Digital Signal Processing for Music

Part 3: Signal Descriptions

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Describing Random Signals

» Ergodic signals do not have a functional description

Other ways of describing these signals have to be found

» Ergodic signal characteristics are not time variant

We are looking for **time-independent descriptions**

» These descriptions might also be convenient to use for some deterministic signals

Probability and Occurence

N: number of overall observations

$N(x_i)$: number of occurences of symbol x_i

» Relative number of occurrences: $\qquad \hat{p}_i = \dfrac{N(x_i)}{N}$

» Probability: $\qquad\qquad\qquad p_i = \lim_{N \to \infty} \dfrac{N(x_i)}{N}$

Properties

$$\sum_i p_i = 1$$

$$0 \leq p_i \leq 1$$

Probability Distribution Example

Value	⚀	⚁	⚂	⚃	⚄	⚅
p(x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Probability Distribution for the roll of two dice

2	3	4	5	6	7	8	9	10	11	12
$\dfrac{1}{36}$	$\dfrac{1}{18}$	$\dfrac{1}{12}$	$\dfrac{1}{9}$	$\dfrac{5}{36}$	$\dfrac{1}{6}$	$\dfrac{5}{36}$	$\dfrac{1}{9}$	$\dfrac{1}{12}$	$\dfrac{1}{18}$	$\dfrac{1}{36}$
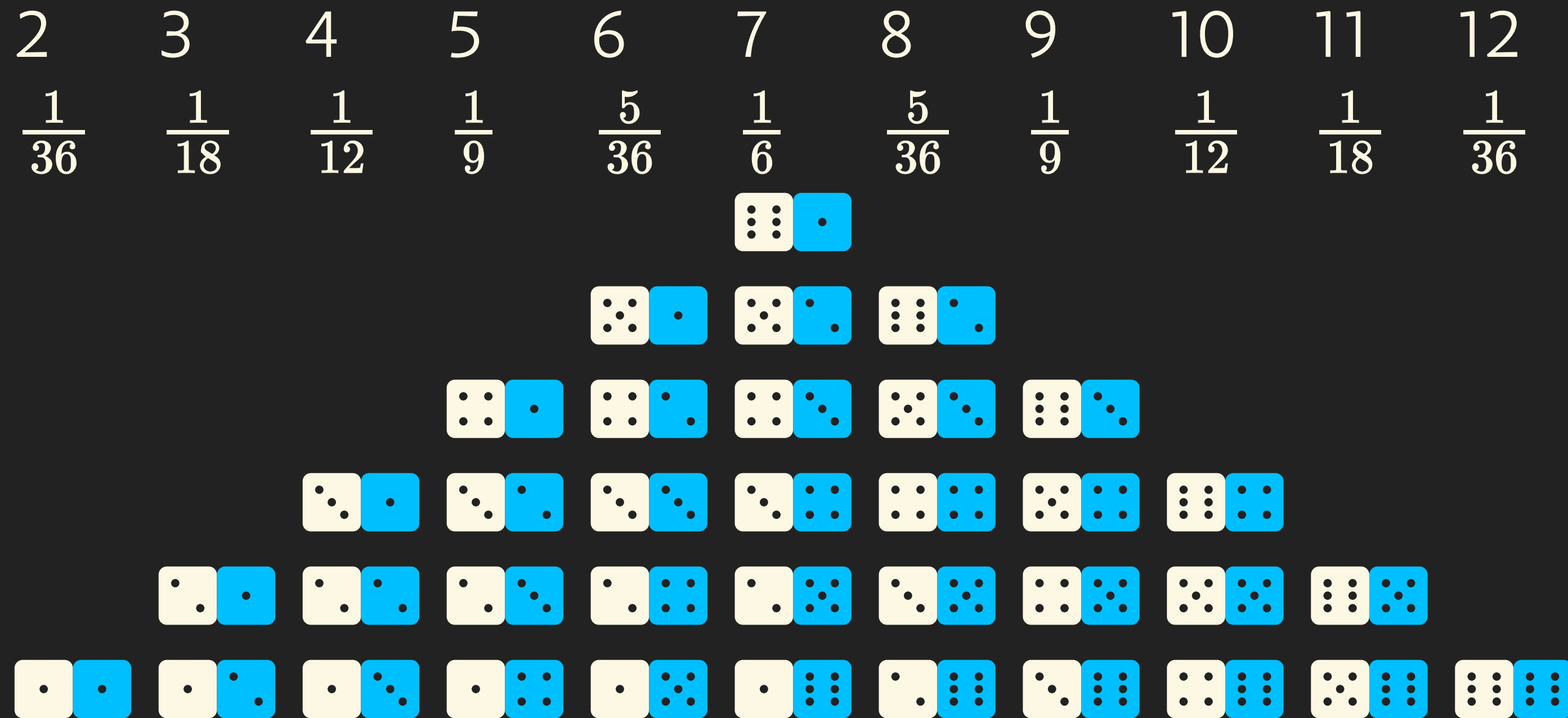
Continuous Probability Density Distribution

