Mecánica

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Contents

1 Transformaciones de Lorentz

$$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \implies \begin{matrix} t' = \gamma\left(t - \frac{vx}{c^2}\right) \\ x' = \gamma\,(x - vt) \\ y' = y \\ z' = z \end{matrix} \implies \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{u}{c}\gamma & 0 & 0 \\ -\frac{u}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \implies \begin{cases} \Delta t' = \Delta t/\gamma \\ \Delta x' = \gamma \Delta x \end{cases}$$

1.1 Boost de Lorentz

$$B = \left(\begin{array}{c|c} \gamma & -\gamma\frac{\bar{u}^T}{c} \\ \hline -\gamma\frac{\bar{u}}{c} & I + \frac{\gamma-1}{u^2}\bar{u}\bar{u}^T \end{array}\right)$$

2 Cinemática Relativista

$$D = 1 - \frac{uv^x}{c^2} \implies \begin{cases} dt' = \gamma(dt - \frac{u}{c^2}dx) \\ dx' = \gamma(dx - udt) \\ dy' = dy \\ dz' = dz \end{cases} \implies \begin{cases} v'^x = \frac{v^x - u}{D} \\ v'^y = \frac{v^y}{\gamma D} \\ v'^z = \frac{v^z}{\gamma D} \end{cases} \implies \begin{cases} a'^x = \frac{a^x}{\gamma^3 D^3} \\ a'^y = \frac{a^y}{\gamma^2 D^2} + \frac{a^x v^y u}{c^2\gamma^2 D^3} \\ a'^y = \frac{a^z}{\gamma^2 D^2} + \frac{a^x v^z u}{c^2\gamma^2 D^3} \end{cases}$$

Por tanto la fórmula de suma de velocidades es

$$\begin{cases} v'^x = \frac{v^x + u}{D(-u)} \\ v'^y = \frac{v^y}{\gamma D(-u)} \\ v'^z = \frac{v^z}{\gamma D(-u)} \end{cases}$$

2.1 Aceleración propia

$$u = v \implies D = \gamma^{-2}(v) \implies \alpha = \gamma^3(v)a = \frac{d}{dt}(\gamma(v)v) \xRightarrow[MRUA]{} v = \frac{\alpha t}{\sqrt{1+\frac{\alpha^2 t^2}{c^2}}} \implies x = \frac{c^2}{\alpha}\sqrt{1+\frac{\alpha^2 t^2}{c^2}} - \frac{c^2}{\alpha}$$

$$\tau = \frac{c}{\alpha}\ln\left(\frac{\alpha t}{c} + \sqrt{1+\frac{\alpha^2 t^2}{c^2}}\right)$$

3 Espacio-tiempo y 4-vectores

En relatividad se cumple un invariante llamado la *Identidad fundamental*

$$\Delta s'^2 = \Delta s^2 \iff \Delta s'^T \eta \Delta s' = \Delta s^T \eta \Delta s \iff \Delta ct' - \Delta x' - \Delta y' - \Delta z' = \Delta ct - \Delta x - \Delta y - \Delta z$$

Los cuadrivectores cumplen

$$A'^\mu = \Lambda^\mu_\nu A^\nu, \qquad A^2 = \eta_{\mu\nu}A^\mu A^\nu, \qquad A\cdot B = \eta_{\mu\nu}A^\mu B^\nu \implies A^2 = A'^2, \qquad A\cdot B = A'\cdot B$$

Son cuadrivectores:

$$U := \frac{dx}{d\tau} = \gamma(v)\begin{pmatrix} c \\ v^x \\ v^y \\ v^z \end{pmatrix}, \quad U^2 = c^2, \qquad A := \frac{dU}{d\tau} = \begin{pmatrix} \frac{\gamma^4}{c}\bar{v}\cdot\bar{a} \\ \gamma^4\frac{\bar{v}\cdot\bar{a}}{c^2}\bar{v} + \gamma^2\bar{a} \end{pmatrix}, \quad A^2 = -\alpha^2, \quad U\cdot A = 0$$

$$P = mU = m\gamma\begin{pmatrix} c \\ v^x \\ v^y \\ v^z \end{pmatrix} \implies \sum P_i = \sum P_f, \qquad P^2 = m^2c^2$$

