Bernstein-Sato theory in prime power characteristic

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May 2025

A thesis submitted in partial fulfillment of the requirements for
Bachelor's Degree in Mathematics
Bachelor's Degree in Engineering Physics

Acknowledgements

First and foremost, I want to thank my parents. Their unconditional support and optimism have always been, and will always be, the most important source of motivation in my life.

I am also grateful to my friends from Barcelona for making this journey so much more enjoyable. Late-night Discord calls, messages in our chaotic group chats (*patos?*), and road-trips with them helped me get through the challenges of living abroad for nine months.

I owe a big thanks to everyone in the academic world who supported me along the way. I am especially thankful to my advisor, Eamon Quinlan-Gallego, for his patience during our weekly meetings and for his continuous support throughout these months. It has been an incredible experience, and I could not have asked for a better advisor. I also want to thank my local advisor, Josep Àlvarez Montaner, for all his help and support during the writing of this thesis, as well as for the opportunities he gives to all his students by organizing reading groups and courses.

My gratitude also goes to CFIS and the incredible environment I have found there. The program has been demanding at times, but overall, a truly rewarding experience. I am thankful to all the professors and staff for their help and support over the years. Professors like Miguel Ángel Barja, Juanjo Rue, and Josep Àlvarez himself have been sources of knowledge, inspiration, and opportunities.

I would also like to thank everyone at the University of Utah, who welcomed me so warmly. Special thanks to Professor Srikanth Iyengar, whose guidance and conversations about my career have been especially fruitful.

Last, but definitely not least, thank you to all the people I met at the University of Utah tennis club. Hitting after a long, stressful day was the perfect way to clear my mind and protect my mental health. Thanks for the hits, the laughs, (the rides home) and all the good times.

Abstract

Given an ideal $\mathfrak{a}$ of a ring R, Bernstein-Sato theory associates to it a set of invariants called Bernsteio-Sato roots. Originally introduced in positive characteristic by Mustaţă, Takagi, and Watanabe in the early 2000s, these invariants have been actively studied since then.

Building on the recent work of Bitoun and Quinlan-Gallego (2024), this thesis develops a systematic construction of Bernstein–Sato roots in prime power characteristic (p^{m+1}), extending classical tools from prime characteristic such as Frobenius endomorphism, ν-invariants, differential operators, and Morita equivalence to this broader context.

In this setting, we study and analyze a natural generalization of Bernstein–Sato roots, termed the *strength*, which captures finer invariants of singular hypersurfices. As an application, we compute both the Bernstein–Sato roots and strengths for diagonal hypersurfaces, illustrating the effectiveness of the extended framework.

Keywords: Commutative algebra, Algebraic geometry, Characteristic $p > 0$, Bernstein-Sato, strength.

MSC: 13A35, 14B05, 14F10, 14H20, 14J17, 16D90.

Resumen

Dado un ideal $\mathfrak{a}$ de un anillo R, la teoría de Bernstein-Sato le asocia un conjunto de invariantes llamados raíces de Bernstein-Sato. La construcción de dichos invariantes en característica positiva fue originalmente introducida por Mustaţă, Takagi y Watanabe a principios de la década de los 2000 y ha sido objeto de estudio desde entonces.

Construyendo sobre el trabajo reciente de Bitoun y Quinlan-Gallego (2024), esta tesis desarrolla una construcción sistemática de raíces de Bernstein-Sato en característica potencia-prima (p^{m+1}), extendiendo herramientas clásicas de la característica primo como el endomorfismo de Frobenius, los ν-invariantes, los operadores diferenciales y la equivalencia de Morita a este contexto más amplio.

En este contexto, estudiamos y analizamos una generalización natural de las raíces de Bernstein-Sato, denominada *strength*, que captura invariantes más finos de las hipersuperficies singulares. Como aplicación, calculamos tanto las raíces de Bernstein-Sato como las strengths para hipersuperficies diagonales, ilustrando la efectividad del marco ampliado.

Palabras clave: Álgebra conmutativa, Geometría algebraica, Característica $p > 0$, Bernstein-Sato, *strength*.

Resum

Donat un ideal $\mathfrak{a}$ d'un anell R, la teoria de Bernstein-Sato li associa un conjunt d'invariants anomenats arrels de Bernstein-Sato. La construcció d'aquests invariants en característica positiva va ser originalment introduïda per Mustaţă, Takagi i Watanabe a principis de la dècada dels 2000 i ha estat objecte d'estudi des de llavors.

Basant-se en el treball recent de Bitoun i Quinlan-Gallego (2024), aquesta tesi desenvolupa una construcció sistemàtica d'arrels de Bernstein-Sato en característica potència-prima (p^{m+1}), ampliant eines clàssiques de la característica prima com l'endomorfisme de Frobenius, els ν-invariants, els operadors diferencials i l'equivalència de Morita a aquest context més ampli.

En aquest context, estudiem i analitzem una generalització natural de les arrels de Bernstein-Sato, anomenada *strength*, que captura invariants més fins de les hipersuperfícies singulars. Com aplicació, calculem tant les arrels de Bernstein-Sato com les strengths per a hipersuperfícies diagonals, il·lustrant l'efectivitat del marc ampliat.

Paraules clau: Àlgebra commutativa, Geometria algebraica, Característica $p > 0$, Bernstein-Sato, *strength*.

Contents

Introduction

The study of singularities within the contexts of algebraic geometry and commutative algebra has constituted a central theme of mathematical research for several decades. A fundamental problem in these areas, as well as in numerous other branches of mathematics, is the classification of mathematical objects up to a given notion of equivalence. Among the most common techniques employed in such classification problems is the use of invariants. In algebraic geometry, an invariant refers to a mathematical object such as a number, set, ring, or module, associated to an algebraic variety or, more generally, to a scheme. These invariants serve as tools for distinguishing and classifying the objects under consideration, often encoding essential geometric or algebraic information.

Classical invariants in algebraic geometry include the log canonical threshold, the Milnor number, and the Bernstein–Sato polynomial in characteristic zero, as well as the F-pure threshold, the F-signature, and the Bernstein–Sato roots in positive characteristic. These invariants have been the subject of extensive study, particularly in the context of singularity theory in characteristic zero. In recent years, there has been a growing interest in studying singularities in positive characteristic, which has led to new methods and theoretical approaches. In this thesis, we take a closer look at Bernstein–Sato roots in positive characteristic, aiming to better understand their structure and how they can be used in this setting.

The development of Bernstein–Sato theory originates with its construction in the setting of a polynomial ring over a field of characteristic zero. The initial formulation of the b-function (later known as the Bernstein–Sato polynomial) was introduced by Bernstein in the early 1970s in [Ber71] and [Ber72]. The foundational question can be stated as follows: given a polynomial $f \in \mathbb{C}[\underline{x}]$, defining a hypersurface in $\mathbb{C}^n$, determine whether there exist a nonzero polynomial $b_f(s) \in \mathbb{C}[s]$ and a differential operator $P(s) \in D[s]$ such that the following functional equation holds:

$$P(s) \cdot f^{s+1} = b_f(s) \cdot f^s$$

where D denotes the Weyl algebra, i.e., the ring of differential operators $D = \mathbb{C}[\underline{x}]\langle \underline{\partial} \rangle$.

The answer to the problem mentioned above is not only affirmative, but $b_f(s)$ it is also unique up to multiplication by scalar. If one requires $b_f(s)$ to be monic, then it is uniquely determined and is referred to as the Bernstein–Sato polynomial associated to f. Moreover, fundamental results concerning the nature of its roots were established by Kashiwara, who proved their rationality and negativity in [Kas77] and [Kas06].

The foundational developments in extending Bernstein–Sato theory to positive characteristic were performed by Mustaţă, Takagi, and Watanabe in the early 2000s. In [MTW04], the authors introduced analogues of several key notions from characteristic zero, including test ideals and F-jumping numbers, mirroring the roles of multiplier ideals and jumping coefficients, respectively. In addition, this work introduced the concept of ν-invariants, a powerful collection of invariants associated to an ideal. These ν-invariants can serve as a tool in the computation of Bernstein–Sato roots in positive characteristic and play a crucial role in the formulation of a parallel theory to that of characteristic zero.

The first formal construction of Bernstein–Sato roots in positive characteristic was presented by Mustaţă in [Mta09], who introduced a characteristic p analog of the Bernstein–Sato polynomial using F-modules. This construction was later refined by Bitoun in [Bit18], where a more robust framework was developed to define and study these roots. Finally, the various constructions within the theory were explained in detail by Quinlan-Gallego in [Qui21], who also introduced the computation of Bernstein–Sato roots as a p-adic limit.

For instance, we have two central results regarding the Bernstein-Sato roots in prime characteristic and the roots of the Bernstein-Sato polynomial of the zero characteristic case.

Theorem 0.0.1. [Kas77, Corollary 5.2] The roots of the Bernstein-Sato polynomial of a polynomial $f \in \mathbb{C}[\underline{x}]$ are strictly negative rational numbers.

Theorem 0.0.2. [Bit18, Theorem 2.4.1] Let k be a perfect field of prime characteristic p, R a smooth k-algebra and let $f \in R$ not contained in k. The Bernstein-Sato roots are rational and contained in the interval $[-1, 0)$.

We will see that this is not exactly the case in the prime power characteristic case. However, one can find similar results under certain conditions.

The structure of the thesis is as follows:

In [Chapter 1], we provide a comprehensive overview of all the necessary machinery needed for the development of the thesis. This overview will cover the basics notions for working with algebra in prime characteristic.

In [Chapter 2], we present the construction of Bernstein–Sato roots, first in the classical prime characteristic case (first introduced in [MTW04]), and secondly in the case of prime power characteristic, as introduced in [BQG24]. We discuss some properties of Bernstein-Sato roots and introduce the concept of strength. Finally, we discuss some results about the connection between Bernstein-Sato roots of the zero, prime and prime power characteristic cases.

Finally, in [Chapter 3], we compute the Bernstein–Sato roots of diagonal hypersurfaces, which are a special class of hypersurfaces defined by a polynomial of the form $f = c_1x_1^{\alpha_1} + c_2x_2^{\alpha_2} + \cdots + c_nx_n^{\alpha_n} \in \mathbb{Z}/p^{m+1}\mathbb{Z}[\underline{x}]$. We also discuss the behavior of these roots within the three characteristic cases.

At the end of the thesis, we include an appendix with some background on p-adic numbers and base p expansions, as well as some theorems and propositions that are used throughout the thesis.

Chapter 1

Background on algebra in characteristic $p > 0$

1.1 The Frobenius endomorphism

In the study of commutative algebra in positive characteristic, the Frobenius endomorphism emerges as a central tool. This morphism captures deep arithmetic and geometric properties of the ring and serves as a foundational ingredient in the construction of numerous concepts, such as test ideals and F-singularities, among other important invariants.

Definition 1.1.1 (Ring of characteristic $p > 0$)**.** Let R be a commutative ring. We say R has characteristic $p > 0$ if it contains as a subring the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, that is, R is an $\mathbb{F}_p$-algebra.

One of the main tools used in the study of these type of rings is the Frobenius endomorphism, defined as follows.

Definition 1.1.2 (Frobenius endomorphism)**.** Let R be a ring of characteristic $p > 0$. For every $e \in \mathbb{Z}_{\geq 0}$ the endomorphism $F^e : R \to R$ defined by $r \to r^{p^e}$ is called the eth Frobenius endomorphism.

To see that the map F^e is, indeed, a morphism, we only have to check for $e = 1$, since F^e is the e-fold composition of F^1. We check that it respect products

$$F^1(rs) = (rs)^p = r^p s^p = F^1(r)F^1(s)$$

and sums

$$F^1(r+s) = (r+s)^p = \sum_{i+j=p} \binom{p}{i} r^i s^j = r^p + s^p = F^1(r) + F^1(s)$$

where the previous follows from $\binom{p}{i} \equiv 0 \mod p$ for all $1 \leq i \leq p-1$.

From now on, we will denote F^1 simply as F.

Lemma 1.1.3. Let R be a ring of characteristic $p > 0$. Then F^e fixes the elements of the subring $\mathbb{F}_p \subseteq R$.

Proof. Let $r \in \mathbb{F}_p \subseteq R$. By Fermat's little theorem [Thm A.2.12]

$$r^{p-1} \equiv 1 \mod p \quad \Rightarrow \quad r^p \equiv r \mod p \quad \Rightarrow \quad F(r) = r^p = r$$

Thus, F fixes $\mathbb{F}_p$ and also F^e by composition. □

Observation 1.1.4. The endomorphism F is not necessarily injective nor surjective.

Example 1.1.5. See the following counterexamples for the previous observation

1. Let $R = \mathbb{F}_p[x]/(x^2)$. Then F is not injective.
2. Let $R = \mathbb{F}_p[T]$, with T transcendental over $\mathbb{F}_p$. Then F is not surjective.

(1) The nonzero element $[x] \in \mathbb{F}_p[x]/(x^2)$ maps to $F([x]) = [x^p] = 0$, showing F is not injective.

(2) Suppose that $T \in \mathrm{Im}(F)$. Then, there must exist $r = a_0 + a_1 T + \cdots a_n T^n$ with $a_i \in \mathbb{F}_p$ such that

$$T = F(r) = F(a_0) + F(a_1)F(T) + \cdots + F(a_n)F(T)^n = a_0 + a_1 T^p + \cdots + a_n T^{np}$$

since the elements of the field $a_i \in \mathbb{F}_p$ are fixed by the Frobenius endomorphism by [Lem 1.1.3]. This implies

$$a_0 + a_1 T^p + \cdots + a_n T^{p^n} - T = 0$$

which is a contradiction since T is transcendental over $\mathbb{F}_p$. Hence, F is not surjective.

Ideally, for some purposes we want our Frobenius endomorphism to be an automorphism (i.e. bijective). We introduce some extra conditions that guarantee that.

Definition 1.1.6 (Perfect field)**.** A field k is perfect if every irreducible polynomial is separable (i.e. it has distinct roots) in any field extension L/k.

Lemma 1.1.7. Let k be a field and R a ring. A morphism of rings $f : k \to R$ is always injective.

Proof. The kernel of the map $\ker(f) \subseteq k$ must be an ideal. However, the only ideals of a field are $\{0\}$ and k. Since f is not the zero map ($1_k \mapsto 1_R$), we must have $\ker(f) = \{0\}$, proving the injectivity. □

Observe that the lemma immediately implies that every morphism of fields is injective.

Lemma 1.1.8. Let k be a field of characteristic $p > 0$ and $f \in k[x]$. Then the formal derivative f' is zero if and only if f is inseparable.

Proof. Suppose $f' = 0$. Then, clearly, there must exist $g \in k[t]$ such that $f(t) = g(t^p)$. Let α be a root of $f(t)$. Then,

$$f(\alpha) = 0 \quad \Rightarrow \quad g(\alpha^p) = 0$$

Thus, if $(x - \alpha)$ is a factor of f, then $t^p - \alpha^p = (t - \alpha)^p$ is a factor of $g(t^p) = f(t)$.

Conversely, assume that f is irreducible. If f is inseparable, then $\deg(\gcd(f, f')) \geq 1$, since f and f' have a common root in the splitting field. Since f is irreducible, then it follows $f \mid f'$. By degree considerations, the only way that this is possible is if $f' = 0$, proving the statement. □

Proposition 1.1.9. If k is a field of characteristic $p > 0$, then k is perfect if and only if $F : k \to k$ is an automorphism.

Proof. First we consider the $\Rightarrow$ implication. Suppose k is perfect. Since F is a morphism of fields, by [Lem 1.1.7], we have that it is injective. For the surjectivity, let $a \in k$. We must show that the polynomial $f = x^p - a \in k[x]$ has a root in k. Let $\alpha \in \overline{k}$ such that $\alpha^p - a = 0$, meaning $(\alpha - a)^p = 0$. Since $k[x]$ is a unique factorization domain (see [AK13, Proposition 2.5]), we can decompose f in $k[x]$ in the following way

$$f = (x - \alpha)^p = (x - \alpha)^l \cdots (x - \alpha)^l$$

where $(x - \alpha)^l$ is irreducible in $k[x]$. Then either $l = 1$ or $l = p$, since p is prime. The latter case is not possible, since that would mean that f is inseparable in the extension $\overline{k}/k$, contradicting the hypothesis of k perfect. Thus, $l = 1$ and clearly $\alpha \in k$, proving the surjectivity.

For the converse, suppose F is an automorphism. Suppose that $f \in k[x]$ is irreducible and inseparable, that is, it has $\alpha_1 = \alpha_2$ in the following expression

$$f = a_0 + a_1 x + \cdots + a_n x^n = A \prod_{i=1}^{n} (x - \alpha_i)$$

By [Lem 1.1.8], we have that $f' = 0$. Thus, we have that f is of the form

$$f = a_0 + a_1 x^p + \cdots + a_n x^{np}$$

Finally, we have, by surjectivity of F that there exist $c_i \in k$ such that $c_i^p = a_0$, and rewriting the expression

$$f = c_0^p + c_1^p x^p + \cdots + c_n^p x^{np} = (c_0 + c_1 x + \cdots + c_n x^n)^p$$

contradicting the fact that f is irreducible. Thus, f must be separable. □

Definition 1.1.10. Let R be a ring of characteristic $p > 0$, M an R-module and $e \in \mathbb{Z}_{\geq 0}$. We construct the R-module $F^e_* M$ as the abelian group canonically isomorphic to M (i.e. $m \in M \leftrightarrow F^e_* m \in F^e_* M$) endowed with the action

$$r \cdot F^e_* m := F^e_* r^{p^e} m$$

for $r \in R$ and $F^e_* m \in F^e_* M$

Definition 1.1.11 (F-finite ring)**.** Let R be a Noetherian ring of characteristic $p > 0$. We say R is F-finite if F_*R is finitely generated as an R-module.

Proposition 1.1.12. Let R be a Noetherian ring of characteristic $p > 0$. The following are equivalent:

1. R is $F-$finite.
2. F_*R is finitely generated as R-module.
3. F_*^eR is finitely generated as R-module for all $e \geq 1$.
4. F_*^eR is finitely generated as R-module for some $e \geq 1$.

Proof. (1) $\iff$ (2) by definition. For (2) $\Rightarrow$ (3) note that the composition of finite morphism is a finite morphism. Thus, F^e is a finite morphism, concluding F_*^eR is finitely generated $\forall e \in \mathbb{Z}_{\geq 1}$.

Clearly (3) $\Rightarrow$ (4).

For (4) $\Rightarrow$ (2) let $\{F_*^e x_1, \dots, F_*^e x_n\}$ be a set of generators of the R-module F_*^eR. Then if $r \in R$ we can express $r = \sum_{i=1}^n r_i^{p^e} x_i = \sum_{i=1}^n (r_i^{p^{e-1}})^p x_i$, meaning $\{F_*^1 x_1, \dots, F_*^1 x_n\}$ is a set of generators of the R-module $F_*^1 R$. □

Proposition 1.1.13. Let k be a perfect field of characteristic $p > 0$. Then k is F-finite.

Proof. By [Prop 1.1.9] we have that F is an automorphism and by [Prop 1.1.12] we only have to prove that F_*k is finitely generated. Pick an element $a \in k$, and notice that we have $c \in k$ such that $F(c) = a$ for the surjectivity of F. Then

$$F_*a = c \cdot F_*1$$

and clearly F_*1 generates F_*k. □

Given an F-finite ring of characteristic $p > 0$ one can construct more F-finite rings via the following proposition.

Proposition 1.1.14. Let R be an F-finite ring. Then

1. $R[\underline{x}]$ is F-finite.
2. $R[[\underline{x}]]$ is F-finite.
3. The quotient R/I is F-finite for any proper ideal I.
4. The localization $W^{-1}R$ is F-finite for any multiplicative set S.

Proof.
(1) Clearly $\mathbb{F}_p \subseteq R \subseteq R[x]$, meaning $R[x]$ has characteristic $p > 0$. We now show that

$$\{F_*^e f_1, \dots, F_*^e f_m\} \text{ generates } F_*^eR \quad \Rightarrow \quad \{F_*^e f_i x^j : 1 \leq i \leq m, 0 \leq j < p\} \text{ generates } F_*^eR[x]$$

Given $r \in R[x]$ we can write the element as

$$r = r_0 + r_1x + \cdots r_nx^n = (r_0 + \cdots + r_{p-1}x^{p-1}) + x^p(r_p + \cdots + r_{2p-1}x^{p-1}) + \cdots$$

grouping the expression so that the degree of the factored expression does not exceed $p-1$. Applying the Frobenius we get

$$F^e_* r = (F^e_* r_0 + \cdots + F^e_*(r_{p-1}x^{p-1})) + x \cdot (F^e_* r_p + F^e_*(r_{2p-1}x^{p-1}))$$

Now we decompose into the generators each $r_i = \sum_j r_i^j F^e_* f_j$ and we finally get

$$F^e_* r = \left(\sum_{k=0}^{p-1} \sum_{j=1}^{m} r_k^j F^e_*(f_j x^k) \right) + \left(\sum_{k=0}^{p-1} \sum_{j=1}^{m} r_{k+p}^j x F^e_*(f_j x^k) \right)$$

showing that the proposed set is, indeed, a generator set of $F^e_* R[x]$. By induction in the number of variables we get that $F^e_* R[x_1, \ldots, x_n]$ is finitely generated.

(2) The proof is almost identical to (1). The generating set is the same, but now considering infinite degree instead of arbitrary.

(3) Notice that the kernel of the map $\pi : \mathbb{F}_p \subseteq R \to R/I$ is

$$\ker(\pi) = \mathbb{F}_p \cap I = \{0\}$$

by [Lem 1.1.7]. Then $\mathbb{F}_p \subseteq R/I$ and clearly R/I has characteristic $p > 0$. We now see that

$$\{F^e_* f_1, \ldots, F^e_* f_m\} \text{ generates } F^e_* R \quad \Rightarrow \quad \{F^e_*[f_1], \ldots, F^e_*[f_m]\} \text{ generates } F^e_*(R/I)$$

since clearly given $F^e_* r = \sum_i r_i F^e_* f_i \in R$ with $r_i \in R$, we have $F^e_*[r] = \sum_i [r_i] F^e_*[f_i] \in F^e_*(R/I)$.

(4) The composition of the maps $\mathbb{F}_p \hookrightarrow R \to W^{-1}R$ gives a map $\mathbb{F}_p \hookrightarrow W^{-1}R$ which is injective by [Lem 1.1.7]. Thus, the ring $W^{-1}R$ has characteristic $p > 0$. We now see that

$$\{F^e_* f_1, F^e_* f_m\} \text{ generates } F^e_* R \quad \Rightarrow \quad \{F^e_* f_1, F^e_* f_m\} \text{ generates } F^e_*(W^{-1}R)$$

that is, the generating set is the same. To see this let $r \in R, w \in W$, then there exist $r_i \in R$ such that $F^e_*(w^{p^e-1}r) = \sum_i r_i F^e_* f_i$. Thus

$$F^e_* \left(\frac{r}{w} \right) = F^e_* \left(\frac{1}{w^{p^e}} w^{p^e-1} r \right) = \frac{1}{w} F^e_*(w^{p^e-1}r) = \sum_{i=0}^{m} \frac{r_i}{w} F^e_* f_i$$

proving the statement. □

The prototypical examples of F-finite rings are rings of polynomials. Great part of the thesis will cover this case for both simplicity as well as ubiquity.

Example 1.1.15. Some examples of F-finite rings.

1. If $R = k[x_1, \ldots, x_n]$ with k a perfect field, then $F^e_* R$ has the generating set $\{F^e_* x_1^{i_1} \cdots x_n^{i_n} : 0 \le i_1, \ldots, i_n \le p^e - 1\}$.
2. In particular, $R = \mathbb{F}_p[x_1, \ldots, x_n]$ since $\mathbb{F}_p$ is perfect.

We have so far discussed whether a ring is F-finite or not, that is, if $F_*^e R$ is finitely generated as a module. At this point, the natural question of when the module $F_*^e R$ is free arises. It turns out that regularity and locality play an important role in this phenomenon.

Definition 1.1.16 (Regular ring)**.** The ring R is regular if every localization at prime ideal is a regular local ring (i.e. the minimum number of generators of the maximal ideal is the Krull dimension [Def A.2.1]).

Theorem 1.1.17 (Kunz)**.** Let R be a Noetherian ring of characteristic $p > 0$. Then

$$R \text{ regular} \quad \Leftrightarrow \quad F_*^e R \text{ flat for some (equivalently all) } e \in \mathbb{Z}_{\geq 1}$$

For a detailed proof of Kunz's theorem see [SS24].

If we add hypotheses, we come up with more specific results. Some reasonable assumptions for the ring R are F-finiteness and localness. We prove two results that extend Kunz's theorem to these cases.

Proposition 1.1.18 (Kunz 2)**.** Let R be a Noetherian, F-finite ring of characteristic $p > 0$. Then

$$R \text{ regular} \quad \Leftrightarrow \quad F_*^e R \text{ projective for some (equivalently all) } e \in \mathbb{Z}_{\geq 1}$$

Proof. Notice that R is regular if and only if $F_*^e R$ is flat by [Thm 1.1.17]. Since R is Noetherian, by [Prop A.2.4] $F_*^e R$ finitely generated is equivalent to $F_*^e R$ finitely presented, and by [Prop A.2.5], $F_*^e R$ flat and finitely presented is equivalent to $F_*^e R$ projective and finitely generated. Schematically:

$$R \text{ regular} \quad \Leftrightarrow \quad F_*^e R \begin{cases} \text{flat} \\ \text{fin. gen} \end{cases} \quad \Leftrightarrow \quad F_*^e R \begin{cases} \text{flat} \\ \text{fin. pres.} \end{cases} \quad \Leftrightarrow \quad F_*^e R \begin{cases} \text{proj.} \\ \text{fin. gen} \end{cases}$$

□

Proposition 1.1.19 (Kunz 3)**.** Let R be a Noetherian, F-finite, local ring of characteristic $p > 0$. Then

$$R \text{ regular} \quad \Leftrightarrow \quad F_*^e R \text{ free for some (equivalently all) } e \in \mathbb{Z}_{\geq 1}$$

Proof. Notice that by [Prop 1.1.18] we have that R is regular if and only if $F_*^e R$ is projective. But in a local ring projective is equivalent to free by [Prop A.2.6]. □

We will see in the following sections that having properties such as regularity, projectivity and freeness of $F_*^e R$ will be crucial in the development of the theory. For that reason, we will work mainly with regular rings.

One of the main properties of a regular ring of positive characteristic is that the Frobenius map splits. This property will become crucial in Morita theory and the connection between the two ways to computing the Bernstein-Sato roots.

Proposition 1.1.20. Let M, N be finitely generated projective R-modules. Then the following conditions are equivalent

1. $M \xrightarrow{f} N$ splits.
2. There exist a section $s : N \to M$ such that $s \circ f = \mathrm{Id}_M$.
3. The map $f^* : \mathrm{Hom}_R(N, R) \twoheadrightarrow \mathrm{Hom}_R(M, R)$ is surjective.

Proof. (1) $\Leftrightarrow$ (2) by definition. For (2) $\Rightarrow$ (3) we have that, given $\varphi \in \mathrm{Hom}_R(M, R)$, then $\varphi \circ s \in \mathrm{Hom}_R(N, R)$ lies in the preimage of f^* since

$$f^*(\varphi \circ s) = \varphi \circ s \circ f = \varphi \circ \mathrm{Id}_M = \varphi$$

For (3) $\Rightarrow$ (2), dualize again the map f^* to get the following map

$$f^{**} : \mathrm{Hom}_R(\mathrm{Hom}_R(N, R), R) \cong N \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R) \cong M$$

because N and R are projective. Then, $f^{**} = f$ and it is infective because f^* is surjective. Injective maps between projective modules always split, so we have that f splits. $\square$

Proposition 1.1.21. Let R be a regular ring of characteristic $p > 0$. Then the Frobenius map $R \xrightarrow{F^e} F^e_* R$ splits. This is, there exists a section $s : F^e_* R \to R$ such that $s \circ F^e = \mathrm{Id}_R$.

Proof. See [Qui21, Proposition II.6] $\square$

One of the immediate consequences of the existence of a section is the injectivity of the Frobenius map.

Proposition 1.1.22. Let R be a regular F-finite ring of characteristic $p > 0$. Then the Frobenius map is injective.

Proof. By [Prop 1.1.21] we have that the Frobenius map splits, meaning that there exists a section $s : F^e_* R \to R$ such that $s \circ F^e = \mathrm{Id}_R$. Suppose F^e is not injective, that is, there exist $r, t \in R$ different such that $F^e(r) = F^e(t)$. Composing with s gives

$$s \circ F^e(r) = s \circ F^e(t) \quad \Rightarrow \quad \mathrm{Id}_R(r) = \mathrm{Id}_R(t) \quad \Rightarrow \quad r = t$$

which is a contradiction. Thus, F^e is injective. $\square$

1.2 Cartier operators

Cartier operators and Cartier ideals play an important role in the computation of Bernstein-Sato roots of a variety. In this section we perform the construction of Cartier ideals, as well as providing a survey of its main properties and results.

Throughout this section let R be a Noetherian ring of characteristic $p > 0$. When needed, we may assume R regular, in which case it will be specified.

Definition 1.2.1 (Frobenius power)**.** Given an ideal $I \subseteq R$ and an integer $e \geq 0$, we define the e-th Frobenius power as

$$I^{[p^e]} := (f^{p^e} : f \in I)$$

Proposition 1.2.2. If R is Noetherian, then I is finitely generated and

$$I = (f_1, \dots, f_n) \quad \Rightarrow \quad I^{[p^e]} = (f_1^{p^e}, \dots, f_n^{p^e})$$

Proof. Clearly $(f_1^{p^e}, \dots, f_n^{p^e}) \subseteq I^{[p^e]}$ since $f_i^{p^e} \in I^{[p^e]}$. For the other inclusion let $f \in I^{[p^e]}$. We can write f as

$$f = \sum_j r_j g_j^{p^e} = \sum_j r_j \left(\sum_{i=1}^{n} s_i f_i \right)^{p^e} = \sum_j r_j \sum_{i=1}^{n} s_i^{p^e} f_i^{p^e} = \sum_{i=1}^{n} \left(s_i^{p^e} \sum_j r_i \right) f_i^{p^e}$$

where $g_i = \sum_i s_i f_i$, concluding with the proof. □

We have so far two ways of *exponentiate* an ideal I when working in a ring of characteristic $p > 0$. The first one is the classical honest power $I^k = (f_1 f_2 \cdots f_k : f_i \in I)$, and the other is the Frobenius power just introduced in [Def 1.2.1]. The natural question to ask here is how related they are, and one satisfactory answer is that they are cofinal.

Proposition 1.2.3. The families of ideals $\{I^k\}_{k\geq 0}$ and $\{I^{[p^k]}\}_{k\geq 0}$ are cofinal.

Proof. Let $I = (f_1, \dots, f_m)$. The containment $I^{[p^k]} \subseteq I^{p^k}$ is clear. The first ideal is generated by $(f_i^{p^k})$, and f^{p^k} is clearly contained in I^{p^k}.

For the other containment we prove that $I^{m(p^k-1)+1} \subseteq I^{[p^k]}$, where m is the minimal number of generators of I. We know

$$I^{m(p^k-1)+1} = (f_1^{a_1} \cdots f_m^{a_m} : \sum a_i = m(p^m - 1) + 1)$$

but the sum condition tells us that in each generator element there must be an index i such that $a_i \geq p^k$ by pigeonhole principle. Thus, each generator is guaranteed to be contained in $I^{[p^k]}$. □

We now introduce the set of Cartier operators.

Definition 1.2.4 (Cartier operators)**.** Let R be an $F-$finite ring and an integer $e \geq 0$. The set of Cartier operators of level e is

$$\mathcal{C}_R^e := \mathrm{Hom}_R(F_*^e R, R)$$

Each $\mathcal{C}_R^e$ has also an $R-$module structure given by the action $(r \cdot \varphi)(F_*^e f) = r\varphi(F_*^e f)$

Rather than the modules of Cartier operators by themselves, we would like to focus on the ideals of the form

$$\mathcal{C}_R^e \cdot I = (\varphi(F_*^e f) : \varphi \in \mathcal{C}_R^e, f \in I)$$

where $I \subseteq R$. We call these ideals *Cartier ideals*. Later we will see how the *jumps* in these Cartier ideals (i.e. Cartier ideals that are not the same under certain setting) provide some useful set of invariants of our variety, called ν-invariants.

When $F_*^e R$ is a free module, such as in the case of $R = \mathbb{F}_p[\underline{x}]$, the Cartier ideal $\mathcal{C}_R^e \cdot I$ can be computed via the following proposition.

Proposition 1.2.5. Let R be a regular $F-$finite ring and $F_*^e R$ a free R-module with basis $\{F_*^e \alpha_1, \ldots, F_*^e \alpha_n\}$. Suppose

$$I = (f_1, \ldots, f_m) \quad \text{and} \quad F_*^e f_i = \sum_{j=1}^{n} f_{ij} F_*^e \alpha_j$$

Then, one has $\mathcal{C}_R^e \cdot I \subseteq (f_{ij} : 1 \leq i \leq m, 1 \leq j \leq n)$.

Proof. Let $\varphi_i : F_*^e \alpha_j \mapsto \delta_{ij}\ \forall i = 1, \ldots, n$, where δ_{ij} is the Kronecker delta function. Then, each Cartier operator is of the form $\varphi = \sum_{i=1}^{n} r_i \varphi_i$ for some $r_i \in R$. Hence,

$$\mathcal{C}_R^e \cdot I = (\varphi_i(F_*^e f_j) : 1 \leq i \leq n, 1 \leq j \leq m) = (\varphi_i(\sum_{k=1}^{n} f_{jk} F_*^e f_k) : 1 \leq i \leq n, 1 \leq j \leq m) =$$
$$= (f_{ij} : 1 \leq i \leq n, 1 \leq j \leq m)$$

concluding the proof. $\square$

The following example depicts how to compute explicitly a Cartier ideal with the aid of the latter proposition.

Example 1.2.6. Let $R = \mathbb{F}_3[x, y]$. Clearly $\{x^i y^j : 0 \leq i, j < 3\}$ is a basis for $F_*^1 R$. Given the element $f = (x^2 + y^3)^2 \in \mathbb{F}_3[x, y]$, we can write it like

$$F_*^1 f = F_*^1 x^4 + F_*^1 2y^3 x^2 + F_*^1 y^6 = x \cdot F_*^1 x + 2y \cdot F_*^1 x^2 + y^2 \cdot F_*^1 1$$

meaning that the ideal formed by the elements acting on the basis is the Cartier ideal

$$\mathcal{C}_R^1 \cdot (x^2 + y^3)^2 = (x, 2y, y^2) = (x, y)$$

Let us discuss now some properties that will allow us to have a better understanding of how Cartier operators behave. We do not provide the proof of this proposition, since a more general version will be proved later in the thesis.

Proposition 1.2.7. Let R be an F-finite regular ring, $I, J \subseteq R$ ideals, $e \geq 0$ and $\lambda \in \mathbb{R}_{\geq 0}$. Then

1. $I \subseteq (\mathcal{C}_R^e \cdot I)^{[p^e]}$.
2. $I \subseteq J \quad \Rightarrow \quad \mathcal{C}_R^e \cdot I \subseteq \mathcal{C}_R^e \cdot J$.
3. $\mathcal{C}_R^e \cdot I = \mathcal{C}_R^{e+1} \cdot I^{[p]}$.
4. $\mathcal{C}_R^e \cdot I \subseteq J \quad \Leftrightarrow \quad I \subseteq J^{[p^e]}$.
5. $\mathcal{C}_R^e \cdot I^{\lceil \lambda p^e \rceil} \subseteq \mathcal{C}_R^{e+1} \cdot I^{\lceil \lambda p^{e+1} \rceil}$.

