$F-$módulos

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Endomorfismo de Frobenius

Sea R un anillo con característica $p > 0$. Definimos el endomorfismo de Frobenius como el mapa

$$\begin{aligned} f : R &\to R \\ r &\to r^p \end{aligned}$$

Observación

Este morfismo en general no es inyectivo ni exhaustivo.

Module with Frobenius action

Given M an $R-$Module, we define the module $M^{(e)}$ induced by $f^{(e)}$ as the abelian group M endowed with the action

$$r \cdot m = f^{(e)}(r)m = r^{p^e} m$$

Notation

For simplicity we will write $M^{(1)}$ as M' and $R^{(1)}$ as R'.

Functor de Frobenius

Functor de Frobenius

Definimos el functor de Frobenius como el el functor $F : \mathbf{R} - \mathbf{Mod} \to \mathbf{R} - \mathbf{Mod}$ que envía

$$M \longmapsto R' \otimes_R M, \qquad (M \xrightarrow{\phi} N) \longmapsto R' \otimes_R M \xrightarrow{id \otimes_R \phi} R' \otimes_R N$$

Frobenius of a complex

Given the complex $M^\bullet$, we define its induced complex $F(M^\bullet)$ as the complex

$$\begin{tikzcd} \cdots \arrow[r] & M_{k-1} \arrow[r, "h_{k-1}"] \arrow[d, "F"] & M_k \arrow[r, "h_k"] \arrow[d, "F"] & M_{k+1} \arrow[r] \arrow[d, "F"] & \cdots \\ \cdots \arrow[r] & F(M_{k-1}) \arrow[r, "F(h_{k-1})"] & F(M_k) \arrow[r, "F(h_k)"] & F(M_{k+1}) \arrow[r] & \cdots \end{tikzcd}$$

Exactly the same construction works for $F^{(e)}$.

Properties

Properties of Frobenius functor

1. F is right exact. Furthermore, if R is regular, then R' is flat and F is exact.
2. F commutes with direct sums.
3. F commutes with localization.
4. F commutes with direct limits.
5. F preserves finitely generation of modules.
6. If R is regular, then F commutes with cohomology of complexes.

Properties

Frobenius power ideal

Given $I = (x_1, \dots, x_n)$ an ideal of R, we define its Frobenius $e-$power ideal as

$$I_{p^e} := (x_1^{p^e}, \dots, x_n^{p^e})R$$

Some examples of transformations

- $F(R) \cong R$
- $F(I) \cong I_{p^e}$
- $F(R/I) \cong R/I_{p^e}$

$F-$module

Definition of $F-$module

An $F-$ *module* is an $R-$module M equipped with an $R-$isomorphism $\theta : M \to F(M)$ called the structure morphism.

Morphism of $F-$modules

Given two $F-$ modules (M, θ_M) and (N, θ_N), we say $f : M \to N$ is a morphism of $F-$modules if the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "\theta_M"] & N \arrow[d, "\theta_N"] \\ F(M) \arrow[r, "F(g)"] & F(N) \end{tikzcd}$$

An alternative form

$F-$modules can also be thought as a module over the ring $R[F]$, that is, the ring R in which we have adjoined the non-commutative variable F with the relations $r^pF = Fr \; \forall r \in R$. This characterization is presented in [Bli04], and the notation $R[F]-$module taken in the thesis is very suggestive once we know where it comes from.

Two important cases

In the case $M = R$ is the ring itself with $R-$module structure, we have a natural isomorphism $\theta : R \to F(R)$, which makes (R, θ) an $F-$module. This isomorphism is given by

$$\begin{aligned} \theta : R &\to F(R) \cong R' \otimes_R R \\ r &\mapsto r \otimes 1 \end{aligned}$$

Let $M = S^{-1}R$, then we have the isomorphism of $R-$modules $F(S^{-1}R) \cong S^{-1}R$. This is shown from the commutativity of the Frobenius functor with localization $F(S^{-1}R) \cong S^{-1}F(R) \cong S^{-1}R$. The natural isomorphism is given by

$$\begin{aligned} \theta : S^{-1}R &\to R' \otimes_R S^{-1}R \\ \frac{r}{s} &\mapsto rs^{p-1} \otimes \frac{1}{s} \end{aligned}$$

$F-$finite modules

Generating morphism

Given an $F-$module (M, θ) we define its generating morphism $\theta_0 : M_0 \to F(M_0)$ as the morphisms in the direct system

$$\begin{array}{ccccccccc}
M_0 & \xrightarrow{\theta_0} & F(M_0) & \xrightarrow{F(\theta_0)} & F^2(M_0) & \xrightarrow{F^2(\theta_0)} & \cdots & & M \\
\downarrow{\scriptstyle \theta_0} & & \downarrow{\scriptstyle F(\theta_0)} & & \downarrow{\scriptstyle F(\theta_0)} & & & & \downarrow{\scriptstyle \theta} \\
F(M_0) & \xrightarrow{F(\theta_0)} & F^2(M_0) & \xrightarrow{F^2(\theta_0)} & F^3(M_0) & \xrightarrow{F^3(\theta_0)} & \cdots & & F(M)
\end{array}$$

whose limit is the module M and the morphism θ

$F-$finite module

We say that the module M is $F-$finite if M has a generating morphism $\theta_0 : M_0 \to F(M_0)$ with M a finitely generated $R-$module.

Local cohomology

LC via torsion functor

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