

Digital Signal Processing for Music

Part 15: Digital Filters II

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Z-Transform: Introduction

The z-transform is

- » A generalization of DFT,
- » Widely used in DSP as analysis,
- » A useful tools to describe systems,
- » The discrete-time counterpart of the Laplace transform

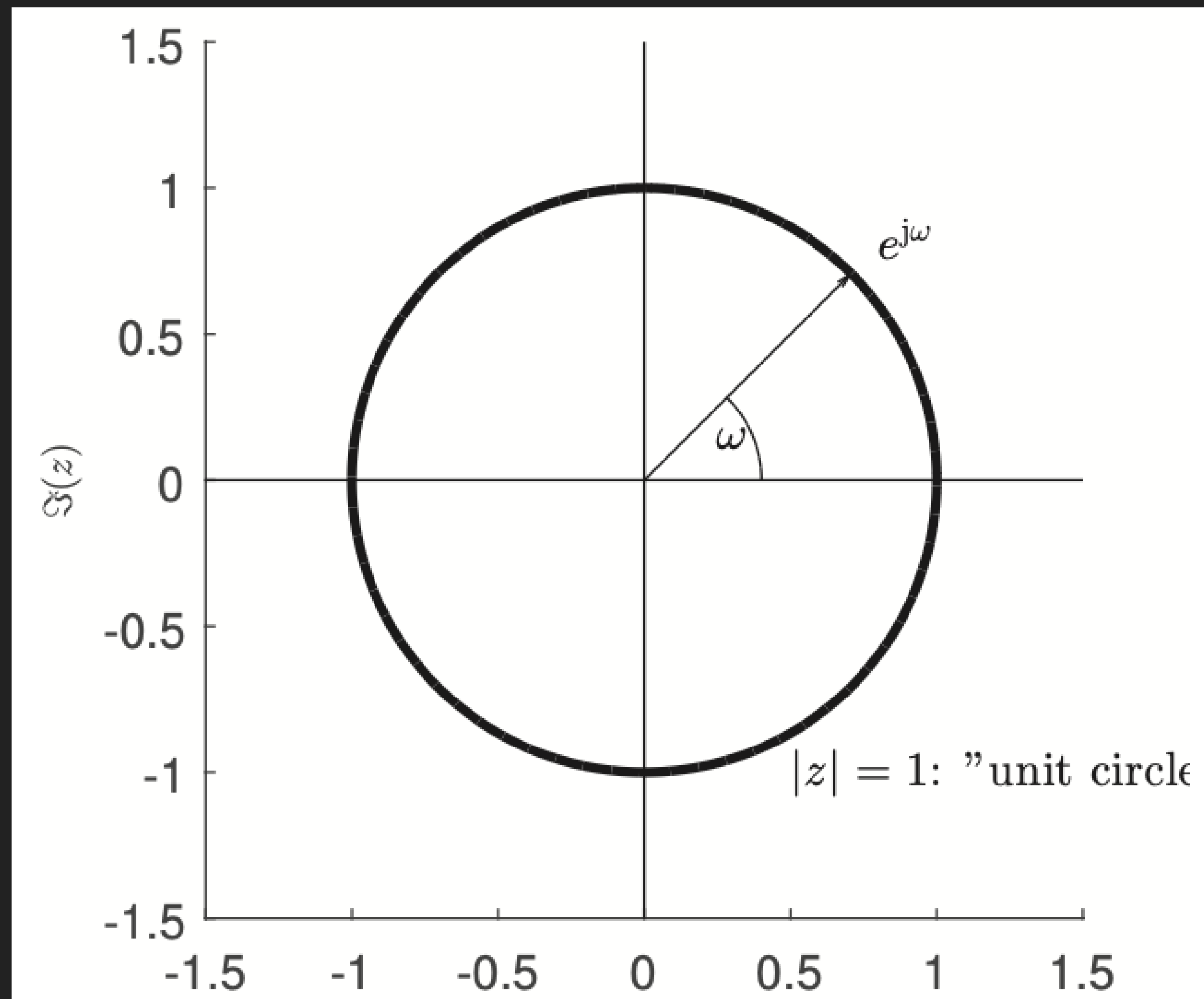
