Problems Abstract Algebra
Second List

Abel Doñate Muñoz

1 *Nakayama's lemma.* Let M be a finitely generated $A-$module and I an ideal of A contained in the Jacobson radical. Prove:

$$IM = M \Rightarrow M = 0$$

We will use a characterization of the elements of J, the Jacobson radical: $x \in J \iff 1 - xy$ is a unity for all $y \in A$.

We suppose $M \neq 0$. Let $x_1, x_2, \dots, x_n$ be a minimal set of generators of the module M. Because $M = IM$ we can express the element $x_1 = a_1x_1 + a_2x_2 + \cdots + a_nx_n$, where $a_i \in I$. Then let b the inverse of $1 - a_1$ (whose existence has been previously proved).

$$(1 - a_1)x_1 = a_2x_2 + \cdots + a_nx_n = 0 \Rightarrow b(a_1 - 1)x_1 = x_1 = ba_2x_2 + \cdots + ba_nx_n$$

entering in contradiction with $\{x_i\}$ being a minimal set unless $x_i = 0\ \forall i$, thus $M = 0$.

2 Under the previous hypothesis, prove:

1. $A/I \otimes_A M = 0 \Rightarrow M = 0$
2. If $N \subseteq M$ is a submodule, $M = IM + N \Rightarrow M = N$
3. If $f : N \to M$ is a homomorphism, $\overline{f} : N/IN \to M/IM$ surjective $\Rightarrow f$ surjective

(1) Using the exercise 10 we have $0 = A/I \otimes_A M \cong M/IM$, so by Nakayama's lemma since $IM = M \Rightarrow M = 0$.

(2) We start by considering the following equality in the quotient module

$$I(M/N) = \sum a_i(m_i + N) = \sum a_im_i + N = \frac{IM + N}{N} = \frac{M}{N}$$

Therefore, since $I(M/N) = M/N$, by Nakayama's lemma we have $M/N = 0 \Rightarrow M = N$ as desired.

(3) We consider the following surjective map induced by $f : N \to M$ such that $f(n) = m$:

$$N \xrightarrow{\pi} N/IN \xrightarrow{\bar{f}} M/IM$$

$$n \longmapsto n + IN \longmapsto f(n) + IM$$

In order of $\overline{f}$ to be surjective it must be accomplished $f(N) + IM = M$. By the last exercice if $M = IM + f(N) \Rightarrow M = f(N)$, since $f(N)$ is a submodule of M so f is surjective.

3 Let A be a non-local ring. Prove that the $A-$module A has two minimal system of generators with a different number of generators.

Obviously $1 \in A$ generates the module, so we have a minimal set with one generator.

However, as A is non-local we can choose two different maximal ideals $\mathfrak{m}, \mathfrak{n}$, and by maximality $\mathfrak{m} + \mathfrak{n} = A \Rightarrow \exists x \in \mathfrak{m}, y \in \mathfrak{n} : x + y = 1$. So trivially $\{\mathfrak{m}, \mathfrak{n}\}$ generates the module, and is minimal because we can choose an element of $\mathfrak{m}$ $\mathfrak{n}$ which is not of the form ay for some $a \in A$.

Thus we have found two minimal sets of generators: $\{1\}$ and $\{x, y\}$:

4 Let (diagram) be a short exact sequence of $A-modules$. Prove that if M' and M'' are finitely generated, then M is finitely generated.

We start by fixing the set of generators of M' as $x_1, \dots, x_n$ and of M'' as $z_1, \dots, z_m$.

Since g is surjective, we can find elements $y_1, \dots, y_m$ such that $g(y_i) = z_i$. Now we select an arbitrary element $y \in M$. Then we have

$$g(y) = b_1 z_1 + \dots + b_m z_m = g(b_1 y_1) + \dots + g(b_m y_m) \Rightarrow g(y - \sum b_i y_i) = 0 \Rightarrow y - \sum b_i y_i \in \ker(g)$$

for some $b_i \in A$. By exactness of the sequence we have $y - \sum b_i y_i \in \operatorname{Im}(f)$, so

$$y - \sum b_i y_i = f(\sum a_i x_i) = \sum a_i f(x_i) \Rightarrow y = \sum a_i f(x_i) + \sum b_i y_i$$

for some $a_i \in A$. Thus, a set of generators of M is $f(x_1), \dots, f(x_n), y_1, \dots, y_m$

5 Prove that $\mathbb{Z}[\sqrt{d}]$ is a Noetherian ring

This is equivalent to prove that $M = \mathbb{Z}[\sqrt{d}]$ is a Noetherian module. Since every submodule of M is finitely generated (by 1 and $\sqrt{d}$), then the module is Noetherian.