6. $\mathcal{C}_R^{e+d} \cdot I = \mathcal{C}_R^e \cdot (\mathcal{C}_R^d \cdot I)$.

Another useful property is that localization commutes with Cartier operator when we are working with a regular F-finite ring.

Proposition 1.2.8. Let R be a regular F-finite ring, $e \geq 0$ and $W \subseteq R$ be a multiplicative set. Then

$$(W^{-1}I)^{[p^e]} = W^{-1}I^{[p^e]}$$

this is, localization commutes with Frobenius power.

Proof. The inclusion $\subseteq$ is clear. For the other inclusion, notice that if $f \in I$ and $g \in W$, then

$$\frac{f^{p^e}}{g} \in W^{-1}I^{[p^e]} \quad \Rightarrow \quad \frac{f^{p^e}}{g} = \frac{f^{p^e} g^{p^e-1}}{g^{p^e}} \in (W^{-1}I)^{[p^e]}$$

concluding the proof. $\square$

Proposition 1.2.9. Let R be a regular F-finite ring, $e \geq 0$ and $W \subseteq R$ be a multiplicative set. Then

$$\mathcal{C}_{W^{-1}R}^e \cdot W^{-1}I = W^{-1}\mathcal{C}_R^e \cdot I$$

this is, localization commutes with Cartier operators.

Proof. By [Prop 1.2.8], localization commutes with frobenius power. Hence, by [Prop 1.2.7] (1) we have,

$$I \subseteq (\mathcal{C}_R^e \cdot I)^{[p^e]} \quad \Rightarrow \quad W^{-1}I \subseteq W^{-1}\left(\mathcal{C}_R^e \cdot I\right)^{[p^e]} = \left(W^{-1}\mathcal{C}_R^e \cdot I\right)^{[p^e]}$$

Applying cartier operators to both sides we have by [Prop 1.2.7] (2) that

$$\mathcal{C}_{W^{-1}R}^e \cdot W^{-1}I \subseteq W^{-1}\mathcal{C}_R^e \cdot I$$

For the other inclusion, notice that every element of $W^{-1}\mathcal{C}_R^e \cdot I$ is a finite sum of elements of the form

$$\frac{\varphi(f)}{w} \quad \text{with} \quad \varphi \in \mathcal{C}_R^e, \quad f \in I, \quad w \in W$$

Hence, since

$$\frac{\varphi(f)}{w} = \left(\frac{\varphi}{w}\right)\left(\frac{f}{1}\right) \in \mathcal{C}_{W^{-1}R}^e \cdot W^{-1}I$$

we have the other inclusion

$$W^{-1}\mathcal{C}_R^e \cdot I \subseteq \mathcal{C}_{W^{-1}R}^e \cdot W^{-1}I$$

concluding the proof. $\square$

Proposition 1.2.10. Let R be an F-finite regular ring and $I \subseteq R$ be an ideal containing a nonzero divisor. Then there exists $e \geq 0$ large enough such that $\mathcal{C}_R^e \cdot I = R$.

Proof. Since localization is compatible with Cartier ideal construction $\mathcal{C}_R^e \cdot I$ by [Prop 1.2.9], we may assume $(R, \mathfrak{m})$ is local. Pick a nonzerodivisor $f \in I$. By Krull's intersection Theorem [Thm A.2.2], there exist some integer d such that $f \notin \mathfrak{m}^d$. Since $\{\mathfrak{m}^k\}_k$ and $\{\mathfrak{m}^{[p^k]}\}_k$ are cofinal by [Prop 1.2.3], then there exist an integer e such that $\mathfrak{m}^{[p^e]} \subseteq \mathfrak{m}^d$. Thus

$$f \notin \mathfrak{m}^{[p^e]} \quad \Rightarrow \quad \mathcal{C}_R^e \cdot f \nsubseteq \mathfrak{m} \quad \Rightarrow \quad \mathcal{C}_R^e \cdot I \nsubseteq \mathfrak{m} \quad \Rightarrow \quad \mathcal{C}_R^e \cdot I = R$$

proving the statement. □

A direct consequence of this proposition is the following corollary, which will be used in the construction of the test ideals in the following section.

Corollary 1.2.11. Let R be an F-finite regular ring. Then, there exist $e \gg 0$ large enough, $f \in I$ nonzerodivisor and $\varphi \in \mathcal{C}_R^e$ such that $\varphi(F_*^e f) = 1$.

Proof. By proposition [Prop 1.2.10], since $\mathcal{C}_R^e \cdot I = R$ generates the whole ring, in particular there must be an element $\varphi \in \mathcal{C}_R^e$ mapping some element $f \in I$ into 1. □

1.3 ν-invariants and test ideals

We have discussed so far in the last section that Cartier operators behave nicely under the partial ordering of inclusion, that is, $\mathcal{C}_R^e \cdot$ preserve inclusion by [Prop 1.2.7] (2). We consider now the infinite chain of ideals induced by exponentiation

$$I \supseteq I^2 \supseteq I^3 \supseteq \cdots$$

Applying Cartier operators at this chain, a new chain is obtained

$$\mathcal{C}_R^e \cdot I \supseteq \mathcal{C}_R^e \cdot I^2 \supseteq \mathcal{C}_R^e \cdot I^3 \supseteq \cdots$$

The powers in which this chain has a jump (i.e. $\mathcal{C}_R^e \cdot I^l \neq \mathcal{C}_R^e \cdot I^{l+1}$) are called the ν-invariants of level e. A precise definition is given below.

Definition 1.3.1 (ν-invariants)**.** Let R be an F-finite ring and fix an ideal $\mathfrak{a} \subseteq R$. The set of ν-invariants of $\mathfrak{a}$ at level $e \geq 0$ is defined as

$$\nu_{\mathfrak{a}}^{\bullet}(p^e) = \{l \geq 0 : \mathcal{C}_R^e \cdot \mathfrak{a}^l \neq \mathcal{C}_R^e \cdot \mathfrak{a}^{l+1}\}$$

We also have an equivalent characterization which, in some cases, will be more useful to approach.

Proposition 1.3.2. Let R be an $F-$finite ring and fix an ideal $\mathfrak{a}$. Given a proper ideal J such that $\mathfrak{a} \subseteq \sqrt{J}$, the ν-invariants with base in J of $\mathfrak{a}$ can be computed as

$$\nu_{\mathfrak{a}}^J(p^e) \coloneqq \max\left\{l \geq 0 : \mathfrak{a}^l \nsubseteq J^{[p^e]}\right\}$$

If we range over all possible such ideals J, we recover all the ν-invariants

$$\nu_{\mathfrak{a}}^{\bullet}(p^e) \coloneqq \left\{\nu_{\mathfrak{a}}^J(p^e) : \mathfrak{a} \subseteq \sqrt{J}\right\}$$

Proof. Fix an ideal J such that $\mathfrak{a} \subseteq \sqrt{J}$ and let $l = \nu_{\mathfrak{a}}^J(p^e)$. This is, $\mathfrak{a}^l \not\subseteq J^{[p^e]}$ and $\mathfrak{a}^{l+1} \subseteq J^{[p^e]}$. Applying the previous proposition this is equivalent to

$$\mathcal{C}_R^e \cdot \mathfrak{a}^l \not\subseteq J, \qquad \mathcal{C}_R^e \cdot \mathfrak{a}^{l+1} \subseteq J$$

Thus, clearly $\mathcal{C}_R^e \cdot \mathfrak{a}^l \neq \mathcal{C}_R^e \cdot \mathfrak{a}^{l+1}$.

Conversely, let l be an integer such that $\mathcal{C}_R^e \cdot \mathfrak{a}^l \neq \mathcal{C}_R^e \cdot \mathfrak{a}^{l+1}$. Define $J := \mathcal{C}_R^e \cdot \mathfrak{a}^{l+1}$. In particular

$$\mathcal{C}_R^e \cdot \mathfrak{a}^{l+1} \subseteq J \quad \Rightarrow \quad \mathfrak{a}^{l+1} \subseteq J^{[p^e]}$$

Thus, if $f \in \mathfrak{a}$, then $f^{l+1} \in \mathfrak{a}^{l+1} \subseteq J^{[p^e]} \subseteq J$, meaning that J satisfies the condition $\mathfrak{a} \subseteq \sqrt{J}$. Finally

$$\mathcal{C}_R^e \cdot \mathfrak{a}^l \neq J \quad \Rightarrow \quad \mathcal{C}_R^e \cdot \mathfrak{a}^l \not\subseteq J \quad \Rightarrow \quad \mathfrak{a}^l \not\subseteq J^{[p^e]}$$

concluding the proof. □

Proposition 1.3.3. Let R be a Noetherian ring of positive characteristic, $\mathfrak{a}$ be an ideal and fix a level $e \geq 0$. Given an admissible ideal J such that $a \subseteq \sqrt{J}$, then the ν-invariant $\nu_{\mathfrak{a}}^J(p^e)$ is finite.

Proof. Since R is Noetherian $\mathfrak{a}$ and J are finitely generated, meaning there exist an $n \geq 0$ such that $\mathfrak{a}^n \subseteq J$. By cofinality of $\{J^k\}_{k\geq 0}$ and $\{J^{[p^k]}\}_{k\geq 0}$ in [Prop 2.2.9], there is a $k \geq 0$ such that $J^k \subseteq J^{[p^e]}$. Hence

$$a \subseteq \sqrt{J} \quad \Rightarrow \quad \mathfrak{a}^n \subseteq J \Rightarrow \mathfrak{a}^{nk} \subseteq J^k \subseteq J^{[p^e]}$$

concluding the proof. □

The following proposition allows us to restrict our computations at one interval.

Proposition 1.3.4. Let $\mathfrak{a}$ be a principal ideal $\mathfrak{a} = (f)$. Then $\mathcal{C}_R^e \cdot \mathfrak{a}^{l+p^e} = \mathfrak{a} \cdot \mathcal{C}_R^e \cdot \mathfrak{a}^l$.

Proof. We write

$$F_*^e f^{l+p^e} = F_*^e f^{p^e} f^l = f \cdot F_*^e f^l$$

Thus, we know that the Cartier ideal $\mathcal{C}_R^e \cdot \mathfrak{a}^{l+p^e}$ must be of the form $(f \cdot \alpha_i)$ for $(\alpha_i) \in \mathcal{C}_R^e \cdot \mathfrak{a}^l$, meaning $\mathcal{C}_R^e \cdot \mathfrak{a}^{l+p^e} = \mathfrak{a} \cdot \mathcal{C}_R^e \cdot \mathfrak{a}^l$. □

One can visualize the data of the Cartier ideals and jumping numbers for each $e \geq 1$ in the following diagram:

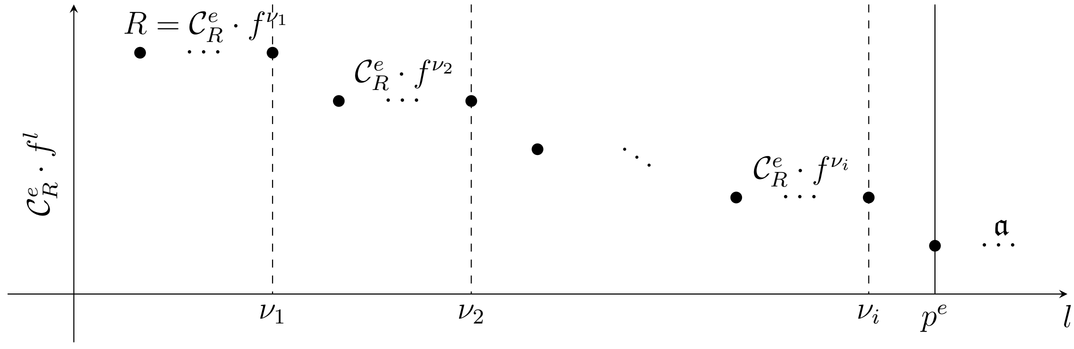


Figure 1.1: Jumps in Cartier ideals

These ν-invariants, however, depend on the level e of the Cartier ideals. Ideally, we would like to gather all this information in such a way that give us only a set of invariants that does not depend on the level e. This is the idea behind Test ideals, which we introduce now.

Definition 1.3.5 (Test ideals)**.** Let R be a regular $F-$finite ring and $\mathfrak{a}$ be an ideal. We define the test ideals as the collection of Cartier ideals parameterized by $\lambda \in \mathbb{R}_{\leq 0}$ as

$$\tau(\mathfrak{a}^{-}) : \mathbb{R}_{\geq 0} \to \{I \subseteq R\}, \qquad \tau(\mathfrak{a}^{\lambda}) = \bigcup_{e=0}^{\infty} \mathcal{C}_R^e \cdot \mathfrak{a}^{\lceil \lambda p^e \rceil}$$

However, by [Prop 1.2.7] (5) we notice that the ascending chain

$$\mathcal{C}_R^0 \cdot \mathfrak{a}^{\lceil \lambda p^0 \rceil} \subseteq \mathcal{C}_R^1 \cdot \mathfrak{a}^{\lceil \lambda p^1 \rceil} \subseteq \cdots \subseteq \mathcal{C}_R^t \cdot \mathfrak{a}^{\lceil \lambda p^t \rceil} = \mathcal{C}_R^{t+1} \cdot \mathfrak{a}^{\lceil \lambda p^{t+1} \rceil} = \cdots$$

stabilizes at some t because R is Noetherian. Hence, we can write $\tau(\mathfrak{a}^{\lambda}) = \mathcal{C}_R^e \cdot \mathfrak{a}^{\lceil \lambda p^e \rceil}$ for a sufficiently large e.

We prove some useful properties of these test ideals

Proposition 1.3.6. Let R be a regular F-finite ring and $\mathfrak{a}$ be an ideal. Then

1. If $\lambda \leq \lambda' \Rightarrow \tau(\mathfrak{a}^{\lambda}) \supseteq \tau(\mathfrak{a}^{\lambda'})$.
2. $\forall \lambda \in \mathbb{R}_{\geq 0}\ \exists \varepsilon : \tau(\mathfrak{a}^{\lambda}) = \tau(\mathfrak{a}^{\lambda+\varepsilon})$.
3. If $\mathfrak{a} = (f_1, \ldots, f_r)$ is generated by r elements, then $\tau(\mathfrak{a}^{\lambda}) = \mathfrak{a}\tau(\mathfrak{a}^{\lambda-1})$ for $\lambda \geq r$.
4. $\exists \varepsilon > 0$ such that $\tau(\mathfrak{a}^{\varepsilon}) = R$.

Proof.
(1) Let $e \gg 0$ such that we can write $\tau(\mathfrak{a}^{\lambda}) = \mathcal{C}_R^e \cdot \mathfrak{a}^{\lceil \lambda p^e \rceil}$ and $\tau(\mathfrak{a}^{\chi}) = \mathcal{C}_R^e \cdot \mathfrak{a}^{\lceil \chi p^e \rceil}$. Then, since $\mathfrak{a}^{\lceil \chi p^e \rceil} \subseteq \mathfrak{a}^{\lceil \lambda p^e \rceil}$, by proposition [Prop 1.2.7] (2) $\mathcal{C}_R^e \cdot \mathfrak{a}^{\lceil \chi p^e \rceil} \subseteq \mathcal{C}_R^e \cdot \mathfrak{a}^{\lceil \lambda p^e \rceil}$.

(2) We only need to check that there exists a positive integer b such that $\tau(\mathfrak{a}^{\lambda}) = \tau(\mathfrak{a}^{\lambda + \frac{1}{p^b}})$. Fix an element $c \in \mathfrak{a}$. Since R is regular, by [Prop 1.1.21], setting $a \gg 0$ there exists an element $\varphi \in \mathcal{C}_R^a$ such that $\varphi(F_*^a c) = 1$ because of the splitting of the Frobenius. Taking $e \gg 0$ we have the following inclusions

$$\begin{aligned} &\tau(\mathfrak{a}^{\lambda + \frac{1}{p^{e+a}}}) = \mathcal{C}_R^{e+a} \cdot \mathfrak{a}^{\lceil \lambda p^{e+a} \rceil + 1} \overset{(i)}{\supseteq} \mathcal{C}_R^{e+a} \cdot c\mathfrak{a}^{\lceil \lambda p^{e+a} \rceil} \overset{(ii)}{\supseteq} \mathcal{C}_R^{e+a} \cdot c\mathfrak{a}^{\lceil \lambda p^e \rceil p^a} \overset{(iii)}{\supseteq} \\ &\overset{(iii)}{\supseteq} \mathcal{C}_R^{e+a} \cdot c \left(\mathfrak{a}^{\lceil \lambda p^e \rceil}\right)^{[p^a]} \overset{(iv)}{=} \mathcal{C}_R^e \cdot \left(\mathcal{C}_R^a \cdot c \left(\mathfrak{a}^{\lceil \lambda p^e \rceil}\right)^{[p^a]}\right) \overset{(v)}{\supseteq} \mathcal{C}_R^e \cdot \varphi\left(F_*^a c \left(\mathfrak{a}^{\lceil \lambda p^e \rceil}\right)^{[p^a]}\right) \overset{(vi)}{=} \\ &\overset{(vi)}{=} \mathcal{C}_R^e \cdot \mathfrak{a}^{\lceil \lambda p^e \rceil} \varphi(F_*^a c) \overset{(vii)}{=} \mathcal{C}_R^e \cdot \mathfrak{a}^{\lceil \lambda p^e \rceil} = \tau(\mathfrak{a}^{\lambda}) \end{aligned}$$

where (i) follows from [Prop 1.2.7] (2) using $c\mathfrak{a}^k \subseteq \mathfrak{a}^{k+1}$, (ii) from [Prop 1.2.7] (2) using $\mathfrak{a}^{\lceil \alpha\beta \rceil} \supseteq \mathfrak{a}^{\lceil \alpha \rceil \beta}$, (iii) from $I^{[p^e]} \subseteq I^{p^e}$, (iv) follows from [Prop 1.2.7] (3), (v) from restricting to the ideal generated only by φ, (vi) from $\varphi(F_*^e c I^{[p^e]}) \supseteq I\varphi(F_*^e c)$ and finally (vii) by the initial assumption $\varphi(F_*^a c) = 1$

(3) The $\supseteq$ inclusion is clear, since $(I^m)^{[p^e]} \subseteq (I^m)^{p^e} = I^{mp^e}$. To prove $\subseteq$, first we have to prove the following statement.

$$\text{If } n \geq mp^e + (r-1)(p^e - 1), \quad \Rightarrow \quad \mathfrak{a}^n = \mathfrak{a}^{n - mp^e}(\mathfrak{a}^m)^{[p^e]}$$

Then, choosing $n = \lceil(\lambda-1)p^e\rceil + p^e$ and $m = 1$, we get

$$\tau(\mathfrak{a}^\lambda) = \mathcal{C}_R^e \cdot \mathfrak{a}^{\lceil(\lambda-1)p^e\rceil+p^e} = \mathcal{C}_R^e \cdot \left(\mathfrak{a}^{\lceil(\lambda-1)p^e\rceil}\mathfrak{a}^{[p^e]}\right) = \mathfrak{a}\mathcal{C}_R^e \cdot \mathfrak{a}^{\lceil(\lambda-1)p^e\rceil} = \mathfrak{a}\tau(\mathfrak{a}^{\lambda-1})$$

(4) Mimicking the same argument as in (2), by [Cor 1.2.11] one can choose $e \gg 0$ such that there exists $c \in \mathfrak{a}, \varphi \in \mathcal{C}_R^e \cdot \mathfrak{a}$ such that $\varphi(F_*^e c) = 1$. Then

$$1 \in \mathcal{C}_R^e \cdot \mathfrak{a} = \tau(\mathfrak{a}^{\frac{1}{p^e}})$$

and clearly $\tau(\mathfrak{a}^{\frac{1}{p^e}}) = R$. □

Observation 1.3.7. Notice that when $\mathfrak{a} = (f)$ is a principal ideal, the property [Prop 1.3.6] (3) is telling us that it suffices to study the test ideal for $\lambda \in [0, 1]$ because of its periodicity.

All these properties allow us to visualize the test ideals in the real line with *jumps* in some numbers as shown in the picture. In the literature, they are called F-jumping numbers or F-jumping exponents (see [BMS08]). The chain of ideals starts with the whole ring R and it gets smaller by containment in each jump.

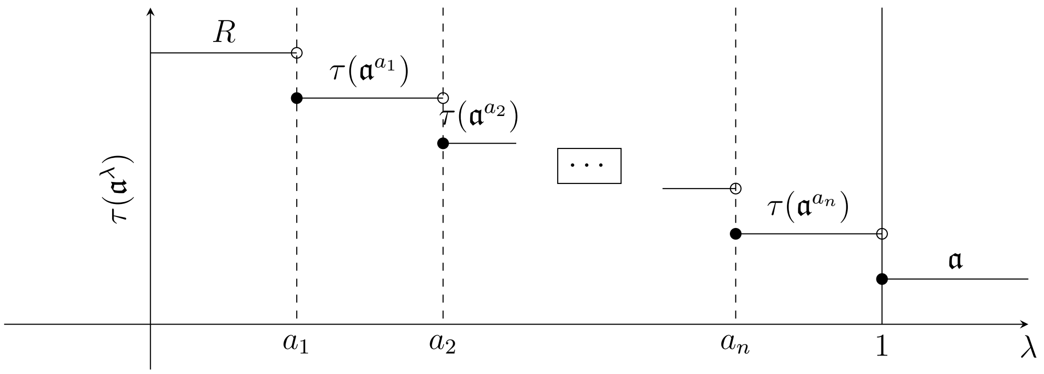


Figure 1.2: Jumps in test ideals

Proposition 1.3.8. In the case of a principal ideal $I = (f)$ there is a direct relationship between the test ideal and the Cartier ideal given by

$$\tau(f^{\frac{n}{p^e}}) = \mathcal{C}_R^e \cdot f^n$$

Proof. For each e, set $t > e$ such that the union of ideals stabilizes. Then

$$\tau(f^{\frac{n}{p^e}}) = \mathcal{C}_R^t \cdot f^{\lceil \frac{n}{p^e}p^t \rceil} = \mathcal{C}_R^t \cdot (f^n)^{p^{t-e}} = \mathcal{C}_R^e \cdot f^n$$

concluding the proof. □

Example 1.3.9. We now compute the sets $\nu_{\mathfrak{a}}^\bullet(p^e)$ of the principal ideal $\mathfrak{a} = (f)$ generated by $f = x^2 + y^3 \in \mathbb{F}_p[x, y]$ for $p > 2$.

First, notice that the module $F_*^e R$ is free and have a basis $\{x^i y^j : 0 \le i, j < p^e\}$. Since $\mathfrak{a}$ is principal, by [Obs 1.3.7] we only need to focus on the jumps for the range of values $l \in [0, p^e]$.

Trivially $\mathcal{C}_R^e \cdot \mathfrak{a}^{p^e} = \mathcal{C}_R^e \cdot f^{p^e} = (x^2+y^3)$. On the other hand we now show that $\mathcal{C}_R^e \cdot \mathfrak{a}^{p^e-1} = (x,y)$. Expanding the expression we have

$$F_*^e(x^2+y^3)^{p^e-1} = \sum_{i+j=p^e-1} \binom{p^e-1}{i} F_*^e x^{2i} y^{3j}$$

Notice the sum of $i+j = p^e-1$ in base p does not carry digits. Thus, by [Thm A.2.3], $c_i \neq 0$ for all i. If we consider the monomial where $i = j = \frac{1}{2}(p^e-1)$, this is

$$F_*^e x^{p^e-1} y^{\frac{3}{2}(p^e-1)} = y \cdot F_*^e x^{p^e-1} y^{\frac{1}{2}(p^e-3)}$$

we get that $y \in \mathcal{C}_R^e \cdot \mathfrak{a}^{p^e-1}$. If we consider the monomial where $j = \frac{1}{3}(p^e-1), i = \frac{2}{3}(p^e-1)$, this is

$$F_*^e x^{\frac{4}{3}(p^e-1)} y^{p^e-1} = x \cdot F_*^e y^{p^e-1} x^{\frac{1}{3}(p^e-4)}$$

we get that $x \in \mathcal{C}_R^e \cdot \mathfrak{a}^{p^e-1}$. Notice that there is no choice of i, j such that the monomial

$$x^{2i}y^{3j} \quad \text{with} \quad i+j = p^e-1$$

belongs to the basis of $F_*^e R$. Thus, the element acting on $F_*^e R$ is never a unit, and we conclude that $\mathcal{C}_R^e \mathfrak{a}^{p^e-1} = (x,y)$ by maximality. Hence, there is one more jump from R to (x,y), which we compute now.

In order to get $\mathcal{C}_R^e \cdot f^k = R$ we need a choice of $i, j : i+j = k$ such that $x^{2i}y^{3j}$ is an element of the basis of $F_*^e R$. Now we need to distinguish two cases: $p \equiv 1 \mod 6$ and $p \equiv 5 \mod 6$.

Case $p \equiv 1 \mod 6$: We have

$$\begin{cases} 2i < p^e \Rightarrow i \leq \frac{1}{2}(p^e-1) = \left[\frac{p-1}{2}\frac{p-1}{2}\dots\frac{p-1}{2}\right]_p \\ 3j < p^e \Rightarrow j \leq \frac{1}{3}(p^e-1) = \left[\frac{p-1}{3}\frac{p-1}{3}\dots\frac{p-1}{3}\right]_p \end{cases}$$

where the notation $[a_n a_{n-1} \dots a_0]_p$ represents the base p expansion of a number (see [Not A.1.6]). In this case we see that we do not carry in base p when the equality is achieved. Thus

$$k = i+j = \left[\frac{5p-5}{6}\frac{5p-5}{6}\dots\frac{5p-5}{6}\right]_p = \frac{5}{6}(p^e-1)$$

and we finally get $\nu(p^e) = \{\frac{5}{6}(p^e-1), p^e-1\} + \mathbb{Z}_{\geq 0}$.

Case $p \equiv 5 \mod 6$: We will suppose e is even, the other case can be computed analogously. Then $p^e \equiv 1 \mod 6$ and we have

$$\begin{cases} 2i < p^e \Rightarrow i \leq \frac{1}{2}(p^e-1) = \left[\frac{p-1}{2}\frac{p-1}{2}\dots\frac{p-1}{2}\right]_p \\ 3j < p^e \Rightarrow j \leq \frac{1}{3}(p^e-1) = \left[\frac{p-2}{3}\frac{2p-1}{3}\frac{p-2}{3}\dots\frac{2p-1}{3}\right]_p \end{cases}$$

Now we must find the following value

$$\max\{i + j : i \leq \frac{1}{2}(p^e - 1), j \leq \frac{1}{3}(p^e - 1) \text{ and } i + j \text{ does not carry}\}$$

Notice there are a lot of choices for $i + j$, but the sum is invariant. One solution is the following

$$\begin{cases} i = \left[\frac{p-1}{2}\frac{p-1}{2}\dots\frac{p-1}{2}\right]_p \\ j = \left[\frac{p-2}{3}\frac{p-1}{2}\dots\frac{p-1}{2}\right]_p \end{cases} \Rightarrow \quad k = \left[\frac{5p-7}{6}(p-1)(p-1)\dots(p-1)\right]_p = \frac{5}{6}p^e - \frac{1}{6}p^{e-1} - 1$$

It can be easily checked by the same method that for the case e odd we get the same ν-invariant. Thus, $\nu(p^e) = \{\frac{5}{6}(p^e) - \frac{1}{6}p^{e-1} - 1, p^e - 1\} + \mathbb{Z}_{\geq 0}$.

There is no need to compute for a number bigger than p^e, since by [Prop 1.3.4], all the jumps are cyclic with period p^e.

In summary, the ν-invariants are

$$\begin{cases} \nu_f^\bullet(p^e) = \{\frac{5}{6}(p^e - 1), p^e - 1\} + \mathbb{Z}_{\geq 0} & \text{if} \quad p \equiv 1 \mod 6 \\ \nu_f^\bullet(p^e) = \{\frac{5}{6}p^e - \frac{1}{6}p^{e-1} - 1, p^e - 1\} + \mathbb{Z}_{\geq 0} & \text{if} \quad p \equiv 5 \mod 6 \end{cases}$$

1.4 F-jumping numbers and F-thresholds

In the last section we discussed the construction of test ideals from Cartier ideals and proved some properties. However, as mentioned before, there is a special relation between the ν-invariants (i.e. the *jumps* in the Cartier ideals) for each level e and the F-jumping numbers (i.e. the *jumps* in the test ideal). One should think about the test ideal as a *limit* of the Cartier ideals, and the F-jumping numbers as a *limit* of the ν-invariants. This notion is poorly defined, but the goal of this section is building a precise framework for defining this limit.

All these concepts were introduced by Mustaţă, Takagi and Watanabe in the paper [MTW04], where they were trying to find an analogue of jumping coefficients and multiplier ideals. It is also worth to mention a survey in this topics including some other interesting results in the work of Blickle, Mustaţă and Smith [BMS08].

Definition 1.4.1 (F-jumping number)**.** Let R be a regular F-finite ring and $\mathfrak{a} \subseteq R$ be an ideal. A real number $\lambda \in \mathbb{R}_{\geq 0}$ is an F-jumping number if $\tau(\mathfrak{a}^\lambda) \neq \tau(\mathfrak{a}^{\lambda-\varepsilon})$ $\forall 0 < \varepsilon \ll 1$. We will denote the set of F-jumping numbers as

$$\mathrm{FJN}(\mathfrak{a}) := \{\lambda \in \mathbb{R}_{\geq 0} : \tau(\mathfrak{a}^\lambda) \neq \tau(\mathfrak{a}^{\lambda-\varepsilon}) \ \forall 0 < \varepsilon \ll 1\}$$

This definition builds the set of F-jumping numbers FJN directly from the test ideal. However, we will see that we can recover this set from the ν-invariants $\nu_{\mathfrak{a}}^\bullet(p^e)$ of each level e.

Proposition 1.4.2. Let R be a regular F-finite ring of characteristic $p > 0$ and $J \subseteq R$ be an ideal. Then

$$f \notin J \quad \Rightarrow \quad f^p \notin J^{[p]}$$

Proof. We prove this by contrapositive. Suppose $f^p \in J^{[p]}$. Then there exists a $g \in J$ such that $f^p = g^p$. Since F is injective by [Prop 1.1.22], we have that $f = g$, and thus $f \in J$. □

Proposition 1.4.3. Let R be a regular F-finite ring and $J, \mathfrak{a} \subseteq R$ be ideals such that $\mathfrak{a} \subseteq \sqrt{J}$. The sequence $\{\frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}\}_{e \geq 0}$ is increasing and bounded.

Proof. By [Prop 1.4.2] we have that

$$\mathfrak{a}^{\nu_{\mathfrak{a}}^J(p^e)} \not\subseteq J^{[p^e]} \Rightarrow \mathfrak{a}^{p\nu_{\mathfrak{a}}^J(p^e)} \not\subseteq J^{[p^{e+1}]}$$

meaning $\nu_{\mathfrak{a}}^J(p^{e+1}) \geq p\nu_{\mathfrak{a}}^J(p^e)$, which proves

$$\frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e} \leq \frac{\nu_{\mathfrak{a}}^J(p^{e+1})}{p^{e+1}}$$

For the boundedness, suppose $\mathfrak{a}$ is generated by s elements and consider $l \in \mathbb{Z}_{\geq 0}$ such that $\mathfrak{a}^l \subseteq J$, which is finite by [Prop 1.3.3]. Then

$$\mathfrak{a}^{l(s(p^e-1)+1)} \subseteq (\mathfrak{a}^l)^{[p^e]} \subseteq J^{[p^e]}$$

Thus, $\nu_{\mathfrak{a}}^J(p^e) \leq l(s(p^e-1)+1) - 1$ for all $e \in \mathbb{Z}_{\geq 0}$ and

$$\frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e} \leq \frac{l(s(p^e-1)+1)-1}{p^e} \leq ls$$

concluding the proof. □

Hence, we can consider the limit of this sequence.

Definition 1.4.4 (F-threshold)**.** Let R be a regular F-finite ring and $\mathfrak{a}, J \subseteq R$ be ideals such that $\mathfrak{a} \subseteq \sqrt{J}$. We define the F-threshold of $\mathfrak{a}$ with respect to J as

$$c^J(\mathfrak{a}) := \lim_{e \to \infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}$$

This limit is, essentially, extracting information purely from the ν-invariants. However, as discussed before, the set of F-thresholds running for every admissible ideal J is equal to the set of F-jumping numbers constructed via the test ideal. In detail, we have the following equality.

Proposition 1.4.5. Let R be a regular F-finite ring and $\mathfrak{a} \subseteq R$ be an ideal. The set of F-jumping numbers of $\mathfrak{a}$ is equal to the set of F-thresholds of $\mathfrak{a}$. This is

$$\mathrm{FJN}(\mathfrak{a}) = \{c^J(\mathfrak{a}) : \mathfrak{a} \subseteq \sqrt{J}\}$$

Proof. Fix an ideal J such that $\mathfrak{a} \subseteq \sqrt{J}$. This gives a sequence of ν-invariants

$$\{\nu_e := \nu_{\mathfrak{a}}^J(p^e)\}_{e\geq 0} \quad \text{such that} \quad \mathcal{C}_R^e \cdot f^{\nu_e} \neq \mathcal{C}_R^e \cdot f^{\nu_e+1}\ \forall e \geq 0$$

We consider the limit $L = \lim \frac{\nu_e}{p^e} = c^J(\mathfrak{a})$. Then, by definition

$$\tau(\mathfrak{a}^\lambda) = \bigcup_{e\geq 0} \mathcal{C}_R^e \cdot \mathfrak{a}^{\lceil \lambda p^e \rceil}$$

and since the sequence $\{\frac{\nu_e}{p^e}\}$ is increasing by [Prop 1.4.3], we have that

$$\mathcal{C}_R^e \cdot \mathfrak{a}^{\lceil Lp^e \rceil} \subseteq \mathcal{C}_R^e \cdot \mathfrak{a}^{\nu_e}$$

Fix $e \geq 0$. Then there exists an $\varepsilon > 0$ such that $L \geq \frac{\nu_e}{p^e} + \frac{\varepsilon}{p^e}$ and

$$\mathcal{C}_R^e \cdot \mathfrak{a}^{\lceil Lp^e \rceil} \subseteq \mathcal{C}_R^e \cdot \mathfrak{a}^{\lceil \nu_e+\varepsilon \rceil} = \mathcal{C}_R^e \cdot \mathfrak{a}^{\nu_e+1}$$

On the other hand, fix $\varepsilon > 0$. Then there exist e_0 such that for all $e \geq e_0$ then $\frac{\nu_e}{p^e} > L - \varepsilon$ and

$$\mathcal{C}_R^e \cdot \mathfrak{a}^{\lceil (L-\varepsilon)p^e \rceil} \supseteq \mathcal{C}_R^e \cdot \mathfrak{a}^{\nu_e}$$

Thus, for all $\varepsilon > 0$ one can find an e_0 such that for all $e \geq e_0$ we have

$$\mathcal{C}_R^e \cdot \mathfrak{a}^{\lceil (L-\varepsilon)p^e \rceil} = \supseteq \mathcal{C}_R^e \cdot f^{\nu_e} \supsetneq \mathcal{C}_R^e \cdot f^{\nu_e+1} \supseteq \mathcal{C}_R^e \cdot f^{\lceil Lp^e \rceil}$$

and finally we have that for all $\varepsilon > 0$, then $\tau(\mathfrak{a}^{L-\varepsilon}) \neq \tau(\mathfrak{a}^L)$, so $L \in \mathrm{FJN}(\mathfrak{a})$.

