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1 Tema 1: Señales

Types of signals

$$\Pi(t) = 1 \text{ in } t \in (-\frac{1}{2}, \frac{1}{2})$$
$$\Delta(t) = 1 - |t| \text{ in } |t| < 1$$
$$sinc(t) = \frac{\sin(\pi t)}{\pi t}$$

Types of systems

- Linear: $ax_1 + bx_2 \to ay_1 + ay_2$
- Invariant: $x(t-\Delta) \to y(t-\Delta)$
- Casual: depende de las anteriores
- Stable: $|x(t)| < M_x \Rightarrow |y(t)| < M_y$

Energy and power

$$E = \int |x(t)|^2 dt, \qquad P = \lim \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt$$

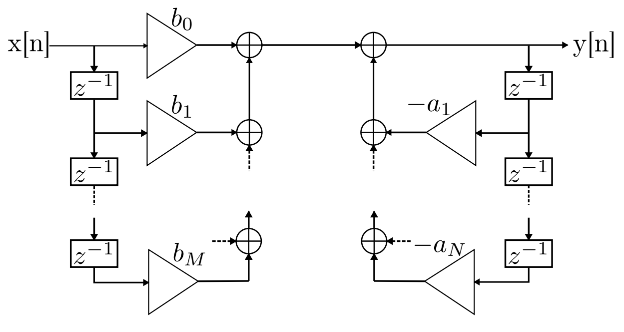
Linear time-invariant (LTI)

$$y(t) = x(t) * h(t) = \int x(\tau)h(t-\tau)d\tau = \sum x[i]h[n-i]$$

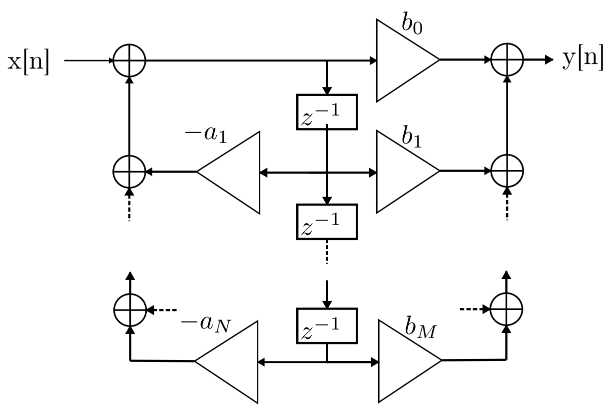
Differences equation LTI

$$y[n] = \sum_0^M b_j x[n-j] - \sum_1^N a_i y[n-i] \Rightarrow H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}$$

Forma directa 1



Forma directa 2 ($N = M$)



2 Tema 2: Transformadas

Transforms

$$x(t) \to X(\omega) = \int x(t)e^{-j\omega t}dt \qquad x(t) = \frac{1}{2\pi}\int X(\omega)e^{j\omega t}$$
$$x[n] \to X(z) = \sum x[n]z^{-n}$$

Fourier series

$$\begin{aligned} x(t) &= \frac{a_0}{2} + \sum (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)) \\ &= A_0 + \sum A_n \cos(n\omega_0 t + \varphi_n) = \sum c_n e^{jn\omega_0 t} \\ a_n &= \frac{2}{T}\int_T x(t)\cos(n\omega_0 t)dt, b_n = \frac{2}{T}\int_T x(t)\sin(n\omega_0 t)dt \\ A_n &= \sqrt{a_n^2 + b_n^2}, \varphi_n = \arctan\left(\frac{-b_n}{a_n}\right) \\ c_n &= \frac{a_n - jb_n}{2}, \quad c_{-n} = c_n^* \\ P_m &= \frac{1}{T}\int_T |x|^2 dt = \sum |c_n|^2 = A_0^2 + \sum A_n^2 \end{aligned}$$

LTI response to $x(t) = \sum A_n \cos(n\omega_0 t + \varphi_n)$

$$y(t) = \sum A_n |H(n\omega_0)| \cos(n\omega_0 t + \varphi + \angle H(n\omega_0))$$

Fourier transform properties

$$x(at) \to \frac{1}{|a|}X(\frac{\omega}{a}), \qquad x(t) \to\to 2\pi x(\omega)$$
$$x(t-\Delta) \to X(\omega)e^{-j\omega\Delta}, \qquad x(t)e^{j\omega_0 t} \to X(\omega - \omega_0)$$
$$\frac{dx(t)}{dt} \to j\omega X(\omega), \qquad \int_{-\infty}^t x(\tau)d\tau \to \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$$
$$f(t) * g(t) \to F(\omega)G(\omega), \qquad f(t)g(t) \to \frac{1}{2\pi}F(\omega) * G(\omega)$$

Signal	**Fourier transform**
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\Pi(\frac{t}{T})$	$Tsinc(\frac{\omega T}{2\pi})$
$\Delta(\frac{t}{T})$	$Tsinc^2(\frac{\omega T}{2\pi})$
$\cos(\omega_0 t)$	$\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$
$\sin(\omega_0 t)$	$-j\pi\delta(\omega - \omega_0) + j\pi\delta(\omega + \omega_0)$
$sinc(\frac{\omega_0 t}{\pi})$	$\frac{\pi}{\omega_0}\Pi(\frac{\omega}{2\omega_0})$
$sign(t)$	$\frac{2}{j\omega}$
$e^{-at}u(t)$	$\frac{1}{a+j\omega}$
$\frac{1}{t}$	$-j\pi sign(\omega)$

Z-transform properties

$$x[n-n_0] \to z^{-n_0}X(z), \quad x[-n] \to X(\frac{1}{z})$$
$$a^n x[n] = X(\frac{z}{a}), \quad nx[n] \to -z\frac{dX(z)}{dz}$$

Signal	Z-transform
$\delta[n]$	1
$p_N[n]$	$\frac{1-z^{-N}}{1-z^{-1}}$
$u[n]$	$\frac{1}{1-z^{-1}}$
$\cos(\Omega_0 n)u[n]$	$\frac{1-\cos(\Omega_0)z^{-1}}{1-2\cos(\Omega_0)z^{-1}+z^{-2}}$
$\sin(\Omega_0 n)u[n]$	$\frac{\sin(\Omega_0)z^{-1}}{1-2\cos(\Omega_0)z^{-1}+z^{-2}}$

Compute inverse Z-transform

$$X(z) = \frac{d_0 + d_1 z^{-1} + \cdots d_M z^{-M}}{c_0 + c_1 z^{-1} + \cdots c_N z^{-N}}$$

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}$$

$$x[n] = \sum_{r=0}^{M-N} B_r \delta(n-r) + \sum_{k=1}^{N} A_k p_k^n u[n]$$

Connections

Serie: $H = H_1 H_2$ Parallel: $H = H_1 + H_2$

Estabilidad

- Estable si $p_k \in D\ \forall k$
- Marginalmente estable si $p_k \in \overline{D}\ \forall k$
- Inestable si $\exists k : p_k \notin \overline{D}$

DFT

$$X_N[k] = DFT_N[x[n]] = X(e^{j\Omega})|_{\Omega = k\frac{2\pi}{N}} =$$
$$= \sum_{n=0}^{N-1} x[n] e^{-j\Omega n}|_{\Omega = k\frac{2\pi}{N}} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$
$$x_N[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_N[k] e^{j\frac{2\pi}{N}kn}$$

Properties

$$x[n]e^{j\frac{2\pi}{N}k_0 n} \to \tilde{X}_N[k-k_0]p_N[k]$$
$$x[n] \cdot y[n] \to \frac{1}{N} X_N[k] \circ Y_N[k]$$
$$x[n] \circ y[n] \to X[n]Y[n]$$
$$\tilde{x}_N[n-m] \cdot p_N[n] \to X_N[k] e^{-j\frac{2\pi}{N}km}$$

DFT of $A\cos\left(2\pi \frac{k_0}{N}\right)$:

$$X(k) = \begin{cases} \frac{AN}{2} & \text{si } k = k_0, N-k_0 \\ 0 & \text{otherwise} \end{cases}$$

3 Tema 3: Conversores A/D y D/A

$f_s = \frac{1}{T}$	frecuencia de sampleo (Hz)
$\omega_2 = \frac{2\pi}{T} = 2\pi f_s$	frecuencia de sampleo (rad/s)
ω_m	frecuencia máxima

Criterio de Nyquist $\omega_s > 2\omega_m$

Antialiasing filter $H(\omega) = \Pi(\frac{\omega}{\omega_s}),\ \Omega = \omega T$

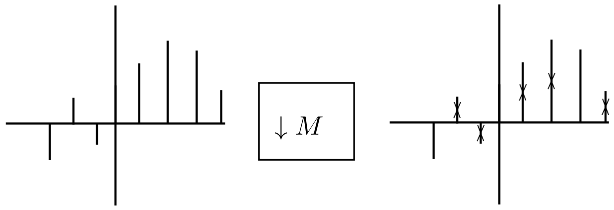
Tipos de interpolación

Linear	$h(t) = \Lambda(\frac{t}{T}),$
ZOH	$h(t) = \Pi(\frac{t-T/2}{T}),\ H(\omega) = T sinc(\frac{\omega}{\omega_s})e^{-j\omega\frac{T}{2}}$
Ideal	$h(t) = sinc\left(\frac{t}{T}\right),\ H(\omega) = \Pi(\frac{\omega}{\omega_s})$

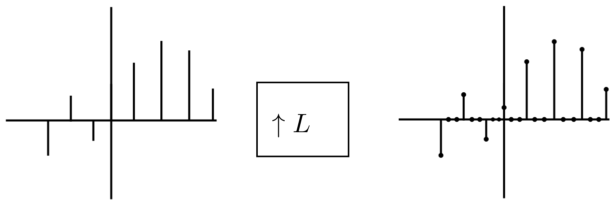
Transformada señal digital

$$p(t) = \sum \delta(t - nT) \Rightarrow P(\omega) = \frac{2\pi}{T} \sum \delta(\omega - \frac{2\pi}{T}k)$$
$$y(t) = x(t)p(t) \Rightarrow Y(\omega) = \frac{1}{T} \sum X(\omega - \frac{2\pi}{T}k)$$

Diezmado



Interpolacion



4 Tema 4: Random signals

Cumulative distrubution function (cdf) and probability density function (pdf)

$$F_X(x_1;t_1) = P(X(t_1) \le x_1), \quad f_X(x_1;t_1) = \frac{\partial F_X(x_1;t_1)}{\partial x_1}$$

Definition (Mean).

$$m_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_X(x,t) dx$$

Definition (Auto-correlation).

$$r_X(t_1,t_2) = E[X(t_1)X^*(t_2)] = \int\int x_1 x_2^* f_X(x_1,x_2;t_1,t_2) dx_1 dx_2$$
$$\overline{r}_X(\tau) = \frac{1}{T}\int_0^T r_X(t+\tau,t)dt$$

Definition (Instantaneous power).

$$P_X(t) = E[|X(t)|^2] = r_X(t,t)$$

Definition (Auto-covariance).

$$c_X(t_1,t_2) = r_X(t_1,t_2) - m_X(t_1)m_X(t_2)$$

Definition (Variance).

$$\sigma_X^2(t) = c_X(t,t) = r_X(t,t) - m_X(t)m_X^*(t)$$

Definition (Cross-correlation). *Orthogonal* $r_{XY} = 0$

$$r_{XY}(t_1, t_2) = E[X(t_1)Y^*(t_2)] = \int\int xy^* f_{XY}(x, y; t_1, t_2)dxdy$$

Definition (Cross-Covariance). *Uncorrelated* $c_{XY} = 0$

$$c_{XY}(t_1, t_2) = r_{XY}(t_1, t_2) - m_X(t_1)m_Y^*(t_2)$$

Independence $\Rightarrow$ Uncorrelation

$$\begin{aligned} \text{Independence} &\iff f_{XY}(x, y) = f_X(x)f_Y(y) \\ \text{Uncorrelation} &\iff E[XY] = E[X]E[Y] \end{aligned}$$

Definition (Stationary). .

- *1st order:* $f_X(x; t) = f(x; t + \Delta)\ \forall t, \Delta$
- *2nd order:* $f_X(x_1, x_2; t_1, t_2) = f(x_1, x_2; t_1 + \Delta, t_2 + \Delta)\ \forall t_1, t_2, \Delta$
- *WSS* $m_X(t) = cte, \quad r_X(t_1, t_2) = r_X(\tau)$
- *jointly WSS* X, Y *WSS,* $\quad r_{XY}(t_1, t_2) = r_X(\tau)$
- *Cyclostationary* $m_x(t), r_X(t, t + \tau$ *periodic*

Properties

$$\begin{aligned} &r_X(\tau) = r_X^*(-\tau), \quad r_{XY}(\tau) = r_{XY}^*(-\tau) \\ &|r_{XY}(\tau)| \leq \sqrt{r_X(0)r_Y(0)} \\ &r_{X+Y} = r_X(\tau) + r_Y(\tau) + r_{XY}(\tau) + r_{YX}(\tau) \end{aligned}$$

Definition (Power spectral density (PSD)).

$$S_X(\omega) = \lim_{T\to\infty} \frac{E[|X_T(\omega, S)|^2]}{2T}$$

Theorem (Wiener-Khinchin). $S_X(\omega) = F[\overline{r}_X(\tau)]$

Properties of PSD

$$\begin{aligned} &S_X(\omega) \geq 0, \quad S_X(\omega) = S_X(-\omega) \\ &P_X = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega \end{aligned}$$

WSS process before filter $H(\omega)$

- Mean $m_Y = m_x H(0)$
- Cross-correlation $r_{YX}(\tau) = r_X(\tau) * h(\tau)$
- Auto-correlation $r_Y(\tau) = r_X(\tau) * h(\tau) * h^*(-\tau)$
- Spectral density $S_Y(\omega) = S_X(\omega)|H(\omega)|^2$

Definition (Auto-correlation matrix).

$$R_X = \begin{pmatrix} r_X[0] & r_X[1] & \cdots & r_X[L-1] \\ r_X^*[1] & r_X[0] & \cdots & r_X[L-2] \\ \vdots & \vdots & \ddots & \vdots \\ r_X^*[L-1] & r_X^*[L-2] & \cdots & r_X[0] \end{pmatrix}$$

Properties of R_X

- $T_X = Q\Lambda Q^H$
- $P_Y = h^H R_X h = \sum \lambda_i |h^H q_i|^2$

Definition (Gaussian random variable).

$$\begin{aligned} f_X(x) &= \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(x-m_x)^2}{2\sigma_X^2}} \\ f_X(x) &= \frac{1}{\pi\sigma_X^2} e^{-\frac{|x-m_x|^2}{\sigma_X^2}} \\ f_X(x) &= \frac{1}{(2\pi)^{L/2}|C_X|^{1/2}} e^{-\frac{1}{2}(x-\mu_X)^T C_X^{-1}(x-\mu_X)} \\ f_X(x) &= \frac{1}{(\pi)^L |C_X|} e^{-(x-\mu_X)^H C_X^{-1}(x-\mu_X)} \end{aligned}$$

5 Tema 5: Estimation theory

Definition (Bias). $b_{\hat{\theta}} = E[\hat{\theta}(x) - \theta] = \mu_{\hat{\theta}} - \theta$

Definition (Covariance).

$$C_{\hat{\theta}} = E[(\hat{\theta}(x) - \mu_{\hat{\theta}})(\hat{\theta}(x) - \mu_{\hat{\theta}})^H], \qquad \sigma_{\hat{\theta}_i}^2 = (C_{\hat{\theta}})_{i,i}$$

Definition (Mean square error (MSE)).

$$M_{\hat{\theta}} = E[(\hat{\theta}(x) - \theta)(\hat{\theta}(x) - \theta)^H] = C_{\hat{\theta}} + b_{\hat{\theta}} b_{\hat{\theta}}^H$$

Definition (Minimum Variance Unbiased Estimator (MVUE)). $b_{\hat{\theta}} = 0, \quad \sigma_{\hat{\theta}}^2|_{min}$

Definition (Sharpness). $S = -E\left[\frac{\partial^2 \ln(f_\theta(x))}{\partial\theta^2}\right]$

Definition (Cramer-Rao Lower Bound (CRLB)).

$$\sigma_{\hat{\theta}}^2 \geq \frac{1}{E\left[\left|\frac{\partial ln(f_\theta(x)}{\partial\theta^*}\right|^2\right]} = \frac{1}{-E\left[\frac{\partial^2 ln(f_\theta(x)}{\partial\theta^*\partial\theta}\right]}$$

Definition (Efficient estimator).

$$\hat{\theta(x)} - \theta = CRLB(\theta) \cdot \frac{\partial \ln(f_\theta(x))}{\partial\theta}$$

Definition (Fisher information matrix).

$$(J(\theta))_{i,j} = E\left[\frac{\partial^2 \ln f_\theta(x)}{\partial\theta_i^*\partial\theta_j}\right] = -E\left[\frac{\partial \ln f_\theta(x)}{\partial\theta_i^*}\frac{\partial \ln f_\theta(x)}{\partial\theta_j}\right]$$

Definition (CRLB for multiple parameters).

$$\sigma_{\hat{\theta}_i}^2 \geq (J^{-1}(\theta))_{i,i}$$

Definition (Maximum Likelihood Estimator (MLE)).

$$\hat{\theta}_{ML}(x) = \arg(\max_\theta f_\theta(x))$$

Definition (Maximum A Posteriori (MAP)).

$$\hat{\theta}_{MAP}(x) = \arg(\max_\theta f_{\theta|x}(\theta|x)) = \arg(\max_\theta f_{x|\theta}(x|\theta)f_\theta(\theta))$$

Definition (Minimum Mean Square Error (MMSE)).

$$\hat{\theta}_{MMSE}(x) = \int \theta f_\theta(\theta) f_{x|\theta}(x|\theta)$$

6 Tema 6: Spectral estimation

Definition (Windowed sequence).

$$x_v[n] = x[n]v[n] = x[n]p_N[n], \quad |V(e^{j\Omega})| = \frac{\sin(N\Omega/2)}{\sin(\Omega/2)}$$

Definition (Triangular window).

$$w[m] = \frac{1}{N}v[m] * v[-m], \quad W(e^{j\Omega}) = \frac{1}{N}\left|V(e^{j\Omega})\right|^2$$

Definition (Biased estimation of auto-correlation).

$$\hat{r}_X[m] = \frac{1}{N}x_v[m] * x_v^*[-m] = \frac{1}{N}\sum x[n+m]x^*[n]v[n+m]v[n]$$

$$E[\hat{r}_X[m]] = \left(1 - \frac{|m|}{N}\right) r_X[m], \ |m| \leq N-1; \ 0 \ otherwise$$

Definition (Periodogram). *Biased*

$$\hat{S}_p(e^{j\Omega}) = F[\hat{r}[m]] = \frac{1}{N}|X_v(e^{j\Omega})|^2$$

$$E[\hat{S}_p(e^{j\Omega})] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_X(e^{j(\Omega-\theta)})\frac{1}{N}W(e^{j\theta})d\theta$$

Definition (Variance). *(Very complicated, approximation)*

$$Var[\hat{S}_p(e^{j\Omega})] \approx (S_x(e^{j\Omega}))^2$$

Definition (Estimation of the power). *Unbiased*

$$\hat{P}_X = \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{S}_p(e^{j\Omega})d\Omega = \frac{1}{N}\sum |x[n]|^2$$

$$E[\hat{P}_X] = E[\hat{r}_X[0]] = P_X$$

Definition (Unbiased estimator of auto-correlation).

$$\check{r}_X[m] = \begin{cases} \frac{1}{N-|m|}\sum_0^{N-|m|-1} x[n+|m|]x^*[n], \ |m| \leq N-1 \\ 0 \ otherwise \end{cases}$$

$E[\check{r}_X[m]] = r_X[m] \ si \ |m| \leq N-1$

Definition (Modified Periodogram). *(Any window)*

$$\hat{P}_X = \frac{1}{N}\sum_0^{N-1} |v[n]|^2|x[n]|^2$$

Definition (Blackman-Tukey). *Biased estimate of the auto-correlation* $(\hat{r}_X[m], |m| \leq N-1) \Rightarrow$ *Windowing* $\hat{r}_X[m]w_a[m], |m| \leq L-1 \Rightarrow$ *Fourier transform* $\hat{S}_{BT}(e^{j\Omega}) = F[\hat{r}_X[m]w_a[m]]$

The estimator is asymptotically unbiased if $N, L \to \infty, w_a[0] = 1$

Definition (Variance BT). E_{W_a} *energy of the window*

$$Var[\hat{S}_{BT}(e^{j\Omega})] \approx \frac{E_{W_a}}{N}S_X^2(e^{j\Omega})$$

(BARLETT-WELCH 36)

Definition (Models). $w[n] \to \boxed{h(z)} \to x[n], r_w[0] = \sigma^2$

$$AR(p) \Rightarrow H(z) = \frac{1}{1+\sum_1^p a_k z^{-k}}$$

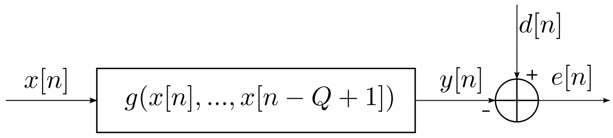
$$MA(q) \Rightarrow H(z) = 1 + \sum_1^q b_k z^{-k}$$

$$ARMA(p,q) \Rightarrow H(z) = \frac{1+\sum_1^q b_k z^{-k}}{1+\sum_1^p a_k z^{-k}}$$

Definition (Yule-Walker equations).

$$R_x \begin{pmatrix} a_1 \\ \vdots \\ a_p \end{pmatrix} = -\begin{pmatrix} r_x[1] \\ \vdots \\ r_x[p] \end{pmatrix}, \quad \sigma^2 = r_x[0] + \sum_1^p a_k r_x[-k]$$

7 Tema 7: Wiener Filtering



$$g = \mathbf{h}^H = [h^*[0], \ldots, h^*[Q-1]], \quad \mathbf{x}[n] = [x[n], \ldots, x[n-Q+1]]^T$$

Definition (Linear MSE estimator).

$$y[n] = \mathbf{h}^H\mathbf{x}[n], \quad e[n] = d[n] - y[n]$$

Theorem (MSE).

$$\xi = E[|e[n]|^2] = P_d + \mathbf{h}^H\mathbf{R}_x\mathbf{h} - \mathbf{h}^H\mathbf{r}_{xd} - \mathbf{r}_{xd}^H\mathbf{h}$$

Definition (Wiener-Hopf equations). *(minimize MSE)*

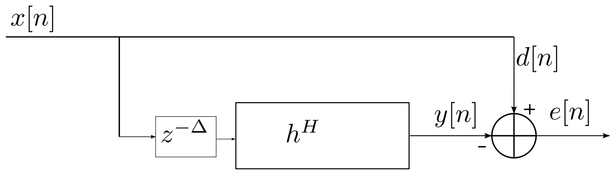
$$\mathbf{R}_x\mathbf{h}_{opt} = \mathbf{r}_{xd} \Rightarrow \mathbf{h}_{opt} = \mathbf{R}_x^{-1}\mathbf{r}_{xd}$$

Proposition. *MSE not using Wiener filter*

$$\xi = \xi_{min} + (\mathbf{h} - \mathbf{h}_{opt})^H\mathbf{R}_x(\mathbf{h} - \mathbf{h}_{opt})$$

$$E[\xi] = \xi_{min} + trace(\mathbf{R}_x E[(\mathbf{h} - \mathbf{h}_{opt})(\mathbf{h} - \mathbf{h}_{opt})^H])$$

Linear prediction



$$y[n] = \hat{x}[n] = \mathbf{h}^H\mathbf{x}[n-\Delta], \quad \xi_{min} = r_x[0] - \mathbf{h}_{opt}^H\mathbf{R}_x\mathbf{h}_{opt}$$

$$\mathbf{r}_{xd} = [r_x[-\Delta], \ldots, r_x[-\Delta - Q + 1]]^T, \quad \mathbf{h}_{opt} = \mathbf{R}_x^{-1}\mathbf{r}_x[-1]$$

Gradient method. Given a filter $h^{(k)}$ we want to find a new one $h^{(k+1)}$ with lower MSE. (Minimize $(h^{(k)} - h_{opt})^H R_x(h^{(k)} - h_{opt})$)

$$h^{(k+1)} = h^{(k)} - \mu\nabla_{h^*}\xi(h^{(k)}) = h^{(k)} - \mu(R_x h^{(k)} - r_{xd})$$

$$z^{(k)} := Q^H(h^{(k)} - h_{opt}) \Rightarrow z^{(k+1)} = (I - \mu\Lambda)z^{(k)}$$

Condiciones de convergencia

$$0 < \mu < \frac{2}{trace(R_x)} \leq \frac{2}{\lambda_{max}}, \quad \mu_{opt} = \frac{2}{\lambda_{min} + \lambda_{max}}$$

$$N_{ite} = \frac{\ln\varepsilon}{\ln|1 - \mu\lambda_i|}, \quad \chi := \frac{\lambda_{max}}{\lambda_{min}}$$

Definition (Instantaneus estimate of gradient (LMS)).

$$\nabla\xi(h^{(n)}) = -x[n]e^*[n] \Rightarrow h^{(n+1)} = h^{(n)} + \mu x[n]e^*[n]$$