Problems Abstract Algebra First List

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- **1** Let f be a morphism in a category \mathcal{C} . Prove the following:
 - (a) If f an isomorphism then f is a monomorphism and an epimorphism.
 - (b) The inclusion of \mathbb{Z} in \mathbb{Q} is a monomorphism and an epimorphism in the category of rings but not an isomorphism.

We begin with the proof of (a). Since $f:A\to B$ is an isomorphism, that means there exist $g:B\to A$ such that both $g\circ f=Id_A$ and $f\circ g=Id_B$.

Let h, k morphisms of the category that fulfill $f \circ h = f \circ k$. Then by composing from the left with g we have

$$g \circ f \circ h = g \circ f \circ k \Rightarrow Id_A \circ h = Id_A \circ k \Rightarrow h = k$$

so we conclude f is a monomorphism.

Let h, k morphisms of the category that fulfill $h \circ f = k \circ f$. Then by composing from the right with g we have

$$h \circ f \circ g = k \circ f \circ g \Rightarrow h \circ Id_B = k \circ Id_B \Rightarrow h = k$$

so we conclude f is an epimorphism.

We move to the proof of (b). Let $i: \mathbb{Z} \to \mathbb{Q}$ the inclusion in \mathbb{Q} $(i: n \mapsto n)$. Now let $h, k \in \text{Hom}_{rings}(A, \mathbb{Z})$ such that $i \circ h = i \circ k$. It is clear that, since $i(n) = n \ \forall n \in \mathbb{Z}$, then $h(a) = k(a) \ \forall a \in A$, concluding h = k and i monomorphism.

Now let $h, k \in \text{Hom}_{rings}(\mathbb{Z}, A)$ such that $h \circ i = k \circ i$. It is clear that, since $i(n) = n \ \forall n \in \mathbb{Z}$, then $h(i(a)) = k(i(a)) \Rightarrow h(a) = k(a) \ \forall a \in A$, concluding h = k and i epimorphism.

Suppose i is an isomorphism. Thus, it must exists $g:\mathbb{Q}\to\mathbb{Z}$ such that $i\circ g=Id_\mathbb{Q}$ and $g\circ i=Id_\mathbb{Z}$. Let $a\in\mathbb{Z}$ such that $g(\frac{1}{2})=a$. Then $i\circ g(\frac{1}{2})=i(a)=a\neq\frac{1}{2}$, so $i\circ g\neq Id_\mathbb{Q}$, concluding f is not an isomorphism.

2 Show that in the category of finite dimensional vector spaces over a field \mathbb{K} we have a natural equivalente of functors between the identity Id and the bidual $(-)^{**}$

We consider the map $\tau: X \to X^{**}$ that sends $v \mapsto \operatorname{eval}(v)$, where $\operatorname{eval}(v) \in \operatorname{Hom}(\operatorname{Hom}(X,\mathbb{R}),\mathbb{R})$ is the evaluation in v morphism.

$$\begin{array}{ccc} X & \stackrel{\tau}{\longrightarrow} X^{**} \\ f \!\!\! \downarrow & & \!\!\! \downarrow^{G(f)} \\ Y & \stackrel{\tau}{\longrightarrow} Y^{**} \end{array}$$

Now we have to see that the diagram commutes. We will see that the construction G(f): eval $(v) \mapsto \text{eval}(f(v))$ works. If we pick $v \in X$, then

$$G(f)(\tau(v)) = \operatorname{eval}(f(v)) = \operatorname{eval}(w) = \tau(f(v)) \quad \Rightarrow \tau \circ f = G(f) \circ \tau$$

Proving the commutativity of the diagram. Now we left to proof that τ is an isomorphism $\forall X$, but this is immediate since we can construct the inverse τ^{-1} : $\operatorname{eval}(v) \to v$ provided every element of X^{**} is of the form $\operatorname{eval}(v)$ (recall we are working in a finite dimensional vector space).

3 Show that two categories \mathcal{B} and \mathcal{C} are naturally equivalent if and only if there exists a fully faithful and essentially surjective covariant functor $F: \mathcal{B} \to \mathcal{C}$.

 \Rightarrow Suppose F defines an equivalence of categories. Then, there exists a $G: \mathcal{C} \to \mathcal{B}$ such that we have $F \circ G \simeq Id_{\mathcal{C}}$ and $G \circ F \simeq Id_{\mathcal{B}}$.

