Problems Abstract Algebra
First List

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1 Let f be a morphism in a category $\mathcal{C}$. Prove the following:

(a) If f an isomorphism then f is a monomorphism and an epimorphism.

(b) The inclusion of $\mathbb{Z}$ in $\mathbb{Q}$ is a monomorphism and an epimorphism in the category of rings but not an isomorphism.

We begin with the proof of (a). Since $f : A \to B$ is an isomorphism, that means there exist $g : B \to A$ such that both $g \circ f = Id_A$ and $f \circ g = Id_B$.

Let h, k morphisms of the category that fulfill $f \circ h = f \circ k$. Then by composing from the left with g we have

$$g \circ f \circ h = g \circ f \circ k \Rightarrow Id_A \circ h = Id_A \circ k \Rightarrow h = k$$

so we conclude f is a monomorphism.

Let h, k morphisms of the category that fulfill $h \circ f = k \circ f$. Then by composing from the right with g we have

$$h \circ f \circ g = k \circ f \circ g \Rightarrow h \circ Id_B = k \circ Id_B \Rightarrow h = k$$

so we conclude f is an epimorphism.

We move to the proof of (b). Let $i : \mathbb{Z} \to \mathbb{Q}$ the inclusion in $\mathbb{Q}$ $(i : n \mapsto n)$. Now let $h, k \in \text{Hom}_{rings}(A, \mathbb{Z})$ such that $i \circ h = i \circ k$. It is clear that, since $i(n) = n \ \forall n \in \mathbb{Z}$, then $h(a) = k(a) \ \forall a \in A$, concluding $h = k$ and i monomorphism.

Now let $h, k \in \text{Hom}_{rings}(\mathbb{Z}, A)$ such that $h \circ i = k \circ i$. It is clear that, since $i(n) = n \ \forall n \in \mathbb{Z}$, then $h(i(a)) = k(i(a)) \Rightarrow h(a) = k(a) \ \forall a \in A$, concluding $h = k$ and i epimorphism.

Suppose i is an isomorphism. Thus, it must exists $g : \mathbb{Q} \to \mathbb{Z}$ such that $i \circ g = Id_{\mathbb{Q}}$ and $g \circ i = Id_{\mathbb{Z}}$. Let $a \in \mathbb{Z}$ such that $g(\frac{1}{2}) = a$. Then $i \circ g(\frac{1}{2}) = i(a) = a \neq \frac{1}{2}$, so $i \circ g \neq Id_{\mathbb{Q}}$, concluding f is not an isomorphism.

4 *Pullbacks in the category of abelian groups:* Let A and B be abelian groups together with homomorphisms $f : A \to S$ and $g : B \to S$. Prove that

$$A \times_S B = \{(a, b) \in A \times B | f(a) = g(b)\}$$

Let $U = \{(a, b) \in A \times B | f(a) = g(b)\}$. We will show that the pullback $A \times_S B$ is, in fact, U. We construct the following diagram:

$$\begin{tikzcd}
C \arrow[dr, "h"] \arrow[drr, bend left, "h_A"] \arrow[ddr, bend right, "h_B"'] & & \\
& U \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & S
\end{tikzcd}$$

We first construct the morphisms π_A and π_B that make the square commute. Those are

$$\begin{cases} \pi_A((a,b)) = a \\ \pi_B((a,b)) = b \end{cases} \Rightarrow f \circ \pi_A((a,b)) = f(a) = g(b) = g \circ \pi_B((a,b)) \ \forall (a,b) \in U$$

thus, the square commutes.

Now we construct h from h_A and h_B. Note that, for the two triangular diagrams to commute, the h must fulfill:

$$\begin{cases} \pi_A(h(c)) = h_A(c) \\ \pi_B(h(c)) = h_B(c) \end{cases} \quad \forall c \in C \quad \Rightarrow \quad h = (h_A, h_B)$$

and the h is unique, concluding the proof.

5 *Pushouts in the category of abelian groups:* Let A and B be abelian groups together with homomorphisms $f : S \to A$ and $g : S \to B$. Prove that

$$A \sqcup_S B = \frac{A \oplus B}{W}$$

where W is the subgroup generated by $(f(s), -g(s))$ with $s \in S$.

Let $U = \frac{A \oplus B}{W}$. We will show that the pushout $A \sqcup_S B$ is, in fact, U. We construct the following diagram:

$$\begin{tikzcd}
C & & \\
& U \arrow[ul, "h"'] & A \arrow[l, "i"'] \arrow[ull, bend right, "h_A"'] \\
& B \arrow[u, "j"] \arrow[uul, bend left, "h_B"] & S \arrow[l, "g"] \arrow[u, "f"']
\end{tikzcd}$$

We first construct the morphisms i, j such that the square diagram commutes. We propose

$$i(a) = [(a, 0)] \qquad j(b) = [(0, b)]$$

and we check for commutativity for all $s \in S$

$$\begin{cases} i \circ f(s) = [(f(s), 0)] \\ j \circ g(s) = [(0, g(s))] \end{cases} \quad \text{but } [(0, g(s))] = [(0, g(s)) + (f(s), -g(s))] = [(f(s), 0)] \Rightarrow i \circ f = j \circ g \ \forall s \in S$$

so we have proved the square commutes.

Now we construct the morphism h through h_A and h_B. We construct it in the following way:

$$\begin{cases} h([(a, 0)]) = h_A(a) \\ h([(0, b)]) = h_B(b) \end{cases} \Rightarrow h([(a, b)]) = h([(a, 0)] + [(0, b)]) = h([(a, 0)]) + h([(0, b)]) = h_A(a) + h_B(b)$$

and clearly this is well defined and unique as morphism in the category of abelian groups, so the other triangular diagrams commute as well and we conclude with the proof.

6 *Inverse limits in the category of sets / groups / abelian groups / modules*: Let $(\{A_i\}, \{f_{ji}\})$ be an inverse system over a preordered set I. Prove that

$$\varprojlim A_i = \{(a_i) \in \prod A_i | f_{ji}(a_j) = a_i \ i \leq j\}$$

Let $U = \{(a_i) \in \prod A_i | f_{ji}(a_j) = a_i \ i \leq j\}$. We will prove that this is, in fact, the inverse limit we are looking for through the following diagram:

$$\begin{tikzcd}
 & & A_i \\
B \arrow[r, "h"] \arrow[urr, bend left, "f_i"] \arrow[drr, bend right, "f_j"'] & U \arrow[ur, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\
 & & A_j \arrow[uu, "f_{ji}"']
\end{tikzcd}$$

We can easily check the commutativity of the right part of the diagram because $f_{ji} \circ \lambda_j((a_k)) = f_{ji}(a_j) = a_i = \lambda_i((a_k))$.

We now prove that the morphism h is unique. h must be of the form $h(c) = (h_k(c))$, but, since the upper and lower part of the diagram must commute, we have $\lambda_i \circ h(c) = f_i(c) \Rightarrow h_i = f_i$, so the morphism we are looking form is $h = (h_k)$, and is unique.