Digital Signal Processing for Music

Part 3: Signal Descriptions

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Describing Random Signals

- Ergodic signals do not have a functional description

  Other ways of describing these signals have to be found

- Ergodic signal characteristics are not time variant

  We are looking for **time-independent descriptions**

- These descriptions might also be convenient to use for some deterministic signals

Probability and Occurence

N: number of overall observations

$N(x_i)$: number of occurences of symbol x_i

- Relative number of occurrences: $\hat{p}_i = \frac{N(x_i)}{N}$
- Probability: $p_i = \lim_{N \to \infty} \frac{N(x_i)}{N}$

Properties

$$\sum_i p_i = 1$$

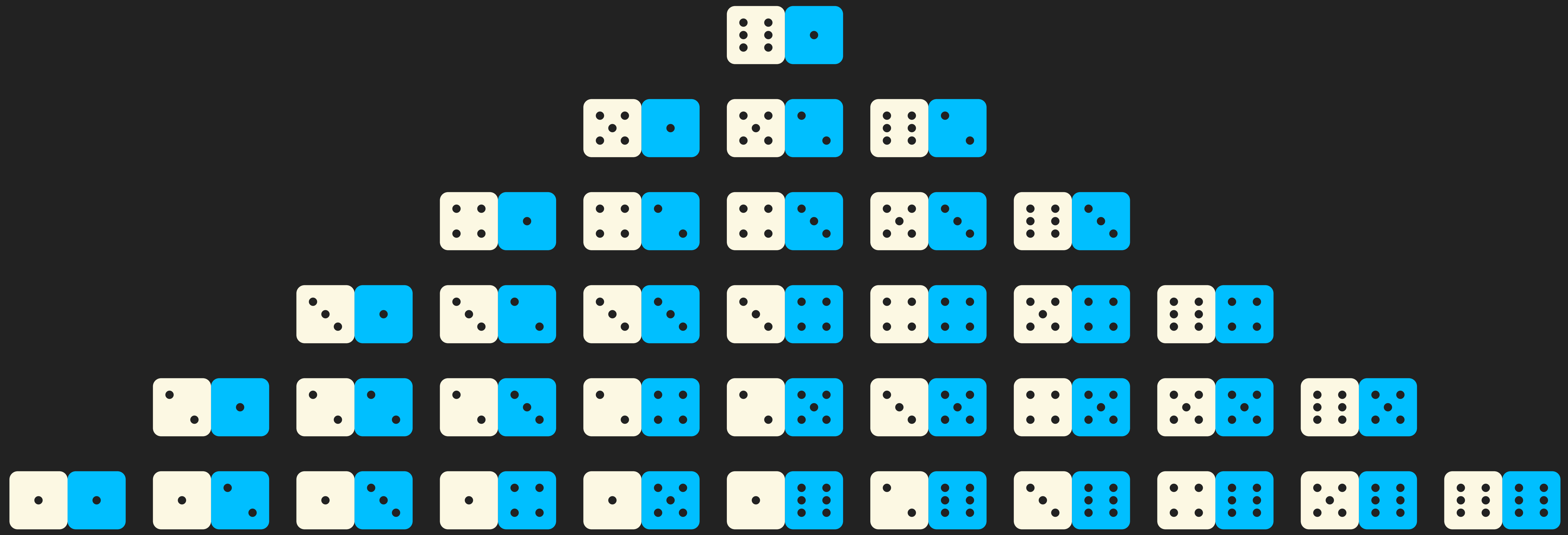
$$0 \leq p_i \leq 1$$

Probability Distribution Example

Value	⚀	⚁	⚂	⚃	⚄	⚅
p(x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Probability Distribution for the roll of two dice

