Problems Abstract Algebra
Second List

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1 *Nakayama's lemma.* Let M be a finitely generated $A-$module and I an ideal of A contained in the Jacobson radical. Prove:

$$IM = M \Rightarrow M = 0$$

We suppose $M \neq 0$. Let $x_1, x_2, \dots, x_n$ be a minimal set of generators of the module M. Because $M = IM$ we can express the element $x_1 = a_1x_1 + a_2x_2 + \cdots + a_nx_n$, where $a_i \in I$. Then

$$(a_1 - 1)x_1 + a_2x_2 + \cdots + a_nx_n = 0 \Rightarrow \begin{cases} a_1 - 1 = 0 \\ \vdots \\ a_n = 0 \end{cases}$$

But if $a_1 = 1 \in I$, that means $I = (1) = A$, which cannot be contained in the Jacobson radical.

4 Let (diagram) be a short exact sequence of $A - modules$. Prove that if M' and M'' are finitely generated, then M is finitely generated.

We start by fixing the set of generators of M' as $x_1, \dots, x_n$ and of M'' as $z_1, \dots, z_m$.

Since g is surjective, we can find elements $y_1, \dots, y_m$ such that $g(y_i) = z_i$. Now we select an arbitrary element $y \in M$. Then we have

$$g(y) = b_1z_1 + \cdots + b_mz_m = g(b_1y_1) + \cdots + g(b_my_m) \Rightarrow g(y - \sum b_iy_i) = 0 \Rightarrow y - \sum b_iy_i \in \ker(g)$$

for some $b_i \in A$. By exactness of the sequence we have $y - \sum b_iy_i \in \mathrm{Im}(f)$, so

$$y - \sum b_iy_i = f(\sum a_ix_i) = \sum a_if(x_i) \Rightarrow y = \sum a_if(x_i) + \sum b_iy_i$$

for some $a_i \in A$. Thus, a set of generators of M is $f(x_1), \dots, f(x_n), y_1, \dots, y_m$