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1 Tema 1: Señales

Types of signals

$$\Pi(t) = 1 \text{ in } t \in (-\frac{1}{2}, \frac{1}{2})$$
$$\Delta(t) = 1 - |t| \text{ in } |t| < 1$$
$$sinc(t) = \frac{\sin(\pi t)}{\pi t}$$

Types of systems

- Linear: $ax_1 + bx_2 \to ay_1 + ay_2$
- Invariant: $x(t-\Delta) \to y(t-\Delta)$
- Casual: depende de las anteriores
- Stable: $|x(t)| < M_x \Rightarrow |y(t)| < M_y$

Energy and power

$$E = \int |x(t)|^2 dt, \qquad P = \lim \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt$$

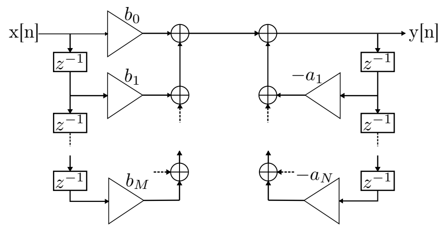
Linear time-invariant (LTI)

$$y(t) = x(t) * h(t) = \int x(\tau) h(t-\tau) d\tau = \sum x[i] h[n-i]$$

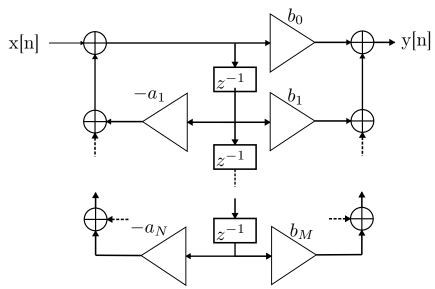
Differences equation LTI

$$y[n] = \sum_0^M b_j x[n-j] - \sum_1^N a_i y[n-i] \Rightarrow H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}$$

Forma directa 1



Forma directa 2 ($N = M$)



2 Tema 2: Transformadas

Transforms

$$x(t) \to X(\omega) = \int x(t) e^{-j\omega t} dt \qquad x(t) = \frac{1}{2\pi} \int X(\omega) e^{j\omega t}$$
$$x[n] \to X(z) = \sum x[n] z^{-n}$$

Fourier series

$$x(t) = \frac{a_0}{2} + \sum (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$$
$$= A_0 + \sum A_n \cos(n\omega_0 t + \varphi_n) = \sum c_n e^{jn\omega_0 t}$$
$$a_n = \frac{2}{T} \int_T x(t) \cos(n\omega_0 t) dt, b_n = \frac{2}{T} \int_T x(t) \sin(n\omega_0 t) dt$$
$$A_n = \sqrt{a_n^2 + b_n^2}, \varphi_n = \arctan\left(\frac{-b_n}{a_n}\right)$$
$$c_n = \frac{a_n - jb_n}{2}, \quad c_{-n} = c_n^*$$
$$P_m = \frac{1}{T} \int_T |x|^2 dt = \sum |c_n|^2 = A_0^2 + \sum A_n^2$$

LTI response to $x(t) = \sum A_n \cos(n\omega_0 t + \varphi_n)$

$$y(t) = \sum A_n |H(n\omega_0)| \cos(n\omega_0 t + \varphi + \angle H(n\omega_0))$$

Fourier transform properties

$$x(at) \to \frac{1}{|a|} X(\frac{\omega}{a}), \qquad x(t) \to\to 2\pi x(\omega)$$
$$x(t-\Delta) \to X(\omega) e^{-j\omega\Delta}, \qquad x(t) e^{j\omega_0 t} \to X(\omega - \omega_0)$$
$$\frac{dx(t)}{dt} \to j\omega X(\omega), \qquad \int_{-\infty}^t x(\tau) d\tau \to \frac{X(\omega)}{j\omega} + \pi X(0) \delta(\omega)$$
$$f(t) * g(t) \to F(\omega) G(\omega), \qquad f(t) g(t) \to \frac{1}{2\pi} F(\omega) * G(\omega)$$