For the other inclusion, let $L \in \mathrm{FJN}(\mathfrak{a})$. Then for every $e \geq e_0$ we have that

$$J := \mathcal{C}_R^e \cdot \mathfrak{a}^{\lceil Lp^e \rceil} \text{ is the same for all } e \geq e_0$$

and setting $\nu_e = \lceil Lp^e \rceil - 1$ we have that $\nu_e \in \nu_{\mathfrak{a}}^J(p^e)$ since

$$\mathcal{C}_R^e \cdot \mathfrak{a}^{\nu_e} \not\subseteq \mathcal{C}_R^e \cdot \mathfrak{a}^{\nu_e+1} \quad \Rightarrow \quad \mathcal{C}_R^e \cdot \mathfrak{a}^{\nu_e} \not\subseteq J \quad \Rightarrow \quad \mathfrak{a}^{\nu_e} \not\subseteq J^{[p^e]}$$

Finally, the limit of the sequence $\frac{\nu_e}{p^e}$ is clearly L, proving the equivalence □

This equivalence gives sense to the previous discussion. The following diagram depicts that fact

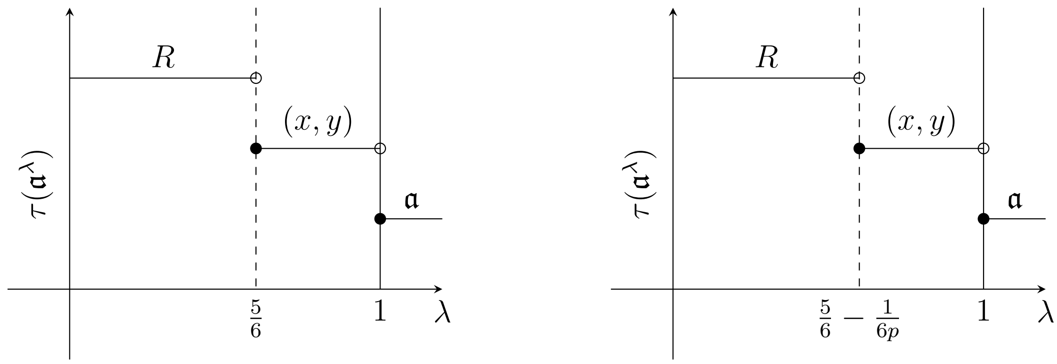
$$\begin{tikzcd}
\nu_e \in \nu_{\mathfrak{a}}^J(p^e) \arrow[r, leftrightarrow] \arrow[d, "\lim_{e\to\infty}"'] & J = \mathcal{C}_R^e \cdot \mathfrak{a}^{\nu_e+1} \arrow[d, "\lim_{e\to\infty}"] \\
c \in c^J(\mathfrak{a}) \arrow[r, leftrightarrow] & J = \tau(\mathfrak{a}^c)
\end{tikzcd}$$

Example 1.4.6. We can compute how the test ideal and the F-jumping numbers of the ideal $\mathfrak{a} = (x^2 + y^3)$ behaves. Simply performing the limits of the ν-invariants computed in [Ex 1.3.9],

$$(p \equiv 1 \mod 6) \qquad \lim_{e\to\infty} \frac{\nu_{\mathfrak{a}}^{(x,y)}(p^e)}{p^e} = \frac{5(p^e-1)}{6p^e} = \frac{5}{6}$$

$$(p \equiv 3 \mod 6) \qquad \lim_{e\to\infty} \frac{\nu_{\mathfrak{a}}^{(x,y)}(p^e)}{p^e} = \frac{5p^e - p^{e-1} - 6}{6p^e} = \frac{5}{6} - \frac{1}{6p}$$

we get the test ideals



(a) Test ideal of $\mathfrak{a}$ for $p \equiv 1 \mod 6$ (b) Test ideal of $\mathfrak{a}$ for $p \equiv 5 \mod 6$

Figure 1.3: Test ideal of $\mathfrak{a} = (x^2 + y^3)$

Now that we have shown the equivalence between the F-jumping numbers and the F-thresholds, we aim to know something more about this set. One of the most important properties is that, under the assumption of regularity, F-finiteness and principality of the ideal, the set of F-jumping numbers is discrete and rational. This result was first shown in [BMS08].

Theorem 1.4.7 (Discretness and regularity of FJN)**.** [BMS08, Theorem 1.1] Let R be a regular F-finite ring and $\mathfrak{a} \subseteq R$ a principal ideal. Then the set of F-jumping numbers $\mathrm{FJN}(\mathfrak{a})$ is discrete and rational.

1.5 Ring of differential operators

Let V be a commutative ring and R be a V-algebra. We construct the $R \otimes_V R$-module $\mathrm{End}_V(R)$ with the following natural action

$$((r \otimes s) \cdot \varphi)(x) = r\varphi(sx) \quad \forall r \otimes s \in R \otimes_V R, \varphi \in \mathrm{End}_V(R), x \in R$$

We also define a multiplication map

$$\eta : R \otimes_V R \to R \qquad \eta : r \otimes s \mapsto rs$$

Definition 1.5.1. We denote the ideal $J_{R|V} \subseteq R \otimes_V R$ as the kernel of the multiplication map

$$J_{R|V} = \ker(\eta : R \otimes_V R \to R)$$

Proposition 1.5.2. Let $r \in R$ and define $dr \coloneqq 1 \otimes r - r \otimes 1 \in R \otimes_v R$. Then $J_{R|V} = (dr : r \in R)$.

Proof. The inclusion $(dr) \subseteq J$ is clear. See that

$$\nu((s \otimes t)(1 \otimes r - r \otimes 1)) = \nu(s \otimes tr - sr \otimes t) = str - str = 0$$

For the other inclusion notice

$$r \otimes s = (r \otimes 1)(1 \otimes s - s \otimes 1) + rs \otimes 1 = (r \otimes 1)ds + rs \otimes 1$$

Then every element of $R \otimes_V R$ can be expressed as $t = \sum_i r_i \otimes s_i = \sum_i((r_i \otimes 1)ds_i + r_is_i \otimes 1) = \sum_i(r_i \otimes 1)ds_i + (\sum_i r_is_i) \otimes 1$. If $t \in J_{R|V}$, then $\eta(t) = \sum r_is_i = 0$, and the last term is zero, giving the result $t = \sum(r_i \otimes 1)ds_i \in J$. $\square$

We can now properly define the ring of differential operators via the Grothendieck construction.

Definition 1.5.3 (Grothendieck construction of differential operators)**.** Let V be a commutative ring and R be a V-algebra. We construct the subring $\mathcal{D}_{R|V} \subseteq \operatorname{End}_V(R)$ as

$$\mathcal{D}_{R|V} = \bigcup_{n\geq 0} \mathcal{D}^n_{R|V}, \qquad \text{where} \qquad \mathcal{D}^n_{R|V} = \{\varphi \in \operatorname{End}_V(R) : J^{n+1}_{R|V} \cdot \varphi = 0\}$$

An element $\varphi \in \mathcal{D}^n_{R|V}$ is called differential operator of order $\leq n$.

Proposition 1.5.4. We have the following ascendant chain of inclusions

$$\mathcal{D}^0_{R|V} \subseteq \mathcal{D}^1_{R|V} \subseteq \cdots \subseteq \mathcal{D}^n_{R|V} \subseteq \cdots \subseteq \mathcal{D}_{R|V}$$

Proof. Let $n \in \mathbb{Z}_{\geq 1}$ and $\varphi \in \mathcal{D}^{n-1}_{R|V}$, then

$$J^n_{R|V} \cdot \varphi = 0 \quad \Rightarrow \quad J^{n+1}_{R_V} \cdot \varphi = J_{R|V} \cdot J^n_{R|V} \cdot \varphi = 0 \quad \Rightarrow \quad \varphi \in \mathcal{D}^n_{R|V}$$

concluding the proof. $\square$

At first, this construction seems arbitrary, but we are now exploring the case of a polynomial ring in characteristic zero for some intuition.

Example 1.5.5. Let $V = \mathbb{Z}$ and $R = \mathbb{Z}[x_1, \ldots, x_n]$, that is, a polynomial ring in n variables. Notice that, apart from multiplication by an element of R, differentiation in the classical sense is also an endomorphism. Defining $\partial_i := \frac{\partial}{\partial x_i}$, we have that if $\sum i_j = l$, then $\partial_1^{i_1}\partial_2^{i_2} \cdots \partial_n^{i_n} \in \mathcal{D}^l_{R|V}$.

Proof. We first see that the endomorphisms given by multiplication by an element, this is

$$\varphi_f : R \to R \quad \text{given by} \quad \varphi_f(r) = fr \quad \text{for} \quad f \in R$$

are in $\mathcal{D}^0_{R|V}$, since

$$[(1 \otimes s - s \otimes 1)\varphi_f]\, r = \varphi_f(s)r - s\varphi_f(r) = fsr - sfr = 0$$

We proceed by induction. Notice that we can rearrange the partial derivative operators ∂_i in any order we want, since they commute. For the base case we see that $\partial_i \in \mathcal{D}^1_{R|V}$.

$$[(1 \otimes s - s \otimes 1)\partial_i]\,(r) = \partial_i(sr) - s\partial_i(r) = \partial_i(s)r + s\partial_ir - s\partial_ir = (\partial_is)(r)$$

since the result is a zero order differential operator, we have that $\partial_i \in \mathcal{D}^1_{R|V}$.

Now suppose that $\partial_1^{i_1} \cdots \partial_n^{i_n - 1} \in \mathcal{D}^{\sum i_j - 1}_{R|V}$. Then we have that

$$\begin{aligned}
&\left[(1 \otimes x_1 - x_1)^{i_1} \cdots (1 \otimes x_n - x_n \otimes 1)^{i_n} \partial_1^{i_1} \cdots \partial_n^{i_n}\right](r) = \\
=&(1 \otimes x_1 - x_1)^{i_1} \cdots (1 \otimes x_n - x_n \otimes 1)^{i_n - 1}(\partial_1^{i_1} \cdots \partial_n^{i_n}(x_n r) - x_n \partial_1^{i_1} \cdots \partial_n^{i_n}(r)) = \\
=&(1 \otimes x_1 - x_1)^{i_1} \cdots (1 \otimes x_n - x_n \otimes 1)^{i_n - 1}(x_n \partial_1^{i_1} \cdots \partial_n^{i_n}(r) + i_n \partial_1^{i_1} \cdots \partial_n^{i_n - 1} r - x_n \partial_1^{i_1} \cdots \partial_n^{i_n}(r)) = \\
=&\left[(1 \otimes x_1 - x_1)^{i_1} \cdots (1 \otimes x_n - x_n \otimes 1)^{i_n - 1}(i_n \partial_1^{i_1} \cdots \partial_n^{i_n - 1})\right](r)
\end{aligned}$$

but we know by induction hypothesis that $\partial_1^{i_1} \cdots \partial_n^{i_n - 1} \in \mathcal{D}^{l-1}_{R|V}$, and thus the last expression is zero.

For the final step notice that $(dx_i : 1 \leq i \leq n) = (dr : r \in R)$. The inclusion $\supseteq$ is trivial, and for $\subseteq$ see that if $r = \sum_\alpha c_\alpha x^\alpha$ with multi-index notation, then $dr = \sum c_\alpha (1 \otimes x^\alpha - x^\alpha \otimes 1)$. We can decompose each summand in the form $x^\alpha = x_i q$ for some polynomial q and index i and show that

$$(1 \otimes xq - xq \otimes 1) = (1 \otimes x - x \otimes 1)(1 \otimes q + q \otimes 1) + (1 \otimes q - q \otimes 1)(1 \otimes x + x \otimes 1)$$

Iterating this process by induction we see that $dr = \sum dx_i g_i$ for some $g_i \in R \otimes_V R$ and thus we have the result.

Hence, we have proved that $\partial_1^{i_1} \cdots \partial_n^{i_n} \in \mathcal{D}^{\sum i_j}_{R|V}$. □

We consider now the construction of this ring of differential operators in positive characteristic. In this case we can make use of the Frobenius map.

Definition 1.5.6. Let R be a ring of characteristic $p > 0$. Then we define the set of differential operators of level $\leq e$ as

$$\mathcal{D}_R^{[e]} = \mathrm{Hom}_R(F^e_* R, F^e_* R)$$

We observe that this $\mathcal{D}_R^{[e]}$ is a non-commutative ring where the addition is given by the usual addition of endomorphisms and the multiplication is given by the composition of endomorphisms. We will study how this ring can be related to R when we talk about Morita theory at the end of the section.

Recall that the construction of the ring of differential operators is performed via the union of the rings of differential operators of order $e \geq 0$, $\mathcal{D}^e_R$. The key observation is that one can also recover the ring of differential operators $\mathcal{D}_R$ as the union of the rings of differential operators $\mathcal{D}_R^{[e]}$.

Proposition 1.5.7. [YS92, Theorem 1.4.9] Let R be a regular F-finite ring of characteristic $p > 0$. Then

$$\mathcal{D}_R = \bigcup_{e \geq 0} \mathcal{D}_R^{[e]}$$

We will prove these statements in the more general case in the next chapter. Notice that in rings of characteristic zero we have a nice characterization, at least in the polynomial case, of a natural order of each differential operator. This is given by the chain

$$\mathcal{D}_R^0 \subseteq \mathcal{D}_R^1 \subseteq \cdots \subseteq \mathcal{D}_R^n \subseteq \cdots \subseteq \mathcal{D}_R$$

However, in characteristic $p > 0$ it is sometimes more useful to consider the chain

$$\mathcal{D}_R^{[0]} \subseteq \mathcal{D}_R^{[1]} \subseteq \cdots \subseteq \mathcal{D}_R^{[e]} \subseteq \cdots \subseteq \mathcal{D}_R$$

In the section of Cartier operators one of the properties we had was that applying the Frobenius power and then Cartier operator to an ideal resulted in the same ideal, but if we reverse the order of the operations we only have one containment. That is

$$I \xrightarrow{[p^e]} I^{[p^e]} \xrightarrow{\mathcal{C}_R^e \cdot} \mathcal{C}_R^e \cdot I^{[p^e]} = I, \quad \text{but} \quad I \xrightarrow{\mathcal{C}_R^e \cdot} \mathcal{C}_R^e \cdot I \xrightarrow{[p^e]} (\mathcal{C}_R^e \cdot I)^{[p^e]} \supseteq I$$

However, this can be fixed considering some ideals as $D_R^{[e]}$-modules. Not only we have this bijection between R-submodules of R and $\mathcal{D}_R^{[e]}$-submodules of R, but we also have the following equivalence of categories.

Theorem 1.5.8 (Morita equivalence). Let R be a regular $F-$finite ring of characteristic $p > 0$. Then the rings R and $\mathcal{D}_R^{[e]}$ are Morita equivalent, that is, there exist an equivalence of categories

$$F^{e*} : \mathbf{R - mod} \to \mathcal{D}_{\mathbf{R}}^{[\mathbf{e}]} - \mathbf{mod}$$

given by $F^{e*}(M) = \operatorname{Hom}_R(R, F_*^e R) \otimes_R M$.

A more general version of this theorem will be proved in the next chapter.

Thus, via this equivalence of categories, we have the following identification

$$\begin{gathered}\{R\text{-submodules of } R\} \leftrightarrow \{\mathcal{D}_R^{[e]}\text{-submodules of } R\} \\ J \mapsto J^{[p^e]} \\ \mathcal{C}_R^e \cdot I \leftarrow\!\shortmid \mathcal{D}_R^{[e]} \cdot I\end{gathered}$$

We will see this in more detail in the next chapter.

Example 1.5.9. Let $R = \mathbb{F}_3[x, y]$ be a polynomial ring. We compute the differential ideal $\mathcal{D}_R^{[e]} \cdot I$ of the principal ideal I generated by $f = (x^2 + y^3)^2$.

Following [Ex 1.2.6], $\{x^i y^j : 0 \leq i, j < 3\}$ is a basis of $F_*^1 R$. We have that $f = x^4 + 2x^2y^3 + y^6$, and thus, for $\varphi \in \mathcal{D}_R^{[1]}$

$$\varphi(F_*^1 f) = x \cdot F_*^1 x + 2y \cdot F_*^1 x^2 + y^2 \cdot F_*^1 1$$

Notice that the map φ is determined by the images of the basis elements of $F_*^1 R$. Then, the ideal $\mathcal{D}_R^{[1]} \cdot I$ is

$$\mathcal{D}_R^{[1]} \cdot (x^2 + y^3)^2 = (x, 2y, y^2) F_*^1 R = (x, y) F_*^1 R$$

Chapter 2

Bernstein-Sato Roots in positive characteristic

In this chapter we make the construction and different characterizations of Bernstein-Sato roots, first in the classical case of prime characteristic, and then in the prime power characteristic. The first section is intended to be a summary of the Bernstein-Sato theory in positive characteristic, covering the most important theorems and constructions in it. The second section covers in detail all the construction of the prime power characteristic case, first performed in [BQG24]. Notice that the usual prime characteristic case is contained in this more general setting. Thus, all the proofs given in this section work directly in the the prime characteristic case.

2.1 Bernstein-Sato in characteristc $p > 0$

This first section is intended to be a summary of the Bernstein-Sato theory in prime characteristic. We will see that Bernstein-Sato roots of a given polynomial (we will stick to the case that the ideal is principal) can be computed via the p-adic limit of the ν-invariants or, additionally, via one construction of an $\mathcal{D}_R$-module with an exotic structure. Another constructions can be found in the literature, for instance, the vanishing of certain modules constructed using local cohomology. For a detailed summary on those constructions, see [JNBQG23].

In this section we will construct a $\mathcal{D}_R$-module structure on the module $R_f \boldsymbol{f}^{\boldsymbol{\alpha}}$ for $\alpha \in \mathbb{Z}_p$, where R_f is the localization of R on $f \in R$. Fixing $f \in R$, this module will depend on the choice of $\alpha \in \mathbb{Z}_p$. The behavior of the resulting module will show whether α is a Bernstein-Sato root or not.

Recall that $\alpha_{<e}$ denotes the truncation of α at e (see [Def A.1.2]).

Definition 2.1.1. Let $\alpha \in \mathbb{Z}_p$, $\delta \in \mathcal{D}_R$ and $f \in R$. Then, the map

$$\Upsilon_{\alpha,f} : \mathcal{D}_R \to \mathcal{D}_{R_f}, \quad \delta \mapsto f^{-\alpha_{<e}} \delta f^{\alpha_{<e}} \quad \text{such that} \quad \delta \in \mathcal{D}_R^{[e]}$$

is well defined and induces maps $\Upsilon_{\alpha,f} : \mathcal{D}_R^{[e]} \to \mathcal{D}_{R_f}^{[e]}$.

Proof. We observe that if $\delta \in \mathcal{D}_R^{[e]}$, then $f^{-\alpha_{<a}}\delta f^{\alpha_{<a}}$ are equal for all $a \geq e$. Indeed if $a, a' \geq e$, then there exists $m \in \mathbb{Z}$ such that $\alpha_{<a'} = \alpha_{<a} + mp^e$ and

$$\left(f^{-\alpha_{<a'}}\delta f^{\alpha_{<a'}}\right)(x) = f^{-\alpha_{<a}}\left(f^m\right)^{-p^e}\delta\left(\left(f^m\right)^{p^e} f^{\alpha_{<a}} x\right) =$$
$$= f^{-\alpha_{<a}}\left(f^m\right)^{-p^e}\left(f^m\right)^{p^e}\delta\left(f^{\alpha_{<a}} x\right) = \left(f^{-\alpha_{<a}}\delta f^{\alpha_{<a}}(x)\right)$$

where we used the fact that $(f^m)^{p^e}$ commutes with δ. Notice that if $\delta \in \mathcal{D}_R^{[e]}$, then

$$\Upsilon_{\alpha,f}(\delta)(t^{p^e}x) = f^{-\alpha_{<e}}\delta\left(f^{\alpha_{<e}}t^{p^e}x\right) = t^{p^e}\left(f^{-\alpha_e}\delta f^{\alpha_{<e}}x\right) = t^{p^e}\Upsilon_{\alpha,f}(\delta)(x)$$

again because of the commutativity of t^{p^e} with δ. Hence, $\Upsilon_{\alpha,f}(\delta) \in \mathcal{D}_{R_f}^{[e]}$. □

We now construct a $\mathcal{D}_R$-module structure via this map. Fix $\alpha \in \mathbb{Z}_p$ and $f \in R$. Then we construct the $\mathcal{D}_R$-module $R_f \boldsymbol{f}^{\boldsymbol{\alpha}}$, which is canonically isomorphic as a set to R_f ($g\boldsymbol{f}^{\boldsymbol{\alpha}} \leftrightarrow g$), and whose action is given by

$$\delta \cdot (g\boldsymbol{f}^{\boldsymbol{\alpha}}) = (\Upsilon_{f,\alpha}(\delta) \cdot g)\boldsymbol{f}^{\boldsymbol{\alpha}}$$

We can now define properly the Bernstein-Sato roots of a nonzerodivisor $f \in R$.

Definition 2.1.2 (Bernstein-Sato root)**.** Let $\alpha \in \mathbb{Z}_p$ and $f \in R$. Then α is a Bernstein-Sato root of f if the following equivalent conditions are satisfied:

1. $\boldsymbol{f}^{\boldsymbol{\alpha}} \notin \mathcal{D}_R \cdot f\boldsymbol{f}^{\boldsymbol{\alpha}}$
2. α is the $p-$adic limit of a sequence $\{\nu_e\}$ such that $\nu_e \in \nu_f^{\bullet}(p^e)$.

We will prove this equivalence in the next section in the more general case.

2.2 Construction of the prime power case

At this point we are aiming for a more general setting. Rather than considering a ring of prime characteristic, we wonder how rings of prime power characteristic (i.e. those containing $\mathbb{Z}/p^{m+1}\mathbb{Z}$ as subring for some $m \in \mathbb{Z}_{\geq 0}$) behave. This section will mimic the construction given by the work done by Bitoun and Quinlan-Gallego in [BQG24].

Notice that in this setting the regular Frobenius we used in characteristic $p > 0$ is not even a morphism. Thus, we need a broader definition of Frobenius, which turns out to be the *lift of Frobenius*.

Setting 2.2.1. For the remainder of the section we set

- $(V, \mathfrak{m})$ is a commutative Artinian local ring such that $V/\mathfrak{m}$ is a field of prime characteristic $p > 0$.
- m is an integer such that $\mathfrak{m}^{m+1} = 0$.
- R is a V-algebra.

2.2.1 The lift of Frobenius

Definition 2.2.2 (Lift of Frobenius)**.** Let $(V, \mathfrak{m})$ be a commutative Artinian local ring such that $V/\mathfrak{m}$ is a field of characteristic $p > 0$. A *lift of Frobenius* is a bijective ring homomorphism $F_V : V \to V$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "F_V"] \arrow[d, two heads] & V \arrow[d, two heads] \\ V/\mathfrak{m} \arrow[r, "F"'] & V/\mathfrak{m} \end{tikzcd}$$

commutes, where F is the Frobenius endomorphism.

Proposition 2.2.3. Suppose there exist a lift of Frobenius $F_V : V \to V$. Then, the field $k := V/\mathfrak{m}$ is perfect.

Proof. By assumption, F_V is bijective and since $\mathfrak{m}$ is formed by all the non-units of V by [Prop A.2.7], then $F_V(\mathfrak{m}) = \mathfrak{m}$. Thus, $F : k \to k$ is an automorphism and, by [Prop 1.1.9], k is perfect. $\square$

Definition 2.2.4 (Compatible lift of Frobenius)**.** Let R be a V-algebra. Given a lift of Frobenius $F_V : V \to V$, we say the lift $F_R : R \to R$ is compatible with F_V if the diagram

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d, "F_V"'] & R \arrow[r, two heads] \arrow[d, "F_R"'] & R/\mathfrak{m}R \arrow[d, "F_{\mathfrak{m}}"'] \\ V \arrow[r, hook] & R \arrow[r, two heads] & R/\mathfrak{m}R \end{tikzcd}$$

commutes, where $F_{\mathfrak{m}}$ is the Frobenius endomorphism.

For language economy we call from now on, if there is no ambiguity, simply *Frobenius* to a *compatible lift of Frobenius.*

Notice that, if there exist a Frobenius $F : R \to R$, then we can stack vertically e diagrams to form by composition the Frobenius of level e $F^e : R \to R$. Thus, fixing a Frobenius, we say that the *Frobenius of level e* is $F^e := F \circ \overset{(e)}{\cdots} \circ F$.

Recall that for a prime characteristic ring we had existence and uniqueness of the Frobenius endomorphism. In this case, we cannot assure the existence of a lift of Frobenius, neither its uniqueness. Thus, every time we are making computations with this Frobenius involved (for instance, ν-invariants, as we will see), we first we need to fix a Frobenius.

Proposition 2.2.5. The main case of interest is setting

$$V = \mathbb{Z}/p^{m+1}\mathbb{Z}, \quad \mathfrak{m} = (p), \quad V/\mathfrak{m} = \mathbb{F}_p \quad \text{and} \quad R = \mathbb{Z}/p^{m+1}\mathbb{Z}[x_1, \dots, x_n]$$

Then, there exists a Frobenius $F : R \to R$ uniquely defined by $F(x_i) = x_i^p$.

Proof. We must prove that the following diagram commutes

$$\begin{tikzcd}
\mathbb{F}_p \arrow[d, "F_p"'] & \frac{\mathbb{Z}}{p^{m+1}\mathbb{Z}} \arrow[l, two heads, "q"'] \arrow[r, hook, "\iota"] \arrow[d, "F_V"'] & \frac{\mathbb{Z}}{p^{m+1}\mathbb{Z}}[\underline{x}] \arrow[r, two heads, "\pi"] \arrow[d, "F"'] & \mathbb{F}_p[\underline{x}] \arrow[d, "F_{\mathfrak{m}}"'] \\
\mathbb{F}_p & \frac{\mathbb{Z}}{p^{m+1}\mathbb{Z}} \arrow[l, two heads, "q"] \arrow[r, hook, "\iota"'] & \frac{\mathbb{Z}}{p^{m+1}\mathbb{Z}}[\underline{x}] \arrow[r, two heads, "\pi"'] & \mathbb{F}_p[\underline{x}]
\end{tikzcd}$$

Notice that we only need to check for commutativity for $a \in \mathbb{Z}/p^{m+1}\mathbb{Z}$ and x_i, since by the fact that F is a ring homomorphism, we have

$$F\left(\sum_{\alpha} a_\alpha \underline{x}^\alpha\right) = \sum_{\alpha} F(a_\alpha)F(\underline{x}^\alpha) = \sum_{\alpha} F(a_\alpha)\underline{F(x)}^\alpha$$

The commutativity of the first square (in addition to the trivial bijectivity of F_V) implies that F_V is a lift of Frobenius. To check the commutativity we notice that $F_p(a) = a \ \forall a \in \mathbb{F}_p$. For the second square we let $a \in \mathbb{Z}/p^{m+1}\mathbb{Z}$

$$F \circ \iota(a) = F(a) = a, \qquad \iota \circ F(a) = \iota(a) = a$$

so, it clearly commutes.

For the third square, since F_V is additive and multiplicative, we only have to show that the square commutes for the elements a and x_i. We denote by $[r]_p \coloneqq \pi(r)$ the class under the projection.

$$F_{\mathfrak{m}} \circ \pi(a) = F_{\mathfrak{m}}([a]_p) = [a]_p^p = [a^p]_p = [a]_p, \qquad \pi \circ F(a) = \pi(a) = [a]_p$$

and

$$F_{\mathfrak{m}} \circ \pi(x_i) = F_{\mathfrak{m}}([x_i]_p) = [x_i]_p^p = [x_i^p]_p, \qquad \pi \circ F(x_i) = \pi(x_i^p) = [x_i^p]_p$$

Hence, the diagram commutes and F is a lift of Frobenius. $\square$

Composing e times this Frobenius we get that the Frobenius of level e is, then, uniquely defined by linearly extending the map $F^e(x_i) = x_i^{p^e}$. for each variable x_i.

At this point we redefine some concepts of the background that are crucial in the development of the theory, adapting the definitions to this new Frobenius. Notice that this notions will depend on the choice of the lift of Frobenius.

Definition 2.2.6. Redefinitions of some objects adapted to a lift of Frobenius

- Frobenius power $J^{[p^e]} \coloneqq (F^e(f) : f \in J)$. We also define the Frobenius of one element $f^{[p^e]} \coloneqq F^e(f)$.
- The module $F_*^e M$ has the action $r \cdot F_*^e f \coloneqq F_*^e F^e(r) f$.
- Cartier operator $\mathcal{C}_R^e \coloneqq \mathrm{Hom}_R(F_*^e R, R)$.
- Differential operator $\mathcal{D}_R^{[e]} \coloneqq \mathrm{Hom}_{F^e(R)}(R, R) = \mathrm{Hom}(F_*^e R, F_*^e R)$.

Notice that in this case we loose, in general, the property $J^{[p^e]} \subseteq J^{p^e}$, which, for instance, proved the cofinality of $\{J^{[p^e]}\}_{e\geq 0}$ and $\{J^e\}_{e\geq 0}$ in the prime characteristic case. However, some properties of Cartier ideals and differential operators still hold. Next propositions aim to show that this lift of Frobenius have some of these properties.

Proposition 2.2.7. If $f, g \in R$ are elements such that $f \equiv g \mod \mathfrak{m}^k R$ for some $k \in \mathbb{Z}_{\geq 0}$, then $f^{p^e} \equiv g^{p^e} \mod \mathfrak{m}^{k+e} R$ for all $e \in \mathbb{Z}_{\geq 0}$.

Proof. We only have to show the case $e = 1$, and $e \geq 1$ follows from induction, since we can consider the chain

$$f \equiv g \mod \mathfrak{m}^k R \quad \Rightarrow \quad f^{p^1} \equiv g^{p^1} \mod \mathfrak{m}^{k+1} R \quad \Rightarrow \cdots \Rightarrow \quad f^{p^e} \equiv g^{p^e} \mod \mathfrak{m}^{k+e} R$$

We first consider $p = 2$. We have that

$$f^2 - g^2 = (f-g)(f+g)$$

Knowing that $f - g \in \mathfrak{m}^k R$ and $f + g = (f-g) + 2g \in \mathfrak{m} R$, then $f^2 - g^2 \in \mathfrak{m}^{k+1} R$.

For $p \neq 2$, notice that if $f - g \in \mathfrak{m}^k R$, then $f^i - g^i = (f-g)(f^{i-1} + \cdots + g^{i-1}) \in \mathfrak{m}^k R \; \forall i \in \mathbb{Z}_{\geq 1}$. Now expand

$$(f-g)^p - (f^p - g^p) = \sum_{i=1}^{p-1} (-1)^i \binom{p}{i} f^i g^{p-i} = \sum_{i=1}^{\frac{p-1}{2}} (-1)^i \binom{p}{i} f^i g^{p-i} + \sum_{j=1}^{\frac{p-1}{2}} (-1)^j \binom{p}{j} f^i g^{p-j} =$$
$$= \sum_{i=1}^{\frac{p-1}{2}} (-1)^i \binom{p}{i} f^i g^{p-i} + \sum_{i=1}^{\frac{p-1}{2}} (-1)^{p-i} \binom{p}{i} f^{p-i} g^{p} = \sum_{i=1}^{\frac{p-1}{2}} (-1)^i \binom{p}{i} f^i g^i (f^{p-2i} - g^{p-2i})$$

getting

$$f^p - g^p = (f-g)^p - \sum_{i=1}^{\frac{p-1}{2}} (-1)^i \binom{p}{i} f^i g^i (f^{p-2i} - g^{p-2i})$$

We know that the term $(f-g)^p \in \mathfrak{m}^{kp} R \subseteq \mathfrak{m}^{k+1} R$, and the terms in the sum

$$(f^{p-2i} - g^{p-2i}) \in \mathfrak{m}^k R \Rightarrow \binom{p}{i} (f^{p-2i} - g^{p-2i}) \in \mathfrak{m}^{k+1} R$$

since $p \in \mathfrak{m}$. Thus, $f^p - g^p \in \mathfrak{m}^{k+1} R$ as desired. □

Proposition 2.2.8. Let m such that $\mathfrak{m}^{m+1} = 0$. Then for all $g \in R, e \in \mathbb{Z}_{\geq 0}$ we have $F^e(g^{p^m}) = g^{p^{m+e}}$ for F a lift of Frobenius.

Proof. Because of the commutativity of the diagram [Def 2.2.4], $F^e(g) - g^{p^e} \in \mathfrak{m} R$. Applying [Prop 2.2.7] with $e = m, k = 1$, it follows that

$$F^e(g)^{p^m} - (g^{p^e})^{p^m} = F^e(g^{p^m}) - g^{p^{e+m}} \equiv 0 \mod \mathfrak{m}^{m+1} R \Rightarrow F^e(g^{p^m}) - g^{p^{e+m}} = 0$$

concluding the proof. □

As discussed previously, the cofinality of the families of ideals $\{J^k\}_{k\geq 0}$ and $\{J^{[p^k]}\}_{k\geq 0}$, one of the important properties we had in characteristic $p > 0$, is not attained. The following proposition uses an extra assumption to achieve cofinality.

Proposition 2.2.9. [BQG24, Lemma 2.17] Let V be as in [Sett 2.2.1], R be a flat V-algebra, J a finitely generated ideal $J \subseteq R$ such that $R/J^k R$ is flat over V for all $k \in \mathbb{Z}_{\geq 1}$ and suppose there exists a Frobenius such that $J^{[p]} \subseteq J$. Then the families of ideals $\{J^i\}$ and $\{J^{[p^i]}\}$ are cofinal.

Proof. See [BQG24, Lemma 2.17] $\square$

2.2.2 Cartier operators and ν-invariants

Proposition 2.2.10. [BQG24, Lemma 2.20] Let V be as in [Sett 2.2.1], R a smooth V-algebra and $F : R \to R$ a lift of Frobenius. Then the R-module $F^e_* R$ is finitely generated and projective.

At this point we have some important checks to perform, mainly that the following properties stated in [Prop 1.2.7] extend well to this more general setting. The first one is the explicit computation of the cartier ideals in the case $F^e_* R$ is a free module.

Proposition 2.2.11. Let V be as in [Sett 2.2.1]. Let R be a smooth $V-$ algebra with a lift of Frobenius $F : R \to R$ such that $F^e_* R$ a free $R-$module with basis $\{F^e_*\alpha_1, \dots, F^e_*\alpha_n\}$. Suppose

$$I = (f_1, \dots, f_m) \quad \text{and} \quad F^e_* f_i = \sum_{j=1}^{n} f_{ij} F^e_* \alpha_j$$

Then, one has $\mathcal{C}^e_R \cdot I = (f_{ij} : 1 \leq i \leq m, 1 \leq j \leq n)$.

