2D Geometric Transformation

Geometric Transformations

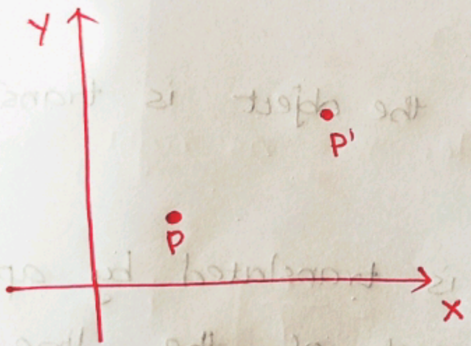
→ changes in orientations, size and shape are accomplished with geometric transformations that alter the coordinate descriptions of objects.

→ The basic geometric transformations are:

* Translation
* Rotation
* Scaling

Translation – Repositioning in a straight line path

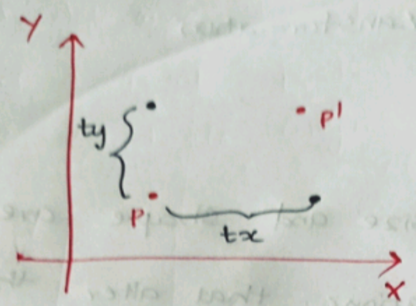
Translation is the repositioning of an object along a straight-line path from one coordinate location to another.



→ To translate a point (x, y) to a new position (x', y'), we have to add translation distance t_x and t_y to the original coordinate (x, y).

$$x' = x + t_x \qquad y' = y + t_y$$

The translation distance pair (t_x, t_y) is called a translation vector or shift vector.

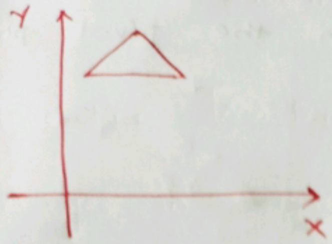


$$P = \begin{bmatrix} x \\ y \end{bmatrix} \quad P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad T = \begin{bmatrix} tx \\ ty \end{bmatrix}$$

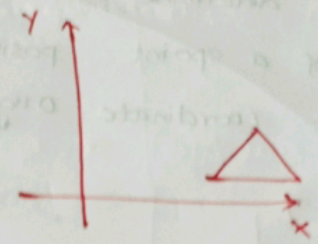
$$P' = P + T$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} tx \\ ty \end{bmatrix}$$

- Translation is a rigid body transformation that moves objects with deformation
- That is, every point on the object is translated by the same amount.
- A straight line segment is translated by applying the transformation equation to each of the line end points and redrawing the line between the new endpoint positions.
- Polygons are translated by adding the translation vector to the coordinate position of each vertex and regenerating the polygon using the new set of vertex coordinates and the current attribute settings.



