Digital Signal Processing for Music

Part 5: LTI Systems & Convolution

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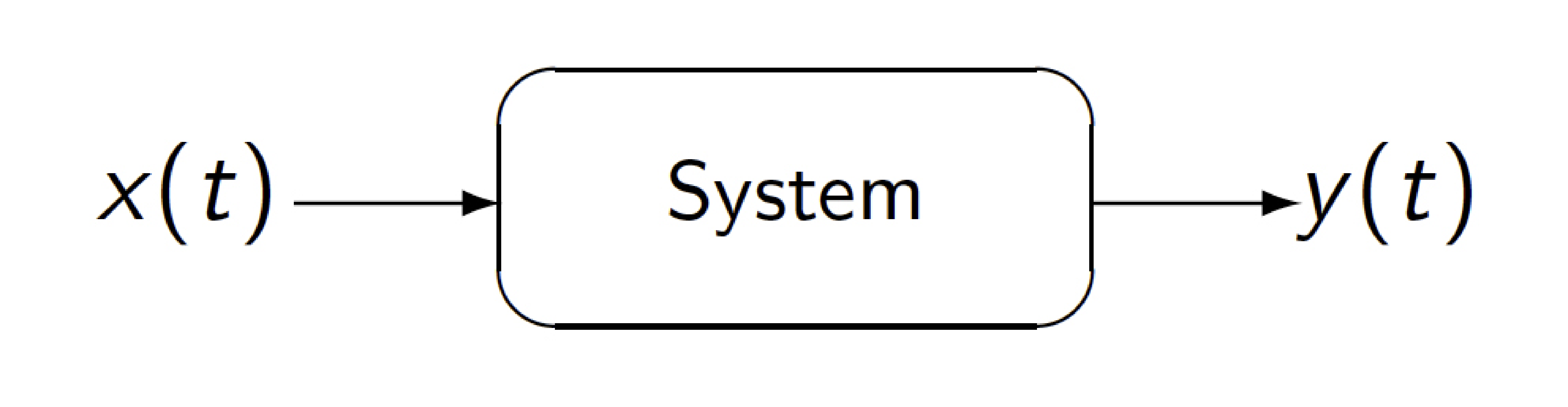
Systems

- Any process producing an output signal in response to an input signal



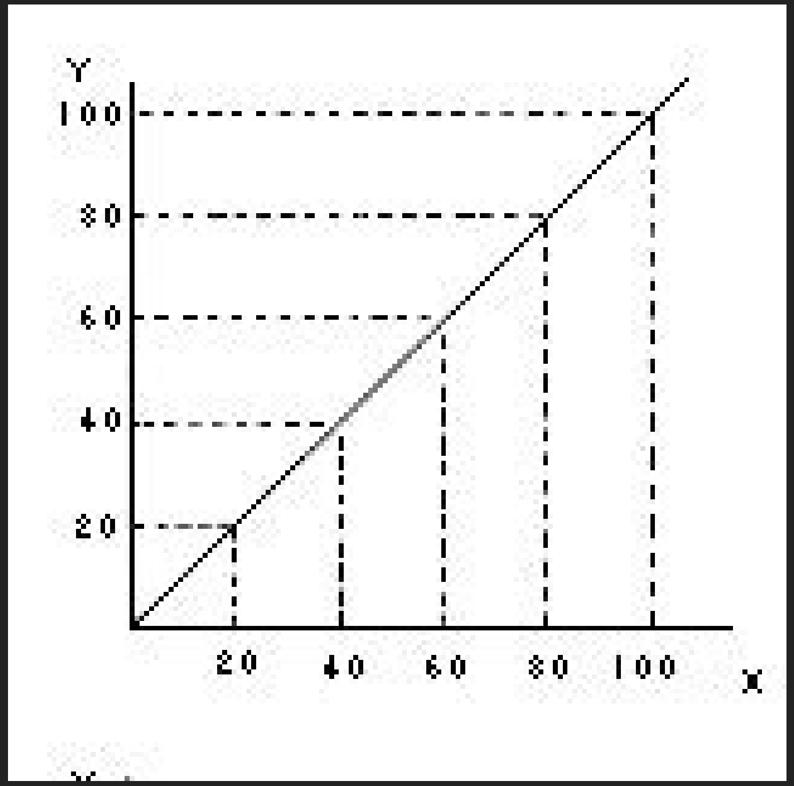
Examples of systems in signal processing

- Filters, Effects
- Vocal Tract
- Room
- Audio cable
- ...

