Chapter 6. Portfolio Optimization in R

1. Theoretical background

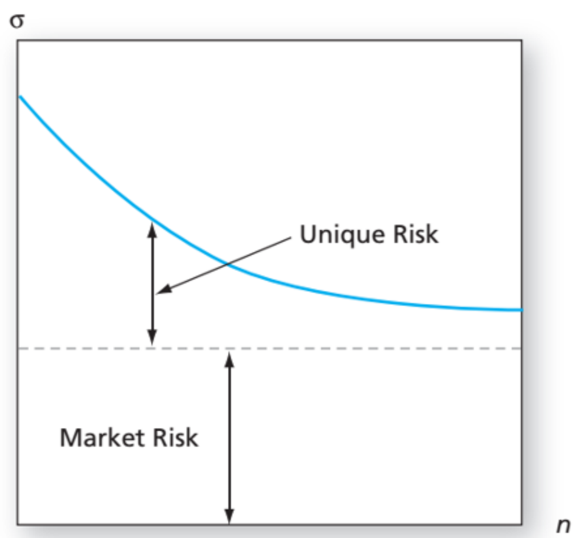
1) Diversification and portfolio

Suppose your portfolio is composed of only one stock, GM. What would be the sources of risk to this "portfolio"? You might think of two broad sources of uncertainty. First, there is the risk that comes from conditions in the general economy, such as the business cycle, inflation, interest rates, and exchange rates. None of these macroeconomic factors can be predicted with certainty, and all affect the rate of return on GM stock. In addition to these macroeconomic factors, there are firm-specific influences, such as GM's success in research and development, and personnel changes. These factors affect GM without noticeably affecting other firms in the economy.

Now consider a naïve **diversification** strategy, in which you include additional securities in your portfolio. For example, place half your funds in Exxon and half in GM. What should happen to portfolio risk? To the extent that the firm-specific influences on the two stocks differ, diversification should reduce portfolio risk. For example, when oil prices fall, hurting Exxon, GM's stock prices might rise. The two effects are offsetting and stabilize portfolio return.

But why end diversification at only two stocks? If we diversify into many more securities, we continue to spread out our exposure to firm-specific factors, and portfolio volatility should continue to fall. Ultimately, however, even with a large number of stocks we cannot avoid risk altogether, since virtually all securities are affected by the common macroeconomic factors. For example, if all stocks are affected by the business cycle, we cannot avoid exposure to business cycle risk no matter how many stocks we hold.

When common sources of risk affect all firms, however, even extensive diversification cannot eliminate risk. In the following figure, portfolio standard deviation falls as the number of securities increases, but it cannot be reduced to zero.



The risk that remains even after extensive diversification is called **market risk,** risk that is attributable to market wide risk sources. Such risk is also called **systemic risk,** or **non-diversifiable risk.** In contrast, the risk that *can* be eliminated by diversification is called **unique risk, firm-specific risk, nonsystematic risk, or diversifiable risk**.

2) Portfolio of two risky assets

It is time to study **efficient** diversification, whereby we construct risky portfolios to provide the lowest possible risk for any given level of expected return.

Portfolios of two risky assets are relatively easy to analyze, and they illustrate the principles and considerations that apply to portfolios of many assets. We will consider a portfolio comprised of two mutual funds, a bond portfolio specializing in long-term debt securities, denoted D, and a stock fund that specializes in equity securities, E. Table below lists the parameters describing the rate-of-return distribution of these funds. These parameters are representative of those that can be estimated from actual funds.

	Debt		Equity
Expected return, E(r)	8%		13%
Standard deviation, σ	12%		20%
Covariance, Cov(r_D, r_E)		0.0072	
Correlation coefficient, ρ_{DE}		0.30	

A proportion denoted by w_D is invested in the bond fund, and the remainder, $1 - w_D$, denoted w_E is invested in the stock fund. The rate of return on this portfolio, r_p, will be

$$r_p = w_D r_D + w_E r_E$$

where r_D is the rate of return on the debt fund and r_E is the rate of return on the equity fund.

The expected return on the portfolio is a weighted average of expected returns on the component securities with portfolio proportions as weights:

$$\mathrm{E}(r_p) = w_D\, E(r_D) + w_E\, E(r_E) \qquad ------ \quad (1)$$

The variance of the two-asset portfolio is

$$\sigma_p^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D w_E\, Cov(r_D, r_E) \qquad ------ \quad (2)$$

Our first observation is that the variance of the portfolio, unlike the expected return, is *not* a weighted average of the individual asset variances. Equation (2) reveals that variance is reduced if the covariance term is negative. It is important to recognize that even if the covariance term is positive, the portfolio standard deviation *still* is less than the weighted average of the individual security standard deviations, unless the two securities are perfectly positively correlated.

