Chapter Three
Data Representation and Computer Arithmetic

- Number Systems and Conversion
- Units of Data Representation
- Coding Methods
- Binary Arithmetic
- Complements
- Fixed and Floating points representation
- (Boolean Algebra) and Logic Circuits

Number systems and conversion

- Number Systems
    - Decimal
    - Binary
    - Octal
    - Hexadecimal
- Conversion

Decimal systems

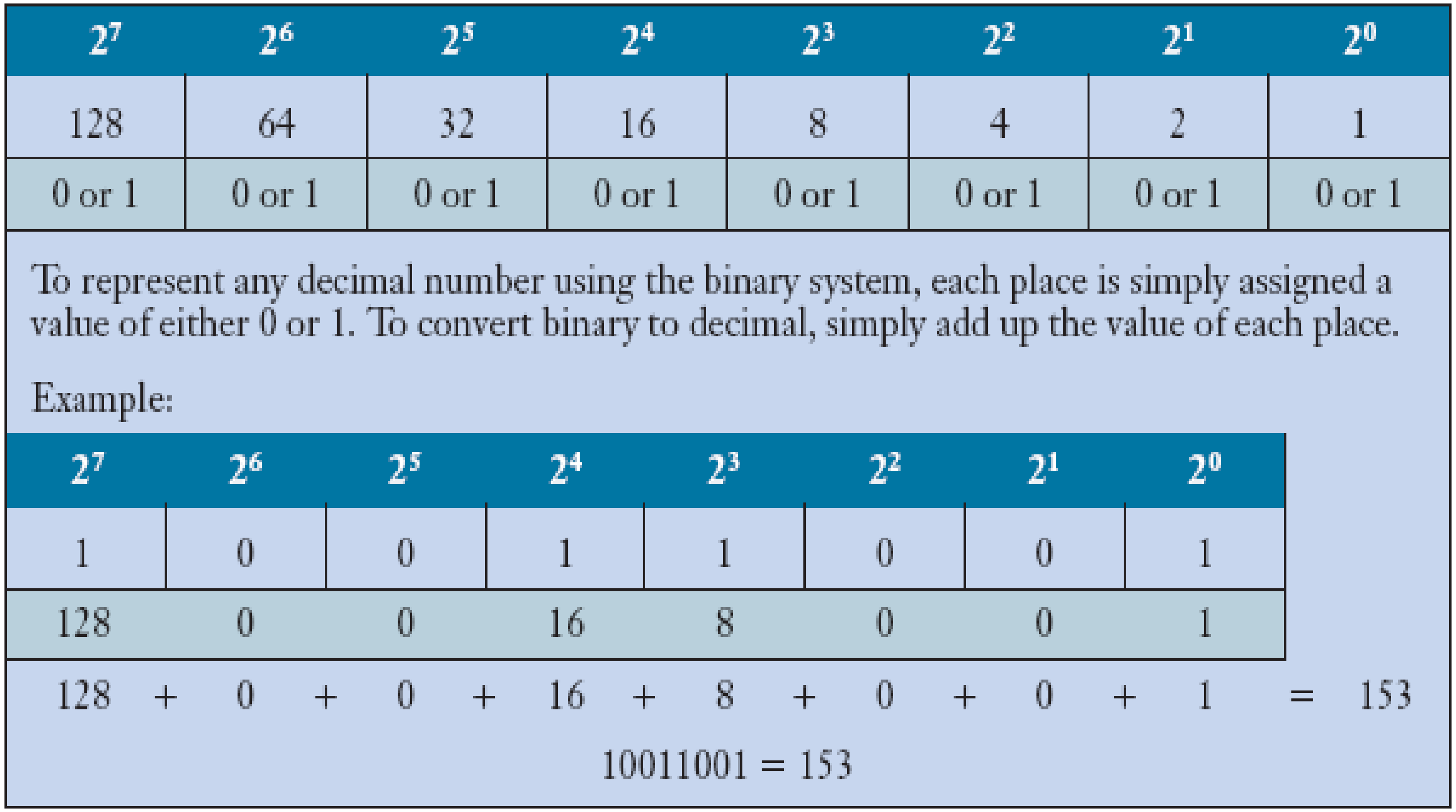
- **The decimal system**
  - ➢ Base 10 with ten distinct digits (0, 1, 2, …, 9)
  - ➢ Any number greater than 9 is represented by a combination of these digits
  - ➢ The weight of a digit based on power of 10

Example:

- ➢ The number 81924 is actually the sum of: $(8X10^4)+(1X10^3)+(9X10^2)+(2X10^1)+(4X10^0)$

Binary systems

Computers use the binary system to store and compute numbers.