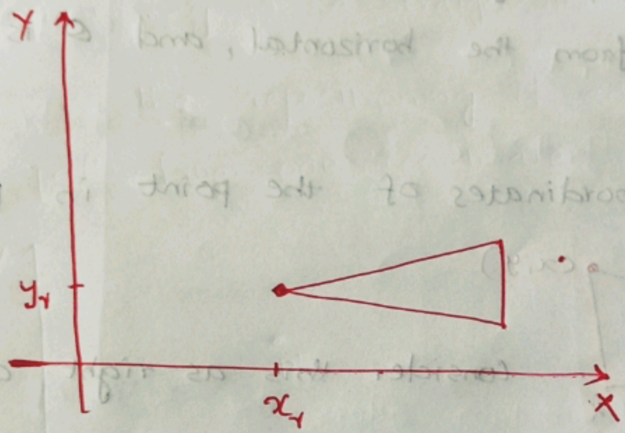
a) Initial position of a polygon



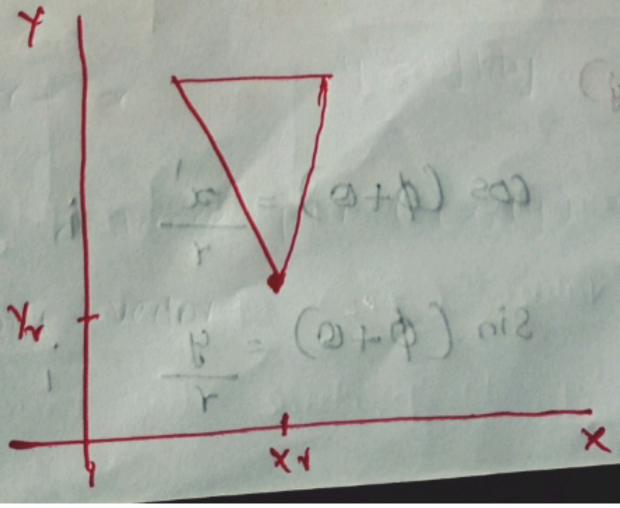
b) New position after translation

Rotation - Repositioning in a circular path.

→ A two dimensional rotation is applied to an object by repositioning it along a circular path in the XY plane.



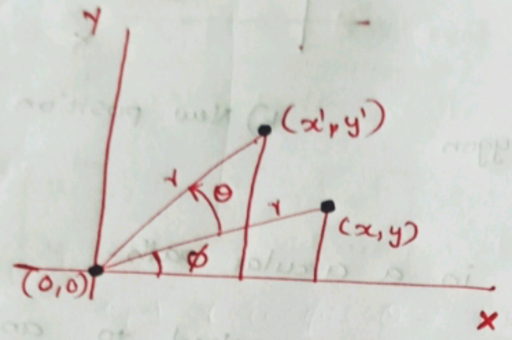
To generate a rotation, we specify a rotation angle θ and the position (x_r, y_r) of the rotation point (or pivot point) about which the object is to be rotated



New position after rotation, here $\theta = 90°$

We first determine the transformation equations for rotation of a point position P. when the pivot point is at the coordinate origin.

Rotation on coordinate origin (0,0)

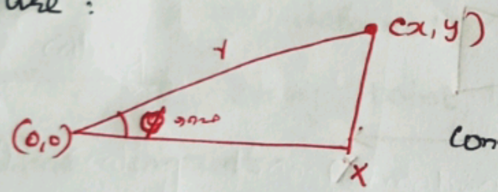


Rotation angle = θ

ϕ is the angle made by X axis initial point

→ In this figure, r is the constant distance of the point from the origin., angle ϕ is the original angular position of the point from the horizontal, and θ is the the rotation angle.

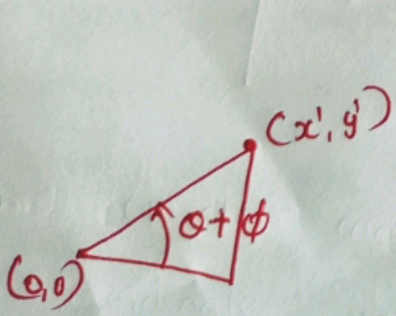
→ The original coordinates of the point in polar coordinates are :



Consider this as right angle triangle.

$$\cos\phi = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{x}{r} \quad ; \quad x = r\cos\phi \quad \text{—(1)}$$

$$\sin\phi = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{y}{r} \quad ; \quad y = r\sin\phi \quad \text{—(2)}$$



$$\cos(\phi+\theta) = \frac{x'}{r} \quad ; \quad x' = r\cos(\phi+\theta)$$

$$\sin(\phi+\theta) = \frac{y'}{r} \quad ; \quad y' = r\sin(\phi+\theta)$$

$$\boxed{\begin{aligned}\cos(A+B) &= \cos A \cos B - \sin A \sin B \\ \sin(A+B) &= \cos A \sin B + \sin A \cos B\end{aligned}}$$

$$x' = r\cos(\phi+\theta)$$

$$= r\cos\phi\cos\theta - r\sin\phi\sin\theta \quad \text{——} ③$$

$$y' = r\sin(\phi+\theta)$$

$$r\cos\phi\sin\theta + r\sin\phi\cos\theta \quad \text{——} ④$$

Substituting ① and ② in ③ and ④

$$\left.\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \end{aligned}\right\} \text{ where } \begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned}$$

We can write the rotation equation in matrix form, where the rotation matrix is.

