Zürcher Hochschule
für Angewandte Wissenschaften

School of Engineering

Chapter 3:

Fourier Transformation

Signals in Frequency Domain

Contents

References

[1] M.J.Roberts, “Signals and Systems_Analysis using Transform Methods and Matlab”, Tata McGraw-Hill, 2005

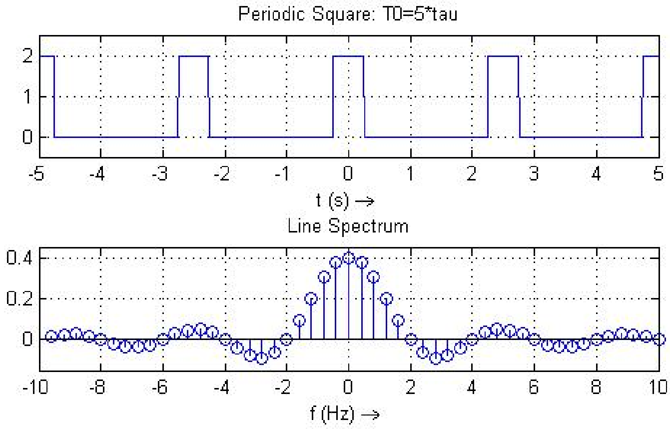
[2] L.F.Chaparro , „Signals and Systems using Matlab“, Academic Press, 2015

1. Transition from Fourier Series to Fourier Transformation

In chapter 2 we discussed the representation of periodic signals in the frequency domain. In this chapter we want to extend this representation to non-periodic signals. In fact we can consider that a non-periodic signal is a limit case of a periodic signal, when the period tends to infinity. So, we can check what happens with the Fourier series when the period of the time function tends to infinity.

This idea is shown in figure 3-1 below, where a periodic square pulse with amplitude $A = (2)$ width $\tau = (0.5)$ and a period starting at $T_0 = (2.5) = (5 \cdot \tau)$ and increasing. From chapter 2 we have the expression of the complex Fourier coefficients c_k.

$$c_k = \frac{A\tau}{T_0} \cdot sinc\left(k\frac{\tau}{T_0}\right)$$



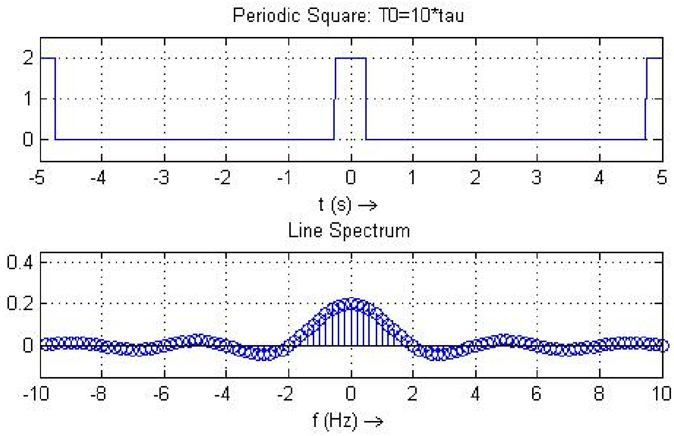
$$A = 2 \quad ; \quad \tau = 0.5 \quad ; \quad T_0 = 5 \cdot \tau \quad ; \quad f_0 = 0.4$$

$$\Rightarrow \quad c_k = \frac{2}{5} \cdot sinc\left(\frac{k}{5}\right)$$

Obs.: the variable used for the horizontal axis here is f (Hz) and not k = f/f0 as in chap-2.
Therefore the zero-crossings happen at:

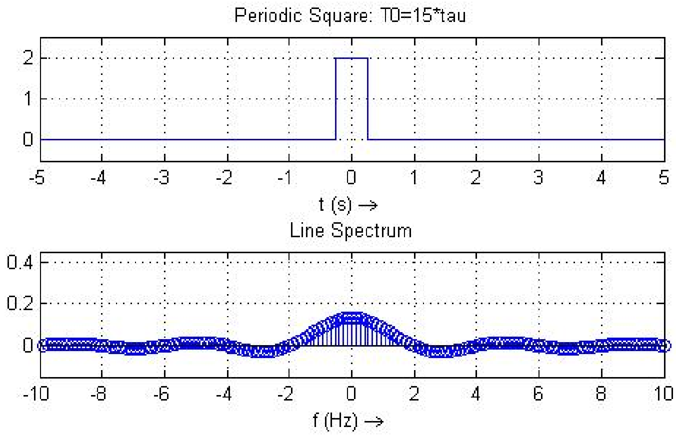
$$k = \pm 5 \cdot n \quad with \quad n \in Z \Rightarrow$$

$$f = k \cdot f_0 = \pm \frac{5 \cdot n}{T_0} = \pm 2 \cdot n$$



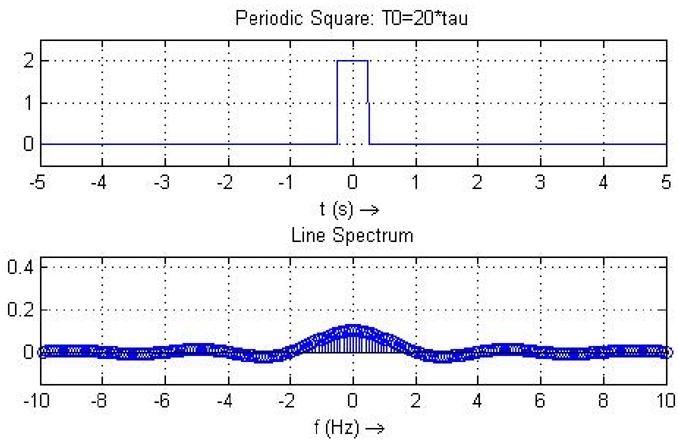
$$A = 2 \quad ; \quad \tau = 0.5 \quad ; \quad T_0 = 10 \cdot \tau \quad ; \quad f_0 = 0.2$$

$$\Rightarrow \quad c_k = \frac{2}{10} \cdot sinc\left(\frac{k}{10}\right)$$



$$A = 2 \quad ; \quad \tau = 0.5 \quad ; \quad T_0 = 15 \cdot \tau \quad ; \quad f_0 = 0.133$$

$$\Rightarrow \quad c_k = \frac{2}{15} \cdot sinc\left(\frac{k}{15}\right)$$



$$A = 2 \quad ; \quad \tau = 0.5 \quad ; \quad T_0 = 20 \cdot \tau \quad ; \quad f_0 = 0.1$$

$$\Rightarrow \quad c_k = \frac{2}{20} \cdot sinc\left(\frac{k}{20}\right)$$

Figure 3-1 Limit Case of a Fourier Series when Period tends to infinity

One can observe following effects on the Fourier series as the $T_0 \rightarrow \infty$:

