Problems Abstract Algebra
First List

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> **1** Let f be a morphism in a category $\mathcal{C}$. Prove the following:
> (a) If f an isomorphism then f is a monomorphism and an epimorphism.
> (b) The inclusion of $\mathbb{Z}$ in $\mathbb{Q}$ is a monomorphism and an epimorphism in the category of rings but not an isomorphism.

We begin with the proof of (a). Since $f : A \to B$ is an isomorphism, that means there exist $g : B \to A$ such that both $g \circ f = Id_A$ and $f \circ g = Id_B$.

Let h, k morphisms of the category that fulfill $f \circ h = f \circ k$. Then by composing from the left with g we have

$$g \circ f \circ h = g \circ f \circ k \Rightarrow Id_A \circ h = Id_A \circ k \Rightarrow h = k$$

so we conclude f is a monomorphism.

Let h, k morphisms of the category that fulfill $h \circ f = k \circ f$. Then by composing from the right with g we have

$$h \circ f \circ g = k \circ f \circ g \Rightarrow h \circ Id_B = k \circ Id_B \Rightarrow h = k$$

so we conclude f is an epimorphism.

We move to the proof of (b). Let $i : \mathbb{Z} \to \mathbb{Q}$ the inclusion in $\mathbb{Q}$ ($i : n \mapsto n$). Now let $h, k \in \mathrm{Hom}_{rings}(A, \mathbb{Z})$ such that $i \circ h = i \circ k$. It is clear that, since $i(n) = n\ \forall n \in \mathbb{Z}$, then $h(a) = k(a)\ \forall a \in A$, concluding $h = k$ and i monomorphism.

Now let $h, k \in \mathrm{Hom}_{rings}(\mathbb{Z}, A)$ such that $h \circ i = k \circ i$. It is clear that, since $i(n) = n\ \forall n \in \mathbb{Z}$, then $h(i(a)) = k(i(a)) \Rightarrow h(a) = k(a)\ \forall a \in A$, concluding $h = k$ and i epimorphism.

Suppose i is an isomorphism. Thus, it must exists $g : \mathbb{Q} \to \mathbb{Z}$ such that $i \circ g = Id_{\mathbb{Q}}$ and $g \circ i = Id_{\mathbb{Z}}$. Let $a \in \mathbb{Z}$ such that $g(\frac{1}{2}) = a$. Then $i \circ g(\frac{1}{2}) = i(a) = a \neq \frac{1}{2}$, so $i \circ g \neq Id_{\mathbb{Q}}$, concluding f is not an isomorphism.

> **2** Show that in the category of finite dimensional vector spaces over a field $\mathbb{K}$ we have a natural equivalence of functors between the identity Id and the bidual $(-)^{**}$

We must prove that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\tau_X"] \arrow[d, "f"'] & X^{**} \arrow[d, "G(f)=f^{**}"] \\ Y \arrow[r, "\tau_Y"] & Y^{**} \end{tikzcd}$$

We will see that for the map $\tau_X(v)(w) = w(v)$ it holds. This is also call evaluation morphism $\tau_X(v) : w \mapsto w(v)$, that maps each element $v \in X$ into the evaluation in itself of the element in the dual space $w \in X^*$.

The morphisms f, f^* and f^{**} are given by

$$\begin{aligned} f : X &\to Y \\ v &\mapsto f(v) \end{aligned} \qquad \begin{aligned} f^* : Y^* &\to X^* \\ w &\mapsto f^*(w) = w \circ f \end{aligned} \qquad \begin{aligned} f^{**} : X^{**} &\to Y^{**} \\ v &\mapsto f^{**}(v) = v \circ f^* \end{aligned}$$

Now we see that the diagram commutes taking both paths. For $v \in X, w \in Y^*$

$$\begin{aligned} (\tau_Y \circ f)(v)(w) &= \tau_Y(f(v))(w) = w(f(v)) \\ (f^{**} \circ \tau_X)(v)(w) &= \tau_X(v) \circ f^*(w) = \tau_X(v)(w \circ f) = w(f(v)) \end{aligned}$$

proving that the diagram commutes. Thus, we have a natural equivalence.