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$P' = R \cdot P$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Rotation of a point about an arbitrary pivot position

$$\cos\phi = \frac{(x-x_r)}{r} \quad \frac{\text{Adjacent side}}{\text{Hypotenuse}}$$

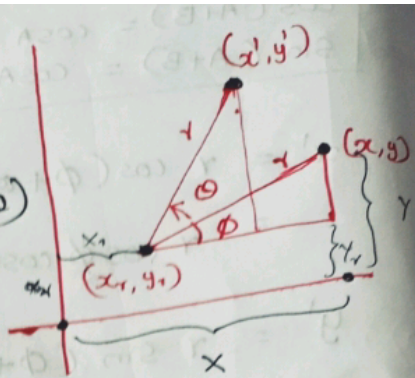
$$r\cos\phi = (x-x_r) \quad \text{——} ①$$

$$\sin\phi = \frac{(y-y_r)}{r} \quad \frac{\text{opposite side}}{\text{Hypotenuse}}$$

$$r\sin\phi = (y-y_r) \quad \text{——} ②$$

$$\sin(\phi+\theta) = \frac{y'-y_r}{r} \qquad \cos(\phi+\theta) = \frac{x'-x_r}{r}$$

$$y'-y_r = r\sin(\phi+\theta) \qquad x'-x_r = r(\cos(\phi+\theta))$$



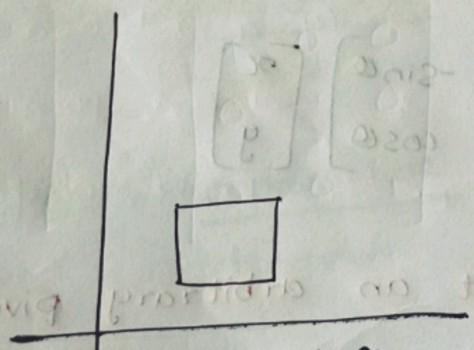
$$r\cos\phi = x - x_r$$
$$r\sin\phi = y - y_r$$

$$x' = x_r + r\cos\phi\cos\theta - r\sin\phi\sin\theta$$
$$y' = y_r + r\cos\phi\sin\theta - r\sin\phi\cos\theta$$

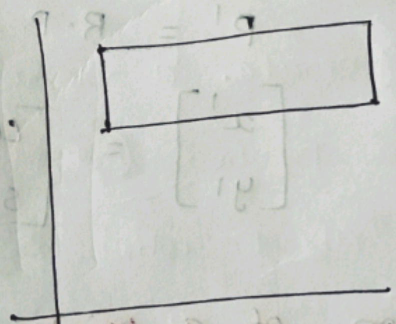
$$x' = x_r + (x-x_r)\cos\theta - (y-y_r)\sin\theta$$
$$y' = y_r + (x-x_r)\sin\theta - (y-y_r)\cos\theta$$

Scaling

→ A scaling transformation alters the size of an object



initial Size



Size after scaling

→ This operation can be carried out for polygons by multiplying the coordinate values (x,y) of each vertex by scaling factors S_x and S_y to produce the transformed coordinates (x', y')

$$x' = x \cdot S_x$$
$$y' = y \cdot S_y$$

the transformation equation can also be written in matrix form :

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$P' = S \cdot P$$

where S is the 2×2 Scaling matrix.

→ Any positive numeric values can be assigned to the scaling factors S_x and S_y.

→ values less than 1 reduce the size of objects.

→ values greater than 1 produce an enlargement

$S_x = S_y = 1$ ⇒ Size of the objects will be unchanged

$S_x = S_y$ ⇒ uniform scaling

$S_x \neq S_y$ ⇒ differential scaling

→ Scaling factors with values less than 1 moves objects closer to the coordinate origin. *

→ While values greater than 1 move coordinate position farther from the origin.

Other Transformations (2D)

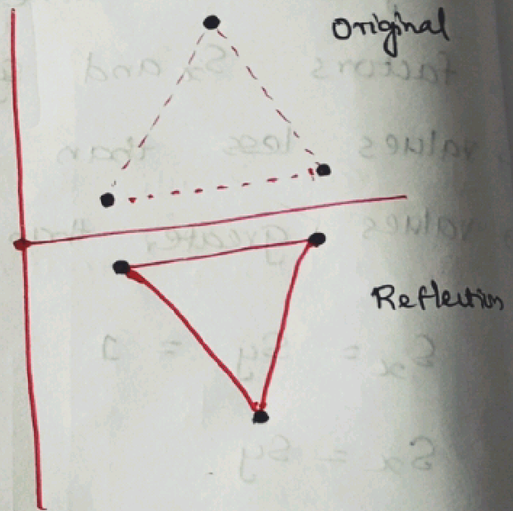
(i) Reflection

(ii) Shear.

