Notes on Coxeter Matroids

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1 Matroids

Definition (Matroid). *A base of a matroid M over a given a ground set $[n]$ is $\binom{\mathcal{B}(M)\subseteq}{[n],r}$, where r is the rank of the matroid. The set $\mathcal{B}$ must fulfill:*

- $A, B \in \mathcal{B}, a \in A - B \Rightarrow \exists b \in B - A : (A - \{a\}) \cup \{b\} \in \mathcal{B}$

2 Permutahedron

2.1 Regular permutahedron

The permutahedron Π_n is generated by the convex hull of the vertices $V = \{(\sigma(1), \ldots \sigma(n)) : \sigma \in S_n\}$

There is a (fancy) bijection between the flags of $[n]$ and the faces of permutahedron Π_n as shown in the picture.

Flags could be interpreted as ordered partitions. One example of the three points of view as follows:
$F = \{\{3\}, \{1, 2, 3, 4\}\} \iff 3|124 \iff$ "the face whose vertices have a 3 in the first position and the other three are free permutations".
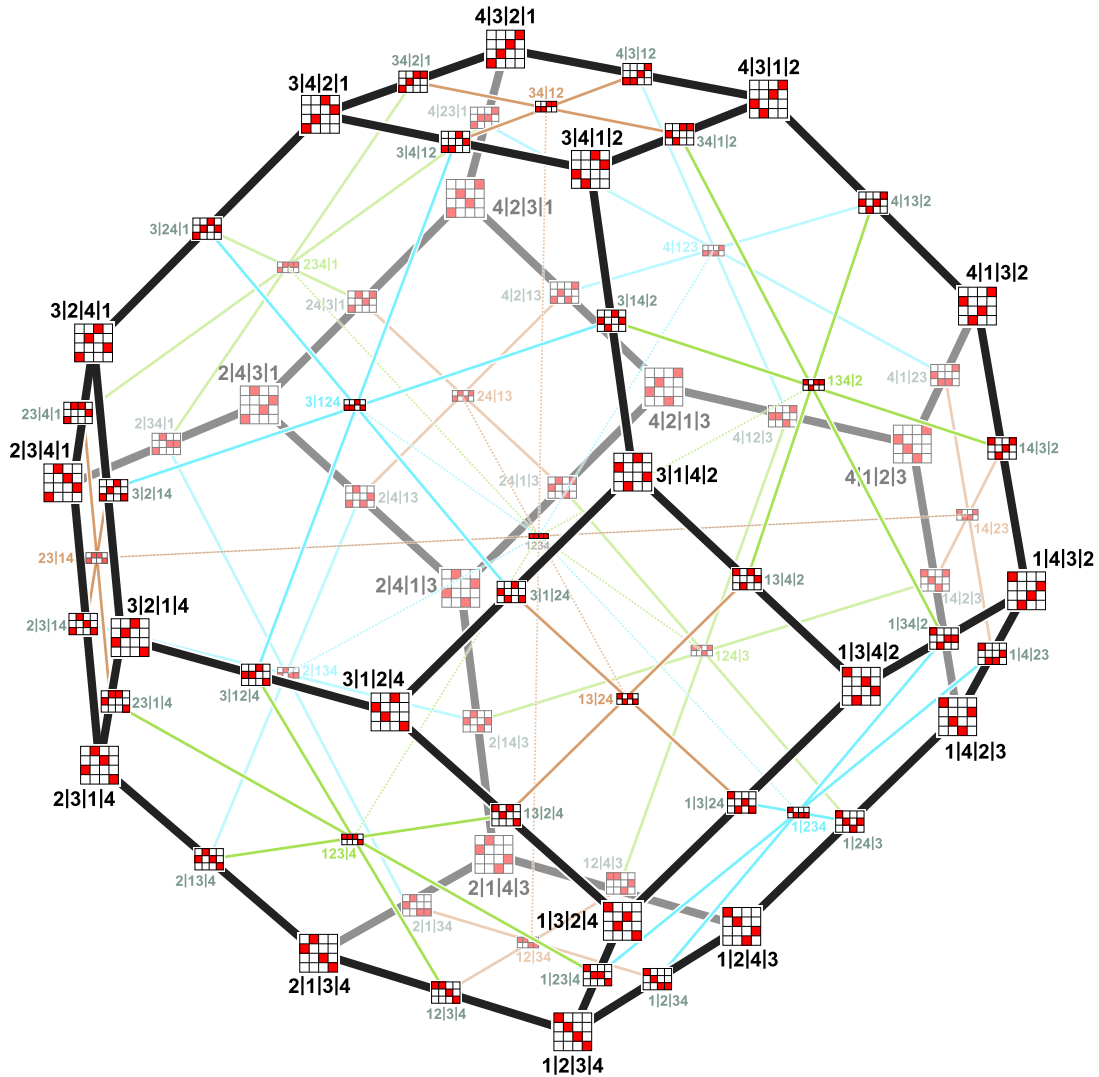
2.2 Generalized permutahedra