To see this, notice that the covariance can be computed from the correlation coefficient, ρ_{DE}, as

$$Cov\ (r_D, r_E) \ = \ \rho_{DE}\sigma_D\sigma_E$$

Therefore,

$$\sigma_p^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_Dw_E\ \sigma_D\sigma_E\rho_{DE} \qquad ------- \qquad (3)$$

Because the covariance is higher, portfolio variance is higher when ρ_{DE} is higher. In the case of perfect correlation, $\rho_{DE} = 1$, the right-hand side of Equation (3) is a perfect square and simplifies to

$$\sigma_p^2 = \ (w_D\sigma_D + w_E\sigma_E)^2$$

or

$$\sigma_p \ = \ w_D\sigma_D + w_E\sigma_E$$

Therefore, the standard deviation of the portfolio with perfect positive correlation is just the weighted average of the component standard deviations. In all other cases, the correlation coefficient is less than 1, making the portfolio standard deviation *less* than the weighted average of the component standard deviations.

A hedge asset has *negative* correlation with the other assets in the portfolio. Equation (3) shows that such assets will be particularly effective in reducing total risk. Moreover, Equation (1) shows that expected return is unaffected by correlation between returns. Therefore, other things equal, we will always prefer to add to our portfolios assets with low or, even better, negative correlation with our existing position.

Because the portfolio's expected return is the weighted average of its component expected returns, whereas its standard deviation is less than the weighted average of the component standard deviations, *portfolios of less than perfectly correlated assets always offer better risk–return opportunities than the individual component securities on their own*. The lower the correlation between the assets, the greater the gain in efficiency. How low can portfolio standard deviation be? The lowest possible value of the correlation coefficient is 1, representing perfect negative correlation. In this case, Equation (3) simplifies to

$$\sigma_p^2 = \ (w_D\sigma_D \ - \ w_E\sigma_E)^2$$

And the portfolio standard deviation is

$$\sigma_p = |\ w_D\sigma_D \ - \ w_E\sigma_E\ |$$

When $\rho \ = \ -1$, a perfectly hedged position can be obtained by choosing the portfolio proportions to solve

$$w_D \sigma_D - w_E \sigma_E = 0$$

The solution to this equation is

$$w_D = \frac{\sigma_E}{\sigma_D + \sigma_E}$$
$$w_E = \frac{\sigma_D}{\sigma_D + \sigma_E} = 1 - w_D$$

These weights drive the standard deviation of the portfolio to zero.

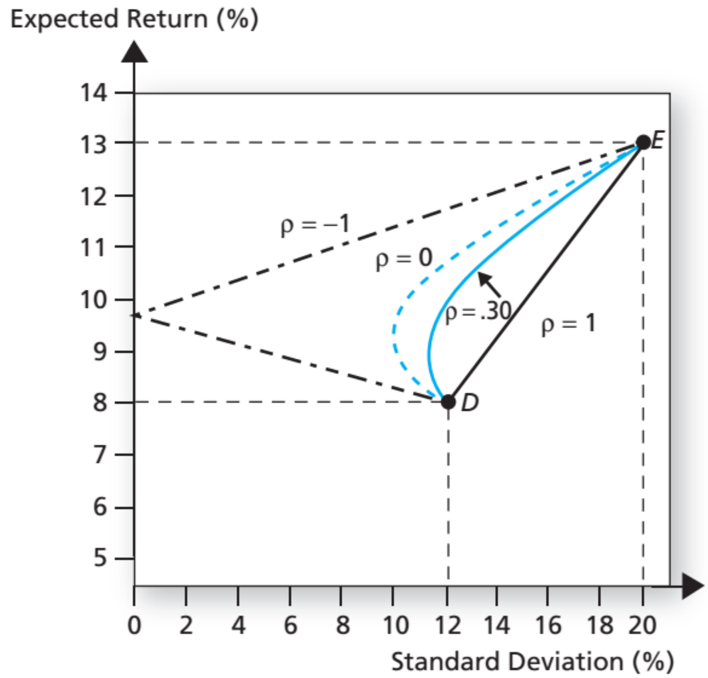
We can experiment with different portfolio proportions to observe the effect on portfolio expected return and variance. Suppose we change the proportion invested in bonds. The effect on expected return is as follows.