Signal	**Fourier transform**
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\Pi(\frac{t}{T})$	$Tsinc(\frac{\omega T}{2\pi})$
$\Delta(\frac{t}{T})$	$Tsinc^2(\frac{\omega T}{2\pi})$
$\cos(\omega_0 t)$	$\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$
$\sin(\omega_0 t)$	$-j\pi\delta(\omega - \omega_0) + j\pi\delta(\omega + \omega_0)$
$sinc(\frac{\omega_0 t}{\pi})$	$\frac{\pi}{\omega_0} \Pi(\frac{\omega}{2\omega_0})$
$sign(t)$	$\frac{2}{j\omega}$
$e^{-at} u(t)$	$\frac{1}{a + j\omega}$
$\frac{1}{t}$	$-j\pi sign(\omega)$

Z-transform properties

$$x[n - n_0] \to z^{-n_0} X(z), \quad x[-n] \to X(\frac{1}{z})$$
$$a^n x[n] = X(\frac{z}{a}), \quad nx[n] \to -z \frac{dX(z)}{dz}$$

Signal	**Z-transform**
$\delta[n]$	1
$p_N[n]$	$\frac{1 - z^{-N}}{1 - z^{-1}}$
$u[n]$	$\frac{1}{1 - z^{-1}}$
$\cos(\Omega_0 n) u[n]$	$\frac{1 - \cos(\Omega_0) z^{-1}}{1 - 2\cos(\Omega_0) z^{-1} + z^{-2}}$
$\sin(\Omega_0 n) u[n]$	$\frac{\sin(\Omega_0) z^{-1}}{1 - 2\cos(\Omega_0) z^{-1} + z^{-2}}$

Compute inverse Z-transform

$$X(z) = \frac{d_0 + d_1 z^{-1} + \cdots d_M z^{-M}}{c_0 + c_1 z^{-1} + \cdots c_N z^{-N}}$$
$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^N \frac{A_k}{1 - p_k z^{-1}}$$
$$x[n] = \sum_{r=0}^{M-N} B_r \delta(n-r) + \sum_{k=1}^N A_k p_k^n u[n]$$

Connections

Serie: $H = H_1 H_2$ Parallel: $H = H_1 + H_2$

Estabilidad

- Estable si $p_k \in D\ \forall k$
- Marginalmente estable si $p_k \in \overline{D}\ \forall k$
- Inestable si $\exists k : p_k \notin \overline{D}$

DFT

$$X_N[k] = DFT_N[x[n]] = X(e^{j\Omega})|_{\Omega = k\frac{2\pi}{N}} =$$
$$= \sum_{n=0}^{N-1} x[n] e^{-j\Omega n}|_{\Omega = k\frac{2\pi}{N}} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$
$$x_N[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_N[k] e^{j\frac{2\pi}{N}kn}$$

Properties

$$x[n] e^{j\frac{2\pi}{N}k_0 n} \to \tilde{X}_N[k - k_0] p_N[k]$$
$$x[n] \cdot y[n] \to \frac{1}{N} X_N[k] \circ Y_N[k]$$
$$x[n] \circ y[n] \to X[n] Y[n]$$
$$\tilde{x}_N[n-m] \cdot p_N[n] \to X_N[k] e^{-j\frac{2\pi}{N}km}$$

DFT of $A\cos(2\pi\frac{k_0}{N})$:

$$X(k) = \begin{cases} \frac{AN}{2} & \text{si } k = k_0, N - k_0 \\ 0 & \text{otherwise} \end{cases}$$

3 Tema 3: Conversores A/D y D/A

$f_s = \frac{1}{T}$	frecuencia de sampleo (Hz)
$\omega_2 = \frac{2\pi}{T} = 2\pi f_s$	frecuencia de sampleo (rad/s)
ω_m	frecuencia máxima

Criterio de Nyquist $\omega_s > 2\omega_m$

Antialiasing filter $H(\omega) = \Pi(\frac{\omega}{\omega_s}),\ \Omega = \omega T$

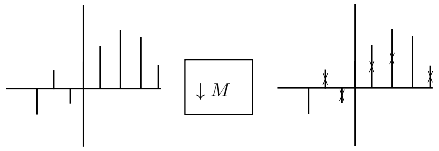
Tipos de interpolación

Linear	$h(t) = \Lambda(\frac{t}{T}),$
ZOH	$h(t) = \Pi(\frac{t - T/2}{T}),\ H(\omega) = Tsinc(\frac{\omega}{\omega_s}) e^{-j\omega\frac{T}{2}}$
Ideal	$h(t) = sinc\left(\frac{t}{T}\right),\ H(\omega) = \Pi(\frac{\omega}{\omega_s})$

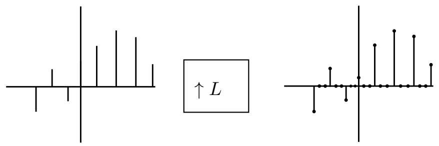
Transformada señal digital

$$p(t) = \sum \delta(t - nT) \Rightarrow P(\omega) = \frac{2\pi}{T} \sum \delta(\omega - \frac{2\pi}{T} k)$$
$$y(t) = x(t) p(t) \Rightarrow Y(\omega) = \frac{1}{T} \sum X(\omega - \frac{2\pi}{T} k)$$

Diezmado



Interpolacion



4 Tema 4: Random signals

Cumulative distrubution function (cdf) and probability density function (pdf)

$$F_X(x_1; t_1) = P(X(t_1) \leq x_1), \quad f_X(x_1; t_1) = \frac{\partial F_X(x_1; t_1)}{\partial x_1}$$

Definition (Mean).

$$m_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_X(x, t) dx$$

Definition (Auto-correlation).

$$r_X(t_1, t_2) = E[X(t_1) X^*(t_2)] = \int \int x_1 x_2^* f_X(x_1, x_2; t_1, t_2) dx_1 dx_2$$
$$\overline{r}_X(\tau) = \frac{1}{T} \int_0^T r_X(t + \tau, t) dt$$

Definition (Instantaneous power).

$$P_X(t) = E[|X(t)|^2] = r_X(t, t)$$

Definition (Auto-covariance).

$$c_X(t_1, t_2) = r_X(t_1, t_2) - m_X(t_1) m_X(t_2)$$

Definition (Variance).

$$\sigma_X^2(t) = c_X(t, t) = r_X(t, t) - m_X(t) m_X^*(t)$$

Definition (Cross-correlation). *Orthogonal* $r_{XY}=0$

$$r_{XY}(t_1,t_2)=E[X(t_1)Y^*(t_2)]=\int\int xy^* f_{XY}(x,y;t_1,t_2)dxdy$$

Definition (Cross-Covariance). *Uncorrelated* $c_{XY}=0$

$$c_{XY}(t_1,t_2)=r_{XY}(t_1,t_2)-m_X(t_1)m_Y^*(t_2)$$

Independence $\Rightarrow$ Uncorrelation

$$\begin{aligned}\text{Independence} &\iff f_{XY}(x,y)=f_X(x)f_Y(y)\\ \text{Uncorrelation} &\iff E[XY]=E[X]E[Y]\end{aligned}$$

Definition (Stationary). .

- *1st order:* $f_X(x;t)=f(x;t+\Delta)\ \forall t,\Delta$
- *2nd order:* $f_X(x_1,x_2;t_1,t_2)=f(x_1,x_2;t_1+\Delta,t_2+\Delta)\ \forall t_1,t_2,\Delta$
- *WSS* $m_X(t)=cte,\quad r_X(t_1,t_2)=r_X(\tau)$
- *jointly WSS* X,Y *WSS,* $\quad r_{XY}(t_1,t_2)=r_X(\tau)$
- *Cyclostationary* $m_x(t),r_X(t,t+\tau$ *periodic*

Properties

$$\begin{aligned}&r_X(\tau)=r_X^*(-\tau),\quad r_{XY}(\tau)=r_{XY}^*(-\tau)\\ &|r_{XY}(\tau)|\le\sqrt{r_X(0)r_Y(0)}\\ &r_{X+Y}=r_X(\tau)+r_Y(\tau)+r_{XY}(\tau)+r_{YX}(\tau)\end{aligned}$$

Definition (Power spectral density (PSD)).

$$S_X(\omega)=\lim_{T\to\infty}\frac{E[|X_T(\omega,S)|^2]}{2T}$$

Theorem (Wiener-Khinchin). $S_X(\omega)=F[\overline{r}_X(\tau)]$

Properties of PSD

$$\begin{aligned}&S_X(\omega)\ge0,\quad S_X(\omega)=S_X(-\omega)\\ &P_X=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_X(\omega)d\omega\end{aligned}$$

WSS process before filter $H(\omega)$

- Mean $m_Y=m_xH(0)$
- Cross-correlation $r_{YX}(\tau)=r_X(\tau)*h(\tau)$
- Auto-correlation $r_Y(\tau)=r_X(\tau)*h(\tau)*h^*(-\tau)$
- Spectral density $S_Y(\omega)=S_X(\omega)|H(\omega)|^2$

Definition (Auto-correlation matrix).

$$R_X=\begin{pmatrix}r_X[0] & r_X[1] & \cdots & r_X[L-1]\\ r_X^*[1] & r_X[0] & \cdots & r_X[L-2]\\ \vdots & \vdots & \ddots & \vdots\\ r_X^*[L-1] & r_X^*[L-2] & \cdots & r_X[0]\end{pmatrix}$$

Properties of R_X

- $T_X=Q\Lambda Q^H$
- $P_Y=h^HR_Xh=\sum\lambda_i|h^Hq_i|^2$

Definition (Gaussian random variable).

$$f_X(x)=\frac{1}{\sqrt{2\pi\sigma_X^2}}e^{-\frac{(x-m_x)^2}{2\sigma_X^2}}$$

$$f_X(x)=\frac{1}{\pi\sigma_X^2}e^{-\frac{|x-m_x|^2}{\sigma_X^2}}$$

$$f_X(x)=\frac{1}{(2\pi)^{L/2}|C_X|^{1/2}}e^{-\frac{1}{2}(x-\mu_X)^TC_X^{-1}(x-\mu_X)}$$

$$f_X(x)=\frac{1}{(\pi)^L|C_X|}e^{-(x-\mu_X)^HC_X^{-1}(x-\mu_X)}$$

5 Tema 5: Estimation theory

Definition (Bias). $b_{\hat\theta}=E[\hat\theta(x)-\theta]=\mu_{\hat\theta}-\theta$

Definition (Covariance).

$$C_{\hat\theta}=E[(\hat\theta(x)-\mu_{\hat\theta})(\hat\theta(x)-\mu_{\hat\theta})^H],\qquad \sigma_{\hat\theta_i}^2=(C_{\hat\theta})_{i,i}$$

Definition (Mean square error (MSE)).

$$M_{\hat\theta}=E[(\hat\theta(x)-\theta)(\hat\theta(x)-\theta)^H]=C_{\hat\theta}+b_{\hat\theta}b_{\hat\theta}^H$$

Definition (Minimum Variance Unbiased Estimator (MVUE)). $b_{\hat\theta}=0,\quad \sigma_{\hat\theta}^2|_{min}$

Definition (Sharpness). $S=-E\left[\frac{\partial^2\ln(f_\theta(x))}{\partial\theta^2}\right]$

Definition (Cramer-Rao Lower Bound (CRLB)).

$$\sigma_{\hat\theta}^2\ge\frac{1}{E\left[\left|\frac{\partial ln(f_\theta(x))}{\partial\theta^*}\right|^2\right]}=\frac{1}{-E\left[\frac{\partial^2 ln(f_\theta(x))}{\partial\theta^*\partial\theta}\right]}$$

Definition (Efficient estimator).

$$\hat{\theta(x)}-\theta=CRLB(\theta)\cdot\frac{\partial\ln(f_\theta(x))}{\partial\theta}$$

Definition (Fisher information matrix).

$$(J(\theta))_{i,j}=E\left[\frac{\partial^2\ln f_\theta(x)}{\partial\theta_i^*\partial\theta_j}\right]=-E\left[\frac{\partial\ln f_\theta(x)}{\partial\theta_i^*}\frac{\partial\ln f_\theta(x)}{\partial\theta_j}\right]$$

Definition (CRLB for multiple parameters).

$$\sigma_{\hat\theta_i}^2\ge(J^{-1}(\theta))_{i,i}$$

Definition (Maximum Likelihood Estimator (MLE)).

$$\hat\theta_{ML}(x)=\arg(\max_\theta f_\theta(x))$$

Definition (Maximum A Posteriori (MAP)).

$$\hat\theta_{MAP}(x)=\arg(\max_\theta f_{\theta|x}(\theta|x))=\arg(\max_\theta f_{x|\theta}(x|\theta)f_\theta(\theta))$$

Definition (Minimum Mean Square Error (MMSE)).

$$\hat\theta_{MMSE}(x)=\int\theta f_\theta(\theta)f_{x|\theta}(x|\theta)$$

6 Tema 6: Spectral estimation

Definition (Windowed sequence).

$$x_v[n]=x[n]v[n]=x[n]p_N[n],\quad |V(e^{j\Omega})|=\frac{\sin(N\Omega/2)}{\sin(\Omega/2)}$$

Definition (Triangular window).

$$w[m]=\frac{1}{N}v[m]*v[-m],\quad W(e^{j\Omega})=\frac{1}{N}\left|V(e^{j\Omega})\right|^2$$

Definition (Biased estimation of auto-correlation).

$$\hat r_X[m]=\frac{1}{N}x_v[m]*x_v^*[-m]=\frac{1}{N}\sum x[n+m]x^*[n]v[n+m]v[n]$$

$$E[\hat r_X[m]]=\left(1-\frac{|m|}{N}\right)r_X[m],\ |m|\le N-1;\ 0\ \textit{otherwise}$$

Definition (Periodogram). *Biased*

$$\hat S_p(e^{j\Omega})=F[\hat r[m]]=\frac{1}{N}|X_v(e^{j\Omega})|^2$$

$$E[\hat S_p(e^{j\Omega})]=\frac{1}{2\pi}\int_{-\pi}^{\pi}S_X(e^{j(\Omega-\theta)})\frac{1}{N}W(e^{j\theta})d\theta$$

Definition (Variance). *(Very complicated, approximation)*

$$Var[\hat S_p(e^{j\Omega})]\approx(S_x(e^{j\Omega}))^2$$

Definition (Estimation of the power). *Unbiased*

$$\hat P_X=\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat S_p(e^{j\Omega})d\Omega=\frac{1}{N}\sum|x[n]|^2$$

$$E[\hat P_X]=E[\hat r_X[0]]=P_X$$

Definition (Unbiased estimator of auto-correlation).

$$\check r_X[m]=\begin{cases}\frac{1}{N-|m|}\sum_0^{N-|m|-1}x[n+|m|]x^*[n], & |m|\le N-1\\ 0\ \textit{otherwise}\end{cases}$$

$E[\check r_X[m]]=r_X[m]$ *si* $|m|\le N-1$

Definition (Modified Periodogram). *(Any window)*

$$\hat P_X=\frac{1}{N}\sum_0^{N-1}|v[n]|^2|x[n]|^2$$

Definition (Blackman-Tukey). *Biased estimate of the auto-correlation* $(\hat r_X[m],|m|\le N-1)\Rightarrow$ *Windowing* $\hat r_X[m]w_a[m],|m|\le L-1\Rightarrow$ *Fourier transform* $\hat S_{BT}(e^{j\Omega})=F[\hat r_X[m]w_a[m]]$

The estimator is asymptotically unbiased if $N,L\to\infty,w_a[0]=1$

Definition (Variance BT). E_{W_a} *energy of the window*

$$Var[\hat S_{BT}(e^{j\Omega})]\approx\frac{E_{W_a}}{N}S_X^2(e^{j\Omega})$$

(BARLETT-WELCH 36)

Definition (Models). $w[n]\to\boxed{h(z)}\to x[n],r_w[0]=\sigma^2$

$$AR(p)\Rightarrow H(z)=\frac{1}{1+\sum_1^pa_kz^{-k}}$$

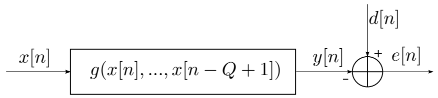
$$MA(q)\Rightarrow H(z)=1+\sum_1^qb_kz^{-k}$$

$$ARMA(p,q)\Rightarrow H(z)=\frac{1+\sum_1^qb_kz^{-k}}{1+\sum_1^pa_kz^{-k}}$$

Definition (Yule-Walker equations).

$$R_x\begin{pmatrix}a_1\\ \vdots\\ a_p\end{pmatrix}=-\begin{pmatrix}r_x[1]\\ \vdots\\ r_x[p]\end{pmatrix},\quad \sigma^2=r_x[0]+\sum_1^pa_kr_x[-k]$$

7 Tema 7: Wiener Filtering



$g=\mathbf{h}^H=[h^*[0],\dots,h^*[Q-1]],\quad \mathbf{x}[n]=[x[n],\dots,x[n-Q+1]]^T$

Definition (Linear MSE estimator).

$$y[n]=\mathbf{h}^H\mathbf{x}[n],\quad e[n]=d[n]-y[n]$$

Theorem (MSE).

$$\xi=E[|e[n]|^2]=P_d+\mathbf{h}^H\mathbf{R}_x\mathbf{h}-\mathbf{h}^H\mathbf{r}_{xd}-\mathbf{r}_{xd}^H\mathbf{h}$$

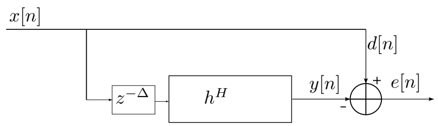
Definition (Wiener-Hopf equations). *(minimize MSE)*

$$\mathbf{R}_x\mathbf{h}_{opt}=\mathbf{r}_{xd}\Rightarrow\mathbf{h}_{opt}=\mathbf{R}_x^{-1}\mathbf{r}_{xd}$$

Proposition. *MSE not using Wiener filter*

$$\begin{aligned}&\xi=\xi_{min}+(\mathbf{h}-\mathbf{h}_{opt})^H\mathbf{R}_x(\mathbf{h}-\mathbf{h}_{opt})\\ &E[\xi]=\xi_{min}+trace(\mathbf{R}_xE[(\mathbf{h}-\mathbf{h}_{opt})(\mathbf{h}-\mathbf{h}_{opt})^H])\end{aligned}$$

Linear prediction



$$\begin{aligned}&y[n]=\hat x[n]=\mathbf{h}^H\mathbf{x}[n-\Delta],\quad \xi_{min}=r_x[0]-\mathbf{h}_{opt}^H\mathbf{R}_x\mathbf{h}_{opt}\\ &\mathbf{r}_{xd}=[r_x[-\Delta],\dots,r_x[-\Delta-Q+1]]^T,\quad \mathbf{h}_{opt}=\mathbf{R}_x^{-1}\mathbf{r}_x[-1]\end{aligned}$$

Gradient method. Given a filter $h^{(k)}$ we want to find a new one $h^{(k+1)}$ with lower MSE. (Minimize $(h^{(k)}-h_{opt})^HR_x(h^{(k)}-h_{opt})$)

$$\begin{aligned}&h^{(k+1)}=h^{(k)}-\mu\nabla_{h^*}\xi(h^{(k)})=h^{(k)}-\mu(R_xh^{(k)}-r_{xd})\\ &z^{(k)}:=Q^H(h^{(k)}-h_{opt})\Rightarrow z^{(k+1)}=(I-\mu\Lambda)z^{(k)}\end{aligned}$$

Condiciones de convergencia

$$\begin{aligned}&0<\mu<\frac{2}{trace(R_x)}\le\frac{2}{\lambda_{max}},\quad \mu_{opt}=\frac{2}{\lambda_{min}+\lambda_{max}}\\ &N_{ite}=\frac{\ln\varepsilon}{\ln|1-\mu\lambda_i|},\quad \chi:=\frac{\lambda_{max}}{\lambda_{min}}\end{aligned}$$

Definition (Instantaneus estimate of gradient (LMS)).

$$\nabla\xi(h^{(n)})=-x[n]e^*[n]\Rightarrow h^{(n+1)}=h^{(n)}+\mu x[n]e^*[n]$$