Problems Abstract Algebra
First List

Abel Doñate Muñoz

> **1** Let f be a morphism in a category $\mathcal{C}$. Prove the following:
> (a) If f an isomorphism then f is a monomorphism and an epimorphism.
> (b) The inclusion of $\mathbb{Z}$ in $\mathbb{Q}$ is a monomorphism and an epimorphism in the category of rings but not an isomorphism.

We begin with the proof of (a). Since $f : A \to B$ is an isomorphism, that means there exist $g : B \to A$ such that both $g \circ f = Id_A$ and $f \circ g = Id_B$.

Let h, k morphisms of the category that fulfill $f \circ h = f \circ k$. Then by composing from the left with g we have

$$g \circ f \circ h = g \circ f \circ k \Rightarrow Id_A \circ h = Id_A \circ k \Rightarrow h = k$$

so we conclude f is a monomorphism.

Let h, k morphisms of the category that fulfill $h \circ f = k \circ f$. Then by composing from the right with g we have

$$h \circ f \circ g = k \circ f \circ g \Rightarrow h \circ Id_B = k \circ Id_B \Rightarrow h = k$$

so we conclude f is an epimorphism.

We move to the proof of (b). Let $i : \mathbb{Z} \to \mathbb{Q}$ the inclusion in $\mathbb{Q}$ ($i : n \mapsto n$). Now let $h, k \in \text{Hom}_{rings}(A, \mathbb{Z})$ such that $i \circ h = i \circ k$. It is clear that, since $i(n) = n \; \forall n \in \mathbb{Z}$, then $h(a) = k(a) \; \forall a \in A$, concluding $h = k$ and i monomorphism.

Now let $h, k \in \text{Hom}_{rings}(\mathbb{Z}, A)$ such that $h \circ i = k \circ i$. It is clear that, since $i(n) = n \; \forall n \in \mathbb{Z}$, then $h(i(a)) = k(i(a)) \Rightarrow h(a) = k(a) \; \forall a \in A$, concluding $h = k$ and i epimorphism.

Suppose i is an isomorphism. Thus, it must exists $g : \mathbb{Q} \to \mathbb{Z}$ such that $i \circ g = Id_{\mathbb{Q}}$ and $g \circ i = Id_{\mathbb{Z}}$. Let $a \in \mathbb{Z}$ such that $g(\frac{1}{2}) = a$. Then $i \circ g(\frac{1}{2}) = i(a) = a \neq \frac{1}{2}$, so $i \circ g \neq Id_{\mathbb{Q}}$, concluding f is not an isomorphism.

> **5** *Pushouts in the category of abelian groups:* Let A and B be abelian groups together with homomorphisms $f : S \to A$ and $g : S \to B$. Prove that
>
> $$A \sqcup_S B = \frac{A \oplus B}{W}$$
>
> where W is the subgroup generated by $(f(s), -g(s))$ with $s \in S$.

Let $U = \frac{A \oplus B}{W}$. We will show that the pushout $A \sqcup_S B$ is, in fact, U.