2	3	4	5	6	7	8	9	10	11	12
$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$



Continuous Probability Density Distribution

i ➞ continuous 👉 **PDF**

$$\int_{-\infty}^{\infty} p_X(x)dx = 1$$

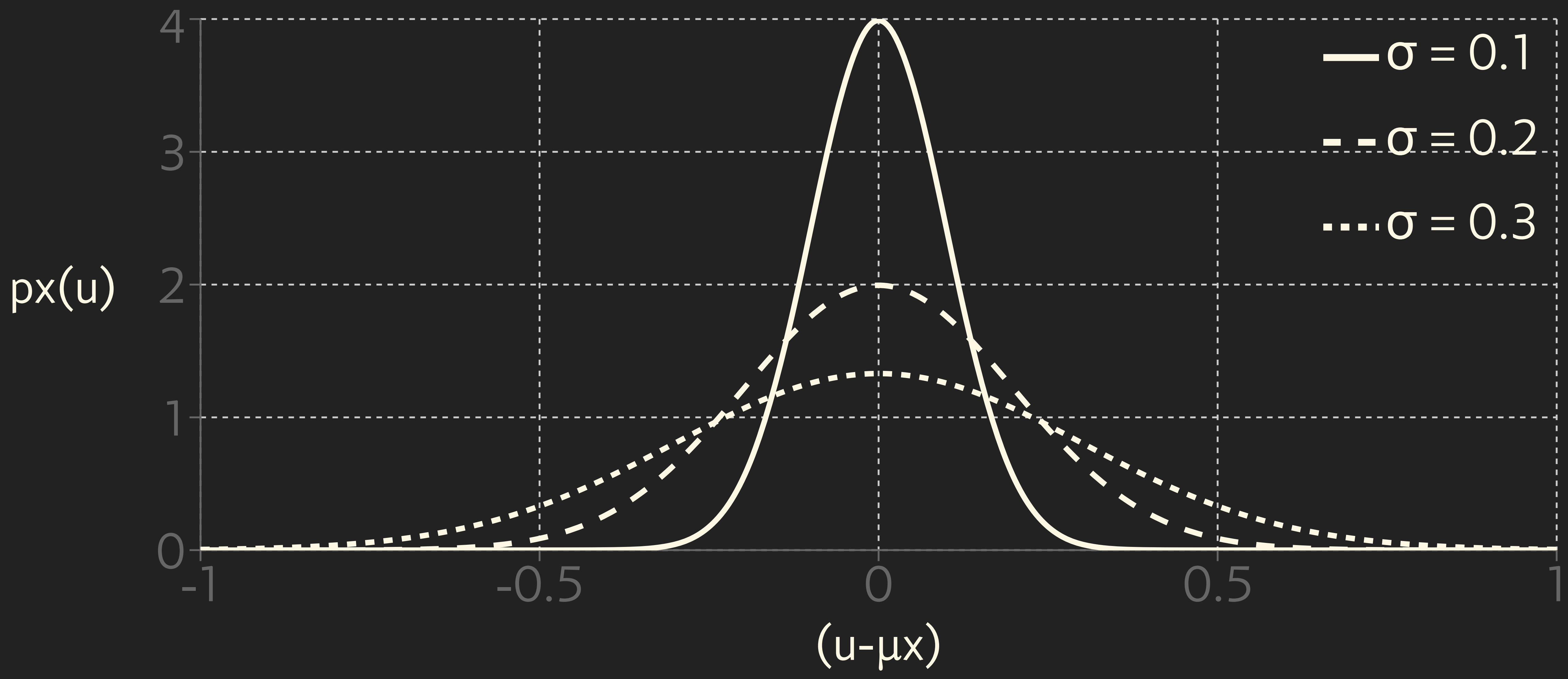