Reflection → (Rotation of an object in 180°)

→ Gives mirror image of the object with respect to the axis of reflection.

Reflection with respect to X axis:



Matrix

no change in X values.

change in Y values only

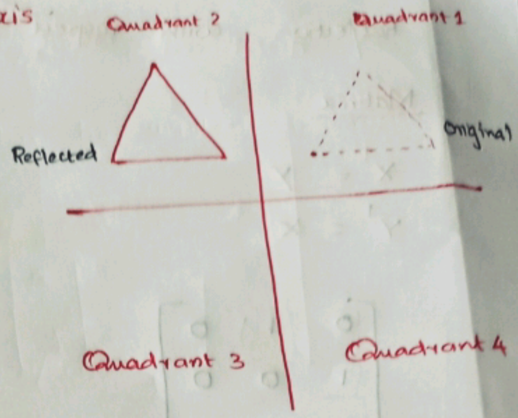
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Take 2x2 matrix

$$\begin{array}{c} x \text{ value} \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ \text{Y value.} \end{array}$$

Reflection with respect to Y axis.



Matrix

no change in Y values.

change in X values only.

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

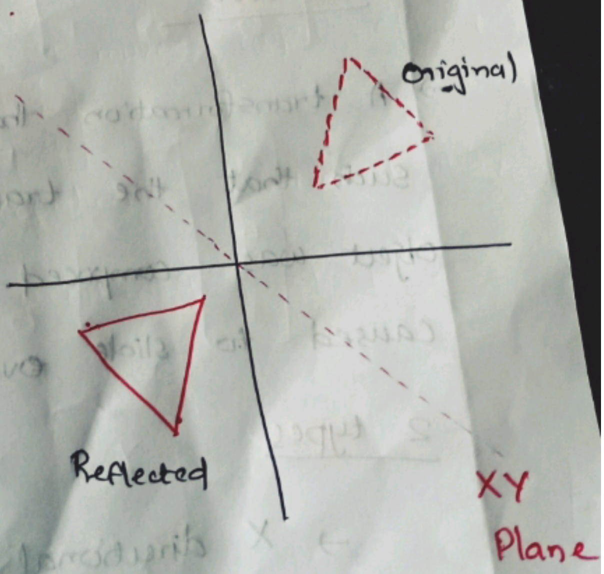
change in x value

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$ no change in Y value.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$P' =$ Reflection Matrix $\times$ P

Reflection with respect to XY Plane.



Matrix

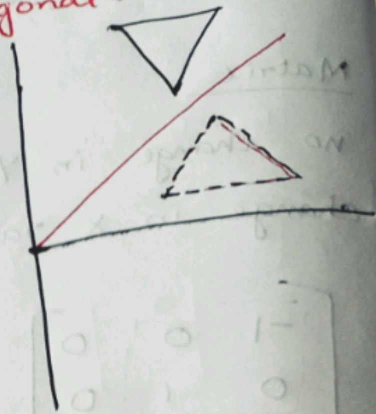
$X' = -X$

$Y' = -Y$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Reflection with respect to $y = x$ diagonal.



Matrix

$x' = y$

$y' = x$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

$$= \begin{bmatrix} y \\ x \\ 1 \end{bmatrix}$$

Shear

→ A transformation that distorts the shape of an object such that the transformed shape appears as if the object was composed of multiple internal layers that are caused to slide over each other.

2 types

→ X directional Shear

→ Y directional Shear.

imagine the object is made with multiple layers and when we are applying shear, the layers will overlap each other and we will get an distored object. This is what shear means.

X Directional Shear Matrix

$$\begin{bmatrix} 1 & Shx & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Shearing factor.

