F-módulos

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F−modules

Endomorfismo de Frobenius

Sea R un anillo con característica p>0. Definimos el endomorfismo de Frobenius como el mapa

$$f: R \to R$$

 $r \to r^p$

Observación

Este morfismo en general no es inyectivo ni exhaustivo.

Module with Frobenius action

Given M an R-Module, we define the module $M^{(e)}$ induced by $f^{(e)}$ as the abelian group M endowed with the action

$$r \cdot m = f^{(e)}(r)m = r^{p^e}m$$

Notation

For simplicity we will write $M^{(1)}$ as M' and $R^{(1)}$ as R'.

Functor de Frobenius

Functor de Frobenius

Definimos el functor de Frobenius como el el functor

 $F: \mathbf{R} - \mathbf{Mod} \to \mathbf{R} - \mathbf{Mod}$ que envía

$$M \mapsto R' \otimes_R M, \qquad (M \stackrel{\phi}{\to} N) \mapsto R' \otimes_R M \stackrel{id \otimes_R \phi}{\to} R' \otimes_R N$$

Frobenius of a complex

Given the complex M^{\bullet} , we define its induced complex $F(M^{\bullet})$ as the complex

$$\cdots \longrightarrow M_{k-1} \xrightarrow{h_{k-1}} M_k \xrightarrow{h_k} M_{k+1} \longrightarrow \cdots$$

$$\downarrow^F \qquad \downarrow^F \qquad \downarrow^F$$

$$\cdots \longrightarrow F(M_{k-1}) \xrightarrow{F(h_{k-1})} F(M_k) \xrightarrow{F(h_k)} F(M_{k+1}) \longrightarrow \cdots$$

Exactly the same construction works for $F^{(e)}$.

Properties

Properties of Frobenius functor

- F is right exact. Furthermore, if R is regular, then R' is flat and F is exact.
- F commutes with direct sums.
- F commutes with localization.
- F commutes with direct limits.
- F preserves finitely generation of modules.
- \bullet If R is regular, then F commutes with cohomology of complexes.

Properties

Frobenius power ideal

Given $I = (x_1, ..., x_n)$ an ideal of R, we define its Frobenius e-power ideal as

$$I_{p^e} := (x_1^{p^e}, \dots, x_n^{p^e})R$$

Some examples of transformations

- $F(R) \cong R$
- $F(I) \cong I_{p^e}$
- $F(R/I) \cong R/I_{p^e}$

F—module

Definition of F-module

An F-module is an R-module M equipped with an R-isomorphism $\theta:M\to F(M)$ called the structure morphism.

Morphism of F—modules

Given two F- modules (M, θ_M) and (N, θ_N) , we say $f: M \to N$ is a morphism of F-modules if the following diagram commutes

$$\begin{array}{ccc}
M & \xrightarrow{g} & N \\
\downarrow^{\theta_M} & & \downarrow^{\theta_N} \\
F(M) & \xrightarrow{F(g)} & F(N)
\end{array}$$

An alternative form

F—modules can also be thought as a module over the ring R[F], that is, the ring R in which we have adjoined the non-commutative variable F with the relations $r^PF = Fr \ \forall r \in R$. This characterization is presented in [Bli04], and the notation R[F]—module taken in the thesis is very suggestive once we know where it comes from.

Two important cases

In the case M=R is the ring itself with R-module structure, we have a natural isomorphism $\theta:R\to F(R)$, which makes (R,θ) an F-module. This isomorphism is given by

$$\theta: R \to F(R) \cong R' \otimes_R R$$
$$r \mapsto r \otimes 1$$

Let $M=S^{-1}R$, then we have the isomorphism of R-modules $F(S^{-1}R)\cong S^{-1}R$. This is shown from the commutativity of the Frobenius functor with localization $F(S^{-1}R)\cong S^{-1}F(R)\cong S^{-1}R$. The natural isomorphism is given by

$$\theta: S^{-1}R \to R' \otimes_R S^{-1}R$$
$$\frac{r}{s} \mapsto rs^{p-1} \otimes \frac{1}{s}$$

F—finite modules

Generating morphism

Given an F-module (M, θ) we define its generating morphism $\theta_0: M_0 \to F(M_0)$ as the morphisms in the direct system

$$M_{0} \xrightarrow{\theta_{0}} F(M_{0}) \xrightarrow{F(\theta_{0})} F^{2}(M_{0}) \xrightarrow{F^{2}(\theta_{0})} \cdots \qquad M$$

$$\downarrow_{\theta_{0}} \qquad \downarrow_{F(\theta_{0})} \qquad \downarrow_{F(\theta_{0})} \qquad \downarrow_{\theta}$$

$$F(M_{0}) \xrightarrow{F(\theta_{0})} F^{2}(M_{0}) \xrightarrow{F^{2}(\theta_{0})} F^{3}(M_{0}) \xrightarrow{F^{3}(\theta_{0})} \cdots \qquad F(M)$$

whose limit is the module M and the morphism θ

F-finite module

We say that the module M is F-finite if M has a generating morphism $\theta_0: M_0 \to F(M_0)$ with M a finitely generated R-module.