Proof. Let $\varphi_i : F^e_*\alpha_j \mapsto \delta_{ij}\ \forall i = 1, \dots, n$, where δ_{ij} is the Kronecker delta function. Then, each Cartier operator is of the form $\varphi = \sum_{i=1}^n r_i\varphi_i$ for some $r_i \in R$. Hence,

$$\mathcal{C}^e_R \cdot I = (\varphi_i(F^e_* f_j) : 1 \leq i \leq n, 1 \leq j \leq m) = (\varphi_i(\sum_{k=1}^{n} f_{jk} F^e_* f_k) : 1 \leq i \leq n, 1 \leq j \leq m) =$$
$$= (f_{ij} : 1 \leq i \leq n, 1 \leq j \leq m)$$

concluding the proof. $\square$

Proposition 2.2.12. Let V be as in [Sett 2.2.1]. Let R be a smooth V-algebra with a lift of Frobenius $F : R \to R$ such that $F^e_* R$ is free with the setting [Sett 2.2.1], ideals $I, J \subseteq R$ and a level $e \geq 1$. Then

1. $\mathcal{C}^e_R \cdot I^{[p^e]} = I$
2. $I \subseteq (\mathcal{C}^e_R \cdot I)^{[p^e]}$.
3. $I \subseteq J \Rightarrow \mathcal{C}^e_R \cdot I \subseteq \mathcal{C}^e_R \cdot J$.
4. $\mathcal{C}^e_R \cdot \mathcal{C}^d_R \cdot I = \mathcal{C}^{e+d}_R \cdot I$

5. $\mathcal{C}_R^d \cdot I = \mathcal{C}_R^{e+d} \cdot I^{[p^e]}$.
6. $\mathcal{C}_R^e \cdot I \subseteq J \iff I \subseteq J^{[p^e]}$.

Proof.
(1) Recall that the morphisms $\varphi \in \mathcal{C}_R^e$ fullfil $\varphi(F_*^e f^{[p^e]}) = f\varphi(F_*^e 1)$. Then

$$\mathcal{C}_R^e \cdot I^{[p^e]} = \mathcal{C}_R^e \cdot (f^{[p^e]} : f \in I) = (\varphi(F_*^e f^{[p^e]}) : f \in I, \varphi \in \mathcal{C}_R^e) = (f\varphi(F_*^e 1) : f \in I, \varphi \in \mathcal{C}_R^e)$$

Notice that by [Prop 1.1.21], there exists a section $s \in \mathcal{C}_R^e$, meaning $s(F_*^e 1) = 1$. Thus, we can reduce the last expression to

$$(f\varphi(F_*^e 1) : f \in I, \varphi \in \mathcal{C}_R^e) = (f \cdot 1 : f \in I) = I$$

(2) Let $I = (f_1, \ldots, f_r)$. Since $F_*^e R$ is finitely generated, then

$$F_*^e f_i = \sum_{j=1}^{n} f_{ij} F_*^e \alpha_j \quad \Rightarrow \quad f_i = \sum_{j=1}^{n} f_{ij}^{p^e} \alpha_j$$

Notice that by [Prop 2.2.11], f_{ij} generate $\mathcal{C}_R^e \cdot I$, in particular $f_{ij} \in I$. Thus, $f_i \in (\mathcal{C}_R^e \cdot I)^{[p^e]}$.

(3) Let $g \in \mathcal{C}_R^e \cdot I$. Then $\exists f \in I : \varphi(F_*^e f) = g$. Since $f \in J$, $g = \varphi(F_*^e f) \in \mathcal{C}_R^e \cdot J$. Conversely, let $f \in I$, then $\varphi(f) \in \mathcal{C}_R^e \cdot I$, and $\varphi(f) \in \mathcal{C}_R^e \cdot J$ as well, meaning $f \in J$.

(4) See [Qui21, Proposition 2.48].

(5) From (4) and (1) we have that

$$\mathcal{C}_R^{d+e} \cdot I^{[p^e]} = \mathcal{C}_R^d \cdot \mathcal{C}_R^e \cdot I^{[p^e]} = \mathcal{C}_R^d \cdot I$$

(6) Suppose $\mathcal{C}_R^e \cdot I \subseteq J$. Then, by (2) is clear $I \subseteq (\mathcal{C}_R^e \cdot I)^{[p^e]} \subseteq J^{[p^e]}$. Conversely, if $I \subseteq J^{[p^e]}$, then by (2) $\mathcal{C}_R^e \cdot I \subseteq \mathcal{C}_R^e \cdot (J^{[p^e]}) = J$, where the last equality follows form (4). $\square$

In this prime power construction, the definition of the ν-invariants of level e is the same as in the prime characteristic case introduced in [Def 1.3.1]. The only difference is that we have to fix a Frobenius $F : R \to R$. Thus, the ν-invariants are defined with respect to this Frobenius, which we usually omit in the notation for visual clarity.

Definition 2.2.13 (ν-invariants)**.** Let R be as in [Sett 2.2.1] and $F : R \to R$ a lift of Frobenius. Fix an ideal $\mathfrak{a} \subseteq R$. The set of ν-invariants of $\mathfrak{a}$ at level $e \geq 0$ is defined as

$$\nu_{\mathfrak{a}}^{\bullet}(p^e) = \{l \geq 0 : \mathcal{C}_R^e \cdot \mathfrak{a}^l \neq \mathcal{C}_R^e \cdot \mathfrak{a}^{l+1}\}$$

We also have a new version of the periodicity of ν-invariants introduced in [Prop 1.3.4]. This time, the period is p^{e+m}.

Proposition 2.2.14. Let R be as in [Sett 2.2.1] and $F : R \to R$ a lift of Frobenius. Let $f \in R$. Then,

$$\nu_f^{\bullet}(p^e) = \nu_f^{\bullet}(p^e) + p^{e+m}\mathbb{Z}$$

Proof. By [Prop 2.2.8], we can write $f^{p^{e+m}} = F^e(f^{p^m})$. Then, we have

$$\mathcal{C}_R^e \cdot f^{l+p^{e+m}} = f^{p^m}\mathcal{C}_R^e \cdot f^l \neq f^{p^m}\mathcal{C}_R^e \cdot f^{l+1} = \mathcal{C}_R^e \cdot f^{l+p^{e+m}+1}$$

Thus, we have that shifting the ν-invariants by p^{e+m} gives the same set of ν-invariants. $\square$

2.2.3 Ring of differential operators $\mathcal{D}_{R|V}$ and Morita equivalence

We performed the Grothendieck construction of the ring of differential operators in [Def 1.5.3] for arbitrary commutative ring V and R a V-algebra. We now show that we also have the same description as a union of $\mathcal{D}_R^{[e]}$ as in [Prop 1.5.7] for this case. The proof differs from the prime characteristic, since in general $\{J^i\}$ and $\{J^{[p^i]}\}$ are not cofinal. Thus we will have to make use of a refined argument.

We have the same description of $\mathcal{D}_{R|V}^{[e]} = \operatorname{Hom}_R(F_*^e R, F_*^e R)$ as a non-commutative ring where the addition is given by the usual addition of endomorphisms and the multiplication is given by the composition of endomorphisms. In the same fashion that the prime characteristic case, we can recover the ring of differential operators as the union of the rings $\mathcal{D}_R^{[e]}$. We now prove some previous propositions.

Definition 2.2.15 (Degree)**.** Let $R = V[x_1, \dots, x_n]$ be a polynomial ring. We define the degree of an element $f = \sum_{\alpha \in I} \underline{x}^\alpha$ as

$$\deg(f) = \max\{|\alpha| : \alpha \in I\}$$

Proposition 2.2.16. *Let V be as in [Sett 2.2.1] and $R = V[x_1, \dots, x_n]$. Let $J = (df : f \in R) \subseteq R \otimes_V R$. Then we have*

$$J = (dx_i : 1 \leq i \leq n)$$

In particular, J is finitely generated and $J^{[p^e]} = (dx_i^p : 1 \leq i \leq n) \subseteq J$.

Proof. Notice we only have to show that each monomial $d\underline{x}^\alpha \in (dx_i : 1 \leq n)$. Thus, consider the element $r = \underline{x}^\alpha$. If $|\alpha| = 0$, then $dr = 0$ and if $|\alpha| = 1$, then $dr = dx_i$ for some i. Consider, then, $|\alpha| \geq 2$. In this case we can decompose $\underline{x}^\alpha = x_i \underline{\hat{x}}^\alpha$ for some i.

Notice that in general, if $r = gh$ with $r, g, h \in R \otimes_V R$, we have the decomposition

$$dr = 1 \otimes gh - gh \otimes = (1 \otimes g - g \otimes 1)(1 \otimes h) + (1 \otimes h - h \otimes 1)(g \otimes 1) = (1 \otimes h)dg + (g \otimes 1)dh$$

In particular if $g = x_i$ and $h = \underline{\hat{x}}^\alpha$, we have

$$d\underline{x}^\alpha = (1 \otimes \underline{\hat{x}}^\alpha)dx_i + (x_i \otimes 1)d\underline{\hat{x}}^\alpha$$

Observe that $(1 \otimes \underline{\hat{x}}^\alpha)dx_i \in (dx_i : 1 \leq i \leq n)$ and $\deg(\underline{\hat{x}}^\alpha) = \alpha - 1$. Then, by induction over $|\alpha|$, we have that $d\underline{x}^\alpha \in (dx_i : 1 \leq i \leq n)$.

Thus, we have shown that $J \subseteq (dx_i : 1 \leq n)$ and the reverse inclusion is clear, giving the equality. The finite generation and the fact $J^{[p^e]} \subseteq J$ follows directly from the equality. $\square$

Proposition 2.2.17. Let V be as in [Sett 2.2.1], R be a V-algebra of finite type and suppose there exist a Frobenius $F : R \to R$. Then

$$\mathcal{D}_{R|V} = \bigcup_{e=0}^{\infty} \mathcal{D}_{R|V}^{[e]}$$

Proof. First we show that $\mathcal{D}_R^{[e]} = \{\varphi \in \mathrm{End}_V(R) : (J^{[p^e]}) \cdot \varphi = 0\}$. Note that $J^{[p^e]} = (F^e \otimes F^e)(J) = (1 \otimes F^e(r) - F^e(r) \otimes 1 : r \in R)$. Thus

$$\begin{aligned} &\{\varphi \in \mathrm{End}_V(R) : (1 \otimes F^e(r) - F^e(r) \otimes 1 : r \in R) \cdot \varphi = 0\} = \\ =&\{\varphi \in \mathrm{End}_V(R) : \varphi(F^e(r)x) = F^e(r)\varphi(x)\ \forall x \in R\} = \mathrm{Hom}_{F^e(R)}(R, R) = \mathcal{D}_R^{[e]} \end{aligned}$$

For the next part of the proof we assume $R := V[x_1, \ldots, x_n]$ is a polynomial ring with the a lift of Frobenius such that $F(x_i) = x_i^p$. Then, by [Prop 2.2.16] the ideal J is generated by differentials of the variables, that is

$$J = (dx_1, \ldots, dx_n) = (1 \otimes x_i - x_i \otimes 1)$$

Next, we must see that the family of ideals $\{J^i\}$ and $\{J^{[p^i]}\}$ are cofinal in $S := R \otimes_V R$. We only need to check the conditions for [Prop 2.2.9]. Clearly J is finitely generated and $J^{[p]} \subseteq J$ by [Prop 2.2.16]. The flatness of S comes from the fact that S is a free V-algebra. The flatness of S/J^kS comes from

$$S/J^kS = \bigoplus_{\alpha} V[x_1, \ldots, x_n] dx_1^{\alpha_1} \cdots dx_n^{\alpha_n} \quad \text{such that} \quad \alpha_1 + \cdots + \alpha_n \leq k - 1$$

is free over V.

For completing the proof we must prove that this behaves well in the case of a finite type algebra, this is, there exist a polynomial ring $P := V[x_1, \ldots, x_n]$ and a surjection $\pi : P \to R$ such that the diagram

$$\begin{tikzcd} P \arrow[r, "F"] \arrow[d, two heads, "\pi"'] & P \arrow[d, two heads, "\pi"] \\ R \arrow[r, "F"] & R \end{tikzcd}$$

commutes. The existence of this morphism $F : P \to P$ is guaranteed because P is a free V-algebra.

Then we have that the families of ideals $\{J_P^i\}$ and $\{J_P^{[p^i]}\}$ are cofinal. By taking the image $J_R^i = (\pi \otimes \pi)(J_P^i)$ and $J_R^{[p^i]} = (\pi \otimes \pi)(J_P^{[p^i]})$, cofinality is preserved, thus completing the proof. $\square$

Thus, we have that the chains

$$\mathcal{D}_{R|V}^0 \subseteq \mathcal{D}_{R|V}^1 \subseteq \cdots \subseteq \mathcal{D}_{R|V}^n \subseteq \cdots \subseteq \mathcal{D}_{R|V}$$

and
$$\mathcal{D}^{[0]}_{R|V} \subseteq \mathcal{D}^{[1]}_{R|V} \subseteq \cdots \subseteq \mathcal{D}^{[e]}_{R|V} \subseteq \cdots \subseteq \mathcal{D}_{R|V}$$
both generate the ring of differential operators.

In [Section 1.2], one of the properties we had was that applying the Frobenius power and then Cartier operator to an ideal resulted in the same ideal, but if we reverse the order of the operations we only have one containment. That is

$$I \xrightarrow{[p^e]} I^{[p^e]} \xrightarrow{\mathcal{C}^e_R \cdot} \mathcal{C}^e_R \cdot I^{[p^e]} = I, \quad \text{but} \quad I \xrightarrow{\mathcal{C}^e_R \cdot} \mathcal{C}^e_R \cdot I \xrightarrow{[p^e]} (\mathcal{C}^e_R \cdot I)^{[p^e]} \supseteq I$$

However, this can be fixed considering some ideals as $D^{[e]}_R$-modules. Not only we have this bijection between R-submodules of R and $\mathcal{D}^{[e]}_R$-submodules of R, but we also have an equivalence of categories.

Now the role of regular F-finite rings we had in the characteristic $p > 0$ is translated into R being a smooth V-algebra in this characteristic p^{m+1}.

We also have the following properties with the differential operators, that allow us to identify $\mathcal{D}^{[e]}_R \cdot I$ with $\mathcal{C}^e_R \cdot I$.

Proposition 2.2.18. Let V be as in [Sett 2.2.1], R a smooth V-algebra and a lift of Frobenius $F : R \to R$. Let $I \subseteq R$ and $e \geq 1$. Then

- $\mathcal{D}^{[e]}_R \cdot I = (\mathcal{C}^e_R \cdot I)^{[p^e]}$
- $I^{[p^e]} = \mathcal{D}^{[e]}_R \cdot I^{[p^e]}$
- $\mathcal{C}^e_R \cdot \left(\mathcal{D}^{[e]}_R \cdot I\right) = \mathcal{C}^e_R \cdot I$

Proof. See [BQG24] □

More precisely, in the same fashion that we had in characteristic $p > 0$ we have the following bijection.

Proposition 2.2.19. Let V be as in [Sett 2.2.1], R a smooth V-algebra and a lift of Frobenius $F : R \to R$. Then the following map is a bijection

$$\begin{aligned} \{R\text{-submodules of } R\} &\leftrightarrow \{\mathcal{D}^{[e]}_R\text{-submodules of } R\} \\ J &\mapsto J^{[p^e]} \\ \mathcal{C}^e_R \cdot I &\leftarrow\!\shortmid I \end{aligned}$$

Proof. One must show that the two compositions are the identity. Let $J \subseteq R$. Going from left to right and then right to left we have

$$J \quad \mapsto \quad \mathcal{C}^e_R \cdot \left(J^{[p^e]}\right) = J$$

by [Prop 2.2.12] (3). Notice that, by [Prop 2.2.18], $\mathcal{D}^{[e]}_R \cdot J^{[p^e]}$ and $J^{[p^e]}$ are interchangeable. For the other composition, we have

$$\mathcal{D}^{[e]}_R \cdot J \quad \mapsto \quad (\mathcal{C}^e_R \cdot I)^{[p^e]} = \mathcal{D}^{[e]}_R \cdot J$$

by [Prop 2.2.18] (1). □

Additionally, the equivalence of categories we discussed in the prime characteristic case also holds in this more general setting. This is, if R is an smooth V-algebra, then $\mathbf{R}-\mathbf{Mod}$ and $\mathcal{D}_{\mathbf{R}}^{[\mathbf{e}]}-\mathbf{Mod}$ are equivalent categories. To prove such statement we will need first some machinery. We start with some definitions.

Definition 2.2.20 (Generator)**.** Let $\mathcal{C}$ be a category. We call $M \in \mathcal{C}$ a generator for the category $\mathcal{C}$ if for any two different parallel morphisms $f, g : A \to B$ in $\mathcal{C}$, then there exists a morphism $h : M \to A$ such that $f \circ h \neq g \circ h$.

Definition 2.2.21 (Progenerator)**.** Let $\mathbf{Mod}-\mathbf{R}$ be the category of right R-modules. We call $P \in \mathbf{Mod}-\mathbf{R}$ a progenerator if it is a finitely generated projective generator.

Theorem 2.2.22 (Morita)**.** Two rings R and S are Morita equivalent if and only if there exists a progenerator P in $\mathbf{Mod}-\mathbf{R}$ such that $S \cong \mathrm{End}_R(P)$.

Proof. See [Bas62]. □

We see that this is exactly the the required machinery for proving the Morita equivalence in our case. Notice that letting R the smooth V-algebra and $P = F^e_* R$ we have

$$R - \mathbf{Mod} \simeq \mathrm{End}_R(F^e_* R) - \mathbf{Mod} = \mathcal{D}_R^{[e]} - \mathbf{Mod}$$

Thus, we only need to show that $F^e_* R$ is a progenerator for the category $\mathbf{Mod}-\mathbf{R}$. We prove first that the Frobenius map $R \xrightarrow{F^e} F^e_* R$ splits.

Proposition 2.2.23. Let V be as in [Sett 2.2.1] and R an smooth V-algebra with a lift of Frobenius $F : R \to T$. Then the Frobenius map $R \xrightarrow{F^e} F^e_* R$ splits. This is, there exists a section $s : F^e_* R \to R$ such that $s \circ F^e = \mathrm{Id}_R$.

Proof. First, notice that by [Prop 1.1.20] the following conditions are equivalent

$$R \xrightarrow{F^e} F^e_* R \text{ splits} \quad \Leftrightarrow \quad \mathrm{Hom}_R(F^e_* R, R) \xrightarrow{-\circ F^e} \mathrm{Hom}_R(R, R) \cong R \text{ is surjective}$$

where the second map is induce by the lift of Frobenius F^e via the $\mathrm{Hom}(-, R)$ contravariant functor.

Thus, it suffices to prove the surjectivity of the second map. Observe that, by [Lem 2.5.8] (that we will prove later), we have the following equivalence:

- $\mathrm{Hom}_R(F^e_* R, R) \xrightarrow{-\circ F^e} \mathrm{Hom}_R(R, R)$ is surjective
- $\mathrm{Hom}_R(F^e_* R, R)_0 \xrightarrow{-\circ F^e} \mathrm{Hom}_R(R, R)_0 \cong R_0$ is surjective

Hence, we only have to prove the surjectivity of the reduction map. Consider the following commutative diagram

$$\begin{tikzcd}
& \mathrm{Hom}_R(F^e_* R, R)_0 \arrow[r, two heads, "(v)"] \arrow[d, two heads, "(iii)"] & R_0 \\
\mathrm{Hom}_R(F^e_* R, R) \arrow[ur, two heads, "(i)"] \arrow[r, two heads, "(ii)"] & \mathrm{Hom}_{R_0}(F^e_* R_0, R_0) \arrow[ur, two heads, "(iv)"] &
\end{tikzcd}$$

Since R_0 is regular, then $R_0 \xrightarrow{F^e} F^e_* R_0$ splits, and (iv) is surjective again by [Prop 1.1.21]. The map (i) is a quotient map, hence surjective. For the map (ii) we consider the diagrams

$$\begin{tikzcd}
F^e_* R \arrow[r, dashed, "\exists\psi"] \arrow[d, "\pi"'] & R \arrow[d, two heads, "\pi"] & \Rightarrow & & R \arrow[d, two heads, "\pi"] \\
F^e_* R_0 \arrow[r, "\varphi"'] & R_0 & \Rightarrow & F^e_* R \arrow[ur, dashed, "\exists\psi"] \arrow[r, "\varphi\circ\pi"'] & R_0
\end{tikzcd}$$

Since $F^e_* R$ is projective by [Prop 2.2.10], then for every $\varphi : F^e_* R_0 \to R_0$, we can lift the map and get a $\psi : F^e_* R \to R$. Thus, the map (ii) is surjective.

Since (i) and (ii) are surjective, by diagram chasing we have that (iii) is surjective as well. Finally, knowing that (iii) and (iv) are surjective, we have that (v) is surjective as well, completing the proof. $\square$

At this point we have all the required properties to prove that $F^e_* R$ is a progenerator of the category $\mathbf{Mod-R}$.

Proposition 2.2.24. Let V be as in [Sett 2.2.1], R a smooth V-algebra with lift of Frobenius $F : R \to R$ and $e \geq 1$. Then $F^e_* R$ is a progenerator for the category $\mathbf{Mod-R}$.

Proof. By [Prop 2.2.10], $F^e_* R$ is finitely generated and projective as an R-module. We only have to show that it is a generator.

Let $M, N \in \mathbf{Mod-R}$ and $f, g : M \to N$ two different morphisms. Let $m \in M$ such that $f(m) \neq g(m)$. Then, by [Prop 1.1.21] there exists a section $s : F^e_* R \to R$ such that $s \circ F^e = \mathrm{Id}_R$. Consider the following commutative diagram

$$\begin{tikzcd}
 & & & N & & & & & f(m) \\
F^e_* R \arrow[r, "s"] & R \arrow[r, "i"] & M \arrow[ur, "f"] \arrow[dr, "g"'] & & & F^e_* 1 \arrow[r, mapsto, "s"] & 1 \arrow[r, mapsto, "i"] & m \arrow[ur, mapsto, "f"] \arrow[dr, mapsto, "g"'] & \\
 & & & N & & & & & g(m)
\end{tikzcd}$$

Then, clearly letting $h = i \circ s$ we have that $f \circ h \neq g \circ h$, meaning that $F^e_* R$ is a generator for the category $\mathbf{Mod-R}$. $\square$

Now we can properly state the theorem of the equivalence of categories.

Theorem 2.2.25 (Morita equivalence)**.** Let V be as in [Sett 2.2.1], R a smooth V-algebra with lift of Frobenius $F : R \to R$ and $e \geq 1$. Then the rings R and $\mathcal{D}^{[e]}_{R|V}$ are Morita equivalent. This is, there exist an equivalence of categories

$$F^{e*} : \mathbf{R-mod} \to \mathcal{D}^{[\mathbf{e}]}_{\mathbf{R}} - \mathbf{mod}$$

Proof. Provided $F^e_* R$ is a progenerator for the category $\mathbf{Mod-R}$, apply the theorem [Thm 2.2.22]. $\square$

Via this Morita equivalence, we have the same properties that we had in the prime characteristic case. In particular, the following relation between the Cartier and the differential ideals holds.

Proposition 2.2.26. Let V be as in [Sett 2.2.1], R a smooth V-algebra with lift of Frobenius $F : R \to R$ and $e \geq 1$. Let $I, J \subseteq R$ be ideals. We have

$$\mathcal{C}_R^e \cdot J = \mathcal{C}_R^e \cdot I \qquad \Longleftrightarrow \qquad \mathcal{D}_R^{[e]} \cdot J = \mathcal{D}_R^{[e]} \cdot I$$

Proof. Suppose $\mathcal{C}_R^e \cdot J = \mathcal{C}_R^e \cdot I$. Then the Frobenius power

$$\mathcal{D}_R^{[e]} \cdot J = (\mathcal{C}_R^e \cdot J)^{[p^e]} = (\mathcal{C}_R^e \cdot I)^{[p^e]} = \mathcal{D}_R^{[e]} \cdot I$$

by [Prop 2.2.18] (1). Conversely, if $\mathcal{D}_R^{[e]} \cdot J = \mathcal{D}_R^{[e]} \cdot I$, then applying Cartier operators we have

$$\mathcal{C}_R^e \cdot J = \mathcal{C}_R^e \cdot \left(\mathcal{D}_R^{[e]} \cdot J\right) = \mathcal{C}_R^e \cdot \left(\mathcal{D}_R^{[e]} \cdot I\right) = \mathcal{C}_R^e \cdot I$$

by [Prop 2.2.18] (3) $\square$

What this fact is telling us is that, given an ideal $\mathfrak{a}$, the jumps in the chain

$$\mathcal{C}_R^e \cdot 1 \supseteq \mathcal{C}_R^e \cdot \mathfrak{a} \supseteq \mathcal{C}_R^e \cdot \mathfrak{a}^2 \supseteq \cdots$$

and

$$\mathcal{D}_R^{[e]} \cdot 1 \supseteq \mathcal{D}_R^{[e]} \cdot \mathfrak{a} \supseteq \mathcal{D}_R^{[e]} \cdot \mathfrak{a}^2 \supseteq \cdots$$

are located in the same position.

2.2.4 The module $R_f \boldsymbol{f^{\alpha}}$

We now construct the module $R_f \boldsymbol{f^{\alpha}}$ in the same fashion that we did in the previous section. Note that we must perform a subtle modification in order for all to work properly.

For the construction of the $\Upsilon_{\alpha,f}$ map, instead of truncating α at level e, we will truncate it at level $e+m$ (recall the meaning of truncating $\alpha_{<e}$ in [Def A.1.2]). Proposition [Prop 2.2.8] plays a crucial role in the construction of this version of the $\Upsilon_{\alpha,f}$ map.

Proposition 2.2.27. Let R be as in [Sett 2.2.1] with lift of Frobenius $F : R \to R$. Let m such that $\mathfrak{m}^{m+1} = 0$, $\alpha \in \mathbb{Z}_p$, $\delta \in \mathcal{D}_R$ and $f \in R$. Then the map

$$\Upsilon_{\alpha,f} : \mathcal{D}_R \to \mathcal{D}_{R_f}, \quad \delta \mapsto f^{-\alpha_{<e+m}} \delta f^{\alpha_{<e+m}} \text{ for } e : \delta \in \mathcal{D}_R^{[e]}$$

is well defined and induces maps $\Upsilon_{\alpha,f} : \mathcal{D}_R^{[e]} \to \mathcal{D}_{R_f}^{[e]}$.

Proof. Let $\delta \in \mathcal{C}_R^e$. In fact, if $a \equiv a' \equiv \alpha_{<m+e} \mod p^{e+m}$ then $a' = kp^{e+m} + a$ for some $k \in \mathbb{Z}$ and

$$\left(f^{-a'} \delta f^{a'}\right)(x) = f^a \left(f^k\right)^{-p^{e+m}} \delta\left(\left(f^k\right)^{p^{e+m}} f^a x\right) = f^a \left(f^k\right)^{-p^{e+m}} \delta\left(F^e(f^{kp^m}) f^a x\right) =$$
$$= f^a \left(f^k\right)^{-p^{e+m}} \left(F^e(f^{kp^m}) \delta f^a x\right) = f^a \left(f^k\right)^{-p^{e+m}} \left(f^k\right)^{p^{e+m}} \delta\left(f^a x\right) = \left(f^a \delta f^a(x)\right)$$

where we have used the fact that $g^{p^{e+m}} = (g^{p^m})^{[p^e]}$ by [Prop 2.2.8] and that $(f^{kp^m})^{[p^e]}$ commutes with δ. Notice that if $\delta \in \mathcal{D}_R^{[e]}$, then

$$\Upsilon_{\alpha,f}(\delta)(t^{[p^e]}x) = f^{-\alpha_{<e+m}}\delta\left(f^{\alpha_{<e+m}}t^{[p^e]}x\right) = t^{[p^e]}\left(f^{-\alpha_{e+m}}\delta f^{\alpha_{<e+m}}x\right) = t^{[p^e]}\Upsilon_{\alpha,f}(\delta)(x)$$

again because of the commutativity of $t^{[p^e]}$ with δ. Hence, $\Upsilon_{\alpha,f}(\delta) \in \mathcal{D}_{R_f}^{[e]}$. □

We proceed in the same way constructing the $\mathcal{D}_R$-module structure via the $\Upsilon_{\alpha,f}$ map. Fix $\alpha \in \mathbb{Z}_p$ and $f \in R$ and we construct the $\mathcal{D}_R$-module $R_f\boldsymbol{f^\alpha}$ whose set is canonically isomorphic to R_f ($g\boldsymbol{f^\alpha} \leftrightarrow g$), and whose action is given by

$$\delta \cdot (g\boldsymbol{f^\alpha}) = (\Upsilon_{f,\alpha}(\delta) \cdot g)\boldsymbol{f^\alpha}$$

We end up with the same definition of Bernstein-Sato roots as in the prime characteristic case.

Definition 2.2.28 (Bernstein-Sato root)**.** Let V be as in [Sett 2.2.1] and R a smooth V-algebra. Then we call $\alpha \in \mathbb{Z}_p$ a Bernstein-Sato root of $f \in R$ if $\boldsymbol{f^\alpha} \notin \mathcal{D}_R \cdot f\boldsymbol{f^\alpha}$.

The Benrstein-Sato roots of an element $f \in R$ form a set which we denote as $\mathrm{BSR}(f)$. Notice that this definition is independent of the choice of the lift of Frobenius. This is because the construction of the ring of differential operators is performed via the Grothendieck construction and does not make use of any lift of Frobenius. Thus, the set of Bernstein-Sato roots is, indeed, an invariant associated just to the element $f \in R$. We will show properties such as rationality, discreteness and negativity of the elements of this set in the next section.

Now we prove that we can recover this set by taking p-adic limits over the set of ν-invariants $\nu_f^\bullet(p^e)$.

Proposition 2.2.29. Let V be as in [Sett 2.2.1] and R a smooth V-algebra with lift of Frobenius $F : R \to R$. Let $\alpha \in \mathbb{Z}_p$ and $f \in R$. Then, the following conditions are equivalent

1. α is a Bernstein-Sato root of f.
2. α is the p-adic limit of a sequence $\{\nu_e\}_{e\geq 0}$ such that $\nu_e \in \nu_f^\bullet(p^e)$.

Proof. First we prove (2) $\Rightarrow$ (1). Suppose $\boldsymbol{f^\alpha} \in \mathcal{D}_R \cdot f\boldsymbol{f^\alpha}$. Fix $e \geq 0$ such that $\boldsymbol{f^\alpha} \in \mathcal{D}_R^{[e]} \cdot f\boldsymbol{f^\alpha}$ and $a \geq 0$ such that $a \equiv \alpha \mod p^{e+m}$ (e.g. the truncation $a = \alpha_{<e+m}$). Then

$$\boldsymbol{f^\alpha} \in \mathcal{D}_R^{[e]} \cdot f\boldsymbol{f^\alpha} \Rightarrow 1 \in f^{-a}\mathcal{D}_R^{[e]} \cdot f^{a+1} \Rightarrow f^a \in \mathcal{D}_R^{[e]} \cdot f^{a+1} \Rightarrow \mathcal{D}_R^{[e]} \cdot f^a = \mathcal{D}_R^{[e]} \cdot f^{a+1}$$

where the last implication follows from the fact that we always have $\mathcal{D}_R^{[e]} \cdot f^a \supseteq \mathcal{D}_R^{[e]} \cdot f^{a+1}$. Thus, $a \notin \nu_f^\bullet(p^e)$.

Now assume that α is the p-adic limit of a ν-sequence $\{\nu_e\}_{e\geq 0}$ of $\nu_f^\bullet$. We can construct a subsequence $\{\gamma_e\}_{e\geq 0}$ of $\{\nu_e\}_{e\geq 0}$ such that $\gamma_e \equiv \alpha \mod p^{e+m}$ $\forall e \in \mathbb{Z}_{\geq 0}$. Thus, clearly $\gamma_e \in \nu_f^\bullet(p^e)$, and by [Prop 2.2.14], $\gamma_e + p^{e+m}\mathbb{Z} \subseteq \nu_f^\bullet(p^e)$, meaning, in particular, $a \in \nu_f^\bullet(p^e)$, which contradicts the assumption.

For the converse, suppose $\boldsymbol{f^\alpha} \notin \mathcal{D}_R \cdot f\boldsymbol{f^\alpha}$. Fix $e \geq 0$ and $a \geq 0$ such that $a \equiv \alpha \mod p^{e+m}$. Then

$$\boldsymbol{f^\alpha} \notin \mathcal{D}_R^{[e]} \cdot f\boldsymbol{f^\alpha} \Rightarrow 1 \notin f^{-a}\mathcal{D}_R^{[e]} \cdot f^{a+1} \Rightarrow f^a \notin \mathcal{D}_R^{[e]} \cdot f^{a+1} \Rightarrow \mathcal{D}_R^{[e]} \cdot f^a \neq \mathcal{D}_R^{[e]} \cdot f^{a+1}$$

Thus, $a \in \nu_f^\bullet(p^e)$. Clearly if we construct a sequence $\{a_e\}_{e\geq 0}$ so that $a_e \equiv \alpha \mod p^{e+m}\ \forall e \geq 0$, then $a_e \in \nu_f^\bullet(p^e)$, and its p-adic limit is α, concluding (1) $\Rightarrow$ (2). □

Corollary 2.2.30. Let $f \in R$ be a nonzerodivisor and $\alpha \in \mathrm{BSR}(f)$. Then $\alpha_{<e+m} \in \nu_f^\bullet(p^e)$.

Proof. Follows trivially from [Prop 2.2.29]. □

2.3 Properties of Bernstein-Sato roots

We explore now some of the basic properties of Bernstein-Sato roots and we compare them with the classical case of prime characteristic. We will present a quick survey of the properties introduced in [BQG24]. Throughout this section we will work with the following setting.

Setting 2.3.1. Let V be as in [Sett 2.2.1] and $R = V[x_1, \dots, x_n]$. Fix the lift of Frobenius that extends linearly $F^e(x_i) = x_i^{p^e}$. Notice that $F_*^e R$ is a free R-module with basis $\{F_*^e x_1^{i_1} \cdots x_n^{i_n} : 0 \leq i_j < p^e\}$.

One of the biggest results in Bernstein-Sato theory is the finiteness and rationality of the roots. One of the aims of this section is prove this result for the case of a polynomial ring. We begin with some previous propositions.

Proposition 2.3.2. Let R as in [Sett 2.3.1]. If $J \subseteq R$ is generated in degrees $\leq d$ (see [Def A.2.10]), then $\mathcal{C}_R^e \cdot J$ is generated in degrees $\leq d/p^e$.

Proof. Let $J = (f_1, \dots, f_r)$, and let $\{\underline{x}^\alpha : 0 \leq \alpha_i < p^e\}$ be a basis of $F_*^e R$ written in multi-index notation. Then, for each f_i we have

$$f_i = \sum_\alpha F^e(g_{i,\alpha})\underline{x}^\alpha$$

Taking degrees both sides we get

$$\deg(f_i) \geq p^e \deg(g_{i,\alpha})$$

meaning

$$d \geq \deg(f_i) \geq p^e \deg(g_{i,\alpha}) \quad \Rightarrow \quad \deg(g_{i,\alpha}) \leq \frac{d}{p^e}$$

as desired. □

Proposition 2.3.3. Let R be as in [Sett 2.3.1]. The set of ν-invariants of $f \in R$ of level e between 0 and p^{e+m} is uniformly bounded. This is, there exist a constant K such that for all $e \geq 0$ we have

$$\#\{\nu \in \nu_f^{\bullet}(p^e) : \nu \in [0, p^{e+m})\} \leq K$$

Proof. Let $d \coloneqq \deg(f)$ as defined in [Def A.2.9]. Then, by [Prop 2.3.2] we have that $\mathcal{C}_R^e \cdot f^k$ is generated in degrees $\leq \frac{dk}{p^e}$. Consider the following chain of inclusions

$$\mathcal{C}_R^e \cdot 1 \supseteq \mathcal{C}_R^e \cdot f \supseteq \cdots \supseteq \mathcal{C}_R^e \cdot f^{p^{e+m}-1} \supseteq \mathcal{C}_R^e \cdot f^{p^{e+m}}$$

Observe that the last module is generated in degrees $\leq \frac{dp^{e+m}}{p^e} = dp^m$.

