Notes on Coxeter Matroids

Abel Doñate Muñoz
abel.donate@estudiantat.upc.edu

Contents

1 Matroids

Definition (Matroid)**.** *The set of bases of a matroid M over a given a ground set $[n]$ is $\binom{\mathcal{B}(M)\subseteq}{[n],r}$, where r is the rank of the matroid. The set $\mathcal{B}$ must fulfill:*

- $A, B \in \mathcal{B}, a \in A - B \Rightarrow \ \exists b \in B - A : (A - \{a\}) \cup \{b\} \in \mathcal{B}$

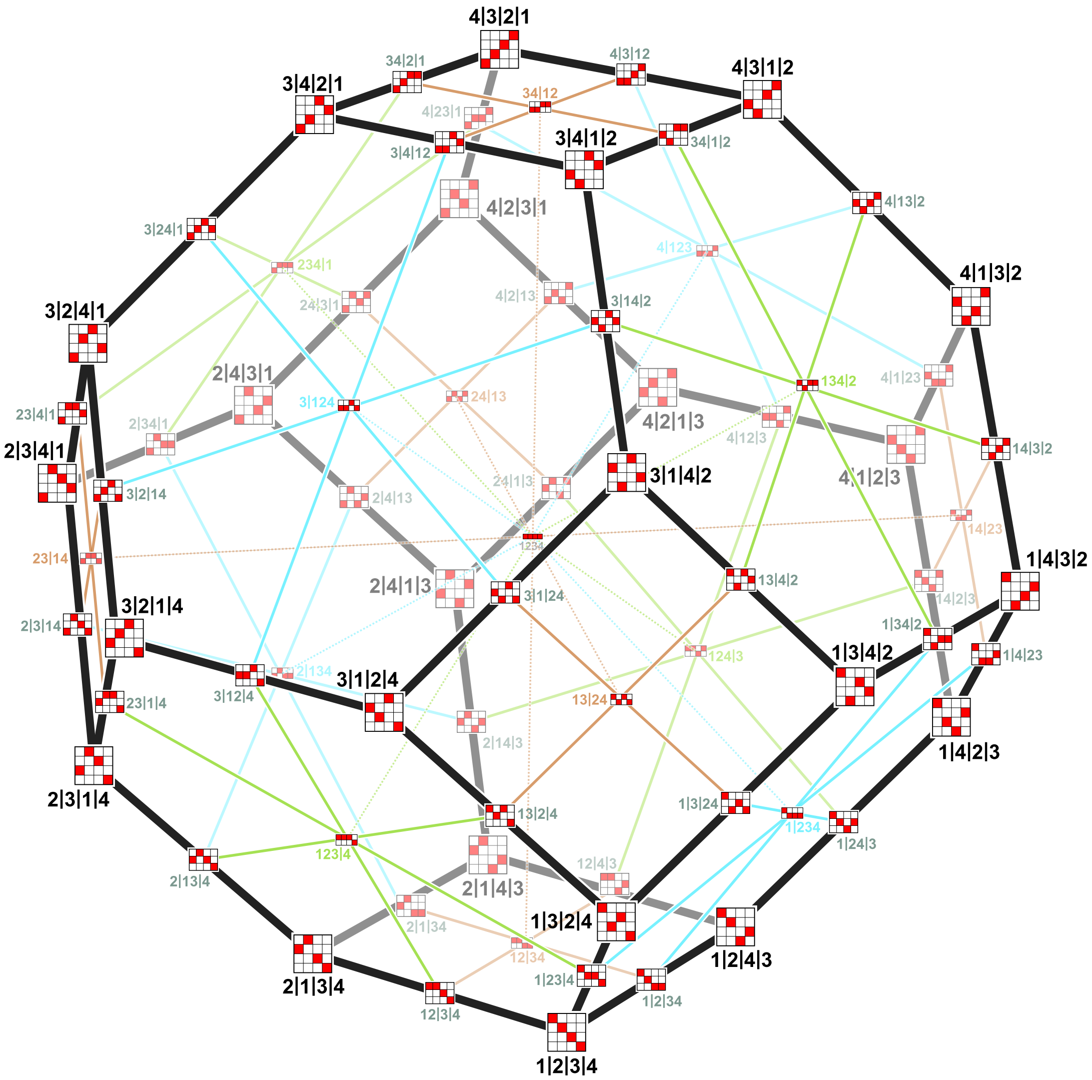
2 Permutahedron

2.1 Regular permutahedron

The permutahedron Π_n is the convex hull of the vertices $V = \{(\sigma(1), \dots \sigma(n)) : \sigma \in S_n\}$

There is a (fancy) bijection between the flags of $[n]$ and the faces of permutahedron Π_n as shown in the picture.

Flags could be interpreted as ordered partitions. One example of the three points of view as follows: $F = \{\{3\}, \{1, 2, 3, 4\}\} \iff 3|124 \iff$ "the face whose vertices have a 3 in the first position and the other three are free permutations".



2.2 Generalized permutahedra

Definition (Hypersimplex). $\Delta(n, k) = \{(x_1, \ldots, x_n) : x_1 + \ldots + x_n = k, 0 \leq x_i \leq 1\}$

The vertices of $\Delta(n, k)$ are vectors with k ones and $n - k$ zeroes.

Definition (Generalized Permutahedron). *Convex polytope with all the edges parallel to $e_i - e_j$*

Vertices of generalized permutahedron came from a subset of the vertices of $\Delta(n, k)$

Definition (Matroid polytope). *The Matroid (base) polytope corresponding to a matroid M is the convex hull of the indicator vectors of each basis of M.*

3 Coxeter Groups

Definition (Coxeter Group). *Given a set of generators $S = \{s_i\}$, a Coxeter group is a group whose presentation $\langle s_1, s_2, \ldots, s_n\rangle$ satisfy*

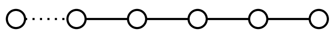
- $(s_i s_j)^{m_{ij}} = 1$
- $m_{ii} = 1$
- $m_{ij} \geq 2 \ \forall i \neq 2$

We can associate every generator with a hyperplane reflection passing through the origin $s_i \leftrightarrow \rho_i$