To prove that is essentially surjective we choose $M' \in \mathbb{C}$ and

$$M = G(M') \Rightarrow F(M) = F \circ G(M') \simeq M'$$

We prove now that is faithful considering $f_1, f_2 : M \to N$ with $M, N \in \mathcal{B}$ such that $F(f_1) = F(f_2)$. Since we have the isomorphisms $u_M : M \to G(F(M))$ and $u_N : N \to G(F(M))$ with the commuting diagram

$$M \xrightarrow{u_M} G(F(M))$$

$$f_i \downarrow \qquad \qquad \downarrow_{G(F(f_i))}$$

$$N \xrightarrow{u_M} G(F(N))$$

So we have $f_i = u_N^{-1} \circ G(F(f_i)) \circ u_M$ and $G(F(f_1)) = G(F(f_2)) \Rightarrow f_1 = u_N^{-1} \circ G(F(f_1)) \circ u_M = u_N^{-1} \circ G(F(f_2)) \circ u_M = f_2$, proving that is faithful.

Last thing we have to prove is that F is full, that is, for all morphism $f':F(M)\to F(N)$ exists $f:M\to N$ that makes the diagram commutative

$$M \xrightarrow{u_M} G(F(M)) \qquad M \xrightarrow{u_M} G(F(M))$$

$$f \downarrow \qquad \qquad \downarrow_{G(f')} \qquad \qquad f \downarrow \qquad \qquad \downarrow_{G(F(f))}$$

$$N \xrightarrow{u_N} G(F(N)) \qquad \qquad N \xrightarrow{u_N} G(F(N))$$

But setting $f = u_N^{-1} \circ G(f') \circ u_M$ we have G(F(f)) = G(f'), but since G is faithful we have F(f) = f'.

 \sqsubseteq We start with $M' \in \mathcal{C}$. Since F is full we have $M \in \mathcal{B}$ such that there exists an isomorphism $u_{M'}: M' \to F(M)$. Let G(M') = M.

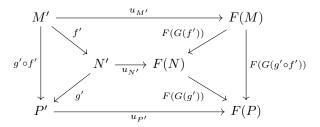
Let now a morphism $f': M' \to N'$. Defining f = G(f') such that the following diagram commutes

$$\begin{array}{ccc} M' & \xrightarrow{u_{M'}} G(F(M')) \\ f' \!\!\! & & \!\!\! \downarrow_{G(F(f')) = F(f)} \\ N' & \xrightarrow{u_{N'}} G(F(N')) \end{array}$$

in such a way that $F(f) = u_{N'} \circ f' \circ u_{M'}^{-1}$ by construction. Since F is fully faithful, there exists a unique morphism $f: M \to N$ accomplishing the condition, and thus G(f') = f is well defined.

We prove now that G behaves like a functor, that is $G(g' \circ f') = G(g') \circ G(f')$. Since F is full, it is

equivalent to show that $F \circ G$ is a functor. Considering the following diagram:



it can be deduced the functorial nature of $F \circ G$ and, thus, of G

The last thing we have to do is the construction of the functorial isomorphisms of $F \circ G$ and $G \circ F$. But considering the first diagram of the exercise, te goal now is to construct the isomorphisms u_M and u_N for the diagram to commute. But, since F is full the only thing to check is

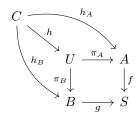
$$F(G(F(f))) \circ F(u_M) = F(u_M) \circ F(f)$$

which is true by definition of F(f).

4 Pullbacks in the category of abelian groups: Let A and B be abelian groups together with homomorphisms $f: A \to S$ and $g: B \to S$. Prove that

$$A \times_S B = \{(a, b) \in A \times B | f(a) = g(b)\}$$

Let $U = \{(a,b) \in A \times B | f(a) = g(b)\}$. We will show that the pullback $A \times_S B$ is, in fact, U. We construct the following diagram:



We first construct the morphisms π_A and π_B that make the square commute. Those are

$$\begin{cases} \pi_A((a,b)) = a \\ \pi_B((a,b)) = b \end{cases} \Rightarrow f \circ \pi_A((a,b)) = f(a) = g(b) = g \circ \pi_B((a,b)) \ \forall (a,b) \in U$$

thus, the square commutes.