Z-Transform Definition

$$X(z) = \sum_{i=-\infty}^{\infty} x(i)z^{-i}, \quad z \in \mathfrak{C}$$

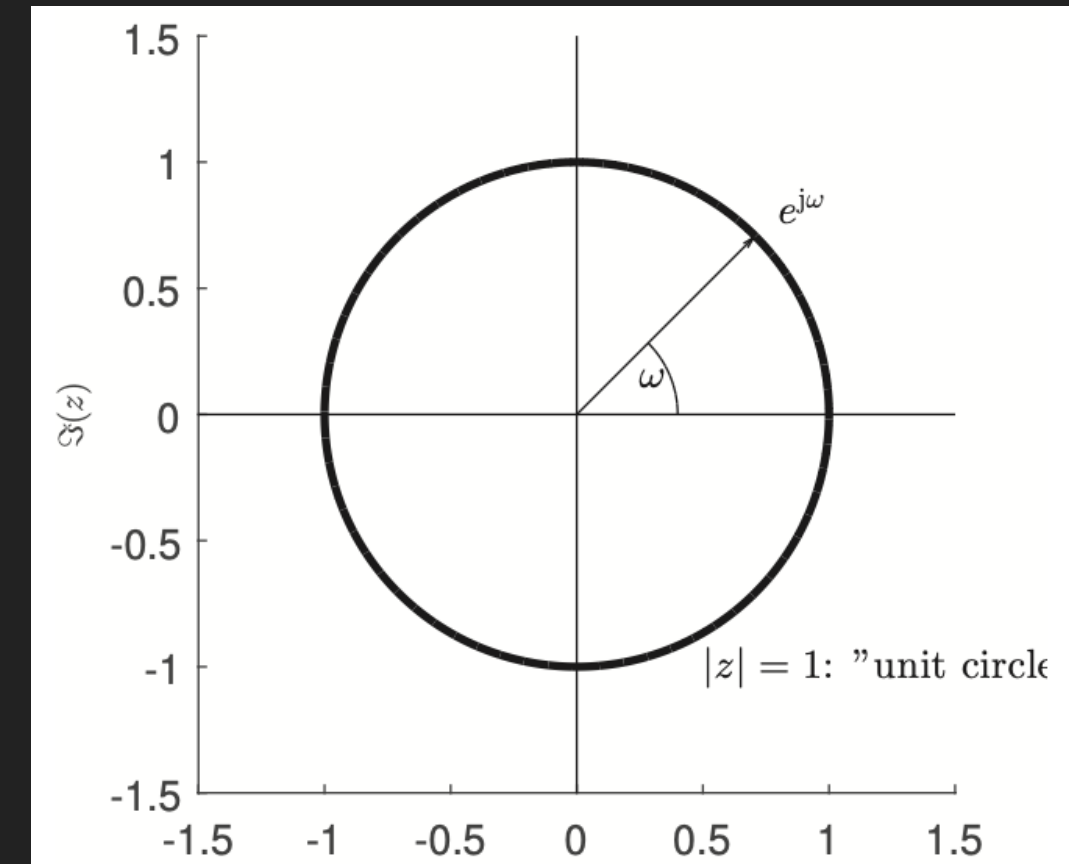
- » $X(z)$: complex function of a complex number
- » Compare Fourier transform $X(j\omega)$: complex function of real-valued ω

$$X(j\omega) = \sum_{i=-\infty}^{\infty} x(i)e^{-j\omega i} \Rightarrow X(j\omega) = X(z) \text{ at } z = e^{j\omega}$$