Suppose we have two generators s_i and s_j. Then we can represent this as two lines in a plane as follows:

If the angle $\angle \rho_i \rho_j = \frac{\pi}{k}$ we have that $s_i s_j$ is a rotation of angle $\frac{2\pi}{k}$. It follows that $m(s_i, s_j) = k$.

Coxeter groups can be represented using a diagram in which each generator is represented by a node and we connect two nodes by a labeled edge with the value of $m(s_i, s_j) \geq 3$. If $m = 3$ it is not needed to label it.

A_n 

There is a correspondence with each flag selecting one variety(?) to a generator, though it is not a bijection. (define before flag and find a word for variety)

The flag consists of a vertex, an edge containing the vertex, a 2-face containing the edge . . . etc. If we focus on one variety, then the corresponding symmetry is the hyperplane ρ that fixes the lower dimension varieties and keeps invariant the higher ones. (By keep invariant I mean the subspace goes to the same subspace, not necessarily point to point, while fix is point to point)

We will see it more clearly in the next subsection.

3.1 Classification of Platonic Solids

(Probably cut this section idk) There exist 5 platonic solids. We can think all the solids in the projective space to make easier computations.

Tetrahedron A_3

Vertices are the indicator vectors of $\binom{[4]}{1}$

We have a flag $\{V_1, V_2, V_3\}$, where V_i are projective varieties that correspond to a vertex, edge and face respectively and for each V_i we have the induced hyperplane ρ_i associated with the reflection s_i. If we compute the normal vectors of the planes:

$$\rho_1^\perp = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \rho_2^\perp = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \rho_3^\perp = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \quad \Rightarrow \quad \angle\rho_1\rho_2 = \frac{\pi}{3}, \quad \angle\rho_1\rho_3 = \frac{\pi}{2}, \quad \angle\rho_2\rho_3 = \frac{\pi}{3}$$

