# $F-$módulos

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## Endomorfismo de Frobenius

Sea $R$ un anillo con característica $p > 0$. Definimos el endomorfismo de Frobenius como el mapa

$$\begin{aligned} f : R &\to R \\ r &\to r^p \end{aligned}$$

## Observación

Este morfismo en general no es inyectivo ni exhaustivo.

## Module with Frobenius action

Given $M$ an $R-$Module, we define the module $M^{(e)}$ induced by $f^{(e)}$ as the abelian group $M$ endowed with the action

$$r \cdot m = f^{(e)}(r)m = r^{p^e} m$$

## Notation

For simplicity we will write $M^{(1)}$ as $M'$ and $R^{(1)}$ as $R'$.

# Functor de Frobenius

## Functor de Frobenius

Definimos el functor de Frobenius como el el functor $F : \mathbf{R} - \mathbf{Mod} \to \mathbf{R} - \mathbf{Mod}$ que envía

$$M \longmapsto R' \otimes_R M, \qquad (M \xrightarrow{\phi} N) \longmapsto R' \otimes_R M \xrightarrow{id \otimes_R \phi} R' \otimes_R N$$

## Frobenius of a complex

Given the complex $M^\bullet$, we define its induced complex $F(M^\bullet)$ as the complex

$$\begin{tikzcd}
\cdots \arrow[r] & M_{k-1} \arrow[r, "h_{k-1}"] \arrow[d, "F"] & M_k \arrow[r, "h_k"] \arrow[d, "F"] & M_{k+1} \arrow[r] \arrow[d, "F"] & \cdots \\
\cdots \arrow[r] & F(M_{k-1}) \arrow[r, "F(h_{k-1})"] & F(M_k) \arrow[r, "F(h_k)"] & F(M_{k+1}) \arrow[r] & \cdots
\end{tikzcd}$$

Exactly the same construction works for $F^{(e)}$.

# Properties

## Properties of Frobenius functor

1. $F$ is right exact. Furthermore, if $R$ is regular, then $R'$ is flat and $F$ is exact.
2. $F$ commutes with direct sums.
3. $F$ commutes with localization.
4. $F$ commutes with direct limits.
5. $F$ preserves finitely generation of modules.
6. If $R$ is regular, then $F$ commutes with cohomology of complexes.

# Properties

## Frobenius power ideal

Given $I = (x_1, \ldots, x_n)$ an ideal of $R$, we define its Frobenius $e-$power ideal as

$$I_{p^e} := (x_1^{p^e}, \ldots, x_n^{p^e})R$$

## Some examples of transformations

- $F(R) \cong R$
- $F(I) \cong I_{p^e}$
- $F(R/I) \cong R/I_{p^e}$

# $F-$module

## Definition of $F-$module

An $F-$ *module* is an $R-$module $M$ equipped with an $R-$isomorphism $\theta : M \to F(M)$ called the structure morphism.

## Morphism of $F-$modules

Given two $F-$ modules $(M, \theta_M)$ and $(N, \theta_N)$, we say $f : M \to N$ is a morphism of $F-$modules if the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{g} & N \\ \downarrow{\scriptstyle \theta_M} & & \downarrow{\scriptstyle \theta_N} \\ F(M) & \xrightarrow{F(g)} & F(N) \end{array}$$

## An alternative form

$F-$modules can also be thought as a module over the ring $R[F]$, that is, the ring $R$ in which we have adjoined the non-commutative variable $F$ with the relations $r^p F = Fr \ \forall r \in R$. This characterization is presented in [Bli04], and the notation $R[F]-$module taken in the thesis is very suggestive once we know where it comes from.

# Two important cases

In the case $M = R$ is the ring itself with $R-$module structure, we have a natural isomorphism $\theta : R \to F(R)$, which makes $(R, \theta)$ an $F-$module. This isomorphism is given by

$$\begin{aligned} \theta : R &\to F(R) \cong R' \otimes_R R \\ r &\mapsto r \otimes 1 \end{aligned}$$

Let $M = S^{-1}R$, then we have the isomorphism of $R-$modules $F(S^{-1}R) \cong S^{-1}R$. This is shown from the commutativity of the Frobenius functor with localization $F(S^{-1}R) \cong S^{-1}F(R) \cong S^{-1}R$. The natural isomorphism is given by

$$\begin{aligned} \theta : S^{-1}R &\to R' \otimes_R S^{-1}R \\ \frac{r}{s} &\mapsto rs^{p-1} \otimes \frac{1}{s} \end{aligned}$$

# $F-$finite modules

## Generating morphism

Given an $F-$module $(M,\theta)$ we define its generating morphism $\theta_0 : M_0 \to F(M_0)$ as the morphisms in the direct system

$$\begin{array}{ccccccccc}
M_0 & \xrightarrow{\theta_0} & F(M_0) & \xrightarrow{F(\theta_0)} & F^2(M_0) & \xrightarrow{F^2(\theta_0)} & \cdots & & M \\
\downarrow{\scriptstyle \theta_0} & & \downarrow{\scriptstyle F(\theta_0)} & & \downarrow{\scriptstyle F(\theta_0)} & & & & \downarrow{\scriptstyle \theta} \\
F(M_0) & \xrightarrow{F(\theta_0)} & F^2(M_0) & \xrightarrow{F^2(\theta_0)} & F^3(M_0) & \xrightarrow{F^3(\theta_0)} & \cdots & & F(M)
\end{array}$$

whose limit is the module $M$ and the morphism $\theta$

## $F-$finite module

We say that the module $M$ is $F-$finite if $M$ has a generating morphism $\theta_0 : M_0 \to F(M_0)$ with $M$ a finitely generated $R-$module.

