Problems Abstract Algebra
Second List

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1 *Nakayama's lemma.* Let M be a finitely generated $A-$module and I an ideal of A contained in the Jacobson radical. Prove:
$$IM = M \Rightarrow M = 0$$

First we prove a characterization of the elements of J, the Jacobson radical: $x \in J \iff 1 - xy$ is a unity for all $y \in A$.

(prove it)

We suppose $M \neq 0$. Let $x_1, x_2, \dots, x_n$ be a minimal set of generators of the module M. Because $M = IM$ we can express the element $x_1 = a_1x_1 + a_2x_2 + \cdots + a_nx_n$, where $a_i \in I$. Then let b the inverse of $1 - a_1$ (that we have previously seen that exists).

$$(1 - a_1)x_1 = a_2x_2 + \cdots + a_nx_n = 0 \Rightarrow b(a_1 - 1)x_1 = x_1 = ba_2x_2 + \cdots ba_nx_n$$

entering in contradiction with $\{x_i\}$ being a minimal set unless $x_i = 0$, thus $M = 0$

(rehacer)

2 Under the previous hypothesis, prove:

1. $A/I \otimes_A M = 0 \Rightarrow M = 0$
2. If $N \subseteq M$ is a submodule, $M = IM + N \Rightarrow M = N$
3. If $f : N \to M$ is a homomorphism, $\overline{f} : N/IN \to M/IM$ surjective $\Rightarrow f$ surjective

3 Let A be a non-local ring. Prove that the $A-$module A has two minimal system of generators with a different number of generators.

4 Let (diagram) be a short exact sequence of $A - modules$. Prove that if M' and M'' are finitely generated, then M is finitely generated.

We start by fixing the set of generators of M' as $x_1, \dots, x_n$ and of M'' as $z_1, \dots, z_m$.

Since g is surjective, we can find elements $y_1, \dots, y_m$ such that $g(y_i) = z_i$. Now we select an arbitrary element $y \in M$. Then we have

$$g(y) = b_1z_1 + \cdots + b_mz_m = g(b_1y_1) + \cdots + g(b_my_m) \Rightarrow g(y - \sum b_iy_i) = 0 \Rightarrow y - \sum b_iy_i \in \ker(g)$$

for some $b_i \in A$. By exactness of the sequence we have $y - \sum b_iy_i \in \mathrm{Im}(f)$, so

$$y - \sum b_iy_i = f(\sum a_ix_i) = \sum a_if(x_i) \Rightarrow y = \sum a_if(x_i) + \sum b_iy_i$$

for some $a_i \in A$. Thus, a set of generators of M is $f(x_1), \dots, f(x_n), y_1, \dots, y_m$

5 Prove that $\mathbb{Z}[\sqrt{d}]$ is a Noetherian ring

This is equivalent to prove that $M = \mathbb{Z}[\sqrt{d}]$ is a Noetherian module. Since every submodule of M is finitely generated (by 1 and $\sqrt{d}$), then the module is Noetherian.

6 Prove that the ring $\mathbb{Z}[2T, 2T^2, 2T^3, \ldots] \subseteq \mathbb{Z}[T]$ is not Noetherian

We search for an ascending chain of ideals $I_1 \subseteq I_2 \subseteq \ldots$ in which for every I_i we have $x_i \in I_i$ but $x_i \notin I_{i-1}$. This chain can be $I_i = (2T, 2T^2, \ldots, 2T^{i-1}, 2T^i + 2T^{i+1} + \ldots)$. Notice that the containments are obvious and $x_i = 2T^{i-1} \in I_i$, but not in I_{i-1}.

7 Let M be an $A-$module and let N_1, N_2 be submodules of M. Prove that if M/N_1 and M/N_2 are Noetherian (Artinian), then $M/(N_1 \cap N_2)$ is Noetherian (Artinian) as well.

8 Let M be an $A-$module, $f : M \to N$ an $A-$endomorphism. Prove:

1. If M is Noetherian and f surjective $\Rightarrow$ f isomorphism
2. If M is Artinian and f injective $\Rightarrow$ f isomorphism

9 Compute:

1. $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$
2. $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q})$
3. $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Q})$

(1) We look for an element in $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$. Let $f(\frac{1}{n}) = x_n$ for n a nonzero integer and $f(1) = C \in \mathbb{Z}$. Then we have

$$C = f(1) = f\left(\frac{n}{n}\right) = nf\left(\frac{1}{n}\right) = nx_n. \quad \Rightarrow \quad x_n = 0 \ \forall |n| > C$$

But if we take into account $C = nx_n$ holds for all nonzero n, then $C = 0$, meaning all the x_n are zero. We end up with $f\left(\frac{a}{b}\right) = af\left(\frac{1}{b}\right) = a \times 0 = 0$. So $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$

(2) We look for an element in $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q})$. Let $f(\frac{1}{n}) = \frac{x_n}{y_n}$ for n a nonzero integer and $f(1) = C \in \mathbb{Q}$. Then we have

$$C = f(1) = f\left(\frac{n}{n}\right) = nf\left(\frac{1}{n}\right) = n\frac{x_n}{y_n}. \quad \Rightarrow \quad \frac{x_n}{y_n} = \frac{C}{n}$$

That means our morphism f_C is uniquely determined by the choice of $C \in \mathbb{Q}$, and is the morphism that sends $1 \to C$ and $\frac{1}{n} \to \frac{c}{n}$ and extends linearly $\frac{a}{b} \to \frac{a}{b}C$. So $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) \simeq \mathbb{Q}$

(3) We look for an element in $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Q})$. Let $f(\overline{1}) = r \in \mathbb{Q}$. Then

$$0 = f(\overline{0}) = f(\overline{m}) = mf(\overline{1}) = mr \Rightarrow r = 0$$

So the only possibility is the morphism 0 and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Q}) = 0$

10 Let A be a ring, M an $A-$module and $I \subseteq A$ an ideal. Prove

$$M/IM \cong A/I \otimes_A M$$

We construct the following maps and see that they are well-defined

$$\begin{aligned} f : M/IM &\to A/I \otimes_A M \\ x + IM &\mapsto (1 + I) \otimes_A x \end{aligned} \qquad \qquad \begin{aligned} g : A/I \otimes_A M &\to M/IM \\ (a + I) \otimes_A y &\mapsto ay + IM \end{aligned}$$

If we pick $x' \sim_{IM} x \Rightarrow x' = x + n$ for $n \in IM$ and we have

$$f(x' + IM) = (1 + I) \otimes_A (x + n)) = (1 + I) \otimes_A x + (1 + I) \otimes_A n = (1 + I) \otimes_A x = f(x + I)$$

Since the second term of the sum vanishes as $n = \sum i_k m_k$ for $i_k \in I, m_k \in M$, so

$$(1+I) \otimes_A \sum i_k m_k = \sum (i_k + I) \otimes_A m_k = \sum (0+I) \otimes_A m_k = 0$$

Therefore the application f is well-defined.

If we pick $a' \sim_A a \Rightarrow a' = a + i$ for $i \in I$ and we have

$$g((a'+I) \otimes_A y) = (a+i)y + IM = ay + iy + IM = ay + IM = g((a+I) \otimes_A y)$$

proving that the application is well defined.

It is trivial to check that f and g satisfy all the necessary conditions in order to be morphisms of modules, since only quotients, multiplications and tensor products are involved.

Finally we see $f \circ g = Id$ and $g \circ f = Id$:

$$\begin{aligned} f(g((a+I) \otimes_A y)) &= f(ay + IM) = (1+I) \otimes_A ay = (a+I) \otimes_A y && \Rightarrow \quad f \circ g = Id \\ g(f(x+IM)) &= g((1+I) \otimes_A x) = x + IM && \Rightarrow \quad g \circ f = Id \end{aligned}$$

finishing the proof.

11 Let A be a ring and $I, J \subseteq A$ ideals. Prove

$$A/I \otimes_A A/J \cong A/(I+J)$$

We construct the following maps and see that they are well-defined

$$\begin{aligned} f : A/I \otimes_A A/J &\to A/(I+J) \\ (x+I) \otimes_A (y+J) &\mapsto xy + (I+J) \end{aligned} \qquad \begin{aligned} g : A/(I+J) &\to A/I \otimes_A A/J \\ z + (I+J) &\mapsto (z+I) \otimes_A (1+J) \end{aligned}$$

If we pick $x' \sim x$ and $y' \sim y$ that means $x' = x + i, y' = y + j$ with $i \in I, j \in J$ and we have

$$f((x'+I) \otimes (y'+J)) = x'y' + (I+J) = xy + iy' + x'j + ij + I + J = xy + (I+J) = f((x+I) \otimes (y+J))$$

so f is well-defined.

If we pick $z' \sim z$ that means $z' = z + i + j$ with $i \in I, j \in J$ and we have

$$\begin{aligned} g(z' + (I+J)) &= (z'+I) \otimes (1+J) = (z+i+j+I) \otimes (1+J) = z \otimes_A (1+J) + (i+I) \otimes (1+J) + \\ &+ (j+I) \otimes_A (1+J) = (z+I) \otimes_A (1+J) + (0+I) \otimes_A (1+J) + (1+I) \otimes_A (0+J) = g(z + (I+J)) \end{aligned}$$

It is trivial to check that f and g satisfy all the necessary conditions in order to be morphisms of modules, since only quotients, multiplications and tensor products are involved.

Finally we see $f \circ g = Id$ and $g \circ f = Id$:

$$\begin{aligned} f(g(z+(I+J))) &= f((z+I) \otimes_A (1+J)) = z + (I+J) && \Rightarrow \quad f \circ g = Id \\ g(f((x+I) \otimes_A (y+J))) &= g(xy + (I+J)) = (xy+I) \otimes (1+J) = (x+I) \otimes_A (y+J) && \Rightarrow \quad g \circ f = Id \end{aligned}$$

finishing the proof.

12 Let A be a ring, M, N finitely generated $A-$ modules. Prove:

1. $M \otimes_A N$ is a finitely generated $A-$module
2. If A is Noetherian, then $\mathrm{Hom}_A(M, N)$ is a finitely generated $A-$module

13 Let A be a local ring, M, N finitely generated $A-$modules. Prove that

$$M \otimes_A N = 0 \iff (M = 0 \quad \text{ó} \quad N = 0)$$

14 Let M be a finitely generated $A-$module and let $S \subseteq A$ be a multiplicatively closed set. Prove that

$$S^{-1}M = 0 \iff \exists s \in S : sM = 0$$

15 Let $S \subseteq A$ be a multiplicatively closed set. Prove that the localization functor $S^{-1}-$ is exact.