- the lines representing the harmonic of the Fourier series get closer and closer to each other so that in the limit case they will become a continuous function in the frequency domain;
- the amplitude of the coefficients get smaller and smaller because they are multiplied with the factor $1/T_0$; so that in the limit case all coefficients will tend to zero.
- the position of the first Null stays constant at 1/tau.

The 1st effect is in principle not a problem, but the 2nd makes it impossible to use this definition to represent non-periodic signals with the Fourier series.

Therefore in order to avoid an all-zero representation we adapt the Fourier series and say that for non-periodic signals the representation in the frequency domain corresponds to $T_0.ck$ or $ck/(1/T_0)$. This corresponds to the definition of the Fourier transformation for a signal in the time domain (see equation (1) below).

Definition as limit case of the Fourier Series

$$\lim_{T_0\to\infty}(c_k \cdot T_0) = \lim_{T_0\to\infty}\left(c_k \cdot \frac{1}{f_0}\right) = X(f)$$

Fourier Transformation (1a)

$$X(f) = \int_{-\infty}^{+\infty} x(t)\cdot e^{-j2\pi f t} dt \qquad ; \qquad X(\omega) = \int_{-\infty}^{+\infty} x(t)\cdot e^{-j\omega t} dt$$

Inverse Fourier Transformation (1b)

$$x(t) = \int_{-\infty}^{+\infty} X(f)\cdot e^{+j2\pi f t} df \qquad ; \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\cdot e^{+j\omega t} d\omega$$

The Fourier transformation associates a pair of functions x(t) and X(f) or X(ω). The function in the time domain x(t) is real valued[1] , and the function in frequency domain X(f) or X(ω) are complex valued. The usual graphical representation for X(f) or X(ω) is also a spectrum, or more precisely a spectrum density, split into an amplitude and a phase diagram.

Pair of associated functions in time and frequency domain

$$x(t) \xleftrightarrow{FT} X(f) \tag{2}$$

The complex valued spectrum density is often split into amplitude and phase components:

$$X(f) = |X(f)| \cdot e^{j\cdot phase\{X(f)\}} \tag{3}$$

The Fourier transformation according to equation (1a) can be calculated for all time functions which are totally integrable, which means for energy signals.

We can also observe that the spectrum for the Fourier series, c_k, represents the distribution of the signal power in the frequency domain, and the spectrum density of the Fourier transformation represents the distribution of the signal energy in the frequency domain; and power is energy over time.

---

[1] During this semester we will only work with real valued functions in time domain, but actually the Fourier transformation is also possible for complex valued functions in the time domain.

Another important characteristic is that periodic signals in time domain have a discrete (or sparse) representation in the frequency domain, and non-periodic signals in the time domain have a continuous representation in the frequency domain. This means that in order to synthesise a periodic signal only certain frequencies are required (see Fourier Synthesis examples in chapter 2).

Time Domain	Frequency Domain
➢ periodic function power signal	discrete spectrum (with lines) power distribution in f (Hz)
➢ non-periodic function energy signal	continuous spectrum energy distribution in f (Hz)

2. Examples of Fourier Transformation using the Definition

Let us use the definition of the Fourier transformation to calculate X(f) for a non-periodic square pulse, such that we can compare it to the limit approach discussed in the previous section and confirm that one gets the same X(f) function.

Example 3-1

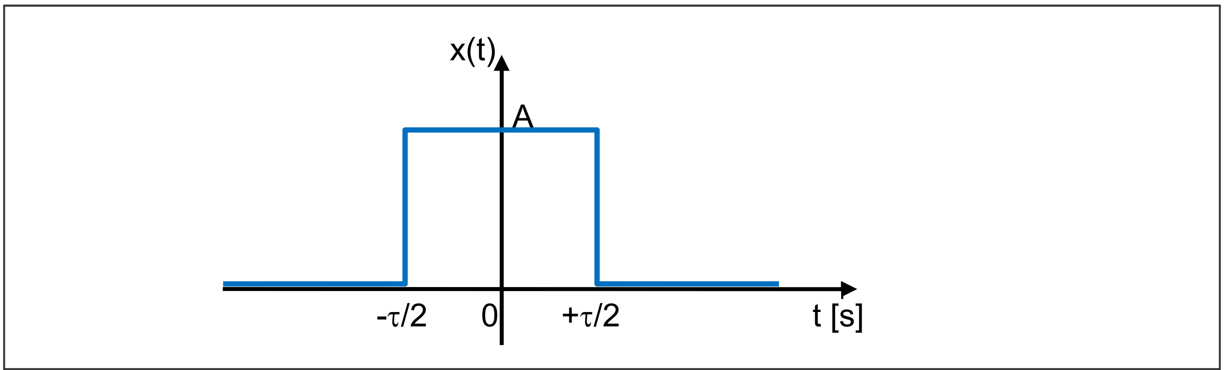


Figure 3-2 Non-periodic square pulse function with width τ and amplitude A

$$X(f) = \underbrace{\int_{-\infty}^{+\infty} x(t) \cdot e^{-j2\pi f t} dt}_{\boxed{A}} = \underbrace{\lim_{T_0 \to \infty} c_k \cdot T_0}_{\boxed{B}}$$