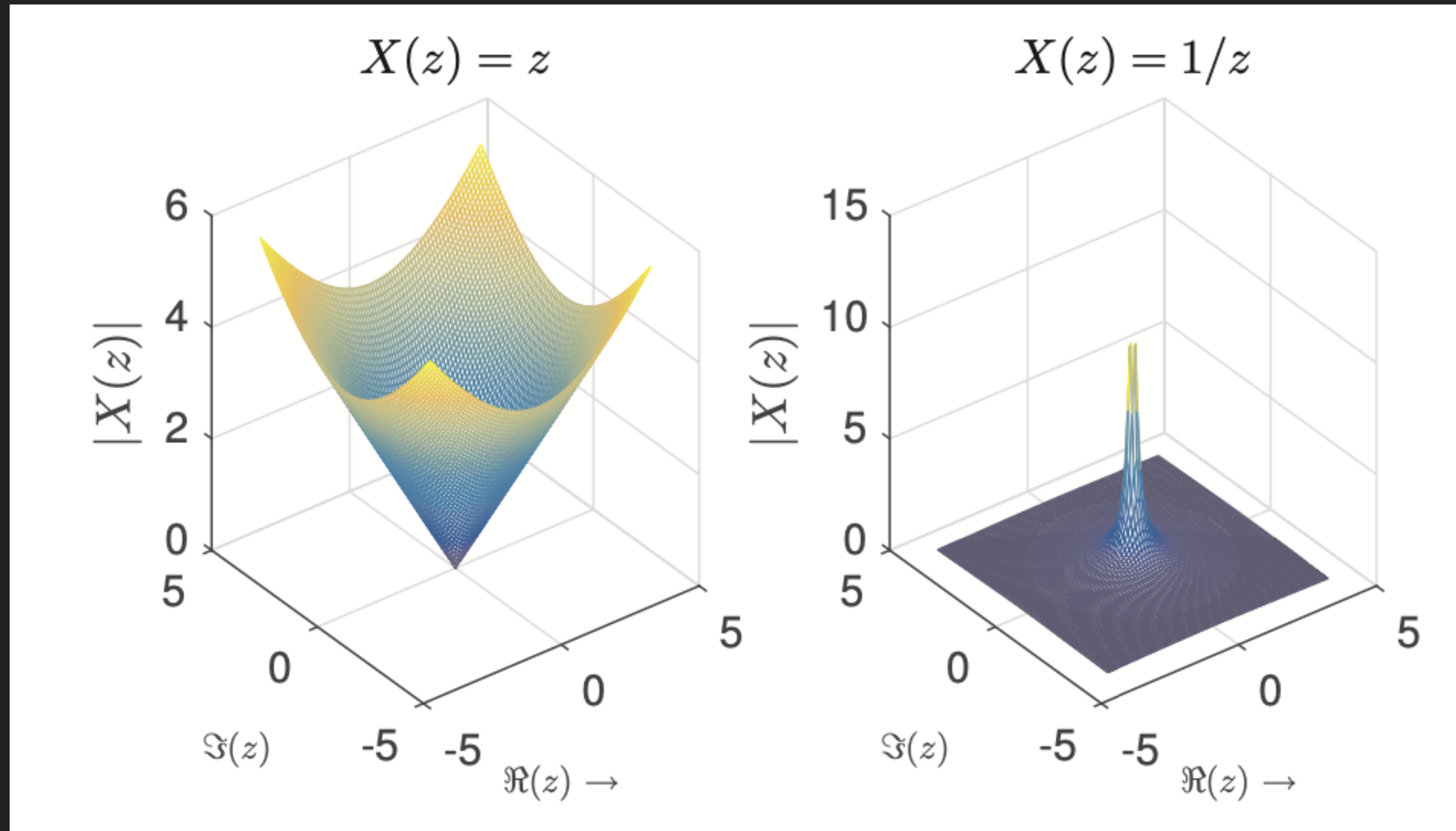
Z-plane



- » $X(z)$ defined on complex plane
- » $X(j\omega)$ defined on unit circle
- » Observation: $X(j\omega)$ is periodic with 2π



Trivial Examples



What is the magnitude for $X(z) = \frac{1}{(z-0.5)}$
Same as $\frac{1}{z}$ but shifted

System Description

Fourier transform and z-transform have largely similar properties, most importantly