We will have a ν-invariant whenever there is a proper inclusion in the chain. In order to bound the quantity of such inclusions, notice that we can consider the same chain but chopping off the degrees by dp^m, since this is the bound on the degree of the generators of $\mathcal{C}_R^e \cdot f^k$ for $0 \leq k \leq p^{e+m}$. Thus, we can state K as the length of the module R_{dp^m} (this is, R chopping off the degrees higher than dp^m) as a V-module. This length is finite because V is Artinian, and does not depend on e. Thus, the number of jumps in the chain is bounded. □

Theorem 2.3.4 (Finiteness of Bernstein-Sato roots)**.** Let R be as in [Sett 2.3.1] and $f \in R$ be a nonzerodivisor. Then the set of Bernstein-Sato roots of f is finite.

Proof. Suppose there are $\alpha_1, \ldots, \alpha_{K+1}$ distinct roots, where K is the bound of [Prop 2.3.3]. Then, by [Prop A.1.5] one can find an integer $a \geq 0$ such that

$$(\alpha_i + p^{a+m}\mathbb{Z}_p) \cap (\alpha_j + p^{a+m}\mathbb{Z}_p) = \emptyset \quad \forall i \neq j$$

Let $\nu_{i,a} = \alpha_{i<a}$ be the truncation of α_i at level a. Observe that $0 \leq \nu_{i,a} < p^{a+m}$. Then, $\nu_{i,a} \in \nu_f^{\bullet}(p^a)$ and by the above observation we have $\nu_{i,a} \neq \nu_{j,a}$ whenever $i \neq j$. Thus, the size of the set

$$\#\{\nu \in \nu_f^{\bullet}(p^e) : \nu \in [0, p^{e+m})\} \geq K+1$$

contradicting [Prop 2.3.3]. □

We now prove the rationality of Bernstein-Sato roots. One can follow the same strategy as in [BMS08], but must refine the arguments for this prime power characteristic case. We first state some previous results.

Proposition 2.3.5. Let R be as [Sett 2.2.1], $f \in R$ be a nonzerodivisor and m such that $\mathfrak{m}^{m+1} = 0$. Then

$$\nu_e \in \nu^{\bullet}_{f^{p^m}}(p^e) \quad \Rightarrow \quad p\nu_e + i_e \in \nu^{\bullet}_{f^{p^m}}(p^{e+1}) \quad \text{for some} \quad 0 \leq i_e < p$$

Proof. Recall that from [Prop 2.2.8] we have that $f^{p^{m+e}} = F^e(g^{p^m})$. Hence, we construct the following chain

$$\begin{aligned} \mathcal{C}_R^{e+1} \cdot \left(f^{p^m}\right)^{p\nu} = \mathcal{C}_R^{e+1} \cdot (f^{\nu})^{p^{m+1}} = \mathcal{C}_R^{e+1} \cdot F^1\left(f^{\nu p^m}\right) = \mathcal{C}_R^{e} \cdot \left(f^{p^m}\right)^{\nu} \neq \\ \neq \mathcal{C}_R^{e} \cdot \left(f^{p^m}\right)^{\nu+1} = \mathcal{C}_R^{e+1} \cdot F^1\left(f^{(\nu+1)p^m}\right) = \mathcal{C}_R^{e+1} \cdot \left(f^{\nu+1}\right)^{p^m} = \mathcal{C}_R^{e+1} \cdot \left(f^{p^m}\right)^{p\nu+p} \end{aligned}$$

which proves that there must be some jump of level $e+1$ located between $p\nu_e e$ and $p\nu_e + p$. □

Proposition 2.3.6. Let R be as in [Sett 2.2.1] and $f \in R$ be a nonzerodivisor. Let $\alpha \in \mathbb{Z}_p$ be a Bernstein-Sato root of $f^{p^m} \in R$ with m such that $\mathfrak{m}^{m+1} = 0$. Then

$$p\alpha + i \in \mathrm{BSR}(f^{p^m}) \quad \text{for some} \quad 0 \le i < p$$

Proof. Consider a sequence of ν-invariants $\{\nu_e\}_{e \ge 0}$ such that $\nu_e \in \nu^{\bullet}_{f^{p^m}}(p^e)$ converging p-adically to α. Then, by [Prop 2.3.5] we can consider a new sequence $\{i_e\}_{e \ge 0}$ such that

$$p\nu_e + i_e \in \nu^{\bullet}_{f^{p^m}}(p^{e+1})$$

in which every $0 \le i_e < p$. Since this sequence can only take a finite number of values, we can find a subsequence that takes one value $0 \le i < p$. Hence, that subsequence converges p-adically to $p\alpha + i$, proving that $p\alpha + i$ is a Bernstein-Sato root of f^{p^m}. □

Proposition 2.3.7. Let $k \in \mathbb{Z}_{\ge 0}$. Then

$$\nu \in \nu^{\bullet}_f(p^e) \quad \Rightarrow \quad \left\lfloor \frac{\nu}{k} \right\rfloor \in \nu^{\bullet}_{f^k}(p^e)$$

Proof. Consider the chain of inclusions

$$\mathcal{C}^e_R \cdot (f^k)^{\lfloor \frac{\nu}{k} \rfloor} \supseteq \mathcal{C}^e_R \cdot f^{\nu} \supsetneq \mathcal{C}^e_R \cdot f^{\nu+1} \supseteq \mathcal{C}^e_R \cdot (f^k)^{(\lfloor \frac{\nu}{k} + 1 \rfloor)}$$

so that the inclusion between the ideals at both extremes is strict. □

Proposition 2.3.8. Let $\alpha \in \mathbb{Z}_p$ be a Bernstein-Sato root of $f \in R$. Then, the left truncation $\alpha_{\ge k}$ is a Bernstein-Sato root of f^{p^k}.

Proof. By [Cor 2.2.30] we have that $\alpha_{<e+m} \in \nu^{\bullet}_f(p^e)$. Then, by [Prop 2.3.7] we deduce

$$\left\lfloor \frac{\alpha_{<e+m}}{p^k} \right\rfloor \in \nu^{\bullet}_{f^{p^k}}(p^e)$$

But this expressions are just the left truncation of $\alpha_{\ge k}$, meaning that its p-adic limit is precisely $\alpha_{\ge k}$. □

We see now that when R is a polynomial ring, the Bernstein-Sato roots are rational, meaning that the digits in the p-adic expansion are eventually periodic.

Theorem 2.3.9 (Rationality of Bernstein-Sato roots)**.** Let V be as in [Sett 2.2.1] and $R = V[x_1, \dots, x_n]$. Let $f \in R$ be a nonzerodivisor. Then the Bernstein-Sato roots of f are rational.

Proof. Fix m such that $\mathfrak{m}^{m+1} = 0$. Notice that it suffices to prove that $\alpha_{\ge m}$ is rational, which correspond to a Bernstein-Sato root of f^{p^m}. Thus, we just need to show that the Bernstein-Sato roots of f^{p^m} are rational. From [Prop 2.3.6] we know that, fixing a Bernstein-Sato root α_0 we can generate a sequence $\{\alpha_i\}_{i \ge 0}$ of elements inside $\mathrm{BSR}(f^{p^m})$ as

$$\alpha_0 \quad \to \quad \alpha_1 = p\alpha_0 + c_1 \quad \to \cdots \to \quad \alpha_k = p\alpha_{k-1} + c_k = \alpha_0 p^k + c_1 p^{k-1} + \cdots + c_k$$

However, since the set $\mathrm{BSR}(f^{p^m})$ is finite by [Thm 2.3.4], we will have the indexes a, b such that $\alpha_a = \alpha_b$, meaning

$$\alpha_a = \alpha_0 p^a + c_1 p^{a-1} + \cdots + c_a = \alpha_0 p^b + c_1 p^{b-1} + \cdots + c_b = \alpha_b$$

Hence, we get

$$\alpha_0 = \frac{c_1 p^{a-1} + \cdots + c_a - (c_1 p^{b-1} + \cdots + c_b)}{p^b - p^a}$$

concluding that α_0 is rational. □

Proposition 2.3.10. Let $R_x = \mathbb{Z}/p^{m+1}\mathbb{Z}[\underline{x}], R_y = \mathbb{Z}/p^{m+1}\mathbb{Z}[\underline{y}]$ and ${}_{x,y} = \mathbb{Z}/p^{m+1}\mathbb{Z}[\underline{x}, \underline{y}]$. Let $g \in R_x$ and $h \in R_y$. Consider $gh \in R_{x,y}$. Then

$$\mathrm{BSR}(gh) = \mathrm{BSR}(g) \cup \mathrm{BSR}(h)$$

Proof. For every integer $l \geq 0$ and level $e \geq 0$ the powers viewed in the Frobenius rings can be written in their basis decomposition and, following [Prop 2.2.11], they generate the cartier ideal as

$$F_*^e g^l = \sum_\alpha g_\alpha F_*^e \underline{x}^\alpha \quad \Rightarrow \quad \mathcal{C}_R^e \cdot \quad g^l = (g_\alpha)$$
$$F_*^e h^l = \sum_\beta h_\beta F_*^e \underline{y}^\beta \quad \Rightarrow \quad \mathcal{C}_R^e \cdot \quad h^l = (h_\beta)$$

Thus, since a basis of $F_*^e R_{x,y}$ is

$$\{x^\alpha y^\beta : 0 \leq \max(\alpha), \max(\beta) < p^e\}$$

the multiplication of g^l and h^l viewed in $F_*^e R_{x,y}$ is

$$F_*^e g^l h^l = F_*^e g^l F_*^e h^l = \sum_\alpha (g_\alpha F_*^e \underline{x}^\alpha) \left(\sum_\beta h_\beta F_*^e \underline{y}^\beta \right) = \sum_{\alpha,\beta} g_\alpha h_\beta F_*^e \underline{x}^\alpha \underline{y}^\beta$$

meaning

$$\mathcal{C}_R^e \cdot (gh)^l = (g_\alpha h_\beta) = (\mathcal{C}_R^e \cdot g^l) \cdot (\mathcal{C}_R^e \cdot h^l)$$

Thus, either if there is a jump in the chain $\mathcal{C}_R^e \cdot g^l$ or $\mathcal{C}_R^e \cdot h^l$, there will be a jump in the chain $\mathcal{C}_R^e \cdot (gh)^l$, proving that the ν-invariants are the union. Taking the p-adic limit one has the result. □

We now prove some interesting facts about the module $R_f \boldsymbol{f^\alpha}$ and its relation with the Bernstein-Sato roots. Specially, it is very relevant to show the relation between these modules when we shift by an integer the value of $\alpha \in \mathbb{Z}_p$ or we multiply it by an integer. This properties are explained in detailed in [BQG24].

Proposition 2.3.11. Let R be a F-finite ring, $f \in R, \alpha \in \mathbb{Z}_p$ and $n \in \mathbb{Z}$. Then

1. The R_f-module isomorphism $\varphi : R_f \boldsymbol{f^{\alpha+1}} \to R_f \boldsymbol{f^\alpha}$ such that $\varphi : \boldsymbol{f^{\alpha+1}} \mapsto f \boldsymbol{f^\alpha}$ is $\mathcal{D}_R$-linear.
2. The R_f-module isomorphism $\varphi : R_f \boldsymbol{f^n} \to R_f$ such that $\varphi : \boldsymbol{f^n} \mapsto f^n$ is $\mathcal{D}_R$-linear.

3. The R_f-module isomorphism $\varphi : R_f(\boldsymbol{f^n})^{\boldsymbol{\alpha}} \to R_f\boldsymbol{f^{n\alpha}}$ such that $\varphi : g(\boldsymbol{f^n})^{\boldsymbol{\alpha}} \mapsto g\boldsymbol{f^{n\alpha}}$ is $\mathcal{D}_R$-linear.

Proof.
(1) Notice that $a \in \mathbb{Z}$ p-adically approximates α at level e if and only if $a+1$ p-adically approximates $\alpha+1$ at level e. Then, fixed α and e we have for $\delta \in \mathcal{D}_R^{[e]}$

$$\varphi\left(\delta \cdot g\boldsymbol{f^{\alpha+1}}\right) = \varphi\left(f^{-1-a}(\delta f^{a+1}g)\boldsymbol{f^{\alpha+1}}\right) = f^{-a}\delta f^a fg\boldsymbol{f^{\alpha}} = \delta \cdot gf\boldsymbol{f^{\alpha}} = \delta \cdot \varphi(g\boldsymbol{f^{\alpha+1}})$$

for all $g \in R_f$, proving the D_{R_f} linearity. For the isomorphism, one can construct the inverse map $\boldsymbol{f^{\alpha}} \mapsto f^{-1}\boldsymbol{f^{\alpha+1}}$.

(2) We apply the isomorphism of (ii) n and get $R_f\boldsymbol{f^n} \to R_f\boldsymbol{f^0}$. But we have the the isomorphism $R_f\boldsymbol{f^0} \to R_f$ defined by $g\boldsymbol{f^0} \to g$ is $\mathcal{D}_R$-linear, since the action by $\delta \in \mathcal{D}_R$ is the usual action because $a = 0$.

(3) Notice that if $a \in \mathbb{Z}$ p-adically approximates α at level e, then an p-adically approximates $n\alpha$ at level e. Then, fixed α and e we have for $\delta \in \mathcal{D}_R^{[e]}$

$$\varphi\left(\delta \cdot g(\boldsymbol{f^n})^{\boldsymbol{\alpha}}\right) = \varphi\left(f^{-a}\delta f^a g(\boldsymbol{f^{\alpha}})^{\boldsymbol{\alpha}}\right) = f^{-a}\delta f^a g\boldsymbol{f^{n\alpha}} = \delta \cdot g\boldsymbol{f^{n\alpha}} = \delta \cdot \varphi(g(\boldsymbol{f^n})^{\boldsymbol{\alpha}})$$

for all $g \in R_f$, proving the D_{R_f} linearity. $\square$

At this point it is natural to consider the following chains

$$\begin{array}{ll}
(C_{\alpha,k}) & \mathcal{D}_R \cdot f^k\boldsymbol{f^{\alpha}} \subseteq \mathcal{D}_R \cdot f^{k-1}\boldsymbol{f^{\alpha}} \subseteq \mathcal{D}_R \cdot f^{k-2}\boldsymbol{f^{\alpha}} \subseteq \cdots \\
\left(C^{\alpha,k}\right) & \mathcal{D}_R \cdot f^k\boldsymbol{f^{\alpha}} \supseteq \mathcal{D}_R \cdot f^{k+1}\boldsymbol{f^{\alpha}} \supseteq \mathcal{D}_R \cdot f^{k+2}\boldsymbol{f^{\alpha}} \supseteq \cdots \\
(C_{\alpha}) & \cdots \subseteq \mathcal{D}_R \cdot f^1\boldsymbol{f^{\alpha}} \subseteq \mathcal{D}_R \cdot 1\boldsymbol{f^{\alpha}} \subseteq \mathcal{D}_R \cdot f^{-1}\boldsymbol{f^{\alpha}} \subseteq \cdots
\end{array}$$

Proposition 2.3.12. Let R be F-finite, $\alpha \in \mathbb{Z}_{(p)}$ such that $\alpha \in [-1,0)$ and f nonzerodivisor. Then

$$R_f\boldsymbol{f^{\alpha}} = \mathcal{D}_R \cdot 1\boldsymbol{f^{\alpha}} \quad \Longleftrightarrow \quad \mathrm{BSR}(f) \cap \{\alpha-1, \alpha-2, \ldots\} = \emptyset$$

Proof. For the implication $\Rightarrow$, as $\mathcal{D}_R \cdot 1\boldsymbol{f^{\alpha}} = \mathcal{D}_R f^{-k}$ $\forall K \in \mathbb{Z}_{\geq 0}$, then the containments in $C_{\alpha,0}$ are equalities. Thus,

$$\mathcal{D}_R \cdot f^{-k}\boldsymbol{f^{\alpha}} = \mathcal{D}_R \cdot f^{-k-1}\boldsymbol{f^{\alpha}} \Longleftrightarrow \mathcal{D}_R \cdot f\boldsymbol{f^{\alpha-k-1}} = \mathcal{D}_R \cdot 1\boldsymbol{f^{\alpha-k-1}}$$

by the isomorphism of [Prop 2.3.11], meaning $\alpha - k - 1$ is not a Bernstein-Sato root. For the converse we can reconstruct the equalities in the chain and then deduce that the modules $\mathcal{D}_R \cdot 1\boldsymbol{f^{\alpha}}$ and $R_f\boldsymbol{f^{\alpha}}$ are equal. $\square$

Taking advantage of the isomorphisms of the proposition [Prop 2.3.11], we can extend this last proposition for all $\alpha \in \mathbb{Z}_{(p)}$ in the following fashion.

Proposition 2.3.13. Let V be as in [Sett 2.2.1] and R a smooth V-algebra. Let $\alpha \in \mathbb{Z}_{(p)}$ and $f \in R$ nonzerodivisor. Then

$$R_f\boldsymbol{f^{\alpha}} = \mathcal{D}_R \cdot f^{-\lfloor\alpha+1\rfloor}\boldsymbol{f^{\alpha}} \quad \Longleftrightarrow \quad \mathrm{BSR}(f) \cap \{\alpha - \lfloor\alpha+1\rfloor - 1, \alpha - \lfloor\alpha+1\rfloor - 2, \ldots\} = \emptyset$$

Proof. We see that $(C_{\alpha,0})$ stabilizes, since all the containments in $(C_{\alpha,-\lfloor\alpha+1\rfloor})$ are equalities. Thus $\alpha-\lfloor\alpha+1\rfloor-k-1$ is not a Bernstein-Sato root $\forall k\in\mathbb{Z}_{\geq 0}$. However, the jumps in the chain

$$\mathcal{D}_R\cdot 1\boldsymbol{f}^{\boldsymbol{\alpha}}\subseteq\cdots\subseteq\mathcal{D}_R\cdot f^{-\lfloor\alpha\rfloor}$$

correspond to the Bernstein-Sato roots in the set $\{\alpha-1,\dots,\alpha-\lfloor\alpha+1\rfloor\}$. □

Finally, we show that -1 is always a Bernstein-Sato root.

Proposition 2.3.14. Let R be as in [Sett 2.2.1], $l\geq 1$ and $f\in R$. Then

$$f^l\in\mathfrak{m}\quad\Rightarrow\quad f\in\mathfrak{m}$$

Proof. Let $f_{\mathfrak{m}}\in R_{\mathfrak{m}}:=R/\mathfrak{m}R$ be the reduction of f modulo $\mathfrak{m}$. Then

$$f^l\in\mathfrak{m}\quad\Rightarrow\quad f_{\mathfrak{m}}^l=0\quad\Rightarrow\quad f_{\mathfrak{m}}=0\quad\Rightarrow\quad f\in\mathfrak{m}$$

where the second implication follows from the fact that $R/\mathfrak{m}R$ is reduced by [Prop A.2.8]. □

Proposition 2.3.15. Let R be as in [Sett 2.2.1], $f\in R$, $l\geq 1$ and fix a level $e\geq 0$. Then

$$f\notin\mathfrak{m}\quad\Rightarrow\quad\mathcal{C}_R^e\cdot f^l\nsubseteq\mathfrak{m}$$

Proof. Suppose $\mathcal{C}_R^e\cdot f^l\subseteq\mathfrak{m}$. Then, by [Prop 1.2.7], we have that

$$f^l\in\mathfrak{m}^{[p^e]}=\mathfrak{m}$$

since the Frobenius fixes all the elements in V. Now, by [Prop 2.3.14] we have that $f\in\mathfrak{m}$, which is a contradiction. □

Finally, we show that that $p^{e+m}-1$ is always a ν-invariant of level e for the polynomial case [Sett 2.3.1]. To prove this, we introduce now a new notion of *degree*, where, roughly speaking, we just degree of the part that is not contained in the maximal ideal.

Definition 2.3.16 ($\mathfrak{m}$-degree)**.** Let R be as in [Sett 2.3.1] and $f\in R$ a nonzerodivisor. The $\mathfrak{m}$-degree of f is defined as the maximum degree among the monomials that are not divisible by p. This is,

$$\text{If}\quad f=f_0+pf_1+\cdots+p^mf_m\quad\text{then}\quad\deg_{\mathfrak{m}}(f):=\deg(f_0)$$

where any f_i is divisible by p.

Proposition 2.3.17. Let $f,g\in R$. The $\mathfrak{m}$-degree fulfils

$$\deg_{\mathfrak{m}}(fg)=\deg_{\mathfrak{m}}(f)+\deg_{\mathfrak{m}}(g)$$

Proof. Just decomposing the elements

$$f=f_0+pf_1+\cdots+p^mf_m\quad\text{and}\quad g=g_0+pg_1+\cdots+p^mg_m$$

gives

$$fg=f_0g_0+p(f_1g_0+f_0g_1)+\cdots\quad\Rightarrow\quad\deg_{\mathfrak{m}}(fg)=\deg(f)+\deg(g)=\deg_{\mathfrak{m}}(f)+\deg_{\mathfrak{m}}(g)$$

concluding the proof. □

Proposition 2.3.18. Let R be as in [Sett 2.3.1]. If $J \subseteq R$ is generated in $\mathfrak{m}$-degrees $\leq d$, then $\mathcal{C}_R^e \cdot J$ is generated in $\mathfrak{m}$-degrees $\leq d/p^e$.

Proof. The proof is similar to the one in [Prop 2.3.2]. Let $J = (f_1, \ldots, f_r)$, and let $\{\underline{x}^\alpha : 0 \leq \alpha_i < p^e\}$ be a basis of $F_*^e R$ written in multi-index notation. Then, for each f_i we have

$$f_i = \sum_\alpha F^e(g_{i,\alpha})\underline{x}^\alpha$$

Taking degrees both sides we get

$$\deg_{\mathfrak{m}}(f_i) \geq p^e \deg_{\mathfrak{m}}(g_{i,\alpha})$$

meaning

$$d \geq \deg_{\mathfrak{m}}(f_i) \geq p^e \deg_{\mathfrak{m}}(g_{i,\alpha}) \quad \Rightarrow \quad \deg_{\mathfrak{m}}(g_{i,\alpha}) \leq \frac{d}{p^e}$$

as desired. □

With this new tool we can now prove that $p^{e+m} - 1$ is a ν-invariant of level e.

Proposition 2.3.19. Let R be as in [Sett 2.3.1], $f \in R$ nonzerodivisor and a level $e \geq 0$. Then $p^{m+e} - 1 \in \nu_f^\bullet(p^e)$.

Proof. By [Prop 2.2.8], we know that $\mathcal{C}_R^e \cdot f^{p^{e+m}} = \mathcal{C}_R^e \cdot F^e(f^{p^m}) = (f^{p^m})$. By [Prop 2.3.2], we can express the following Cartier ideal as

$$\mathcal{C}_R^e \cdot f^{p^{e+m}-1} = (g_1, \ldots, g_s) \quad \text{such that} \quad \deg_{\mathfrak{m}}(g_i) \leq \frac{\deg_{\mathfrak{m}}(f) \cdot (p^{e+m} - 1)}{p^e} < \deg(f) \cdot p^m$$

since $\deg_{\mathfrak{m}}$ fulfils $\deg_{\mathfrak{m}}(f^l) = \deg_{\mathfrak{m}}(f) \cdot l$. Thus, clearly by degree comparison

$$(g_1, \ldots, g_s) = \mathcal{C}_R^e \cdot f^{p^{e+m}-1} \supsetneq \mathcal{C}_R^e \cdot f^{p^{e+m}} = (f^{p^m})$$

concluding that, indeed, there is a jump. □

Proposition 2.3.20. Let R be as in [Sett 2.3.1] and $f \in R$ nonzerodivisor. Then -1 is a Bernstein-Sato root of f.

Proof. By [Prop 2.3.19] we have seen that for each level $e \geq 0$, we have the ν-invariant $p^{e+m} - 1$. Thus, these ν-invariants form a sequence converging p-adically to -1. □

2.4 Strength

In the following section, we introduce a concept developed in [BQG24] called *strength* of a Bernstein-Sato root. This concept can be seen as an extension of the notion of Bernstein-Sato roots invariants, which provides more information about the variety. We will start introducing some notions of strength of ideals and ν-invariants, and see how they relate to the strength of a Bernstein-Sato root.

The most straightforward definition of the strength of a Bernstein-Sato root is obtained via the $R_f \boldsymbol{f}^{\boldsymbol{\alpha}}$ module constructed in the previous section.

Definition 2.4.1 (Strength of Bernstein-Sato root)**.** Let V be as in [Sett 2.2.1], R be a smooth V-algebra and $\alpha \in \mathbb{Z}_p$ be a Bernstein-Sato root of the nonzerodivisor $f \in R$. We define the strength of α as

$$\operatorname{str}(\alpha, f) = \min\{t \geq 0 : \mathfrak{m}^t \boldsymbol{f}^{\boldsymbol{\alpha}} \subseteq \mathcal{D}_R \cdot f\boldsymbol{f}^{\boldsymbol{\alpha}}\}$$

Notice that, for a fixed f, the strength can be seen as a function on the p-adic numbers.

$$\operatorname{str}(-, f) : \mathbb{Z}_p \to \{0, 1, \ldots, m+1\}$$

This function is 0 if α is not a Bernstein-Sato root, and can take the values $\{1, \ldots, m+1\}$ for the Bernstein-Sato roots. Thus, it is a more powerful invariant extending the information of the classical Bernstein-Sato roots.

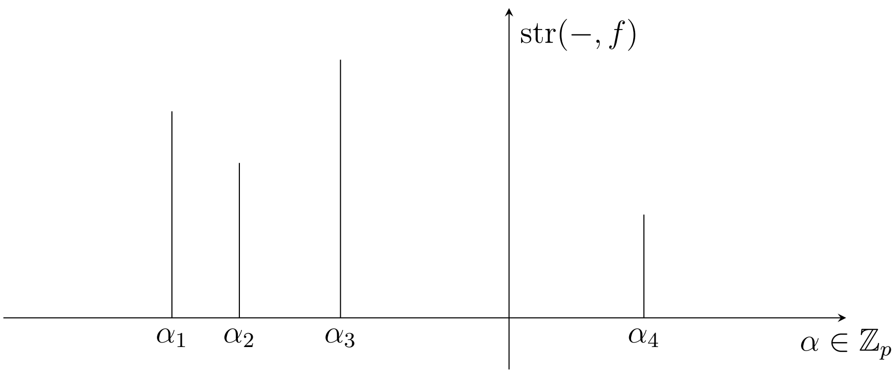


Figure 2.1: Strength function $\operatorname{str}(-, f)$

We will now construct some notions of strength for ν-invariants and see how they can induce the strength of a Bernstein-Sato root.

We start by noticing that multiplication by an ideal contained in V commutes with operators $\mathcal{C}_R^e \cdot$ and $\mathcal{D}_R^{[e]} \cdot$.

Proposition 2.4.2. Let V, R be as in [Sett 2.2.1]. Let $\mathfrak{b} \subseteq V$ and $I \subseteq R$. Then

1. $\mathcal{C}_R^e \cdot \mathfrak{b} I = \mathfrak{b} \mathcal{C}_R^e \cdot I$
2. $\mathcal{D}_R^{[e]} \cdot \mathfrak{b} I = \mathfrak{b} \mathcal{D}_R^{[e]} \cdot I$

Proof. For (1), recall that the elements of $\mathcal{C}_R^e \cdot \mathfrak{b}$ can be computed as

$$\mathcal{C}_R^e \cdot \mathfrak{b} I = (\varphi(F_*^e f) : \varphi \in \mathcal{C}_R^e, f \in \mathfrak{b} I) = (\mathfrak{b}\varphi(F_*^e f) : \varphi \in \mathcal{C}_R^e, f \in I) = \mathfrak{b} \mathcal{C}_R^e \cdot I$$

since φ is V-linear. In the same fashion, for (2) we have

$$\mathcal{D}_R^{[e]} \cdot \mathfrak{b} I = (\varphi(F_*^e f) : \varphi \in \mathcal{D}_R^{[e]}, f \in \mathfrak{b} I) = (\mathfrak{b}\varphi(F_*^e f) : \varphi \in \mathcal{D}_R^{[e]}, f \in I) = \mathfrak{b} \mathcal{D}_R^{[e]} \cdot I$$

for the same reason. $\square$

We now give two definition of strength: the first one is the strength of a ν-invariant (via the Cartier operator), and the second one is the strength of an ideal J such that $f \in \sqrt{J}$. Recall that a ν-invariant can be induced by a family of multiple ideals of such type, but we will see that there is a connection between both strengths.

Definition 2.4.3 (Strength of ν-invariant)**.** Let $\nu \in \nu_f^\bullet(p^e)$. We define the strength of level e of ν as

$$\operatorname{str}(\nu, e, f) = \min\{t \geq 0 : \mathfrak{m}^t \mathcal{C}_R^e \cdot f^\nu \subseteq \mathcal{C}_R^e \cdot f^{\nu+1}\} = \min\{t \geq 0 : \mathfrak{m}^t \mathcal{D}_R^{[e]} \cdot f^\nu \subseteq \mathcal{D}_R^{[e]} \cdot f^{\nu+1}\}$$

Notice that the equality of the two expressions follows from the fact that the minimums can be rewritten as

$$\min\{t \geq 0 : \mathfrak{m}^t \mathcal{C}_R^e \cdot f^\nu \subseteq \mathcal{C}_R^e \cdot f^{\nu+1}\} \quad \text{and} \quad \min\{t \geq 0 : \mathfrak{m}^t \mathcal{D}_R^{[e]} \cdot f^\nu \subseteq \mathcal{D}_R^{[e]} \cdot f^{\nu+1}\}$$

using [Prop 2.4.2], and that they are equal by [Prop 2.2.26].

Definition 2.4.4 (Strength of an ideal)**.** Let J be an ideal such that $f \in \sqrt{J}$. We define the strength of level e of J as

$$\operatorname{str}(J, e, f) := \min\{t \geq 0 : \mathfrak{m}^t f^{\nu_f^J(p^e)} \in J^{[p^e]}\}$$

Different elements of the family of ideals of such type giving the same ν-invariants could, a priori, have different strengths. However, we see that the maximum of this collection of strengths is the strength of the ν-invariant.

Proposition 2.4.5. Let $\nu \in \nu_f^\bullet(p^e)$ and $\{J_\alpha\}$ the set of all ideals such that $\nu = \nu_f^J(p^e)$. Then $J^* := \mathcal{C}_R^e \cdot f^{\nu+1} \in \{J_\alpha\}$ and $J^* \subseteq J$ for all $J \in \{J_\alpha\}$.

Proof. For proving $J^* \in \{J_\alpha\}$, notice that

$$f^\nu \in \left(\mathcal{C}_R^e \cdot f^{\nu+1}\right)^{[p^e]} \quad \Rightarrow \quad \mathcal{C}_R^e \cdot f^\nu \subseteq \mathcal{C}_R^e \cdot f^{\nu+1}$$

contradicting the fact that ν is a ν-invariant. On the other hand we clearly have the inclusion

$$f^{\nu+1} \in \left(\mathcal{C}_R^e \cdot f^{\nu+1}\right)^{[p^e]}$$

by [Prop 2.2.12] (2). Thus, $\nu = \nu_f^{J^*}(p^e)$ and $J^* \in \{J_\alpha\}$.

For proving $J^* \subseteq J$ notice

$$f^{\nu+1} \in J^{[p^e]} \Rightarrow \mathcal{C}_R^e \cdot f^{\nu+1} \subseteq J \Rightarrow J^* \subseteq J$$

which finishes the proof. □

What this proposition is suggesting is the existence of a canonical representative of an ideal of such type for each ν-invariant. It is canonical in the sense that it is minimal under inclusion (i.e. it is contained in all the other ideals). Given a ν-invariant, this canonical representative is $J^* = \mathcal{C}_R^e \cdot f^{\nu+1}$.

At this point, we are ready to prove the claim that the strength of a ν-invariant is the maximum of the strengths of the family of ideals J such that $f \in \sqrt{J}$.

Proposition 2.4.6. Fix a level $e \geq 1$. Let $\nu \in \nu_f^\bullet(p^e)$. Then

$$\operatorname{str}(\nu, e, f) = \max\{\operatorname{str}(J, e, f) : \nu = \nu_f^J(p^e)\}$$

Proof. Let s be the maximum of the right set. We claim that this maximum is attained at $J^* = \mathcal{C}_R^e \cdot f^{\nu+1}$, since for every $J \supseteq J^*$ we have

$$\mathfrak{m}^s f^\nu \subseteq J^{*[p^e]} \subseteq J^{[p^e]}, \quad \text{thus} \quad \operatorname{str}(J, e, f) \leq s$$

Hence, applying [Prop 2.2.12], we have

$$\begin{cases} \mathfrak{m}^{s-1} f^\nu \not\subseteq J^{*[p^e]} \\ \mathfrak{m}^s f^\nu \subseteq J^{*[p^e]} \end{cases} \Rightarrow \begin{cases} \mathfrak{m}^{s-1}\mathcal{C}_R^e \cdot f^\nu \not\subseteq J^* = \mathcal{C}_R^e \cdot f^{\nu+1} \\ \mathfrak{m}^s\mathcal{C}_R^e \cdot f^\nu \subseteq J^* = \mathcal{C}_R^e \cdot f^{\nu+1} \end{cases}$$

showing that s is the minimum integer such that $\mathfrak{m}^s\mathcal{C}_R^e \cdot f^\nu \subseteq \mathcal{C}_R^e \cdot f^{\nu+1}$. □

So far, we have been working with ν-invariants and ideals. The concept of strength has been defined in dependence of e via, for instance, the $\mathcal{D}_R^{[e]}$ operator. However, we need to *glue*, in some sense, all this information to recover the strength of a Bernstein-Sato root.

Now we see that, if we have a sequence of ν-invariants converging p-adically to α, then their strengths induce a strength of α in a certain way. First, we prove some preliminary results.

Proposition 2.4.7. Fix a level $e \geq 0$. Let m such that $\mathfrak{m}^{m+1} = 0$. Then,

1. $\nu_f^\bullet(p^e) + p^{e+m}\mathbb{Z} = \nu_f^\bullet(p^e)$
2. $\nu_f^\bullet(p^e)_{\operatorname{str}=s} + p^{e+m}\mathbb{Z} = \nu_f^\bullet(p^e)_{\operatorname{str}=s}$

where the subscript $\operatorname{str} = s$ means that we are taking the set of ν-invariants with strength s.