(A)

$$X(f) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-j2\pi f t} dt \quad = \int_{-\frac{\tau}{2}}^{+\frac{\tau}{2}} A \cdot e^{-j2\pi f t} dt = \frac{A}{-j2\pi f} \cdot e^{-j2\pi f t}\Big|_{-\frac{\tau}{2}}^{+\frac{\tau}{2}} \quad =$$

$$X(f) = \frac{A}{-j2\pi f} \cdot \left[e^{-j2\pi f \frac{\tau}{2}} - e^{+j2\pi f \frac{\tau}{2}} \right] \quad = \frac{A}{-j2\pi f} \cdot \left[-2j \sin(\pi f \tau) \right] \quad = A\tau \cdot \left[\frac{\sin(\pi f \tau)}{\pi f \tau} \right]$$

$$\boxed{X(f) = \quad A\tau \cdot sinc(f\tau)}$$

(B)

$$X(f) = \quad \lim_{T_0 \to \infty} c_k \cdot T_0$$

with

$$c_k = \frac{A\tau}{T_0} \cdot sinc\left(k \frac{\tau}{T_0} \right) \quad ; \quad T_0 \to \infty \quad ; \quad f_0 \to 0 \quad ; \quad k \cdot f_0 \to f \quad ;$$

$$X(f) = \quad \lim_{T_0 \to \infty} A\tau \cdot sinc\left(k \frac{\tau}{T_0} \right) \quad = \quad \lim_{T_0 \to \infty} A\tau \cdot sinc(kf_0\tau) \quad = \boxed{A\tau \cdot sinc(f\tau)}$$

Question 3-1

Plot the spectrum of the square pulse in Matlab, and control the correctness of your graphic by checking the amplitude and position of the first null (zero crossing of the amplitude). Make three versions of your plot:

- complete X(f) function (here possible cause real valued);
- split X(f) into Amplitude [linear scale] and Phase [°] components;
- split X(f) into Amplitude [dB] and Phase [°] components;

Another four examples for the Fourier transformation of a decaying exponential, a dirac impulse, a step and a cosine wave are shown below:

Question 3-2

Based on example 3-2 (following page), draw a sketch of the spectrum X(f) for the time functions: decaying exponential and Dirac impulse.

Example 3-2

Signal in time domain	**Signal in Frequency Domain**
Decaying exponential $x(t)=\varepsilon(t)\cdot e^{-at}=\begin{cases} e^{-at} & ;t\geq 0 \\ 0 & ;t<0 \end{cases}$ $with \quad a>0$	$X(f)=\int_0^{+\infty} e^{-at}\cdot e^{-j2\pi ft}dt=\int_0^{+\infty} e^{-(a+j2\pi f)t}dt=\frac{-1}{(a+j2\pi f)}\cdot e^{-(a+j2\pi f)t}\Big\vert_0^{\infty}=\frac{+1}{(a+j2\pi f)}=\frac{+1}{(j\omega+a)}$
Dirac impulse $x(t)=\delta(t)$	$X(f)=\int_{-\infty}^{+\infty}\delta(t)\cdot e^{-j2\pi ft}dt=\int_{-\infty}^{+\infty}\delta(t)\cdot 1\,dt=1$
Unit step $x(t)=\varepsilon(t)=\begin{cases} 1 & ;t\geq 0 \\ 0 & ;t<0 \end{cases}$	$X(f)=\int_0^{+\infty} 1\cdot e^{-j2\pi ft}dt=....$ integral does not converge[2]
Cosine wave $x(t)=\cos(\omega_0 t)=\frac{1}{2}\cdot\left[e^{+j\omega_0 t}+e^{-j\omega_0 t}\right]$	$X(\omega)=\frac{1}{2}\cdot\left[\int_{-\infty}^{+\infty} e^{-j(\omega-\omega_0)t}dt+\int_{-\infty}^{+\infty} e^{-j(\omega+\omega_0)t}dt\right]=....$ integral does not converge

As we can see from the examples above only the energy signals can have their Fourier transformation calculated with the definition. But there is a trick that enables us to extend the Fourier transformation concept to other signals than energy signals, and this is to use properties of the Fourier transformation, allowing us to derive X(f) through these properties. In the next section we discuss such properties and related examples.

---

[2] Later on you will learn the Laplace transformation, where one multiplies the time function x(t) with a decaying exponential, and get then a converging integral. The Laplace transform is especially useful for control and automation topics. It allows to consider initial conditions and to check stability boundaries.

$$X(s)=\int_0^{+\infty} x(t)\cdot e^{-\sigma t}\cdot e^{-j2\pi ft}dt=\int_0^{+\infty} x(t)\cdot e^{-st}dt \qquad with \quad s=\sigma+j2\pi f=\sigma+j\omega$$

3. Properties of Fourier Transformation

Table 3-1 below gives an overview of some properties of the Fourier transformation. For a complete list of properties and the proof of them please refer to the bibliography reference [1].

Property	Time Domain	Frequency Domain
Linearity or superposition	$y(t) = A \cdot x_1(t) + B \cdot x_2(t)$	$Y(f) = A \cdot X_1(f) + B \cdot X_2(f)$
Time-Shift	$y(t) = x(t-\lambda)$	$Y(f) = X(f) \cdot e^{-j2\pi f\lambda}$
Time-Scaling or Time-Bandwidth Product	$y(t) = x(a \cdot t)$	$Y(f) = \frac{1}{\lvert a \rvert} \cdot X\left(\frac{f}{a}\right)$
Duality or Symmetry between time and frequency domain	$y(t) = X(f)$	$Y(f) = x(-t)$
Frequency-Shift	$y(t) = x(t) \cdot e^{+j2\pi f_0 t}$	$Y(f) = X(f - f_0)$
Derivation in time domain	$y(t) = \frac{dx(t)}{dt} = \dot{x}(t)$	$Y(f) = (j2\pi f) \cdot X(f)$
	$y(t) = \frac{d^n x(t)}{dt^n}$	$Y(f) = (j2\pi f)^n \cdot X(f)$
Convolution vs Multiplication	$y(t) = x_1(t) * x_2(t)$	$Y(f) = X_1(f) \cdot X_2(f)$
	$y(t) = x_1(t) \cdot x_2(t)$	$Y(f) = X_1(f) * X_2(f)$
Symmetry in the frequency domain	$y(t) \in \Re$	Amplitude $\lvert Y(f) \rvert$ is even Phase $\langle Y(f)$ is odd
Symmetry in the time domain	$y(t) \in \Re$ and even	$Y(f)$ is purely real
	$y(t) \in \Re$ and odd	$Y(f)$ is purely imaginary
Parseval Theorem or Signal Energy	$E_x = \int_{-\infty}^{+\infty} \lvert x(t) \rvert^2 dt$	$E_x = \int_{-\infty}^{+\infty} \lvert X(f) \rvert^2 df$