6 Prove that the ring $\mathbb{Z}[2T, 2T^2, 2T^3, \dots] \subseteq \mathbb{Z}[T]$ is not Noetherian

We search for an ascending chain of ideals $I_1 \subseteq I_2 \subseteq \dots$ in which for every I_i we have $x_i \in I_i$ but $x_i \notin I_{i-1}$. This chain can be $I_i = (2T, 2T^2, \dots, 2T^{i-1}, 2T^i + 2T^{i+1} + \dots)$. Notice that the containments are obvious and $x_i = 2T^{i-1} \in I_i$, but not in I_{i-1}.

7 Let M be an $A-$module and let N_1, N_2 be submodules of M. Prove that if M/N_1 and M/N_2 are Noetherian (Artinian), then $M/(N_1 \cap N_2)$ is Noetherian (Artinian) as well.

Consider the following short exact sequence

$$0 \longrightarrow M/N_1 \xrightarrow{i} M/(N_1 \cap N_2) \xrightarrow{\pi} M/N_2 \longrightarrow 0$$

Where i and π are the natural inclusion and projection maps. Then $\ker \pi = \operatorname{Im} i = N_2/(N_1 \cap N_2)$, so in fact this is a short exact sequence. We recall that in a short exact sequence $0 \to M' \to M \to M'' \to 0$ it holds M', M'' Noetherians $\iff$ M Noetherian. Applying this our case $M/N_1, M/N_2$ are Noetherians $\Rightarrow M/(N_1 \cap N_2)$ is Noetherian.

The proof replicates exactly for Artinian modules.

8 Let M be an $A-$module, $f : M \to M$ an $A-$endomorphism. Prove:

1. If M is Noetherian and f surjective $\Rightarrow f$ isomorphism
2. If M is Artinian and f injective $\Rightarrow f$ isomorphism

(1) First we observe f surjective $\Rightarrow f^n$ surjective. The key observation is that $\ker f^i$ form a chain of submodules ordered by inclusion that, since M is Noetherian must stabilize at some point

$$\ker f \subsetneq \ker f^2 \subsetneq \dots \subsetneq \ker f^n = \ker f^{n+1} = \dots = \ker f^{2n}$$

Suppose $y \in \ker f^n$ is nonzero. Then, since f^n is surjective there exists $x \in M : f^n(x) = y$ and in particular $x \in \ker f^{2n} \setminus \ker f^n$, but since the kernels are equal, the only element is the zero element, and all the kernels must be zero, in particular $\ker f = 0$, so f is an isomorphism.

(2) First we observe f injective $\Rightarrow f^n$ injective. The key observation is that coker f^i form a chain of submodules ordered by inclusion that, since M is Artinian must stabilize at some point

$$\operatorname{coker} f \supsetneq \operatorname{coker} f^2 \supsetneq \cdots \supsetneq \operatorname{coker} f^n = \operatorname{coker} f^{n+1} = \cdots$$

Which means $\operatorname{Im} f^n = \operatorname{Im} f^{n+1}$. Thus for any $x \in M$ there exists a $y : f^n(x) = f^{n+1}(y) \Rightarrow f^n(x - u(y)) = 0$ and by injectivity of f^n finally $x = f(y)$. Since x was chosen arbitrarily, then f is surjective, which makes f an isomorphism.

> **9** Compute:
> 1. $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$
> 2. $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q})$
> 3. $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Q})$

(1) We look for an element in $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$. Let $f(\frac{1}{n}) = x_n$ for n a nonzero integer and $f(1) = C \in \mathbb{Z}$. Then we have

$$C = f(1) = f\left(\frac{n}{n}\right) = nf\left(\frac{1}{n}\right) = nx_n. \quad \Rightarrow \quad x_n = 0 \ \forall |n| > C$$

But if we take into account $C = nx_n$ holds for all nonzero n, then $C = 0$, meaning all the x_n are zero. We end up with $f\left(\frac{a}{b}\right) = af\left(\frac{1}{b}\right) = a \times 0 = 0$. So $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$

