Isogeny representations for cryptography

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Prelude

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- Downsides :
  - Schemes are slow
  - Schemes die when a theorem from 1997 is discovered by cryptographers.
- Upsides :
  - Keys are small
  - There are many different approaches to tackle a specific problem.

Act I – Standard isogeny representations

Act I – The rational maps and kernel representations

Elliptic curves, isogenies, torsion groups and graphs

Elliptic curves and their isogenies

Definition (Elliptic curve)

An elliptic curve E over $\mathbb{F}_q$ is a smooth projective algebraic curve of genus 1. It is described by a cubic equation :

$$E = \left\{(x, y) \in \overline{\mathbb{F}}_q \times \overline{\mathbb{F}}_q,\ \ y^2 = x^3 + ax + b\right\} \cup \{\infty\}$$

with $a, b \in \mathbb{F}_q$ such that $4a^3 + 27b^2 \neq 0 \mod q$.

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E is the elliptic curve; $E(\mathbb{F}_q)$ is the set of rational points over $\mathbb{F}_q$.
$E(\mathbb{F}_q)$ is an abelian group. Its neutral element is ∞.

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Example

Let's take $E : y^2 = x^3 + 1$ over $\mathbb{F}_5$. It has 6 rational points :

$$E(\mathbb{F}_5) = \{(0, 1), (0, 4), (2, 2), (2, 3), (4, 0), \infty\}$$

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$$\varphi : (x, y) \mapsto \left(\frac{x^2 + x - 2}{x + 1}, \frac{x^2 + 2x - 2}{x^2 + 2x + 1} y \right)$$

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The kernel of φ is $\{(4, 0), \infty\} \leftrightsquigarrow \deg(\varphi) = 2$.

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Being efficient is not an intrinsic property !
Efficiency can depend on :

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- The height of the field extension involved,
- What data is available to us
- The co/domain curves having special properties.

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[Robert24] : **Damien Robert**, *On the efficient representation of isogenies*, eprint : 2024/1071

1.1 : The Rational maps representation

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- Evaluation is completely explicit
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But factoring φ into smaller pieces $\rightsquigarrow$ efficient for smooth-degree isogenies !

1.2 : The kernel representation

Definition (n-torsion group)

Let n be a positive integer. The n-torsion group of E is defined as

$$E[n] = \left\{ P \in \overline{\mathbb{F}}_p, \quad nP = \infty \right\}$$

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Theorem

Let E be an elliptic curve and ℓ a prime. There is a 1-to-1 correspondence

$$\left\{ \begin{array}{c} \text{Cyclic subgroups} \\ \text{of } E[\ell] \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Isogenies of degree } \ell \\ \text{emanating from } E \end{array} \right\}$$

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$$\{\infty\} \longrightarrow \langle P \rangle \longrightarrow E \xrightarrow{\varphi} E/\langle P \rangle \longrightarrow \{\infty\}$$

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Efficient only if $E[n]$ is defined over a small field extension
$\rightsquigarrow$ Imposes $n \mid (p+1)^2$.
Evaluation requires translating back to rational maps
$\rightsquigarrow$ n needs to be smooth, in the end

ℓ-isogeny graphs

$$\text{Vertices : } \left\{ \begin{array}{c} \text{Isomorphism classes} \\ \text{of supersingular} \\ \text{elliptic curves} \end{array} \right\}, \quad \text{Edges : } \left\{ \begin{array}{c} \ell\text{-isogenies} \\ \text{between the} \\ \text{curves} \end{array} \right\}$$

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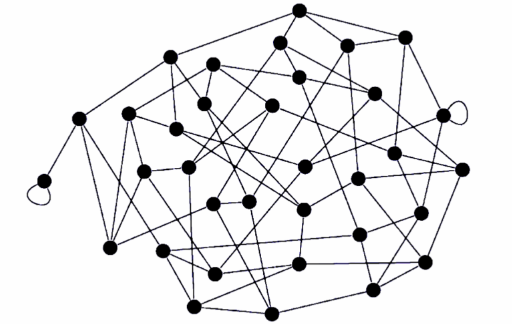