Table 3-1 Properties of the Fourier Transformation

Question 3-3

Use the properties above to calculate the Fourier transformation X(f) for the following time functions:

- Unity step function : $x(t) = \varepsilon(t)$
- Constant DC-value : $x(t) = K$
- Cosine wave : $x(t) = \cos(\omega_0 t) = \cos(2\pi\, f_0 t)$
- Sine wave : $x(t) = \sin(\omega_0 t) = \sin(2\pi\, f_0 t)$
- Sinc function in time domain : $x(t) = A\tau \cdot sinc(t\,\tau)$
- Square function with width tau/2

Draw for each of the functions above a sketch of the signal in time and in frequency domain.

Question 3-4

Use the derivation property to calculate the transfer function of a passive RLC low pass filter (as you probably already know from electricity courses). Start your calculation with the differential equation of the RLC filter.
Do you know how to determine the order of this low pass filter (LPF) ?

Question 3-5

Use the derivation property to calculate the complex impedance of a capacitor and of an inductance. Then use this result to calculate the transfer function of the passive RLC filter from the previous question.

4. Fourier Transformation Application Example: Modulation

The frequency shift property is very important for applications in telecommunications, where the information signal is often modulated into a carrier signal of much higher frequency. The frequency of the carrier signal corresponds to the channel allocated within the available frequency range (also called the available spectrum for the application).

Let us look at the example of an amplitude modulated radio signal. The name of the modulation technique indicates the characteristic of the carrier signal which will be modified by the information signal.

The AM radio application (MW medium waves) has an allocated available spectrum within the range from 500 kHz to 1600 KHz (exact frequencies vary from country to country) divided into several channels of 9 kHz.

Figure 3-3 shows a carrier signal modulated by an information signal s(t). We can use the frequency shift property of the Fourier transformation to deduce how the modulation modifies the spectrum of the carrier signal.

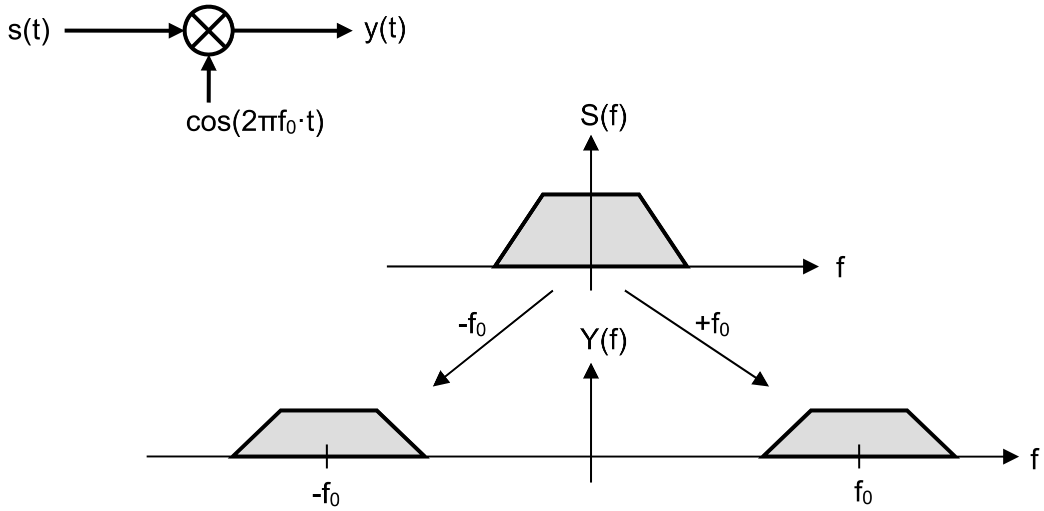


Figure 3-3 Multiplication of a time signal s(t) with a cosine carrier signal

$$y(t) = s(t) \cdot \cos(2\pi f_0 t) = \frac{1}{2} \cdot s(t) \cdot e^{j2\pi f_0 t} + \frac{1}{2} \cdot s(t) \cdot e^{-j2\pi f_0 t}$$

Which implies in the frequency domain:

$$Y(f) = \frac{1}{2} \cdot S(f + f_0) + \frac{1}{2} \cdot S(f - f_0)$$

Normally, though, the information signal is unknown and we need to filter it before the modulation in order to ensure that the bandwidth limit is fulfilled.

Question 3-6

Given the channel width of the AM radio application, what is the allowed bandwidth for the information signal ?

5. Fourier Transformation for Periodic Signals

In question 3-3 we see an example showing that it is possible to calculate the Fourier transformation of a periodic signal by using the properties instead of the definition with the integral calculation. We want to generalise this approach to other periodic functions.

If we compare the spectrum X(f) of the cosine wave with the Fourier series ck of the same cosine function, we see that they are very similar. In fact X(f) has discrete lines in the same frequency as c_k, only that these lines are now two Dirac Impulses, which mathematically enable us to describe a discrete spectrum with a continuous frequency variable f. Furthermore the spectrum X(f) is a density spectrum, and the Dirac impulse represents in this case the concentration of the whole signal power into two single frequencies $+f_0$ and $-f_0$.