Definition (Hypersimplex). $\Delta(n, k) = \{(x_1, \ldots, x_n) : x_1 + \ldots + x_n = k\}$

The basis of $\Delta(n, k)$ (vertices of the polytope) is formed by vectors with k ones and $n - k$ zeroes.

Definition (Generalized Permutahedron). *Convex polytope with all the edges parallel to $e_i - e_j$*

Permutahedron vertices came from a subset of the vertices of $\Delta(n, k)$

Definition (Matroid polytope). *Matroid generated by the permutahedron whose vertices are a subset of $\Delta(n, k)$*

3 Coxeter Groups

A coxeter group is a finite group that has a set of generators $S = \{s_1, \ldots s_n\}$ and a function $m : S \times S \to \mathbb{N}$ with $m(s_i, s_i) = 1$ and $m(s_i, s_j) = m(s_j, s_i)$ such that the group is described with its presentation:

$$G = \{: s_i^2 = e; \ (s_i s_j)^{m(s_i, s_j)} = e\}$$

We can associate every generation with a hyperplane reflection passing through the origin $s_i \leftrightarrow \rho_i$

Suppose we have two generators s_i and s_j. Then we can represent this as two lines in a plane as follows:

If the angle $\angle \rho_i \rho_j = \frac{\pi}{k}$ we have that $s_i s_j$ is a rotation of angle $\frac{2\pi}{k}$. It follows that $m(s_i, s_j) = k$.

Coxeter groups have a fancy representation into a diagram. We represent each generator as a node, and we connect two nodes by a labeled edge with the value of $m(s_i, s_j) \geq 3$. If $m = 3$ it is not needed to label it.

$$A_n \quad \text{⚬} \cdots \text{⚬}\!-\!\!-\text{⚬}\!-\!\!-\text{⚬}\!-\!\!-\text{⚬}\!-\!\!-\text{⚬}$$

There is a correspondence with each flag selecting one variety to a generator, though it is not a bijection.

The flag is composed by a vertex, an edge containing the vertex, a face containing the edge ... etc. If we focus on one variety, then the corresponding symmetry is the hyperplane ρ that fixes the lower dimension varieties and keeps invariant the higher ones.

We will see it more clear in the next subsection.

3.1 Classification of Platonic Solids

There exist 5 platonic solids. We can think all the solids in the projective space to make easier computations.