And then we deduce the diagram ① —— ② —— ③

Cube B_3

Octahedron B_3

Vertices are the indicator vectors of $\binom{[4]}{2}$

We have a flag $\{V_1, V_2, V_3\}$, where V_i are projective varieties that correspond to a vertex, edge and face respectively and for each V_i we have the induced hyperplane ρ_i associated with the reflection s_i. If we compute the normal vectors of the planes:

$$\rho_1^\perp = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \rho_2^\perp = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \rho_3^\perp = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \quad \Rightarrow \quad \angle\rho_1\rho_2 = \frac{\pi}{3}, \quad \angle\rho_1\rho_3 = \frac{\pi}{2}, \quad \angle\rho_2\rho_3 = \frac{\pi}{4}$$

And then we deduce the diagram ① —— ② $\overset{4}{\text{——}}$ ③

Dodecahedron H_3

Icosahedron H_3

4 Description of Coxeter Groups

4.1 Group A_n

$\circ \longrightarrow \circ \longrightarrow \cdots \longrightarrow \circ \longrightarrow \circ$

We can think type A_n Coxeter Groups as groups generated by the reflections $s_1, \ldots, s_{n-1}$ whose associated hyperplanes are defined by

$$\rho_1^\perp = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \rho_2^\perp = \begin{pmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots \quad \rho_{n-1}^\perp = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix},$$

It is easy to see that the angles between the hyperplanes are

$$\angle\rho_i\rho_{i+1} = \frac{\pi}{3}, \quad \angle\rho_i\rho_j = \frac{\pi}{2} \; if \; i - j \neq \pm 1$$

so that corresponds to the Dynkin diagram, and thus to the Coxeter group

4.2 Group B_n

$\circ \xrightarrow{4} \circ \longrightarrow \cdots \longrightarrow \circ \longrightarrow \circ$

We can think type B_n Coxeter groups as the group generated by the reflections $\tau, s_1, \dots, s_n$, whose associated hyperplanes are defined by

$$\rho_\tau^\perp = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \rho_1^\perp = \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \rho_2^\perp = \begin{pmatrix} 0 \\ 1 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots \quad \rho_n^\perp = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 1 \end{pmatrix},$$

One can check that the angles between the hyperplanes are

$$\angle \rho_\tau \rho_1 = \frac{\pi}{4}, \quad \angle \rho_\tau \rho_i = \frac{\pi}{2} \; if \; i > 1, \quad \angle \rho_i \rho_{i+1} = \frac{\pi}{3}, \quad \angle \rho_i \rho_j = \frac{\pi}{2} \; if \; i - j \neq \pm 1$$

so the construction correspond to the Dynkin diagram, and thus to the coxeter group.

4.3 Group D_n

$$\begin{tikzcd}
\circ \arrow[dr] & & & & & \\
& \circ \arrow[r] & \circ \arrow[r] & \cdots \arrow[r] & \circ \arrow[r] & \circ \\
\circ \arrow[ur] & & & & &
\end{tikzcd}$$

We can think type D_n Coxeter groups as the group generated by the reflections $\tau_1, \tau_2, s_1, \dots, s_n$, whose associated hyperplanes are defined by

$$\rho_{\tau_1}^\perp = \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \rho_{\tau_2}^\perp = \begin{pmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \rho_1^\perp = \begin{pmatrix} 0 \\ 1 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots \quad \rho_n^\perp = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 1 \end{pmatrix},$$

One can check that the angles between the hyperplanes are

$$\angle \rho_{\tau_1} \rho_{\tau_2} = \frac{\pi}{2}, \quad \angle \rho_{\tau_1} \rho_1 = \frac{\pi}{3}, \quad \angle \rho_{\tau_2} \rho_1 = \frac{\pi}{3}, \quad \angle \rho_i \rho_{i+1} = \frac{\pi}{3}, \quad \angle \rho_i \rho_j = \frac{\pi}{2} \; if \; i - j \neq \pm 1$$

so the construction correspond to the Dynkin diagram, and thus to the coxeter group.