Figure: A representation of a supersingular 3-isogenies graph

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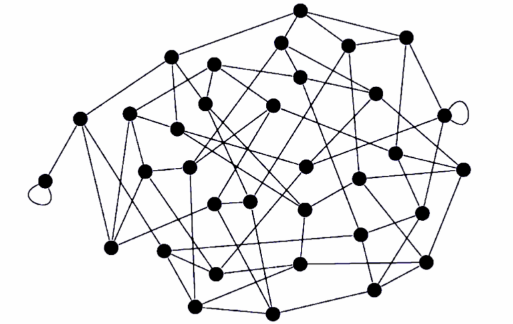

Figure: A representation of a supersingular 3-isogenies graph

- Isogeny graphs are connex and contain $\approx p/12$ vertices,
- The ℓ-isogeny graph is $(\ell + 1)$-regular,
- Those graphs are Ramanujan.

Isogenies hard problems

Isogeny Problem

Given two isogenous curves E_1 and E_2,
find an efficient representation of an isogeny $\varphi : E_1 \to E_2$.

ℓ-Isogeny Path Problem

Given two isogenous curves E_1 and E_2,
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No sub-expomential time algorithm for those problems, even with access to a quantum computer.

Act II – Modern representations

Act II – The Quaternion and HD representations

Deuring, KLPT and Kani diagrams

New results

Using the quaternion representation, we can prove :

Theorem

The ℓ-isogeny path problem between E_1 and E_2 can be solved in polynomial time, assuming we know $\mathrm{End}(E_1)$, $\mathrm{End}(E_2)$.

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Using the HD representation, we can prove :

Theorem

An isogeny $\varphi : E_1 \to E_2$ with non-smooth degree can be efficiently represented, assuming we know $\varphi(E_1[m])$ for m big enough.

Quaternion algebras and orders

Definition (The quaternion algebra ramified at p and ∞)

We will make use of the quaternion algebra $B_{p,\infty}$ defined as :

$$B_{p,\infty} = \mathbb{Q} + i\mathbb{Q} + j\mathbb{Q} + k\mathbb{Q}$$

with $i^2 = -1$, $j^2 = -p$, $k := ij = -ji$.

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$\mathcal{O} = \mathbb{Z} + i\mathbb{Z} + j\mathbb{Z} + k\mathbb{Z}$ is an order.
$\mathcal{O}_0 = \mathbb{Z} + i\mathbb{Z} + \frac{i+j}{2}\mathbb{Z} + \frac{1+k}{2}\mathbb{Z}$ is a maximal order.

The Deuring Correspondence in one slide

Theorem (Deuring)

$$\left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{(supersingular) elliptic curves} \\ \text{over } \mathbb{F}_{p^2} \text{ and their isogenies} \end{array} \right\} \overset{\text{2-to-1}}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Maximal orders of } B_{p,\infty} \\ \text{and their connecting ideals} \end{array} \right\}$$

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The canonical example

Take $E_0 : y^2 = x^3 + x$ over $\mathbb{F}_{p^2}$, with $p = 3 \mod 4$.
Then, we have

$$\begin{array}{ccc} \mathrm{End}(E_0) & = & \mathbb{Z} + \iota\mathbb{Z} + \frac{\iota+\pi}{2}\mathbb{Z} + \frac{1+\pi\iota}{2}\mathbb{Z} \\ \wr\| & & \\ \mathcal{O}_0 & = & \mathbb{Z} + i\mathbb{Z} + \frac{i+j}{2}\mathbb{Z} + \frac{1+k}{2}\mathbb{Z} \end{array}$$

Translating the ℓ-isogeny path problem

The ℓ-isogeny path problem

Let E_1, E_2 be two elliptic curves over $\mathbb{F}_{p^2}$. Let ℓ be a small prime.