Proof. We have to prove that for all $k \in \mathbb{Z}$, if ν is a ν-invariant, so is $\nu + kp^{e+m}$. By proposition [Prop 2.2.8], we have

$$\mathcal{C}_R^e \cdot f^{\nu+kp^{e+m}} = \mathcal{C}_R^e \cdot F(f^{kp^m})f^\nu = f^{kp^m}\mathcal{C}_R^e \cdot f^\nu \neq f^{kp^m}\mathcal{C}_R^e \cdot f^{\nu+1} = \mathcal{C}_R^e \cdot f^{\nu+kp^{e+m}+1}$$

Similarly let $s = \operatorname{str}(\nu, e, f)$ and $s' = \operatorname{str}(\nu + kp^{e+m}, e, f)$. Then

$$\mathfrak{m}^s\mathcal{C}_R^e \cdot f^{\nu+kp^{e+m}} = \mathfrak{m}^s f^{kp^m}\mathcal{C}_R^e \cdot f^\nu \subseteq f^{kp^m}\mathcal{C}_R^e \cdot f^{\nu+1} = \mathcal{C}_R^e \cdot f^{\nu+kp^{e+m}+1}$$

Thus $s' \leq s$. By symmetry, making the transformation $k \mapsto -k$, we have $s' \geq s$, and the equality holds. □

Proposition 2.4.8. Fix $e \geq 0$, m such that $\mathfrak{m}^{m+1} = 0$ and $1 \leq s_0 \leq m+1$. Then $\nu_f^\bullet(p^e)_{\operatorname{str}\geq s_0} \supseteq \nu_f^\bullet(p^{e+1})_{\operatorname{str}\geq s_0}$ where the subscript $\operatorname{str} \geq s_0$ means that we are taking the set of ν-invariants with strength $\geq s_0$. In particular, we also have $\nu_f^\bullet(p^e) \supseteq \nu_f^\bullet(p^{e+1})$.

Proof. Let $\nu \in \nu_f^\bullet(p^{e+1})$ with strength (of level $e+1$) $> s_0$. Suppose that $\nu \notin \nu_f^\bullet(p^e)$. Then

$$\mathcal{C}_R^e \cdot f^\nu = \mathcal{C}_R^e \cdot f^{\nu+1} \quad \Rightarrow \quad \mathcal{C}_R^{e+1} \cdot f^\nu = \mathcal{C}_R^{e+1} \cdot f^{\nu+1}$$

which is a contradiction. Thus, $\nu \in \nu_f^\bullet(p^e)$. By the same argument, suppose its strength (of level e) is $\leq s_0$. Then

$$\mathfrak{m}^{s_0}\mathcal{C}_R^e \cdot f^\nu \subseteq \mathcal{C}_R^e \cdot f^{\nu+1} \quad \Rightarrow \quad \mathfrak{m}^{s_0}\mathcal{C}_R^{e+1} \cdot f^\nu \subseteq \mathcal{C}_R^{e+1} \cdot f^{\nu+1}$$

leading again to a contradiction. Clearly, the last statement of the proposition follows setting $s_0 = 1$. □

Proposition [Prop 2.4.8] states that every time we have a ν-invariant $\nu \in \nu_f^\bullet(p^e)$, the specification of the level e is needed to talk about a strength, since the computation depends on e. However, one can consider the strength of the same ν-invariant, but with a lower level, since $\nu_f^\bullet(p^{e+1}) \subseteq \nu_f^\bullet(p^e)$ by [Prop 2.4.8].

Proposition 2.4.9. Fix a level $e \geq 1$ and m such that $\mathfrak{m}^{m+1} = 0$. For all $t \geq 0$ and $\nu_1 \in \nu_f^\bullet(p^e), \nu_2 \in \nu_f^\bullet(p^{e+t})$ we have

1. $\mathrm{str}(\nu_2, e+t, f) \leq \mathrm{str}(\nu_2, e, f)$
2. If $\nu_1 \equiv \nu_2 \mod p^{e+m}$, then $\mathrm{str}(\nu_2, e, f) = \mathrm{str}(\nu_1, e, f)$

Proof.
For (1), fixing s, we have the chain

$$\nu_f^\bullet(p^{e+t})_{\mathrm{str}=s} \subseteq \nu_f^\bullet(p^{e+t})_{\mathrm{str}\geq s} \subseteq \cdots \subseteq \nu_f^\bullet(p^{e+1})_{\mathrm{str}\geq s} \subseteq \nu_f^\bullet(p^e)_{\mathrm{str}\geq s}$$

where the first inclusion is clear and the other ones follow from [Prop 2.4.8].

The statement (2) follows directly from [Prop 2.4.7]. □

Once we have computed the ν-invariants of f, suppose we have a sequence that p-adically converge to $\alpha \in \mathbb{Z}_p$. This α is a Bernstein-Sato root of f with certain strength s.

Definition 2.4.10 (Indexing function)**.** Let $\{\nu_e\}_{e\geq 0}$ be a sequence of ν-invariants whose p-adic limit is $\alpha \in \mathbb{Z}_p$. We define the indexing function $e_0 : \mathbb{Z} \to \mathbb{Z}$ as

$$e_0(e) = \min\{i : \nu_a \equiv \nu_b \mod p^{e+m} \ \forall a, b \geq i\}$$

This function represents the minimum index such that the congruence modulo p^{e+m} stabilizes for each e. Notice that $e_0(e) < \infty$ is bounded for all e because of the p-adic convergence of $\{\nu_e\}_{\geq 0}$.

Hence, we can represent these ν-invariants $\{\nu_e\}_{e\geq 0}$ in the following diagram

$$\begin{array}{c|cccccccccc|c}
0 & \nu_0 & \nu_1 & \nu_2 & \nu_3 & \nu_4 & \nu_5 & \nu_6 & \nu_7 & \cdots & \cdots & s_0 \\
1 & & & \nu_2 & \nu_3 & \nu_4 & \nu_5 & \nu_6 & \nu_7 & \cdots & \cdots & s_1 \\
2 & & & & & & \nu_5 & \nu_6 & \nu_7 & \cdots & \cdots & s_2 \\
\vdots & & & & & & & & \ddots & & & \\
e & & & & & & & & & \nu_{e_0(e)} & \cdots & s_e \\
\hline
\end{array}$$

where:

- The columns represent the ν-invariants. This is, in the column i we will have the ν-invariant ν_i. Recall that if $\nu \in \nu_f^\bullet(p^e)$, then $\nu \in \nu_f^\bullet(p^k)$ for all $0 \leq k \leq e$..
- The rows represent the stabilization of the congruence modulo p^{e+m}. This is, in the row j we will start at $e_0(j)$, where the congruence stabilizes modulo p^{j+m}.
- The last column represent the the strength of ν-invariants of the corresponding row. This strength is the same in each ν-invariant of the row by [Prop 2.4.9] (2).

We state now the correspondence between the strengths of the ν-invariants of a sequence and the strength of the Bernstein-Sato root of its limit.

Proposition 2.4.11. Let $\{\nu_e\}$ be a sequence of ν-invariants of $f \in R$ converging p-adically to $\alpha \in \mathbb{Z}_p$ such that $\nu_e \in \nu_f^\bullet(p^e)$ and consider the diagram [Section 2.4]. Then, the strengths of the ν-invariants converge towards the bottom of the diagram. More precisely, the limit

$$s_\alpha = \lim_{e\to\infty} s_e = \lim_{e\to\infty}\lim_{k\to\infty} \operatorname{str}(\nu_k, e, f)$$

exists.

Proof. Consider the following diagram of strengths induced by [Section 2.4]

$$\begin{array}{c|cccccccccc|c}
0 & s_0 & s_0 & s_0 & s_0 & s_0 & s_0 & s_0 & s_0 & \cdots & \cdots & s_0 \\
1 & & & s_1 & s_1 & s_1 & s_1 & s_1 & s_1 & \cdots & \cdots & s_1 \\
2 & & & & & & s_2 & s_2 & s_2 & \cdots & \cdots & s_2 \\
\vdots & & & & & & & & \ddots & & & \\
e & & & & & & & & & s_e & \cdots & s_e \\
\hline
\end{array}$$

We have proved in [Prop 2.4.9] that the strength decreases towards the bottom always and is stable in each row. Then, clearly the sequence $\{s_e\}$ has a limit, since it is decreasing and bounded by 1. $\square$

Notice that we can pass to a subsequence by deforming the scheme [Section 2.4] so that the staircase is diagonal by considering the subsequence

$$\{\nu_{e_0(i)}\}_{i\geq 0}, \qquad \text{this is} \qquad i \mapsto \nu_{e_0(i)}$$

Then, the form of the diagram would be diagonal, as

$$\begin{array}{c|ccccc|c}
0 & \nu_0 & \nu_1 & \nu_2 & \cdots & \cdots & s_1 \\
1 & & \nu_1 & \nu_2 & \cdots & \cdots & s_2 \\
2 & & & \nu_2 & \cdots & \cdots & s_3 \\
\vdots & & & & \ddots & & \\
e & & & & & \nu_{e_0(e)} & s_e \\
\hline
\end{array}
\quad \Rightarrow \quad
\begin{array}{c|ccccc|c}
0 & s_0 & s_0 & s_0 & \cdots & \cdots & s_1 \\
1 & & s_1 & s_1 & \cdots & \cdots & s_2 \\
2 & & & s_2 & \cdots & \cdots & s_3 \\
\vdots & & & & \ddots & & \\
e & & & & & s_e & s_e \\
\hline
\end{array}$$

If we perform this deformation of passing to this subsequence one can just say that

$$s_\alpha = \lim_{e\to\infty} \operatorname{str}(\nu_e, e, f)$$

Theorem 2.4.12. Let R be as in [Sett 2.2.1], $\alpha \in \mathbb{Z}_p$ be a BS root of $f \in R$ and $\{\nu_e\}_{e\geq 0}$ such that $\nu_e \in \nu_f^\bullet(p^e)$ converging p-adically to α. The following are equivalent:

1. The strength of α is $\operatorname{str}(\alpha, f) = s$
2. $\lim\limits_{e\to\infty} \lim\limits_{k\to\infty} \operatorname{str}(\nu_k, e, f) = s$

Proof. First, notice that by the previous remark, we can pass to a subsequence assuming $e = e_0(e+m)$ and (2) transforms into $\lim_{e\to\infty} \operatorname{str}(\nu_e, e, f) = s$.

Suppose, after passing to a subsequence, $\lim_{e\to\infty} \operatorname{str}(\nu_e, e, f) = s$ and $\operatorname{str}(\alpha, f) = t < s$. Then $\mathfrak{m}^{s-1}\boldsymbol{f}^{\boldsymbol{\alpha}} \subseteq \mathcal{D}_R \cdot f\boldsymbol{f}^{\boldsymbol{\alpha}}$, meaning that we can pick $e \gg 0$ such that, simultaneously, $\mathfrak{m}^{s-1}\boldsymbol{f}^{\boldsymbol{\alpha}} \subseteq \mathcal{D}_R^{[e]} \cdot f\boldsymbol{f}^{\boldsymbol{\alpha}}$ and $\operatorname{str}(\nu_e, e, f) = s$. Then

$$\mathfrak{m}^{s-1}\boldsymbol{f}^{\boldsymbol{\alpha}} \subseteq \mathcal{D}_R^{[e]} \cdot f\boldsymbol{f}^{\boldsymbol{\alpha}} \quad \Rightarrow \quad \mathfrak{m}^{s-1}f^{\nu_e} \subseteq \mathcal{D}_R^{[e]} \cdot f^{\nu_e+1} \quad \Rightarrow \quad \mathfrak{m}^{s-1}\mathcal{D}_R^{[e]} \cdot f^{\nu_e} \subseteq \mathcal{D}_R^{[e]} \cdot f^{\nu_e+1}$$

meaning that $\operatorname{str}(\nu_e, e, f) < s$, which is a contradiction. Now suppose $\operatorname{str}(\alpha, f) = t > s$. Then $\mathfrak{m}^{s}\boldsymbol{f}^{\boldsymbol{\alpha}} \nsubseteq \mathcal{D}_R \cdot f\boldsymbol{f}^{\boldsymbol{\alpha}}$, meaning that for every $e \geq 0$ we have $\mathfrak{m}^{s}\boldsymbol{f}^{\boldsymbol{\alpha}} \nsubseteq \mathcal{D}_R^{[e]} \cdot f\boldsymbol{f}^{\boldsymbol{\alpha}}$. Then

$$\mathfrak{m}^{s}\boldsymbol{f}^{\boldsymbol{\alpha}} \nsubseteq \mathcal{D}_R^{[e]} \cdot f\boldsymbol{f}^{\boldsymbol{\alpha}} \quad \Rightarrow \quad \mathfrak{m}^{s}f^{\nu_e} \nsubseteq \mathcal{D}_R^{[e]} \cdot f^{\nu_e+1} \quad \Rightarrow \quad \mathfrak{m}^{s}\mathcal{D}_R^{[e]} \cdot f^{\nu_e} \nsubseteq \mathcal{D}_R^{[e]} \cdot f^{\nu_e+1}$$

meaning that $\operatorname{str}(\nu_e, e, f) > s$, which is a contradiction. □

Corollary 2.4.13. Let R be as in [Sett 2.2.1], $\alpha \in \mathbb{Z}_p$ be a BS root of $f \in R$ and $\{\nu_e\}_{e\geq 0}$ such that $\nu_e \in \nu_f^\bullet(p^e)$ converging p-adically to α such that $\nu_a \equiv \nu_b \mod p^{e+m}$ for all $a, b \geq e$. The following are equivalent:

1. The strength of α is $\operatorname{str}(\alpha, f) = s$
2. $\lim\limits_{e\to\infty} \operatorname{str}(\nu_e, e, f) = s$

Proof. Follows directly from [Thm 2.4.12]. In the proof we first pass to a subsequence and then proved this statement. Hence, with this extra condition that step is no longer necessary. □

Recall that for each $f \in R$ nonzerodivisor, we have that -1 is a Bernstein-Sato root. The first application of this theorem is to show that the strength of the root -1 is always $m+1$.

Proposition 2.4.14. Let R and m be as in [Sett 2.2.1] and $f \in R$ nonzerodivisor. Then $\operatorname{str}(-1, f) = m+1$.

Proof. Let $\nu_e \coloneqq p^{e+m} - 1$. Recall that, by [Prop 2.3.19], we have that

$$(g_1, \ldots, g_s) = \mathcal{C}_R^e \cdot f^{p^{e+m}-1} \supsetneq \mathcal{C}_R^e \cdot f^{p^{e+m}} = (f^{p^m})$$

with each $\deg_{\mathfrak{m}}(g_i) < \deg_{\mathfrak{m}}(f) \cdot p^m$. Hence, clearly by degree comparison $p^t g_i \in (f^{p^m})$ if and only if $t \geq m+1$. Thus, the strength of the ν-invariant is

$$\operatorname{str}(p^{e+m} - 1, e, f) = \min\{t \geq 0 : p^t(g_1, \ldots, g_s) \in (f^{p^m})\} = m+1$$

and considering that $\nu_{e+k} \equiv \nu_e \mod p^{e+m}$ for all $k \geq 0$, then by [Thm 2.4.12] we have that desired result. □

2.5 Reduction

2.5.1 Connection with the zero characteristic case

We discuss now the relationship between the Bernstein-Sato roots in the zero characteristic, prime characteristic and prime power characteristic cases. In [MTW04] and [BMS05] one can find a detailed explanation on the connection between prime and zero characteristic cases. On the other hand, [BQG24] exposes the connection between the prime power and prime characteristic cases.

For the remainder of this subsection, we will have a zero characteristic polynomial $f \in \mathbb{Z}[\underline{x}]$, and we will consider the projections of this polynomial in the positive characteristic rings as

$$f_p \in \mathbb{Z}/p\mathbb{Z}[\underline{x}], \quad \overline{f} \in \mathbb{Z}/p^{m+1}\mathbb{Z}[\underline{x}]$$

respectively. In the same fashion, we will denote with a p subscript when we refer to the prime characteristic case, and with a bar when we refer to the prime power characteristic case. Notice that the notation for the prime characteristic case the notation does not express the dependence on p, but, if not specified, we will always assume that such prime is the same as the one in f_p.

For the prime power characteristic case, fix the lift of Frobenius as the linear extension of the morphism sending $x_i \mapsto x_i^{p^e}$ for each variable (see [Prop 2.2.5]). Finally, we will denote by $b_f(s)$ the Bernstein-Sato polynomial of f.

In [MTW04] it was shown that, under certain conditions, some roots of $b_f(s)$ can be computed via the ν-invariants of the polynomial f_p in the prime characteristic case. More precisely, we have the following proposition:

Proposition 2.5.1. [MTW04, Remark 3.13] Let f be an element of a ring R of prime characteristic p. Fix some ideal J such that $f \in \sqrt{J}$ and a level $e \geq 1$. Suppose that there exists some integer $N \geq 1$ such that for all $p \gg 0$ there exists polynomials $P_i \in \mathbb{Q}[x]$ such that $\nu_{f_p}^J(p^e) = P_i(p)$ for $i \in \{1, \dots, N\}$. Then

$$P_i(0) \in \mathrm{BSR}(f)$$

The first goal of this section is to generalize this result for the prime power case.

Proposition 2.5.2. Let $\overline{J} \subseteq \overline{R}$ and level $e \geq m+1$. Then $\mathcal{D}_R \cdot \overline{J}^{[p^e]} \subseteq \overline{J}^{[p^e]}$.

Proof. Let $g \in \overline{J}$. Then, since $\mathcal{D}_R$ is linear in the sum, we only have to show that $\mathcal{D}_R \cdot rF^e(g) \in \overline{J}^{[p^e]}$. Recall $\mathcal{D}_R = \mathbb{Z} < x_i, \partial_i >$. We only have to show that $x_i rF^e(g) \in \overline{J}^{[p^e]}$ and $\partial_i(rF^e(g)) \in \overline{J}^{[p^e]}$. The first statement is trivial, and for the second let $g = \sum_\alpha c_\alpha \underline{x}^\alpha$

$$\partial_i(rF^e(g)) = \frac{\partial r}{\partial x_i} F(g) + r\frac{\partial}{\partial x_i}\left(\sum_\alpha c_\alpha \underline{x}^{\alpha p^e}\right) = \frac{\partial r}{\partial x_i} F^e(g) + rp^e \left(\sum_{\alpha'} \alpha'_i c_{\alpha'} x_1^{\alpha'_1 p^e} \cdots x_i^{\alpha'_i p^e} \cdots x_n^{\alpha'_n p^e}\right)$$

where the first term is in $\overline{J}^{[p^e]}$ and the second term is zero provided $e \geq m+1$. ☐

Proposition 2.5.3. Let $f \in R$, a level $e \geq m+1$ and $\nu \in \nu_{\overline{f}}^{\bullet}(p^e)$. Let $s = \mathrm{str}(\nu, e, f)$ be its strength . Then $b_f(\nu) = 0 \mod p^s$ for all $p \gg 0$.

Proof. Let $\overline{J}$ such that $\overline{f} \in \sqrt{\overline{J}}$, $\nu = \nu_{\overline{f}}^{\overline{J}}(p^e)$ and has the maximum strength s (this is, $s = \mathrm{str}(\nu, e, f) = \mathrm{str}(J, e, f)$). Then $\overline{f}^{\nu} \notin \overline{J}^{[p^e]}, p^s \overline{f}^{\nu} \in \overline{J}^{[p^e]}$ and $\overline{f}^{\nu+1} \in \overline{J}^{[p^e]}$. There exists an a such that the Bernstein-Sato equation gives

$$b_f(s) f^s = P(s) f^{s+1}$$

with $b_f(s) \in \frac{1}{a}\mathbb{Z}[s]$ and $P(s) \in \frac{1}{a}\mathcal{D}_R[s]$. Let $p > a$ such that a is a unity in $\mathbb{Z}/p^{m+1}\mathbb{Z}$. Then specializing the Bernstein-Sato equation at $s = \nu$ and contracting modulo p^{m+1} gives

$$\overline{b_f(\nu)}\,\overline{f}^{\nu} = \overline{P(\nu)}\,\overline{f}^{\nu+1} \in \overline{J}^{[p^e]}$$

where the second term is in $\overline{J}^{[p^e]}$ by [Prop 2.5.2]. Since the only nontrivial ideals of $\mathbb{Z}/p^{m+1}\mathbb{Z}$ are $(p^{m+1}) \subseteq (p^m) \subseteq \cdots \subseteq (p)$, then $b_f(\nu) \in (p^s)$ by definition of strength. Thus $b_f(\nu) = kp^s$ and the proposition follows. □

Corollary 2.5.4. Let $f \in R$, $e \geq m+1$ and $\nu \in \nu_{\overline{f}}^{\bullet}(p^e)$. Then $b_f(\nu) = 0 \mod p$.

In all our examples we have seen that the $\nu_f^J(p^e)$ behaves like a polynomial in p depending the congruence of p. This does not hold for the general case, but if this holds, then we can extract some Bernstein-Sato rooots of the zero-characteristic polynomial via the ν-invariants of the positive characteristic case. The discussion about in which cases this property holds is interesting in its own, and one can find some conjectures about it in [MTW04].

Proposition 2.5.5. Let R be as in [Sett 2.2.1] and $f \in R$ be a nonzerodivisor. Fix some ideal J such that $f \in \sqrt{J}$ and a level $e \geq m+1$. Suppose that there exists an integer $N \geq 1$ such that for all $p \gg 0$ there exists polynomials $P_i \in \mathbb{Q}[x]$ such that $\nu_{\overline{f}}^{J}(p^e) = P_i(p)$ for $i \in \{0, \dots, N-1\}$. Then

$$P_i(0) \in \mathrm{BSR}(f)$$

Proof. Notice that there exists an integer $a \geq 1$ such that $P_i \in \frac{1}{a}\mathbb{Z}[x]\ \forall i \in \{0, \dots, N-1\}$. If we choose $p > a$, then a is a unit in R, and we can reduce the equation to

$$(b_f(P_i(p)))_p = 0 \qquad \text{with} \quad b_f \in R_p[t], \quad P_i \in R_p[t]$$

Since $(p-0) \mid (b_f(p))_p - (b_f(0))_p$, then $b_f(0) = 0$. By Dirichlet theorem [Thm A.2.11], we can find infinitely many primes p such that $p \equiv i \mod N$, so the proposition follows. □

Corollary 2.5.6. The original result in [Prop 2.5.1] follows directly by plugging $m = 0$.

2.5.2 Projection to prime characterstic case

We now focus on the relationship between the prime and prime power characteristic cases. For the remainder of this subsection we will denote by R and f the ring and an element of the prime power case, whereas we will denote by a subscript p the projection to the prime characteristic case. We start with a few results.

Definition 2.5.7. We define the endofunctor $(-)_p$ as

$$(-)_p : \mathbf{V}-\mathbf{Mod} \to \mathbf{V}-\mathbf{Mod}, \qquad M_p = M \otimes_V V/\mathfrak{m} = M/\mathfrak{m}M$$

This functor is right exact, since it comes form a tensor product. Notice that the last equality is a consequence of the isomorphism $M \otimes_R R/I \cong M/IM$, and thus this functor is a reduction modulo $\mathfrak{m}$. There is also a quotient map between the modules M and M_p. Hence, we will use interchangeably the notation of the subscript to denote the functor and the quotient map.

Lemma 2.5.8. Let $\varphi : M \to N$ be a module homomorphism. Then φ is surjective if and only if φ_p is surjective.

Proof. Suppose φ is surjective. Then, by right exactness we have

$$M \xrightarrow{\varphi} N \to 0 \quad \Rightarrow \quad M_p \xrightarrow{\varphi_p} N_p \to 0$$

that φ_0 is surjective. For the other direction suppose φ_0 is surjective. Then, if Q is the cokernel of φ we have

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[d, "(-)_p"'] & N \arrow[r] \arrow[d, "(-)_p"'] & Q \arrow[r] \arrow[d, "(-)_p"'] & 0 \\
M_p \arrow[r, two heads] & N_p \arrow[r] & 0 &
\end{tikzcd}$$

and by commutativity of the diagram $Q_p = Q/\mathfrak{m}Q = 0$. Then $Q = \mathfrak{m}Q = \mathfrak{m}^2 Q = \cdots \mathfrak{m}^{m+1}Q = 0$. Hence, φ is sujective. ☐

Lemma 2.5.9. Let R be as in [Sett 2.2.1] and $f \in R$ be a nonzerodivisor. Then

1. There is an isomorphism $(\mathcal{D}_R)_p \cong \mathcal{D}_{R_p}$
2. There is an isomorphism $(R_f \boldsymbol{f}^{\alpha})_p \cong R_{pf_p} \boldsymbol{f_p^{\alpha}}$
3. There is a morphism $(\mathcal{D}_R \cdot \boldsymbol{f^{\alpha}})_p \to \mathcal{D}_{R_p} \cdot \boldsymbol{f_p^{\alpha}}$

Proof.
(1) Consider the following commutative diagram

$$\begin{tikzcd}
 & \mathrm{Hom}_R(F^e_* R, F^e_* R)_p \arrow[d, two heads, "(iii)"] \\
\mathrm{Hom}_R(F^e_* R, F^e_* R) \arrow[ur, two heads, "(i)"] \arrow[r, two heads, "(ii)"] & \mathrm{Hom}_{R_p}(F^e_* R_p, F^e_* R_p)
\end{tikzcd}$$

The map (i) is a quotient map, hence surjective. For the map (ii) we consider the diagrams

$$\begin{tikzcd}
F^e_* R \arrow[r, dashed, "\exists\varphi"] \arrow[d, "\pi"'] & F^e_* R \arrow[d, two heads, "\pi"] & \Rightarrow & & R \arrow[d, two heads, "\pi"] \\
F^e_* R_p \arrow[r, "\varphi_p"'] & F^e_* R_p & \Rightarrow & F^e_* R \arrow[ur, dashed, "\exists\varphi"] \arrow[r, "\varphi_0\circ\pi"'] & F^e_* R_p
\end{tikzcd}$$

Since F^e_*R is projective by [Prop 2.2.10], then for every $\varphi_p : F^e_*R_p \to F^e_*R_p$, we can lift the map and get a $\varphi : F^e_*R \to F^e_*R$. Thus, the map (ii) is surjective. The surjectivity of (iii) follows form the surjectivity of (i) and (ii).

On the other hand, suppose that $\varphi_p \in \mathcal{D}^{[e]}_{R_p}$ is the zero map, meaning

$$\varphi_p(F^e_*r_p) = 0 \quad \forall r_p \in R_p$$

Then,

$$\varphi(F^e_*r) \in \mathfrak{m}R \quad \Rightarrow \quad (\varphi)_p(F^e_*r) = 0 \ \forall r \in R$$

and the map $(\varphi)_p \in (\mathcal{D}^{[e]}_R)_p$ is the zero map. Thus, the map (iii) is injective.

Finally, by [Prop 2.2.17], the isomorphism $(\mathcal{D}^{[e]}_R)_p \cong \mathcal{D}^{[e]}_{R_p}$ induce the isomorphism $(\mathcal{D}_R)_p \cong \mathcal{D}_{R_p}$.

(2) First notice we have the isomorphism of $\mathcal{D}_{R_0}$-modules $(R_f)_0 \cong R_{0_{f_0}}$, since

$$(R_f)_p = \frac{R_f}{\mathfrak{m}R_f} \cong \left(\frac{R}{\mathfrak{m}R}\right)_{f_p} = (R_p)_{f_p}$$

This isomorphism induces a $\mathcal{D}_{R_p}$-module isomorphism

$$(R_f\boldsymbol{f}^{\boldsymbol{\alpha}})_p \cong R_{p_{f_p}}\boldsymbol{f_p}^{\boldsymbol{\alpha}}$$

(3) The morphism $(\mathcal{D}_R)_p \to \mathcal{D}_{R_p}$ induces the morphism of D_{R_p}-modules

$$(\mathcal{D}_R \cdot \boldsymbol{f}^{\boldsymbol{\alpha}})_p = \frac{\mathcal{D}_R\boldsymbol{f}^{\boldsymbol{\alpha}}}{\mathfrak{m}\mathcal{D}_R\boldsymbol{f}^{\boldsymbol{\alpha}}} \to \mathcal{D}_{R_p} \cdot \boldsymbol{f_p}^{\boldsymbol{\alpha}}, \quad [\delta\boldsymbol{f}^{\boldsymbol{\alpha}}] \mapsto \delta_p\boldsymbol{f_p}^{\boldsymbol{\alpha}}$$

□

Theorem 2.5.10. [BMS09, Theorem 2.11]. Let $R_p \coloneqq \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring over a perfect field $\mathbb{K}$. Let $f_p \in R_p$ be a nonzero element and $\alpha \in \mathbb{Z}_{(p),<0}$. Then

$$\mathcal{D}_{R_p} \cdot \boldsymbol{f_p}^{\boldsymbol{\alpha}} = R_{p f_p}\boldsymbol{f_p}^{\boldsymbol{\alpha}}$$

Proof. See [BMS09, Theorem 2.11]. □

There is a direct extension of this theorem to our case.

Corollary 2.5.11. [BQG24, Corollary 4.10] Let V be as in [Sett 2.2.1] and $R = V[x_1, \dots, x_n]$ be a polynomial ring over V. Let $f \in R$ be a nonzerodivisor and $\alpha \in \mathbb{Z}_{(p),<0}$. Then

$$\mathcal{D}_R \cdot \boldsymbol{f}^{\boldsymbol{\alpha}} = R_f\boldsymbol{f}^{\boldsymbol{\alpha}}$$

Proof. Consider the inclusion $\iota : \mathcal{D}_R \cdot \boldsymbol{f}^{\boldsymbol{\alpha}} \hookrightarrow R_f\boldsymbol{f}^{\boldsymbol{\alpha}}$. We must prove that ι is surjective. Considering the reduction

$$\begin{tikzcd}
\mathcal{D}_R \cdot \boldsymbol{f}^{\boldsymbol{\alpha}} \arrow[r, "i"] \arrow[d, "(-)_p"] & R_f\boldsymbol{f}^{\boldsymbol{\alpha}} \arrow[d, "(-)_p"] \\
(\mathcal{D}_R \cdot \boldsymbol{f}^{\boldsymbol{\alpha}})_p \cong \mathcal{D}_{R_p} \cdot \boldsymbol{f_p}^{\boldsymbol{\alpha}} \arrow[r, two heads, "j"] & (R_f\boldsymbol{f}^{\boldsymbol{\alpha}})_p \cong R_{f_p}\boldsymbol{f_p}^{\boldsymbol{\alpha}}
\end{tikzcd}$$

where we have used the isomorphisms of [Lem 2.5.9], we know that j is surjective, and by [Lem 2.5.8] then i is surjective. □

Theorem 2.5.12. [BQG24, Theorem 4.12] Let V be as in [Sett 2.2.1], $R = V[x_1, \ldots, x_n]$ be a polynomial ring over V and $f \in R$ be a nonzerodivisor. Then $\alpha \in \mathbb{Z}_{(p),<0}$ is a Bernstein-Sato root of f if and only if it is a Bernstein-Sato root of f_p.

Proof. Consider the following diagram

$$\begin{tikzcd}
\mathcal{D}_R \cdot f\boldsymbol{f^{\alpha}} \arrow[r, two heads] \arrow[d, hook, "i"'] & (\mathcal{D}_R \cdot f\boldsymbol{f^{\alpha}})_p \arrow[r, two heads] \arrow[d, hook, "j"'] & D_{R_p} \cdot f_p\boldsymbol{f_p^{\alpha}} \arrow[d, hook, "k"'] \\
\mathcal{D}_R\boldsymbol{f^{\alpha}} \cong R_f\boldsymbol{f^{\alpha}} \arrow[r, two heads] & (\mathcal{D}_R\boldsymbol{f^{\alpha}})_p \cong (R_f\boldsymbol{f^{\alpha}})_p \arrow[r, "\cong"] & \mathcal{D}_{R_p}\boldsymbol{f_p^{\alpha}} \cong R_{pf_p}\boldsymbol{f_p^{\alpha}}
\end{tikzcd}$$

where the isomorphisms come from [Cor 2.5.11] and [Lem 2.5.9] (2).

We have to prove, then, that i is surjective if and only if k is surjective. For one direction, if i is surjective, then by diagram chasing we inmediately get the surjectivity of j and k by the surjectivity of the lower morphisms. Similarly, if k is surjective, we get from the lower left isomorphism that j is surjective. Finally, by [Lem 2.5.8] we have that i is surjective. □

Theorem 2.5.13 (Translation)**.** [BQG24, Corollary 4.16] Let R be as in [Sett 2.2.1], $f \in R$ be a nonzerodivisor and $f_p \in R_p$ be the reduction of this element. Then

$$\mathrm{BSR}(\overline{f}) + \mathbb{Z} = \mathrm{BSR}(f_p) + \mathbb{Z}$$

Proof. By [Thm 0.0.2] and [Thm 2.5.12] we know that

$$\mathrm{BSR}(f) \cap [-1, 0) = \mathrm{BSR}(f_p) \subseteq [-1, 0) \cap \mathbb{Z}_{(p)}$$

Thus, we just need to show that if $\alpha \notin \mathrm{BSR}(f_0)$ with $\alpha \in \mathbb{Z}_{(p)}$, then $\alpha + k \notin \mathrm{BSR}(f)$ for every $k \in \mathbb{Z}$, which follows from [Prop 2.3.12] □

These lasts results show that the Bernstein-Sato roots have plenty of similarities between the prime and prime power characteristic. Mainly, we have that the negative roots are the same, rational and confined in the interval $[-1, 0)$ (recall Kashiwara's theorem [Thm 0.0.1]). However, in the prime power characteristic, some positive roots arise, and they are integer shifts of some root in prime characteristic.

Chapter 3

Bernstein-Sato roots of diagonal hypersurfaces

Let $\{\alpha_1, \dots, \alpha_n\} \in \mathbb{Z}_{\geq 1}$ and $p > \max\{\alpha_1, \dots, \alpha_n\}$. Consider the diagonal hypersurface defined by $f = c_1 x_1^{\alpha_1} + \cdots + c_n x_n^{\alpha_n} \in R = \mathbb{Z}/p^{m+1}\mathbb{Z}[\underline{x}]$ with c_i units. The aim of this section is to compute the Bernstein-Sato roots and its strengths of such polynomial.

Observe that we can express the collection of exponents of the polynomial f as a matrix

$$A = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{pmatrix}$$

This matrix is square and diagonal because of the form of f as a diagonal hypersurface.

Notation 3.0.1. For a vector $\underline{i} = (i_1, \dots, i_n)$ with $l = |\underline{i}|$, we denote the multinomial coefficient as

$$\binom{l}{\underline{i}} := \binom{l}{i_1, \dots, i_n} = \frac{l!}{i_1! \cdots i_n!}$$

First we compute the ν-invariants of the polynomial. Turns out that computing all of them is rather cumbersome in general, but we can prove that they behave in a certain way if they exist. Let us begin with some preliminary results.

Conditions 3.0.2 (Noncrossing condition NCC)**.** Let $\underline{i} \in \mathbb{Z}^n_{\geq 0}$ and a level $e \geq 0$. We say that $\underline{i}$ satisfies the NCC if there exists an index $1 \leq j \leq n$ such that

$$\left\lfloor \frac{\alpha_j(i_j+1)}{p^e} \right\rfloor = \left\lfloor \frac{\alpha_j i_j}{p^e} \right\rfloor$$

Proposition 3.0.3. Fix a level $e \geq 1$ and let $\underline{i} \in \mathbb{Z}^n_{\geq 0}$. Suppose that for each choice of $(r_1, \dots, r_n) \in \mathbb{Z}^n_{\geq 1}$, we have that

$$|\underline{i}| \neq \left\lfloor \frac{r_1 p^e - 1}{\alpha_1} \right\rfloor + \cdots + \left\lfloor \frac{r_n p^e - 1}{\alpha_n} \right\rfloor$$

Then, $\underline{i}$ satisfies [Cond 3.0.2].

