# A Two-Level Logic Approach to Reasoning About Typed Specification Languages

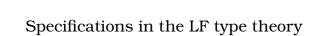
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2014-12-17 FSTTCS

#### Outline

- Specifications in the LF type theory
- Reasoning about specifications
- The Abella/LF branch
- The type encoding problem
- Perspectives



Running example: natural numbers

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sum/s : {M:nat} {N:nat} {K:nat}
sum M N K -> sum (s M) N (s K).
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nat : type.

lsum/emp : lsum emp z.

lsum/cons : lsum (cons N L) K <-</pre>

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sum : nat -> nat -> nat -> type.
sum/z : {M:nat} sum z M M.
sum/s : {M:nat} {N:nat} {K:nat}
        sum M N K \rightarrow sum (s M) N (s K).
lsum : list -> nat -> type.
```

lsum L M <- sum M N K.

#### LF: HOAS

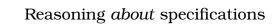
### Intrinsically encoded simply typed $\lambda$ -terms

```
ty : type.
i : ty.
arr : ty -> ty -> ty.

tm : ty -> type.
app : tm (arr A B) -> tm A -> tm B.
abs : (tm A -> tm B) -> tm (arr A B).
```

The term  $\lambda x$ .  $\lambda y$ . x y ( $\lambda z$ . y) is encoded as:

```
abs [x] abs [y] app (app x y) (abs [z] y)
```



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proof : {M:nat} {N:nat} {K1:nat} {K2:nat}
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```

```
proof/z : proof z M M M (sum/z M) (sum/z M) eq/refl.
proof/s : ...
```

## Reasoning (meta) logic

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Let  $\langle M:A \rangle$  stand for: the LF judgement M:A is valid.

We must be able to show:

#### Transformed theorem

```
\forall \texttt{M}, \texttt{N}, \texttt{K1}, \texttt{K2}.  \left( <\texttt{M}: \texttt{nat} > \land <\texttt{N}: \texttt{nat} > \land <\texttt{K1}: \texttt{nat} > \land <\texttt{K2}: \texttt{nat} > \land <\_: \texttt{sum M N K1} > \land <\_: \texttt{sum M N K2} > \right)   \supset \texttt{K1} = \texttt{K2}
```

### Transformed theorem

```
<N:nat> \equiv (N = z)
                               \vee (\exists M. N = s M \land \langle M:nat \rangle)
                                   (\mathbf{M} = \mathbf{z} \wedge \mathbf{N} = \mathbf{K})
<_:sum M N K> \equiv
                               \lor (\existsM1, K1. M = s M1 \land K = s K1
                                                \forallM, N, K1, K2.
        (<M:nat> \land <N:nat> \land <K1:nat> \land <K2:nat>
           \land <:sum M N K1>\land <:sum M N K2>)
              \supset K1 = K2
```

### Transformed theorem

```
<N:nat> \equiv_{\mu} (\mathbf{N}=\mathbf{z})
                                       \lor (\exists \mathtt{M}.\,\mathtt{N} = \mathtt{s}\,\,\mathtt{M} \land \lt \mathtt{M}:\mathtt{nat} \gt)
                                       (\mathtt{M} = \mathtt{z} \wedge \mathtt{N} = \mathtt{K})
<_:sum M N K> \equiv_{\mu}
                                        \lor (\existsM1, K1. M = s M1 \land K = s K1
                                                              \wedge < :sum M1 N K1>)
       \forallM, N, K1, K2.
           (<M:nat> \land <N:nat> \land <K1:nat> \land <K2:nat>
               \land <_:sum M N K1>\land <_:sum M N K2>)
                   \supset K1 = K2
```

Abella/LF

#### Abella

http://abella-prover.org

- Interactive tactics-based theorem prover
- Supports:
  - First-order intuitionistic logic
  - Inductive and co-inductive predicate definitions
  - Intensional (structural) equality
  - Generic reasoning and nominal abstraction
- Philosophy:
  - Simple and rigorous proof theory
  - Relational view
  - Pattern unification part of the kernel

### Abella/LF

http://abella-prover.org/lf

- Ability to "load" arbitrary LF signatures.
- LF typing judgement <\_:\_> is an inductive definition.

### Abella/LF Example

#### nat.elf

```
nat : type.
z : nat.
s : nat -> nat.

sum : nat -> nat -> nat -> type.
sum/z : {M:nat} sum z M M.
sum/s : {M:nat} {N:nat} {K:nat}
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2.3