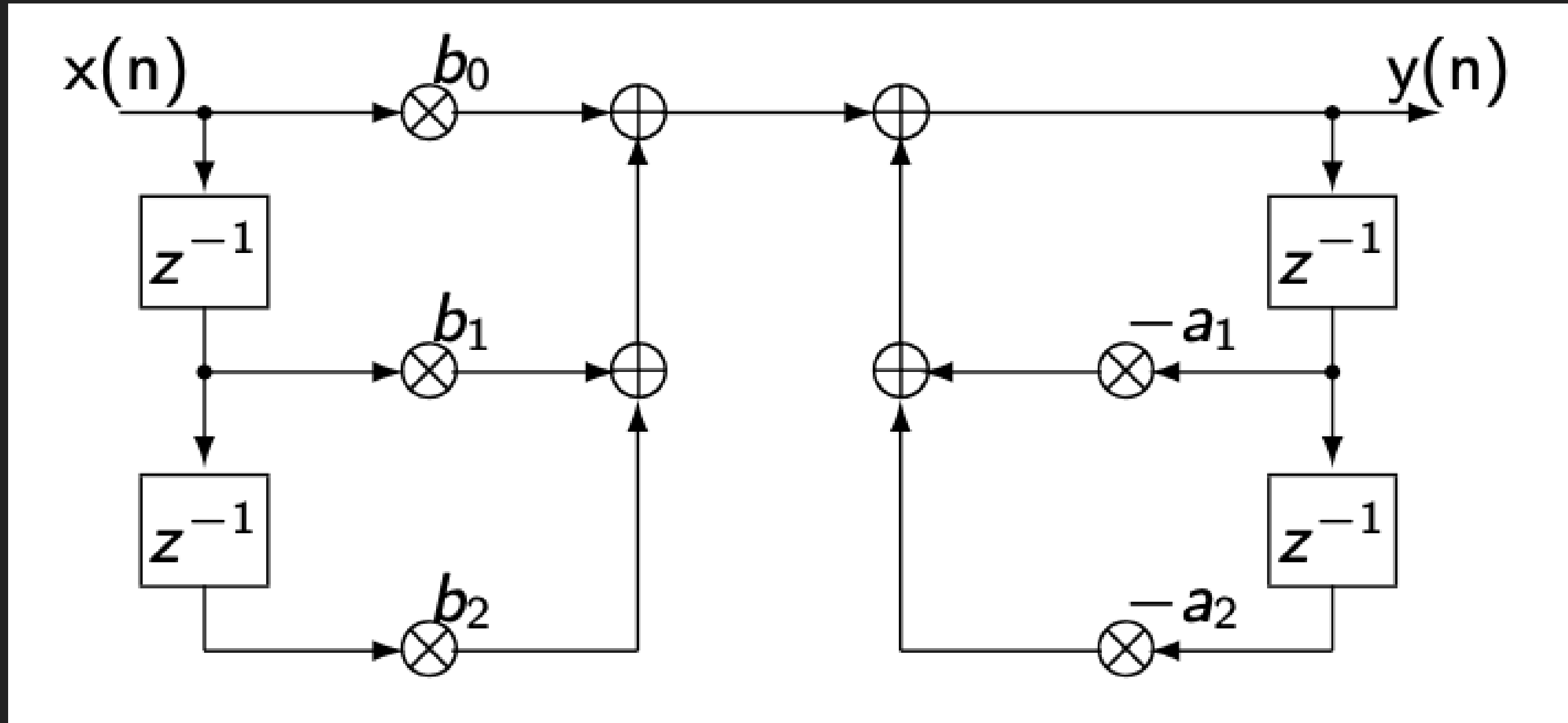
» Linearity

$$\begin{aligned}y(i) = c_1 x_1(i) + c_2 x_2(i) &\Rightarrow Y(j\omega) = c_1 X_1(j\omega) + c_2 X_2(j\omega) \\ &\Rightarrow Y(z) = c_1 X_1(z) + c_2 X_2(z)\end{aligned}$$