(2) We look for an element in $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q})$. Let $f(\frac{1}{n}) = \frac{x_n}{y_n}$ for n a nonzero integer and $f(1) = C \in \mathbb{Q}$. Then we have

$$C = f(1) = f\left(\frac{n}{n}\right) = nf\left(\frac{1}{n}\right) = n\frac{x_n}{y_n}. \quad \Rightarrow \quad \frac{x_n}{y_n} = \frac{C}{n}$$

That means our morphism f_C is uniquely determined by the choice of $C \in \mathbb{Q}$, and is the morphism that sends $1 \to C$ and $\frac{1}{n} \to \frac{c}{n}$ and extends linearly $\frac{a}{b} \to \frac{a}{b}C$. So $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) \simeq \mathbb{Q}$

(3) We look for an element in $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Q})$. Let $f(\overline{1}) = r \in \mathbb{Q}$. Then

$$0 = f(\overline{0}) = f(\overline{m}) = mf(\overline{1}) = mr \Rightarrow r = 0$$

So the only possibility is the morphism 0 and $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Q}) = 0$

> **10** Let A be a ring, M an $A-$module and $I \subseteq A$ an ideal. Prove
>
> $$M/IM \cong A/I \otimes_A M$$

We construct the following maps and see that they are well-defined

$$\begin{aligned} f : M/IM &\to A/I \otimes_A M \\ x + IM &\mapsto (1 + I) \otimes_A x \end{aligned} \qquad\qquad \begin{aligned} g : A/I \otimes_A M &\to M/IM \\ (a + I) \otimes_A y &\mapsto ay + IM \end{aligned}$$

If we pick $x' \sim_{IM} x \Rightarrow x' = x + n$ for $n \in IM$ and we have

$$f(x' + IM) = (1 + I) \otimes_A (x + n)) = (1 + I) \otimes_A x + (1 + I) \otimes_A n = (1 + I) \otimes_A x = f(x + I)$$

Since the second term of the sum vanishes as $n = \sum i_k m_k$ for $i_k \in I, m_k \in M$, so

$$(1 + I) \otimes_A \sum i_k m_k = \sum (i_k + I) \otimes_A m_k = \sum (0 + I) \otimes_A m_k = 0$$

Therefore the application f is well-defined.

If we pick $a' \sim_A a \Rightarrow a' = a + i$ for $i \in I$ and we have

$$g((a' + I) \otimes_A y) = (a + i)y + IM = ay + iy + IM = ay + IM = g((a + I) \otimes_A y)$$

proving that the application is well defined.

It is trivial to check that f and g satisfy all the necessary conditions in order to be morphisms of modules, since only quotients, multiplications and tensor products are involved.

Finally we see $f \circ g = Id$ and $g \circ f = Id$:

$$\begin{aligned} f(g((a+I) \otimes_A y)) &= f(ay + IM) = (1+I) \otimes_A ay = (a+I) \otimes_A y && \Rightarrow \quad f \circ g = Id \\ g(f(x+IM)) &= g((1+I) \otimes_A x) = x + IM && \Rightarrow \quad g \circ f = Id \end{aligned}$$

finishing the proof.

11 Let A be a ring and $I, J \subseteq A$ ideals. Prove

$$A/I \otimes_A A/J \cong A/(I+J)$$

We construct the following maps and see that they are well-defined

$$\begin{aligned} f : A/I \otimes_A A/J &\to A/(I+J) \\ (x+I) \otimes_A (y+J) &\mapsto xy + (I+J) \end{aligned} \qquad \begin{aligned} g : A/(I+J) &\to A/I \otimes_A A/J \\ z + (I+J) &\mapsto (z+I) \otimes_A (1+J) \end{aligned}$$

If we pick $x' \sim x$ and $y' \sim y$ that means $x' = x + i, y' = y + j$ with $i \in I, j \in J$ and we have

$$f((x'+I) \otimes (y'+J)) = x'y' + (I+J) = xy + iy' + x'j + ij + I + J = xy + (I+J) = f((x+I) \otimes (y+J))$$

so f is well-defined.

If we pick $z' \sim z$ that means $z' = z + i + j$ with $i \in I, j \in J$ and we have

$$\begin{aligned} g(z' + (I+J)) &= (z'+I) \otimes (1+J) = (z+i+j+I) \otimes (1+J) = z \otimes_A (1+J) + (i+I) \otimes (1+J) + \\ &+ (j+I) \otimes_A (1+J) = (z+I) \otimes_A (1+J) + (0+I) \otimes_A (1+J) + (1+I) \otimes_A (0+J) = g(z + (I+J)) \end{aligned}$$

It is trivial to check that f and g satisfy all the necessary conditions in order to be morphisms of modules, since only quotients, multiplications and tensor products are involved.