$$0 \leq p_X(x)$$

Probability of x being a value smaller than or equal to x_c

$$\int_{-\infty}^{x_c} p_X(x)dx$$

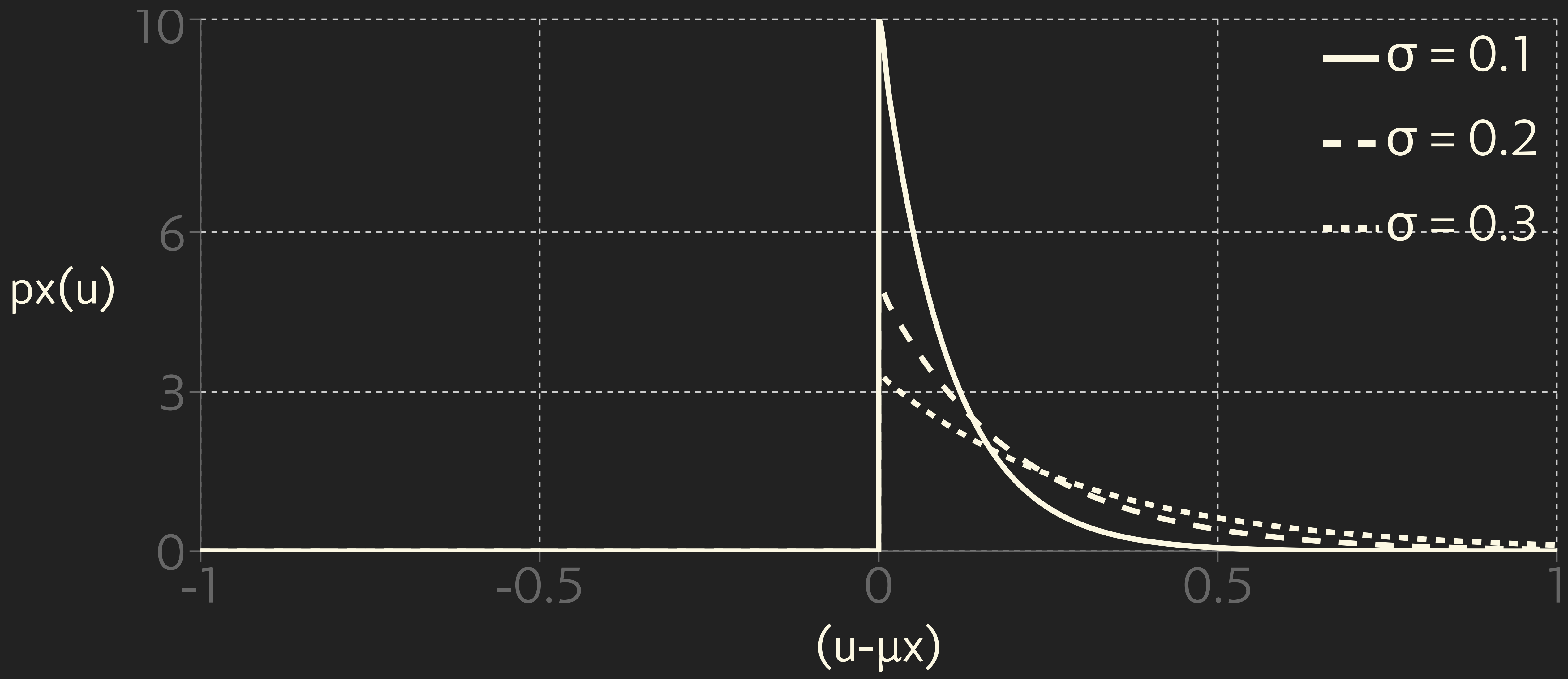
Example PDF: Gaussian

$$p_X(x) = \frac{1}{\sigma_X\sqrt{2\pi}} e^{-(\frac{x-\mu_X}{2\sigma_X})^2}$$



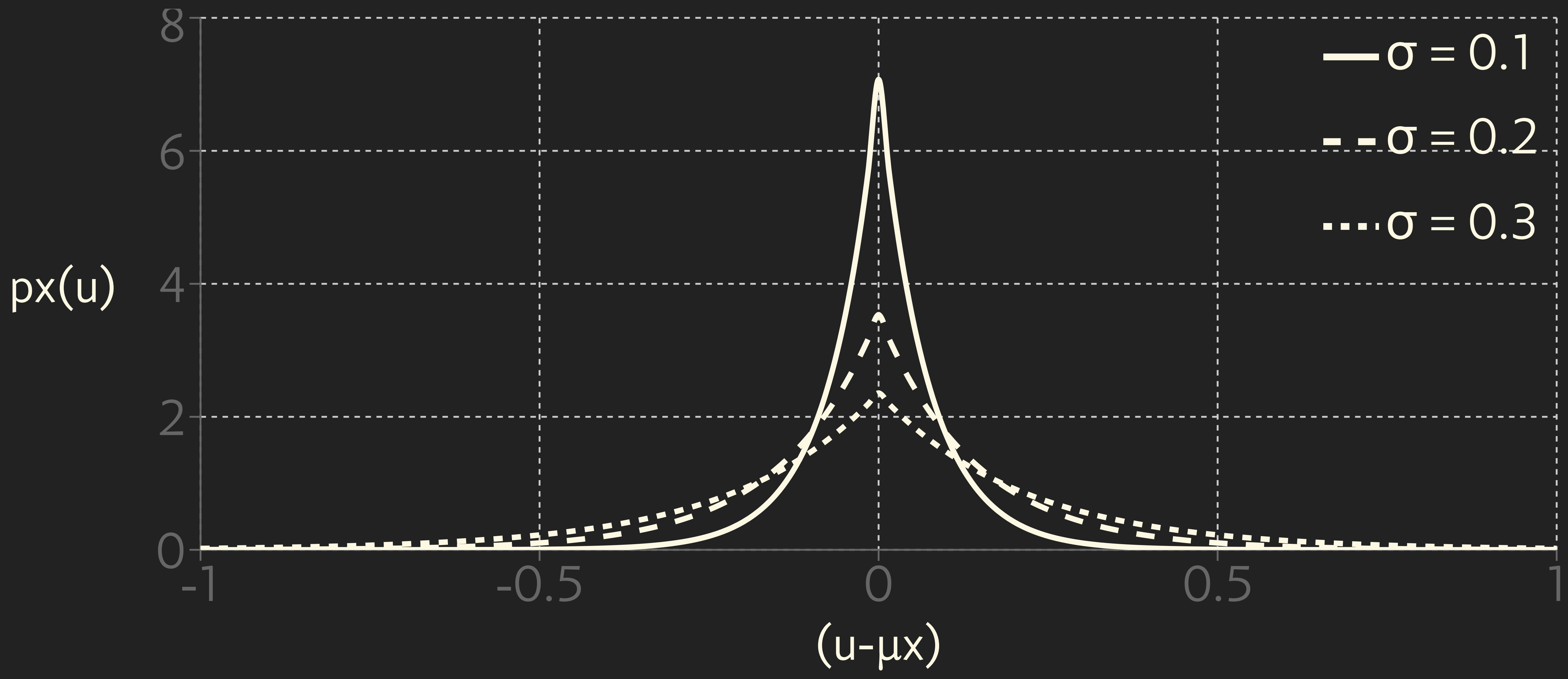
Example PDF: Exponential

$$p_X(x) = \begin{cases} \frac{1}{\sigma_X} e^{-\frac{x}{\sigma_X}} & \text{if x} > 0 \\ 0 & \text{else} \end{cases}$$