			Portfolio Standard Deviation for Given Correlation			
w_D	w_E	$E(r_P)$	$\rho = -1$	$\rho = 0$	$\rho = .30$	$\rho = 1$
0.00	1.00	13.00	20.00	20.00	20.00	20.00
0.10	0.90	12.50	16.80	18.04	18.40	19.20
0.20	0.80	12.00	13.60	16.18	16.88	18.40
0.30	0.70	11.50	10.40	14.46	15.47	17.60
0.40	0.60	11.00	7.20	12.92	14.20	16.80
0.50	0.50	10.50	4.00	11.66	13.11	16.00
0.60	0.40	10.00	0.80	10.76	12.26	15.20
0.70	0.30	9.50	2.40	10.32	11.70	14.40
0.80	0.20	9.00	5.60	10.40	11.45	13.60
0.90	0.10	8.50	8.80	10.98	11.56	12.80
1.00	0.00	8.00	12.00	12.00	12.00	12.00

When the proportion invested in debt (bonds) varies from zero to 1 (so that the proportion in equity varies from 1 to zero), the portfolio expected return goes from 13% to 8%. What happens when $w_D > 1$ and $w_E < 0$? In this case, portfolio strategy would be to sell the equity fund short and invest the proceeds of the short sale in the debt fund. This will decrease the expected return of the portfolio. For example, when $w_D = 2$ and $w_E = -1$, expected portfolio return falls to $2 \times 0.08 + (-1) \times 0.13 = 0.03$. At this point, the value of the bond fund in the portfolio is twice the net worth of the account. This extreme position is financed in part by short selling stocks equal in value to the portfolio's net worth.

The reverse happens when $w_D < 0$ and $w_E > 1$. This strategy calls for selling the bond fund short and using the proceeds to finance additional purchases of the equity fund. Of course, varying investment proportions also has an effect on portfolio standard deviation.

Now, we demonstrate the relationship between portfolio risk (standard deviation) and expected return – given the parameters of the available assets. For any pair of investments proportions, w_D, w_E, we read the resulting pairs of expected return and standard deviation shown in the table above.



The solid blue curve above shows **the efficient frontier** (**portfolio opportunity set)** for $\rho = 0.3$. We call it **the efficient frontier** or **portfolio opportunity set** because it shows all combinations of portfolio expected return and standard deviation that can be constructed from the two available assets. The other lines show the portfolio opportunity set for other values of the correlation coefficient. The solid black line connecting the two funds shows that there is no benefit from diversification when the correlation between the two is positive ($\rho = 1$). The opportunity set is not “pushed” to the northwest. The dashed blue line demonstrates the greater benefit from diversification when the correlation coefficient is lower than 0.3. Finally, for $\rho = -1$, the portfolio opportunity set is linear, but now it offers a perfect hedging opportunity and the maximum advantage from diversification.

To summarize, although the expected return of any portfolio is simply the weighted average of the asset expected returns, this is not true of the standard deviation. Potential benefits from diversification arise when correlation is less than perfectly positive. The lower the correlation, the greater the potential benefit from diversification. In the extreme case of perfect negative correlation, we have a perfect hedging opportunity and can construct a zero-variance portfolio.

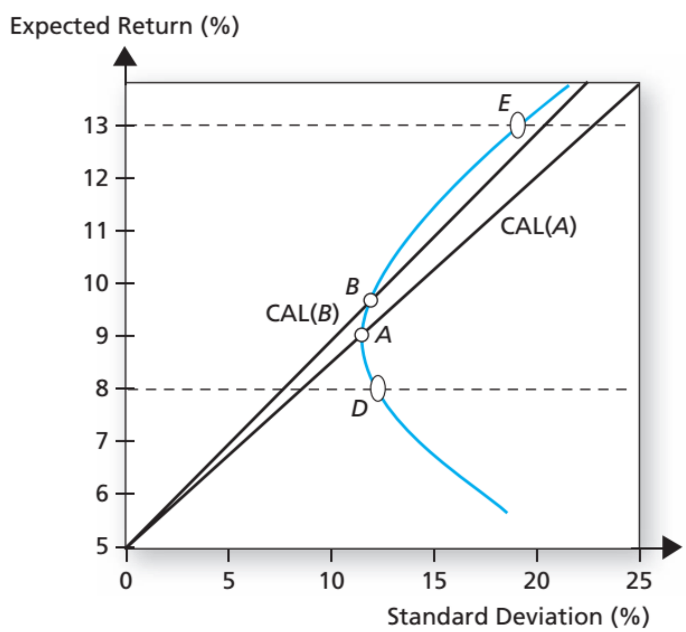
Suppose now an investor wishes to select the optimal portfolio from the opportunity set. The best portfolio will depend on risk aversion. Portfolios to the northeast provide higher rates of return but impose greater risk. The best trade-off among these choices is a matter of personal preference. Investors with greater risk aversion will prefer portfolios to the southwest, with lower expected return but lower risk.

3) Asset allocation with stocks, bonds, and bills

We have seen the capital allocation decision, the choice of how much of the portfolio to leave in debt funds versus in equity funds. Now, we have taken a further step, specifying that the risky portfolio comprises a stock and a bond fund. We still need to show how investors can decide on the proportion of their risky portfolios to allocate to the stock versus the bond market. This is an asset allocation decision.

