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Subvarieties of Moduli Spaces

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In this paper we try to decide along algebraic lines whether moduli spaces of abelian varieties or of algebraic curves contain complete subvarieties. In Theorem (1.1) we consider abelian varieties and curves of genus three in characteristic $p \neq 0$. In Section 2 we use monodromy in order to show certain families of abelian varieties up to a purely inseparable isogeny are isotrivial; probably the results (2.1) and (2.2) are special cases of more general facts about l-adic monodromy. In Section 4 we answer a question raised by Manin concerning a possible generalization of the fact that two supersingular elliptic curves over an algebraically closed field (of characteristic p) are isogenous. We abbreviate abelian variety(ies) by AV; we use X^t for the dual, and $\hat{X}$ for the formal group of an AV X.

1. Moduli Spaces in Characteristic p which Contain Complete Curves

All moduli spaces in this section considered will be moduli spaces in the sense of [21] with a field k as base ring.

Suppose $k = \mathbb{C}$, the field of complex numbers; the coarse moduli scheme A_g of principally polarized abelian varieties of dimension g contains a projective subscheme of dimension $g-1$; this easily follows from the existence and the properties of the Satake compactification of A_g as for example Shafarevich remarked (cf. [32], p. 111).

As Mumford pointed out to me, along the same lines it follows that if $k = \mathbb{C}$, and $g \geqq 3$, then the coarse moduli scheme M_g or irreducible, non-singular, complete algebraic curves of genus g contains a complete algebraic curve; this can be seen as follows: let M_g' be the coarse moduli scheme of good curves of genus g (stable curves of genus g whose Jacobian variety is an abelian variety), and

$$j\colon\ M_g' \to A_g$$

the Jacobi mapping; let $A_g \subset \bar{A}_g \subset \mathbb{P}^N$ be the Satake compactification, and denote by C the closure of $j(M_g)$ in $\bar{A}_g$; each component of $C \setminus j(M_g)$ has codimension at least two in C: this is seen to be true for components of $j(M_g') \setminus j(M_g)$ by counting moduli, and for components of $C \setminus j(M_g')$ it follows from $\bar{A}_g \setminus A_g = \bigcup_{h<g} A_h$; thus we can intersect $j(M_g) \subset \mathbb{P}^N$ with a

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convenient linear space of dimension $N-3g+4$ which does not meet $C \smallsetminus j(M_g)$.

By a result of Deligne and Mumford we know a natural compactification D of M_g exists by the use of stable curves, cf. [2]; note that $D \smallsetminus M'_g$ and $M'_g \smallsetminus M_g$ both have at least one component of codimension one (consider irreducible curves of genus g having one node, or consider a component U coming from curves consisting of an elliptic curve and a smooth curve of genus $g-1>1$ connected by one normal crossing); thus it is not clear the method above can be applied to this or another compactification of M'_g (and note that j "contracts" U to a lower dimensional $j(U) \subset A_g$).

Theorem (1.1). *Let k be field*, $\operatorname{char}(k)=p \neq 0$.

a) *The coarse moduli scheme $A_{g,d}$ of abelian varieties of dimension g plus a polarization of degree d^2 has a projective subscheme of dimension at least $\frac{1}{2}g(g-1)$.*

b) *Suppose k is algebraically closed. The coarse moduli scheme M_3 of complete irreducible non-singular algebraic curves of genus 3 contains a complete algebraic curve.*

Definition (1.2). Let X be an AV over a field k, $\operatorname{char}(k)=p$; we say the p-rank of X equals f, notation: $\operatorname{pr}(X)=f$, if

$$|{}_pX(K)|=p^f,$$

where K is an algebraically closed field containing k, and

$${}_pX = \operatorname{Ker}(p\colon\ X \to X);$$

we say X is *ordinary* if $f=\dim X$, and we say X is *very special* if $f=0$.