$i \rightarrow$ continuous 👉 **PDF**

$$\int\limits_{-\infty}^{\infty} p_X(x)dx = 1$$

$$0 \leq p_X(x)$$

Probability of x being a value smaller than or equal to x_c

$$\int\limits_{-\infty}^{x_c} p_X(x)dx$$

Example PDF: Gaussian

$$p_X(x) = \frac{1}{\sigma_X \sqrt{2\pi}} e^{-\left(\frac{x - \mu_X}{2\sigma_X}\right)^2}$$
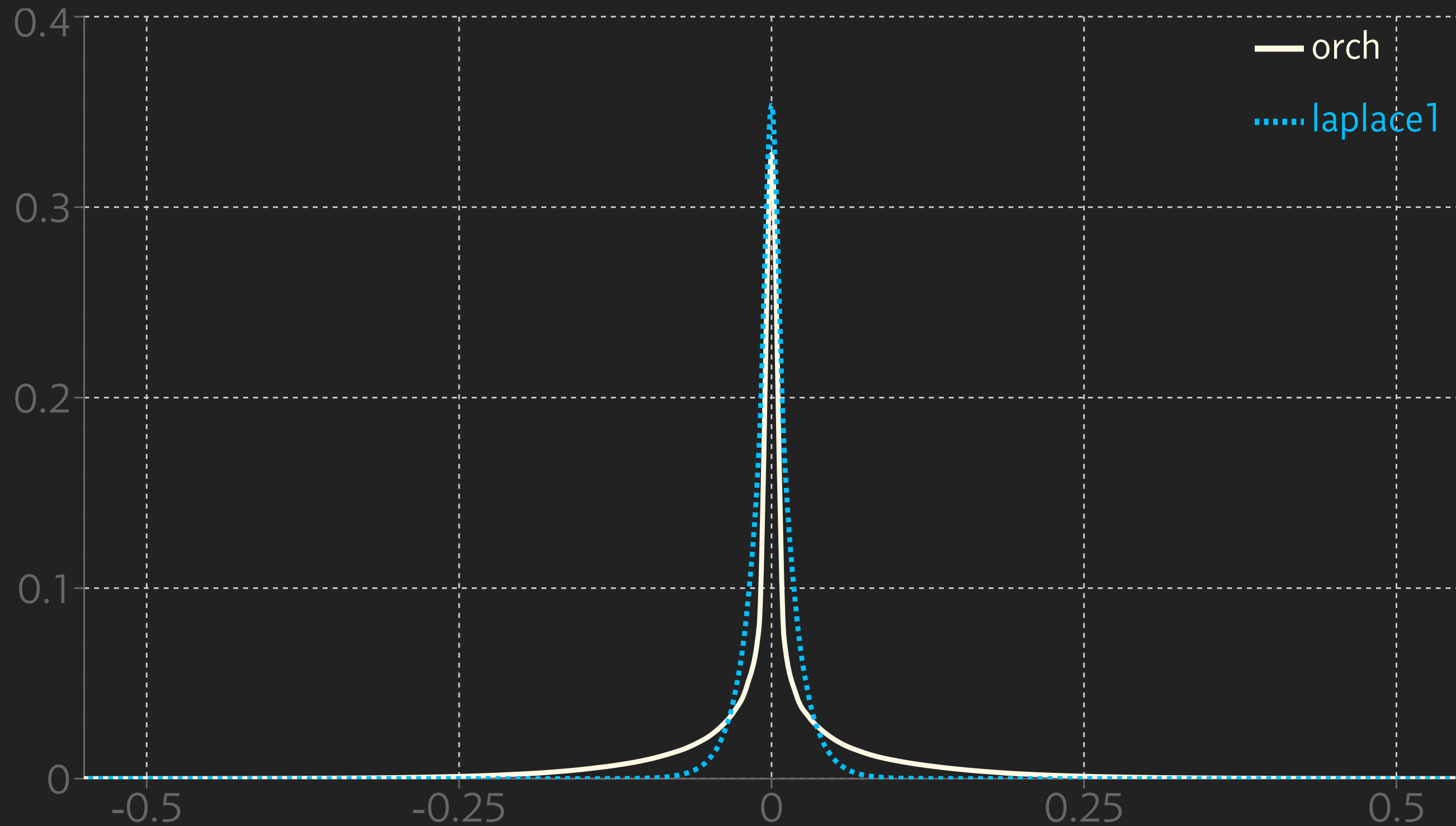
Example PDF: Exponential

