Coxeter Matroids

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Overview

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What is a matroid?

Abstract definition (Basis)

Given a ground set $[n]$, a subset $\mathcal{B}(M) \subseteq [n]$ is the set of basis of a matroid M if satisfies the exchange axiom:

$$\forall A, B \in \mathcal{B}(M), a \in A - B \ \exists b \in B - A \ : \ A - \{a\} \cup b \in \mathcal{B}(M)$$

Rank

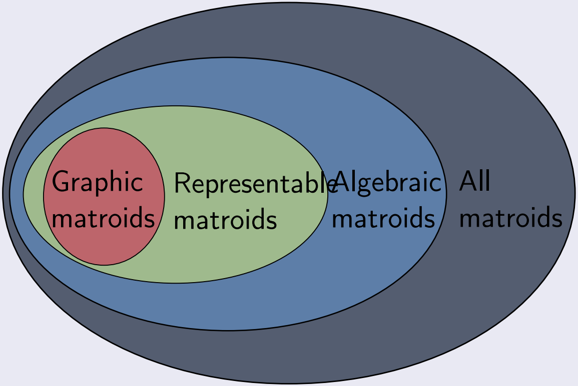
It can be seen that the cardinal of the elements of $\mathcal{B}(M)$ must be the same. We will call it $k = rank(M)$

The elements of $\mathcal{B}(M)$ are called *Maximal independent sets*. We will notice the intuition of this naming in the next slide.

Representable Matroids

Matroids can be represented in several ways. The following diagram shows the different types of matroids and its relations.

Diagram of matroids

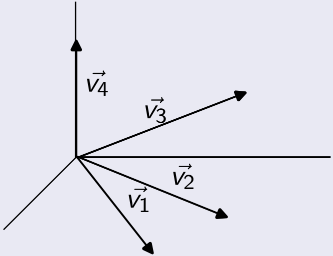


Representable Matroids

Definition

In a representable matroid we assign a vector to every element of the ground set $i \in [n] \to v_i \in V$. Now all the maximal independent sets of the matroids can be thought as sets of vectors that are a basis of a vector subspace $W \subseteq V$ of rank k.

Example



Independent sets

We notice that v_1, v_2, v_3 lie all in the same plane, so they are linearly dependent. We can make a basis of $\mathbb{R}^3$ in the following ways

$$\langle v_1, v_2, v_4\rangle, \langle v_1, v_3, v_4\rangle, \langle v_2, v_3, v_4\rangle$$

So the basis of the matroid are

$$124, 134, 234$$

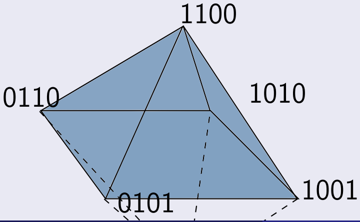
Matroid base polytope

Definition

A matroid (base) polytope is a geometrical representation of the basis of a matroid in a vector space.
Given a matroid M of rank k and the set of basis $\mathcal{B}(M) \subseteq \binom{[n]}{k}$, we assign each basis to its indicator vector $A \in \mathcal{B}(M) \to e_A = e_{i_1} + \ldots + e_{i_k} \in V$.
Thus, the convex hull of such vectors is a polytope $P \subseteq \Delta(n, k)$

Example

$$\mathcal{B}(M) = \{12, 13, 14, 23, 24\}$$



Matroid polytope

The polytope will have the following vertices:

$$12 \to (1, 1, 0, 0)$$
$$13 \to (1, 0, 1, 0)$$
$$14 \to (1, 0, 0, 1)$$
$$23 \to (0, 1, 1, 0)$$

Coxeter Groups

Definition

Given a set of generators $S = \{s_i\}$, a Coxeter group is a group described by its presentation

$$\langle s_1, s_2, \ldots, s_n | (s_i s_j)^{m_{ij}=1} \rangle, \quad m_{ii} = 1, \quad m_{ij} \geq 2 \ \forall i \neq j$$

Coxeter groups are usually thought as finite groups of reflections into a vector space V. Each generator s_i is associated with a hyperplane ρ_i such as s_i is the reflection by ρ_i.

Dynkin diagram

We can assign every Coxeter group a diagram in the following way:

- Nodes are the set of generators S
- We connect nodes s_i, s_j by an edge if $m_{ij} \geq 3$
- We label the edges with m_{ij} if it is ≥ 4

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