Let Y be a scheme over the prime field $\mathbb{F}_p$; we denote by $F=F_Y$: $Y \to Y$ the morphism obtained by raising all sections of $\mathcal{O}_Y$ to the p-th power; the induced homomorphism

$$F^* = h\colon\ H^1(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y)$$

is usually called the *Hasse-Witt transformation.*

Lemma (1.3). *Let X be an* AV *over an algebraically closed field of characteristic p of p-rank f; then the number of elements $v \in H^1(X, \mathcal{O}_X)$ such that*

$$hv=v \quad \textit{equals } p^f.$$

Proof. By duality of abelian varieties, the p-rank of X equals f if and only if the semi-simple rank of h equals f, which, by [10], p. 488, Satz 10, yields the result (also cf. [22], p. 143, Corollary).

Lemma (1.4). *Let S be an irreducible algebraic k-scheme, and $\mathsf{X} \to S$ an abelian scheme over S; let f be the p-rank of the generic fibre; for any field K, and for any $s \in S(K)$,*

$$\mathrm{pr}(\mathsf{X}_s) \leqq f.$$

Corollary (1.5). *Let $\mathsf{X} \to T$ be an abelian scheme over a locally noetherian k-scheme T, and n an integer; the set of points s of T with $\mathrm{pr}(\mathsf{X}_s) \leqq n$ is a closed set in T.*

Lemma (1.6). *Assumptions as in (1.4); let W be the closed subset of S over which the fibre has p-rank at most $f-1$ (closed because of 1.5); then either W is empty or each component of W has codimension one in S.*

Proof of (1.4), (1.5), (1.6). The sheaf $\mathscr{H} = R^1(\mathsf{X} \to S)(\mathcal{O}_X)$ is locally free over S, thus each point of S has a neighborhood $\mathrm{Spec}(R) \hookrightarrow S$ over which $\mathscr{H} \,|\, \mathrm{Spec}(R)$ is a free coherent sheaf; let $G = (\mathbb{G}_{a,R})^g$ be the affine space of dimension $g = \dim(\mathsf{X}/S)$ and note $\mathscr{H}$ can be identified with the sheaf of germs of sections in G; thus the Hasse-Witt transformation can be viewed as a group scheme homomorphism; its kernel $N := \mathrm{Ker}(h\colon G \to G)$ can be given by the equations $(h\,v - v)_i = 0$, and if $v = \sum x_i\, v_i$, then

$$(h\,v - v)_i = \sum_j x_j^p\, h_{ij} - x_i;$$

N is quasi-finite over $\mathrm{Spec}(R)$, and N is smooth over $\mathrm{Spec}(R)$ because the derivations $\partial/\partial x_j$ of the defining equations have value 1, if $i = j$, or 0, if $i \neq j$, which implies smoothness. At a geometric point s of $\mathrm{Spec}(R)$, the rank of the finite group scheme equals p^n if and only if the p-rank of X_s equals n, thus (1.4) is proved.

This implies (1.5).

For the proof of (1.6) we restrict to $\mathrm{Spec}(R) \subset S$ as above; suppose k perfect and S reduced, let K be the field of fractions of R, and choose $L \supset K$, a separable finite extension, so that $N \otimes_R L$ is a constant L-group scheme; let S' be the normalization of $\mathrm{Spec}(R)$ in L, and $N' = N \times_S S'$, $G' = G \times_S S'$; choose some coordinate system for G' over S', let $v_i^{(m)} \in L$ be all the coordinates of the non-zero points of $N \otimes_R L$, and let W' be the union of all divisors defined by (the poles of the) $v_i^{(m)}$ on S'; because S' is normal each component of W' has codimension one in S', and for $s' \in S'$ the p-rank of the fibre of $\mathsf{X}' = \mathsf{X} \times_S S'$ is smaller than f if and only if $s' \in W'$; thus W' has the required properties, and because each fibre of $S' \to \mathrm{Spec}(R)$ is non-empty and finite, Lemma (1.6) is proved.