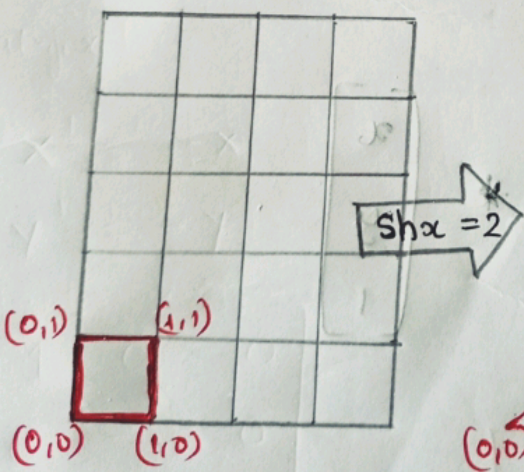
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & shx & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

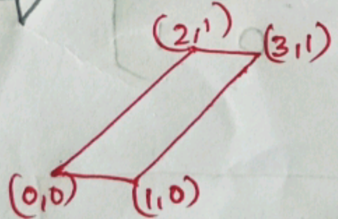
$$\boxed{\begin{aligned} x' &= x + shx * y \\ y' &= y \end{aligned}}$$

⇒ change in X value

⇒ No change in Y value.

Example





$(0,0) \Rightarrow (0,0)$

$(1,0) \Rightarrow 1 + 2*0$

$1 + 0 = 1 \Rightarrow (1,0)$

$(0,1) \Rightarrow 0 + 2*1 \Rightarrow (2,1)$

$0 + 2 = 2$

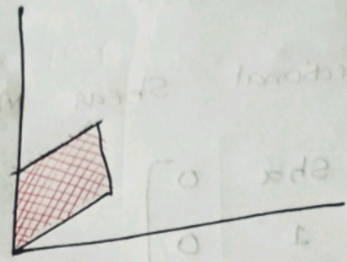
$(1,1) \Rightarrow 1 + 2*1 = 1+2 = 3$

$(3,1)$

No change in Y values

change for X values only.

Y directional shear



no change in X values.

change in Y values.

Shear factor shy determine the Y displacement in Y directional shear.

Unit square to a parallelogram

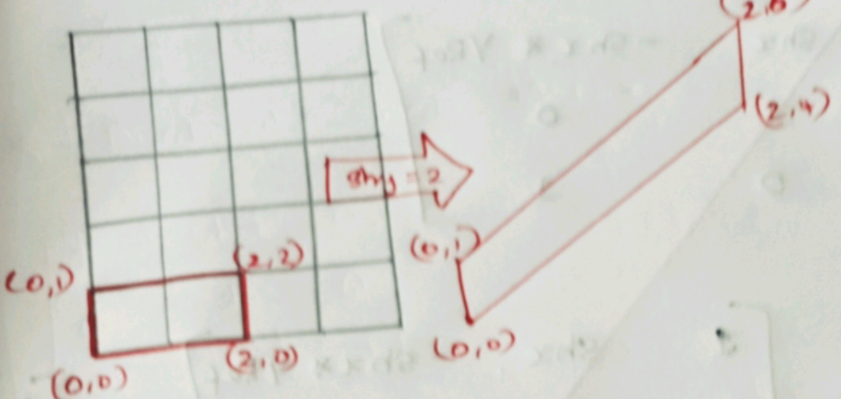
Y directional shear matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ shy & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ shy & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\boxed{\begin{aligned} Y' &= Y + shy * x \\ x' &= X \end{aligned}}$$

Example



$(0,0) = (0,0)$

$(2,0) =$ $x = 2$

$y = y + shy * x$

$= 0 + 2 * 2$

$y = 0 + 4 = 4$

$(2,0) \Rightarrow (2,4)$

$(0,1) \Rightarrow$ $x = 0$

$y = y + shy * x$

$= 1 + 2 * 0$

$= 1 + 0$

$y = 1$

$(0,1) \Rightarrow (0,1)$

$\cancel{(2,2)} \Rightarrow$ $\cancel{x = 2}$

$\cancel{y = y + shy * x}$

$\cancel{= 5 + 2 \times 2}$

$\cancel{= 5 +}$

$(2,2) \Rightarrow$ $x = 2$

$y = y + shy * x$

$= 2 + 2 * \cancel{2}$

$= 2 + 4 = 6$

$(2,2) \Rightarrow (2,6)$