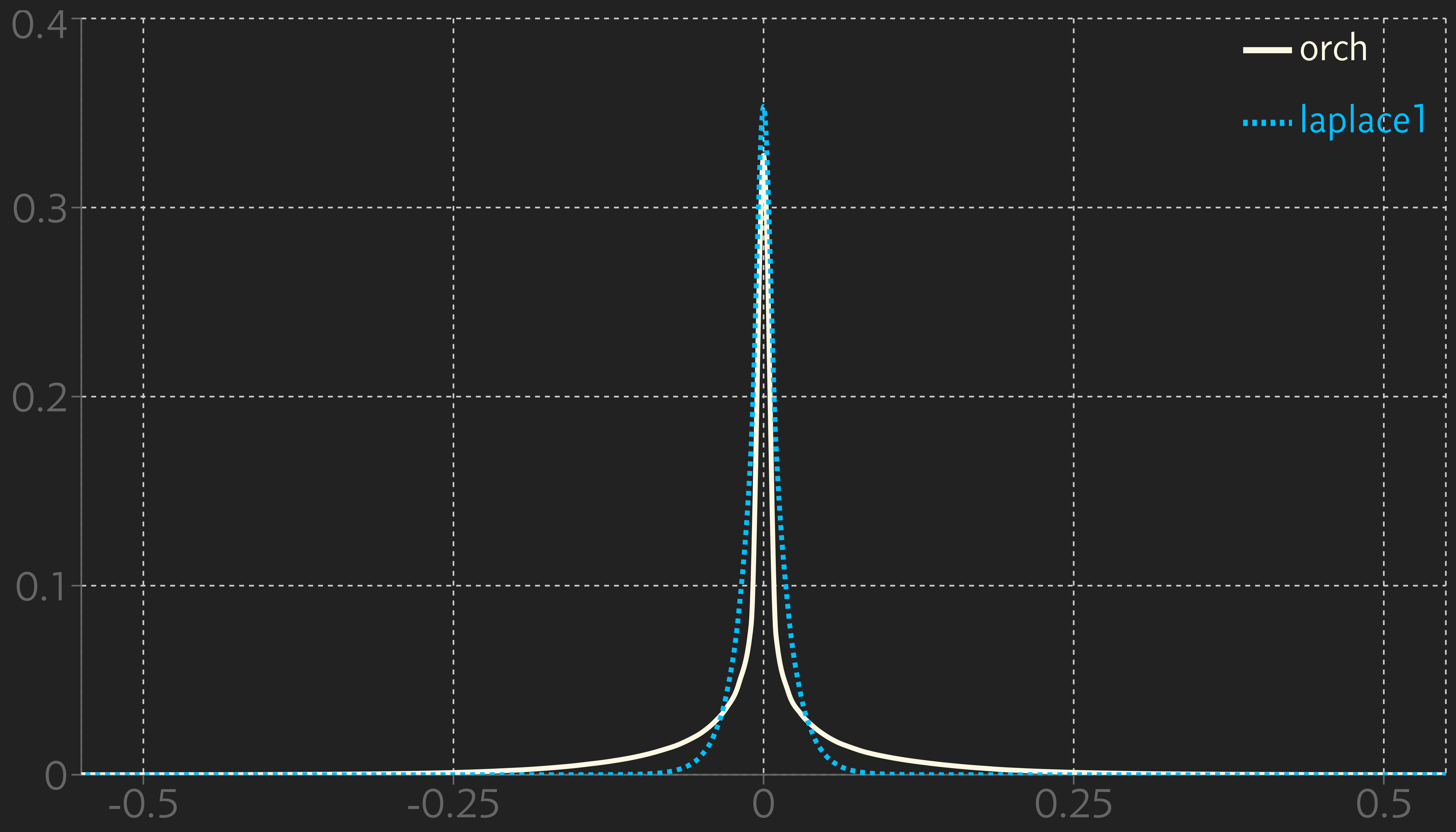


Example PDF: Laplace (2-sided exp)

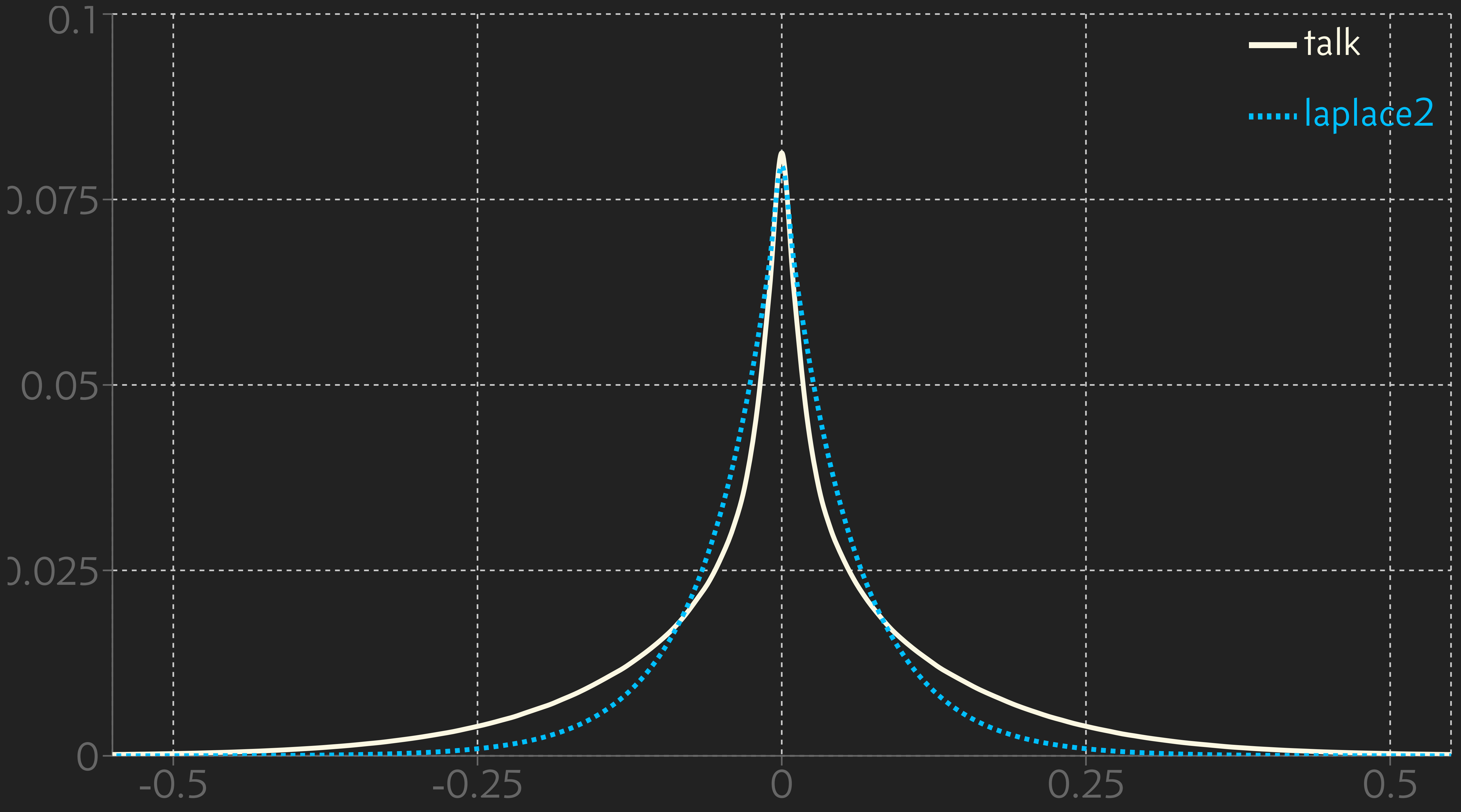
$$p_X(x) = \frac{1}{\sigma_X\sqrt{2}} e^{-\sqrt{2}\frac{|x-\mu_X|}{\sigma_X}}$$