Remark (1.7). We should like to make (1.6) more precise in the following sense. Consider the set of 2×2-matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(consider the entries as unknown over some field k of characteristic p, i.e. A can be considered as the generic point of four-dimensional affine space), and consider those matrices, for which

$$AA^{(p)}:=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}=0;$$

is the set of such points given by two equations? (and generalize: A is a $g\times g$-matrix, is the set of points $AA^{(p)}\dots A^{(p^{g-1})}=0$ given by g-equations?). For example

$$c\neq 0, \qquad a^{p+1}+b\,c^p=0=c\,a^p+d\,c^p$$

define a closed set in $(c\neq 0)$ for which $AA^{(p)}=0$ (if $d=0$ it easily follows, if $d\neq 0$, then $a\,d=b\,c$ follows etc.), but if $a=0=c$, and $d\neq 0$, then $AA^{(p)}\neq 0$, although the matrix satisfies the two equations; in particular we were not able to check [19], p. 79, lines 11 and 12 (probably the lower line should read $b_{2p-1}\,b_{p-1}^p+b_{2p-2}\,b_{2p-1}^p=0$, a misprint both in the Russian version and the translation): consider $p=5$, and the curve given by

$$Y^2=X^5+X^3+1;$$

according to [19], p. 79, its Hasse-Witt matrix equals

$$A=\begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$$

thus $AA^{(p)}\neq 0$, although the coefficients of the curve satisfy the equations which are supposed to define the formal type $2G_{1,1}$; however, basically the claims in the middle of p. 79, [19], are correct: at every point of the moduli space M_2 the closed set corresponding to curves with very special Jacobian variety *locally* is given by 2 equations.

Proof of Theorem (1.1 a). Let $W_{g,d}$ be the subset of points in $A_{g,d}$ corresponding to very special abelian varieties; as there exists a proper morphism from a fine moduli scheme to $A_{g,d}$, by (1.5) we conclude $W_{g,d}$ is closed in $A_{g,d}$. Clearly $W_{g,d}$ is non-empty ($W_{g,1}$ is non-empty, contains e.g. a point corresponding to C^g, where C is a supersingular elliptic curve: take an α_p-covering of an AV corresponding to a point in $W_{g,d}$, this gives a point in $W_{g,dp}$, etc.). As each component of $A_{g,d}$ has dimension at least $\frac{1}{2}g(g+1)$ (cf. [24], Theorem 2.3.3), by (1.6) it follows that each component of $W_{g,d}$ has at least dimension $\frac{1}{2}g(g+1)-g=\frac{1}{2}g(g-1)$; as $A_{g,d}$ is quasi-projective and $W_{g,d}$ is closed in $A_{g,d}$, it follows $W_{g,d}$ is quasi-projective. Let R be a discrete valuation ring, which is a k-algebra, with field of fractions K; we want to show

$$\mathrm{Mor}_k(\mathrm{Spec}(R), W_{g,d}) \xrightarrow{\sim} \mathrm{Mor}_k(\mathrm{Spec}(K), W_{g,d}); \qquad (*)$$

this map is injective; suppose given $w \in W_{g,d}(K)$; then there exists a finite extension $K' \supset K$ and a (very special) abelian variety X over $\mathrm{Spec}(K')$ plus a polarization which (over $\bar{K}$) defines the point w; by the stable reduction theorem (cf. [9], I.6; or cf. [25] for other references), there exists a finite extension $L \supset K'$, a discrete valuation ring $U \subset L$ and a smooth stable group scheme Y over $\mathrm{Spec}(U)$ with generic fibre $\mathsf{Y} \otimes L = X \otimes L$; consider the p-Lie algebra $\mathrm{Lie}(\mathsf{Y})$ of the group scheme $\mathsf{Y} \to \mathrm{Spec}(U)$ (cf. [3], II.7.2); the p-operation on $\mathrm{Lie}(\mathsf{Y})$ is nilpotent because $\mathrm{Lie}(\mathsf{Y}) \hookrightarrow \mathrm{Lie}(\mathsf{Y}) \otimes_U L = \mathrm{Lie}(X \otimes L)$, and because X is very special; thus the p-operation on the Lie-algebra $\mathrm{Lie}(Y_0) = \mathrm{Lie}(\mathsf{Y}) \otimes l$ (l is the residue class field of U) is nilpotent; as Y is stable over $\mathrm{Spec}(U)$ this implies Y is an abelian scheme over $\mathrm{Spec}(U)$. Thus the polarization on $X \otimes L$ can be extended to Y, and a commutative diagram results