Now we construct h from h_A and h_B . Note that, for the two triangular diagrams to commute, the h must fulfill:

$$\begin{cases} \pi_A(h(c)) = h_A(c) \\ \pi_B(h(c)) = h_B(c) \end{cases} \quad \forall c \in C \quad \Rightarrow \quad h = (h_A, h_B)$$

and the h is unique. We also note that the resulting set U has an abelian group strucutre, since it is close under the operation, every element has an inverse and is commutative

$$(a_1, b_1), (a_2, b_2) \in A \times_S B \Rightarrow f(a_1) = g(b_1), f(a_2) = g(b_2) \Rightarrow f(a_1 + a_2) = g(b_1 + b_2)$$

 $(a, b) \in A \times_S B \Rightarrow f(a) = g(b) \Rightarrow f(a^{-1}) = g(b^{-1}) \Rightarrow (a, b)^{-1} := (a^{-1}, b^{-1}) \in A \times_S B$

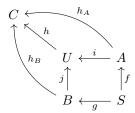
concluding the proof.

5 Pushouts in the category of abelian groups: Let A and B be abelian groups together with homomorphisms $f: S \to A$ and $g: S \to B$. Prove that

$$A \sqcup_S B = \frac{A \oplus B}{W}$$

where W is the subgroup generated by (f(s), -g(s)) with $s \in S$.

Let $U = \frac{A \oplus B}{W}$. We will show that the pushout $A \sqcup_S B$ is, in fact, U. We construct the following diagram:



We first construct the morphisms i, j such that the square diagram commutes. We propose

$$i(a) = [(a, 0)]$$
 $j(b) = [(0, b)]$

and we check for commutativity for all $s \in S$

$$\begin{cases} i \circ f(s) = [(f(s), 0)] \\ j \circ g(s) = [(0, g(s))] \end{cases} \text{ but } [(0, g(s))] = [(0, g(s)) + (f(s), -g(s))] = [(f(s), 0)] \Rightarrow i \circ f = j \circ g \ \forall s \in S \end{cases}$$

so we have proved the square commutes.

Now we construct the morphism h through h_A and h_B . We construct it in the following way:

$$\begin{cases} h([(a,0)]) = h_A(a) \\ h([(0,b)]) = h_B(b) \end{cases} \Rightarrow h([(a,b)]) = h([(a,0)] + [(0,b)]) = h([(a,0)]) + h([(0,b)]) = h_A(a) + h_B(b)$$

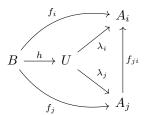
and clearly this is well defined and unique as morphism in the category of abelian groups, so the other triangular diagrams commute as well.

Finally we check that U is, in fact, an abelian group. Obviously is closed, has an inverse and is commutative by inheritance of A and B, so it is abelian.

6 Inverse limits in the category of sets / groups / abelian groups / modules: Let $(\{A_i\}, \{f_{ji}\})$ be an inverse system over a preordered set I. Prove that

$$\varprojlim A_i = \{(a_i) \in \prod A_i | f_{ji}(a_j) = a_i \ i \le j\}$$

Let $U = \{(a_i) \in \prod A_i | f_{ji}(a_j) = a_i \ i \leq j\}$. We will prove that this is, in fact, the inverse limit we are looking for through the following diagram:



We can easily check the commutativity of the right part of the diagram because $f_{ji} \circ \lambda_j((a_k)) = f_{ji}(a_j) = a_i = \lambda_i((a_k))$.

We now prove that the morphism h is unique. h must be of the form $h(c) = (h_k(c))$, but, since the upper and lower part of the diagram must commute, we have $\lambda_i \circ h(c) = f_i(c) \Rightarrow h_i = f_i$, so the morphism we are looking form is $h = (h_k)$, and is unique.