3.1 Energía

$$E_0 = mc^2 \implies T = E - E_0 = mc^2(\gamma - 1), \quad E^2 = m^2c^4 + c^2p^2, \qquad |p| = \frac{h\nu}{c} = \frac{h}{\lambda}$$

4 Mecánica Newtoniana

$$\sum m_i \ddot{r}_a = \frac{d}{dt}P = F^{ext}, \qquad r_G = \frac{1}{m}\sum m_i r_i \implies v_G = P/m, \qquad \frac{d}{dt}(r_G - \frac{t}{m}P)) - \frac{t}{m}F^{ext}$$

$$L = m_i r_i \times \ddot{r}_i \implies \frac{d}{dt}L_A = M_A^{ext}$$

$$E_{mec} = T + V = \frac{1}{2}mv^2 - \int_{x_0}^{x} F(y)dy, \qquad \frac{d}{dt}T = F \cdot v = \mathcal{P}, \qquad \Delta T = W_{1\to 2}, \qquad F = -\nabla V$$

4.1 Shifts de T y L

$$T_G = \frac{1}{2}\sum m_i(v_i - V_G)^2 \implies T = \frac{1}{2}mv_G^2 + T_G, \qquad L_O = r_G \times P + L_G$$

4.2 Sistemas en Rotación

Dos sistemas de Referencia S, S', ω velocidad angular de S' respecto de S. R posición de O' desde S.

$$\frac{du}{dt_s} = \frac{du}{dt_{S'}} + \omega \times u \implies \begin{cases} r = R + r' \\ v = V + v' + \omega \times r' \\ a = A + a' + \alpha \times r' + 2\omega \times v' + \omega \times (\omega \times r') \end{cases}$$

$$ma' = F_{real} + F_{tran} +_{Eul} + F_{Cor} + F_{cent} \implies \begin{cases} F_{trans} = -mA \\ F_{Eul} = -m\alpha \times r' \\ F_{Cor} = -2m\omega \times v' \\ F_{cen} = -m\omega \times (\omega \times r') \end{cases}$$

4.3 Sólido rígido

$$v_P = v_Q + \omega \times QP, \qquad a_P = a_Q + \alpha \times QP + \omega \times (\omega \times QP)$$

$$F = ma_G, \qquad I = \sum m_i d_i^2 = I_G + md^2, \qquad M = I\alpha, \qquad L = I\omega, \qquad T = \frac{1}{2}mv_G^2 + \frac{1}{2}I\omega^2$$

4.4 Coordenadas polares

$$\begin{cases} \bar{r} = r\hat{r} = x\hat{i} + y\hat{j} \\ \dot{\bar{r}} = \dot{r}\hat{r} + t\dot{\theta}\hat{\theta} \end{cases} \implies \begin{cases} \hat{\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j} \\ \hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j} \\ \dot{\hat{\theta}} = -\dot{\theta}\hat{r} \\ \dot{\hat{r}} = \dot{\theta}\hat{\theta} \end{cases}$$

5 Lagrangiano

$$Q_k(q_j, \dot{q}_j, t) = \sum F_a \frac{\partial r_a}{\partial q_k} \implies Q_k = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}, \qquad \mathcal{L} = T - V \implies \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = Q_k^{nc}$$

5.1 Potenciales generalizados

$$V \text{ potencial generalizado por } Q_k = \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_k}\right) - \frac{\partial V}{\partial q_k}$$

Campos electromagnéticos

$$\bar{F}_{em} = e(\bar{E} + \bar{v} \times \bar{B}), \quad \begin{cases} \bar{E} = -\frac{\partial \bar{A}}{\partial t} - \nabla\phi \\ \bar{B} = \nabla \times \bar{A} \end{cases} \implies V_{em}(\bar{r}, \bar{v}, t) = e(\phi - \bar{v} \cdot \bar{A})$$