3 Show that two categories $\mathcal{B}$ and $\mathcal{C}$ are naturally equivalent if and only if there exists a fully faithful and essentially surjective covariant functor $F : \mathcal{B} \to \mathcal{C}$.

$\boxed{\Rightarrow}$ Suppose F defines an equivalence of categories. Then, there exists a $G : \mathcal{C} \to \mathcal{B}$ such that we have $F \circ G \simeq Id_{\mathcal{C}}$ and $G \circ F \simeq Id_{\mathcal{B}}$.

To prove that is essentially surjective we choose $M' \in \mathbb{C}$ and

$$M = G(M') \Rightarrow F(M) = F \circ G(M') \simeq M'$$

We prove now that is faithful considering $f_1, f_2 : M \to N$ with $M, N \in \mathcal{B}$ such that $F(f_1) = F(f_2)$. Since we have the isomorphisms $u_M : M \to G(F(M))$ and $u_N : N \to G(F(M))$ with the commuting diagram

$$\begin{tikzcd} M \arrow[r, "u_M"] \arrow[d, "f_i"'] & G(F(M)) \arrow[d, "G(F(f_i))"] \\ N \arrow[r, "u_N"'] & G(F(N)) \end{tikzcd}$$

So we have $f_i = u_N^{-1} \circ G(F(f_i)) \circ u_M$ and $G(F(f_1)) = G(F(f_2)) \Rightarrow f_1 = u_N^{-1} \circ G(F(f_1)) \circ u_M = u_N^{-1} \circ G(F(f_2)) \circ u_M = f_2$, proving that is faithful.

Last thing we have to prove is that F is full, that is, for all morphism $f' : F(M) \to F(N)$ exists $f : M \to N$ that makes the diagram commutative

$$\begin{tikzcd} M \arrow[r, "u_M"] \arrow[d, "f"'] & G(F(M)) \arrow[d, "G(f')"] \\ N \arrow[r, "u_N"] & G(F(N)) \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "u_M"] \arrow[d, "f"'] & G(F(M)) \arrow[d, "G(F(f))"] \\ N \arrow[r, "u_N"] & G(F(N)) \end{tikzcd}$$

But setting $f = u_N^{-1} \circ G(f') \circ u_M$ we have $G(F(f)) = G(f')$, but since G is faithful we have $F(f) = f'$.

$\boxed{\Leftarrow}$ We start with $M' \in \mathcal{C}$. Since F is full we have $M \in \mathcal{B}$ such that there exists an isomorphism $u_{M'} : M' \to F(M)$. Let $G(M') = M$.

Let now a morphism $f' : M' \to N'$. Defining $f = G(f')$ such that the following diagram commutes

$$\begin{tikzcd} M' \arrow[r, "u_{M'}"] \arrow[d, "f'"'] & G(F(M')) \arrow[d, "G(F(f'))=F(f)"] \\ N' \arrow[r, "u_{N'}"'] & G(F(N')) \end{tikzcd}$$

in such a way that $F(f) = u_{N'} \circ f' \circ u_{M'}^{-1}$ by construction. Since F is fully faithful, there exists a unique morphism $f : M \to N$ accomplishing the condition, and thus $G(f') = f$ is well defined.

We prove now that G behaves like a functor, that is $G(g' \circ f') = G(g') \circ G(f')$. Since F is full, it is equivalent to show that $F \circ G$ is a functor. Considering the following diagram:

$$\begin{tikzcd}
M' \arrow[rrr, "u_{M'}"] \arrow[dr, "f'"] \arrow[dd, "g' \circ f'"'] & & & F(M) \arrow[dl, "F(G(f'))"'] \arrow[dd, "F(G(g' \circ f'))"] \\
& N' \arrow[r, "u_{N'}"'] \arrow[dl, "g'"] & F(N) \arrow[dr, "F(G(g'))"] & \\
P' \arrow[rrr, "u_{P'}"'] & & & F(P)
\end{tikzcd}$$

it can be deduced the functorial nature of $F \circ G$ and, thus, of G.

