## Problems Abstract Algebra First List

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- 1 Let f be a morphism in a category C. Prove the following:
  - (a) If f an isomorphism then f is a monomorphism and an epimorphism.
  - (b) The inclusion of  $\mathbb{Z}$  in  $\mathbb{Q}$  is a monomorphism and an epimorphism in the category of rings but not an isomorphism.

We begin with the proof of (a). Since  $f: A \to B$  is an isomorphism, that means there exist  $g: B \to A$  such that both  $g \circ f = Id_A$  and  $f \circ g = Id_B$ .

Let h, k morphisms of the category that fulfill  $f \circ h = f \circ k$ . Then by composing from the left with g we have

$$g \circ f \circ h = g \circ f \circ k \Rightarrow Id_A \circ h = Id_A \circ k \Rightarrow h = k$$

so we conclude f is a monomorphism.

Let h, k morphisms of the category that fulfill  $h \circ f = k \circ f$ . Then by composing from the right with g we have

$$h \circ f \circ g = k \circ f \circ g \Rightarrow h \circ Id_B = k \circ Id_B \Rightarrow h = k$$

so we conclude f is an epimorphism.

We move to the proof of (b). Let  $i: \mathbb{Z} \to \mathbb{Q}$  the inclusion in  $\mathbb{Q}$   $(i: n \mapsto n)$ . Now let  $h, k \in \text{Hom}_{rings}(A, \mathbb{Z})$  such that  $i \circ h = i \circ k$ . It is clear that, since  $i(n) = n \ \forall n \in \mathbb{Z}$ , then  $h(a) = k(a) \ \forall a \in A$ , concluding h = k and i monomorphism.

Now let  $h, k \in \text{Hom}_{rings}(\mathbb{Z}, A)$  such that  $h \circ i = k \circ i$ . It is clear that, since  $i(n) = n \ \forall n \in \mathbb{Z}$ , then  $h(i(a)) = k(i(a)) \Rightarrow h(a) = k(a) \ \forall a \in A$ , concluding h = k and i epimorphism.

Suppose i is an isomorphism. Thus, it must exists  $g:\mathbb{Q}\to\mathbb{Z}$  such that  $i\circ g=Id_\mathbb{Q}$  and  $g\circ i=Id_\mathbb{Z}$ . Let  $a\in\mathbb{Z}$  such that  $g(\frac{1}{2})=a$ . Then  $i\circ g(\frac{1}{2})=i(a)=a\neq\frac{1}{2}$ , so  $i\circ g\neq Id_\mathbb{Q}$ , concluding f is not an isomorphism.

**2** Show that in the category of finite dimensional vector spaces over a field  $\mathbb{K}$  we have a natural equivalente of functors between the identity Id and the bidual  $(-)^{**}$ 

We start by noticing if we have a finite dimensional vector space we can think as morphisms as matrices. Let  $X^* = Hom(X, \mathbb{R})$  and  $Y^* = Hom(Y, \mathbb{R})$ . Let F be the contravariant functor that dualizes. We have the following diagrams:

(diagram)

Let x, y be the dimensions of X and Y respectively. We can express f as an  $y \times x$  matrix. For the contravariance of the functor we have that the morphism  $Y^* \to X^*$  must be  $A^T$ . If we repeat this process we are given that  $F(F(A)) = (A^T)^T = A \ \forall A$ .

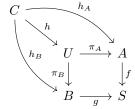
(terminar)

## **3** Show that two categories $\mathcal B$ and $\mathcal C$

4 Pullbacks in the category of abelian groups: Let A and B be abelian groups together with homomorphisms  $f: A \to S$  and  $g: B \to S$ . Prove that

$$A \times_S B = \{(a, b) \in A \times B | f(a) = g(b)\}$$

Let  $U = \{(a,b) \in A \times B | f(a) = g(b)\}$ . We will show that the pullback  $A \times_S B$  is, in fact, U. We construct the following diagram:



We first construct the morphisms  $\pi_A$  and  $\pi_B$  that make the square commute. Those are

$$\begin{cases} \pi_A((a,b)) = a \\ \pi_B((a,b)) = b \end{cases} \Rightarrow f \circ \pi_A((a,b)) = f(a) = g(b) = g \circ \pi_B((a,b)) \ \forall (a,b) \in U$$

thus, the square commutes.

Now we construct h from  $h_A$  and  $h_B$ . Note that, for the two triangular diagrams to commute, the h must fulfill:

$$\begin{cases} \pi_A(h(c)) = h_A(c) \\ \pi_B(h(c)) = h_B(c) \end{cases} \quad \forall c \in C \quad \Rightarrow \quad h = (h_A, h_B)$$
uding the proof

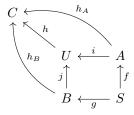
and the h is unique, concluding the proof.

**5** Pushouts in the category of abelian groups: Let A and B be abelian groups together with homomorphisms  $f: S \to A$  and  $g: S \to B$ . Prove that

$$A \sqcup_S B = \frac{A \oplus B}{W}$$

where W is the subgroup generated by (f(s), -g(s)) with  $s \in S$ .

