$F-$módulos

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Endomorfismo de Frobenius

Sea R un anillo con característica $p > 0$. Definimos el endomorfismo de Frobenius como el mapa

$$\begin{aligned} f : R &\to R \\ r &\to r^p \end{aligned}$$

Observación

Este morfismo en general no es inyectivo ni exhaustivo.

Module with Frobenius action

Given M an $R-$Module, we define the module $M^{(e)}$ induced by $f^{(e)}$ as the abelian group M endowed with the action

$$r \cdot m = f^{(e)}(r)m = r^{p^e} m$$

Functor de Frobenius

Functor de Frobenius

Definimos el functor de Frobenius como el el functor $F : \mathbf{R} - \mathbf{Mod} \to \mathbf{R} - \mathbf{Mod}$ que envía

$$M \longmapsto R' \otimes_R M, \qquad (M \xrightarrow{\phi} N) \longmapsto R' \otimes_R M \xrightarrow{id \otimes_R \phi} R' \otimes_R N$$

Frobenius of a complex

Given the complex $M^\bullet$, we define its induced complex $F(M^\bullet)$ as the complex

$$
\begin{tikzcd}
\cdots \arrow[r] & M_{k-1} \arrow[r, "h_{k-1}"] \arrow[d, "F"] & M_k \arrow[r, "h_k"] \arrow[d, "F"] & M_{k+1} \arrow[r] \arrow[d, "F"] & \cdots \\
\cdots \arrow[r] & F(M_{k-1}) \arrow[r, "F(h_{k-1})"] & F(M_k) \arrow[r, "F(h_k)"] & F(M_{k+1}) \arrow[r] & \cdots
\end{tikzcd}
$$

Exactly the same construction works for $F^{(e)}$.

Properties

Properties of Frobenius functor

1. F is right exact. Furthermore, if R is regular, then R' is flat and F is exact.
2. F commutes with direct sums.
3. F commutes with localization.
4. F commutes with direct limits.
5. F preserves finitely generation of modules.
6. If R is regular, then F commutes with cohomology of complexes.

Properties

Frobenius power ideal

Given $I = (x_1, \dots, x_n)$ an ideal of R, we define its Frobenius $e-$power ideal as

$$I_{p^e} := (x_1^{p^e}, \dots, x_n^{p^e})R$$

Some examples of transformations

- $F(I) \cong I_{p^e}$
- $F(R/I) \cong R/I_{p^e}$

$F-$module

Definition of $F-$module

An $F - module$ is an $R-$module M equipped with an $R-$isomorphism $\theta : M \to F(M)$ called the structure morphism.

Morphism of $F-$modules

Given two $F-$ modules (M, θ_M) and (N, θ_N), we say $f : M \to N$ is a morphism of $F-$modules if the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{g} & N \\ \downarrow{\scriptstyle \theta_M} & & \downarrow{\scriptstyle \theta_N} \\ F(M) & \xrightarrow{F(g)} & F(N) \end{array}$$