Finally we see $f \circ g = Id$ and $g \circ f = Id$:

$$\begin{aligned} f(g(z + (I+J))) &= f((z+I) \otimes_A (1+J)) = z + (I+J) && \Rightarrow \quad f \circ g = Id \\ g(f((x+I) \otimes_A (y+J))) &= g(xy + (I+J)) = (xy+I) \otimes (1+J) = (x+I) \otimes_A (y+J) && \Rightarrow \quad g \circ f = Id \end{aligned}$$

finishing the proof.

12 Let A be a ring, M, N finitely generated $A-$ modules. Prove:

1. $M \otimes_A N$ is a finitely generated $A-$module
2. If A is Noetherian, then $\mathrm{Hom}_A(M, N)$ is a finitely generated $A-$module

(1) Let $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ sets of generators of M and N respectively. Then every element $a \in M, b \in N$ can be expressed as $a = \sum r_i x_i, b = \sum r_i y_i$ with $r_i \in A$. An element of the tensor product is, thus

$$a \otimes_A b = \left(\sum_{i=1}^{m} r_i x_i \right) \otimes_A \left(\sum_{j=1}^{n} r_j y_j \right) = \sum_{i=1}^{m} \sum_{j=1}^{n} r_i r_j (x_i \otimes_A y_i)$$

Then $\{x_i \otimes_A y_j\}$ is a set of generators of $M \otimes_A N$.

(2) Notice that since A is Noetherian, every submodule of M and N are finitely generated. Let $M \cong A^m/I, N \cong A^n/J$, then clearly we have the isomorphism

$$\mathrm{Hom}_A(A^m, N) \cong N^m \quad \text{since} \quad \mathrm{Hom}_A(A, N) \cong N$$

with $\{x_i\}$ a set of generators of N. Knowing there exists an injection

$$\mathrm{Hom}_A(M, N) \hookrightarrow \mathrm{Hom}(A^m, N) \cong N^m$$

and since N^n is Noetherian, thus every submodule is finitely generated, in particular $\mathrm{Hom}_A(M,N)$.

13 Let A be a local ring, M, N finitely generated $A-$modules. Prove that

$$M \otimes_A N = 0 \iff (M = 0 \quad \text{ó} \quad N = 0)$$

$\boxed{\Leftarrow}$ Trivial

$\boxed{\Rightarrow}$ Let $k := A/\mathfrak{m}$ a field. We make use of the following facts:

(1) $k \otimes_A M \cong M/\mathfrak{m}M$ (exercise 10 with $I = \mathfrak{m}$)

(2) $k \otimes_A (M \otimes_A N) \cong (k \otimes_A M) \otimes_k (k \otimes_A N)$

To prove (2) we can consider the following applications

$$\begin{aligned} f : k \otimes_A (M \otimes_A N) &\to (k \otimes_A M) \otimes_k (k \otimes_A N) \\ a \otimes_A (m \otimes_A n) &\mapsto ((a \otimes_A m) + \mathfrak{m}) \otimes_k ((1 \otimes_A n) + \mathfrak{m}) \end{aligned}$$

and

$$\begin{aligned} g : (k \otimes_A M) \otimes_k (k \otimes_A N) &\to k \otimes_A (M \otimes_A N) \\ ((a \otimes_A m) + \mathfrak{m}) \otimes_k ((b \otimes_A n) + \mathfrak{m}) &\mapsto (ab) \otimes_A (m \otimes_A n) \end{aligned}$$

It can be seen in a similar way as problems 10 and 11, that f and g are morphisms of modules that are well-defined and are inverses, so this defines an isomorphism.

We have the following implications

$$M \otimes_A N = 0 \Rightarrow k \otimes_A (M \otimes_A N) = 0 \Rightarrow (k \otimes_A M) \otimes_k (k \otimes_A N) = 0 \Rightarrow M/\mathfrak{m}M \otimes_k N/\mathfrak{m}M$$

And since $M/\mathfrak{m}M$ and $N/\mathfrak{m}N$ are both finite vector spaces with dimension m and n respectively, then the tensor product is the usual tensor product of vector spaces, with dimension nm, that only vanishes if $M/\mathfrak{m}M$ or $N/\mathfrak{m}N$ is zero.