5 Regular Subdivision and height functions

5.1 Hypersymplex $\Delta(k, n)$

Given a set of points T, we define a height function as $h : T \to \mathbb{R}$.

The height function h is said to be *M-convex* if the regular subdivision induced by h is permutahedral.

A subset S of points is a (lower) regular subdivision induced by h if the convex hull of S is described by the lower convex hull of the polytope $T \times h(T)$ (not well explained)

Definition (3-Term Plücker Relations). *Let ω be a height function on $\Delta(d, n)$. We say that 3TPR holds if for each $S \in \binom{[n]}{d-2}$ and $i, j, k, l \notin S$ the minimum*

$$\min(h(S_{ij}) + h(S_{kl}), h(S_{ik}) + h(S_{jl}), h(S_{il}) + h(S_{jl}))$$

is attained at least twice.

Theorem. *A height function induces a permutahedral regular division if the **3-Term Plücker Relations** (3TPR) holds.*

Proof

We abbreviate $h_{ij} = h(S_{ij})$

Suppose the 3TPR holds. Given the height function h we fix a linear functional $\varphi = (\varphi_1, \dots, \varphi_n, \varphi_{n+1})$ with $\varphi_{n+1} > 0$ such that the regular division polytope $Q \subseteq P$ is given by

$$Q = \{A \in M : \varphi(A, h(A)) \text{ is minimum}\}$$

We now can relabel the indexes of the elements of Q and φ removing all the terms that are 1 or zero allways (when applying the functional the result is the same inside Q) and relabeling the remaining from 1 to m.

In the cases $m = 2$ and $m = 3$ the subpolytope Q is trivially permutahedral. For case $m \geq 4$ we pick 4 indexes i, j, k, l of elements of Q and φ.

We can assume without loss of generality that $S_{ij}, S_{kl} \in Q$ now we want to prove that exchange axiom is satisfied: i.e. whithout loss of generality $S_{ik}, S_{jl} \in Q$.

We can express the 3TPR in the following way:

$$3TPR \iff h_{ij} + h_{kl} = h_{ik} + h_{jl} \leq h_{il} + h_{jk}$$

The condition for S_{ij}, S_{kl} to lie in a regular subdivision is that

$$\varphi_i + \varphi_j + \varphi_{n+1} h_{ij} = \varphi_k + \varphi_l + \varphi_{n+1} h_{kl} \leq \begin{cases} \varphi_i + \varphi_k + \varphi_{n+1} h_{ik} (1) \\ \varphi_j + \varphi_l + \varphi_{n+1} h_{jl} (2) \end{cases}$$

This leads to the equations

$$\begin{aligned} \varphi_j - \varphi_k &\leq \varphi_{n+1}(h_{ik} - h_{ij}) \\ \varphi_i - \varphi_l &\leq \varphi_{n+1}(h_{lj} - h_{ij}) \\ \varphi_l - \varphi_i &\leq \varphi_{n+1}(h_{ik} - h_{kl}) \\ \varphi_k - \varphi_j &\leq \varphi_{n+1}(h_{jl} - h_{kl}) \end{aligned}$$

Summing up the equations we get

$$0 \leq 2\varphi_{n+1}(h_{ik} + h_{jl} - h_{ij} - h_{kl}) \Rightarrow h_{ij} + h_{kl} \leq h_{ik} + h_{jl}$$

But this inequality should be an equality under de assumption of 3TPR, so all the above inequalities should be equalities, and (1) and (2) implies $S_{ik}, S_{jl} \in Q$

□

5.2 Hypercube $\square_n$

Now we switch from type A_n coxeter groups to type B_n

Definition (Hypercube)**.** $\square_n = conv\{x \in \mathbb{R}^n : x_i = \pm 1\}$