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[rr] \arrow[dr, dashed] \arrow[dd, "w"'] & & \mathrm{Spec}(R) \arrow[d] \\
& \mathrm{Spec}(L) \arrow[r] \arrow[dl, dashed] & \mathrm{Spec}(U) \arrow[d] \\
W_{g,d} \arrow[rr, hook] & & A_{g,d};
\end{tikzcd}$$

thus w can be extended to $w'\colon \mathrm{Spec}(R) \to A_{g,d}$ and as $W_{g,d}$ is closed in $A_{g,d}$ this morphism factors through $W_{g,d}$; thus we have proved (*) to be an isomorphism; by the valuative criterion for properness (cf. EGA, II.7.3.8), this implies $W_{g,d}$ is proper over $\mathrm{Spec}(k)$, and (1.1a) is proved.

Proof of Theorem (1.1b). Denote by A_g the coarse moduli scheme of principally polarized abelian varieties $A_g = A_{g,1}$, and by

$$I_g = j(M_g), \qquad R_g = j(M'_g) \setminus j(M_g)$$

the subsets corresponding to Jacobians of irreducible curves, respectively corresponding to Jacobians of reducible curves. We know $I_g \cup R_g$ is closed in A_g; moreover $I_1 = A_1$, and $I_g \cup R_g = A_g$ for $g = 2, 3$ for dimension reasons (cf. [27]). Note that R_2 is irreducible and has dimension 2, because it is contained in the image of the natural morphism $A_1 \times A_1 \to A_2$, and that each component of R_3 has dimension at most 4, because R_3 is contained in the images of $A_1 \times A_1 \times A_1$ and $A_1 \times A_2$ in A_3, and $\dim(A_1) = 1$, $\dim(A_2) = 3$. Moreover W_1 is zero-dimensional (the points of W_1 correspond to supersingular elliptic curves, $W_1 \subsetneqq A_1$), and each component of W_2 has dimension at most one: suppose $V \subset W_2$, V irreducible of dimension 2; V cannot be contained in R_2, because the generic point of R_2 corresponds to an ordinary AV; moreover I_2 is affine (this follows from the fact that a curve of genus 2 is hyperelliptic, or, cf. [13]); thus $V \setminus (V \cap I_r)$ would be a closed set of dimension one, but $W_2 \cap R_2$

has dimension zero, contradiction, hence each component of W_2 has dimension at most one. From this we deduce that each component of $W_3 \cap R_3$ has dimension at most one: it is contained in the image of $W_1 \times W_2 \to A_3$. But each component of W_3 is proper and has at least dimension $\frac{1}{2} \cdot 3(3-1)-3=3$ (use (1.1a)); as A_3 is quasi-projective, and k algebraically closed we can intersect with a linear space of dimension $N-2$, $L \hookrightarrow \mathbb{P}_k^N$, $A_3 \hookrightarrow \mathbb{P}_k^N$ such that $L \cap W_3$ is a (complete!) algebraic curve, and $L \cap W_3 \cap R_3 = \emptyset$; then $\overset{-1}{j}(L \cap W_3) \subset M_3$ is a complete algebraic curve, and (1.1b) is proved.

Remarks (1.8). If we could prove each component of W_3 has dimension three, and if we could prove $W_4 \cap I_4$ is non-empty (which looks very plausible, true e.g. if $\operatorname{char}(k)=3$, cf. [19], p. 78, Example 2), then it would follow that M_4 contains a complete algebraic curve.

It seems plausible that M_3 contains a complete rational algebraic curve.

Note that M_6 in case $k=\mathbb{C}$ contains a complete curve of genus 129 as was proved by Kodaira (cf. [15], $n=2$, $m=2$, $m(2n-1)=6$).

