Notes on Coxeter Matroids

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1 Matroids

Definition (Matroid)**.** *A base of a matroid M over a given a ground set $[n]$ is $\binom{\mathcal{B}(M)\subseteq}{[n],r}$, where r is the rank of the matroid. The set $\mathcal{B}$ must fulfill:*

- $A, B \in \mathcal{B}, a \in A - B \Rightarrow \; \exists b \in B - A : (A - \{a\}) \cup \{b\} \in \mathcal{B}$

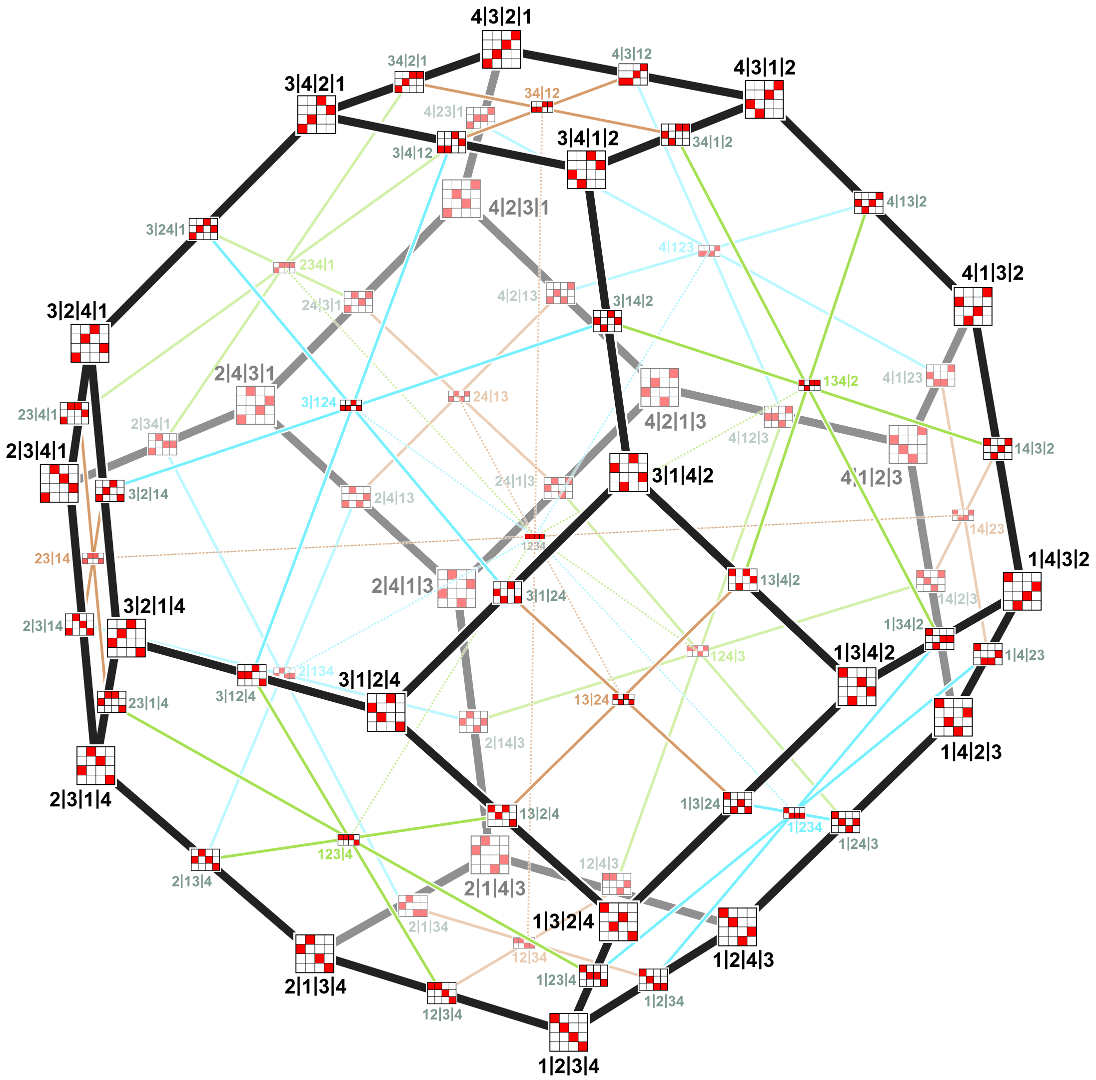
2 Permutahedron

2.1 Regular permutahedron

The permutahedron Π_n is generated by the convex hull of the vertices $V = \{(\sigma(1), \ldots \sigma(n)) : \sigma \in S_n\}$

There is a (fancy) bijection between the flags of $[n]$ and the faces of permutahedron Π_n as shown in the picture.

Flags could be interpreted as ordered partitions. One example of the three points of view as follows: $F = \{\{3\}, \{1, 2, 3, 4\}\} \iff 3|124 \iff$ "the face whose vertices have a 3 in the first position and the other three are free permutations".



