Parámetro arco

$$s(t)=\int_{t_0}^{t}\|\beta'(u)\|du$$

Triedro de Frenet (bon)

$$T(t)=\frac{\beta'(t)}{\|\beta'(t)\|}\quad N(t)=\frac{T'(t)}{\|T'(t)\|}\quad B=T\times N$$

Curvatura y torsión

$$k(s)=\|T'(s)\|=\langle T'(s),N(s)\rangle$$

$$\tau(s)=\langle N'(s),B(s)\rangle$$

Para parámetro arbitrario

$$k=\frac{\|\gamma'\times\gamma''\|}{\|\gamma'\|^3},\quad \tau=\frac{\det\left(\gamma',\gamma'',\gamma^{3)}\right)}{\|\gamma'\times\gamma''\|^2}$$

Si la curva es bidimensional

$$k=\frac{\det(\gamma',\gamma'')}{\|\gamma'\|^3}$$

Fórmulas de Frenet

$$\begin{aligned}T'&=kN\\N'&=-kT+\tau B\\B'&=-\tau N\end{aligned}$$

Superficie parametrizada

$$\sigma(u,v)=\begin{pmatrix}x(u,v)\\y(u,v)\\z(u,v)\end{pmatrix},\quad D\sigma=\begin{pmatrix}x_u & x_v\\y_u & y_v\\z_u & z_v\end{pmatrix}$$

Primera forma fundamental

$$P=\begin{pmatrix}E & F\\F & G\end{pmatrix}=\begin{pmatrix}\langle\sigma_u,\sigma_u\rangle & \langle\sigma_u,\sigma_v\rangle\\\langle\sigma_u,\sigma_v\rangle & \langle\sigma_v,\sigma_v\rangle\end{pmatrix}$$

Cálculo de longitudes:

$$\begin{pmatrix}u(t)\\v(t)\end{pmatrix}\implies L=\int_a^b\sqrt{Eu'^2+2Fu'v'+Gv'^2}dt$$

Cálculo de áreas

$$A=\iint_X\|\sigma_u\times\sigma_v\|dudv=\iint_X\sqrt{EG-F^2}dudv$$

Cálculo de ángulos

$$\cos\alpha=\frac{\langle u,v\rangle}{\|u\|\|v\|}=\frac{u^TPv}{\sqrt{u^TPu}\sqrt{v^TPv}}$$

Aplicación de Gauss

$$N:S\to\mathbb{S}^2\quad N(q)=\frac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|}$$

Aplicación de Weingarten.

$$DN \text{ matriz } A=\begin{pmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{pmatrix}\text{ en base }\sigma_u,\sigma_v$$

Segunda forma fundamental

$$S=\begin{pmatrix}e & f\\f & g\end{pmatrix}=\begin{pmatrix}-\langle\sigma_u,N_u\rangle & -\langle\sigma_u,N_v\rangle\\-\langle\sigma_v,N_u\rangle & -\langle\sigma_v,N_v\rangle\end{pmatrix}=$$

$$=\begin{pmatrix}\langle\sigma_{uu},N\rangle & \langle\sigma_{uv},N\rangle\\\langle\sigma_{uv},N\rangle & \langle\sigma_{vv},N\rangle\end{pmatrix}=-PA=-A^TP$$

$$II(w)=-\langle w,DNw\rangle=w^TSw\ (\text{base }\sigma_u,\sigma_v)$$

$$k_1=-\lambda_1\quad k_2=-\lambda_2\ (\text{VAPs de A})$$

$$k_i^2-2Hk_i+K=0$$

Curvatura de Gauss

$$\det(DN)=\det(A)=\frac{\det(S)}{\det(P)}=k_1k_2$$

$K>0$ elíptico, $K<0$ hiperbólico
$K=0,k_1\neq 0$ parabólico, $K=0,k_i=0$ plano

Curvatura Media

$$H=\frac{k_1+k_2}{2}=\frac{-a_{11}-a_{22}}{2}=\frac{eG-2Ff+Eg}{2(EG-F^2)}$$

(fórmulas de aij)