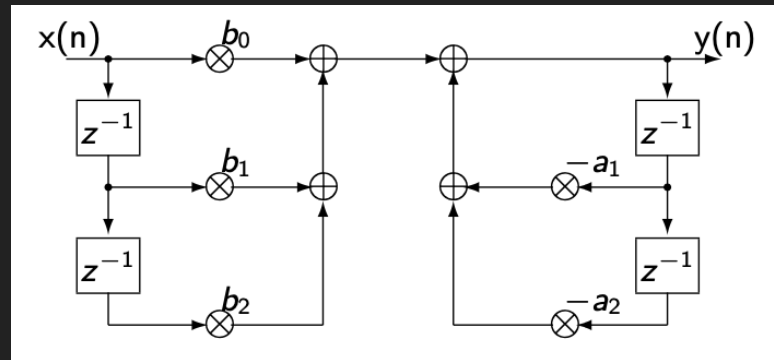
» Time Shift

$$\begin{aligned}y(i) = x(i - n) &\Rightarrow Y(j\omega) = e^{-j\omega n} X(j\omega) \\ &\Rightarrow Y(z) = z^{-n} X(z)\end{aligned}$$

Biquad: Difference Equation



Biquad: Difference Equation



$$y(i) = \sum_{j=0}^2 b_j x(i-j) - \sum_{k=1}^2 a_k y(i-k)$$

$$Y(z) = \sum_{j=0}^2 b_j X(z) z^{-j} - \sum_{k=1}^2 a_k Y(z) z^{-k}$$

$$Y(z) \left(1 + \sum_{j=1}^2 a_j z^{-j} \right) = X(z) \sum_{j=0}^2 b_j z^{-j}$$

Biquad: Transfer Function

$$H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{\sum_{j=0}^2 b_j z^{-j}}{1 + \sum_{j=1}^2 a_j z^{-j}}$$

$$= \frac{b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2}}{1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}}$$