Linearity & Non-Linearity

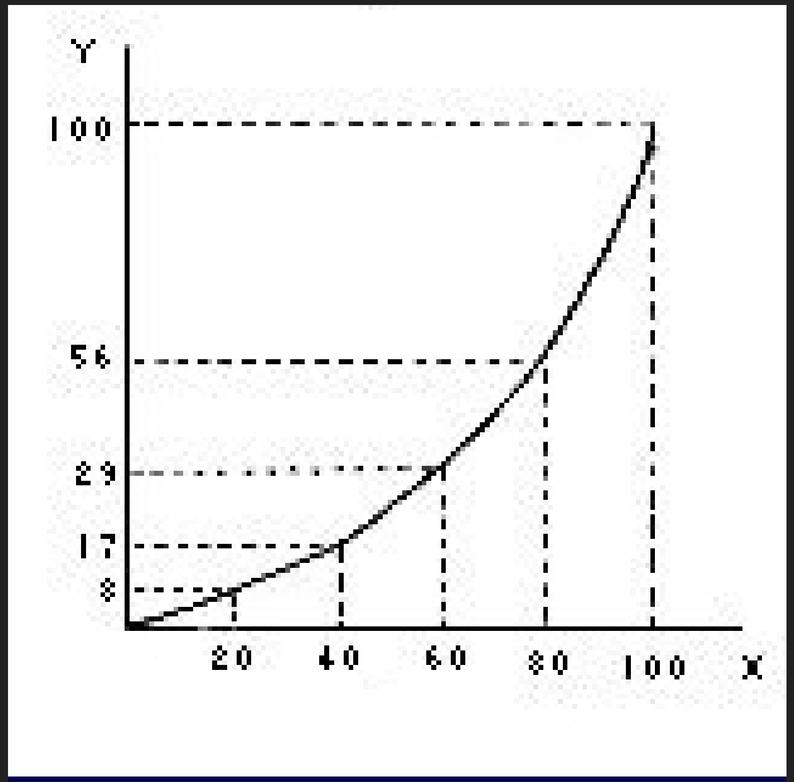
Examples for (mostly) linear systems

- Room
- EQ
- Echo
- Envelope



Examples for non-linear systems

- Diode
- Vacuum Tube
- Optical Compressor
- Distortion



Properties of Linear Systems

1. **Homogeneity**

$$f(ax) = af(x)$$

2. **Superposition** (additivity)

$$(x + y) = f(x) + f(y)$$

Properties of Time Invariant Systems

Systems do not change with time

$$f(x(t-\tau)) = f(x)(t-\tau)$$

LTI: Linear Time-Invariant Systems

Systems with these constraints are a great simplification for many real-world systems we would like to model:

- Circuits
- Spring-Mass-Damper
- Reverbs
- Resonance
- etc...

LTI System Example

1. Hammer gives *impulse*
2. System *responds* with velocity

Linearity:

Double force, double velocity, multiple strikes add up

Time Invariance:

System reacts the same whether I do it now or tomorrow

Other LTI system characteristics

- **Causality:**

  Output depends only on past and present input
- **BIBO Stability:**

  Output is bounded for bounded input

Convolution

It's easy to visualize how a system reacts to an impulse, but what about a more complex input signal?'

- Assume that the signal is constructed from many densely packed impulses (impulse train)
- Output is then a superposition of all individual responses
- For discrete systems this is literal, use integration

Convolution:

$$y(t) = (x * h)(t) := \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$y(t) = (x * h)(t) := \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Steps

1. Flip one signal
2. Multiply the two signals
3. Integrate the result
4. Shift
5. Go to step 2

Convolution Example

x -1, 0, 1 h 1, 1, 1

$x * h$:

0 $x(-2) * h(0) + x(-1) * h(1) + x(0) * h(2) = -1$

1 $x(-1) * h(0) + x(0) * h(1) + x(1) * h(2) = -1$

2 $x(0) * h(0) + x(1) * h(1) + x(2) * h(2) = 0$

3 $x(1) * h(0) + x(2) * h(1) + x(3) * h(2) = 1$

4 $x(2) * h(0) + x(3) * h(1) + x(4) * h(2) = 1$

$x * h$ -1, -1, 0, 1, 1

Convolution Animation

Click here for convolution animation example

Convolution as Echo

Steps

1. Scale
2. Delay
3. Sum
4. Repeat

Convolution as Echo Example

x -1, 0, 1 h 1, 1, 1

$x * h$:

$x(0)$	-1,	-1,	-1			
$x(1)$		0,	0,	0		
$x(2)$			1,	1,	1	
$x * h$	-1,	-1,	0,	1,	1	

Identity and Impulse Response

$$x(t) = \delta(t) * x(t)$$

$$h(t) = \delta(t) * h(t)$$

- Describes the response of a system to an impulse as a function of time
- As an impulse includes all frequency, the resulting IR defines the response for all frequencies
- The convolution of $\delta(t)$ with a signal/impulse response results in that impulse response

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau) \cdot x(t - \tau) d\tau$$

Convolution - Properties

- **Commutativity**

$$h(t) * x(t) = x(t) * h(t)$$

- **Associativity**

$$\big(g(t) * h(t)\big) * x(t) = g(t) * \big(h(t) * x(t)\big)$$

- **Distributivity**

$$g(t) * \big(h(t) + x(t)\big) = \big(g(t) * h(t)\big) + \big(g(t) * x(t)\big)$$

Derivation: Commutativity

$$h(t) * x(t) = x(t) * h(t)$$

Substituting $\tau' = t - \tau$

$$x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau) \cdot x(t - \tau) d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau) \cdot x(t - \tau) d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau') \cdot h(t - \tau') d\tau'$$

$$= h(t) * x(t)$$

Derivation: Associativity

$$\big(g(t) * h(t)\big) * x(t) = g(t) * \big(h(t) * x(t)\big)$$

Changing the order of sums and shifting the operands as shown below

$$\big(g(t) * h(t)\big) * x(t) = \int_{\tau=-\infty}^{\infty} \big(g(\tau) * h(\tau)\big) \cdot x(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\xi) \cdot h(\tau - \xi) \cdot x(t-\tau) d\tau d\xi$$

$$= \int_{-\infty}^{\infty} g(\xi) \cdot \int_{-\infty}^{\infty} h(\tau - \xi) \cdot x(t-\tau) d\tau d\xi$$

$$= \int_{-\infty}^{\infty} g(\xi) \cdot \int_{-\infty}^{\infty} h(\tau') \cdot x(t - \xi - \tau') d\tau' d\xi$$

$$= \int^{\infty} g(\xi) \cdot \big(h(t-\xi) * x(t-\xi)\big) d\xi$$

Derivation: Distributivity

$$g(t) * \big(h(t) + x(t)\big) = g(t) * h(t) + g(t) * x(t)$$

$$g(t) * \big(h(t) + x(t)\big) = \int_{-\infty}^{\infty} g(\tau) \cdot \big(h(t-\tau) + x(t-\tau)\big) d\tau$$

$$= \int_{-\infty}^{\infty} g(\tau) \cdot h(t-\tau) + g(\tau) \cdot x(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} g(\tau) \cdot h(t-\tau) d\tau + \int_{-\infty}^{\infty} g(\tau) \cdot x(t-\tau) d\tau$$

$$= g(t) * h(t) + g(t) * x(t)$$

Importance of convolution for audio DSP

- Ability to model LTI systems
- Used both as a runtime technique and mathematical tool

Uses

- FIR Filters
- Reverbs
- Windowing effects
- Modeling analog systems

Summary - LTI

- Many real-world systems can be approximated by an **LTI system**
- Properties of an LTI system:
    - Linearity 1: Homogeneity (Scaling)
    - Linearity 2: Superposition (additivity)
    - Time Invariance (system doesn't change')
- Additional Properties
    - Causality (no future input)
    - BIBO - Bounded input bounded output
- Impulse response is a **complete** description of an LTI system

Summary - Convolution

Convolution:

- Describes the process of generating the output of an LTI system from the input
- Is commutative
- Is associative
- Is distributive