$$a_{11}=\frac{Ff-eG}{EG-F^2},\quad a_{12}=\frac{gF-Fg}{EG-F^2}$$

$$a_{21}=\frac{eF-Ef}{EG-F^2},\quad a_{22}=\frac{fF-Eg}{EG-F^2}$$

Curvatura normal (T vector tangente)

$$k_n=T^TST$$

Símbolos de Cristoffel

$$\begin{cases}\sigma_{uu}=\Gamma_{11}^1\sigma_u+\Gamma_{11}^2\sigma_v+eN\\\sigma_{uv}=\Gamma_{12}^1\sigma_u+\Gamma_{12}^2\sigma_v+fN\\\sigma_{vv}=\Gamma_{22}^1\sigma_u+\Gamma_{22}^2\sigma_v+gN\end{cases}$$

Los calculamos como

$$\begin{pmatrix}\Gamma_{11}^1 & \Gamma_{12}^1 & \Gamma_{22}^1\\\Gamma_{11}^2 & \Gamma_{12}^2 & \Gamma_{22}^2\end{pmatrix}=P^{-1}\begin{pmatrix}\frac{1}{2}E_u & \frac{1}{2}E_v & F_v-\frac{1}{2}G_u\\F_u-\frac{1}{2}E_v & \frac{1}{2}G_u & \frac{1}{2}G_v\end{pmatrix}$$

Fórmulas de Gauss

1. $EK=(\Gamma_{11}^2)_v-(\Gamma_{12}^2)_u+\Gamma_{11}^1\Gamma_{12}^2+\Gamma_{11}^2\Gamma_{22}^2-\Gamma_{12}^1\Gamma_{11}^2-\Gamma_{12}^2\Gamma_{12}^2$
2. $FK=(\Gamma_{12}^1)_u-(\Gamma_{11}^1)_v+\Gamma_{12}^1\Gamma_{12}^2-\Gamma_{11}^2\Gamma_{22}^1$
3. $GK=(\Gamma_{22}^1)_u-(\Gamma_{12}^1)_v+\Gamma_{11}^1\Gamma_{22}^1+\Gamma_{12}^1\Gamma_{22}^2-\Gamma_{12}^1\Gamma_{12}^1-\Gamma_{12}^2\Gamma_{22}^1$

Codazzi-Mainardi

1. $e_v-f_u=e\Gamma_{12}^1+f(\Gamma_{12}^2-\Gamma_{11}^1)-g\Gamma_{11}^2$
2. $f_v-g_u=e\Gamma_{22}^1+f(\Gamma_{22}^2-\Gamma_{12}^1)-g\Gamma_{12}^2$

Campo tangente

$$X(u(t),v(t)) = a(u(t),v(t))\sigma_u + b(u(t),v(t))\sigma_v$$

Derivada Covariante

$$\nabla_w X = \left[a' + \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 \\ \Gamma_{12}^1 & \Gamma_{22}^1 \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}\right] \sigma_u + \left[b' + \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} \Gamma_{11}^2 & \Gamma_{12}^2 \\ \Gamma_{12}^2 & \Gamma_{22}^2 \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}\right] \sigma_v$$

Geodésicas

$$\begin{cases} u'' + \begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 \\ \Gamma_{12}^1 & \Gamma_{22}^1 \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = 0 \\ v'' + \begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} \Gamma_{11}^2 & \Gamma_{12}^2 \\ \Gamma_{12}^2 & \Gamma_{22}^2 \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = 0 \end{cases}$$

Relación de Clairut. En una superficie de revolución

$$\begin{cases} x = \varphi(u)\cos v \\ y = \varphi(u)\sin v \\ z = \psi(u) \end{cases} \quad \text{con } \varphi'^2 + \psi'^2 = 1$$

Entonces una geodésica $(u(t), v(t))$ satisface $\varphi(u)\cos\theta = \text{const}$