Definitions, results and examples

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1 Rings

2 Modules

Definition 1 (Projective module). *P is projective if and only if for every surjective homomorphism $f : N \to M$ and every homomorphism $g : P \to M$, there exists a lifting $h : P \to N$ with the diagram commuting:*

$$\begin{tikzcd} & N \arrow[d, two heads, "f"] \\ P \arrow[ur, dashed, "h"] \arrow[r, "g"] & M \end{tikzcd}$$

Proposition 1 (Characterizations of projective modules). *The following are equivalent:*

1. *P is projective.*
2. *The SES $0 \to A \to B \to P \to 0$ splits.*
3. *$Hom(P, -)$ is an exact functor.*
4. *P is the direct sum of free modules.*

Definition 2 (Flat module). *M is flat if and only if for every injective homomorphism $f : K \to L$, the map $f \otimes_R id : K \otimes_R M \to L \otimes_R M$ is injective, that is:*

$$\begin{tikzcd} K \arrow[d, hook, "f"] & \Rightarrow & K \otimes_R M \arrow[d, hook, "f \otimes id"] \\ L & \Rightarrow & L \otimes_R M \end{tikzcd}$$

Proposition 2 (Characterizations of flat modules). *The following are equivalent:*

1. *M is flat.*
2. *$\otimes_R M$ is an exact functor.*

Definition 3 (Torsion-free module). *M is torsion free if and only if its torsion submodule (the module with all the zero-divisors) is {0}:*

Proposition 3. *In general we have the following implications of modules*

$$\textit{Free} \Rightarrow \textit{Projective} \Rightarrow \textit{Flat} \Rightarrow \textit{Torsion-free}$$

Example 1 (Counterexamples of implications). *Some counterexamples*

- *Projective $\not\Rightarrow$ Free. $\mathbb{Z}/2\mathbb{Z}$ as $\mathbb{Z}/6\mathbb{Z}$-module.*
- *Flat $\not\Rightarrow$ Projective. $\mathbb{Q}$ as $\mathbb{Z}-$module.*
- *Torsion-free $\not\Rightarrow$ Flat. The ideal $I = (x, y)$ as $K[x, y]-$module.*

3 D-modules

4 F-modules

For all this section R is a commutative Noetherian ring with prime characteristic p.

Definition 4 (Frobenius endomorphism). *homomorphism $f : R \to R$ where $f(r) = r^p$*