Finally we must prove that the structure of U is the same than the structure in A_i . For sets is trivial. For groups

$$(a_i), (b_i) \in U \Rightarrow f_{ji}(a_j) = a_i, f_{ji}(b_j) = b_i \Rightarrow f_{ji}(a_j + b_j) = a_i + b_i \Rightarrow (a_i) + (b_i) \in U$$

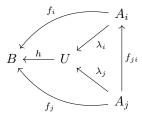
 $(a_i) \in U \Rightarrow f_{ji}(a_j) = a_i \Rightarrow f_{ji}(a_j^{-1}) = a_i^{-1} \Rightarrow (a_i)^{-1} \in U$

7 Direct limits in the category of sets / groups / abelian groups / modules / rings with unit: Let $(\{A_i\}, \{f_ij\})$ be a direct system over a directed set I. Prove that

$$\lim_{i \to \infty} A_i = |A_i| \sim$$

where $a_i \sim a_j \iff f_{il}(a_i) = f_{jl}(a_j)$ for $i, j \leq l$

Let $U = \bigsqcup A_i / \sim$. We will prove that this is, in fact, the direct limit we are looking for through the following diagram:



We have to check first that the right triangle of the diagram commutes. Since $\lambda_i : a_i \mapsto [a_i]$, we have that

$$\lambda_i \circ f_{ij}(a_i) = \lambda_i(a_j) = [a_i] = [a_i] = \lambda_i(a_i) \Rightarrow \lambda_i \circ f_{ij} = \lambda_i$$

where we have set l = j, so $f_{ij}(a_i) = a_j = f_{jj}(a_j) \Rightarrow a_i \sim a_j \Rightarrow [a_j] = [a_i]$.

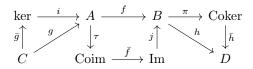
Now we check the commutativity of the $h:[a_i]\mapsto f_i(a_i)$. First we have to ensure that it is well defined. That is, if $a_i\sim a_j\Rightarrow f_i(a_i)=f_j(a_j)$. We suppose by symmetry that j>i, then by definition $a_i\sim a_j\iff f_{il}(a_i)=f_{jl}(a_j)$ for $i,j\leq l$. Setting l=j we get $f_{ij}(a_i)=f_{jj}(a_j)=a_j$. Now, for the big diagram $f_j(f_{ij}(a_i))=f_i(a_i)\Rightarrow f_j(a_j)=f_i(a_i)$ as desired.

Note that $h([a_i]) = f_i(a_i)$ is the only possible choice we could have done in order to assure the commutativity of the upper diagram.

8 Show that in an abelian category we have:

- (a) f is a monomorphism $\iff \ker(f) = 0$
- (b) f is an epimorphism \iff Coker(f) = 0
- (c) A monomorphism is the kernel of its cokernel
- (d) An epimorphism is the cokernel of its kernel
- (e) Every morphism can be expressed as the composition of an epimorphism and a monomorphism
- (f) f is an isomorphism $\iff f$ is an epimorphism and a monomorphism

We draw our commutative diagram. For being abelian the morphism \overline{g} must be unique for all g and the morphism \overline{h} must be unique for all h. Furthermore \overline{f} must be an isomorphism for all f.



(a)

 \implies If f is mono, that means $f \circ k = f \circ l \Rightarrow k = l$. Let k = i, l = 0, then $0 = f \circ i = f \circ 0 \Rightarrow i = 0$. Since there exists a unique morphism $\overline{g}: C \to \ker f$, then $\ker f$ is the unique terminal element of the abelian category 0.

 \Leftarrow

$$(f \circ g = f \circ h \Rightarrow f \circ (g - h)) \quad \Longleftrightarrow \quad (f \circ (g - h) = 0 \Rightarrow g - h = 0) \quad \Longleftrightarrow \quad (f \circ k = 0 \Rightarrow k = 0)$$

So proving that f is mono is the same as proving that $f \circ g = 0 \Rightarrow g = 0$. But if $\ker f = 0$, that means the morphism i = 0 is the zero morphism, and since the category is abelian $\forall g : C \to A$ such that $f \circ g = 0$ there exists a \overline{g} such that $i \circ \overline{g} = g$. But since $i = 0 \Rightarrow g = 0$ proving $f \circ g = 0 \Rightarrow g = 0$ and then f is mono.

(b)

 \implies If f is mono, that means $k \circ f = l \circ f \Rightarrow k = l$. Let $k = \pi, l = 0$, then $0 = \pi \circ f = 0 \circ f \Rightarrow \pi = 0$. Since there exists a unique morphism \overline{h} : Coker $f \to D$, then Coker f is the unique initial element of the abelian category 0.

 \leftarrow it is completely analogous to (a).

(e)

Since τ is epi, j is mono and \overline{f} is iso, then $\overline{f} \circ \tau$ is epi, and following the diagram $f = j \circ \overline{f} \circ \tau = j \circ (\overline{f} \circ \tau)$, which is the desired composition.