Homogenous Coordinate representation of Geometric transformation

Pre requisites

Translation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix} \Rightarrow \boxed{P' = P + T}$$

Rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \boxed{P' = P * R(\theta)}$$

Scaling

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \boxed{P' = P * S}$$

Basics

Basic geometric transformations can be represented in General matrix form

$$\boxed{P' = M_1 * P + M_2}$$

↳ P' and P are coordinate positions represented as column vectors.

↳ M_1 is a 2x2 matrix containing multiplicative terms

↳ M_2 is a 2 element column vector containing translational terms identity matrix.

→ For translation $M1$ is the identity matrix.

→ for rotation and scaling M_2 contains translational terms associated with pivot point or scaling fixed point.

Matrix Representation and Homogenous coordinates

Any transformation can be written using a general equation i,e

M_1 = Multiplicative matrix

M_2 = Additive Matrix.

P = old coordinates

P' = New coordinades.

$$\boxed{P' = M_1 * P + M_2}$$ General Equation

(i) In case of translation.

Translation $P' = T + P$

$P' = M_1 * P + M_2$

M_2 → Translation Matrix.

M_1 → Identity matrix.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} tx \\ ty \end{bmatrix}$$

Any matrix * Identity matrix = Same Matrix.

(ii) In case of rotation

Rotation $\Rightarrow$ $P' = R(\theta) * P$

$$P' = M_1 * P + M_2$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} + M_2$$

$M_2 \Rightarrow$ Pivot Point / fixed point

(iii) In case of Scaling :

Scaling $\Rightarrow$ $P' = S * P$

$$P' = M_1 * P + M_2$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} + M_2$$

$M_2 \rightarrow$ Pivot point / Fixed point.

→ We can combine the multiplicative and translational terms for two-dimensional geometric transformations into a single matrix representation by expanding the 2 by 2 matrix representation by 3 x 3 matrices.

→ for that express all transformation equations as matrix multiplications.

→ To express any two dimensional transformation as a matrix multiplication, we represent each cartesian coordinate position (x, y) with the homogeneous coordinate triple

(x_h, y_h, h) where.

$$x = \frac{x_h}{h} \qquad y = \frac{y_h}{h}$$

$\boxed{x_h = h \cdot x}$ $\boxed{y_h = h \cdot y}$

→ Thus a general homogeneous coordinate representation can also be written as $(h*x, h*y, h)$

→ for two-dimensional geometric transformations, we can choose the homogeneous parameter h to be any non zero value.

→ We choose as $h = 1$

→ Each two dimensional position is then represented with homogeneous coordinates $(x, y, 1)$

→ coordinates are represented with three-element column vectors, and transformation operations are written as 3x3 matrices.

→ for translation we have.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

which we can write in the abbreviated form

$$P' = T(tx, ty) \cdot P$$

where $T(tx, ty)$ is the 3x3 translation matrix.

→ The inverse of the translation matrix is obtained by replacing the translation parameters tx and ty with negatives $-tx$ and $-ty$.

→ Similiarly, rotation transformation equations about the coordinate origin are now written as

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

or as $P' = R(\theta) \cdot P$

→ The rotation transformation operator $R(\theta)$ is the 3 by 3 matrix with the rotation parameter θ

→ the inverse rotation matrix is obtained when θ is replaced with $-\theta$

→ Finally, a Scaling transformation relative to the coordinate origin is now expressed as the matrix multiplication.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

or

$$P' = S(Sx \cdot Sy) \cdot P$$

where $S(Sx, Sy)$ is the 3 by 3 matrix with parameters Sx and Sy.

→ Replacing these parameters with their multiplicative inverses ($1/Sx$ and $1/Sy$) yields the inverse Scaling matrix.