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Definition 1 (System)**.** *A system is a transformation of an input signal*

- *Analog* $T : x(t) \mapsto y(t) = T[x(t)]$
- *Discrete* $T : x[n] \mapsto y[n] = T[x[n]]$

Some examples of transformations are

Amplitude gain	$y(t) = ax(t)$
Temporal delay	$y(t) = x(t - \Delta)$
Time rotation	$y(t) = x(-t)$
Time scaling	$y(t) = x(at)$
Integrator or accumulator	$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$
Differentiator	$y(t) = \frac{dx(t)}{dt}$
Averaging	$y(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)d\tau$

Systems can be classified into the following categories:

Linear	$T[ax_1(t) + bx_2(t)] = ay_1(t)by_2(t)$
Time-Invariant	$T[x(t - \Delta)] = y(t - \Delta)$
Static (no memory)	$y(t) = f(x(t))$
Causal	Does not depend on future values of the input
Stable (bounded)	$\forall x(t) : \lvert x(t)\rvert \le M_x \Rightarrow \ \exists M_y : \lvert y(t)\rvert \le M_y$
Invertible	$\exists U : y(t) \mapsto x(t)$

Now we focus on Linear Time-Invariant systems (LTI). Because of the properties, we only need a impulse response in order to describe completely the system. This input signal will be

$$T : x(t) = \delta(t) \mapsto h(t) \qquad \text{and} \qquad T : x[n] = \delta(n) \mapsto h[n]$$

Then, if we know this function $h(t)$ we can compute the transformation of an arbitrary signal as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t) \qquad \text{and} \qquad y[n] = \sum x[m]h[n-m] = x[n] * h[n]$$

Now we can classify the types of LTI systems

- Casual $\iff$ $h(t) = 0 \ \forall t < 0$
- Stable $\iff$ $\int_{\mathbb{R}} \lvert h(t)\rvert dt < \infty$
- Invertible $\iff$ $\exists h_1(t) : h(t) * h_1(t) = \delta(t)$

3 Transform domains

3.1 Transforms

Fourier series

$$x(t) = \sum_{\mathbb{Z}} c_n e^{jn\omega_0 t} = \sum_{\mathbb{Z}} \lvert c_n\rvert e^{j(n\omega_0 t + \varphi_n)} \qquad \text{where} \qquad c_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt$$

Power (Parseval's inequality) $P_{med} = \dfrac{1}{T_0}\displaystyle\int_{T_0} |x(t)|^2 dt = \sum_{\mathbb{Z}} |c_n|^2$

Laplace transform

$$x(t) \mapsto X(s) = L[x(t)] = \int_0^{\infty} x(t)e^{-st}dt$$

Fourier transform

$$x(t) \mapsto X(\omega) = F[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Z transform

$$x[n] \mapsto X(z) = Z[x[n]] = \sum_{\mathbb{Z}} x[n]z^{-n}, \qquad Z^{-1}[X(z)] = \frac{1}{2\pi j}\oint_{ROC} X(z)z^{n-1}dz$$

Para calcular las inversas de la z-transform

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1-p_k z^{-1}} \Rightarrow x[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k p_k^n u[n]$$

3.2 Analog systems

Definition 2 (Transference function). $H(\omega) = F(h(t)) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$

Input	**Output**
$x(t) = e^{j\omega t}$	$y(t) = e^{j\omega t}H(\omega)$
$x(t) = A\cos(\omega t + \varphi)$	$y(t) = A\lvert H(\omega)\rvert \cos(\omega t + \varphi + \angle H(\omega))$
$x(t) = \sum A_n \cos(n\omega_0 t + \varphi_n)$	$y(t) = \sum A_n \lvert H(n\omega_0)\rvert \cos(\omega t + \varphi_n + \angle H(n\omega_0))$

Properties of the Fourier transform

- Conovlution $F[f(t) * g(t)]F(\omega)G(\omega)$
- Product $F[f(t)g(t)]F(\omega) * G(\omega)$
- Energy preservation $\int_{-\infty}^{infty} x(t)y^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{infty} X(\omega)Y^*(\omega)d\omega$
- Time diferentiation $F[\frac{dx(t)}{dt}] = j\omega X(\omega)$
- Time integration $F[\int_{-\infty}^{t} x(\tau)d\tau] = \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$

(Important transformations)

Given two transference functions H_1, H_2 if we connect them in series $H_{eq} = H_1 \cdot H_2$ and in parallel $H_{eq} = H_1 + H_2$

3.3 Digital systems

(Important transformations) **IIR Case**

$$a_0 y[n] + \ldots + a_N y[n-N] = b_0 x[n] + \ldots + b_M x[n-M] \Rightarrow \boxed{H(z) = z^{N-M}\frac{b_0 z^M + \ldots + b_M}{a_0 z^n + \ldots + a_N}}$$

The impulse response is given by

$$h[n] = K_1 p_1^n u[n] + K_2 p_2^n u[n] + \ldots + K_N p_N^n u[n]$$