Without loss of generality say $M/\mathfrak{m}M = 0$, meaning $\mathfrak{m}M = M$. Since A is local, then it only has $\mathfrak{m}$ as maximal ideal, and thus the Jacobson ideal is precisely $\mathfrak{m}$ and we can apply Nakayama's lemma, meaning $\mathfrak{m}M = M \Rightarrow M = 0$, finishing the proof.

14 Let M be a finitely generated $A-$module and let $S \subseteq A$ be a multiplicatively closed set. Prove that

$$S^{-1}M = 0 \iff \exists s \in S : sM = 0$$

Suppose $\{x_1, \ldots, x_n\}$ is a set of generators of M. We prove both implications.

$\boxed{\Leftarrow}$ Say $\exists s^* \in S$ such that $sM = 0$. Then $\frac{m}{t} \sim \frac{m'}{t'} \iff \exists s \in S : smt' = sm't$. Setting $s = s^*$ we have zero in both sides, as $sM = 0$, concluding the only element is 0.

$\boxed{\Rightarrow}$ Since the module $S^{-1}M = 0$, then any fraction of the form $\frac{x_i}{1} \sim \frac{0}{1}$. From this fact for each x_i we can find an s_i such that $s_i x_i = 0$. Considering the element $s^* = \prod_{i=1}^n s_i \in S$, we have that $s^* x_i = 0$, and thus, because every element of M can be expressed as the sum $m = \sum_{i=1}^n a_i x_i$, then $s^* m = \sum_{i=1}^n a_i s^* x_i = 0$, and $s^* M = 0$.

15 Let $S \subseteq A$ be a multiplicatively closed set. Prove that the localization functor $S^{-1}(-)$ is exact.

Exactness means that for every short exact sequence, the sequence induced by the functor $S^{-1}(-)$

$f'(\frac{m}{s}) := \frac{f(m)}{s}$ is also exact.

$$0 \longrightarrow M' \xrightarrow{\quad f \quad} M \xrightarrow{\quad g \quad} M'' \longrightarrow 0$$

$$0 \longrightarrow S^{-1}M' \xrightarrow{f'} S^{-1}M \xrightarrow{g'} S^{-1}M'' \longrightarrow 0$$

This is, if f is injective, g surjective and $\mathrm{Im}(f) = \ker(g)$, then f' is injective, g' is surjective and $\mathrm{Im}(f') = \ker(g')$.

$\boxed{f' \text{ injective}}$ We prove that $\ker(f') = 0$. Indeed

$$f'(\frac{m}{s}) = 0 \iff \frac{f(m)}{s} = 0 \iff \exists t \in S : tf(m) = 0$$

But since f is an $A-$module morphism and $t \in S \subseteq A$, then $0 = tf(m) = f(tm)$, which means $tm = 0$ by injectivity of f. Thus $\frac{m}{s} = \frac{0}{1}$ and the kernel is the zero module.

$\boxed{g' \text{ surjective}}$ Let $\frac{m''}{s} \in S^{-1}M''$, we want to prove the existence of $\frac{m}{s} \in M$ such that $g(\frac{m}{s}) = \frac{m''}{s}$. But this trivially holds setting m such that $g(m) = m''$, which is well-defined by surjectivity of the g. We check indeed $g(\frac{m}{s}) = \frac{g(m)}{s} = \frac{m''}{s}$.

$\boxed{\mathrm{Im}(f') \subseteq \ker(g')}$ We consider $f'(\frac{m}{s}) = \frac{f(m)}{s} \in \mathrm{Im}(f')$, then $g'(f'(\frac{m}{s})) = \frac{g(f(m))}{s} = \frac{0}{s} = \frac{0}{1}$.

$\boxed{\ker(g') \subseteq \mathrm{Im}(f')}$ We will prove that if $g'(\frac{m}{s}) = 0$, then $\frac{m}{s}$ is element of the image of f'.

$$g'(\frac{m}{s}) = \frac{0}{1} \Rightarrow \ \exists t \in S : tg(m) = g(tm) = 0 \Rightarrow \ \exists m' : f(m') = tm$$

where the last condition arises from the exactness of the original sequence. Considering the element $\frac{m'}{ts} \in S^{-1}M'$, we see that $f'(\frac{m'}{ts}) = \frac{f(m')}{ts} = \frac{tm}{ts} = \frac{m}{s}$ concluding the proof.