2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
128	64	32	16	8	4	2	1
0 or 1	0 or 1	0 or 1	0 or 1	0 or 1	0 or 1	0 or 1	0 or 1

To represent any decimal number using the binary system, each place is simply assigned a value of either 0 or 1. To convert binary to decimal, simply add up the value of each place.

Example:

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	0	0	1	1	0	0	1
128	0	0	16	8	0	0	1

128 + 0 + 0 + 16 + 8 + 0 + 0 + 1 = 153

10011001 = 153

Binary systems

- **The binary system (0 & 1)**
  - The two digits representation is called a binary system
  - Two electrical states – **on (1) & off (0)**
  - The position weights are based on the **power of 2**
  - The various combination of the two digits representation gives us the final value

Examples :

i) 1011011 in binary = 91 in decimal

ii) 1101.01 in binary = 13.25 in decimal

Binary Fractions

Binary fractions can also be represented:

Position Value:	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	etc.
Fractions:	**1/2**	**1/4**	**1/8**	**1/16**	**1/32**	
Decimal:	**.5**	**.25**	**.125**	**.0625**	**.03125**	

Binary into Decimal conversion

$5^{th}4^{th}3^{rd}2^{nd}1^{st}0^{th}$

$$1\,0\,1\,1\,1\,1_2 = (1X2^5)+(0X2^4)+(1X2^3)+(1X2^2)+(1X2^1)+(1X2^0)$$

$$= 32+8+4+2+1$$

$$= 47_{10}$$

$1011001_2 = 89_{10}$

Exercise :

Convert the following binary numbers into their decimal equivalent

- $1110100_2 = (?)_{10}$
- $101101.1101_2 = (?)_{10}$

Conversion of Decimal to Binary

Divide by 2 (remainder division) till the dividend is zero and read remainders in reverse order. The right column shows result of integer division

Read answer in this direction, Write answer left to right

mod 2	637/2
1	318
0	159
1	79
1	39
1	19
1	9
1	4
0	2
0	1
1	0

$\mathbf{637_{10} = 1001111101_2}$

- Convert 789_{10} to base 2

To Binary Fractions - Conversions

Multiply by 2 till enough digits are obtained, say 8, or a product is zero.

$\overline{.637_{10}}$

	1.274
Read answer	0.548
in this direction	1.096
write it left	0.192
to right	0.384
	0.768
	1.536
	1.072

Ans= 0.10100011_2

- Convert 0.325_{10} to base 2

Octal system

- Octal system
  - Base 8 systems (0, 1, 2, ..., 7)
  - Used to give shorthand ways to deal with the long strings of 1 & 0 created in binary
  - Numbers 0 .. 7 can be represented by three binary digits

❖ Examples :-

i) $(3137)_8 = 1631_{10}$

ii) 134 in octal = 1011100 in binary

iii) $(6)_8 = (110)_2$

iv) 432.2 in octal = 282.25 in decimal

v) 123.45 in octal = 001010011.100101 in binary

Hexadecimal systems

- The Hexadecimal system
  - Base 16 system
  - 0 .. 9 and letters A .. F for sixteen place holders needed
  - A = 10, B = 11, …, F = 15
  - Used in programming as a short cut to the binary number systems
  - Can be represented by four binary digits

❖ Examples :-

i) $(1D7F)_{16} = 7551_{\mathbf{10}}$

ii) 6B2 in hexadecimal = 011010110010 in binary

iii) 101000010111 in binary = A17 in hexadecimal

Exercise – Convert ...

