

Problems Abstract Algebra

First List

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1 Let f be a morphism in a category \mathcal{C} . Prove the following:

- (a) If f an isomorphism then f is a monomorphism and an epimorphism.
- (b) The inclusion of \mathbb{Z} in \mathbb{Q} is a monomorphism and an epimorphism in the category of rings but not an isomorphism.

We begin with the proof of (a). Since $f : A \rightarrow B$ is an isomorphism, that means there exist $g : B \rightarrow A$ such that both $g \circ f = Id_A$ and $f \circ g = Id_B$.

Let h, k morphisms of the category that fulfill $f \circ h = f \circ k$. Then by composing from the left with g we have

$$g \circ f \circ h = g \circ f \circ k \Rightarrow Id_A \circ h = Id_A \circ k \Rightarrow h = k$$

so we conclude f is a monomorphism.

Let h, k morphisms of the category that fulfill $h \circ f = k \circ f$. Then by composing from the right with g we have

$$h \circ f \circ g = k \circ f \circ g \Rightarrow h \circ Id_B = k \circ Id_B \Rightarrow h = k$$

so we conclude f is an epimorphism.

We move to the proof of (b). Let $i : \mathbb{Z} \rightarrow \mathbb{Q}$ the inclusion in \mathbb{Q} ($i : n \mapsto n$). Now let $h, k \in \text{Hom}_{\text{rings}}(A, \mathbb{Z})$ such that $i \circ h = i \circ k$. It is clear that, since $i(n) = n \forall n \in \mathbb{Z}$, then $h(a) = k(a) \forall a \in A$, concluding $h = k$ and i monomorphism.

Now let $h, k \in \text{Hom}_{\text{rings}}(\mathbb{Z}, A)$ such that $h \circ i = k \circ i$. It is clear that, since $i(n) = n \forall n \in \mathbb{Z}$, then $h(i(a)) = k(i(a)) \Rightarrow h(a) = k(a) \forall a \in A$, concluding $h = k$ and i epimorphism.

Suppose i is an isomorphism. Thus, it must exists $g : \mathbb{Q} \rightarrow \mathbb{Z}$ such that $i \circ g = Id_{\mathbb{Q}}$ and $g \circ i = Id_{\mathbb{Z}}$. Let $a \in \mathbb{Z}$ such that $g(\frac{1}{2}) = a$. Then $i \circ g(\frac{1}{2}) = i(a) = a \neq \frac{1}{2}$, so $i \circ g \neq Id_{\mathbb{Q}}$, concluding f is not an isomorphism.

2 Show that in the category of finite dimensional vector spaces over a field \mathbb{K} we have a natural equivalence of functors between the identity Id and the bidual $(-)^{**}$

We consider the map $\tau : X \rightarrow X^{**}$ that sends $v \mapsto \text{eval}(v)$, where $\text{eval}(v) \in \text{Hom}(\text{Hom}(X, \mathbb{R}), \mathbb{R})$ is the evaluation in v morphism.

$$\begin{array}{ccc} X & \xrightarrow{\tau} & X^{**} \\ f \downarrow & & \downarrow G(f) \\ Y & \xrightarrow{\tau} & Y^* \end{array}$$

Now we have to see that the diagram commutes. We will see that the construction $G(f) : \text{eval}(v) \mapsto \text{eval}(f(v))$ works. If we pick $v \in X$, then

$$G(f)(\tau(v)) = \text{eval}(f(v)) = \text{eval}(w) = \tau(f(v)) \Rightarrow \tau \circ f = G(f) \circ \tau$$

Proving the commutativity of the diagram.

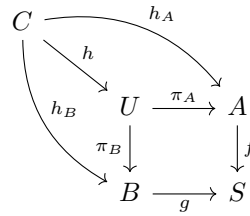
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3 Show that two categories \mathcal{B} and \mathcal{C} are naturally equivalent if and only if there exists a fully faithful and essentially surjective covariant functor $F : \mathcal{B} \rightarrow \mathcal{C}$.