Proof. Suppose [Cond 3.0.2] does not hold. Then, for all the indices $1 \leq j \leq n$ we have

$$\left\lfloor \frac{\alpha_j(i_j+1)}{p^e} \right\rfloor = \left\lfloor \frac{\alpha_j i_j}{p^e} \right\rfloor + 1$$

For each j let an integer r_j such that

$$(r_j - 1)p^e \leq \alpha_j i_j < r_j p^e$$

Then, $r_j p^e \leq \alpha_j(i_j+1)$ and we have

$$\frac{r_j p^e - \alpha_j}{\alpha_j} \leq i_j < \frac{r_j p^e}{\alpha_j}$$

Therefore,

$$i_j = \left\lfloor \frac{r_j p^e - 1}{\alpha_j} \right\rfloor \qquad \text{for some } r_j \in \mathbb{Z}_{\geq 1}$$

thus, contradicting the hypothesis. □

Proposition 3.0.4. Fix a level $e \geq 1$, let $\underline{i} \in \mathbb{Z}^n_{\geq 0}$ and consider $\underline{b} = \underline{i}\%p^e$. Then $\underline{i}$ satisfies the NCC with index $1 \leq j \leq n$, if and only if $\underline{b}$ satisfies the NCC with index j.

Proof. Suppose $\underline{b}$ satisfies the NCC with index j. Then, let $\underline{i}' = (i_1, \ldots, i_j + 1, \ldots, i_n)$. We have that there exist a $k \geq 0$ such that

$$\left\lfloor \frac{(i_j+1)\alpha_j}{p^e} \right\rfloor = \left\lfloor \frac{(kp^e + b_j + 1)\alpha_j}{p^e} \right\rfloor = k\alpha_j + \left\lfloor \frac{(b_j+1)\alpha_j}{p^e} \right\rfloor = k\alpha_j + \left\lfloor \frac{b_j\alpha_j}{p^e} \right\rfloor = \left\lfloor \frac{i_j\alpha_j}{p^e} \right\rfloor$$

For the converse, suppose $\underline{i}$ satisfies the NCC with index j. Then, let $\underline{b}' = (b_1, \ldots, b_j + 1, \ldots, b_n)$. We have that there exists a $k \leq 0$ such that

$$\left\lfloor \frac{(b_j+1)\alpha_j}{p^e} \right\rfloor = \left\lfloor \frac{(kp^e + i_j + 1)\alpha_j}{p^e} \right\rfloor = k\alpha_j + \left\lfloor \frac{(i_j+1)\alpha_j}{p^e} \right\rfloor = k\alpha_j + \left\lfloor \frac{i_j\alpha_j}{p^e} \right\rfloor = \left\lfloor \frac{b_j\alpha_j}{p^e} \right\rfloor$$

concluding the proof □

We introduce now some results on the periodicity of p-adic expressions.

Proposition 3.0.5. Let $r, \alpha \in \mathbb{Z}_{\geq 1}$. The expression of $\left\lfloor \frac{rp^e-1}{\alpha} \right\rfloor$ in base p is eventually periodic. That is

$$\left\lfloor \frac{rp^e - 1}{\alpha} \right\rfloor = [\ldots uu \ldots u]_p$$

for some chunk of digits u of length $\varphi(\alpha)$, where φ is the Euler's totient function defined in [Thm A.2.12].

Proof. We consider two cases. First, if $\alpha \mid r$, then we have that for some k such that $r = k\alpha$

$$\left\lfloor \frac{rp^e - 1}{\alpha} \right\rfloor = \left\lfloor kp^e - \frac{1}{\alpha} \right\rfloor = [\ldots (p-1)(p-1)\ldots(p-1)]_p$$

giving a periodic expression on the last e digits.

If $\alpha \nmid r$, then the expression $\left\lfloor \frac{rp^e-1}{\alpha} \right\rfloor$ is equivalent to computing the expression of

$$\frac{r}{\alpha} = k + \frac{r\%\alpha}{\alpha} = [\dots, uu\dots]_p$$

which is periodic with a period $\varphi(\alpha)$, then shifting the digits e times to the left and finally cutting all the digits after the decimal point. $\square$

Proposition 3.0.6. Fix $\{\alpha_1, \dots, \alpha_n\} \in \mathbb{Z}^n_{\geq 1}$ and let $t := \operatorname{lcm}(\varphi(\alpha_1), \dots, \varphi(\alpha_n))$. Then, we can express

$$\left\lfloor \frac{r_i p^e - 1}{\alpha_i} \right\rfloor = [\dots u_i \dots u_i]_p \quad \text{such that} \quad \operatorname{length}(u_i) = t$$

Proof. By [Prop 3.0.5], a period of $\left\lfloor \frac{r_i p^e-1}{\alpha_i} \right\rfloor$ in base p is $\varphi(\alpha_i)$. Taking the least common multiple of all the periods we get a period t compatible with all the periods. $\square$

For the remaining of the section, the letter u will be reserved for periodic chunks of digits in base p.

Notation 3.0.7. Let $l \geq 0$ be an integer and $\underline{b} \in \{0, \dots, p^e - 1\}^n$. We define the set of vectors

$$I_l^{\underline{b}} = \left\{ \underline{i} \in \mathbb{Z}^n_{\geq 0} : |\underline{i}| = l, i_j\%p^e = b_j \right\}$$

Proposition 3.0.8. Fix a level $e \geq 1$, let $l \geq 0$ be an integer, $\underline{b} \in \{0, \dots, p^e - 1\}^n$ and $\underline{d} \in \mathbb{Z}^n$ with $|\underline{d}| = 1$. If $\underline{b} + \underline{d} \in \{0, \dots, p^e - 1\}$, then there is a bijection

$$I_l^{\underline{b}+\underline{d}} \longleftrightarrow I_{l+|\underline{d}|}^{\underline{b}+\underline{d}} \quad \text{such that} \quad \underline{i} \leftrightarrow \underline{i} + \underline{d}$$

Proof. For showing the bijection, let $\underline{i} \in I_l^{\underline{b}}$. Then

$$|\underline{i}| = l, \quad i_k\%p^e = b_k \quad \Rightarrow \quad |\underline{i} + \underline{d}| = l + 1, \quad (i_j + d_j)\%p^e = b_j + d_j \quad \text{for } 1 \leq j \leq n$$

meaning $\underline{i} + \underline{d} \in I_{l+1}^{\underline{b}+\underline{d}}$. Now consider $\underline{i} + \underline{d} \in I_{l+1}^{\underline{b}+\underline{d}}$, and notice that $0 \leq (b_j + d_j) \leq p^e - 1$ by assumption. Then

$$|\underline{i} + \underline{d}| = l + 1, \quad (i_k + d_k)\%p^e = b_k + d_k \quad \Rightarrow \quad |\underline{i}| = l, \quad i_k\%p^e = b_k$$

meaning $\underline{i} \in I_l^{\underline{b}}$. $\square$

Proposition 3.0.9. Fix a level $e \geq m+1$ and an index set $\underline{i} \in \mathbb{Z}^n_{\geq 0}$ such that $|\underline{i}| = l$. Let $\underline{d} = (d_1, \dots, d_n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}^{n-1}_{\leq 0}$ with $|\underline{d}| = 1$ and $d_1 = \min\{d \geq 0 : \nu_p(i_1 + d) \geq \nu_p(l + 1)\}$. Then

$$\binom{l}{\underline{i}} \equiv \frac{\binom{l}{\underline{b}}}{\binom{l}{\underline{b}+\underline{d}-(1,0,\dots,0)}} \frac{b_1 + d_1}{l+1} \binom{l+1}{\underline{i} + \underline{d}} \mod p^{m+1}$$

Proof. Rearranging the expression of the multinomial coefficient, we have

$$\binom{l}{\underline{i}} \equiv \frac{l!}{i_1!\cdots i_n!} \equiv \frac{(i_1+1)\cdots(i_1+1)}{[i_2\cdots(i_2+d_2+1)]\cdots[i_n\cdots(i_n+d_n+1)]}\binom{l}{\underline{i}+\underline{d}-(1,0,\dots)} \equiv$$
$$\equiv \frac{(b_1+1)\cdots(b_1+1)}{[b_2\cdots(b_2+d_2+1)]\cdots[b_n\cdots(b_n+d_n+1)]}\binom{l}{\underline{i}+\underline{d}-(1,0,\dots)} \equiv$$
$$\equiv \frac{\binom{l}{\underline{b}}}{\binom{l}{\underline{b}+\underline{d}-(1,0,\dots,0)}}\frac{b_1+d_1}{l+1}\binom{l+1}{\underline{i}+\underline{d}} \mod p^{m+1}$$

where we have taken into account that $\nu_p(b_1+d_1)=\nu_p(i_1+d_1)\geq\nu_p(l+1)$ by definition of d_1. □

Notice that in the previous proposition the choice of the index we raise is arbitrary.

Proposition 3.0.10. Let $\{\alpha_1,\dots,\alpha_n\}\in\mathbb{Z}^n_{\geq 1}$, $p>\max\{\alpha_i\}$, a level $e\geq 1$, and an integer $l\geq 0$. Then

$$F^e_* f^l = \sum_{|\underline{b}|\equiv l \mod p^e} g_l^{\underline{b}} F^e_*(x_1^{b_1\alpha_1\% p^e}\cdots x_n^{b_n\alpha_n\% p^e})$$

where

$$g_l^{\underline{b}} = \sum_{\underline{i}\in I_l^{\underline{b}}}\binom{l}{\underline{i}} c_1^{i_1}\cdots c_n^{i_n} x_1^{\left\lfloor\frac{i_1\alpha_1}{p^e}\right\rfloor}\cdots x_n^{\left\lfloor\frac{i_n\alpha_n}{p^e}\right\rfloor}$$

Proof. We begin with the expression of f^l viewed in the ring F^e_*R.

$$F^e_* f^l = \sum_{|\underline{i}|=l}\binom{l}{\underline{i}} c_1^{i_1}\cdots c_n^{i_n} x_1^{\left\lfloor\frac{i_1\alpha_1}{p^e}\right\rfloor}\cdots x_n^{\left\lfloor\frac{i_n\alpha_n}{p^e}\right\rfloor} F^e_*\left(x_1^{i_1\alpha_1\% p^e}\cdots x_n^{i_n\alpha_n\% p^e}\right)$$

Consider now all the possible vectors $\underline{b}\in\{0,\dots,p^e-1\}$ such that $|\underline{b}|\equiv l \mod p^e$. Observe that $i_j\alpha_j\% p^e = i'_j\alpha_j\% p^e \Leftrightarrow i_j\equiv i'_j \mod p^e$. Thus, we can arrange the sum as

$$F^e_* f^l = \sum_{\substack{\underline{b}\in\{0,\dots,p^e-1\}\\ |\underline{b}|\equiv l \mod p^e}}\sum_{\underline{i}\in I_l^{\underline{b}}}\binom{l}{\underline{i}} c_1^{i_1}\cdots c_n^{i_n} x_1^{\left\lfloor\frac{i_1\alpha_1}{p^e}\right\rfloor}\cdots x_n^{\left\lfloor\frac{i_n\alpha_n}{p^e}\right\rfloor} F^e_*\left(x_1^{b_1\alpha_1\% p^e}\cdots x_n^{b_n\alpha_n\% p^e}\right)$$

getting the result. □

Proposition 3.0.11. Fix a level $e\geq m+1$ and an integer $l\geq 0$ such that $l\neq\left\lfloor\frac{r_1p^e-1}{\alpha_1}\right\rfloor+\cdots+\left\lfloor\frac{r_np^e-1}{\alpha_n}\right\rfloor$ for any choice of $(r_1,\dots,r_n)\in\mathbb{Z}^n_{\geq 1}$ and $\nu_p(l+1)=0$. Let $\underline{b}\in\{0,\dots,p^e-1\}$ such that $|\underline{b}|\equiv l \mod p^e$. Then, there exists a constant $C\in\mathbb{Z}/p^{m+1}\mathbb{Z}$ such that

$$g_l^{\underline{b}} = C g_{l+1}^{\underline{b}'}$$

where $\underline{b}'=(b_1,\dots,b_j+1,\dots,b_n)$ for some index $1\leq j\leq n$.

Proof. Let $\underline{b} = \underline{i}\%p^e$. Since $\underline{i}$ satisfies the NCC by [Prop 3.0.3], then, we consider the index $1 \le j \le n$ such that $\underline{i}' = (i_1, \ldots, i_j + 1, \ldots, i_n)$. By [Prop 3.0.9], letting $\underline{d} = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is located in the index j, we have

$$\binom{l}{\underline{i}} \equiv \frac{b_j + 1}{l+1}\binom{l+1}{\underline{i}'} \quad \mod p^{m+1}$$

and by [Prop 3.0.8] we have a bijection $I_l^{\underline{b}} \longleftrightarrow I_{l+1}^{\underline{b}'}$. Then, we can write

$$\begin{aligned} g_l^{\underline{b}} =& \sum_{\underline{i} \in I_l^{\underline{b}}} \binom{l}{\underline{i}} c_1^{i_1} \cdots c_n^{i_n} x_1^{\left\lfloor \frac{i_1\alpha_1}{p^e}\right\rfloor} \cdots x_n^{\left\lfloor \frac{i_n\alpha_n}{p^e}\right\rfloor} = \\ =& c_j^{-1}\frac{b_j+1}{l+1} \sum_{\underline{i}' \in I_{l+1}^{\underline{b}'}} \binom{l+1}{\underline{i}'} c_1^{i_1} \cdots c^{i_j+1} \cdots c_n^{i_n} x_1^{\left\lfloor \frac{i_1\alpha_1}{p^e}\right\rfloor} \cdots x_n^{\left\lfloor \frac{i_n\alpha_n}{p^e}\right\rfloor} = c_j^{-1}\frac{b_j+1}{l+1} g_{l+1}^{\underline{b}'} \end{aligned}$$

concluding the proof, since c_j is a unit in R. □

Proposition 3.0.12. Fix a level $e \ge m+1$, fix $t = \operatorname{lcm}(\varphi(\alpha_1), \ldots, \varphi(\alpha_2))$ and an integer $l \ge 0$ such that $l \ne \left\lfloor \frac{r_1 p^e - 1}{\alpha_1} \right\rfloor + \cdots + \left\lfloor \frac{r_n p^e - 1}{\alpha_n} \right\rfloor$ for any choice of $(r_1, \ldots, r_n) \in \mathbb{Z}^n_{\ge 1}$ and $1 \le \nu_p(l+1) \le e - t(m+1)$. Let $\underline{b} \in \{0, \ldots, p^e - 1\}$ such that $|\underline{b}| \equiv l \mod p^e$. Then, either $g_l^{\underline{b}} = 0$ or there exists a vector $\underline{d} = (d_1, \ldots, d_n)$ such that $|\underline{d}| =$ an element $C \in \mathbb{Z}/p^{m+1}\mathbb{Z}[\underline{x}]$ such that

$$g_l^{\underline{b}} = C g_{l+1}^{\underline{b}+\underline{d}}$$

Proof. Fix $\underline{i} \in I_l^{\underline{b}}$. Since $\nu_p(l+1) \ge 1$, then we can write the base p expansion of the vectors $\underline{i}$ and $\underline{i} + \underline{d}$ where the bar indicates the position of the valuation (see [Not A.1.7]) as

$$\begin{array}{lcl} i_1 = [\cdots \mid i_{1\nu}]_p & & i_1 + d_1 = [\ldots \mid i_{1\nu} + d_{1\nu}]_p \\ i_2 = [\cdots \mid i_{2\nu}]_p & \xrightarrow{\underline{i}+\underline{d}} & i_2 + d_2 = [\ldots \mid i_{2\nu} + d_{2\nu}]_p \\ \vdots & & \vdots \\ i_n = [\cdots \mid i_{n\nu}]_p & & i_n + d_n = [\ldots \mid i_{n\nu} + d_{n\nu}]_p \\ \hline |\underline{i}| = [\cdots \mid (p-1)\cdots(p-1)]_p & & |\underline{i}+\underline{d}| = [\cdots \mid 0\cdots 0]_p \end{array}$$

Case 1: Suppose there exist an index j, take $j = 1$ without loss of generality, such that

$$\left\lfloor \frac{\alpha_1(i_1)}{p^e} \right\rfloor = \left\lfloor \frac{\alpha_i(i_1 + d_1)}{p^e} \right\rfloor \quad \text{where} \quad d_1 = \min\{d \ge 0 : \nu_p(i_1 + d_1) \ge \nu_p(l+1)\}$$

Then, we claim that there exist $d_2, \ldots, d_n \in \mathbb{Z}_{\le 0}$ such that $|\underline{d}| = 1$ such that $g_l^{\underline{b}} = C g_{l+1}^{\underline{b}+\underline{d}}$. To prove the claim, first observe that given d_1 defined as previously, one can choose $d_2, \ldots, d_n$ can be chosen such that $b_i + d_i \ge 0$ for all $2 \le i \le n$. In fact,

$$i_{1\nu} + i_{2\nu} + \cdots + i_{n\nu} \ge p^{\nu_p(l+1)} - 1 \quad \Rightarrow \quad i_{2\nu} + \cdots + i_{n\nu} \ge p^{\nu_p(l+1)} - 1 - i_{1\nu}$$

but, since $p^{\nu_p(l+1)} = i_{1\nu} + d_1 = i_{1\nu} + 1 - d_2 - \cdots - d_n$, then finally

$$i_{2\nu} + \cdots + i_{n\nu} \geq -d_2 - \cdots - d_n \quad \Rightarrow \quad (i_{2\nu} + d_2) + \cdots + (i_{n\nu} + d_n) \geq 0$$

and one can choose such d_i.

Next, we see that, by [Prop 3.0.9], we have the following identity

$$\binom{l}{\underline{i}} \equiv \frac{\binom{l}{\underline{b}}}{\binom{l}{\underline{b}+\underline{d}-(1,0,\ldots,0)}} \frac{b_1+d_1}{l+1} \binom{l+1}{\underline{i}+\underline{d}} \mod p^{m+1}$$

The resulting sets $\underline{i}$ and $\underline{i}+\underline{d}$ can be written as follows, where the bar indicates the position of the valuation

$$\begin{array}{lcl}
i_1 = [\cdots \mid i_{1\nu}]_p & & i_1 + d_1 = [\cdots \mid 0 \cdots 0]_p \\
i_2 = [\cdots \mid i_{2\nu}]_p & \Rightarrow & i_2 + d_2 = [\cdots \mid i_{2\nu} + d_2]_p \\
\quad\vdots & & \vdots \\
i_n = [\cdots \mid i_{n\nu}]_p & & i_n + d_n = [\cdots \mid i_{n\nu} + d_n]_p \\
\hline
|\underline{i}| = [\cdots \mid (p-1)\cdots(p-1)]_p & & |\underline{i}+\underline{d}| = [\cdots \mid 0 \cdots 0]_p
\end{array}$$

But now we can assure that each sum $i_k + d + k$ for $2 \leq k \leq n$ only affects the part after the valuation bar $\underline{i} + \underline{d}$ after the valuation bar.

By [Prop 3.0.8], there exist a bijection between the sets $I_l^{\underline{b}}$ and $I_{l+1}^{\underline{b}+\underline{d}}$, so we have

$$\begin{aligned}
g_l^{\underline{b}} = &\sum_{\underline{i} \in I_l^{\underline{b}}} \binom{l}{\underline{i}} c_1^{i_1} \cdots c_n^{i_n} x_1^{\left\lfloor \frac{i_1\alpha_1}{p^e} \right\rfloor} \cdots x_n^{\left\lfloor \frac{i_n\alpha_n}{p^e} \right\rfloor} = c_1^{-d_1} \cdots c_n^{-d_n} x_2^{\left\lfloor \frac{b_2\alpha_2}{p^e} \right\rfloor - \left\lfloor \frac{(b_2+d_2)\alpha_2}{p^e} \right\rfloor} \cdots \\
&\cdots x_n^{\left\lfloor \frac{b_n\alpha_n}{p^e} \right\rfloor - \left\lfloor \frac{(b_n+d_n)\alpha_n}{p^e} \right\rfloor} \sum_{\underline{j} \in I_{l+1}^{\underline{b}+\underline{d}}} \binom{l+1}{\underline{j}} c_1^{j_1} \cdots c_n^{j_n} x_1^{\left\lfloor \frac{j_1\alpha_1}{p^e} \right\rfloor} \cdots x_n^{\left\lfloor \frac{j_n\alpha_n}{p^e} \right\rfloor} = \\
=&c_1^{-d_1} \cdots c_n^{-d_n} x_2^{\left\lfloor \frac{b_2\alpha_2}{p^e} \right\rfloor - \left\lfloor \frac{(b_2+d_2)\alpha_2}{p^e} \right\rfloor} \cdots x_n^{\left\lfloor \frac{b_n\alpha_n}{p^e} \right\rfloor - \left\lfloor \frac{(b_n+d_n)\alpha_n}{p^e} \right\rfloor} g_{l+1}^{\underline{b}+\underline{d}} = C g_{l+1}^{\underline{b}+\underline{d}}
\end{aligned}$$

<u>Case 2:</u> Suppose that for all the indices $1 \leq j \leq n$ we have that

$$\left\lfloor \frac{\alpha_j(i_j)}{p^e} \right\rfloor < \left\lfloor \frac{\alpha_i(i_j + d_j)}{p^e} \right\rfloor$$

Then, the expression of the sets $\underline{i}$ and $\underline{i}+\underline{d}$ can be written as follows, where the bar indicates the position of the valuation

$$\begin{cases} i_1 = [\ldots u_1 \ldots u_1 u_{1h} \mid i_{1\nu}]_p \\ i_2 = [\ldots u_2 \ldots u_2 u_{2h} \mid i_{2\nu}]_p \\ \vdots \\ i_n = [\ldots u_n \ldots u_n u_{nh} \mid i_{n\nu}]_p \end{cases} \quad \text{such that} \quad \begin{cases} i_{1\nu} \leq [u_{1t} u_1 \ldots u_1]_p \\ i_{2\nu} \leq [u_{2t} u_2 \ldots u_2]_p \\ \vdots \\ i_{n\nu} \leq [u_{nt} u_n \ldots u_n]_p \end{cases}$$

where u_i is a periodic chunk of digits in base p of length t of some expression of the form $\left\lfloor \frac{r_i p^e - 1}{\alpha_i} \right\rfloor$ for $r_i \in \mathbb{Z}_{\geq 1}$.

Subsubcase 2.1: $\operatorname{carry}([u_1]_p, [u_2]_p, \ldots, [u_n]_p) \geq 1$. Then, clearly $\operatorname{carry}(i_1, \ldots, i_n) \geq m+1$, since $e - \nu(l+1) \geq t(m+1)$ assures to have at least $m+1$ chunks of periods. Hence, the binomial coefficient $\binom{l}{\underline{i}}$ is zero and $g_l^{\underline{b}} = 0$.

Subsubcase 2.2: $\operatorname{carry}([u_1]_p, [u_2]_p, \ldots, [u_n]_p) = 0$. Then, summing the inequalities we have

$$p^{\nu(l+1)} - 1 \leq i_{1\nu} + \cdots + i_{n\nu} \leq [u_{1t} u_1 \ldots u_1]_p + \cdots + [u_{nt} u_n \ldots u_n]_p \leq p^{\nu_p(l+1)} - 1$$

Thus, we have equalities in all the inequalities, meaning that

$$i_j = [u_{jt} u_{jh} \ldots u_j]_p = \left\lfloor \frac{r_j p^e - 1}{\alpha_j} \right\rfloor \qquad \text{for some } r_j \in \mathbb{Z}_{\geq 1}$$

contradicting the assumption that $l \neq \left\lfloor \frac{r_1 p^e - 1}{\alpha_1} \right\rfloor + \cdots + \left\lfloor \frac{r_n p^e - 1}{\alpha_n} \right\rfloor$. □

Proposition 3.0.13. Let $p > \max\{\alpha_1, \ldots, \alpha_n\}$, $f = x_1^{\alpha_1} + \cdots + x_n^{\alpha_n} \in \mathbb{Z}/p^{m+1}\mathbb{Z}[\underline{x}]$ with c_i units. Let $t = \operatorname{lcm}(\varphi(\alpha_1), \ldots, \varphi(\alpha_n))$. Then, if $l \neq \lfloor \frac{r_1 p^e - 1}{\alpha_1} \rfloor + \cdots + \lfloor \frac{r_n p^e - 1}{\alpha_n} \rfloor$ for any choice of $(r_1, \ldots, r_n) \in \mathbb{Z}^n_{\geq 1}$ and $\nu_p(l+1) \leq e - t(m+1)$, then $l \notin \nu_f^{\bullet}(p^e)$.

Proof. If $\nu_p(l+1) = 0$, then by [Prop 3.0.11] we have that for all choice of $\underline{b}$ there exist a $C_{\underline{b}}$ such that

$$g_l^{\underline{b}} = C_{\underline{b}} g_{l+1}^{\underline{b}'}$$

since $g_l^{\underline{b}}$ generates $\mathcal{C}_R^e \cdot f^l$, then $\mathcal{C}_R^e \cdot f^l \subseteq \mathcal{C}_R^e \cdot f^l$, meaning that there is no jump at l.

If $\nu_p(l+1) \geq 1$, then by [Prop 3.0.12] we have that for all choice of $\underline{b}$ there exist a $C_{\underline{b}}$ such that

$$g_l^{\underline{b}} = C_{\underline{b}} g_{l+1}^{\underline{b}+\underline{d}}$$

where $|\underline{d}| = 1$. Since $g_l^{\underline{b}}$ generates $\mathcal{C}_R^e \cdot f^l$, then $\mathcal{C}_R^e \cdot f^l \subseteq \mathcal{C}_R^e \cdot f^l$, meaning that there is no jump at l. □

Proposition 3.0.14. Let $p > \max\{\alpha_1, \ldots, \alpha_n\}$, $f = c_1 x_1^{\alpha_1} + \cdots + c_n x_n^{\alpha_n} \in \mathbb{Z}/p^{m+1}\mathbb{Z}[\underline{x}]$ with c_i units. Let $t = \operatorname{lcm}(\varphi(\alpha_1), \ldots, \varphi(\alpha_n))$. Let $e \geq t(m+1)$. Then, if $l = \lfloor \frac{r_1 p^e - 1}{\alpha_1} \rfloor + \cdots + \lfloor \frac{r_n p^e - 1}{\alpha_n} \rfloor$ for some choice of $(r_1, \ldots, r_n) \in \mathbb{Z}^n_{\geq 1}$ and $\nu_p(l+1) \leq e - t(m+1)$, then

$$l \in \nu_f^{\bullet}(p^e) \quad \Leftrightarrow \quad \operatorname{carry}\left(\left\lfloor \frac{r_1 p^e - 1}{\alpha_1} \right\rfloor, \ldots, \left\lfloor \frac{r_n p^e - 1}{\alpha_n} \right\rfloor \right) = 0$$

Proof. We will work with the p-adic expression $\left\lfloor \frac{r_i p^e - 1}{\alpha_i} \right\rfloor = [\ldots u_i \ldots u_i]_p$, where u_i is a chunk of period t. Notice that

$$\operatorname{carry}\left(\left\lfloor \frac{r_1 p^e - 1}{\alpha_1} \right\rfloor, \ldots, \left\lfloor \frac{r_n p^e - 1}{\alpha_n} \right\rfloor \right) = 0 \quad \Leftrightarrow \quad \operatorname{carry}([u_1]_p, [u_2]_p, \ldots, [u_n]_p) = 0$$

Case 1: If carry $([u_1]_p, [u_2]_p, \ldots, [u_n]_p) \geq 1$, by [Prop 3.0.12] we have linear relations between the elements of $\mathcal{C}_R^e \cdot f^l$ and $\mathcal{C}_R^e \cdot f^{l+1}$ except, perhaps, for the indices

$$\begin{cases} i_1' = [\ldots u_1 \ldots u_1]_p \\ i_2' = [\ldots u_2 \ldots u_2]_p \\ \vdots \\ i_n' = [\ldots u_n \ldots u_n]_p \end{cases}$$

But clearly, the sum of them carries $\geq m+1$, since $e \geq t(m+1)$ assures to have at least $m+1$ chunks of periods. Hence, the binomial coefficient $\binom{l}{\underline{i}}$ is zero and the term does not contribute to $\mathcal{C}_R^e \cdot f^l$. Thus, $\mathcal{C}_R^e \cdot f^l = \mathcal{C}_R^e \cdot f^{l+1}$ and $l \notin \nu_f^{\bullet}(p^e)$.

Case 2: If carry$([u_1]_p, [u_2]_p, \ldots, [u_n]_p) = 0$, then notice that it is enough to study the reduction $f_p \in \mathbb{Z}/p\mathbb{Z}[\underline{x}]$ for $l < p^e$, since by [Prop 2.2.14], we have that the ν-invariants are periodic with period p^e. When we perform the reduction, none of the coefficients vanish, since they are units. We denote the reduction modulo p to the polynomial f as $f_p = c_1 x^{\alpha_1} + \ldots + c_n x^{\alpha_n}$, where by abuse of notation the coefficients c_i are really the reduction mod p of the honest c_1. Then, we may assume that $1 \leq r_i \leq \alpha_i - 1$.

We claim that $g_l^{\underline{i}} \notin \mathcal{C}_R^e \cdot f^{l+1}$ where

$$\underline{i} = \left(\left\lfloor \frac{r_1 p^e - 1}{\alpha_1} \right\rfloor, \ldots, \left\lfloor \frac{r_n p^e - 1}{\alpha_n} \right\rfloor \right)$$

To see this recall that, assuming $l \neq p^e - 1$, since that is a trivial ν-invariant, we have

$$g_l^{\underline{i}} = \binom{l}{\underline{i}} c_1^{i_1} \cdots c_n^{i_n} x_1^{\lfloor \frac{i_1 \alpha_1}{p^e} \rfloor} \cdots x_n^{\lfloor \frac{i_n \alpha_n}{p^e} \rfloor} = \binom{l}{\underline{i}} c_1^{i_1} \cdots c_n^{i_n} x_1^{r_1} \cdots x_n^{r_n}$$
$$g_{l+1}^{\underline{j}} = \binom{l+1}{\underline{j}} c_1^{j_1} \cdots c_n^{j_n} x_1^{\lfloor \frac{(j_1+1)\alpha_1}{p^e} \rfloor} \cdots x_n^{\lfloor \frac{j_n \alpha_n}{p^e} \rfloor}$$

since $l < p^e$ and the sum only has one term.

First, notice that p does not divide $\binom{l}{\underline{i}}$ by [Thm A.2.3], since by assumption $i_1, \ldots, i_n$ do not carry in base p, so $g_l^{\underline{i}}$ does not vanish. In order to have the condition $g_l^{\underline{i}} \in (g_{l+1}^{\bullet})$, there should exist an element $g_{l+1}^{\underline{j}}$ such that the degree of each variable is less than or equal to the degrees of $g_l^{\underline{i}}$. This is

$$\left\lfloor \frac{j_1 \alpha_1}{p^e} \right\rfloor \leq \left\lfloor \frac{i_1 \alpha_1}{p^e} \right\rfloor, \quad \ldots, \quad \left\lfloor \frac{j_n \alpha_n}{p^e} \right\rfloor \leq \left\lfloor \frac{i_n \alpha_n}{p^e} \right\rfloor \qquad \text{such that } |\underline{j}| = |\underline{i}| + 1$$

Then, there should be at least an index of $\underline{j}$ (assume j_1 without loss of generality) such that $j_1 > i_1$, meaning

$$\left\lfloor \frac{j_1 \alpha_1}{p^e} \right\rfloor \geq \left\lfloor \frac{(i_1+1)\alpha_1}{p^e} \right\rfloor = \left\lfloor \frac{\left\lfloor \frac{r_1 p^e - 1}{\alpha_1} \alpha_1 + \alpha_1 \right\rfloor}{p^e} \right\rfloor > \left\lfloor \frac{i_1 \alpha_1}{p^e} \right\rfloor$$

contradicting the hypothesis. Hence, the element $g_l^{\underline{i}}$ does not belong to the ideal $\mathcal{C}_R^e \cdot f^{l+1}$ generated by $g_{l+1}^{\bullet}$, and thus $l \in \nu_f^{\bullet}(p^e)$. Furthermore, by degree comparison, there is not an integer $1 \leq t \leq m$ such that $p^t g_l^{\underline{i}}$, so the strength of the ν-invariant is $m+1$. $\square$

Proposition 3.0.15. Let $p > \max\{\alpha_1, \dots, \alpha_n\}$ and $f = c_1 x_1^{\alpha_1} + \cdots + c_n x_n^{\alpha_n} \in \mathbb{Z}/p^{m+1}\mathbb{Z}[\underline{x}]$ for c_i units. Then, the Bernstein-Sato roots are given by

$$\mathrm{BSR}(f) = \left\{ -1, -\frac{k_1}{\alpha_1} - \cdots - \frac{k_n}{\alpha_n} : \text{carry}\left(\left[\frac{k_1}{\alpha_1}\right]_p, \dots, \left[\frac{k_n}{\alpha_n}\right]_p \right) = 0 \right\}$$

for $1 \le k_i \le \alpha_i - 1$. Furthermore, all the roots have strength $m+1$.

Proof. First, by [Prop 3.0.14], we have computed for each level $e \le t(m+1)$ all the ν-invariants l such that $\nu_p(l+1) \le e - t(m+1)$. All the possible ν-invariants whose valuation is bigger will be of the form

$$l = kp^{e-t(m+1)} - 1 \quad \text{for } k \in \mathbb{Z}_{\ge 0}$$

Then, each sequence of ν-invariants of this kind converges p-adically to -1. Such sequence exists, for instance, considering the ν-invariant $l = p^{e+m} - 1$, which is always a ν-invariant with strength $m+1$ by [Prop 2.4.14].

For the other ν-invariants contemplated in [Prop 3.0.14], we notice that if $1 \le k_i \le \alpha_i$, then

$$\text{carry}\left(\left[\left\lfloor \frac{k_1 p^e - 1}{\alpha_1} \right\rfloor\right]_p, \dots, \left[\left\lfloor \frac{k_n p^e - 1}{\alpha_n} \right\rfloor\right]_p \right) = \text{carry}\left(\left[\frac{k_1}{\alpha_1}\right]_p, \dots, \left[\frac{k_n}{\alpha_n}\right]_p \right)$$

Denote as S the set of the indices $(k_1, \dots, k_n)$ such that that sum does not carry. We claim that S is closed under multiplication by $(p\%\alpha_1, \dots, p\%\alpha_n)$. This is

$$(k_1, \dots, k_n) \in S \quad \Longrightarrow \quad (k_1 p\%\alpha_1, \dots, k_n p\%\alpha_n) \in S$$

This is a consequence of the fact that multiplying by p just shifts to the left the periodic expressions in base p of the numbers $\left\lfloor \frac{k_i}{\alpha_i} \right\rfloor$, and it does not affect the carries.