$$= \frac{\text{numerator polynomial}}{\text{denominator polynomial}}$$

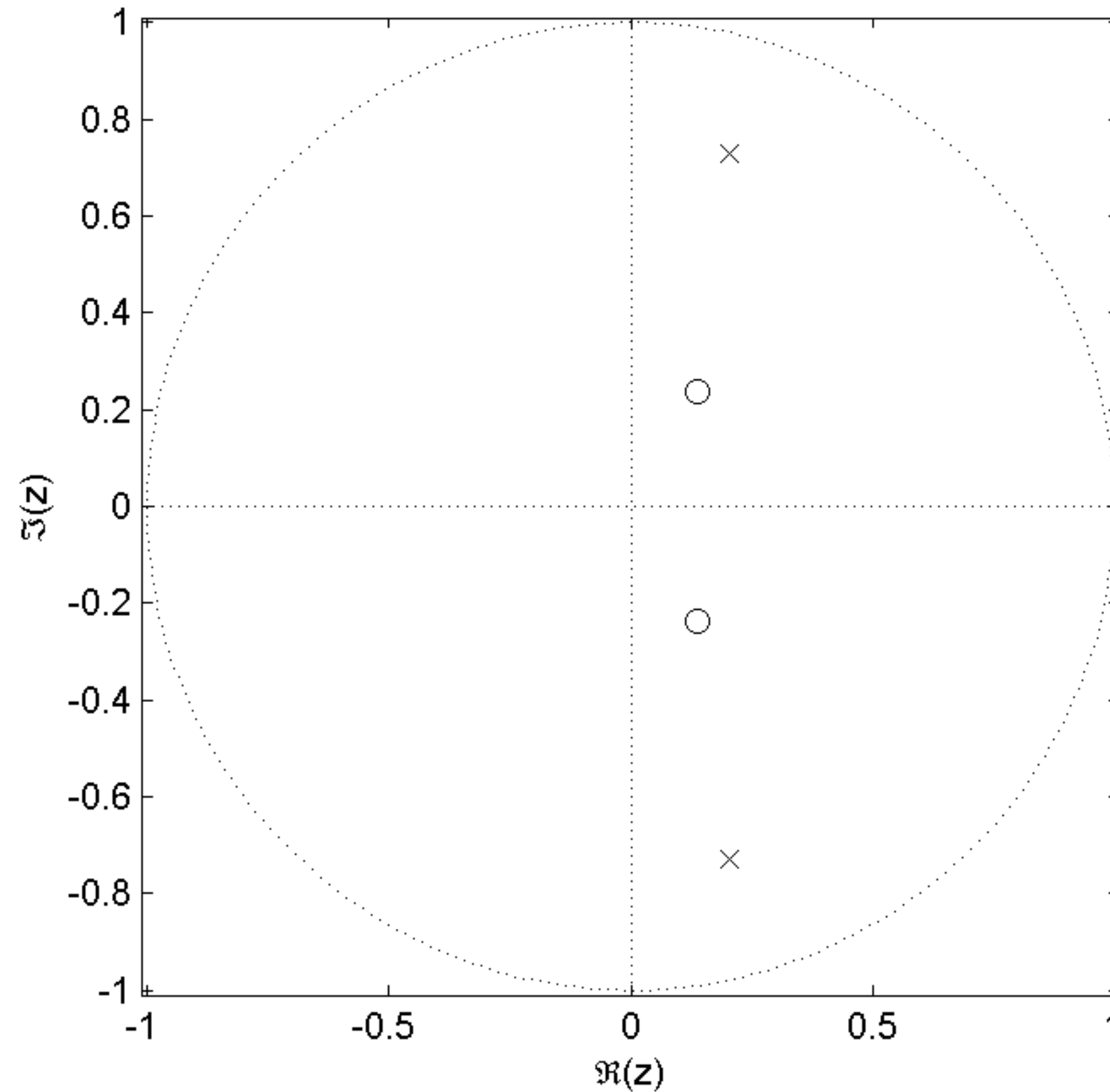
Biquad: Poles and Zeroes

- » Numerator $\rightarrow 0$: Zero