$$p_X(x) = \begin{cases} \frac{1}{\sigma_X} e^{-\frac{x}{\sigma_X}} & \text{if } x > 0 \\ 0 & \text{else} \end{cases}$$
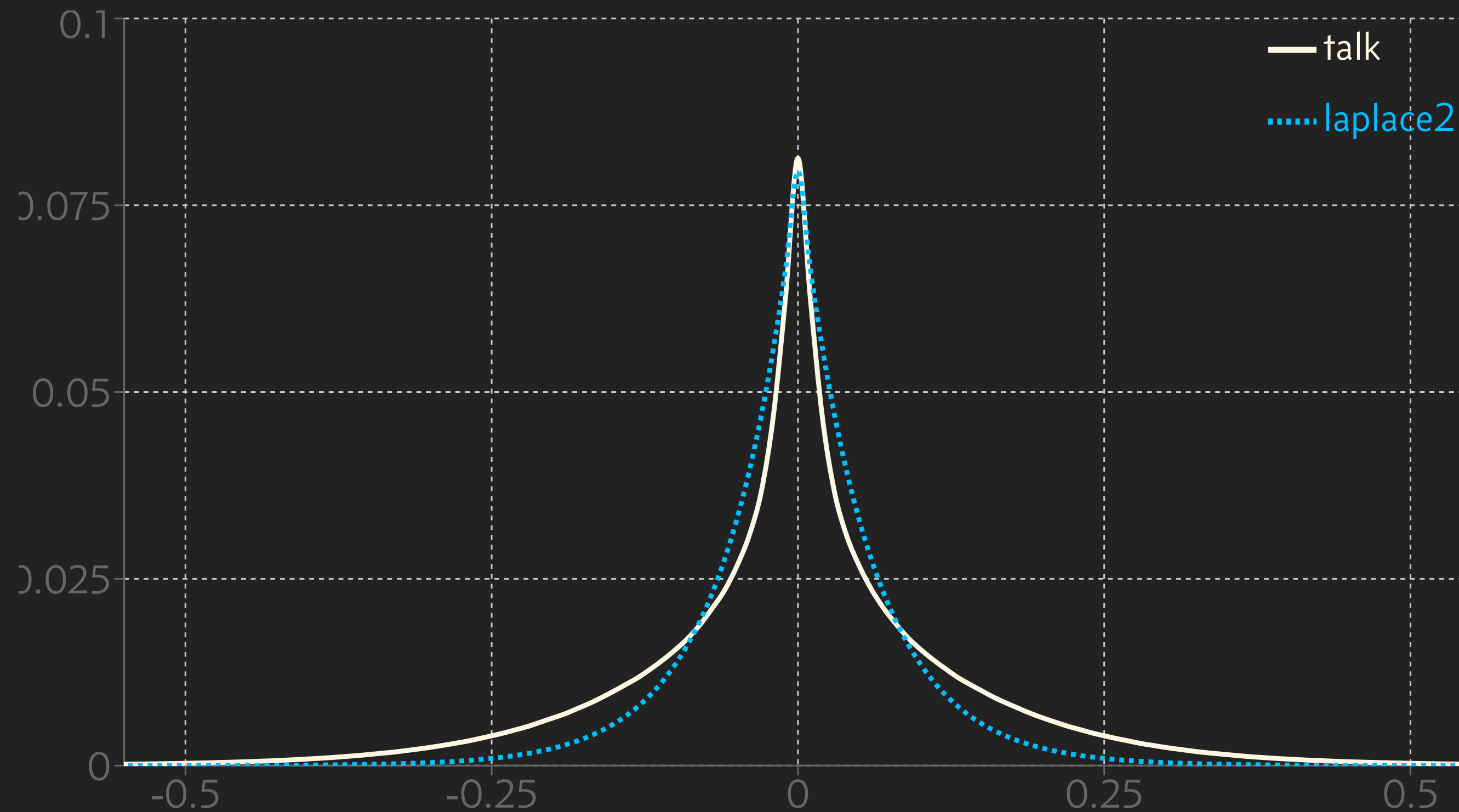
Example PDF: Laplace (2-sided exp)