Tetrahedron A_3

Vertices are the indicator vectors of $\binom{[4]}{1}$

We have a flag $\{V_1, V_2, V_3\}$, where V_i are projective varieties that correspond to a vertex, edge and face respectively and for each V_i we have the induced hyperplane ρ_i associated with the reflection s_i. If we compute the normal vectors of the planes:

$$\rho_1^\perp = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \rho_2^\perp = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \rho_3^\perp = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \quad \Rightarrow \quad \angle\rho_1\rho_2 = \frac{\pi}{3}, \quad \angle\rho_1\rho_3 = \frac{\pi}{2}, \quad \angle\rho_2\rho_3 = \frac{\pi}{3}$$

And then we deduce the diagram ① —— ② —— ③

Cube B_3

Octahedron B_3

Vertices are the indicator vectors of $\binom{[4]}{2}$

We have a flag $\{V_1, V_2, V_3\}$, where V_i are projective varieties that correspond to a vertex, edge and face respectively and for each V_i we have the induced hyperplane ρ_i associated with the reflection s_i. If we compute the normal vectors of the planes:

$$\rho_1^\perp = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \rho_2^\perp = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \rho_3^\perp = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \quad \Rightarrow \quad \angle\rho_1\rho_2 = \frac{\pi}{3}, \quad \angle\rho_1\rho_3 = \frac{\pi}{2}, \quad \angle\rho_2\rho_3 = \frac{\pi}{4}$$

And then we deduce the diagram ① —— ② —4— ③

Dodecahedron H_3

Icosahedron H_3

4 Description of Coxeter Groups

4.1 Group A_n

$$\circ \longrightarrow \circ \longrightarrow \cdots \longrightarrow \circ \longrightarrow \circ$$

We can think type A_n Coxeter Groups as groups generated by the reflections $s_1, \ldots, s_{n-1}$ whose associated hyperplanes are defined by

$$\rho_1^\perp = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \rho_2^\perp = \begin{pmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots \quad \rho_{n-1}^\perp = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix},$$

It is easy to see that the angles between the hyperplanes are

$$\angle\rho_i\rho_{i+1} = \frac{\pi}{3}, \quad \angle\rho_i\rho_j = \frac{\pi}{2} \ if \ i - j \neq \pm 1$$

so that corresponds to the Dynkin diagram, and thus to the Coxeter group

4.2 Group B_n

$$\circ \overset{4}{\longrightarrow} \circ \longrightarrow \cdots \longrightarrow \circ \longrightarrow \circ$$

We can think type B_n Coxeter groups as the group generated by the reflections $\tau, s_1, \ldots, s_n$, whose associated hyperplanes are defined by
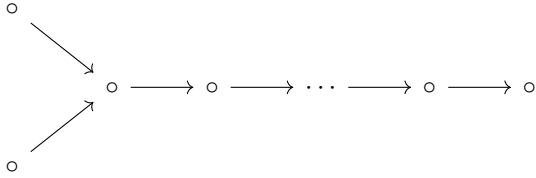
$$
\rho_\tau^\perp = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad
\rho_1^\perp = \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad
\rho_2^\perp = \begin{pmatrix} 0 \\ 1 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots \quad
\rho_n^\perp = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 1 \end{pmatrix},
$$

One can check that the angles between the hyperplanes are

$$
\angle \rho_\tau \rho_1 = \frac{\pi}{4}, \quad \angle \rho_\tau \rho_i = \frac{\pi}{2} \ if \ i > 1, \quad \angle \rho_i \rho_{i+1} = \frac{\pi}{3}, \quad \angle \rho_i \rho_j = \frac{\pi}{2} \ if \ i - j \neq \pm 1
$$

so the construction correspond to the Dynkin diagram, and thus to the coxeter group.

4.3 Group D_n

We can think type D_n Coxeter groups as the group generated by the reflections $\tau_1, \tau_2, s_1, \ldots, s_n$, whose associated hyperplanes are defined by

$$
\rho_{\tau_1}^\perp = \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad
\rho_{\tau_2}^\perp = \begin{pmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad
\rho_1^\perp = \begin{pmatrix} 0 \\ 1 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots \quad
\rho_n^\perp = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 1 \end{pmatrix},
$$

One can check that the angles between the hyperplanes are

$$
\angle \rho_{\tau_1} \rho_{\tau_2} = \frac{\pi}{2}, \quad \angle \rho_{\tau_1} \rho_1 = \frac{\pi}{3}, \quad \angle \rho_{\tau_2} \rho_1 = \frac{\pi}{3}, \quad \angle \rho_i \rho_{i+1} = \frac{\pi}{3}, \quad \angle \rho_i \rho_j = \frac{\pi}{2} \ if \ i - j \neq \pm 1
$$

so the construction correspond to the Dynkin diagram, and thus to the coxeter group.