The last thing we have to do is the construction of the functorial isomorphisms of $F \circ G$ and $G \circ F$. But considering the first diagram of the exercise, te goal now is to construct the isomorphisms u_M and u_N for the diagram to commute. But, since F is full the only thing to check is

$$F(G(F(f))) \circ F(u_M) = F(u_M) \circ F(f)$$

which is true by definition of $F(f)$.

4 *Pullbacks in the category of abelian groups:* Let A and B be abelian groups together with homomorphisms $f : A \to S$ and $g : B \to S$. Prove that

$$A \times_S B = \{(a, b) \in A \times B | f(a) = g(b)\}$$

Let $U = \{(a, b) \in A \times B | f(a) = g(b)\}$. We will show that the pullback $A \times_S B$ is, in fact, U. We construct the following diagram:

$$\begin{tikzcd}
C \arrow[dr, "h"] \arrow[drr, bend left, "h_A"] \arrow[ddr, bend right, "h_B"'] & & \\
& U \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & S
\end{tikzcd}$$

We first construct the morphisms π_A and π_B that make the square commute. Those are

$$\begin{cases} \pi_A((a, b)) = a \\ \pi_B((a, b)) = b \end{cases} \quad \Rightarrow f \circ \pi_A((a, b)) = f(a) = g(b) = g \circ \pi_B((a, b)) \ \forall (a, b) \in U$$

thus, the square commutes.

Now we construct h from h_A and h_B. Note that, for the two triangular diagrams to commute, the h must fulfill:

$$\begin{cases} \pi_A(h(c)) = h_A(c) \\ \pi_B(h(c)) = h_B(c) \end{cases} \quad \forall c \in C \quad \Rightarrow \quad h = (h_A, h_B)$$

and the h is unique. We also note that the resulting set U has an abelian group strucutre, since it is close under the operation, every element has an inverse and is commutative

$$(a_1, b_1), (a_2, b_2) \in A \times_S B \Rightarrow f(a_1) = g(b_1), f(a_2) = g(b_2) \Rightarrow f(a_1 + a_2) = g(b_1 + b_2)$$
$$(a, b) \in A \times_S B \Rightarrow f(a) = g(b) \Rightarrow f(a^{-1}) = g(b^{-1}) \Rightarrow (a, b)^{-1} := (a^{-1}, b^{-1}) \in A \times_S B$$

concluding the proof.

5 *Pushouts in the category of abelian groups:* Let A and B be abelian groups together with homomorphisms $f : S \to A$ and $g : S \to B$. Prove that

$$A \sqcup_S B = \frac{A \oplus B}{W}$$

where W is the subgroup generated by $(f(s), -g(s))$ with $s \in S$.

Let $U = \frac{A \oplus B}{W}$. We will show that the pushout $A \sqcup_S B$ is, in fact, U. We construct the following diagram:

$$\begin{tikzcd}
C & & \\
& U \arrow[ul, "h"'] & A \arrow[l, "i"'] \arrow[llu, bend right, "h_A"'] \\
& B \arrow[u, "j"] \arrow[luu, bend left, "h_B"] & S \arrow[l, "g"] \arrow[u, "f"']
\end{tikzcd}$$

We first construct the morphisms i, j such that the square diagram commutes. We propose

$$i(a) = [(a, 0)] \qquad j(b) = [(0, b)]$$

and we check for commutativity for all $s \in S$

$$\begin{cases} i \circ f(s) = [(f(s), 0)] \\ j \circ g(s) = [(0, g(s))] \end{cases} \quad \text{but } [(0, g(s))] = [(0, g(s)) + (f(s), -g(s))] = [(f(s), 0)] \Rightarrow i \circ f = j \circ g \ \forall s \in S$$

so we have proved the square commutes.