$$p_X(x) = \frac{1}{\sigma_X\sqrt{2}} e^{-\sqrt{2}\frac{|x-\mu_X|}{\sigma_X}}$$
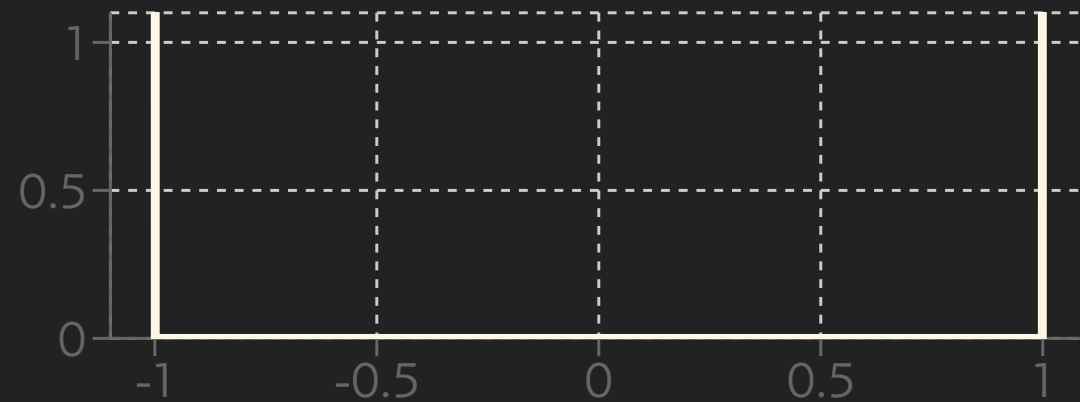
Measured RDF - Orchestra

Measured RDF – Podcast Conversation

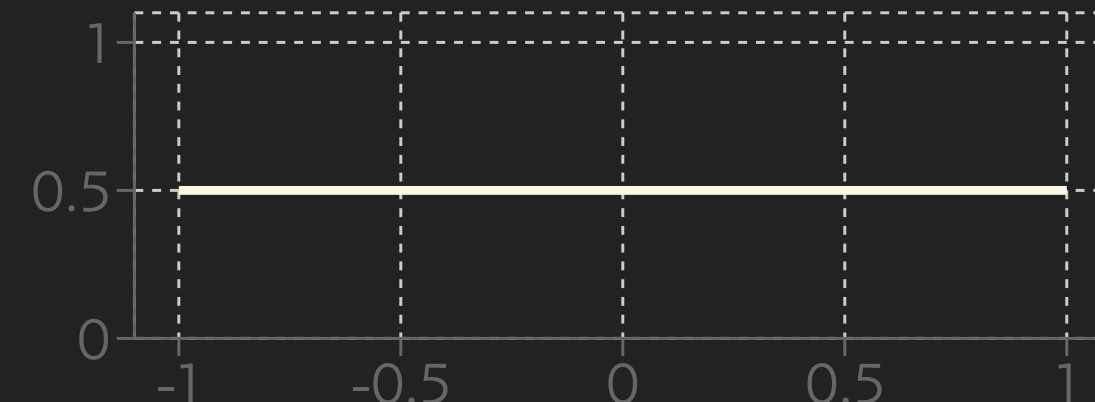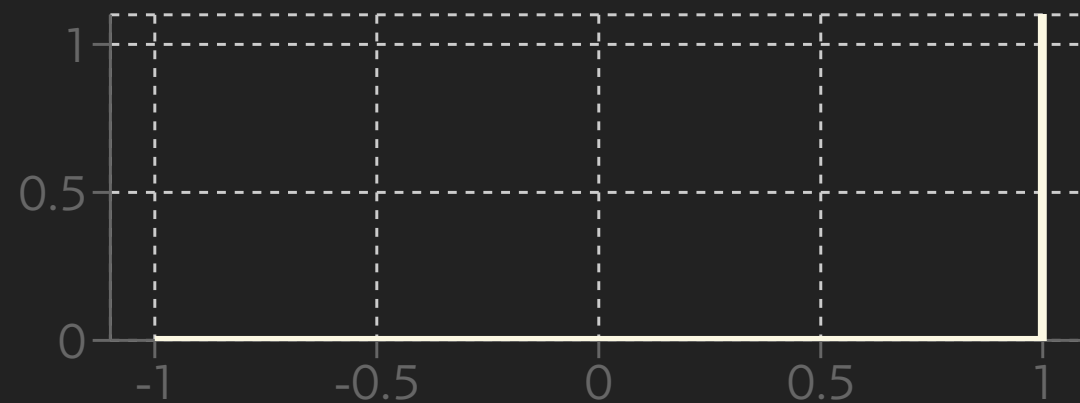# Distributions of Generated Signals