2.2 Generalized permutahedra

Definition (Hypersimplex)**.** $\Delta(n, k) = \{(x_1, \ldots, x_n) : x_1 + \ldots + x_n = k\}$

The basis of $\Delta(n, k)$ (vertices of the polytope) is formed by vectors with k ones and $n - k$ zeroes.

Definition (Generalized Permutahedron)**.** *Convex polytope with all the edges parallel to* $e_i - e_j$

Permutahedron vertices came from a subset of the vertices of $\Delta(n, k)$

Definition (Matroid polytope)**.** *Matroid generated by the permutahedron whose vertices are a subset of* $\Delta(n, k)$

3 Coxeter Groups

A coxeter group is a finite group that has a set of generators $S = \{s_1, \ldots s_n\}$ and a function $m : S \times S \to \mathbb{N}$ with $m(s_i, s_i) = 1$ and $m(s_i, s_j) = m(s_j, s_i)$ such that the group is described with its presentation:

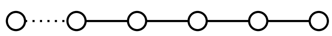
$$G = \{: s_i^2 = e;\ (s_i s_j)^{m(s_i, s_j)} = e\}$$

We can associate every generation with a hyperplane reflection passing through the origin $s_i \leftrightarrow \rho_i$

Suppose we have two generators s_i and s_j. Then we can represent this as two lines in a plane as follows:

If the angle $\angle \rho_i \rho_j = \frac{\pi}{k}$ we have that $s_i s_j$ is a rotation of angle $\frac{2\pi}{k}$. It follows that $m(s_i, s_j) = k$.

Coxeter groups have a fancy representation into a diagram. We represent each generator as a node, and we connect two nodes by a labeled edge with the value of $m(s_i, s_j) \geq 3$. If $m = 3$ it is not needed to label it.

A_n 

There is a correspondence with each flag selecting one variety to a generator, though it is not a bijection.

The flag is composed by a vertex, an edge containing the vertex, a face containing the edge . . . etc. If we focus on one variety, then the corresponding symmetry is the hyperplane ρ that fixes the lower dimension varieties and keeps invariant the higher ones.

We will see it more clear in the next subsection.

3.1 Classification of Platonic Solids

There exist 5 platonic solids. We can think all the solids in the projective space to make easier computations.

Tetrahedron A_3

Vertices are the indicator vectors of $\binom{[4]}{1}$

We have a flag $\{V_1, V_2, V_3\}$, where V_i are projective varieties that correspond to a vertex, edge and face respectively and for each V_i we have the induced hyperplane ρ_i associated with the reflection s_i. If we compute the normal vectors of the planes:

$$\rho_1^\perp = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \rho_2^\perp = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \rho_3^\perp = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \quad \Rightarrow \quad \angle\rho_1\rho_2 = \frac{\pi}{3}, \quad \angle\rho_1\rho_3 = \frac{\pi}{2}, \quad \angle\rho_2\rho_3 = \frac{\pi}{3}$$

And then we deduce the diagram ① —— ② —— ③

Cube B_3

Octahedron B_3

Vertices are the indicator vectors of $\binom{[4]}{2}$

We have a flag $\{V_1, V_2, V_3\}$, where V_i are projective varieties that correspond to a vertex, edge and face respectively and for each V_i we have the induced hyperplane ρ_i associated with the reflection s_i. If we compute the normal vectors of the planes:

$$\rho_1^\perp = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \rho_2^\perp = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \rho_3^\perp = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \quad \Rightarrow \quad \angle\rho_1\rho_2 = \frac{\pi}{3}, \quad \angle\rho_1\rho_3 = \frac{\pi}{2}, \quad \angle\rho_2\rho_3 = \frac{\pi}{4}$$

And then we deduce the diagram ① —— ② $\overset{4}{\text{——}}$ ③

Dodecahedron H_3

Icosahedron H_3

4 Description of Coxeter Groups

4.1 Group A_n

$\circ \longrightarrow \circ \longrightarrow \cdots \longrightarrow \circ \longrightarrow \circ$

We can think type A_n Coxeter Groups as groups generated by the reflections $s_1, \ldots, s_{n-1}$ whose associated hyperplanes are defined by

$$\rho_1^\perp = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \rho_2^\perp = \begin{pmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots \quad \rho_{n-1}^\perp = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix},$$

It is easy to see that the angles between the hyperplanes are

$$\angle\rho_i\rho_{i+1} = \frac{\pi}{3}, \quad \angle\rho_i\rho_j = \frac{\pi}{2} \ if \ i - j \neq \pm 1$$

so that corresponds to the Dynkin diagram, and thus to the Coxeter group

4.2 Group B_n

$\circ \xrightarrow{4} \circ \longrightarrow \cdots \longrightarrow \circ \longrightarrow \circ$