Decimal	Binary	Octal	Hexa-decimal
29.8			
	101.1101		
		3.07	
			DF.E8

Exercise – Convert …

Answer

Decimal	Binary	Octal	Hexa-decimal
29.8	11101.110011…	35.63…	1D.CC…
5.8125	101.1101	5.64	5.D
3.109375	11.000111	3.07	3.1C
12.5078125	1100.10000010	14.404	C.82

Bits

- How many bits does a computer use to store an integer?
  - Some models = 32 bits
  - Other models = 64 bits
- What if we try to compute or store a larger integer?
  - If we try to compute a value larger than the computer can store, we get an <u>arithmetic overflow</u> error.

Representing Unsigned Integers

- How does a 16-bit computer represent the value 14?

0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0

- What is the largest 16-bit integer?

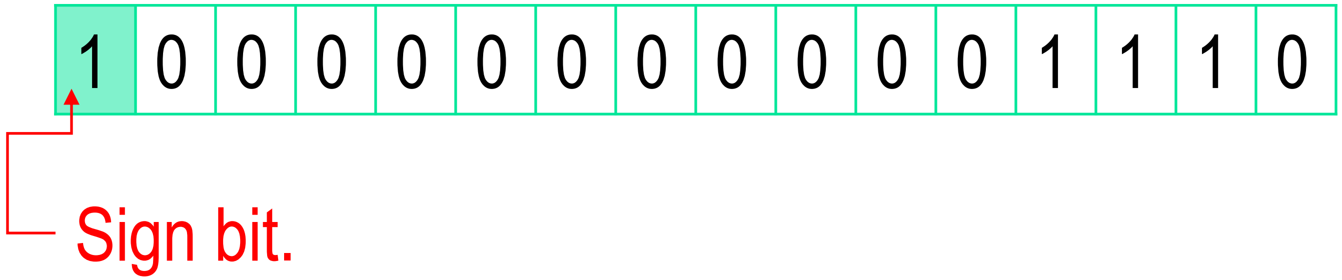
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

$= 1\text{x}2^{15} + 1\text{x}2^{14} + \ldots + 1\text{x}2^{1} + 1\text{x}2^{0} = 65{,}535$

If we just add 1 to this number, for example, overflow will occur. We will get a wrong result.

Representing Signed Integers

- How does a 16 bit computer represent the value -14?



- What is the largest 16-bit signed integer?

0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

$= 1x2^{14} + 1x2^{13} + \ldots + 1x2^{1} + 1x2^{0} = 32{,}767$

- Problem ➔ the value 0 is represented twice!
  - Most computers use a different representation, called <u>two's complement</u>.

Signed-magnitude representation

- Also called, “sign-and-magnitude representation”
- A number consists of a magnitude and a symbol representing the sign
- Usually 0 means positive, 1 negative
  - Sign bit
  - The number is represented with 1 sign bit to the left, followed by magnitude bits

Machine arithmetic with signed-magnitude representation

- Takes several steps to add a pair of numbers
  - Examine signs of the addends
  - If same, add magnitudes and give the result the same sign as the operands
  - If different, must…
    - Compare magnitude of the two operands
    - Subtract smaller number from larger
    - Give the result the sign of the larger magnitude operand
  - If magnitudes are equal and sign is different; two representations of zero problem
- For this reason the signed-magnitude representation is not as popular as one might think because of its “naturalness”

Complement number systems

- Negates a number by *taking its complement* instead of negating the sign
- Not natural for humans, but better for machine arithmetic
- Will describe 2 complement number systems
  - *Radix complement – very popular in real computers*
    - Must first decide how many bits to represent the number – say n.
    - Complement of a number = r^n – number
    - Example: 2's Complement
  - *Diminished radix-complement – not very useful, may skip it or not spend much time on it*
    - *Example: 1's Complement:* r^n - number - 1

Two's-complement representation

- Just radix-complement when radix = 2
- The most used representation of integers in computers and other digital arithmetic circuits
- 0 and positive numbers: leftmost bit = 0
- Negative numbers: leftmost bit = 1
- Representation of zero
  - i.e. 0 is represented as 0000 using 4-bit binary sequence.
- To find a number's 2's-complement – just flip all the bits and add 1

Properties of Two's Complement Notation

- Relationship between +n and –n.