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Local cohomology

Torsion functor

Let $\Gamma_I = \{m \in M : I^n m = 0 \text{ for some } n \in \mathbb{N}\}$. One can check this induces the so-called functor that transform the functions in the following natural way

$$\begin{array}{c}
M \xrightarrow{g} N \\
\downarrow \Gamma_{I} & \downarrow \Gamma_{I} \\
\Gamma_{I}(M) \xrightarrow{\Gamma_{I}(g)} \Gamma_{I}(N)
\end{array}$$

LC via torsion functor

Taking an injective resolution E^{\bullet} of M, we define the j-th local cohomology module of M with support in I as the j-th right derived functor of Γ_I , that is

$$H^j_I(M)=H^j(\Gamma_I(E^\bullet))$$

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Local cohomology

LC via Čech complex

Let $I = (x_1, \ldots, x_n) \subseteq R$. We define the Čech complex $\check{C}^{\bullet}(M, I)$ on the ideal / as

$$0 \longrightarrow M \xrightarrow{d_0} \bigoplus_{1 \le i \le n} M_{x_i} \xrightarrow{d_1} \bigoplus_{1 \le i < j \le n} M_{x_i x_j} \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} M_{x_1 \cdots x_n}$$

where de differential maps d_i are defined via the canonical localization morphism and alternating the sign in order to have $d_i \circ d_{i-1} = 0$. Explicitly we have the morphisms of every component $d_p:M_{x_{i_1}\cdots x_{i_p}} o M_{x_{j_1}\cdots x_{j_{n+1}}}$ as

$$d_p(m) = \begin{cases} (-1)^{k+1} \frac{m}{1} & \text{if } \{i_1, \dots, i_p\} = \{j_1, \dots, \hat{j}_k, \dots, j_{p+1}\} \\ 0 & \text{otherwise} \end{cases}$$

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Properties of LC

Two important properties of LC

- $\bullet \ H_I^j(M) = H_{\sqrt{I}}^j(M)$
- Let N be an A-module and the flat morphism $f: R \to A$. Then $A \otimes_R H^j_I(N) \cong H^j_{IA}(A \otimes_R N)$

If the ring R is regular, then for every ideal $I\subseteq R$ we have $F(H_I^j(R))\cong H_I^j(R)$

F—finiteness

F-finiteness of LC modules

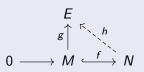
Given an ideal I of R, if M is F-finite, then $H_I^j(M)$ is F-finite.

Observation

Observe this is not the classical behaviour of finitely generated R-modules, since in general for finitely generated module M we will have non-finitely generated $H^j_l(M)$.

Injective module

We say the R-module E is injective if for all R-modules M,N and morphisms $f:M\to N$ injective and $g:M\to E$ arbitrary there exists a morphism $h:N\to E$ such that $h\circ f=g$. This is, the following diagram commutes:



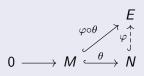
Equivalent characterizations

There are three equivalent characterizations, meaning that the following statements are equivalent

- E is an injective module.
- Any short exact sequence $0 \to E \to M \to N \to 0$ splits.
- If E is a submodule of M, then there exists another submodule $N \subseteq M$ such that $E \oplus N = M$.
- The functor Hom(-, E) is exact.

Injective hull

Given a module M, we define its injective hull as the maximal essential extension $N = E_R(M)$. That is, given an injective $\theta : M \to N$, if $\varphi \circ \theta$ is injective, then φ is also injective.



The structure theorem

Every injective R-module E is the direct sum of indecomposable injective modules with the form

$$E\cong igoplus_{\mathfrak{p}\in \operatorname{\mathsf{Spec}}(R)} E_R(R/\mathfrak{p})^{\mu_\mathfrak{p}}$$

with the Bass numbers $\mu_{\mathfrak{p}}$ independent of the decomposition.

Computing Bass numbers

Bass numbers can be computed as the rank of the Hom sets of residue fields in the following way:

$$\mu_{\mathfrak{p}} = \mathsf{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), E_{\mathfrak{p}})$$

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If R is regular, and E is an injective R-module then $F(E) \cong E$.

(Huneke,-Sharp)

Let (R, \mathfrak{m}) a regular local ring of characteristic p. Then the Bass numbers $\mu_i(\mathfrak{p}, H_I^j(R))$ are finite.

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