In case $\operatorname{char}(k) \neq 2$ the result (1.1a) (and probably also 1.1b) follows from an algebraic construction by Mumford of the Satake compactification (cf. Inventiones math. **3** (1967), p. 236, Main Theorem).

2. Families of AV over Curves

We denote by K^s the separable closure of a field K, and by $T_l X$ the l-Tate-group of an AV X.

Theorem (2.1). *Let k be a perfect field (no restriction on its characteristic), C a complete, smooth irreducible algebraic curve over k and X an* AV *over $K=k(C)$. Let l be a prime number, $l \neq \operatorname{char}(k)$, and suppose that*

$$\rho: \operatorname{Gal}(K^s/K) \to \operatorname{Aut}(T_l X)$$

has the property that

$$\rho[\operatorname{Gal}(K^s/K k^s)] \text{ is a commutative subgroup}$$

of Aut $(T_l X)$. *Then there exists a finite separable extension $L \supset K$, an* AV *Y over the algebraic closure of k in L, and a purely inseparable isogeny*

$$t: Y \otimes L \to X \otimes_K L;$$

the extension $L \supset K$ can be chosen in such a way that it is unramified at all places of C where X has good reduction. If moreover X *is an abelian scheme over C such that $\mathsf{X} \otimes K = X$ and C has a k-rational point, i.e. $C(k) \neq \emptyset$, there exists an unramified k-covering $D \to C$, an* AV *Y over k, and a purely inseparable isogeny t: $Y \otimes_k D \to \mathsf{X} \times_C D$.*

Corollary (2.2). *Suppose* X *is an abelian scheme over* C, *a curve with all the properties stated in* (2.1); *suppose moreover* $\mathrm{genus}(C) \leqq 1$; *then for a suitable k-covering* $D \to C$ *the last conclusion of* (2.1) *holds; if* $\mathrm{genus}(C) = 0$, *then* $D \xrightarrow{\sim} C$.

Corollary (2.3). *Let k be a field of characteristic* zero; *any abelian scheme* $\mathsf{X} \to C$, *with* $\mathrm{genus}(C) \leqq 1$ *becomes constant over a suitable unramified covering* $D \to C$, *and hence any* fine *moduli scheme of abelian schemes in characteristic zero does not contain a complete curve whose normalization has a component of genus zero or one.*

Remark (2.4). Because of the existence of Nmm (= Néron minimal model) the existence of an abelian scheme X over a smooth k-curve C is the same as the existence of an AV $X = \mathsf{X} \otimes K$ over the function field $K = k(C)$ having good reduction at all discrete k-valuations of K. The condition that $Y \otimes D$ and $X \times_C D$ are isogenous is the same as $Y \otimes L$ and $\mathsf{X} \otimes_K L$ being isogenous, where $L = k(D)$. The t as indicated in the last part of the theorem is an isogeny if it is an epimorphism with finite kernel, and it is called purely inseparable if $\mathrm{Ker}(t)$ is an infinitesimal D-group scheme (which is the same as $t \otimes L$ being a purely inseparable isogeny of L-group varieties). Coverings will be considered between smooth, irreducible curves; note that if $D \to C$ is an *unramified* covering with $\mathrm{genus}(C) \leqq 1$, then

$$\mathrm{genus}(C) = \mathrm{genus}(D),$$

because in the unramified case

$$2 \cdot \mathrm{genus}(D) - 2 = (2 \cdot \mathrm{genus}(D) - 2) \cdot n,$$

where n is the degree of the covering (the Hurwitz formula, e.g. cf. [5], p. 215).

Proof of (2.2). The (étale part of the pro-finite completion of the) fundamental group of an algebraic curve C with $\mathrm{genus}(C) \leqq 1$ is known to be commutative (in characteristic p, use a result by Grothendieck, cf. [6], Exposé X, Theorem 2.6), thus if $\mathrm{Gal}(K^s/K k^s)$ acts in an unramified way on $T_l X$ (which is equivalent to X having good reduction everywhere on C: the Néron-Ogg-Shafarevich criterion, cf. [31], Theorem 1 on p. 493) and if moreover $\mathrm{genus}(C) \leqq 1$, the main condition of (2.1) is satisfied because $\pi_1(C) = \mathrm{Gal}(K^s/K k^s)_{\mathrm{unram}}$, thus (2.2) follows from (2.1).