We notice the following injection, labeling the indices of $\Delta(n, 2n)$ as $-n, -n+1, \dots -1, 1, 2, \dots, n$: (explain more)

$$\begin{aligned} V(\square_n) &\to V(\Delta(n, 2n)) \\ x_i = 1 &\mapsto y_i = 1, y_{-i} = 0 \\ x_i = -1 &\mapsto y_{-i} = 1, y_i = 0 \end{aligned}$$

If we restrict the posible values of the vertices of $\Delta(n, 2n)$ such that they fulfill $y_i + y_{-i} = 1$ we get the bijection

$$\square_n \cong \Delta(n, 2n)|_{y_i+y_{-i}=1}$$

Observation. *Edges parallel to e_j in $\square_n \iff$ edges parallel to $e_j - e_{-j}$ in $\Delta(n, 2n)$*

Definition (Cubohedral)**.** *The cubohedral is the convex hull of a subset of vertices of the hypercube $C \subseteq \square_n$ whose edges are parallel to e_j*

We now want to find a relation similar to 3TPR in the case of Coxeter groups of type B_n.

Since the edges of C are be identified with the vertices of a subset $P \subseteq \Delta(n, 2n)$, and the edges of this subset are parallel to $e_j - e_{-j}$ by the observation, then P is permutahedral.

Now we consider a height function on $h : \square_n \to \mathbb{R}$ which we can identify as a height funcion h' : $\Delta(n, 2n) \Big/ y_i + i_{-i} = 1$. Thus, we can apply 3TPR to

Notation: Given a height function h, we say $h_{i-j} := h(T_{i-j})$, where T_{i-j} is the element T such that $x_i = 1, x_j = -1$.

Definition (New Relations)**.** *Let h be an height function on $\square_n$. We say that New Relations (NR) holds if for each $T \in \binom{[n]}{d-2}$ and i, j*

$$h(T_{ij}) + h(T_{-i-j}) = h(T_{-ij}) + h(T_{i-j})$$

Theorem. *A height function induces a cubohedral regular division if the **New Relations** (NR) hold.*

Proof.

Suppose NR holds. Given the height function h we fix a linear functional $\varphi = (\varphi_1, \ldots, \varphi_n, \varphi_{n+1})$ with $\varphi_{n+1} > 0$ such that the vertices of the regular division polytope $C \in \square_n$ is given by

$$C = \{A \in M : \varphi(A, h(A)) \text{ is minimum}\}$$

We now can relabel the indexes of the elements of C and φ removing all the terms that are the same in all vectors and relabeling the remaining from 1 to m.

If $m = 1$ the result is trivially cubohedral (1 point). If $m \geq 2$ we can pick two indexes i, j and assume without loss of generality $T_{ij}, T_{-i-j} \in A$. Now out work is to proof that $T_{i-j}, T_{-ij} \in A$.

The conditions for T_{ij}, T_{-i-j} to lie in the regular subdivision is that

$$\varphi_i + \varphi_j + \varphi_{n+1} h_{ij} = -\varphi_i - \varphi_j + \varphi_{n+1} h_{-i-j} \leq \begin{cases} \varphi_i - \varphi_j + \varphi_{n+1} h_{i-j} \\ -\varphi_i + \varphi_j + \varphi_{n+1} h_{-ij} \end{cases}$$

This leads to the equations

$$\begin{aligned} 2\varphi_j &\leq \varphi_{n+1}(h_{i-j} - h_{ij}) \\ 2\varphi_i &\leq \varphi_{n+1}(h_{-ij} - h_{ij}) \\ -2\varphi_i &\leq \varphi_{n+1}(h_{i-j} - h_{-i-j}) \\ -2\varphi_j &\leq \varphi_{n+1}(h_{-ij} - h_{-i-j}). \end{aligned}$$

Summing up the equations we get

$$0 \leq 2\varphi_{n+1}(h_{i-j} + h_{-ij} - h_{ij} - h_{-i-j}) \Rightarrow h_{ij} + h_{-i-j} \leq h_{i-j} + h_{-ij}$$

But this inequality should be an equality by NR, so all the inequalities before must be equalities, meaning $T_{i-j}, T_{-ij} \in A$

$\square$