Fix now a index set $(k_1, \dots, k_n)$ such that the expression do not carry. Fix the common period $t = \mathrm{lcm}(\varphi(\alpha_1), \dots, \varphi(\alpha_n))$. Then, at level ct for an integer c we have the following associated ν-invariants

$$l_{ct} = \frac{k_1 p^{ct} - k_1}{\alpha_1} + \cdots + \frac{k_n p^{ct} - k_n}{\alpha_n} = \left(\frac{k_1}{\alpha_1} + \cdots + \frac{k_n}{\alpha_n} \right) p^{tc} - \left(\frac{k_1}{\alpha_1} + \cdots + \frac{k_n}{\alpha_n} \right)$$

since $p^t \equiv 1 \mod \alpha_i$. Thus, the subsequence $\{l_{ct}\}_{c \ge 0}$ converges p-adically to the limit

$$-\left(\frac{k_1}{\alpha_1} + \cdots + \frac{k_n}{\alpha_n} \right) \in \mathrm{BSR}(f)$$

We now prove that if a subsequence converges p-adically , it must do it to an expression of this form for some $(k_1, \dots, k_n) \in S$. The expressions of the ν-invariants of level i for some integer $i \ge 0$ are

$$\frac{k_1 p^i - k_1 p^i\%\alpha_1}{\alpha_1} + \cdots + \frac{k_n p^i - k_n p^i\%\alpha_n}{\alpha_n} = \left(\frac{k_1}{\alpha_1} + \cdots + \frac{k_n}{\alpha_n} \right) p^i - \left(\frac{k_1 p^i\%\alpha_1}{\alpha_1} + \cdots + \frac{k_n p^i\%\alpha_n}{\alpha_n} \right)$$

However, since we have proven that S is stable under multiplication by $(p\%\alpha_1, \dots, p\%\alpha_n)$, then there exist a choice $(k_1', \dots, k_n') \in S$ such that

$$(k_1', \dots, k_n') = (k_1 p^i \% \alpha_1, \dots, k_n p^i \% \alpha_n)$$

completing the proof of the roots. For their strength, notice that all the ν-invariants have strength $m+1$, so the roots must have strength $m+1$ as well. $\square$

Example 3.0.16. Let $f = x^5 + y^7 \in \mathbb{Z}/11^{m+1}\mathbb{Z}[x, y]$. Then the Bernstein-Sato roots are

$$\mathrm{BSR}(f) = \left\{ -1, -\frac{34}{35}, -\frac{27}{35}, -\frac{24}{35}, -\frac{19}{35}, -\frac{17}{35}, -\frac{12}{35} \right\}$$

Proof. First we observe that the period in all the expressions $\left[\frac{k_1}{5}\right]_p$ is $t_1 = 1$, whereas the period in all the expressions $\left[\frac{k_2}{7}\right]_p$ is $t_2 = 3$. Thus, we set $t = 3$.

Following [Prop 3.0.15], we compute the general expressions of $\left[\frac{k_i}{\alpha_i}\right]$ in base p and select the pairs that do not carry in base 11. The relevant ones are

$$\begin{aligned} &\frac{1}{5} = [222]_p, \quad \frac{2}{5} = [444]_p \\ &\frac{1}{7} = [163]_p, \quad \frac{2}{7} = [316]_p, \quad \frac{4}{7} = [631]_p \end{aligned}$$

giving the following combinations (k_1, k_2) such that the expressions do not carry

$$(1,1), \quad (1,2), \quad (1,4), \quad (2,1), \quad (2,2), \quad (2,4)$$

Thus, the set of Bernstein-Sato roots follows. $\square$

Proposition 3.0.17. Let $f = c_1 x_1^{\alpha_1} + \cdots + c_n x_n^{\alpha_n} \in \mathbb{Z}[\underline{x}]$. Let $q \gg 0$ and $p \gg 0$ such that $p \equiv 1 \mod \alpha_1 \cdots \alpha_n$ and define the polynomials f_q and f_p as the reduction

$$f_q = c_1 x_1^{\alpha_1} + \cdots + c_n x_n^{\alpha_n} \in \mathbb{Z}/q^{m+1}\mathbb{Z}[\underline{x}] \quad \text{and} \quad f_p = c_1 x_1^{\alpha_1} + \cdots + c_n x_n^{\alpha_n} \in \mathbb{Z}/p^{m+1}\mathbb{Z}[\underline{x}]$$

Then, we have

$$\mathrm{BSR}(f_q) \subseteq \mathrm{BSR}(f_p) \subseteq \mathrm{BSR}(f)$$

Proof. We focus first on the first inclusion. Let $(k_1, \dots, k_n)$ with $1 \le k_i \le \alpha_i - 1$. Following [Prop 3.0.15] suppose $-\frac{k_1}{\alpha_1} - \cdots - \frac{k_n}{\alpha_n}$ is a Bernstein-Sato root of f_q. Then

$$\mathrm{carry}\left(\left[\frac{k_1}{\alpha_1}\right]_q, \dots, \left[\frac{k_n}{\alpha_n}\right]_q \right) = 0 \quad \Rightarrow \quad \mathrm{carry}\left(\left\lfloor \frac{q^e k_1}{\alpha_1} \right\rfloor + \cdots + \left\lfloor \frac{q^e k_n}{\alpha_n} \right\rfloor \right) = 0 \ \forall e \ge 0$$

Choose $e = \mathrm{lcm}(\varphi(\alpha_1), \dots, \varphi(\alpha_n))$. Then $q^e \equiv p^e \equiv 1 \mod \alpha_i$ and in particular

$$\mathrm{carry}\left(\left\lfloor \frac{k_1}{\alpha_1} \right\rfloor, \dots, \left\lfloor \frac{k_n}{\alpha_n} \right\rfloor \right) = 0$$

Therefore, since $p^e \equiv 1 \mod \alpha_i$ for all $e \geq 0$, then

$$\text{carry}\left(\left\lfloor \frac{k_1 p^e}{\alpha_1} \right\rfloor, \dots, \left\lfloor \frac{k_n p^e}{\alpha_n} \right\rfloor\right) = 0 \quad \forall e \geq 0 \quad \Rightarrow \quad \text{carry}\left(\left[\frac{k_1}{\alpha_1}\right]_p, \dots, \left[\frac{k_n}{\alpha_n}\right]_p\right) = 0$$

which means that $-\frac{k_1}{\alpha_1} - \cdots - \frac{k_n}{\alpha_n}$ is a Bernstein-Sato root of f_p.

For the second inclusion, we have that by [ÀMJNB21, Theorem 4.4] the roots of the Bernstein-Sato polynomial $b_f(s)$ are

$$\mathrm{BSR}(f) = \left\{1, -\frac{k_1}{\alpha_1} - \cdots - \frac{k_n}{\alpha_n} : 1 \leq k_i \leq \alpha_i - 1\right\}$$

This is, the same as the ones for characteristic p, but dropping the carry condition. Thus, we trivially have the second inclusion. □

Notice that, in the case of a prime p such that $p \equiv 1 \mod \alpha_i$, the Bernstein-Sato roots are the roots of the Bernstein-Sato polynomial $b_f(s)$ contained in the interval $[-1, 0)$.

Example 3.0.18. Let $f = x^5 + y^7 \in \mathbb{Z}[x, y]$. Let $p = 71$ and $q = 11$. We compute the roots of $b_f(s)$, and we remark the Bernstein-Sato roots of f_p and f_q in blue and red respectively.

$$-\mathrm{BSR}(f) = \left\{1, \frac{12}{35}, \frac{17}{35}, \frac{19}{35}, \frac{22}{35}, \frac{24}{35}, \frac{26}{35}, \frac{27}{35}, \frac{29}{35}, \frac{31}{35}, \frac{32}{35}, \frac{33}{35}, \frac{34}{35}, \frac{36}{35}, \frac{37}{35}, \frac{38}{35}, \frac{39}{35}, \frac{41}{35}, \frac{43}{35}, \frac{44}{35}, \frac{46}{35}, \frac{48}{35}, \frac{51}{35}, \frac{53}{35}, \frac{58}{35}\right\}$$

The larger the number of variables and the smaller the exponents, the less quantity of Bernstein-Sato roots one will find because of the carry condition. In the following example, for a given prime there are not nontrivial Bernstein-Sato roots, for the optimal prime there are only two, while there are up to 24 nontrivial roots of $b_f(s)$.

Example 3.0.19. Let $f = x^3 + y^4 + z^5 \in \mathbb{Z}[x, y, z]$. Let $p = 61$ and $q = 11$. We compute the roots of $b_f(s)$, and we remark the Bernstein-Sato roots of f_p and f_q in blue and red respectively.

$$-\mathrm{BSR}(f) = \left\{1, \frac{47}{60}, \frac{59}{60}, \frac{62}{60}, \frac{67}{60}, \frac{71}{60}, \frac{74}{60}, \frac{77}{60}, \frac{79}{60}, \frac{82}{60}, \frac{83}{60}, \frac{86}{60}, \frac{89}{60}, \frac{91}{60}, \frac{94}{60}, \frac{97}{60}, \frac{98}{60}, \frac{101}{60}, \frac{103}{60}, \frac{106}{60}, \frac{109}{60}, \frac{113}{60}, \frac{118}{60}, \frac{121}{60}, \frac{133}{60}\right\}$$

Finally, we illustrate some examples where we drop the hypothesis of the coefficients being units. We see that this hypothesis is necessary for the expected behavior of the Bernstein-Sato roots. In the following example, we have that the Bernstein-Sato roots are not contained in the interval $[-1, 0)$.

Example 3.0.20. Let $f = x^2 + py^3 \in \mathbb{Z}/p^2\mathbb{Z}[x,y]$. Then, the ν-invariants are

$$\nu_f^\bullet(p^e) = \left\{\frac{p^e-1}{2}, \frac{p^e+1}{2}, p^e-1\right\} + p^e\mathbb{Z}$$

and the strengths of the nontrivial ones is 1. Thus, the Bernstein-Sato roots are

$$\mathrm{BSR}(f) = \left\{-1, -\frac{1}{2}, \frac{1}{2}\right\}$$

and the strength of the nontrivial ones is 1.

Proof. Notice that when raising to a power f^l, we have

$$f^l = x^{2l} + lpx^{2(l-1)}y^3$$

Thus, we can easily compute the ν-invariants via [Prop 2.2.11]. Let $k \geq 0$ be an integer, then

- <u>If $l = kp^e$:</u>

$$F_*^e l = x^{2k} \cdot F_*^e 1 + kp^{e+1}x^{2k-1} \cdot F_*^e x^{p^e-1}y^3 \Rightarrow \mathcal{C}_R^e \cdot f^l = (x^{2k})$$

- <u>If $kp^e + 1 \leq l \leq kp^e + \frac{p^e-1}{2}$:</u>

$$F_*^e l = x^{2k} \cdot F_*^e x^{2(l-kp^e)} + lpx^{2k} \cdot F_*^e x^{2(l-1-kp^e)}y^3 \Rightarrow \mathcal{C}_R^e \cdot f^l = (x^{2k})$$

- <u>If $l = kp^e + \frac{p^e+1}{2}$:</u>

$$F_*^e l = x^{2k+1} \cdot F_*^e x + p(kp^e + \frac{p^e+1}{2})x^{2k} \cdot F_*^e x^{p^e-1}y^3 \Rightarrow \mathcal{C}_R^e \cdot f^l = (px^{2k}, x^{2k+1})$$

- <u>If $kp^e + \frac{p^e+3}{2} \leq l \leq (k+1)p^e - 1$:</u>

$$F_*^e l = x^{2k+1} \cdot F_*^e x^{2(l-kp^e-\frac{p^e-3}{2})} + lpx^{2k+1} \cdot F_*^e x^{2(l-1-kp^e-\frac{p^3+1}{2})}y^3 \Rightarrow \mathcal{C}_R^e \cdot f^l = (x^{2k+1})$$

Then the set of ν-invariants of level e is

$$\nu_f^\bullet(p^e) = \left\{\frac{p^e-1}{2}, \frac{p^e+1}{2}, p^e-1\right\} + p^e\mathbb{Z}$$

and whose strengths are

$$\mathrm{str}\left(\frac{p^e-1}{2}, e, f\right) = 1, \quad \mathrm{str}\left(\frac{p^e+1}{2}, e, f\right) = 1, \quad \mathrm{str}(p^e-1, e, f) = 2$$

This can be visualized in the following diagram

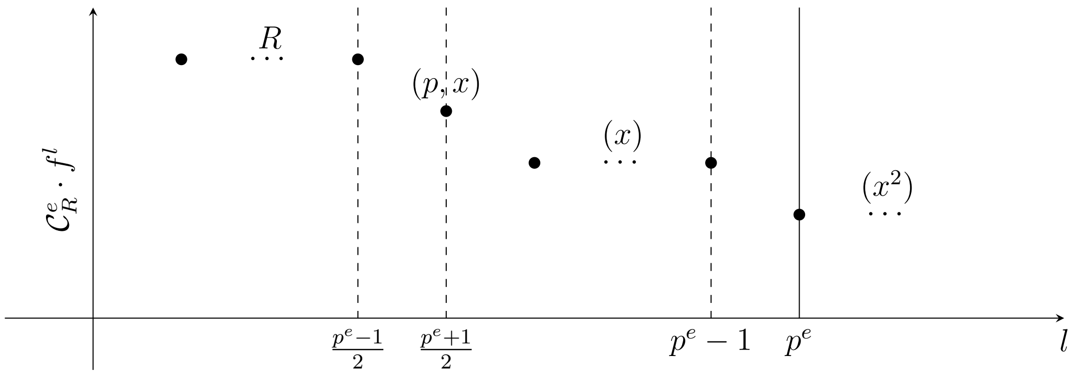


Figure 3.1: The ν-invariants of $f = x^2 + py^3$

where these jumps are periodic with period p^e. Thus, performing the limits, one concludes that the Bernstein-Sato roots are

$$\mathrm{BSR}(f) = \left\{-\frac{1}{2}, \frac{1}{2}, -1\right\}$$

and whose strengths are

$$\mathrm{str}\left(-\frac{1}{2}, f\right) = 1, \quad \mathrm{str}\left(\frac{1}{2}, f\right) = 1, \quad \mathrm{str}\left(-1, f\right) = 2$$

$\square$

Appendix A

Background on commutative algebra

A.1 Basics on p-adic numbers and base p expansions

We introduce some useful definitions and theorems of p-adic numbers that will be used throughout the thesis.

Definition A.1.1 (Ring of p-adic integers)**.** Fixing a prime p, we define the ring of p-adic integers $\mathbb{Z}_p := \hat{\mathbb{Z}}_{(p)}$ as the (p)-adic completion of $\mathbb{Z}_{(p)}$. One can express every element $\alpha \in \mathbb{Z}_p$ as a sequence of integers $\{\alpha_e\}_{e\geq 0}$ such that

$$\alpha = \sum_{i=0}^{\infty} \alpha_i p^i \quad \text{such that} \quad 0 \leq \alpha_i < p$$

We call α_e the e-th digit of α.

Definition A.1.2 (Truncation)**.** Let $\alpha \in \mathbb{Z}_p$. Then, the e-th truncation of α is defined as

$$\alpha_{<e} = \alpha_0 + \alpha_1 p + \cdots + \alpha_{e-1}p^{e-1}$$

Definition A.1.3 (Rational p-adic integer)**.** Let $\alpha \in \mathbb{Z}_p$. We say that α is a rational p-adic integer if its p-adic expansion is eventually periodic. That is, if there exists some $N \geq 1$ such that $\{\alpha_i\}_{i\geq N}$ is periodic.

Definition A.1.4 (p-adic limit)**.** Let $\{a_i\}_{i\geq 0}$ be a sequence of integers. Then $\{a_i\}_{i\geq 0}$ converges p-adically to $\alpha \in \mathbb{Z}_p$ if for every $m \in \mathbb{Z}_{\geq 0}$ there exists some $N \in \mathbb{Z}_{\geq 0}$ such that

$$a_i \equiv \alpha \mod p^m\mathbb{Z}_p \text{ for all } i \geq N$$

Proposition A.1.5. Let $\alpha, \beta \in \mathbb{Z}_p$. Then we have

$$\alpha \neq \beta \Leftrightarrow \exists a \geq 0 \text{ such that } (\alpha + p^a\mathbb{Z}_p) \cap (\beta + p^a\mathbb{Z}_p) = \emptyset$$

Proof. The set $(\alpha+p^a\mathbb{Z}_p)$ represents all the p-adic integers α' such that $\alpha'_{<a} = \alpha_{<a}$. Thus, given two p-adics $\alpha, \beta \in \mathbb{Z}_p$, we have $\alpha \neq \beta$ if and only if there exist some $a \geq 0$ such that $\alpha_{a-1} \neq \beta_{a-1}$. But this last condition is equivalent to $\alpha_{<a} \neq \beta_{<b}$, meaning the intersection is clearly the empty set. □

Throughout the thesis, we will perform some computations in base p expansions. Here, we fix the notation we are going to work with.

Notation A.1.6 (Base p expansions)**.** Let $a \in \mathbb{Z}$. We express a in base p as

$$a = [a_n, a_{n-1}, \dots, a_0]_p = a_n p^n + a_{n-1} p^{n-1} + \dots + a_0$$

where $a_i \in \{0, 1, \dots, p-1\}$ are the digits of a in base p.

Notation A.1.7 (Valuation bar)**.** Let $a \in \mathbb{Z}$. Sometimes we want to focus on the behavior of the base p expansion of a in the last k digits. Then, fixing a k, we express the base p expansion of a as

$$a = [a_n, a_{n-1} \dots a_k \mid a_{k-1} \dots a_0]_p$$

for visual purposes.

Notation A.1.8 (Head and tail of a periodic chunk)**.** We often need to deal with base p expansions that are periodic. If we want to focus on the last digits, the valuation bar may brake one of the periodic chunks. Then, we express the tail and the head of the periodic chunk as

$$a = [\dots uu \dots u]_p = [\dots uu_h \mid u_t u \dots u]_p$$

where u is the periodic chunk, u_h is the head of the periodic chunk and u_t is the tail of the periodic chunk broken by the valuation bar.

A.2 Some theorems and propositions

We introduce some useful definitions and theorems of commutative algebra and algebraic geometry that will be used throughout the thesis.

Definition A.2.1 (Krull dimension)**.** The Krull dimension of the ring R is defined as

$$\dim(R) := \sup\{n : \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq p_n\}$$

with $\mathfrak{p}_i \in \mathrm{Spec}(R)$

Theorem A.2.2 (Krull's intersection)**.** Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then

$$\bigcup_{n\geq 1} \mathfrak{m}^n = (0)$$

Theorem A.2.3 (Kummer)**.** Let $\nu_p(a)$ be the p-adic valuation of $a \in \mathbb{Z}$ (i.e. the highest exponent of p that divides a) and $S_p(b)$ the sum of the digits of b in base p. Then

$$\nu_p\left(\binom{i+j}{i}\right) = \frac{S_p(i) + S_p(j) - S_p(i+j)}{p-1}$$

that is, the number of *carries* of the sum $i + j$ in base p.

Proposition A.2.4. Let R be a Noetherian ring and M be an R-module. Then the following are equivalent:

1. M is finitely generated
2. M is finitely presented

Proof. See [AK13, Theorem 16.15] □

Proposition A.2.5. Let R be a regular ring and M be an R-module. Then the following are equivalent:

1. M is projective and finitely generated
2. M is flat and finitely presented

Proof. See [AK13, Theorem 13.15] □

Proposition A.2.6. Let R be a local ring and M a finitely generated R-module. Then we have

$$M \text{ is free} \quad \Leftrightarrow \quad M \text{ is projective} \quad \Leftrightarrow \quad M \text{ is flat}$$

Proof. See [Mat70, Proposition 3.1] □

Proposition A.2.7. Let $(R, \mathfrak{m})$ be a local ring. Then the maximal ideal $\mathfrak{m}$ is the set of non-units of R.

Proof. Follows directly from the fact that every non-unit element is contained in a maximal ideal. Since the maximal ideal is unique in a local ring, the result follows. □

Proposition A.2.8. Let R be a commutative ring and $I \subseteq R$ a radical ideal. Then R/I is a reduced ring, that is, it does not have nonzero nilpotent elements.

Proof. Suppose that there exists $r \in R \backslash I$ such that $[r]^n = 0$ for some $n \geq 1$. Then $r^n \in I$ and, since I is radical, we have $r \in I$, which is a contradiction. □

Additionally, at some point we will have to deal with the notion of degree of a polynomial. We will use the following definitions:

Definition A.2.9 (Degree)**.** Let $R = V[\underline{x}]$ be a Noetherian polynomial ring. The degree of a polynomial $f = \sum_{i_1}^{m} \underline{x}^{\underline{\alpha_i}}$ is defined as

$$\deg(f) = \max\{|\underline{\alpha_i}| : 1 \leq i \leq m\}$$

Definition A.2.10 (Generation degree of an ideal)**.** Let $I \subseteq R$ be an ideal of a Noetherian polynomial ring $R = V[\underline{x}]$. Then, we say that I is generated in degrees $\leq d$ if we can find $f_1, \ldots, f_r \in R$ such that

$$I = (f_1, \ldots, f_r) \quad \text{with} \quad \deg(f_i) \leq d$$

where the degree of f_i is defined as in [Def A.2.9].

For the connection between the positive and the zero characteristic case, we need some results about the existence of primes fixing a congruence. The foundational theorem is Dirichlet's theorem.

Theorem A.2.11 (Dirichlet)**.** There are infinitely many primes p such that $p \equiv a \mod b$ for any $a, b \in \mathbb{Z}$ with $\gcd(a, b) = 1$.

Finally, we will need some results about how the congruence of a number behaves under exponentiation.

Theorem A.2.12 (Euler's totient function)**.** We define $\varphi(n)$ the number of positive integers less than n and coprime with n. Then if b coprime with a we get

$$b^{\varphi(a)} \equiv 1 \mod a$$

In particular, if $a = p$ is prime, then $\varphi(p) = p - 1$ and we get

$$b^{p-1} \equiv 1 \mod p$$

Appendix B

Code

B.1 Code for the computation of Bernstein-Sato roots

```
loadPackage("TestIdeals", Reload => true)


p = 7;
m = 0; --m = 1 means working over Z/p^2
e = 2;
R = ZZ[x,y]/(p^(m+1));
f = x^2+y^3;

cartierIdeal = (p, m, e, f, R) ->(
    n := numgens R;
    Rvars := R_*;
    Y := local Y;
    T := ZZ(monoid[(Rvars | toList(Y_1..Y_n)), MonomialOrder=>
       ProductOrder{n,n},MonomialSize=>64]);
    S:= T/p^(m+1);
    Svars := S_*;
    J := ideal(apply(n,i->Svars#(n+i) - Svars#i^(p^e)))*S;
    h := (substitute(f,S)) % J;
    L:=ideal((coefficients(h,Variables => Rvars))#1);
    L = first entries mingens L;
    subRelations := apply(n,i->Svars#(n+i) => Svars#i);
    L = apply(L, g ->substitute(g,subRelations));
    substitute(ideal L, R)
); -- This function calculates the ideal C^e.f

findJumps = (p,m,e,f,R) ->(
    g := f;
    ideal1 := cartierIdeal(p, m, e, substitute(1,R), R);
    ideal2 := cartierIdeal(p, m, e, g, R);
    jumps := {};
    for s from 0 to p^(e+m)-1 do(
```

```
  if ideal1 != ideal2 then(
      jumps = append(jumps, s);
      );
      g = f*g;
      ideal1 = ideal2;
      ideal2 = cartierIdeal(p,m,e,g,R);
  print s;
      );
    jumps
); --- this function is more direct, and perhaps a bit faster,
    than the creation of ideals list and  computation of jumps done
     above

cI = new MutableList from toList (p^(e+m)+1:ideal(0_R));
fRaised = new MutableList from toList (p^(e+m)+1:0_R);

-- Precompute all the powers
fRaised#0=1_R;
for i from 0 to p^(e+m) do(
    fRaised#(i+1) = f*fRaised#i;
);


cartierIdealPow = (p, m, e, a, f, R) ->(
    if cI#a != ideal(0_R) then return cI#a;
    n := numgens R;
    Rvars := R_*;
    Y := local Y;
    T := ZZ(monoid[(Rvars | toList(Y_1..Y_n)), MonomialOrder=>
       ProductOrder{n,n},MonomialSize=>64]);
    S:= T/p^(m+1);
    Svars := S_*;
    J := ideal(apply(n,i->Svars#(n+i) - Svars#i^(p^e)))*S;
    h := (substitute(fRaised#a,S)) % J;
    L:=ideal((coefficients(h,Variables => Rvars))#1);
    L = first entries mingens L;
    subRelations := apply(n,i->Svars#(n+i) => Svars#i);
    L = apply(L, g ->substitute(g,subRelations));
    cI#a = substitute(ideal L, R);
    return cI#a;
); -- This function calculates the ideal C^e.f




findJumpsbtw = (a,b,p,m,e,f,R) ->(
    print (a,b);
    if cartierIdealPow(p,m,e,a,f, R) == cartierIdealPow(p,m,e,b,f,
       R) then return {};
    if a+1 == b then return {a};
```

```
    return join( findJumpsbtw(a,(a+b)//2, p, m, e, f, R),
       findJumpsbtw((a+b)//2, b, p, m, e, f, R) );
);



jumps = findJumpsbtw(1, p^(e+m), p,m,e,f,R);
print(jumps);

file = "sheets/sheetcuspp" | toString(p) | "e" | toString(e) | "m"
    | toString(m) | ".txt";
file << "p = " << p << endl << close;
file << "m = " << m << endl;
file << "e = " << e << endl;
file << "f = " << f << endl;
file << "R = " << R << endl;
file << "Jumps = " << jumps << endl;


for g in jumps do(
    print g;
    file << g << endl;
    print adicExpansion(p, g);
    file << adicExpansion(p, g) << endl;
    print(trim(cartierIdealPow(p,m,e,g,f,R)));
    file << trim(cartierIdealPow(p,m,e,g,f,R)) << endl;
    print "----";
    file << "----" << endl;
    );

file << close;
end
```

B.2 Code for the computation of Bernstein-Sato roots

The following code computes the Bernstein-Sato roots of general diagonal hypersurfaces $f = x_1^{\alpha_1} + \ldots + x_n^{\alpha_n} \in \mathbb{Z}/p^{m+1}\mathbb{Z}[\underline{x}]$ for a chosen prime p.

```
import math
from sympy.functions.combinatorial.numbers import totient
from fractions import Fraction
from itertools import product

#In all the code G_a=<p%a> is subgroup of (Z/aZ)* and S is the set
    {1, ..., a-1}

# Compute the orbit of k under the action of G_a
def orbit(k,a, p):
    if math.gcd(p,a) != 1:
```

```
        raise ValueError("p and a must be coprime")
    if k >= a:
        raise ValueError("k must be less than a")

    current_orbit = list()
    current_orbit.append(k)
    last_element = (k*p)%a
    while last_element != k:
        current_orbit.append(last_element)
        last_element = (last_element*p)%a
    return current_orbit

# Compute all the orbits of S the action of G
def orbits(a, p):
    if math.gcd(p,a) != 1:
        raise ValueError("p and a must be coprime")

    orbits = list()
    check_for_list = [True] * a
    for k in range(1,a):
        if check_for_list[k]:
            orbits.append(orbit(k, a, p))
            for i in orbit(k, a, p):
                check_for_list[i] = False
    return orbits

# Express in base p (reverse order)
def decToBase(n, base):
    if n == 0:
        return [0]
    digits = list()
    while n:
        digits.append(int(n % base))
        n //= base
    return digits

# Returns the period of the expression of the threshold in base p
def periodThreshold(k,a,p):
    u = orbit(k, a, p)
    v = list()
    for ui in u:
        v.append(math.floor(p*ui/a))
    return v

# checks if the sum k_i/a_i carries
def periodsCarry(k : list[int], a : list[int], p : int):
    if len(k) != len(a):
        raise ValueError("k and a must have the same length")
    u = list()
    for i in range(len(k)):
        u.append(periodThreshold(k[i], a[i], p))
```

```
    l = math.lcm(*[len(ui) for ui in u])
    for i in range(l):
        if sum([u[j][i%len(u[j])] for j in range(len(k))]) >= p:
            return True
    return False

# Return a list of ((k_1, k_2), BSR), where BSR is a Bernstein-Sato root induced by the pair (k_1, k_2)
def BSR(a : list[int], p):
    v = list()
    for i in range(len(a)):
        v.append(list(range(1, a[i])))

    kpow = list(product(*v))
    bs = list()
    for k in kpow:
        if not periodsCarry(k, a, p):
            bsr = -sum([Fraction(k[i], a[i]) for i in range(len(k))])
            bs.append(bsr)
    return bs

# Example computing the BSR of x^3 + y^5 + z^7 in char 211^{m+1}
print(BSR([3, 5, 7], 211))
```

Appendix C

Sustainability report and Methodology

C.1 Sustainability Matrix

The following matrix provides a qualitative overview of the sustainability aspects considered throughout the life cycle of this theoretical mathematics thesis. The analysis is structured across three sustainability dimensions—environmental, economic, and social—and is evaluated along three stages: project development, project execution, and potential risks or limitations. Although the project primarily relies on intellectual and digital work, sustainability remains relevant, particularly in areas such as resource use, accessibility, and long-term academic impact.

Aspect	**Project Development**	**Project Execution**	**Risk and Limitations**
Environmental	Use of digital tools to minimize paper waste.	Minimal environmental impact. Use of paper, chalk, and limited computation.	Energy consumption for computational resources; carbon emissions from travel.
Economic	Low cost due to non-experimental nature and reliance on existing infrastructure.	No specialized materials or equipment required.	Limited funding for extended research stays or international collaboration.
Social	Supports academic and professional development. Involves mentorship and peer collaboration.	Contributes to the advancement of theoretical knowledge; potential future applications.	Accessibility of results to a broader audience. Introduction and background chapters are written for readers with general mathematical knowledge.

Table C.1: Sustainability Matrix

C.1.1 Environmental

The environmental footprint of this theoretical mathematics project is relatively low, given its minimal reliance on physical materials or energy-intensive processes. Nevertheless, we present an estimate of the CO_2 equivalent (CO_2e) emissions associated with various aspects of the project.

Aspect	CO_2e per unit	CO_2e total
Paper (200 sheets)	4.5g CO_2e / sheet	900g CO_2e
Chalk (10 units)	3.75g CO_2e / chalk	37.5g CO_2e
Computation (200 days)	75g CO_2e / day	15kg CO_2e
Flights (round-trip)	564kg + 588kg CO_2e	1,152kg CO_2e
Total	-	**1,168kg CO_2e**

Table C.2: Estimated CO_2e emissions from project activities

For computation, we estimated daily CO_2e emissions from the use of a standard office laptop for document editing, LaTeX compilation, and occasional software execution. Assuming a duration of approximately 200 working days and an average of 75g CO_2e per day, the total computational impact is around 15kg CO_2e.

The most significant contributor to the project's environmental footprint is air travel. Although not intrinsic to the theoretical work itself, travel was necessary for academic collaboration or conference attendance. The round-trip flight emitted an estimated 1,152kg CO_2e (564kg outbound and 588kg return).

Excluding air travel, the CO_2e emissions amount to approximately 16kg—an exceptionally low figure compared to most research projects. This underscores the inherently sustainable nature of theoretical mathematics in terms of resource consumption and carbon footprint. Nonetheless, the inclusion of travel highlights how even low-material projects can carry substantial environmental costs when mobility is involved.

C.1.2 Economic

Aspect	Economic Cost
Paper (200 sheets)	\$10.00
Chalk (10 units)	\$6.25
Computation (laptop usage)	\$9.00
Flights (round-trip)	\$1,100.00
Total	\$1,125.25

Table C.3: Summary of Economic Costs Related to the Project

Although this thesis belongs to the field of theoretical mathematics, which traditionally entails low material consumption, a minimal economic footprint still exists and is worth quantifying.

The first set of costs corresponds to consumable materials. We estimate the usage of 200 sheets of paper for note-taking, sketches, and printing drafts, which results in an approximate cost of \$10. Likewise, around 10 pieces of chalk were used for blackboard work, leading to an estimated cost of \$6.25.

Computation costs are estimated based on the electricity consumption of a laptop used throughout the research. Assuming an average rate of \$0.0025 per hour, this results in an estimated cost of \$9.00.

Lastly, a significant cost comes from travel. Although not strictly required for theoretical work, attending academic events or relocating temporarily to a research institution contributed to the project's context. The round-trip flight, essential for participating in collaborative academic activities, is estimated at \$1,100.00.

Overall, the total economic cost of the project is approximately \$1,125.25. While this figure is relatively modest, it helps highlight that even low-resource academic work has an economic and potentially environmental impact.

C.1.3 Social

Despite its abstract nature, this thesis contributes meaningfully to the social dimension of academia by fostering intellectual development, promoting critical thinking, and reinforcing the collaborative nature of mathematical research. The work has been conducted within a broader academic community, supported by interactions with advisors, peers, and researchers, each contributing to knowledge exchange and mutual growth.

Furthermore, efforts have been made to ensure that the content of the thesis is accessible to a wider audience within the mathematics community. The introductory and background chapters are written with clarity in mind, aimed at readers with a general mathematical education, rather than specialists alone. This approach supports inclusivity in academia by lowering entry barriers to advanced topics and encouraging the participation of a broader group of students and researchers. The dissemination of ideas in an understandable form contributes to the democratization of knowledge and reinforces the social responsibility inherent in scholarly work.

C.2 Ethical implications

The ethical implications of this thesis are minimal, largely due to the abstract and non-experimental nature of the work. Unlike research involving human subjects, sensitive data, or environmental intervention, theoretical mathematics typically does not raise direct ethical concerns related to safety, privacy, or societal harm. The research is based entirely on logical reasoning, existing mathematical literature, and computational tools that do not pose risks to individuals or communities.

Nevertheless, academic integrity and responsible authorship remain important ethical considerations. Proper attribution of ideas, transparency in methodology, and respect for the collaborative nature of the academic process are upheld throughout the thesis. While the content itself may not lead to immediate real-world applications, maintaining these ethical standards ensures the credibility of the work and supports a culture of honesty and accountability in research. In this sense, ethical responsibility is still present, even if the potential for direct ethical impact is low.

C.3 Relationship with the sustainability development goals

Although this thesis is situated in the realm of theoretical mathematics and does not have immediate practical applications, it still aligns with some Sustainable Development Goals (SDGs), particularly in the area of Quality Education (SDG 4). By contributing to the body of fundamental mathematical knowledge, the project supports the advancement of education and research capacity within universities and academic institutions. The rigorous logical thinking and abstract reasoning cultivated through such work are essential skills that underpin innovation across disciplines, from engineering and computer science to economics and data analysis.

C.4 Methodology

We display the chronological organization of the tasks in a Gantt chart. The chart illustrates the timeline of the project, including the background research, reading, writing, and attending to classes and seminars in University of Utah. The tasks are organized into groups, with each group representing a specific phase of the project. The chart also includes a timeline for each task, roughly indicating the timing of each activity.

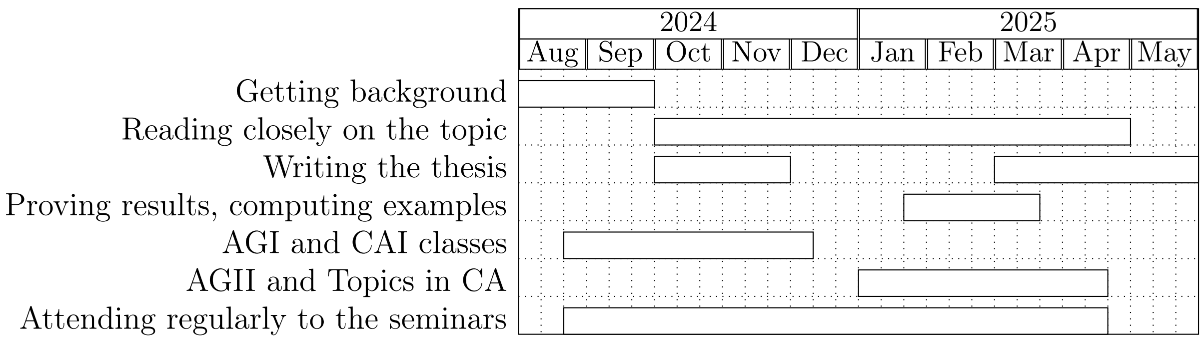


Figure C.1: Gantt chart of the organization of the tasks during the stay

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