Now we construct the morphism h through h_A and h_B. We construct it in the following way:

$$\begin{cases} h([(a, 0)]) = h_A(a) \\ h([(0, b)]) = h_B(b) \end{cases} \Rightarrow h([(a, b)]) = h([(a, 0)] + [(0, b)]) = h([(a, 0)]) + h([(0, b)]) = h_A(a) + h_B(b)$$

and clearly this is well defined and unique as morphism in the category of abelian groups, so the other triangular diagrams commute as well.

Finally we check that U is, in fact, an abelian group. Obviously is closed, has an inverse and is commutative by inheritance of A and B, so it is abelian.

6 *Inverse limits in the category of sets / groups / abelian groups / modules*: Let $(\{A_i\}, \{f_{ji}\})$ be an inverse system over a preordered set I. Prove that

$$\varprojlim A_i = \{(a_i) \in \prod A_i | f_{ji}(a_j) = a_i \ i \leq j\}$$

Let $U = \{(a_i) \in \prod A_i | f_{ji}(a_j) = a_i \ i \leq j\}$. We will prove that this is, in fact, the inverse limit we are looking for through the following diagram:

$$\begin{tikzcd}
& & A_i \\
B \arrow[r, "h"] \arrow[urr, bend left, "f_i"] \arrow[drr, bend right, "f_j"'] & U \arrow[ur, "\lambda_i"] \arrow[dr, "\lambda_j"'] & \\
& & A_j \arrow[uu, "f_{ji}"']
\end{tikzcd}$$

We can easily check the commutativity of the right part of the diagram because $f_{ji} \circ \lambda_j((a_k)) = f_{ji}(a_j) = a_i = \lambda_i((a_k))$.

We now prove that the morphism h is unique. h must be of the form $h(c) = (h_k(c))$, but, since the upper and lower part of the diagram must commute, we have $\lambda_i \circ h(c) = f_i(c) \Rightarrow h_i = f_i$, so the morphism we are looking form is $h = (h_k)$, and is unique.

Finally we must prove that the structure of U is the same than the structure in A_i. For sets is trivial. For groups

$$(a_i), (b_i) \in U \Rightarrow f_{ji}(a_j) = a_i, f_{ji}(b_j) = b_i \Rightarrow f_{ji}(a_j + b_j) = a_i + b_i \Rightarrow (a_i) + (b_i) \in U$$
$$(a_i) \in U \Rightarrow f_{ji}(a_j) = a_i \Rightarrow f_{ji}(a_j^{-1}) = a_i^{-1} \Rightarrow (a_i)^{-1} \in U$$

7 *Direct limits in the category of sets / groups / abelian groups / modules / rings with unit*: Let $(\{A_i\}, \{f_ij\})$ be a direct system over a directed set I. Prove that

$$\varinjlim A_i = \bigsqcup A_i / \sim$$

where $a_i \sim a_j \iff f_{il}(a_i) = f_{jl}(a_j)$ for $i, j \leq l$

Let $U = \bigsqcup A_i / \sim$. We will prove that this is, in fact, the direct limit we are looking for through the following diagram:

$$\begin{tikzcd}
& & A_i \arrow[dll, bend right, "f_i"'] \arrow[dl, "\lambda_i"] \\
B & U \arrow[l, "h"'] & \\
& & A_j \arrow[ul, "\lambda_j"'] \arrow[uu, "f_{ji}"'] \arrow[ull, bend left, "f_j"]
\end{tikzcd}$$

We have to check first that the right triangle of the diagram commutes. Since $\lambda_i : a_i \mapsto [a_i]$, we have that

$$\lambda_j \circ f_{ij}(a_i) = \lambda_j(a_j) = [a_j] = [a_i] = \lambda_i(a_i) \Rightarrow \lambda_j \circ f_{ij} = \lambda_i$$

where we have set $l = j$, so $f_{ij}(a_i) = a_j = f_{jj}(a_j) \Rightarrow a_i \sim a_j \Rightarrow [a_j] = [a_i]$.

