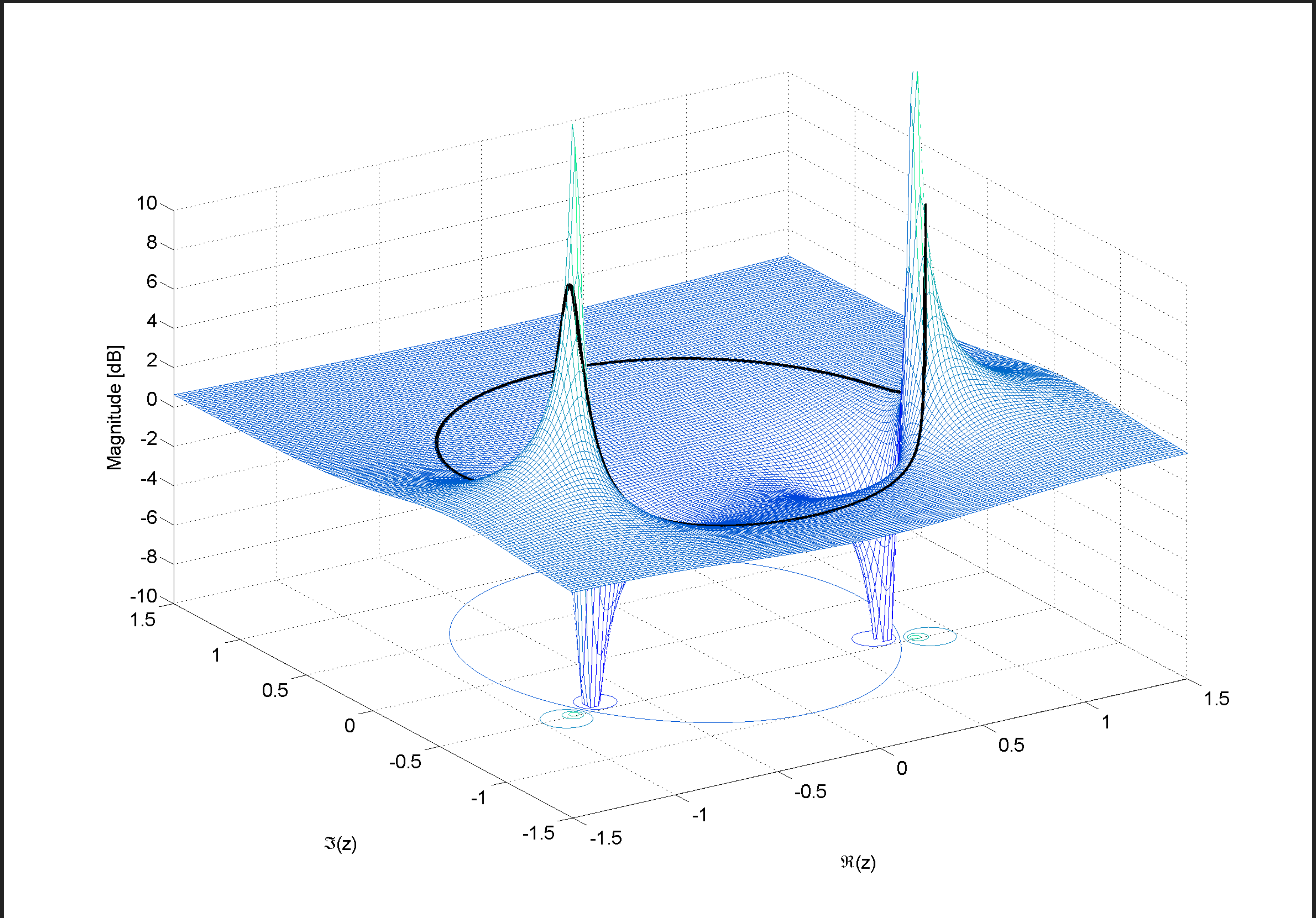
$$1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2} = 0$$

$$\implies z_{\infty 1,2} = \frac{a_1}{2} \pm \frac{1}{2}\sqrt{a_1^2 - 4a_2}$$

Biquad: Z-Plane Example



Biquad: Z-Plane Example



Animation

Filters: Z-Plane Characteristics

- **Stability:**
  Poles within unit circle
- **Zero points and poles:**
  Are either real or complex conjugate
- **Minimal phase systems:**
  No zero points outside of unit circle
- **All pass system:**
  Poles and zeroes symmetric wrt unit circle
- **Linear phase:**
  Zero points within and outside unit circle symmetric wrt unit circle

Filters: Filter Design

- **Impulse invariance**: sample impulse response
  - If continuous system is band-limited, frequency response will be approximately equal (below $f_S/2$)
  - Special case: No filter definition available ➞ FIR coefficients
- **Bi-Linear transform**
  - Map filter from (analogue) Laplace-plane to (digital) z-plane

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}$$

$$z = \frac{1 + s\frac{T_\mathrm{S}}{2}}{1 - s\frac{T_\mathrm{S}}{2}}$$

  - Introduces frequency warping (increasing towards Nyquist frequency)

Filters: Filter Design

- **Frequency Transformation**
  - Transfrom a (low-pass) prototype filter
  - Usually via all-pass mapping filter
- **Iterative approximation** of the magnitude response
- **Intuitive methods**
  - Manually move zeros and poles in z-plane
  - Draw magnitude response in frequency domain

Effects of Word Length

- Quantization of filter coefficients can lead to problems
- Effects depend on filter type and structure
  - Changes of transfer function
  - Instability
  - Quantization noise → SNR

Summary

	FIR	IIR
IR Length	Finite	Infinite
Structure	Non-Recursive	Recursive
Phase Linearity	Possible	Impossible
Ratio Steepness/Workload	Low	High
Stability	Guaranteed	Possibly Unstable

- Every LTI system is **completely described** either by
  - Its complex transfer function,
  - Its impulse response, or
  - Its pole and zero position in the z-plane