The Fourier transformed for periodic functions:

$$X(f) = \sum_{k=-\infty}^{+\infty} c_k \cdot \delta(f - kf_0) \tag{4}$$

The frequency spectrum (FT) of a periodic signal is a series of impulse functions at the harmonics frequencies. Each impulse is weighted by the complex Fourier series coefficient c_k of that harmonic.

Therefore from now on, when we look for the Fourier transformation of a periodic signal, we just calculate its complex Fourier series ck, and take as corresponding Fourier transformed X(f) the same shape of the double sided line spectrum ck, but with Dirac impulses instead [3].

Example 3-3

Let us use the method described above to calculate the spectrum of a periodic train of impulses, also called a comb function, p(t). The comb function is a periodic function with regularly spaced Dirac impulses.

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_0) = \sum_{k=-\infty}^{+\infty} c_k \cdot \exp(j2\pi k f_0 t)$$

Since p(t) is periodic we can calculate the ck coefficients, and the corresponding Fourier transformed P(f), which is interestingly also a comb function :

$$c_k = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{+\frac{T_0}{2}} \left[\sum_{n=-\infty}^{+\infty} \delta(t - nT_0) \right] \exp(-j2\pi k f_0 t) dt = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{+\frac{T_0}{2}} \delta(t) dt = \frac{1}{T_0}$$

The comb function:

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_0) \quad \leftrightarrow \quad P(f) = \frac{1}{T_0} \sum_{k=-\infty}^{+\infty} \delta(f - kf_0) \tag{5}$$

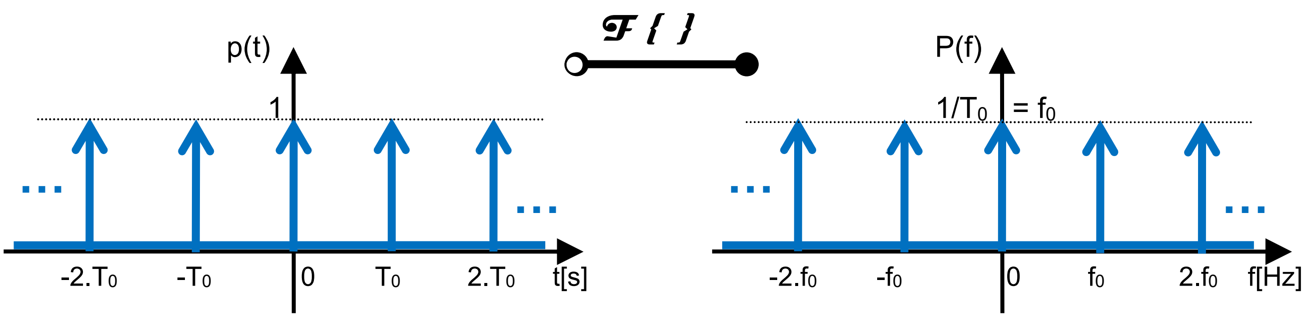


Figure 3-4 The comb function in time domain has a spectrum which is also a comb.

[3] The mathematical proof of this idea is shown in Annex 3-C.

Example 3-4

Let us compare now the spectrum of a non-periodic square pulse and a periodic square pulse, with the same pulse form (amplitude and width).

We call the non-periodic function the generating function $x_g(t)$ with a corresponding continuous spectrum $X_g(f)$. By repeating this function with a time-shift of $k.T_0$ we get the periodic function $x_p(t)$. This repetition and shift in time can be produced by convoluting the generating function $x_g(t)$ with a comb function p(t); which means that the spectrum $X_p(f)$ can be calculated as the product of $X_g(f)$ with the comb function P(f) (according to convolution property in table 3-1).

$$x_p(t) = x_g(t) * p(t) \quad \leftrightarrow \quad X_p(f) = X_p(f) \cdot P(f) \tag{6a}$$

$$X_p(f) = X_g(f) \cdot \frac{1}{T_0} \sum_{k=-\infty}^{+\infty} \delta(f - kf_0) = \sum_{k=-\infty}^{+\infty} \frac{X_g(kf_0)}{T_0} \cdot \delta(f - kf_0) \tag{6b}$$

