# Definitions, results and examples

Abel Doñate Muñoz

## Contents

# 1 Rings

**Definition 1** (Krull dimension)**.** *Supremum of the lengths of all chains of prime ideals.*

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_n \Rightarrow \dim R = n$$

**Definition 2** (Regular ring)**.** *The minimal number of generators of its maximal ideal is the Krull dimension.*

**Definition 3** (Simple ring)**.** *Ring with no two-sided ideal besides zero and itself.*

# 2 Modules

**Definition 4** (Projective module)**.** *$P$ is projective if and only if for every surjective homomorphism $f : N \to M$ and every homomorphism $g : P \to M$, there exists a lifting $h : P \to N$ with the diagram commuting:*

$$\begin{tikzcd} & N \arrow[d, two heads, "f"] \\ P \arrow[ur, dashed, "h"] \arrow[r, "g"'] & M \end{tikzcd}$$

**Proposition 1** (Characterizations of projective modules)**.** *The following are equivalent:*

1. *$P$ is projective.*
2. *The SES $0 \to A \to B \to P \to 0$ splits.*
3. *$Hom(P, -)$ is an exact functor.*
4. *$P$ is the direct sum of free modules.*

**Definition 5** (Flat module)**.** *$M$ is flat if and only if for every injective homomorphism $f : K \to L$, the map $f \otimes_R id : K \otimes_R M \to L \otimes_R M$ is injective, that is:*

$$\begin{tikzcd} K \arrow[d, hook, "f"] & \Rightarrow & K \otimes_R M \arrow[d, hook, "f\otimes id"] \\ L & \Rightarrow & L \otimes_R M \end{tikzcd}$$

**Proposition 2** (Characterizations of flat modules)**.** *The following are equivalent:*

1. *$M$ is flat.*
2. *$\otimes_R M$ is an exact functor.*

**Definition 6** (Torsion-free module)**.** *$M$ is torsion free if and only if its torsion submodule (the module with all the zero-divisors) is {0}:*

**Proposition 3.** *In general we have the following implications of modules*

$$\textit{Free} \Rightarrow \textit{Projective} \Rightarrow \textit{Flat} \Rightarrow \textit{Torsion-free}$$

**Example 1** (Counterexamples of implications)**.** *Some counterexamples*

- *Projective $\not\Rightarrow$ Free. $\mathbb{Z}/2\mathbb{Z}$ as $\mathbb{Z}/6\mathbb{Z}$-module.*
- *Flat $\not\Rightarrow$ Projective. $\mathbb{Q}$ as $\mathbb{Z}-$module.*
- *Torsion-free $\not\Rightarrow$ Flat. The ideal $I = (x, y)$ as $K[x, y]-$module.*

# 3 D-modules

**Definition 7** (Ring / Module/ Weyl algebra)**.** *$A_n = \{\mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle\}$ that has the structure of a ring a module or an algebra.*

**Proposition 4.** *$A_n$ is:*

- *A simple ring*
- *Noetherian*

**Proposition 5.** *Set of monomials $\mathcal{B} = \{x^\alpha \partial^\beta : \alpha, \beta \in \mathbb{N}^n\}$ is a basis of $A_n$. Then we can write every element as*

$$P = \sum_{\alpha,\beta} p_{\alpha\beta} x^\alpha \partial^\beta = \sum_\beta p_\beta(x) \partial^\beta$$

We denote $|\beta| = \sum \beta_i$.

**Definition 8** (Order and total order)**.** .

$$\begin{array}{ccc} \textit{Order} & ord(P) = \max \beta & \sigma(P) = \sum_{|\beta| = ord(P)} p_\beta(x) \xi^\beta \\ \textit{Total Order} & ord^T = \max |\alpha + \beta| & \sigma(P) = \sum_{|\alpha+\beta| = ord^T(P)} p_{\alpha\beta} x^\alpha \xi^\beta \end{array}$$

**Definition 9** (Filtrations)**.** *Respectively Order and Total order filtrations:*

$$F_k(A_n) = \{P \in A_n : ord(P) \leq k\}, \qquad B_k(A_n) = \{P \in A_n : ord^T(P) \leq k\}$$

# 4 F-modules

For all this section $R$ is a commutative Noetherian ring with prime characteristic $p$.

**Definition 10** (Frobenius endomorphism)**.** *homomorphism $f : R \to R$ where $f(r) = r^p$*

**Definition 11** (Frobenius module)**.** *$M^{(e)}$ is the $R$-module $M$ endowed with the action $r \cdot m = f^e(r)m$. We denote $M' := M^{(1)}$.*

**Definition 12** (Frobenius functor)**.** *The application $F$ that sends $M \to R' \otimes_R M$ and $\varphi : M \to N$ to $Id \otimes \varphi : R' \otimes_R M \to R' \otimes_R N$ is a functor.*

**Definition 13** (Frobenius powers)**.** *If $I = (x_1, \ldots, x_n)$ is an ideal or $R$. We define*

$$I^{[p^e]} := (x_1^{p^e}, \ldots, x_n^{p^e})R$$

**Proposition 6.** $F^e(R/I) \cong R/I^{[p^e]}$