- » Denominator $\rightarrow 0$: Pole

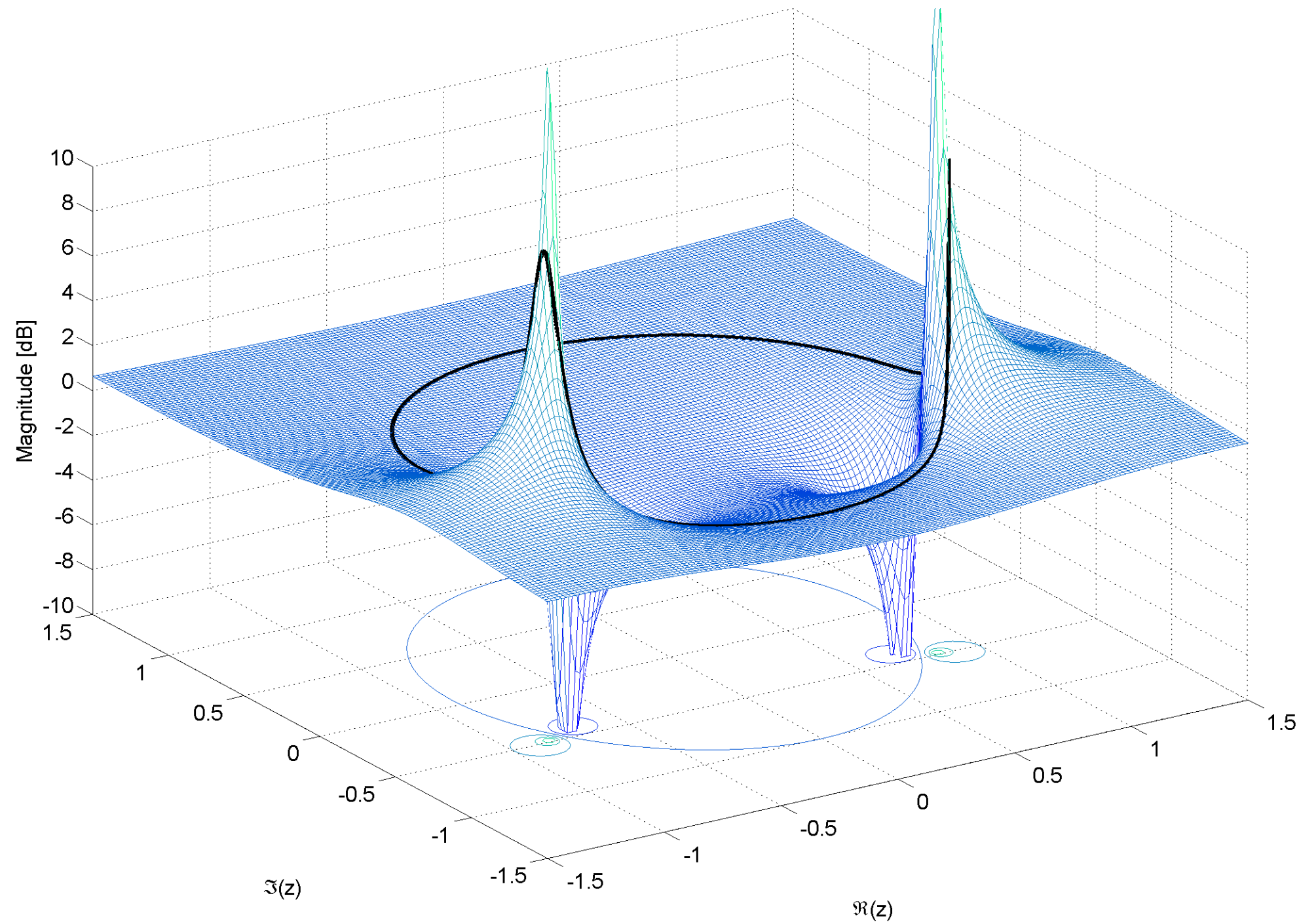
$$1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2} = 0$$