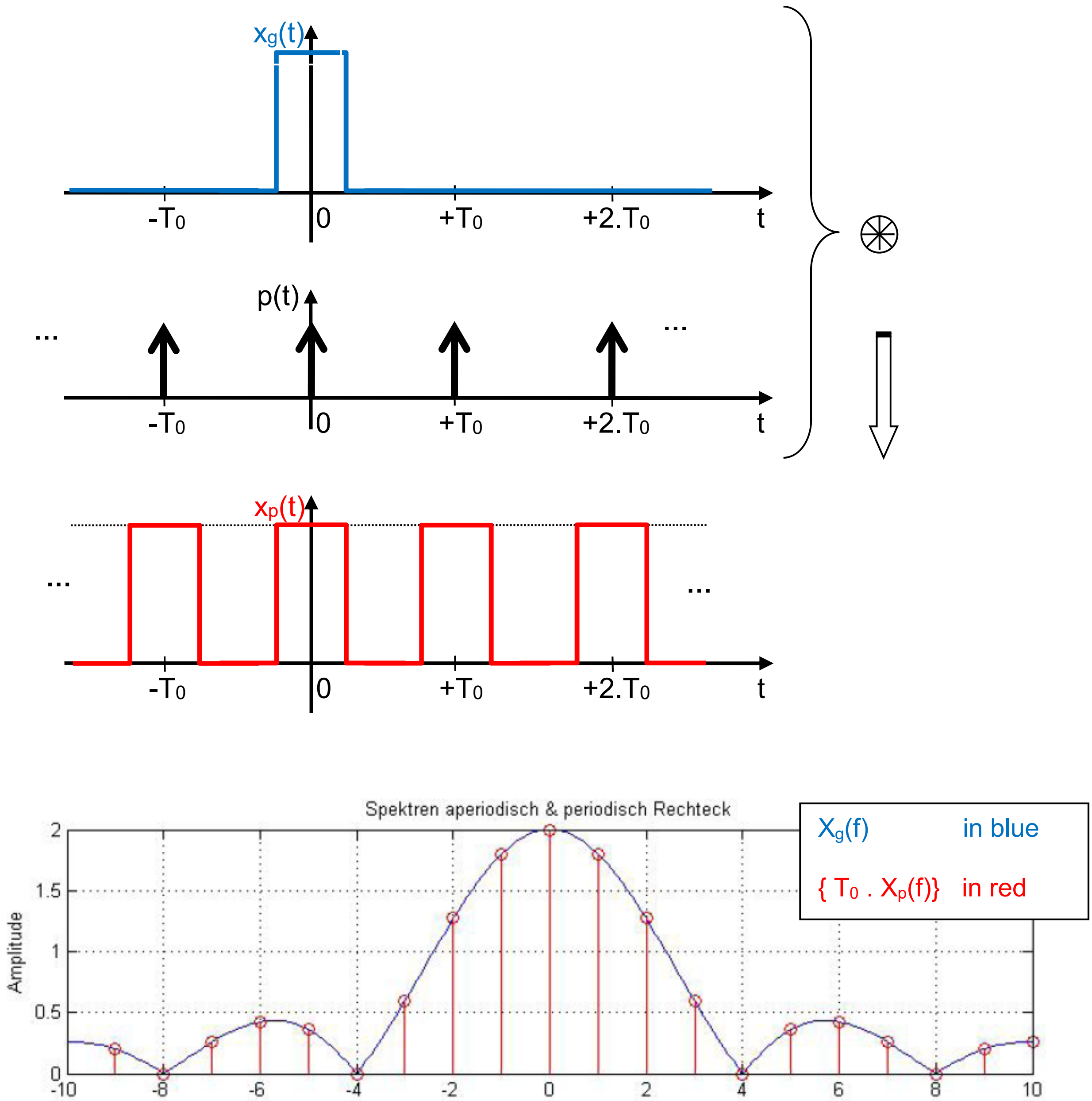


Figure 3-5 **Generating a periodic function through convolution with a comb**

In figure 3-5 we see that the spectrum $X_p(f)$ has an envelope curve with the same form as $X_g(f)$, and is a discrete spectrum with only non-zero values at certain frequencies where the Dirac impulses occur.

Now, if we remember that the FFT provides a numerical method to approximate the Fourier series coefficients, we see that in fact one can use the FFT to approximately calculate the spectrum of any function (and not only the periodic ones). But the result of the FFT is always discrete, which means that the calculated spectrum describes a periodic time function. If your original function in the time domain is not periodic you should consider that the spectrum of it is rather the continuous function corresponding to the envelope of the FFT coefficients.

Question 3-7

Re-calculate the spectrum of the comb function using the idea of shifting and repeating a non-periodic generating function. The generating function in this case is a single Dirac impulse.

6. Fourier Transformation for Discrete Signals

The comb function discussed in example 3-3 can be used to make the ideal sampling of a continuous function in time and get a corresponding discrete function in time. We can use now the duality of the multiplication and convolution operations to calculate the spectrum of the discrete function.

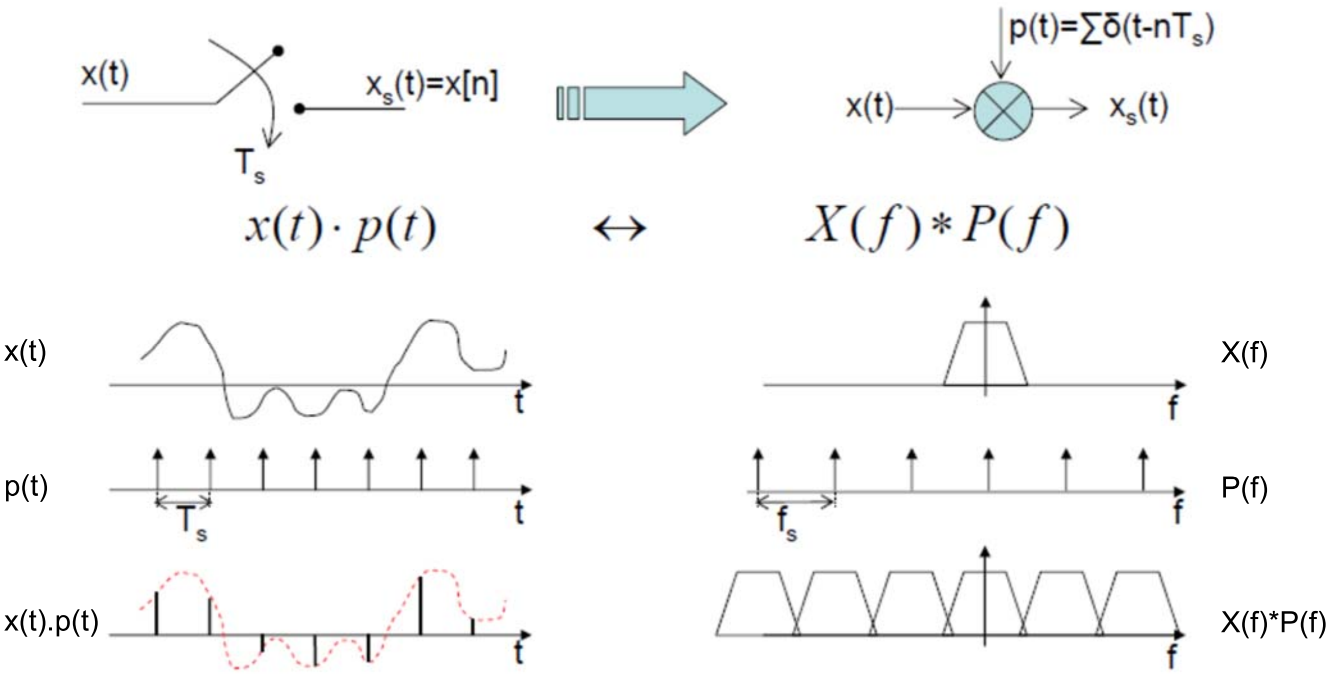


Figure 3-6 The ideal sampling corresponds to multiplication with train of impulses in time domain, which is equivalent to convolution with train of impulses in frequency domain.