4 *Pullbacks in the category of abelian groups:* Let A and B be abelian groups together with homomorphisms $f : A \rightarrow S$ and $g : B \rightarrow S$. Prove that

$$A \times_S B = \{(a, b) \in A \times B \mid f(a) = g(b)\}$$

Let $U = \{(a, b) \in A \times B \mid f(a) = g(b)\}$. We will show that the pullback $A \times_S B$ is, in fact, U . We construct the following diagram:



We first construct the morphisms π_A and π_B that make the square commute. Those are

$$\begin{cases} \pi_A((a, b)) = a \\ \pi_B((a, b)) = b \end{cases} \Rightarrow f \circ \pi_A((a, b)) = f(a) = g(b) = g \circ \pi_B((a, b)) \quad \forall (a, b) \in U$$

thus, the square commutes.

Now we construct h from h_A and h_B . Note that, for the two triangular diagrams to commute, the h must fulfill:

$$\begin{cases} \pi_A(h(c)) = h_A(c) \\ \pi_B(h(c)) = h_B(c) \end{cases} \quad \forall c \in C \Rightarrow h = (h_A, h_B)$$

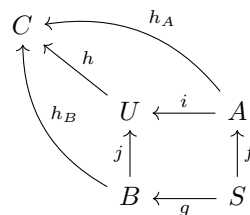
and the h is unique, concluding the proof.

5 *Pushouts in the category of abelian groups:* Let A and B be abelian groups together with homomorphisms $f : S \rightarrow A$ and $g : S \rightarrow B$. Prove that

$$A \sqcup_S B = \frac{A \oplus B}{W}$$

where W is the subgroup generated by $(f(s), -g(s))$ with $s \in S$.

Let $U = \frac{A \oplus B}{W}$. We will show that the pushout $A \sqcup_S B$ is, in fact, U . We construct the following diagram:



We first construct the morphisms i, j such that the square diagram commutes. We propose

$$i(a) = [(a, 0)] \quad j(b) = [(0, b)]$$

and we check for commutativity for all $s \in S$

$$\begin{cases} i \circ f(s) = [(f(s), 0)] \\ j \circ g(s) = [(0, g(s))] \end{cases} \quad \text{but } [(0, g(s))] = [(0, g(s)) + (f(s), -g(s))] = [(f(s), 0)] \Rightarrow i \circ f = j \circ g \quad \forall s \in S$$

so we have proved the square commutes.

Now we construct the morphism h through h_A and h_B . We construct it in the following way:

$$\begin{cases} h([(a, 0)]) = h_A(a) \\ h([(0, b)]) = h_B(b) \end{cases} \Rightarrow h([(a, b)]) = h([(a, 0)] + [(0, b)]) = h([(a, 0)]) + h([(0, b)]) = h_A(a) + h_B(b)$$

and clearly this is well defined and unique as morphism in the category of abelian groups, so the other triangular diagrams commute as well and we conclude with the proof.

6 Inverse limits in the category of sets / groups / abelian groups / modules: Let $(\{A_i\}, \{f_{ji}\})$ be an inverse system over a preordered set I . Prove that

$$\varprojlim A_i = \{(a_i) \in \prod A_i \mid f_{ji}(a_j) = a_i \text{ } i \leq j\}$$

Let $U = \{(a_i) \in \prod A_i \mid f_{ji}(a_j) = a_i \text{ } i \leq j\}$. We will prove that this is, in fact, the inverse limit we are looking for through the following diagram:

$$\begin{array}{ccccc} & & f_i & \rightarrow & A_i \\ & \nearrow & & & \uparrow f_{ji} \\ B & \xrightarrow{h} & U & \xrightarrow{\lambda_i} & A_i \\ & \searrow & & & \downarrow \\ & & f_j & \rightarrow & A_j \end{array}$$

We can easily check the commutativity of the right part of the diagram because $f_{ji} \circ \lambda_j((a_k)) = f_{ji}(a_j) = a_i = \lambda_i((a_k))$.

We now prove that the morphism h is unique. h must be of the form $h(c) = (h_k(c))$, but, since the upper and lower part of the diagram must commute, we have $\lambda_i \circ h(c) = f_i(c) \Rightarrow h_i = f_i$, so the morphism we are looking for is $h = (h_k)$, and is unique.