- The optimal risky portfolio with two risky assets and a risk-free asset

  We start with a graphical solution. The blue curve in the following graph shows the opportunity set (efficient frontier) based on the properties of the bond and stock funds, using the data from previous table (suppose that $\rho = 0.3$.)



Then, what if our risky assets are still confined to the bond and stock funds, but now we can also invest in risk-free T-bills yielding 5%? In other words, we can invest in risk-free T-bills and a portfolio containing both the bond funds and stock funds. In that case, the new portfolio opportunity set becomes black line, which is linear combination of risk-free T-bills and a portfolio that lies in blue curve. Those black lines are called Capital Asset Lines (CALs).

Two possible CALs are drawn from the risk-free rate ($r_f = 5\%$) to two feasible portfolios. The first possible CAL is drawn through the minimum variance portfolio A, which is invested, for example, 82% in bonds and 18% in stocks. Then, the portfolio A's expected return is 8.9%, and its standard deviation is 11.45%. With a T-bill rate of 5%, the **reward-to-volatility (Sharpe) ratio**, which is the slope of the CAL combining T-bills and the minimum variance portfolio, is

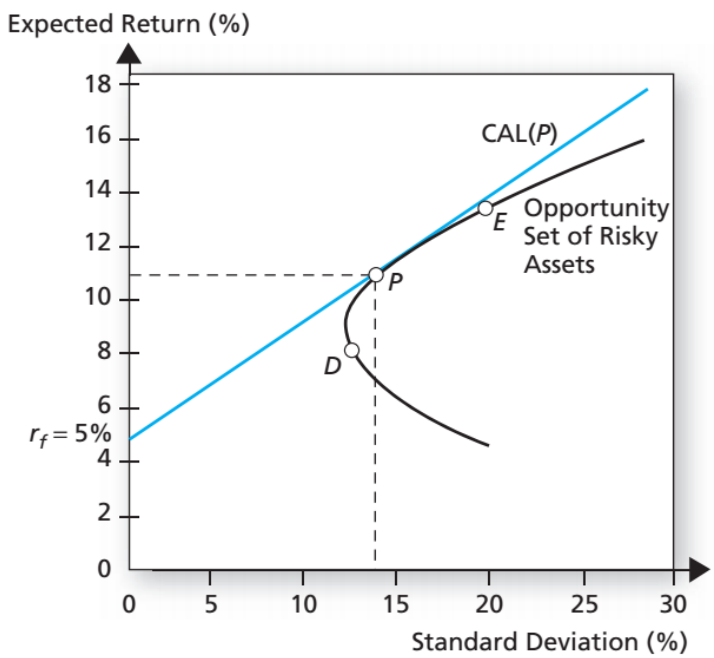
$$S_A = \frac{E(r_A) - r_f}{\sigma_A} = \frac{0.089 - 0.05}{0.1145} = 0.34$$

Now, consider the CAL that uses portfolio B instead of A. Let us assume that portfolio B invests 70% in bonds and 30% in stocks, and its expected return and standard deviation are 9.5% and 11.70%, respectively. Thus, the **reward-to-volatility (Sharpe) ratio** on the CAL that is supported by portfolio B is

$$S_A = \frac{0.095 - 0.05}{0.117} = 0.38$$

which is higher than the **reward-to-volatility (Sharpe) ratio** of the CAL that we obtained using the minimum variance portfolio and T-bills. Hence, portfolio B dominates A.

But why stop at portfolio B? We can continue to ratchet the CAL upward until it ultimately reaches the point of tangency with the investment opportunity set. This must yield the CAL with the highest feasible reward-to-volatility ratio. Therefore, the tangency portfolio, labeled P, is the optimal risky portfolio to mix with T-bills. We can read the expected return and standard deviation of portfolio P from the following graph.



The objective is to find the weights w_D and w_E that result in the highest slope of the CAL (i.e., the weights that result in the risky portfolio with the highest **reward-to-volatility ratio**). Therefore, the objective is to maximize the slope of the CAL for any possible portfolio, P. Thus, our *objective function* is the slope (equivalently, the **Sharpe ratio**) S_p:

$$S_p = \frac{E(r_p) - r_f}{\sigma_p}$$

When we maximize the objective function, S_p, we have to satisfy the constraint that the portfolio weights sum to 1.0 (100%), that is, $w_D + w_E = 1$. Therefore, we solve an optimization problem formally written as

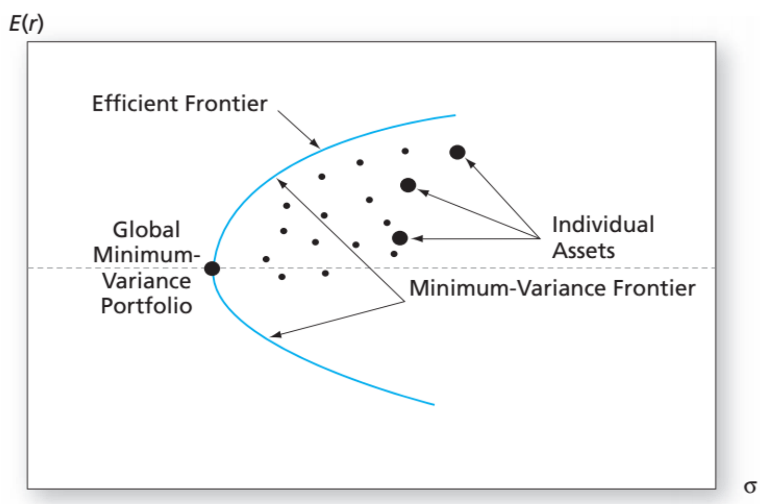
$$\underset{w_i}{\text{Max}}\, S_p = \frac{E(r_p) - r_f}{\sigma_p}$$

subject to $\sum w_i = 1$. This is a maximization problem that can be solved using standard tools of calculus.

4) The Markowitz portfolio selection model

We can generalize the portfolio construction problem to the case of many risky securities and a risk-free asset. As in the two risky assets example, the problem has three parts. First, we identify the risk–return combinations available from the set of risky assets. Next, we identify the optimal portfolio of risky assets by finding the portfolio weights that result in the steepest CAL. Finally, we choose an appropriate complete portfolio by mixing the risk-free asset with the optimal risky portfolio. Before describing the process in detail, let us first present an overview.

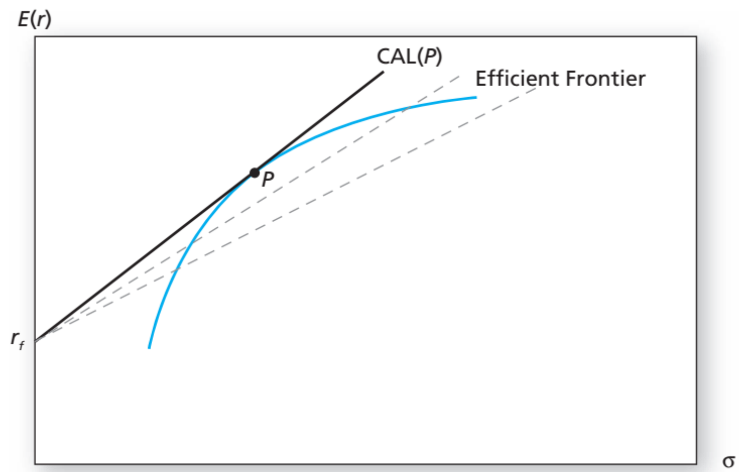
The first step is to determine the risk–return opportunities available to the investor. These are summarized by the **minimum-variance frontier** of risky assets. This frontier is a graph of the lowest possible variance that can be attained for a given portfolio expected return. Given the input data for expected returns, variances, and covariances, we can calculate the minimum-variance portfolio for any targeted expected return. The plot of these expected return–standard deviation pairs is presented in the following graph.



Notice that all the individual assets lie to the right inside the frontier, at least when we allow short sales in the construction of risky portfolios. This tells us that risky portfolios comprising only a single asset are inefficient. Diversifying investments leads to portfolios with higher expected returns and lower standard deviations.

All the portfolios that lie on the minimum-variance frontier from the global minimum-variance portfolio and upward provide the best risk–return combinations and thus are candidates for the optimal portfolio. The part of the frontier that lies above the global minimum-variance portfolio, therefore, is called the **efficient frontier of risky assets.** For any portfolio on the lower portion of the minimum-variance frontier, there is a portfolio with the same standard deviation and a greater expected return positioned directly above it. Hence the bottom part of the minimum-variance frontier is inefficient.

The second part of the optimization plan involves the risk-free asset. As before, we search for the capital allocation line with the highest reward-to-volatility ratio (that is, the steepest slope) as shown in the following graph.



The CAL that is supported by the optimal portfolio, *P*, is tangent to the efficient frontier. This CAL dominates all alternative feasible lines (the broken lines that are drawn through the frontier). Portfolio *P*, therefore, is the optimal risky portfolio.

Finally, in the last part of the problem the individual investor chooses the appropriate mix between the optimal risky portfolio *P* and T-bills. Now let us consider each part of the portfolio construction problem in more detail. In the first part of the problem, risk–return analysis, the portfolio manager needs as inputs a set of estimates for the expected returns of each security and a set of estimates for the covariance matrix.

The portfolio manager is now armed with the n estimates of $E(r_i)$ and $n \times n$ estimates of the covariance matrix in which the n diagonal elements are estimates of the variances, σ_i^2, and the $n^2 - n = n(n - 1)$ off-diagonal elements are the estimates of the covariances between each pair of asset returns.

Once these estimates are compiled, the expected return and variance of any risky portfolio with weights in each security, w_i, can be calculated from the bordered covariance matrix or, equivalently, from the following formulas:

$$E(r_p) = \sum_{i=1}^{n} w_i E(r_i)$$

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov(r_i, r_j)$$

The idea of diversification is age-old. The phrase "don't put all your eggs in one basket" exited long before modern finance theory. It was not until 1952, however, that Harry Markowitz published a formal model of portfolio selection embodying diversification principles, thereby paving the way for his 1990 Nobel Prize in Economics. His model is precisely step one of portfolio management: the identification of the efficient set of portfolio, or the efficient frontier of risky assets.

The principal idea behind the frontier set of risky portfolios is that, for any risk level, we are interested only in that portfolio with the highest expected return. Alternatively, the frontier is the set of portfolios that minimizes the variance for any target expected return.

2. Using R: Mean-Variance efficient portfolio

1) Introduction

In this section, we will use monthly returns of 30 DJIA stocks from Jan. 1991 to Dec. 2015. The DJIA (Dow Jones Industrial Average) is a stock market index that indicates the value of 30 large, publicly owned companies based in the United States. You can download the data file, which is DJIA.cvs, from our course site on Moodle. In my case, the file is located in "C:/users/skim/Google Drive/LSUS/2019 Fall/FIN 740/Module 4/Chapter 6/DJIA.csv". Thus, I created "DJIA" by reading DJIA.csv file.

```
# Create "DJIA" by reading DJIA.csv file
DJIA <- read.csv("C:/users/skim/Google Drive/LSUS/2019 Fall/FIN 740/Module 4/Chapter
6/DJIA.csv")

# Check data type
class(DJIA)
[1] "data.frame"
```

As you can see, this file is data frame. Then, let us convert data frame to xts for our further analysis.

```
# Convert to xts and check data type
returns <- as.xts(DJIA[, -1], order.by = as.Date(DJIA$Date, "%m/%d/%Y"))
class(returns)
```

```
[1] "xts" "zoo"
```

Now, we are ready to go.

First of all, let us briefly review **"returns"**, which contains 300 rows and 30 columns.

```
# Dimensions of returns
dim(returns)
300  30
```

```
# Review the first 6 data
head(returns)
                     AA        AAPL         AXP          BA         BAC          CAT     CVX          DD
1991-01-31  0.128181899  0.29069717  0.10302951  0.08815424  0.22950841  0.065478488 -0.0172 -0.01360543
1991-02-28 -0.003884696  0.03370538  0.05494594 -0.01808654  0.03999988  0.097989291  0.0661  0.04984821
1991-03-31  0.021443165  0.18777315  0.20312426 -0.02590673  0.19835796 -0.123568514  0.0233 -0.01328905
1991-04-30  0.037425309 -0.19117668 -0.12724754 -0.02659574  0.06498192  0.001183947  0.0158  0.12121216
1991-05-31  0.053703048 -0.14326469  0.02500053  0.07676566  0.14237304  0.086614898 -0.0452  0.14639805
1991-06-30 -0.050966017 -0.11702172 -0.11308280 -0.06632650 -0.14381507 -0.045892790 -0.0474 -0.02910058
                    DIS          GE         HD         HPQ        INTC         IBM         JNJ        JPM
1991-01-31  0.068125527  0.11546758 0.11974042  0.21960781  0.18831762  0.12168142  0.06620209 0.22093092
1991-02-28  0.139722408  0.07036994 0.12716859  0.19935467  0.04371621  0.02555987  0.07811810 0.25714325
1991-03-31 -0.035461032  0.02389573 0.10833008  0.07511003 -0.02094210 -0.11553398  0.16742873 0.06807352
1991-04-30 -0.031146064  0.01615821 0.06481483  0.02249901  0.05347219 -0.09549945 -0.01434200 0.17985476
1991-05-31  0.007600458  0.09920135 0.14565206  0.06112591  0.13198281  0.04251293 -0.03683502 0.04268366
1991-06-30 -0.038793029 -0.04207068 0.02751452 -0.06230900 -0.16592306 -0.08480569 -0.07862039 0.01041599
                    KO         MCD      MMM         MRK        MSFT          NKE         PFE           PG
1991-01-31  0.04838711 -0.02145933 -0.01020  0.01808041  0.30398741  0.186340258  0.11300425 -0.079400864
1991-02-28  0.07435965  0.10964919  0.05170  0.11612002  0.05732140  0.005233496  0.17587052  0.025236507
1991-03-31  0.04057835  0.10169560  0.00000  0.03873649  0.02289441 -0.038730634  0.01904693  0.047691906
1991-04-30 -0.02764982 -0.03597123  0.00706  0.02132693 -0.06713910  0.029811491  0.01869271 -0.009020566
1991-05-31  0.08530821  0.04756965  0.07830  0.10324855  0.10858885 -0.163156794  0.08651042  0.014902181
1991-06-30 -0.04385305 -0.06071428 -0.01710 -0.01850996 -0.06890977 -0.090778627 -0.05732508 -0.091041705
               TRV          UTX           VZ      WMT          XOM           T
1991-01-31  0.0159 -0.007833519 -0.085216548  0.09090 -0.002415537 -0.05513746
1991-02-28  0.0392  0.057280320  0.010309155  0.07200  0.081941026  0.02870830
1991-03-31  0.0610 -0.025125772  0.048469330  0.09670  0.061224460  0.04418589
1991-04-30  0.0126 -0.051546412 -0.014740147  0.04520  0.017094051 -0.03675783
1991-05-31 -0.0816  0.029582556 -0.057500172  0.05860 -0.009739177 -0.03044493
1991-06-30 -0.0208 -0.050667412  0.002652633 -0.00192 -0.002145979  0.03381642
```

If you want to compute means column by column, you can use **colMeans(...)** function. Similarly, if you want to compute means per row, you can use **rowMeans(...)** function.

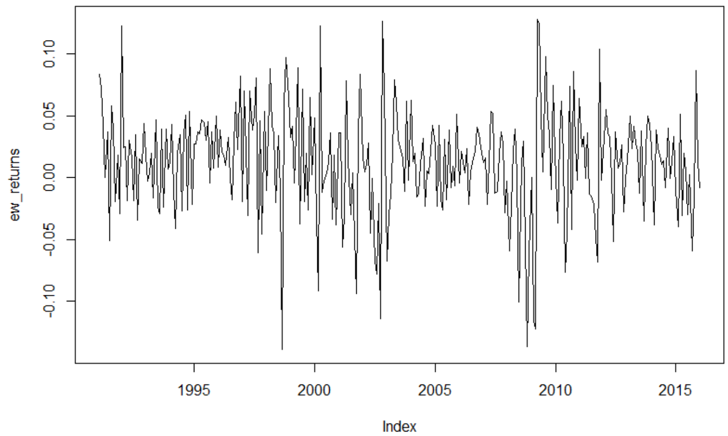
```
# Create a vector of row means
ew_returns <- rowMeans(returns)
```

So, **"ew_returns"** contains means (or averages) for each row. Those means indicate averages on monthly returns for 30 companies. The **"ew_returns"** is numeric vector (you can check it out using **class(...)** function). We convert it to xts as well.

```
# Convert numeric to xts and check data type
ew_returns <- as.xts(ew_returns, order.by = time(returns))
class(ew_returns)
[1] "xts" "zoo"
```

Plot **"ew_returns"**

```
# Plot ew_returns
plot.zoo(ew_returns)
```



2) Finding mean-variance efficient portfolio

The mean-variance efficient portfolio is the portfolio with expected return greater than any other with the same or lesser risk, and lesser risk than any other with the same or greater return. It can be obtained as the solution of minimizing portfolio standard deviation (or variance) under the constraint the portfolio expected return equals a target return. We skip mathematical representation here as it is a bit complex. In **R**, we can easily obtain the solutions, which are the mean-variance efficient portfolio weights under the constraint that the portfolio return equals a target return, using **portfolio.optim(...)** function in the package **tseries**.

```
# Install and load package tseries
install.packages("tseries")
library(tseries)

# Find mean-variance efficient portfolio with target return 5%
eff_port <- portfolio.optim(returns, pm = 0.05, shorts = TRUE)
eff_port$pw
 [1] -1.04665389  0.46143564  0.06760577 -0.11520593 -0.30512078  0.92693417 -0.79690426 -0.58695100
 [9] -0.34164584 -0.58648998  0.88579303 -0.09062575  0.34211091 -0.66861021  0.46730822  0.67272236
[17] -0.07036610  0.16125692  0.39906111  0.01963726  0.76105017  0.65614901 -0.05089402 -0.14256970
[25]  0.05176074  0.65272904 -0.75686800 -0.80108527  0.54149382  0.29294255
```

These are the weights for each company in mean-variance efficient portfolio with target return 5%. The **shorts = TRUE** makes it possible for the weights to be negative. The **"eff_port"** contains four components: 1) **$pw**: the portfolio weights, 2) **$px**: the expected portfolio returns of each row calculated by the portfolio weights **$pw**, 3) **$pm**: the expected return portfolio, and 4) **$ps**: standard deviation of the portfolio returns.

Let us look at the standard deviation of the portfolio.

```
# Standard deviation of mean-variance efficient portfolio with target return 5%
eff_port$ps
[1] 0.1661282
```

3. Using R: Efficient frontier

1) Finding efficient frontier

One of methods to find efficient frontier using **R** is as follows:

First, define the grid of target returns.

Second, find the portfolios that have expected returns equal to the target returns on the grid with the lowest possible standard deviation (or variance).

For the first step, we will use a **for(…)** loop function to compute portfolio weights and standard deviations corresponding to the target returns, which are portfolio expected returns, on the grid.

```
# Calculate mean returns for each stock (means of each column)
mu <- colMeans(returns)
```

We create a sequence of length 60 that begins 0.005 and ends at a higher value than **mu**.

```
# Create a grid of target returns
grid <- seq(0.005, 0.033, length.out = 60)
```

We create empty vectors for **$pm** (target return) and **$psd** (standard deviation) with the same length as grid we created above.

```
# Create empty vectors to store means and standard deviations
vector_pm <- rep(NA, length(grid))
vector_psd <- rep(NA, length(grid))
```

We create empty matrix with 60 rows and 30 columns since we have a sequence of length 60 in the grid and 30 stocks in your portfolio.

```
# Create an empty matrix to store weights
eff_weights <- matrix(NA, 60, 30)
```

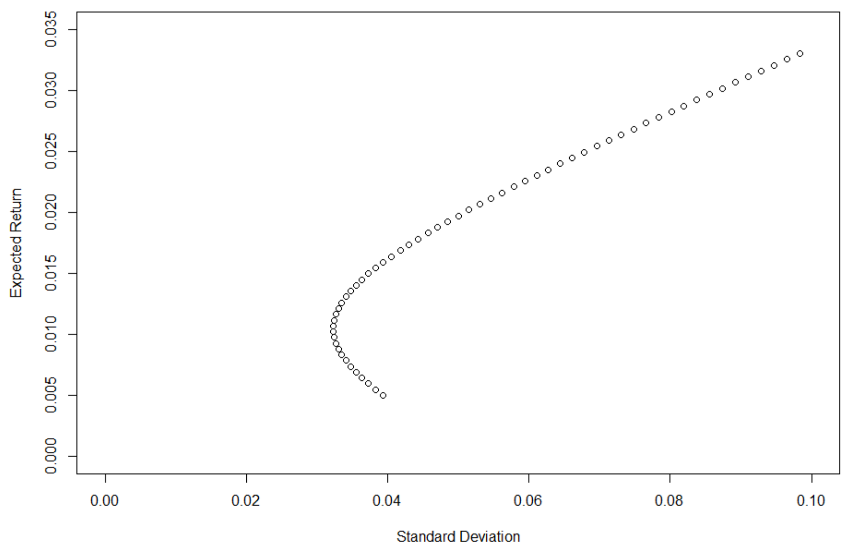
We create for loop to start on the first value of grid and to end on the last.

```
# Create for loop to find efficient frontier
for (i in 1 : length(grid)) {
  eff.port <- portfolio.optim(returns, pm = grid[i], shorts =TRUE)
  vector_pm[i] <- eff.port$pm
  vector_psd[i] <- eff.port$ps
  eff_weights[i, ] <- eff.port$pw
}
```

With each iteration, the loop fills **vector_pm**, **vector_psd**, and **eff_weights** from corresponding values on **eff.port**.

Now, let us plot the efficient frontier.

```
# Plot efficient frontier
# Plot efficient frontier
plot(vector_psd, vector_pm, xlab = 'Standard Deviation', ylab = 'Expected Return', xlim = c(0, 0.1), ylim = c(0, 0.035))
```



2) Finding Capital Asset Line (CAL)

Suppose that risk-free T-bill rate is 0.5%. Now, the investors can invest in both risk-free T-bills and risky portfolio on efficient frontier we have obtained above. First of all, we calculate Sharpe ratio.

```
# Calculate the Sharpe ratio
sharpe <- (vector_pm - 0.005) / vector_psd

# Find maximum Sharpe ratio
max_sharpe <- max(sharpe)

# Combine three vector
psd_pm_sharpe <- cbind(vector_psd, vector_pm, sharpe)

# Find tangency point
tan <- psd_pm_sharpe[sharpe == max_sharpe]
points(tan[1], tan[2], col = "red", lwd = 5)
abline(a = 0.005, b = max_sharpe, col = "blue", lwd = 2)
```

Then, we plot CAL.