Digital Signal Processing for Music

Part 6: Fourier Series

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Overview

1. **Fourier Series**

   Periodic signals as sum of sinusoidals

2. **Fourier Transform**

   Frequency content of any signal

   - Fourier series to transform
   - Properties
   - Windowed Fourier transform

Fourier Series: Introduction

- Periodic signals are **superposition of sinusoidals**
- **Properties**
  - Amplitude
  - Frequency as integer multiple of fundamental of f_0
  - Phase

$$x(t) = \sum_{k=0}^{\infty} a_k \sin(k\omega_0 t + \Phi_k)$$

- **Observations**
  - Time domain is continuous (t)
  - Frequency domain is discrete ($\sum$)

Complex Representation

$$x(t) = \sum_{k=0}^{\infty} a_k \sin(k\omega_0 t + \Phi_k)$$

Trigonometric identity $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$

$$e^{\mathrm{j}\omega t} = \cos(\omega t) + \mathrm{j}\sin(\omega t)$$

$$\mathrm{j} = \sqrt{-1}$$

Phasor representation in complex plane

Real to complex

$$\cos(\omega t) = \frac{1}{2}\left(e^{\mathrm{j}\omega t} + e^{-\mathrm{j}\omega t}\right)$$

$$\sin(\omega t) = \frac{1}{2\mathrm{j}}\left(e^{\mathrm{j}\omega t} - e^{-\mathrm{j}\omega t}\right)$$

$$\begin{aligned}
x(t) &= \sum_{k=0}^{\infty} A_k \cos(k\omega t) + B_k \sin(k\omega t) \\
&= \sum_{k=0}^{\infty} \frac{A_k}{2}\left(e^{\mathrm{j}\omega k t} + e^{-\mathrm{j}\omega k t}\right) - \mathrm{j}\frac{B_k}{2}\left(e^{\mathrm{j}\omega k t} - e^{-\mathrm{j}\omega k t}\right) \\
&= \sum_{k=0}^{\infty} \frac{1}{2}(A_k - \mathrm{j}B_k)e^{\mathrm{j}\omega k t} + \frac{1}{2}(A_k + \mathrm{j}B_k)e^{-\mathrm{j}\omega k t} \\
&= \sum_{k=0}^{\infty} \underbrace{\frac{1}{2}(A_k - \mathrm{j}B_k)}_{c_k}e^{\mathrm{j}\omega k t} + \frac{1}{2}(A_k + \mathrm{j}B_k)e^{-\mathrm{j}\omega k t}
\end{aligned}$$

With $c_{-k} := c_k^* \Rightarrow \quad x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\mathrm{j}\omega_0 k t}$

Coefficients

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\mathrm{j}\omega_0 k t}$$

- Multiply both sides with $e^{-\mathrm{j}\omega_0 n t}$

- Multiply both sides with $e^{-\mathrm{j}\omega_0 nt}$\$: \$$x(t)\cdot e^{-\mathrm{j}\omega_0 nt} = \sum_{k=-\infty}^{\infty} c_k e^{\mathrm{j}\omega_0 (k-n)t}$
- Integrate both sides $\int_0^{T_0} x(t)\cdot e^{-\mathrm{j}\omega_0 nt} dt = \int_0^{T_0} \sum_{k=-\infty}^{\infty} c_k e^{\mathrm{j}\omega_0 (k-n)t} dt$
- Flip sum and integral $\int_0^{T_0} x(t)\cdot e^{-\mathrm{j}\omega_0 nt} dt = \sum_{k=-\infty}^{\infty} c_k \int_0^{T_0} e^{\mathrm{j}\omega_0 (k-n)t} dt$

$$\int_0^{T_0} e^{\mathrm{j}\omega_0 (k-n)t} dt = 0 \qquad k \neq n$$

$$\int_0^{T_0} e^{\mathrm{j}\omega_0 (k-n)t} dt = T_0 \qquad k = n$$

$$\Rightarrow \int_0^{T_0} x(t)\cdot e^{-\mathrm{j}\omega_0 nt} dt = \quad c_n T_0$$

