Problems Abstract Algebra
Second List

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1 *Nakayama's lemma*. Let M be a finitely generated $A-$module and I an ideal of A contained in the Jacobson radical. Prove:

$$IM = M \Rightarrow M = 0$$

We suppose $M \neq 0$. Let $x_1, x_2, \dots, x_n$ be a minimal set of generators of the module M. Because $M = IM$ we can express the element $x_1 = a_1x_1 + a_2x_2 + \cdots + a_nx_n$, where $a_i \in I$. Then

$$(a_1 - 1)x_1 + a_2x_2 + \cdots + a_nx_n = 0 \Rightarrow \begin{cases} a_1 - 1 = 0 \\ \vdots \\ a_n = 0 \end{cases}$$

But if $a_1 = 1 \in I$, that means $I = (1) = A$, which cannot be contained in the Jacobson radical.

(rehacer)

2 Under the previous hypothesis, prove:

1. $A/I \otimes_A M = 0 \Rightarrow M = 0$
2. If $N \subseteq M$ is a submodule, $M = IM + N \Rightarrow M = N$
3. If $f : N \to M$ is a homomorphism, $\overline{f} : N/IN \to M/IM$ surjective $\Rightarrow f$ surjective

4 Let (diagram) be a short exact sequence of $A - modules$. Prove that if M' and M'' are finitely generated, then M is finitely generated.

We start by fixing the set of generators of M' as $x_1, \dots, x_n$ and of M'' as $z_1, \dots, z_m$.

Since g is surjective, we can find elements $y_1, \dots, y_m$ such that $g(y_i) = z_i$. Now we select an arbitrary element $y \in M$. Then we have

$$g(y) = b_1z_1 + \cdots + b_mz_m = g(b_1y_1) + \cdots + g(b_my_m) \Rightarrow g(y - \sum b_iy_i) = 0 \Rightarrow y - \sum b_iy_i \in \ker(g)$$

for some $b_i \in A$. By exactness of the sequence we have $y - \sum b_iy_i \in \mathrm{Im}(f)$, so

$$y - \sum b_iy_i = f(\sum a_ix_i) = \sum a_if(x_i) \Rightarrow y = \sum a_if(x_i) + \sum b_iy_i$$

for some $a_i \in A$. Thus, a set of generators of M is $f(x_1), \dots, f(x_n), y_1, \dots, y_m$

5 Prove that $\mathbb{Z}[\sqrt{d}]$ is a Noetherian ring

This is equivalent to prove that $M = \mathbb{Z}[\sqrt{d}]$ is a Noetherian module. Since every submodule of M is finitely generated (by 1 and $\sqrt{d}$), then the module is Noetherian.

6 Prove that the ring $\mathbb{Z}[2T, 2T^2, 2T^3, \ldots] \subseteq \mathbb{Z}[T]$ is not Noetherian

We search for an ascending chain of ideals $I_1 \subseteq I_2 \subseteq \ldots$ in which for every I_i we have $x_i \in I_i$ but $x_i \notin I_{i-1}$. This chain can be $I_i = (2T, 2T^2, \ldots, 2T^{i-1}, 2T^i + 2T^{i+1} + \ldots)$. Notice that the containments are obvious and $x_i = 2T^{i-1} \in I_i$, but not in I_{i-1}.