Proof of (2.3). The last conclusion of the theorem implies that all geometric fibres of $\mathsf{X} \to C$ are isogenous to $Y \otimes \bar{k}$; this isogeny, being purely inseparable, is an isomorphism if $\mathrm{char}(k) = 0$ because group schemes in characteristic zero are reduced (hence $\mathrm{Ker}(t) \otimes L$, being local, is trivial); moreover a polarization of a constant family is constant (cf. [21], Corollary 6.2), thus (2.3) follows from (2.2).

Remark (2.5). As Ueno pointed out to me, (2.3) can be proved with topological-analytic methods: if C maps to a fine moduli scheme, the universal covering of C (completeness is essential) maps to the universal covering of that fine moduli scheme, which is a Siegel space, and analysis shows this lifted map to be constant (cf. Kas, [14], p. 790).

Remark (2.6). On the other hand the conclusion of the corollary does not hold in case $\mathrm{char}(k)=p\neq 0$: let Z be an AV over k such that

$$(\alpha_p)^3 \hookrightarrow Z^t,$$

where t denotes the dual abelian variety; e.g. Z is the product of three supersingular elliptic curves; for a point $a\in \mathbb{P}_k^2(R)$, where R is a k-algebra, we define

$$\alpha_p \otimes R \xrightarrow{\sim} N_a \subset Z^t \otimes R$$

as follows: let $b_0, b_1, b_2 \in \mathrm{Hom}_k(\alpha_p, (\alpha_p)^3)$ be linearly independent, $a=(a_0:a_1:a_2)$, then

$$N_a := \mathrm{Im}\big((a_0\, b_0,\, a_1\, b_1,\, a_2\, b_2)\colon\ \alpha_p \to Z^t \otimes R\big);$$

we define

$$X_a := (Z^t/N_a)^t,$$

thus we have an exact sequence of group schemes over $\mathbb{P}_k^2$:

$$0 \to \mathsf{N}^D \to \mathsf{X} \xrightarrow{\pi} Z \times \mathbb{P}^2 \to 0$$

(cf. [23], Theorem 19.1; here N^D denotes the dual of N; note $\alpha_p^D \cong \alpha_p$); choose a polarization on Z (and hence on $Z \times \mathbb{P}^2$), and lift it to a polarization λ on X via π (and we can also choose a level n-structure on X); let M be the moduli space of polarized abelian varieties (respectively of polarized abelian varieties with level n-structure); then (X, λ) (plus the level n-structure) defines a k-morphism

$$f\colon\ \mathbb{P}_k^2 \to M,$$

and we claim the image of f has dimension 2: assume k is algebraically closed, suppose $E \subset \mathbb{P}_k^2$ is an irreducible k-algebraic curve such that $f(E)$ is one point on M; this implies there exists a finite k-morphism $g\colon E' \to E$, and an AV Y over k, and an E'-isomorphism

$$Y \otimes_k E' \xrightarrow{\sim} g^*(\mathsf{X} \mid E);$$

this yields a homomorphism

$$h\colon\ Z^t \otimes_k E' \to g^*(\mathsf{X}^t \mid E) \cong Y^t \otimes_k E'$$

and for $e \in E'(k)$, the kernel of $h_e\colon Z^t \to Y^t$

$$\mathrm{Ker}(h_e) = N_{g(e)};$$

take a point $e_0 \in E'(k)$, let $h_0\colon Z^t \to Y^t$ be the fibre of h at e_0; the homomorphisms h and $h_0 \otimes_k E'$ coincide over e_0, thus (cf. [21], Corollary 6.2) $h = h_0 \otimes_k E'$, which implies