# Local cohomology

## Torsion functor

Let $\Gamma_I = \{m \in M : I^n m = 0 \text{ for some } n \in \mathbb{N}\}$. One can check this induces the so-called functor that transform the functions in the following natural way

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "\Gamma_I"] & N \arrow[d, "\Gamma_I"] \\ \Gamma_I(M) \arrow[r, "\Gamma_I(g)"] & \Gamma_I(N) \end{tikzcd}$$

## LC via torsion functor

Taking an injective resolution $E^\bullet$ of $M$, we define the j-th local cohomology module of $M$ with support in $I$ as the $j-$th right derived functor of $\Gamma_I$, that is

$$H^j_I(M) = H^j(\Gamma_I(E^\bullet))$$

## LC via Čech complex

Let $I = (x_1, \ldots, x_n) \subseteq R$. We define the Čech complex $\check{C}^\bullet(M, I)$ on the ideal $I$ as

$$0 \longrightarrow M \xrightarrow{d_0} \bigoplus_{1\le i\le n} M_{x_i} \xrightarrow{d_1} \bigoplus_{1\le i<j\le n} M_{x_i x_j} \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} M_{x_1\cdots x_n}$$

where de differential maps $d_i$ are defined via the canonical localization morphism and alternating the sign in order to have $d_i \circ d_{i-1} = 0$. Explicitly we have the morphisms of every component $d_p : M_{x_{i_1}\cdots x_{i_p}} \to M_{x_{j_1}\cdots x_{j_{p+1}}}$ as

$$d_p(m) = \begin{cases} (-1)^{k+1}\frac{m}{1} & \text{if } \{i_1, \ldots, i_p\} = \{j_1, \ldots, \hat{j}_k, \ldots, j_{p+1}\} \\ 0 & \text{otherwise} \end{cases}$$

# Properties of LC

## Two important properties of LC

- $H^j_I(M) = H^j_{\sqrt{I}}(M)$
- Let $N$ be an $A-$module and the flat morphism $f : R \to A$. Then $A \otimes_R H^j_I(N) \cong H^j_{IA}(A \otimes_R N)$

If the ring $R$ is regular, then for every ideal $I \subseteq R$ we have $F(H^j_I(R)) \cong H^j_I(R)$

# $F-$finiteness

## $F-$finiteness of LC modules

Given an ideal $I$ of $R$, if $M$ is $F-$finite, then $H_I^j(M)$ is $F-$finite.

## Observation

Observe this is not the classical behaviour of finitely generated $R-$modules, since in general for finitely generated module $M$ we will have non-finitely generated $H_I^j(M)$.

# Injective modules

## Injective module

We say the $R-$module $E$ is injective if for all $R-$modules $M, N$ and morphisms $f : M \to N$ injective and $g : M \to E$ arbitrary there exists a morphism $h : N \to E$ such that $h \circ f = g$. This is, the following d This is, the following diagram commutes:

$$\begin{tikzcd}
 & E & \\
0 \arrow[r] & M \arrow[u, "g"] \arrow[r, hook, "f"] & N \arrow[ul, dashed, "h"']
\end{tikzcd}$$

# Injective modules

## Equivalent characterizations

There are three equivalent characterizations, meaning that the following statements are equivalent

- $E$ is an injective module.
- Any short exact sequence $0 \to E \to M \to N \to 0$ splits.
- If $E$ is a submodule of $M$, then there exists another submodule $N \subseteq M$ such that $E \oplus N = M$.
- The functor $\mathrm{Hom}(-, E)$ is exact.

# Injective modules

## Injective hull

Given a module $M$, we define its injective hull as the maximal essential extension $N = E_R(M)$. That is, given an injective $\theta : M \to N$, if $\varphi \circ \theta$ is injective, then $\varphi$ is also injective.

$$
\begin{tikzcd}
 & & E \\
0 \arrow[r] & M \arrow[r, hook, "\theta"] \arrow[ur, hook, "\varphi\circ\theta"] & N \arrow[u, dashed, hook, "\varphi"']
\end{tikzcd}
$$

# Injective modules

## The structure theorem

Every injective $R-$module $E$ is the direct sum of indecomposable injective modules with the form

$$E \cong \bigoplus_{\mathfrak{p} \in \mathrm{Spec}(R)} E_R(R/\mathfrak{p})^{\mu_\mathfrak{p}}$$

with the *Bass numbers* $\mu_\mathfrak{p}$ independent of the decomposition.

## Computing Bass numbers

Bass numbers can be computed as the rank of the Hom sets of residue fields in the following way:

$$\mu_\mathfrak{p} = \mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), E_\mathfrak{p})$$

If $R$ is regular, and $E$ is an injective $R-$module then $F(E) \cong E$.

**(Huneke,-Sharp)**

Let $(R, \mathfrak{m})$ a regular local ring of characteristic $p$. Then the Bass numbers $\mu_i(\mathfrak{p}, H_I^j(R))$ are finite.

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