We can think type B_n Coxeter groups as the group generated by the reflections $\tau, s_1, \dots, s_n$, whose associated hyperplanes are defined by

$$\rho_\tau^\perp = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \rho_1^\perp = \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \rho_2^\perp = \begin{pmatrix} 0 \\ 1 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots \quad \rho_n^\perp = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 1 \end{pmatrix},$$

One can check that the angles between the hyperplanes are

$$\angle \rho_\tau \rho_1 = \frac{\pi}{4}, \quad \angle \rho_\tau \rho_i = \frac{\pi}{2} \ if \ i > 1, \quad \angle \rho_i \rho_{i+1} = \frac{\pi}{3}, \quad \angle \rho_i \rho_j = \frac{\pi}{2} \ if \ i - j \neq \pm 1$$

so the construction correspond to the Dynkin diagram, and thus to the coxeter group.

4.3 Group D_n

$$\begin{tikzcd}
\circ \arrow[dr] & & & & & \\
& \circ \arrow[r] & \circ \arrow[r] & \cdots \arrow[r] & \circ \arrow[r] & \circ \\
\circ \arrow[ur] & & & & &
\end{tikzcd}$$

We can think type D_n Coxeter groups as the group generated by the reflections $\tau_1, \tau_2, s_1, \dots, s_n$, whose associated hyperplanes are defined by

$$\rho_{\tau_1}^\perp = \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \rho_{\tau_2}^\perp = \begin{pmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \rho_1^\perp = \begin{pmatrix} 0 \\ 1 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots \quad \rho_n^\perp = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 1 \end{pmatrix},$$

One can check that the angles between the hyperplanes are

$$\angle \rho_{\tau_1} \rho_{\tau_2} = \frac{\pi}{2}, \quad \angle \rho_{\tau_1} \rho_1 = \frac{\pi}{3}, \quad \angle \rho_{\tau_2} \rho_1 = \frac{\pi}{3}, \quad \angle \rho_i \rho_{i+1} = \frac{\pi}{3}, \quad \angle \rho_i \rho_j = \frac{\pi}{2} \ if \ i - j \neq \pm 1$$

so the construction correspond to the Dynkin diagram, and thus to the coxeter group.

5 Regular Subdivision and height functions

Given a set of points T, we define a height function as $h : T \to \mathbb{R}$.

The height function h is said to be *M-convex* if the regular subdivisions induced by h are permutahedral.

A subset S of points is a (lower) regular subdivision induced by h if the convex hull of S is described by the lower convex hull of the polytope $T \times h(T)$

Definition (3-Term Plücker Relations)**.** *Let ω be a height function. For each $S \in \binom{[n]}{d-2}$ and $i, j, k, l \notin S$ the minimum*

$$\min(h(S_{ij}) + h(S_{kl}), h(S_{ik}) + h(S_{jl}), h(S_{il}) + h(S_{jl}))$$

is attained at least twice.

Theorem. *A height function induces a permutahedral regular division if the* ***3-Term Plücker Relations*** *(3TPR) holds.*

Proof. We first translate what does it mean for the set of vertices $S_{ij}, S_{ik}, S_{il}, S_{jk}, S_{jl}, S_{kl}$ to be a regular subdivision induced by h. Trivially these 6 vertices form a subpermutahedron of P.

Now we have to see that the convex hull of the vertices is a lower convex hull of all the points. This means we can assign it a linear functional $\varphi = (\varphi_1, \ldots, \varphi_n, \varphi_{n+1}),\ \varphi_{n+1} > 0$ that minimizes the set of 6 points over all the points. The consequence is that the functional evaluated in each of the 6 vertices should be equal. We abbreviate $h_{ij} := h(S_{ij})$

$$\varphi_{i_1} + \varphi_{i_2} + \varphi_{n+1} h_{i_1 i_2} = \varphi_{i_3} + \varphi_{i_4} + \varphi_{n+1} h_{i_3 i_4}\ \forall i_1, i_2, i_3, i_4 \in i, j, k, l \tag{1}$$

$$\varphi_{i_1} - \varphi_{i_3} = \varphi_{n+1}(h_{i_3 i_2} - h_{i_1 i_2}) \tag{2}$$

$$(\varphi_{i_2} - \varphi_{i_1})(h_{i_5 i_7} - h_{i_6 i_7}) = (\varphi_{i_6} - \varphi_{i_5})(h_{i_1 i_3} - h_{i_2 i_3}) \tag{3}$$

Making now $i_1 = i_5$ and $i_2 = i_6$ yields

$$h_{i_5 i_7} - h_{i_6 i_7} = h_{i_1 i_3} - h_{i_2 i_3} \quad or \quad \varphi_{i_2} = \varphi_{i_6} \tag{4}$$

We observe that if all φ_i are equal, then all the h_{ij} should be equal by (1)