The consequence is that the sampling of a continuous time function causes its spectrum to be mirrored around multiples of the sampling frequency. Can this be a problem? It depends! If we are able to limit the bandwidth of the continuous signal before the sampling, such that the highest frequencies are below Fs/2 we can avoid it. But if we have part of the signal energy in frequencies above Fs/2, this will be distorted by the sampling and appear «folded» or aliased to a lower frequency.

The aliasing effect and the idea of an anti-aliasing filter (AAF) are shown in figure 3-7 . The aliasing effect can be avoided by either increasing the sampling frequency (which is not always possible), or by low pass filtering the continuous signal before sampling.

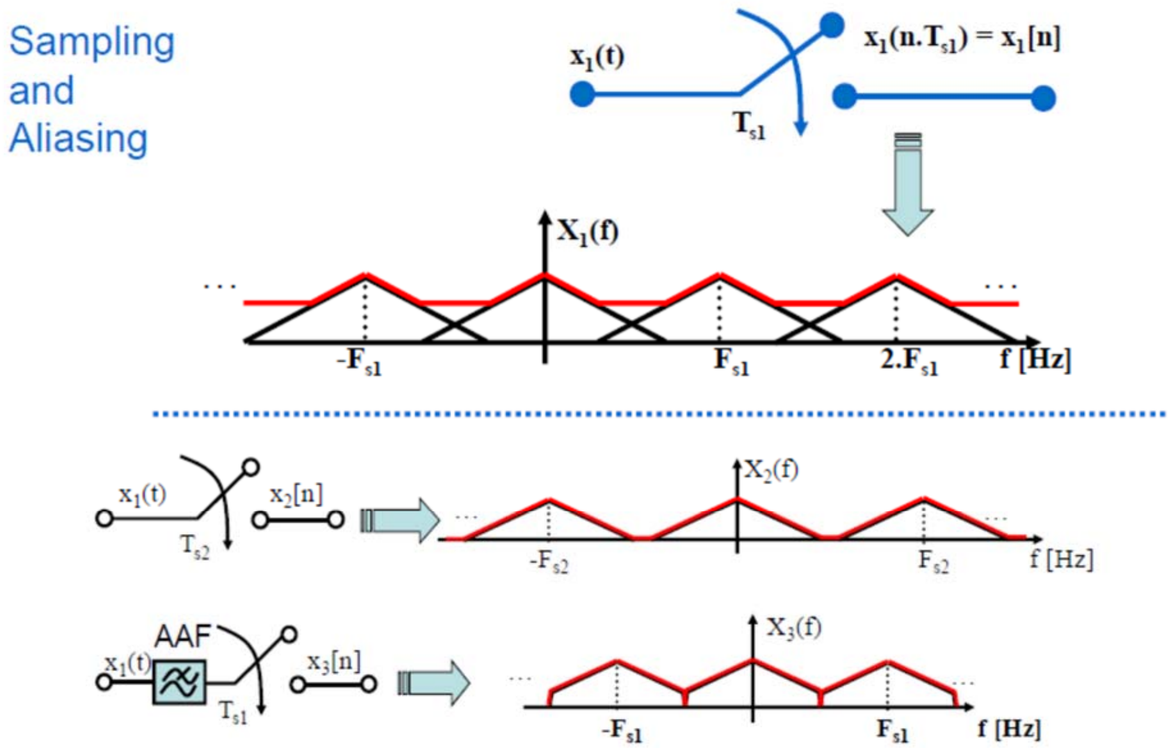


Figure 3-7 Sampling, aliasing effect and anti-aliasing filter

In figure 3-7 we see that a sampled or discrete function in the time domain has a periodic spectrum.

Let us once more recall the result of the FFT; in fact with the FFT we calculate the spectrum of a time function which is periodic and discrete, which means the resulting spectrum is discrete and periodic. But both in time and frequency domain the N points describe an interval with only one period, and the user has to keep in mind the periodicity of the functions.

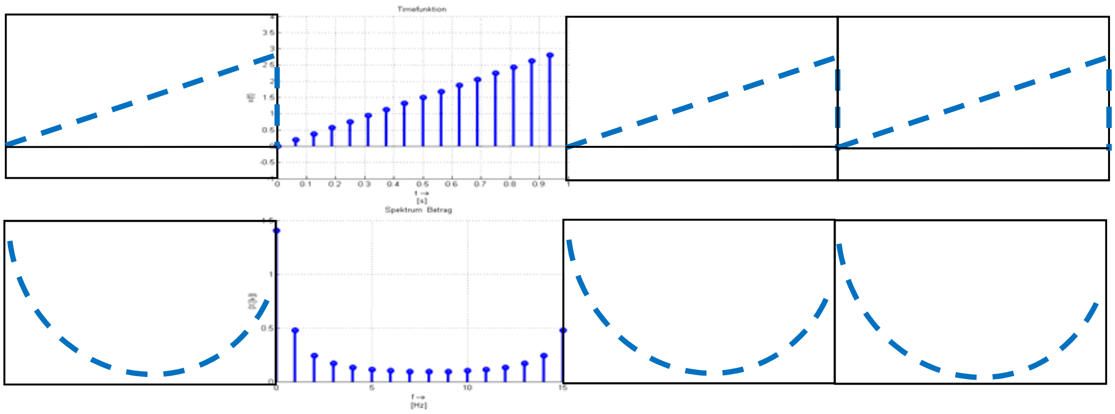


Figure 3-8 The periodicity of the FFT in time and frequency domains

The table in annex 3-B summarises the correspondances of the signal properties in the time and frequency domain discussed in the last sections.

