

# Problems Abstract Algebra

## First List

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**1** Let  $f$  be a morphism in a category  $\mathcal{C}$ . Prove the following:

- (a) If  $f$  an isomorphism then  $f$  is a monomorphism and an epimorphism.
- (b) The inclusion of  $\mathbb{Z}$  in  $\mathbb{Q}$  is a monomorphism and an epimorphism in the category of rings but not an isomorphism.

We begin with the proof of (a). Since  $f : A \rightarrow B$  is an isomorphism, that means there exist  $g : B \rightarrow A$  such that both  $g \circ f = Id_A$  and  $f \circ g = Id_B$ .

Let  $h, k$  morphisms of the category that fulfill  $f \circ h = f \circ k$ . Then by composing from the left with  $g$  we have

$$g \circ f \circ h = g \circ f \circ k \Rightarrow Id_A \circ h = Id_A \circ k \Rightarrow h = k$$

so we conclude  $f$  is a monomorphism.

Let  $h, k$  morphisms of the category that fulfill  $h \circ f = k \circ f$ . Then by composing from the right with  $g$  we have

$$h \circ f \circ g = k \circ f \circ g \Rightarrow h \circ Id_B = k \circ Id_B \Rightarrow h = k$$

so we conclude  $f$  is an epimorphism.

We move to the proof of (b). Let  $i : \mathbb{Z} \rightarrow \mathbb{Q}$  the inclusion in  $\mathbb{Q}$  ( $i : n \mapsto n$ ). Now let  $h, k \in \text{Hom}_{\text{rings}}(A, \mathbb{Z})$  such that  $i \circ h = i \circ k$ . It is clear that, since  $i(n) = n \forall n \in \mathbb{Z}$ , then  $h(a) = k(a) \forall a \in A$ , concluding  $h = k$  and  $i$  monomorphism.

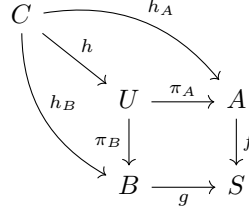
Now let  $h, k \in \text{Hom}_{\text{rings}}(\mathbb{Z}, A)$  such that  $h \circ i = k \circ i$ . It is clear that, since  $i(n) = n \forall n \in \mathbb{Z}$ , then  $h(i(a)) = k(i(a)) \Rightarrow h(a) = k(a) \forall a \in A$ , concluding  $h = k$  and  $i$  epimorphism.

Suppose  $i$  is an isomorphism. Thus, it must exists  $g : \mathbb{Q} \rightarrow \mathbb{Z}$  such that  $i \circ g = Id_{\mathbb{Q}}$  and  $g \circ i = Id_{\mathbb{Z}}$ . Let  $a \in \mathbb{Z}$  such that  $g(\frac{1}{2}) = a$ . Then  $i \circ g(\frac{1}{2}) = i(a) = a \neq \frac{1}{2}$ , so  $i \circ g \neq Id_{\mathbb{Q}}$ , concluding  $f$  is not an isomorphism.

**4** *Pullbacks in the category of abelian groups:* Let  $A$  and  $B$  be abelian groups together with homomorphisms  $f : A \rightarrow S$  and  $g : B \rightarrow S$ . Prove that

$$A \times_S B = \{(a, b) \in A \times B \mid f(a) = g(b)\}$$

Let  $U = \{(a, b) \in A \times B \mid f(a) = g(b)\}$ . We will show that the pullback  $A \times_S B$  is, in fact,  $U$ . We construct the following diagram:



We first construct the morphisms  $\pi_A$  and  $\pi_B$  that make the square commute. Those are

$$\begin{cases} \pi_A((a, b)) = a \\ \pi_B((a, b)) = b \end{cases} \Rightarrow f \circ \pi_A((a, b)) = f(a) = g(b) = g \circ \pi_B((a, b)) \quad \forall (a, b) \in U$$

thus, the square commutes.

Now we construct  $h$  from  $h_A$  and  $h_B$ . Note that, for the two triangular diagrams to commute, the  $h$  must fulfill:

$$\begin{cases} \pi_A(h(c)) = h_A(c) \\ \pi_B(h(c)) = h_B(c) \end{cases} \quad \forall c \in C \Rightarrow h = (h_A, h_B)$$

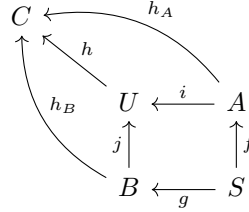
and the  $h$  is unique, concluding the proof.

**5 Pushouts in the category of abelian groups:** Let  $A$  and  $B$  be abelian groups together with homomorphisms  $f : S \rightarrow A$  and  $g : S \rightarrow B$ . Prove that

$$A \sqcup_S B = \frac{A \oplus B}{W}$$

where  $W$  is the subgroup generated by  $(f(s), -g(s))$  with  $s \in S$ .

Let  $U = \frac{A \oplus B}{W}$ . We will show that the pushout  $A \sqcup_S B$  is, in fact,  $U$ . We construct the following diagram:



We first construct the morphisms  $i, j$  such that the square diagram commutes. We propose

$$i(a) = [(a, 0)] \quad j(b) = [(0, b)]$$

and we check for commutativity for all  $s \in S$

$$\begin{cases} i \circ f(s) = [(f(s), 0)] \\ j \circ g(s) = [(0, g(s))] \end{cases} \quad \text{but } [(0, g(s))] = [(0, g(s)) + (f(s), -g(s))] = [(f(s), 0)] \Rightarrow i \circ f = j \circ g \quad \forall s \in S$$

so we have proved the square commutes.

Now we construct the morphism  $h$  through  $h_A$  and  $h_B$ . We construct it in the following way:

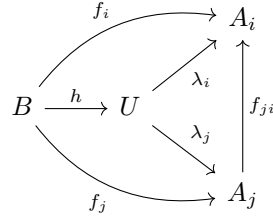
$$\begin{cases} h([(a, 0)]) = h_A(a) \\ h([(0, b)]) = h_B(b) \end{cases} \Rightarrow h([(a, b)]) = h([(a, 0)] + [(0, b)]) = h([(a, 0)]) + h([(0, b)]) = h_A(a) + h_B(b)$$

and clearly this is well defined and unique as morphism in the category of abelian groups, so the other triangular diagrams commute as well and we conclude with the proof.

**6** *Inverse limits in the category of sets / groups / abelian groups / modules:* Let  $(\{A_i\}, \{f_{ji}\})$  be an inverse system over a preordered set  $I$ . Prove that

$$\varprojlim A_i = \{(a_i) \in \prod A_i \mid f_{ji}(a_j) = a_i \text{ } i \leq j\}$$

Let  $U = \{(a_i) \in \prod A_i \mid f_{ji}(a_j) = a_i \text{ } i \leq j\}$ . We will prove that this is, in fact, the inverse limit we are looking for through the following diagram:



We can easily check the commutativity of the right part of the diagram because  $f_{ji} \circ \lambda_j((a_k)) = f_{ji}(a_j) = a_i = \lambda_i((a_k))$ .

We now prove that the morphism  $h$  is unique.  $h$  must be of the form  $h(c) = (h_k(c))$ , but, since the upper and lower part of the diagram must commute, we have  $\lambda_i \circ h(c) = f_i(c) \Rightarrow h_i = f_i$ , so the morphism we are looking for is  $h = (h_k)$ , and is unique.