$$\operatorname{Ker}(h_e) = \operatorname{Ker}(h_{e_0}) \qquad \text{for all } e \in E'(h);$$

as $\{b_0, b_1, b_2\}$ was a basis, different geometric points of $\mathbb{P}_k^2$ yield different subgroup schemes $N_a \subset Z^t$, a contradiction; thus $f(E)$ is not a point in M, and $f(\mathbb{P}_k^2)$ has dimension two; conclusion: *M contains curves E_0, E_1 such that the normalization of E_i has genus i* (it is not difficult to choose $E_i' \subset \mathbb{P}^2$, genus$(E_i') = i$ such that $f \mid E_i'$ is birational). This example seems to contradict [32], Theorem 5, in case $\operatorname{char}(k) = p \neq 0$ and $\dim(A) = d > 2$ (in order to construct E_0 we can do the same procedure with $(\alpha_p)^2 \subset Z^t$, so in that case we can take $d \geqq 2$); in the proof of [32], Theorem 5 in case of characteristic $p \neq 0$ the trace A^* need not be a subvariety of A (cf. [16], VIII.3, Corollary 2 on p. 216).

Proof of (2.1). Suppose k', C', X' as in the theorem. Because C' is of finite type over k', and X' is of finite type over K (and $\mathbf{X}'$ of finite type over C'), there exist $k, C, \mathbf{X}$ such that $k \subset k'$, $C' = C \otimes k'$, $X' = \mathbf{X} \otimes k'(C')$, and such that under the identification $T_l X \cong T_l X'$ (choose a k-embedding $k^s \subset k'^s$) the groups $\operatorname{Gal}(K^s/K k^s)$ and $\operatorname{Gal}(K'^s/K' k'^s)$ have the same image, and such that k is the perfect closure of a field finitely generated over its prime field. Thus it suffices to prove (2.1) in case k has this property.

In order to prove the first conclusion of the theorem, we replace k by a finite extension, again denoted by k, so that all places of C where X has bad reduction become rational over the new field, and such that at least one more point of C is rational over k (note that these properties are satisfied already in the second part of the theorem). We choose $N = l^2$; the group scheme

$$_N X = \operatorname{Ker}(\times N\colon\ X \to X)$$

is constant over an extension $L \supset K$, $L = k(D)$; because $(l, \operatorname{char}(p)) = 1$ this extension (or: the covering $D \to C$, $k(D) = L$) can be chosen such that it is separable and unramified at all places where X has good reduction; we denote the new fields again by k, K, with $K = k(C)$, and we arrive at a situation where $_N X$ is a K-constant group scheme, i.e. $\operatorname{Gal}(K^s/K)$ acts trivially on $_N X(K^s)$.

First Step. For every $a \in \operatorname{Gal}(K^s/K k^s)$ all eigenvalues of $\rho\, a \in \operatorname{Aut}(T_l X)$ are equal to one. This we prove as follows. Let $M \subset C(k)$ be the set of points where X has bad reduction. Let

$$K k^s \subset L \subset K^s,$$

where L is the union of all abelian extension of Kk^s, of degree a power of l, unramified at all places (discrete valuations) of Kk^s/k^s outside M; clearly $\mathrm{Gal}(K^s/L)$ is a characteristic subgroup of $\mathrm{Gal}(K^s/Kk^s)$ (i.e. invariant under every automorphism). Let

$$J = \mathrm{Jac}_M(C)$$

be the generalized Jacobian variety of C constructed with M as conductor (all multiplicities of points in M equal to one), cf. [30], Chap. 5; this is a k-group variety, there is an exact sequence of k-group varieties

$$0 \to (\mathbb{G}_m)^{|M|-1} = H \to J \to \mathrm{Jac}(C) \to 0$$

where $\mathrm{Jac}(C)$ is the (ordinary) Jacobian variety of C (and H has this form because all points in M have multiplicity one and $M \subset C(k)$). Because J is a k-group scheme, $T_l J$ is a $\mathrm{Gal}(k^s/k)$-module in a natural way. Because $\mathrm{Gal}(K^s/L)$ is characteristic in $\mathrm{Gal}(K^s/Kk^s)$, and abelian,