7. Vocabulary

convolution:	Faltung
line spectrum:	Linienspektrum
sampling:	Abtastung
spectrum density:	Spektrumdichte
sampling property or sifting property:	Ausblendeigenschaft
transfer function:	Übertragungsfunktion

8. Annexes

3-A Operations with Dirac Impulse

- **Ideal sampling or sifting property of the Dirac Impulse (Ausblendeneingenschaft)**

In time domain

$$\int_{-\infty}^{+\infty} x(t)\cdot\delta(t-t_0)dt = x(t_0)$$

In frequency domain

$$\int_{-\infty}^{+\infty} X(f)\cdot\delta(f-f_0)df = X(f_0)$$

- **Neutral element of the convolution**

$$x(t)*\delta(t)=\int_{-\infty}^{+\infty} x(\lambda)\cdot\delta(t-\lambda)d\lambda = x(t) \quad \text{or} \quad X(f)*\delta(f)=\int_{-\infty}^{+\infty} X(\lambda)\cdot\delta(f-\lambda)d\lambda = X(f)$$

- **Shifting with convolution**

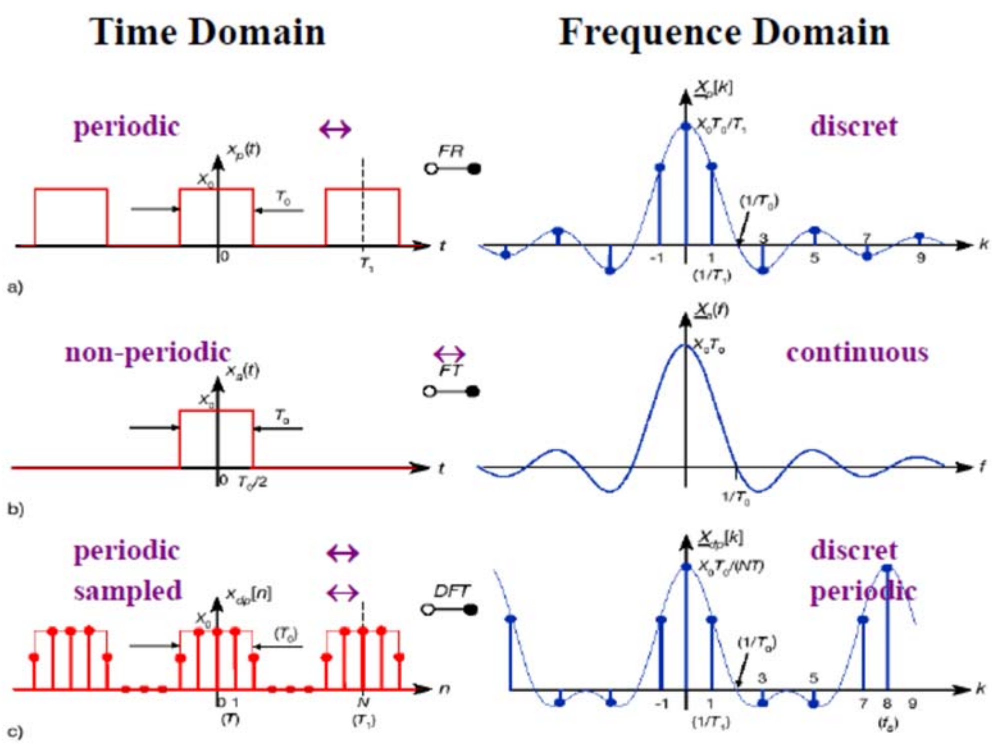
$$x(t)*\delta(t-t_0)=x(t-t_0) \quad \text{or} \quad X(f)*\delta(f-f_0)=X(f-f_0)$$

3-B Correspondances of Signal properties in Time and Frequency domains

Signale & Transformationen

Zeitbereich	Frequenzbereich
x(t) **periodisches Signal**	FR c_k Koeff.: **diskretes Spektrum** (Linienspektrum) FT X(ω) mit δ (ω) : **diskretes Spektrum** (Spektrumdichte mit Diracstoss)
x(t) **aperiodisches Signal**	FR : nicht möglich FT X(ω) : **kontinuierliches Spektrum** (Spektrumdichte)
x[n] **abgetastetes Signal** **(Mit Zeitfenster und Annahme zeitfunktion periodisch) →**	Abtastung (FT) : **periodisches Spektrum** DFT X[k] : **diskretes und periodisches Spektrum** Bmk: wird in SiSy2 und DSV1 ins Detail gelernt

Signals & Transfor-mations



3-C Fourier-Transformation of Periodic Signals

- Given a periodic function x(t) and its Fourier series (FR)

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k \cdot \exp\left(j2\pi\, f_0 t\right)$$

we look for its Fourier Transform (FT)

$$X(f) = \int_{-\infty}^{+\infty}\left[\sum_{k=-\infty}^{+\infty} c_k \cdot \exp\left(j2\pi k f_0 t\right)\right]\exp\left(-j2\pi\, f t\right)dt$$

$$X(f) = \sum_{k=-\infty}^{+\infty} c_k \cdot \int_{-\infty}^{+\infty} \exp\left[-j2\pi\left(f - kf_0\right)t\right]dt = \sum_{k=-\infty}^{+\infty} c_k \cdot \delta\left(f - kf_0\right)$$

→ **The frequency spectrum (FT) of a periodic signal is a series of impulse functions at the harmonics frequencies. Each impuls is weighted by the complex Fourier series (FR) coefficient of that harmonic.** (vide comparison FT-FR sisy1)

3-D Some FT Reference Signals

Sketch Time-D	Equation Time-D	Equation Freq-D	Sketch Freq-D
	δ(t)	1	
	1	δ(f)	
	A.rect(t / τ)	A. τ .sinc(f.τ)	
	A. τ .sinc(t.τ)	A.rect(f / τ)	
	$e^{j2\pi f0t}$	$\delta(f-f_0)$	
	$\cos(2\pi f_0 t)$	$1/2 \,.\, [\delta(f-f_0) + \delta(f+f_0)]$	

3-E Übersicht ¨uber die verschiedenen Varianten der Fourier-Transformation