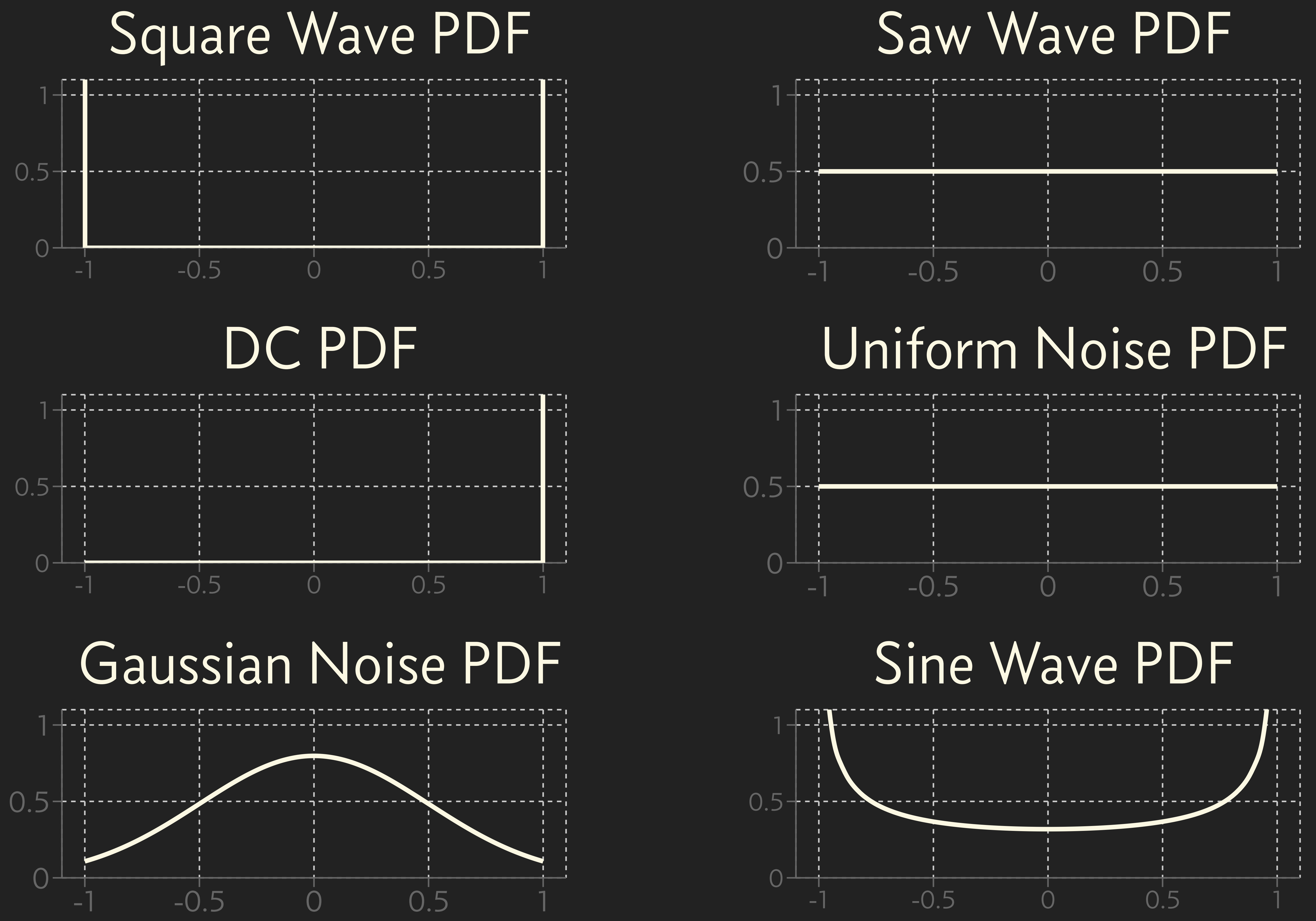
Measured RDF - Orchestra



Measured RDF - Podcast Conversation



Distributions of Generated Signals



Expected Value

Example: Average grade, five students, grades: 1, 2, 1, 3, 5

$$\hat{\mu}_X = \frac{1+2+1+3+5}{5} = 2.4$$

Grade	# of Occurrences	Relative Frequency
1	2	2/5
2	1	1/5
3	1	1/5
4	0	0/5
5	1	1/5

Expected Value

$$\mu = \frac{2}{5} \cdot 1 + \frac{1}{5} \cdot 2 + \frac{1}{5} \cdot 3 + \frac{0}{5} \cdot 4 + \frac{1}{5} \cdot 5 = 2.4$$

$$\mu_X = \sum_{\forall x} p(x) \cdot x$$

$$\mu_X = \mathcal{E}\{X\} = \int_{-\infty}^{+\infty} x p_X(x) dx$$

Expected Value

Generalization:

$$\mathcal{E}\{f(X)\} = \sum_i f(x)p(x)$$

Examples

- Mean: $f(x) = x$
- Quad. Mean: $f(x) = x^2$

- kth moment:

$$\mathcal{E}\{X^k\} = \int_{-\infty}^{+\infty} x^k p_X(x)dx$$

- kth central moment:

$$\mathcal{E}\{(X-\mu_X)^k\} = \int_{-\infty}^{+\infty} (x-\mu_X)^k p_X(x)dx$$

- Example: 2nd order central moment: **Variance**

$$\sigma_X^2 = \mathcal{E}\{(X-\mu_X)^2\} = \int_{-\infty}^{+\infty} (x-\mu_X)^2 p_X(x)dx$$

Calculation of Moments

(Central) moments (mean, power, variance, etc.) can be computed from:

- The signal
- The signal's PDF

Central Moments Summary

Order	Name	Time (Continuous)	PDF (Continuous)
1	μ_X	$\frac{1}{T}\int_{-T/2}^{T/2} x(t)dt$	$\int_{-\infty}^{\infty} xp_X(x)dx$
2	σ_X^2	$\frac{1}{T}\int_{-T/2}^{T/2} (x(t)-\mu_X)^2 dt$	$\int_{-\infty}^{\infty} (x-\mu_X)^2 p_X(x)dx$

Order	Name	Time (Discrete)	PDF (Discrete)
1	μ_X	$\frac{1}{N}\sum_{i=0}^{N} x(i)$	$\sum_{\forall x} xp(x)$
2	σ_X^2	$\frac{1}{N}\sum_{i=0}^{N}(x(i)-\mu_X)^2$	$\sum_{\forall x}(x-\mu_X)^2 p(x)$

Standard deviation $\sigma_X = \sqrt{\sigma_X^2}$

Summary

- PDF can tell us many important details about a signal
- Statistical measures can be used to describe signal properties
- Statistical measures can be derived from both the time domain signal and it's PDF
- Often-used measures are:

Mean and Median

Variance and Standard Deviation

Higher Order moments less frequently (Skewness, Kurtosis)

Other PDF descriptions possible (quartile-distances, etc)