Square Wave PDF

Saw Wave PDF

DC PDF

Uniform Noise PDF
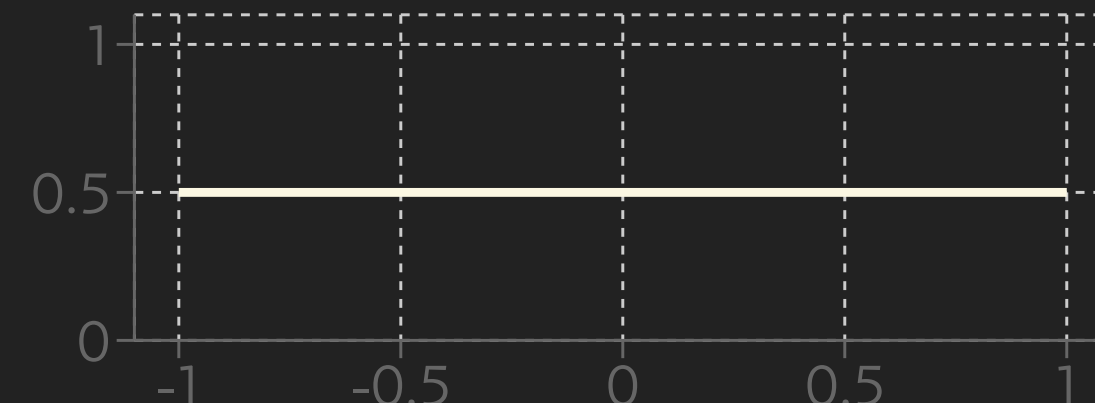
Gaussian Noise PDF
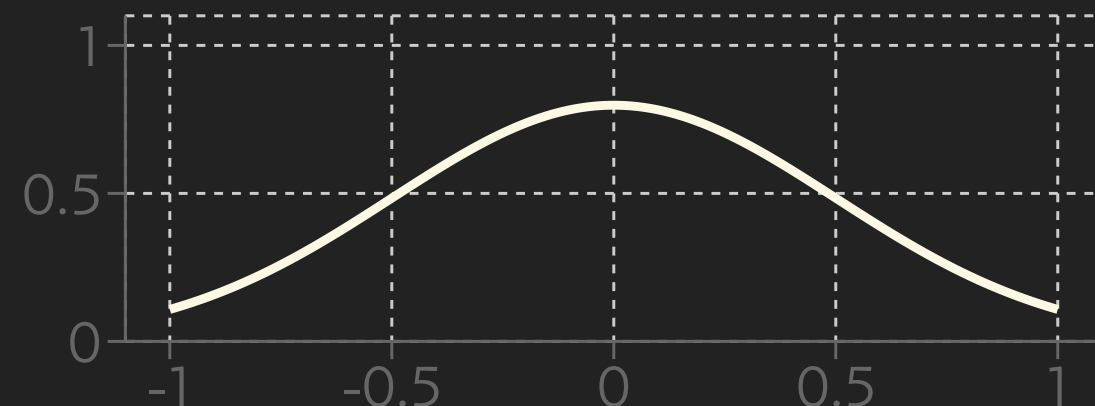
Sine Wave PDF
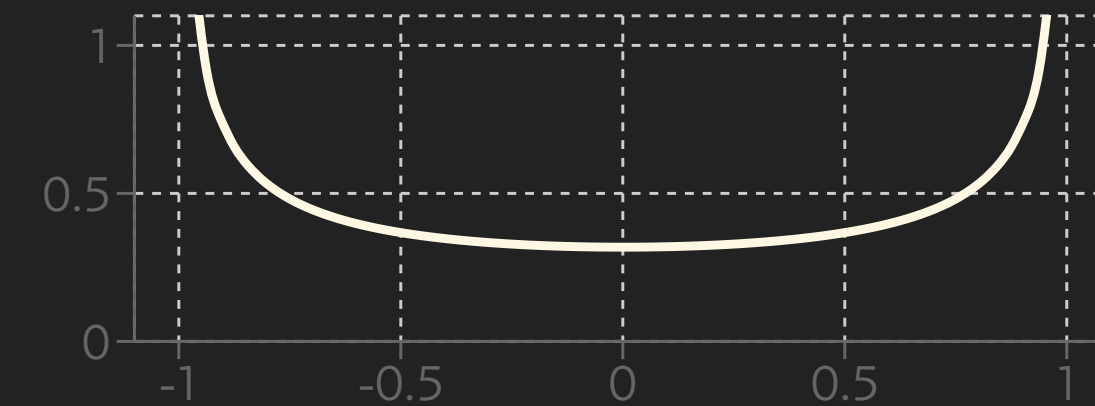
Expected Value