$$\implies z_{\infty 1,2} = \frac{a_1}{2} \pm \frac{1}{2} \sqrt{a_1^2 - 4a_2}$$

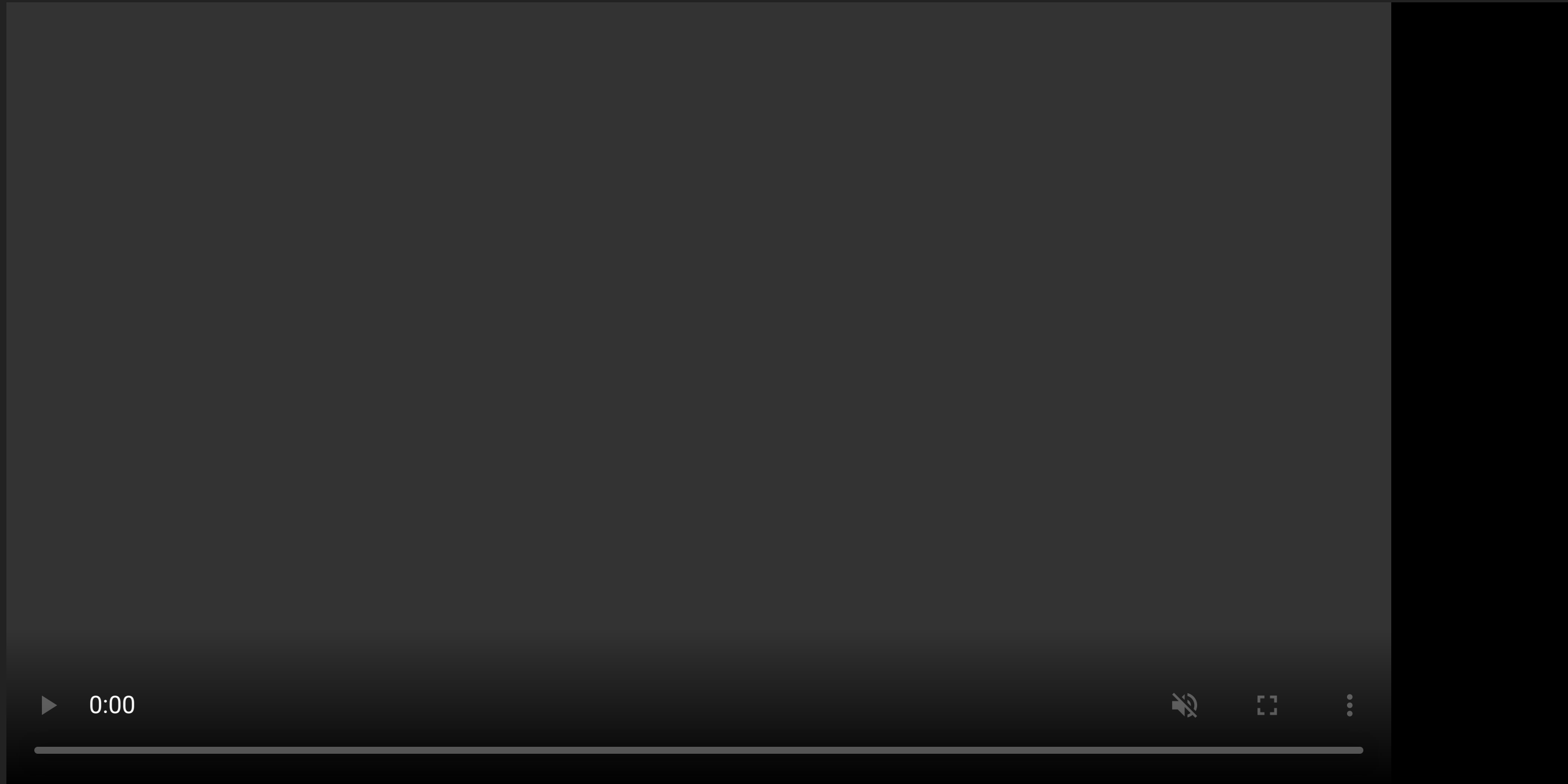
Biquad: Z-Plane Example



Biquad: Z-Plane Example



Animation



Filters: Z-Plane Characteristics

» **Stability:**

Poles within unit circle

» **Zero points and poles:**

Are either real or complex conjugate

» **Minimal phase systems:**

No zero points outside of unit circle

» **All pass system:**

Poles and zeroes symmetric wrt unit circle

» **Linear phase:**

Zero points within and outside unit circle symmetric wrt unit circle

Filters: Filter Design

- » **Impulse invariance:** sample impulse response
 - » If continuous system is band-limited, frequency response will be approximately equal (below $f_s/2$)
 - » Special case: No filter definition available → FIR coefficients
- » **Bi-Linear transform**
 - » Map filter from (analogue) Laplace-plane to (digital) z-plane

$$s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}$$
$$z = \frac{1 + s \frac{T_s}{2}}{1 - s \frac{T_s}{2}}$$

- » Introduces frequency warping (increasing towards Nyquist frequency)

Filters: Filter Design

»» **Frequency Transformation**

- »» Transform a (low-pass) prototype filter
- »» Usually via all-pass mapping filter

»» **Iterative approximation** of the magnitude response

»» **Intuitive methods**

- »» Manually move zeros and poles in z -plane
- »» Draw magnitude response in frequency domain

Effects of Word Length

- » Quantization of filter coefficients can lead to problems
- » Effects depend on filter type and structure
 - » Changes of transfer function
 - » Instability
 - » Quantization noise \rightarrow SNR

Summary

	FIR	IIR
IR Length	Finite	Infinite
Structure	Non-Recursive	Recursive
Phase Linearity	Possible	Impossible
Ratio Steepness/Workload	Low	High
Stability	Guaranteed	Possibly Unstable

- » Every LTI system is **completely described** either by
 - » Its complex transfer function,
 - » Its impulse response, or
 - » Its pole and zero position in the z-plane