Compute an isogeny $\varphi : E_1 \to E_2$ of degree ℓ^e.

$$E_1 \xrightarrow{\varphi} E_2$$

The quaternion ℓ-isogeny path problem

Let $\mathcal{O}_1, \mathcal{O}_2$ be two maximal orders in the quaternion algebra $B_{p,\infty}$.

Compute an ideal $I : \mathcal{O}_1 \to \mathcal{O}_2$ of norm ℓ^e.

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The translation requires the knowledge of $\mathrm{End}(E_1)$ and $\mathrm{End}(E_2)$.

Quaternions (hard ?) problems

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Given two maxima orders $\mathcal{O}_1$ and $\mathcal{O}_2$,
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Both problem can be solved in polynomial time :

- The Quaternion Isogeny Problem is straightforward,
- The Quaternion ℓ-Isogeny Path Problem requires the KLPT algorithm.

2.1 : Quaternion representation

The quaternion ideal representation

One can represent an isogeny $\varphi : E_1 \to E_2$ as a connecting quaternion ideal $I : \mathcal{O}_1 \to \mathcal{O}_2$.

Properties of the representation

- The evaluation algorithm is split into two main steps :

$$I \xrightarrow{\text{Ideal-to-Iso}} \ker(\varphi) \xrightarrow{\text{Vélu}} \varphi$$

The evaluation requires translating back to kernel and rational maps
$\rightsquigarrow$ I must have powersmooth norm.

Solving the ℓ-Isogeny Path Problem

Instance of
the problem

Solution of
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Geometric
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$\boxed{E_1 \quad E_2}$

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Geometric world

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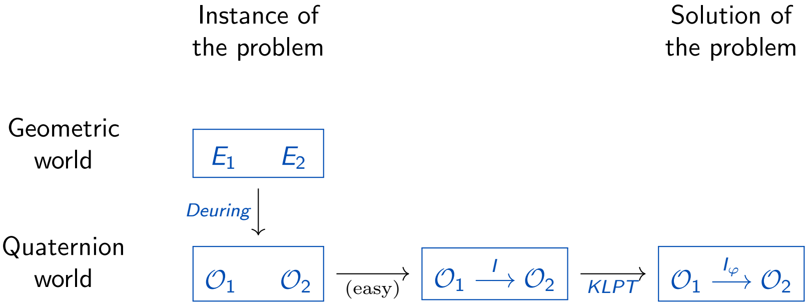
Solving the ℓ-Isogeny Path Problem



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Deuring

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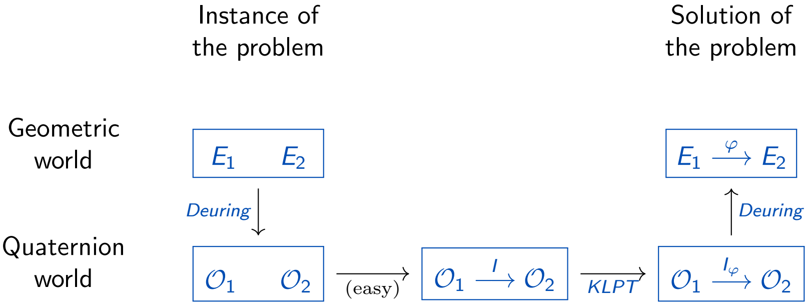

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$\rightsquigarrow$ From the kernel, we evaluate Φ as a chain of (ℓ, ℓ)-isogenies.

2.2 : The HD representation

The HD representation

One can efficiently represent an isogeny $\varphi : E_1 \to E_2$ of arbitrary degree n by embedding it into an isogeny of dimension at most 8, given $\varphi(E_1[m])$ is known, for $m > \sqrt{n}$.

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Properties of the representation

- Efficiency highly depends on the context,
- The evaluation algorithm computes $\hat{\varphi}(P)$ as the first component of

$$\Phi(P, 0) = \begin{pmatrix} \hat{\varphi} & \hat{\alpha} \\ -\alpha' & \varphi' \end{pmatrix} (P, 0)^t = (\hat{\varphi}(P), -\alpha'(P))^t$$

Then, we easily "reverse" $\hat{\varphi}$ to obtain φ. $\rightsquigarrow$ We never explicitely use the rational maps defining φ !

Act III – Recent representations

Act III – HD-quaternions and Hermitian modules representations

IKO, GSS, KLPT² and $\otimes$-MIKE

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Elliptic curves are abelian varieties *of dimension 1*.
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A PPAS is either :

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Polarisations

A polarisation on A is a special isogeny $\lambda : A \to A^\vee$.
When λ is an isomorphism, we say it is principal.

Overview of KLPT^2

	Instance of the problem	Solution of the problem
Geometric world	$\boxed{(A_1, \lambda_1) \qquad (A_2, \lambda_2)}$	$\boxed{(A_1, \lambda_1) \xrightarrow{\Phi} (A_2, \lambda_2)}$

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Research for efficient PPAS/quaternion translations is a rather new subfield.
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Act IV – Abel Jugendtraum representation

Act IV – A Drinfeld module representation ?

Drinfeld modules, Wesolowski's algorithm, Hope & Dreams

Motivation and history

Drinfeld modules in one sentence

Drinfeld modules are to function fields what elliptic curves are to number fields.

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It doesn't sound good... but

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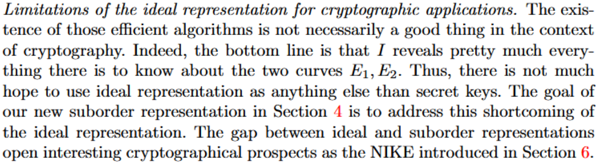

Limitations of the ideal representation for cryptographic applications. The existence of those efficient algorithms is not necessarily a good thing in the context of cryptography. Indeed, the bottom line is that I reveals pretty much everything there is to know about the two curves E_1, E_2. Thus, there is not much hope to use ideal representation as anything else than secret keys. The goal of our new suborder representation in Section 4 is to address this shortcoming of the ideal representation. The gap between ideal and suborder representations open interesting cryptographical prospects as the NIKE introduced in Section 6.

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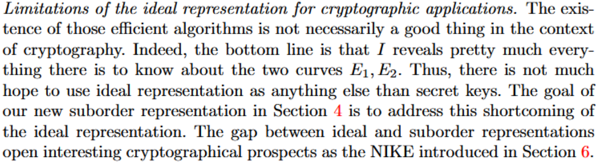

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- Usual hard problems are easy in the quaternion world
- pSIDH $\rightsquigarrow$ Ad-hoc isogeny representation with limited information

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- KLPT is replaced by (an adaptation of) Wesolowski's algorithm.

The mathematics of Drinfeld modules

Choosing an underlying setting

	Generic Drinfeld Modules	Special Drinfeld Modules	Elliptic Curves
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Base ring			
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Residue field	$A/\mathfrak{p}$		
Finite extension	$A/\mathfrak{p} \hookrightarrow K$		

- $\mathrm{Frac}(A)$ is the function field of some curve over $\mathbb{F}_p$.
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Choosing an underlying setting

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- The prime ideal $\mathfrak{p}$ defines the field of coefficients.
- The degree d defines the extension we are working with.

4.1 : The Drinfeld representation

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TBD

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TBD

Properties of the representation

- Possibly cannot exist.
- Would leverage the analogy between elliptic curves and Drinfeld modules
- Could take advantage of Wesolowski's method for computing Drinfeld isogenies
  $\rightsquigarrow$ Some tools are possibly already there.

Conclusion ?

What do we know about isogeny-based cryptography ?

Elliptic curves and their isogenies are very rich mathematical objects !

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  - Schemes are slow
  - Schemes die when a theorem from 1997 is discovered by cryptographers.
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Thank you for your attention !