Example: Average grade, five students, grades: 1, 2, 1, 3, 5

$$\hat{\mu}_X = \frac{1 + 2 + 1 + 3 + 5}{5} = 2.4$$

Grade	# of Occurrences	Relative Frequency
1	2	2/5
2	1	1/5
3	1	1/5
4	0	0/5
5	1	1/5

Expected Value

$$\mu = \frac{2}{5} \cdot 1 + \frac{1}{5} \cdot 2 + \frac{1}{5} \cdot 3 + \frac{0}{5} \cdot 4 + \frac{1}{5} \cdot 5 = 2.4$$

$$\mu_X = \sum_{\forall x} p(x) \cdot x$$

$$\mu_X = \mathcal{E}\{X\} = \int_{-\infty}^{+\infty} x p_X(x) dx$$

Expected Value

Generalization:

$$\mathcal{E}\{f(X)\} = \sum_i f(x)p(x)$$

Examples

» Mean: $f(x) = x$

» Quad. Mean: $f(x) = x^2$

>> kth moment:

$$\mathcal{E}\{X^k\} = \int\limits_{-\infty}^{+\infty} x^k p_X(x) dx$$

>> kth central moment:

$$\mathcal{E}\{(X - \mu_X)^k\} = \int\limits_{-\infty}^{+\infty} (x - \mu_X)^k p_X(x) dx$$

>> Example: 2nd order central moment: **Variance**

$$\sigma_X^2 = \mathcal{E}\{(X - \mu_X)^2\} = \int\limits_{-\infty}^{+\infty} (x - \mu_X)^2 p_X(x) dx$$

Calculation of Moments

(Central) moments (mean, power, variance, etc.) can be computed from:

» The signal

» The signal's PDF

Central Moments Summary

Order	Name	Time (Continuous)	PDF (Continuous)
1	μ_X	$\frac{1}{T} \int\limits_{-T/2}^{T/2} x(t)dt$	$\int\limits_{-\infty}^{\infty} x p_X(x)dx$
2	σ_X^2	$\frac{1}{T} \int\limits_{-T/2}^{T/2} (x(t) - \mu_X)^2 dt$	$\int\limits_{-\infty}^{\infty} x p_X(x)dx$

Order	Name	Time (Discrete)	PDF (Discrete)
1	μ_X	$\frac{1}{N} \sum\limits_{i=0}^{N} x(i)$	$\sum\limits_{\forall x} x p(x)$
2	σ_X^2	$\frac{1}{N} \sum\limits_{i=0}^{N} (x(i) - \mu_X)^2$	$\sum\limits_{\forall x} (x - \mu_X)^2 p(x)$

Standard deviation $\sigma_X = \sqrt{\sigma_X^2}$

Summary

» PDF can tell us many important details about a signal

» Statistical measures can be used to describe signal properties

» Statistical measures can be derived from both the time domain signal and it's PDF

» Often-used measures are:

Mean and Median

Variance and Standard Deviation

Higher Order moments less frequently (Skewness, Kurtosis)

Other PDF descriptions possible (quartile-distances, etc)