5.2 Magnitudes conservadas

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \qquad \text{si } \mathcal{L} \text{ no depende de } q_k \implies p_x \text{ se conserva}$$

$$\mathcal{L} \text{ no depende de } L \implies \mathcal{H} = \sum p_k \dot{q}_k - \mathcal{L} \text{ se conserva.} \qquad \frac{d\mathcal{H}}{dt} = -\frac{\partial \mathcal{L}}{\partial t}$$

5.3 Teorema de Noether 1

$$\begin{cases} Q_i = q_i + \epsilon \mathbb{X}_i(q,t) \\ \tilde{\mathcal{L}}(Q,\dot{Q},t) = \mathcal{L}(Q,\dot{Q},t) - \epsilon \frac{dG}{dt} \\ G = \sum \mathbb{X}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} + F \end{cases} \qquad \text{trasf. sim.} \iff \exists\ F(q,t) : G \text{ se conserva}$$

5.4 Teorema de Noether 1

$$\begin{cases} Q_i = q_i + \epsilon \mathbb{X}_i(q,t) \\ T = t + \epsilon J(q,t) \\ \tilde{\mathcal{L}}(Q,\dot{Q},t) = \mathcal{L}(Q,\dot{Q},t) - \epsilon \frac{dG}{dt} \\ G = \sum \mathbb{X}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} + F - \mathcal{H}J \end{cases} \qquad \text{trasf. sim.} \iff \exists\ F(q,t) : G \text{ se conserva}$$

6 Oscilaciones pequeñas

$$\mathcal{L} = T - V = \frac{2}{2}mv^2 - \frac{1}{2}kx^2 \implies x = A\cos(\omega t + \phi), \quad \omega = \sqrt{\frac{k}{m}}, \quad E = \frac{1}{2}m\omega^2 A^2$$

$$\mathcal{L} = \frac{1}{2}\dot{q}^T M \dot{q} - \frac{1}{2}q^T K q, \implies M\ddot{q} + Kq = 0, (K - \omega^2 M)a = 0, \begin{cases} M = Adiag(m_i^*)A^T \\ K = Adiag(K_i^*)A^T \end{cases}$$

$$\ddot{z}_i + \omega_i^2 \zeta_i = 0 \implies \zeta_i = A_i \cos(\omega_i t + \phi_i), \qquad q = A\zeta = (a_1, \ldots, a_n)\begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_n \end{pmatrix}$$

Amortiguamiento:

$$m\ddot{x} = -kx - b\dot{x} \implies \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0, \quad \omega_0 = \sqrt{\frac{k}{m}}, \gamma = \frac{b}{2m} \implies x = Ae^{-\gamma t}\cos(\omega t + \phi), \quad \omega = \sqrt{\omega_o^2 - \gamma^2}$$

Forzados:

$$m\ddot{x} + kx = F_0\cos(\Omega t) \implies x = A\cos(\omega_0 t + \delta) + \frac{F_0/m}{\omega_0^2 - \Omega^2}\cos(\Omega t)$$

Si hay resonancia ($\Omega = \omega_0$):

$$x = \frac{F_0/m}{2\omega_0} t \cos(\omega_0 t)$$

Forzados y amortiguados:

$$m\ddot{x} + b\dot{x} + kx = F_0\cos(\Omega t) \implies x = \frac{F_0/m}{\sqrt{(\omega_0^2 - \Omega^2)^2 + 4\gamma^2\Omega^2}}\cos(\Omega t - \delta), \quad \tan\delta = \frac{2\gamma\Omega}{\omega_0^2 - \Omega^2}$$

La expresión general para un oscilador con fricción y forzado es:

$$M\ddot{q} + B\dot{q} + Kq = F_0\cos(\Omega t)$$