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Gal}(K^s/Kk^s) \arrow[r] \arrow[d] & \mathrm{Gal}(K^s/K) \arrow[r] \arrow[d, "\rho"] & \mathrm{Gal}(k^s/k) \arrow[r] & 0 \\
& \mathrm{Gal}(L/Kk^s) \cong T_l J \arrow[r, dotted, "\varrho"] & \mathrm{Aut}(T_l X) & &
\end{tikzcd}$$

the action by $\mathrm{Gal}(k^s/k)$ via inner conjugation inside $\mathrm{Gal}(K^s/K)$ on $\mathrm{Gal}(K^s/Kk^s)$ induces an action on $\mathrm{Gal}(L/Kk^s)$; there exists a natural isomorphism of $\mathrm{Gal}(k^s/k)$-modules

$$\mathrm{Gal}(L/Kk^s) \cong T_l J;$$

this can be seen as follows: a finite abelian extension of Kk^s is induced by an isogeny of a generalized Jacobian of C (cf. [30], VI.11, Prop. 9); if such an extension is contained in L, we can choose the support of the conductor to be contained in M (cf. [30], VI.12, Lemma 1); in fact in that case, the degree of the extension is a power of l, hence prime to $\mathrm{char}(k)$, thus the conductor can be chosen contained in M (cf. [30], p. 128, Example 1: if $M \subset N$, $\mathrm{support}(N) = \mathrm{support}(M)$, then

$$\mathrm{Ker}\big(\mathrm{Jac}_N(C) \to \mathrm{Jac}_M(C)\big)$$

is a unipotent group scheme, hence has no l-torsion points); the isogenies $\times l^i \colon J \to J$ are cofinal in the set of all isogenies over J of degree a power of l (if $G \to J$ is an isogeny with kernel annihilated by $\times l^i$, then its factors multiplication by l^i on J), thus the natural action of l-power torsion points on J on the corresponding coverings of C induce an isomorphism (use [30], VI.11, Proposition 10) between $\mathrm{Gal}(L/Kk^s)$ and $T_l J$; note that the canonical morphism $C \setminus M \to J$ is defined over k, because moreover $C(k) \setminus M \neq \emptyset$, and it follows that the action of $\mathrm{Gal}(k^s/k)$

commutes with the isomorphism. Thus we obtain a commutative diagram as indicated above.

Because k is the perfect closure of a field finitely generated over its prime field, there exists an element $\sigma \in \mathrm{Gal}(k^s/k)$ such that its action on $T_l J$

$$r\colon \mathrm{Gal}(k^s/k) \to \mathrm{Aut}(T_l J)$$

has the property:

$$r(\sigma) \in \mathrm{Aut}(T_l J \otimes \mathbb{Q}_l)$$

has no eigenvalues equal to a root of unity and the characteristic polynomial (of any power) of $r(\sigma)$ has integral coefficients (cf. [25], Lemma 3.2). Let $\sigma' \in \mathrm{Gal}(K^s/K)$ be such that

$$\mathrm{Gal}(K^s/K) \to \mathrm{Gal}(K k^s/K) \cong \mathrm{Gal}(k^s/k), \qquad \sigma' \mapsto \sigma,$$

and write $\rho(\sigma') =: S \in \mathrm{Aut}(T_l X)$; let $a \in T_l J$, $\rho\, a =: A \in \mathrm{Aut}(T_l X)$; choose $Q \supset \mathbb{Q}_l$ containing all eigenvalues of A and of S.

Consider $a_i := (r(\sigma^{-i}))(a)$, and $A_i := \rho\, a_i$; because of the hypothesis made, *the matrices A_i all commute;* because of the way σ acts on $\mathrm{Gal}(L/K k^s) \cong T_l J$,

$$A_i = S^{-i} A S^i, \qquad i \in \mathbb{Z}$$

hence any *two of these matrices have the same set of eigenvalues*. