Notes on Bounded Cohomology of Groups

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Contents

1 The setting

We briefly introduce the object we are about to study. Basic definitions are taken from [3]

In a similar way we define cochain complexes in usual topological cohomology we can define in the same way cochain complexes in group cohomology

Definition (Cochain complex)**.** *Given a discrete group* Γ *and an abelian group* A*, the cochain complex is defined as*

$$C^n(\Gamma, A) = \{f : \Gamma^{n+1} \to A\}, \qquad C_b^n(\Gamma, A) = \{f : \Gamma^{n+1} \to A | f \textit{ is bounded}\}$$

What makes a function bounded is the existence of a constant C such that $f(\gamma_0, \dots, \gamma_n) < C \ \forall \gamma_i \in \Gamma$

Definition (Coboundary operator)**.** *This operator "raises" the degree of the cochains* $\delta : C^n(\Gamma, A) \to C^{n+1}(\Gamma, A)$

$$\delta f(\gamma_0, \dots, \gamma_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(\gamma_0, \dots, \hat{\gamma}_i, \dots, \gamma_{n+1})$$

It is straightforward to check $\delta^{k+1} \circ \delta^k = 0$

However, if we define directly the cohomology from the above cochain complex, the information encoded in the structure of the group will be lost, treating the group like a set. For that reason we introduce the $\Gamma-$ invariant cochain complex.

Definition (Cochain complex of $\Gamma-$invariants)**.** *This is the subset of the cochain complex* $C^n(\Gamma, A)^\Gamma \subseteq C^n(\Gamma, A)$ *whose elements fulfill*

$$f(\gamma_0, \dots, \gamma_n) = f(\gamma\gamma_0, \dots, \gamma\gamma_n) \quad \forall \gamma \in \Gamma$$

This is what endows the cochains with the required structure to properly study a group. For convenience we will shorten the notation and we will make use of $\Gamma-$invariant cochain complex when not specified.

With this in mind we can define now the usual cohomology with the diagram in mind

$$0 \longrightarrow C^0(\Gamma, A) \xrightarrow{\delta^0} C^1(\Gamma, A) \xrightarrow{\delta^1} C^2(\Gamma, A) \xrightarrow{\delta^2} C^3(\Gamma, A) \xrightarrow{\delta^3} \cdots$$

Definition (Group Cohomology)**.** *We define the group cohomology from the Cochain complex and the coboundaty operator as* $H^n(\Gamma, A) = \dfrac{\ker(\delta^n)}{\mathrm{Im}(\delta^{n-1})}$

This is well defined in both cases, the bounded and the unbounded. We realize that the fact $C_b^n(\Gamma, A) \subseteq C^n(\Gamma, A)$ induces a map called comparison map

$$c : H_b^n(\Gamma, A) \to H^n(\Gamma, A)$$

The study of this comparison map is fundamental to understand how boundedness can change the setting of the problem.

2 Inhomogeneous formulation

To make computations in low degree we usually make use of inhomogeneous complex to transform the trivial action condition in something easier to work with.

Values of f can be calculated on tuples starting with 1.

$$f(\gamma_0, \dots, \gamma_n) = f(1, \gamma_0^{-1}\gamma_1, \gamma_0^{-1}\gamma_2, \dots, \gamma_0^{-1}\gamma_n)$$

and we let

$$\begin{cases} g_1 = \gamma_0^{-1}\gamma_1 \\ g_2 = \gamma_1^{-1}\gamma_2 \\ \vdots \\ g_n = \gamma_{n-1}^{-1}\gamma_n \end{cases} \quad \Rightarrow \quad f(\gamma_0, \dots, \gamma_n) \leftrightarrow f(1, g_1, g_1 g_2, \dots, g_1 g_2 \cdots g_n) := h(g_1, g_2, \dots, g_n)$$

And this defines a correspondence between the homogeneous and inhomogeneous complexes, which we denote by $\overline{C}^n$.

It is easy to check that the coboundary operator transforms in the following way

Definition (Inhomogeneous coboundary operator)**.**

$$\overline{\delta}^n h(g_1, \dots, g_{n+1}) = h(g_2, \dots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i h(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) + (-1)^{n+1} h(g_1, \dots, g_n)$$

Henceforth we will write $h(g_1, \dots, g_n)$ when working with inhomogeneous complexes and $f(\gamma_0, \dots, \gamma_n)$ with homogeneous. We forget the bar notation, since could be derived from the context.

Two classical computation follow, the groups H^0 and H^1:

Proposition. *For any discrete group* $H^0(\Gamma, A) = H^0_b(\Gamma, A) = A$

Proposition. *For any discrete group* $H^1(\Gamma, A) = Hom(\Gamma, A)$ *and* $H^1_b(\Gamma, \mathbb{Z}) = 0$

We might be surprised by the different result that boundedness gives us on the computation of H^1_b, but this is the consequence of the fact that there are no bounded homomorphisms from Γ to $\mathbb{Z}$ or $\mathbb{R}$.

3 Quasimorphisms

Computing H^2_b is far more complicated than H^0_b and H^1_b. There is a classical result that asserts H^2 is in one-to-one correspondence with the isomorphism classes of central extensions of Γ by A. For the case of bounded cohomology we introduce the idea of quasimorphism.

Definition (Quasimorphisms)**.** *Let* Γ *a group. The space of quasimorphisms is defined as follows*

$$QM(\Gamma) = \{f : \Gamma \to \mathbb{R} : \ \exists C > 0 \text{ such that } |f(g_1) + f(g_2) - f(g_1 g_2)| < C \ \forall g_1, g_2 \in \Gamma\}$$

The study of the following map is crucial for the understanding of the relationship between quasimorphisms and bounded cohomology of degree 2.

$$QM(\Gamma) \xrightarrow{A} \ker(H^2_b(\Gamma, \mathbb{R}) \xrightarrow{c} H^2(\Gamma, \mathbb{R}))$$

that sends each quasimorphism $\varphi \in QM(\Gamma)$ to $[\delta^1\varphi] \in H^2_b$. This is trivially well defined, since $\delta^2 \circ \delta^1 \varphi = 0$, in such a way that $\delta^1\varphi \in \ker(\delta^2)$, and, thus, is mapped to zero in $H^2(\Gamma, \mathbb{R})$. Taking into account the following diagram:

$$\begin{tikzcd} C^1 \arrow[r, "\delta^1"] & C^2 \arrow[r, "\delta^2"] & C^3 \\ C^1_b \arrow[u] \arrow[r, "\delta^1_b"] & C^2_b \arrow[u] \arrow[r, "\delta^1_b"] & C^3_b \arrow[u] \end{tikzcd} \quad \Rightarrow \quad \begin{cases} \ker(\delta^2_b) \subseteq \ker(\delta^2) \\ \mathrm{Im}(\delta^1_b) \subseteq \mathrm{Im}(\delta^1) \end{cases}$$

what suggest that there are some coboundaries $\delta^1\varphi$ of C^1 which are not coboundaries of C^1_b. This is the case of unbounded φ that leads to a quasimorphism (there are a few examples later developed such as Brook's or Rolli's quasimorphisms).

Thus, we can decompose every quasimorphism that maps to zero under the map A as a sum of homomorphism and a bounded function. $\ker(A) = B(\Gamma,\mathbb{R}) \oplus Hom(\Gamma,\mathbb{R})$. By the surjectivity of the map finally we have the isomorphism

$$\ker(A : H_b^2(\Gamma,\mathbb{R}) \to H^2(\Gamma,\mathbb{R})) \cong \frac{QM(\Gamma)}{Hom(\Gamma,\mathbb{R}) \oplus B(\Gamma,\mathbb{R})}$$

4 The free group

We apply now the techniques of bounded cohomology to the study of the free group. We start with the free group of two elements.

Definition (Free group on two elements)**.** *We call $F_2 = \langle a, b\rangle$ the group of reduced words generated by the alphabet $\{a, b, a^{-1}, b^{-1}\}$*

Now we recall that $H^2(F,\mathbb{R}) = 0$, and the kernel of the comparison map is the whole space $H_b^2(F,\mathbb{R})$, giving the isomorphism

$$H_b^2(F,\mathbb{R}) \cong \frac{QM(F)}{Hom(F,\mathbb{R}) \oplus B(\Gamma,\mathbb{R})}$$

What means that we can express every nontrivial element $[\delta\varphi] \in H_b^2$, being φ a quasimorphism which is not bounded nor a homomorphism.

The main advantage of working with cohomology instead of homology is that cohomology is endowed with a cup product defined in the following way:

Definition (Cup product)**.** *We define the cup product as the map*

$$\begin{aligned} \cup : H^n(G,\mathbb{R}) \times H^m(G,\mathbb{R}) &\to H^{n+m}(G,\mathbb{R}) \\ ([f],[g]) &\mapsto [f] \cup [g] \end{aligned}$$

where $(f \cup g)(g_1,\dots,g_n,g_{n+1},\dots,g_{n+m}) := f(g_1,\dots,g_n) \cdot g(g_{n+1},\dots,g_{n+m})$

Two main open questions are now natural to ask:

Open Problem. *Can we characterise the group $H_b^2(F,\mathbb{R})$?*

Open Problem. *Let $k > 0$ and $\alpha \in H_b^k(F,\mathbb{R})$ arbitrary. Let φ be a quasimorphism. Is the map*

$$\begin{aligned} \cup : H_b^2(F,\mathbb{R}) \times H_b^k(F,\mathbb{R}) &\to H_b^{k+2}(F,\mathbb{R}) \\ ([\delta^1\varphi],\alpha) &\mapsto \beta \end{aligned}$$

trivial? (i.e. $\beta = [0]$)

5 Known quasimorphisms

Although Brools, Rolli and $\Delta-$decomposable quasimorphisms are the most well-known quasimorphisms, we can extend the brooks quasimorphisms by summing them to form Calegari quasimorphisms.

5.1 Calegari Quasimorphisms

Calegari Quasimorphisms are a generalization of small Brooks quasimorphisms. This type of quasimorphisms can be thought as a weighted sum of small Brooks quasimorphisms

$$\varphi_\alpha := \sum_{w\in\mathcal{N}^+} \alpha_w h_w$$

where α is an alternating function $\alpha : F \to \mathbb{R}$ and the sum runs for the set of non-self-overlapping words.

Definition. *Let*

$$\kappa_\alpha(1) := \sup\left(\sum_{w\in C} |\alpha_w|\right)$$

be the supremum over all compatible families. This is an intrinsic characteristic of the function α.

Definition (Calegari Quasimorphisms)**.** *We say φ_α is a Calegari quasimorphism if $\kappa_\alpha(1) < \infty$*

5.2 Classification of Calegari quasimorphisms

In [2] the following containment map is proven.

$$\begin{tikzcd}
 & & Cal & & \\
 & l^1_{ind} \arrow[ur] & & \kappa(c_0) \arrow[ul] & \\
\Sigma_{Ind} \arrow[ur] & & l^1_{Br} \arrow[ul] \arrow[ur] & & \kappa(l^1) \arrow[ul] \\
 & & & wl^1_{Br} \arrow[ul] \arrow[ur] & \\
 & & \Sigma_{Br} \arrow[uull] \arrow[ur] & &
\end{tikzcd}$$

Definition (Σ_{Br} quasimorphism)**.** *We say $\varphi \in \Sigma_{Br}$ if is a finite sum of finite Brooks quasimorphisms*

Definition (wl^1_{Br} quasimorphism)**.** *We say $\varphi \in wl^1_{Br}$ if*

$$\sum_{w\in\mathcal{N}^+} |w||\alpha_\omega| < \infty$$

We will show that Theorem A (b) made in [1] for brooks quasimorphisms can be extended for wl^1_{Br} quasimorphisms.

Theorem. *If $\varphi_\alpha \in wl^1_{Br}$, then $[\delta\varphi_\alpha] \cup \omega = [0]$*

Proof. We can proof the following statement in the same way Amontova and Bucher did it in Theorem A (b) [1].

Let $\eta = \sum_{w\in\mathcal{N}^+} \eta_w$, with η_w defined as:

$$\eta_w(g, h_1, \ldots, h_{k-1}) = \sum_{j=1}^{m-l+1} \chi_w(x_j \cdot \ldots \cdot x_{j+l-1})\omega(z_{j+l}(g), h_1, \ldots, h_{k-1})$$

where $z_j(g) = x_j \cdot \ldots \cdot x_m$ with $g = x_1 \cdot \ldots \cdot x_m$

Now we let $\beta = \varphi_\alpha \cup \omega + \delta\eta = \sum_{w\in\mathcal{N}^+} (\alpha_w h_w \cup \omega + \delta\eta_w)$. One can check that $\delta\beta = (\delta\varphi_\alpha) \cup \omega$, and we only need to show that β is bounded.

From the original theorem we know that $\|\beta_w\| \leq (|w|-1)\|\omega\|_\infty$, so $\|\beta_w\| \leq \sum_{w\in\mathcal{N}^+}((|w|-1)\alpha_w)\|\omega\|_\infty < \infty$, because $\varphi_\alpha \in wl^1_{Br}$ $\square$

References

[1] Sofia Amontova and Michelle Bucher. Trivial cup products in bounded cohomology of the free group via aligned chains. In *Forum Mathematicum*, volume 34, pages 933–943. De Gruyter, 2022.

[2] Francesco Fournier-Facio. Infinite sums of brooks quasimorphisms and cup products in bounded cohomology. *arXiv preprint arXiv:2002.10323*, 2020.

[3] Roberto Frigerio. *Bounded cohomology of discrete groups*, volume 227. American Mathematical Soc., 2017.