7 Direct limits in the category of sets / groups / abelian groups / modules / rings with unit: Let $(\{A_i\}, \{f_{ij}\})$ be a direct system over a directed set I . Prove that

$$\varinjlim A_i = \bigsqcup A_i / \sim$$

where $a_i \sim a_j \iff f_{il}(a_i) = f_{jl}(a_j)$ for $i, j \leq l$

Let $U = \bigsqcup A_i / \sim$. We will prove that this is, in fact, the direct limit we are looking for through the following diagram:

$$\begin{array}{ccccc} & & f_i & \rightarrow & A_i \\ & \nwarrow & & & \uparrow f_{ji} \\ B & \xleftarrow{h} & U & \xleftarrow{\lambda_i} & A_i \\ & \swarrow & & & \downarrow \\ & & f_j & \rightarrow & A_j \end{array}$$

We have to check first that the right triangle of the diagram commutes. Since $\lambda_i : a_i \mapsto [a_i]$, we have that

$$\lambda_j \circ f_{ij}(a_i) = \lambda_j(a_j) = [a_j] = [a_i] = \lambda_i(a_i) \Rightarrow \lambda_j \circ f_{ij} = \lambda_i$$

where we have set $l = j$, so $f_{ij}(a_i) = a_j = f_{jj}(a_j) \Rightarrow a_i \sim a_j \Rightarrow [a_j] = [a_i]$.

Now we check the commutativity of the $h : [a_i] \mapsto f_i(a_i)$. First we have to ensure that it is well defined. That is, if $a_i \sim a_j \Rightarrow f_i(a_i) = f_j(a_j)$. We suppose by symmetry that $j > i$, then by definition $a_i \sim a_j \iff f_{il}(a_i) = f_{jl}(a_j)$ for $i, j \leq l$. Setting $l = j$ we get $f_{ij}(a_i) = f_{jj}(a_j) = a_j$. Now, for the big diagram $f_j(f_{ij}(a_i)) = f_i(a_i) \Rightarrow f_j(a_j) = f_i(a_i)$ as desired.

Note that $h([a_i]) = f_i(a_i)$ is the only possible choice we could have done in order to assure the commutativity of the upper diagram.

8 Show that in an abelian category we have:

- (a) f is a monomorphism $\iff \ker(f) = 0$
- (b) f is an epimorphism $\iff \text{Coker}(f) = 0$
- (c) A monomorphism is the kernel of its cokernel
- (d) An epimorphism is the cokernel of its kernel
- (e) Every morphism can be expressed as the composition of an epimorphism and a monomorphism
- (f) f is an isomorphism $\iff f$ is an epimorphism and a monomorphism

We draw our commutative diagram. For being abelian the morphism \bar{g} must be unique for all g and the morphism \bar{h} must be unique for all h . Furthermore \bar{f} must be an isomorphism for all f .

$$\begin{array}{ccccccc} \ker & \xrightarrow{i} & A & \xrightarrow{f} & B & \xrightarrow{\pi} & \text{Coker} \\ \bar{g} \uparrow & & \downarrow \tau & & \uparrow j & & \downarrow \bar{h} \\ C & \xrightarrow{g} & \text{Coim} & \xrightarrow{\bar{f}} & \text{Im} & \xrightarrow{h} & D \end{array}$$

(a)

\Rightarrow If f is mono, that means $f \circ k = f \circ l \Rightarrow k = l$. Let $k = i, l = 0$, then $0 = f \circ i = f \circ 0 \Rightarrow i = 0$. Since there exists a unique morphism $\bar{g} : C \rightarrow \ker f$, then $\ker f$ is the unique terminal element of the abelian category \mathcal{A} .

(b)

\Rightarrow If f is epi, that means $k \circ f = l \circ f \Rightarrow k = l$. Let $k = \pi, l = 0$, then $0 = \pi \circ f = 0 \circ f \Rightarrow \pi = 0$. Since there exists a unique morphism $\bar{h} : \text{Coker } f \rightarrow D$, then $\text{Coker } f$ is the unique initial element of the abelian category \mathcal{A} .