Let  $U = \frac{A \oplus B}{W}$ . We will show that the pushout  $A \sqcup_S B$  is, in fact, U. We construct the following diagram:



We first construct the morphisms i, j such that the square diagram commutes. We propose

$$i(a) = [(a, 0)]$$
  $j(b) = [(0, b)]$ 

and we check for commutativity for all  $s \in S$ 

$$\begin{cases} i \circ f(s) = [(f(s), 0)] \\ j \circ g(s) = [(0, g(s))] \end{cases} \text{ but } [(0, g(s))] = [(0, g(s)) + (f(s), -g(s))] = [(f(s), 0)] \Rightarrow i \circ f = j \circ g \ \forall s \in S \end{cases}$$

so we have proved the square commutes.

Now we construct the morphism h through  $h_A$  and  $h_B$ . We construct it in the following way:

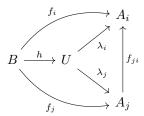
$$\begin{cases} h([(a,0)]) = h_A(a) \\ h([(0,b)]) = h_B(b) \end{cases} \Rightarrow h([(a,b)]) = h([(a,0)] + [(0,b)]) = h([(a,0)]) + h([(0,b)]) = h_A(a) + h_B(b)$$

and clearly this is well defined and unique as morphism in the category of abelian groups, so the other triangular diagrams commute as well and we conclude with the proof.

**6** Inverse limits in the category of sets / groups / abelian groups / modules: Let  $(\{A_i\}, \{f_{ji}\})$  be an inverse system over a preordered set I. Prove that

$$\varprojlim A_i = \{(a_i) \in \prod A_i | f_{ji}(a_j) = a_i \ i \le j\}$$

Let  $U = \{(a_i) \in \prod A_i | f_{ji}(a_j) = a_i \ i \leq j\}$ . We will prove that this is, in fact, the inverse limit we are looking for through the following diagram:



We can easily check the commutativity of the right part of the diagram because  $f_{ji} \circ \lambda_j((a_k)) = f_{ji}(a_j) = a_i = \lambda_i((a_k))$ .

We now prove that the morphism h is unique. h must be of the form  $h(c) = (h_k(c))$ , but, since the upper and lower part of the diagram must commute, we have  $\lambda_i \circ h(c) = f_i(c) \Rightarrow h_i = f_i$ , so the morphism we are looking form is  $h = (h_k)$ , and is unique.

7 Direct limits in the category of sets / groups / abelian groups / modules / rings with unit: Let  $(\{A_i\}, \{f_ij\})$  be a direct system over a directed set I. Prove that

$$\varinjlim A_i = \bigsqcup A_i / \sim$$

where  $a_i \sim a_j \iff f_{il}(a_i) = f_{jl}(a_j)$  for  $i, j \leq l$ 

Let  $U = \bigcup A_i / \sim$ . We will prove that this is, in fact, the direct limit we are looking for through the following diagram:

(diagram)

We have to check first that the right triangle of the diagram commutes. Since  $\lambda_i : a_i \mapsto [a_i]$ , we have that

$$\lambda_j \circ f_{ij}(a_i) = \lambda_j(a_j) = [a_j] = [a_i] = \lambda_i(a_i) \Rightarrow \lambda_j \circ f_{ij} = \lambda_i$$

where we have set l = j, so  $f_{ij}(a_i) = a_i = f_{ij}(a_i) \Rightarrow a_i \sim a_i \Rightarrow [a_i] = [a_i]$ .

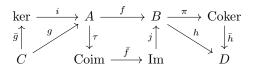
Now we check the commutativity of the  $h:[a_i]\mapsto f_i(a_i)$ . First we have to ensure that it is well defined. That is, if  $a_i\sim a_j\Rightarrow f_i(a_i)=f_j(a_j)$ . We suppose by symmetry that j>i, then by definition  $a_i\sim a_j\iff f_{il}(a_i)=f_{jl}(a_j)$  for  $i,j\leq l$ . Setting l=j we get  $f_{ij}(a_i)=f_{jj}(a_j)=a_j$ . Now, for the big diagram  $f_j(f_{ij}(a_i))=f_i(a_i)\Rightarrow f_j(a_j)=f_i(a_i)$  as desired.

Note that  $h([a_i]) = f_i(a_i)$  is the only possible choice we could have done in order to assure the commutativity of the upper diagram.

prob8Show that in an abelian category we have:

- (a) f is a monomorphism  $\iff \ker(f) = 0$
- (b) f is an epimorphism  $\iff$  Coker(f) = 0
- (c) A monomorphism is the kernel of its cokernel
- (d) An epimorphism is the cokernel of its kernel
- (e) Every morphism can be expressed as the composition of an epimorphism and a monomorphism
- (f) f is an isomorphism  $\iff f$  is an epimorphism and a monomorphism

We draw our commutative diagram. For being abelian the morphism  $\overline{g}$  must be unique for all g and the morphism  $\overline{h}$  must be unique for all h. Furthermore  $\overline{f}$  must be an isomorphism for all f.



(a)

 $\implies$  If f is mono, that means  $f \circ k = f \circ l \Rightarrow k = l$ . Let k = i, l = 0, then  $0 = f \circ i = f \circ 0 \Rightarrow i = 0$ . Since there exists a unique morphism  $\overline{g}: C \to \ker f$ , then  $\ker f$  is the unique terminal element of the abelian category 0.

(b)

 $\implies$  If f is mono, that means  $k \circ f = l \circ f \Rightarrow k = l$ . Let  $k = \pi, l = 0$ , then  $0 = \pi \circ f = 0 \circ f \Rightarrow \pi = 0$ . Since there exists a unique morphism  $\overline{h}$ : Coker  $f \to D$ , then Coker f is the unique initial element of the abelian category 0.