### Abella/LF Example

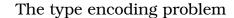
#### nat.thm

```
Specification "nat.elf".
Theorem sum det3 : forall M N K1 K2,
  <M:nat> -> <N:nat> -> <K1:nat> -> <K2:nat> ->
  (exists P1, <P1:sum M N K1>) ->
  (exists P2, <P2:sum M N K2>) ->
 K1 = K2
induction on 1.
intros Mnat Nnat K1nat K2nat sum1 sum2.
sum1 : case sum1. sum2 : case sum2.
Mcase: case Mnat. % cases for <M:nat>
 % case of M = z
 sum1 : case sum1. sum2 : case sum2.
   search. % goal was K2 = K2
 % case of M = s M1
 sum1 : case sum1. sum2 : case sum2.
  Knat : case K1nat. K3nat : case K2nat.
   apply IH to Mcase Nnat Knat K3nat .
   search. % goal was s K3 = s K3
```

### Abella/LF Example

#### nat.thm (contd.)

```
Theorem lsum det2 : forall L M1 M2,
 <L:list> -> <M1:nat> -> <M2:nat> ->
  (exists P1, <P1:lsum L M1>) ->
  (exists P2, <P2:1sum L M2>) ->
 M1 = M2
induction on 1.
intros Llist M1nat M2nat lsum1 lsum2.
lsum1 : case lsum1. lsum2 : case lsum2.
Lcase : case Llist.
 case lsum1, case lsum2, search.
lsum1c : case lsum1. lsum2c : case lsum2.
 apply IH to Lcase1 lsum1c2 lsum2c2 _ _.
 apply sum det3 to Lcase lsum1c2 lsum1c3 lsum2c3 .
 search.
```



```
Theorem sum_det3 : forall M N K1 K2,

<M:nat> -> <N:nat> -> <K1:nat> -> <K2:nat> ->

(exists P1, <P1:sum M N K1>) ->

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• "nat": this would mean that Abella/LF's types must at least contain all LF types.

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What are the types of k1 and k2 in the meta-logic?

- "nat": this would mean that Abella/LF's types must at least contain all LF types.
- "something else": but what?

### Why not "nat"?

- Type systems are not as canonical as logic:
  - There are zillions of type systems
  - There is only one intuitionistic first-order logic (up to variations in the proof system)
  - Specializing logic to particular type systems therefore seems like a mistake
- Think: two incompatible typing judgements in the same theorem.
  - Happens all the time in translations from one typed language to another
  - Would still like to prove properties of such translations

### Un(i)typed encodings

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### Un(i)typed encodings

- Insight: assumptions of the form <m:nat> already contain all the typing information necessary.
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- Abella's reasoning logic remains simply typed.

Forgetful type mapping  $\phi(-)$ :

$$\phi\left(A o\!B
ight)\stackrel{\mathrm{def}}{=}\phi\left(A
ight) o\phi\left(B
ight) \ \phi\left(\{x\!:\!A\}B
ight)\stackrel{\mathrm{def}}{=}\phi\left(A
ight) o\phi\left(B
ight) \ \phi\left(\mathtt{a}\;M_1\;\cdots\;M_k
ight)\stackrel{\mathrm{def}}{=}\mathtt{lfobj} \ \phi\left(\mathtt{type}
ight)\stackrel{\mathrm{def}}{=}\mathtt{lftype}$$

### Translating terms

• To match tye type mapping, we also map dependently typed terms to simply typed terms.

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Term mapping  $\langle - \rangle$ :

$$\langle [x:A]M \rangle \stackrel{\text{def}}{=} \lambda x:\phi(A). \langle M \rangle$$

$$\langle M N \rangle \stackrel{\text{def}}{=} \langle M \rangle \langle N \rangle$$

$$\langle x \rangle = x$$

### Typing as a logic program

 The final ingredient is to recover the typing information that was forgotten in the form of the predicate hastype:

hastype : lfobj ightarrow lftype ightarrow o

 $\mathtt{istype}: \mathtt{lftype} \to \mathtt{o}$ 

# Typing as a logic program

 The final ingredient is to recover the typing information that was forgotten in the form of the predicate hastype:

```
istype: lftype \rightarrow o
Typing program \{\{-\}\}:
                   \{\{\{x:A\}B\}\} \stackrel{\text{def}}{=} \lambda m: (\phi(A) \to \phi(B)).
                                                    pix:\phi(A).\{\{A\}\}x \Rightarrow \{\{B\}\}\}(m x)
        \{\{a\ M_1\ \cdots\ M_k\}\}\stackrel{\mathrm{def}}{=} \lambda m:lfobj.
                                                    hastype m (a \langle M_1 \rangle \cdots \langle M_k \rangle)
                         \{\{\text{type}\}\} \stackrel{\text{def}}{=} \lambda t : \text{lftype. istype } t
```

 $\texttt{hastype}: \texttt{lfobj} \to \texttt{lftype} \to \texttt{o}$ 

### Importing LF signatures

For every signature constant of the form

c:P

(where *P* is a type or a kind):

**1** Add this to the meta-signature:

$$\mathtt{c}:\phi\left(P\right)$$

2 Add a new program clause:

$$\{\!\{P\!\}\!\}\mathbf{c}.$$

#### In action

```
Abella < Specification "nat.elf".
Reading specification "nat.elf"
sig nat.
  type nat lftype.
 type z lfobj.
 type s lfobj -> lfobj.
 type sum lfobj -> lfobj -> lfobj -> lftype.
 type sum/z lfobj -> lfobj.
 type sum/s lfobi -> lfobi -> lfobi -> lfobi -> lfobi.
end.
module nat.
  (* nat:type *)
 istype nat.
  (* z:nat *)
 hastype z nat.
  (* s:nat -> nat *)
  pi lf_1\ hastype lf_1 nat => hastype (s lf_1) nat.
  (* sum:nat -> nat -> type *)
 pi lf_1\ hastype lf_1 nat =>
   pi lf_2\ hastype lf_2 nat =>
     pi 1f_3\ hastype 1f_3 nat => istype (sum 1f_1 1f_2 1f_3).
  (* sum/z:{M:nat} sum z M M *)
 pi M\ hastype M nat => hastype (sum/z M) (sum z M M).
  (* sum/s:{M:nat} {N:nat} sum M N K -> sum (s M) N (s K) *)
 pi M\ hastype M nat =>
   pi N\ hastype N nat =>
     pi K\ hastype K nat =>
       pi lf 1\ hastype lf 1 (sum M N K) =>
         hastype (sum/s M N K lf 1) (sum (s M) N (s K)).
end.
```

#### Adequacy: from Abella back to LF

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Every LP derivation can therefore be inverted into an LF typing derivation.

Hence, the encoding is invisible for the end user.



# Larger Abella/LF examples

http://abella-prover.org/lf

A (growing) number of examples of the use of Abella/LF are available:

- · Cut-elimination for focused intuitionistic logic
- Bijection of the HOAS and De Bruijn representations of  $\lambda$ -terms
- Typing-uniqueness for simply typed Church-encoded  $\lambda$ -terms
- Co-inductive reasoning: divergence of  $\Omega$ , partitioning of untyped  $\lambda$ -terms into normal and diverging terms
- Type-preservation for big-step and small-step evaluation for the pure  $\lambda$ -calculus.

#### Comparing Abella/LF to Twelf

- Abella/LF proofs are *much* easier to understand and maintain.
- Twelf has a number of trusted algorithmic checks that an inductive type family denotes a theorem. In Abella/LF, theorems are written using standard logical inference, and therefore do not need such checks.
  - Of course, this depends on the correctness of the Abella kernel, which performs unification and checks (co-)inductive proofs.
- Twelf has a very high performance LF type checker, which Abella/LF lacks. (We just use Twelf as our type-checker.)
- Twelf can handle implicit syntax, which Abella/LF lacks. (We use Twelf as our elaborator.)

### Comparing Abella/LF to Beluga

http://complogic.cs.mcgill.edu/beluga

- Beluga is also a multilevel system, but its meta level is a functional programming language with unrestricted recursion.
- Both levels of Beluga are dependently typed, which makes different tradeoffs:
  - + encoded terms have very precise types, but
  - many properties are harder to state because of type-interference (e.g., placing the same term in two different contexts)
- Abella has next to no automation in its tactics, so it sometimes requires tedious manual proofs. On the other hand, its trusted codebase is considerably simpler.

#### Summary

#### Our results:

- Abella/LF can be used to reason about LF specifications.
- It is both backward and forward compatible with the standard Abella/HH.
- In particular, the reasoning logic G of Abella is left completely undisturbed.
- Our approach is fairly generic with respect to particular type systems. (The typing derivations must be representable as intuitionistic logic programs.) We have a system in development that can use any functional PTS.

#### Near future:

- Adding native specification-language type checkers to Abella would reduce a dependency on external tools.
- We next plan to look at intersection and refinement types.