Now we check the commutativity of the $h : [a_i] \mapsto f_i(a_i)$. First we have to ensure that it is well defined. That is, if $a_i \sim a_j \Rightarrow f_i(a_i) = f_j(a_j)$. We suppose by symmetry that $j > i$, then by definition $a_i \sim a_j \iff f_{il}(a_i) = f_{jl}(a_j)$ for $i, j \leq l$. Setting $l = j$ we get $f_{ij}(a_i) = f_{jj}(a_j) = a_j$. Now, for the big diagram $f_j(f_{ij}(a_i)) = f_i(a_i) \Rightarrow f_j(a_j) = f_i(a_i)$ as desired.

Note that $h([a_i]) = f_i(a_i)$ is the only possible choice we could have done in order to assure the commutativity of the upper diagram.

8 Show that in an abelian category we have:

(a) f is a monomorphism $\iff$ $\ker(f) = 0$

(b) f is an epimorphism $\iff$ $\mathrm{Coker}(f) = 0$

(c) A monomorphism is the kernel of its cokernel

(d) An epimorphism is the cokernel of its kernel

(e) Every morphism can be expressed as the composition of an epimorphism and a monomorphism

(f) f is an isomorphism $\iff$ f is an epimorphism and a monomorphism

We draw our commutative diagram. For being abelian the morphism $\overline{g}$ must be unique for all g and the morphism $\overline{h}$ must be unique for all h. Furthermore $\overline{f}$ must be an isomorphism for all f.

$$\begin{tikzcd}
\ker \arrow[r, "i"] & A \arrow[r, "f"] \arrow[d, "\tau"] & B \arrow[r, "\pi"] \arrow[dr, "h"] & \mathrm{Coker} \arrow[d, "\bar{h}"] \\
C \arrow[u, "\bar{g}"] \arrow[ur, "g"] & \mathrm{Coim} \arrow[r, "\bar{f}"] & \mathrm{Im} \arrow[u, "j"] & D
\end{tikzcd}$$

(a)

$\boxed{\Rightarrow}$ If f is mono, that means $f \circ k = f \circ l \Rightarrow k = l$. Let $k = i, l = 0$, then $0 = f \circ i = f \circ 0 \Rightarrow i = 0$. Since there exists a unique morphism $\overline{g} : C \to \ker f$, then $\ker f$ is the unique terminal element of the abelian category 0.

$\boxed{\Leftarrow}$

$$(f \circ g = f \circ h \Rightarrow f \circ (g - h)) \quad \Longleftrightarrow \quad (f \circ (g - h) = 0 \Rightarrow g - h = 0) \quad \Longleftrightarrow \quad (f \circ k = 0 \Rightarrow k = 0)$$

So proving that f is mono is the same as proving that $f \circ g = 0 \Rightarrow g = 0$. But if $\ker f = 0$, that means the morphism $i = 0$ is the zero morphism, and since the category is abelian $\forall g : C \to A$ such that $f \circ g = 0$ there exists a $\overline{g}$ such that $i \circ \overline{g} = g$. But since $i = 0 \Rightarrow g = 0$ proving $f \circ g = 0 \Rightarrow g = 0$ and then f is mono.

(b)

$\boxed{\Rightarrow}$ If f is mono, that means $k \circ f = l \circ f \Rightarrow k = l$. Let $k = \pi, l = 0$, then $0 = \pi \circ f = 0 \circ f \Rightarrow \pi = 0$. Since there exists a unique morphism $\overline{h} : \mathrm{Coker}\, f \to D$, then $\mathrm{Coker}\, f$ is the unique initial element of the abelian category 0.

$\boxed{\Leftarrow}$ it is completely analogous to (a).

(e)

Since τ is epi, j is mono and $\overline{f}$ is iso, then $\overline{f} \circ \tau$ is epi, and following the diagram $f = j \circ \overline{f} \circ \tau = j \circ (\overline{f} \circ \tau)$, which is the desired composition.