5 Regular Subdivision and height functions

Given a set of points T, we define a height function as $h : T \to \mathbb{R}$.

The height function h is said to be *M-convex* if the regular subdivisions induced by h are permutahedral.

A subset S of points is a (lower) regular subdivision induced by h if the convex hull of S is described by the lower convex hull of the polytope $T \times h(T)$

Definition (3-Term Plücker Relations)**.** *Let ω be a height function. For each $S \in \binom{[n]}{d-2}$ and $i, j, k, l \notin S$ the minimum*

$$
\min(h(S_{ij}) + h(S_{kl}), h(S_{ik}) + h(S_{jl}), h(S_{il}) + h(S_{jl}))
$$

is attained at least twice.

Theorem. *A height function induces a permutahedral regular division if the **3-Term Plücker Relations** (3TPR) holds.*

Proof

We abbreviate $h_{ij} = h(S_{ij})$

Suppose the 3TPR holds. Given the height function h we fix a linear functional $\varphi = (\varphi_1, \ldots, \varphi_n, \varphi_{n+1})$ with $\varphi_{n+1} > 0$ such that the regular division polytope $Q \subseteq P$ is given by

$$Q = \{A \in M : \varphi(A, h(A)) \text{ is minimum}\}$$

We now can relabel the indexes of the elements of Q and φ removing all the terms that are 1 or zero allways (when applying the functional the result is the same inside Q) and relabeling the remaining from 1 to m.

In the cases $m = 2$ and $m = 3$ the subpolytope Q is trivially permutahedral. For case $m \geq 4$ we pick 4 indexes i, j, k, l of elements of Q and φ.

We can assume without loss of generality that $S_{ij}, S_{kl} \in Q$ now we want to prove that exchange axiom is satisfied: i.e. whithout loss of generality $S_{ik}, S_{jl} \in Q$.

We can express the 3TPR in the following way:

$$3TPR \iff h_{ij} + h_{kl} = h_{ik} + h_{jl} \leq h_{il} + h_{jk}$$

The condition for S_{ij}, S_{kl} to lie in a regular subdivision is that

$$\varphi_i + \varphi_j + \varphi_{n+1} h_{ij} = \varphi_k + \varphi_l + \varphi_{n+1} h_{kl} \leq \begin{cases} \varphi_i + \varphi_k + \varphi_{n+1} h_{ik} \, (1) \\ \varphi_j + \varphi_l + \varphi_{n+1} h_{jl} \, (2) \end{cases}$$

This leads to the equations

$$\varphi_j - \varphi_k \leq \varphi_{n+1}(h_{ik} - h_{ij})$$
$$\varphi_i - \varphi_l \leq \varphi_{n+1}(h_{lj} - h_{ij})$$
$$\varphi_l - \varphi_i \leq \varphi_{n+1}(h_{ik} - h_{kl})$$
$$\varphi_k - \varphi_j \leq \varphi_{n+1}(h_{jl} - h_{kl})$$

Summing up the equations we get

$$0 \leq 2\varphi_{n+1}(h_{ik} + h_{jl} - h_{ij} - h_{kl}) \Rightarrow h_{ij} + h_{kl} \leq h_{ik} + h_{jl}$$

But this inequality should be an equality under de assumption of 3TPR, so all the above inequalities should be equalities, and (1) and (2) implies $S_{ik}, S_{jl} \in Q$