  - 0 1 0 0 +4 0 0 0 1 0 0 1 0 +18
  - 1 1 0 0 -4 1 1 1 0 1 1 1 0 -18

Two’s Complement Notation with 4-bits

Binary Pattern	Value in 2’s complement.
0 1 1 1	7
0 1 1 0	6
0 1 0 1	5
0 1 0 0	4
0 0 1 1	3
0 0 1 0	2
0 0 0 1	1
0 0 0 0	0
1 1 1 1	-1
1 1 1 0	-2
1 1 0 1	-3
1 1 0 0	-4
1 0 1 1	-5
1 0 1 0	-6
1 0 0 1	-7
1 0 0 0	-8

Advantages of Two’s Complement Notation

- It is easy to add two numbers.

$$\begin{array}{crr} & 0\,0\,0\,1 & +1 \\ + & 0\,1\,0\,1 & +5 \\ \hline & \mathbf{0\,1\,1\,0} & \mathbf{+6} \end{array} \qquad \begin{array}{crr} & 1\,0\,0\,0 & -8 \\ + & 0\,1\,0\,1 & +5 \\ \hline & \mathbf{1\,1\,0\,1} & \mathbf{-3} \end{array}$$

- Subtraction is 2’s complement addition
- Multiplication is just a repeated addition
- Division is just a repeated 2’s complement addition
- Two’s complement is widely used in ***ALU***

Comparison of decimal and 4-bit numbers Complements and other Notations

Decimal	Two's Complement	Ones' Complement	Signed Magnitude	Excess 2^{m-1}
−8	1000	—	—	0000
−7	1001	1000	1111	0001
−6	1010	1001	1110	0010
−5	1011	1010	1101	0011
−4	1100	1011	1100	0100
−3	1101	1100	1011	0101
−2	1110	1101	1010	0110
−1	1111	1110	1001	0111
0	0000	1111 or 0000	1000 or 0000	1000
1	0001	0001	0001	1001
2	0010	0010	0010	1010
3	0011	0011	0011	1011
4	0100	0100	0100	1100
5	0101	0101	0101	1101
6	0110	0110	0110	1110
7	0111	0111	0111	1111

Excess is $2^{4-1} = 8$; Thus, retrieved value=stored value-8

Decimal numbers, their two's complements, ones' complements, signed magnitude and excess 2^{m-1} binary codes

Existence of two zeros!

EXPLAIN

Two's-Comp Addition and Subtraction Rules

- Starting from 1000 (-8) on up, each successive 2's comp number all the way to 0111 (+7) can be obtained by adding 1 to the previous one, ignoring any carries beyond the 4th bit position
- Since addition is just an extension of ordinary counting, 2's comp numbers can be added by ordinary binary addition!
- No different cases based on operands' signs!
- Overflow possible
  - ➢ Occurs if result is out of range
  - ➢ if operands are the same sign but **sum is a different sign** of that of the operands, overflow has occurred.

Storing an integer in two's complement format:

- Convert the integer to an n-bit binary.
- If it is **positive** or **zero**, it is stored as it is. If it is **negative**, take the two's complement and then store it.

Retrieving an integer in two's complement format:

- If the **leftmost bit** is 1, the computer applies the two's complement operation to the n-bit binary. If the leftmost bit is 0, no operation is applied.
- The computer changes the binary to decimal (integer) and corresponding sign is added.

1000	1001	1010	1011	1100	1101	1110	1111	0000	0001	0010	0011	0100	0101	0110	0111
−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7

Example:

Retrieve the integer that is stored as 11100110 in memory using two’s complement format.

Solution:

The leftmost bit is 1, so the integer is negative. The integer needs to be two’s complemented before changing to decimal.

Leftmost bit is 1. The sign is negative	1	1	1	0	0	1	1	0
	↓	↓	↓	↓	↓	↓	↓	↓
Apply two’s complement operation	0	0	0	1	1	0	1	0
Integer changed to decimal								26
Sign is added								−26

Comparison

Summary of integer representations

Contents of memory	*Unsigned*	*Sign-and-magnitude*	*Two's complement*
0000	0	0	+0
0001	1	1	+1
0010	2	2	+2
0011	3	3	+3
0100	4	4	+4
0101	5	5	+5
0110	6	6	+6
0111	7	7	+7
1000	8	–0	–8
1001	9	–1	–7
1010	10	–2	–6
1011	11	–3	–5
1100	12	–4	–4
1101	13	–5	–3
1110	14	–6	–2
1111	15	–7	–1