Limited number of coefficients

Reconstruction of periodic signals with a limited number of sinusoidals

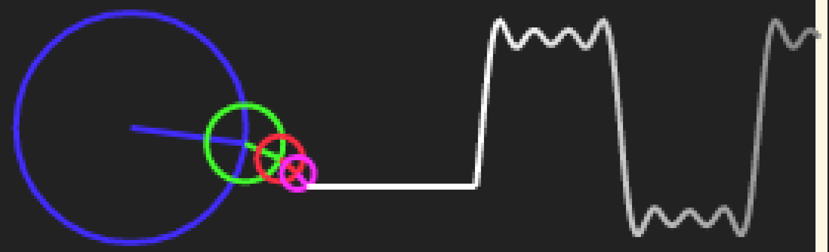
$$\hat{x}(t) = \sum_{k=-\mathcal{K}}^{\mathcal{K}} c_k e^{\mathrm{j}\omega_0 k t}$$

$$\frac{4sin\theta}{\pi}$$

$$\frac{4sin3\theta}{3\pi}$$

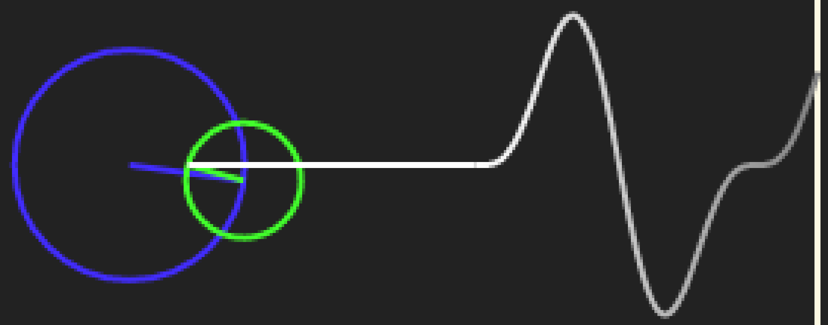
$$\frac{4sin5\theta}{5\pi}$$

$$\frac{4sin7\theta}{7\pi}$$

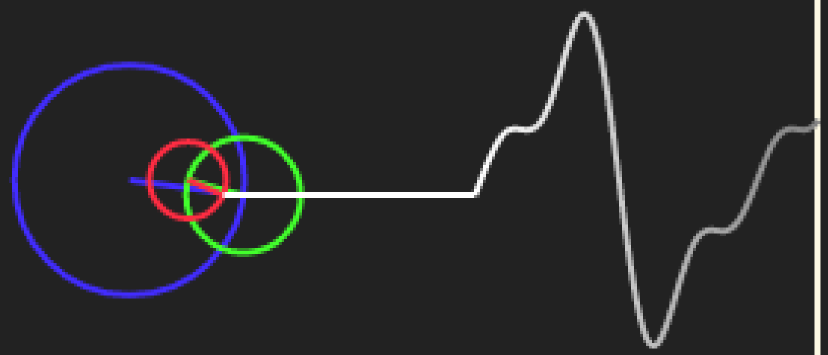


$$\frac{2sin\theta}{\pi}$$

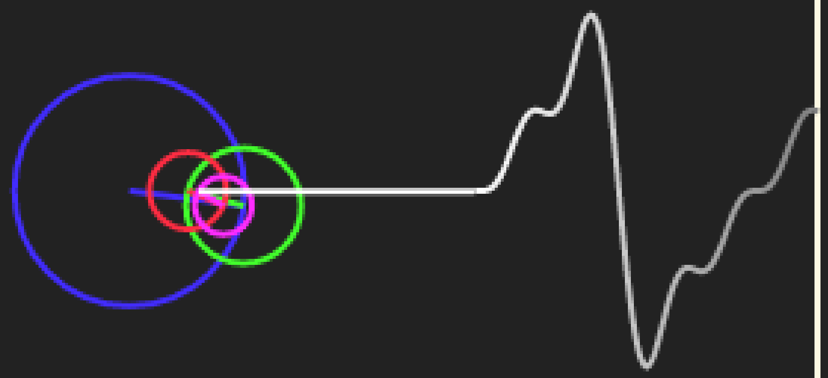
$$\frac{2sin2\theta}{-2\pi}$$



$$\frac{2sin3\theta}{3\pi}$$



$$\frac{2sin4\theta}{-4\pi}$$



Num Harmonics

1

Summary

Any periodic signal can be represented in **Fourier Series**

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{j}\omega_0 k t}$$

Key Components

Summary

- Complex coefficients are a useful hack to represent both phase and amplitude in one value
- To derive the coefficients from a signal we need:
  - Fundamental frequency
  - Functional description
- "Frequency domain" of Fourier Series is discrete (integer multiples)
- "Time domain" can be continuous or discrete (discrete may be a pain to integrate, though)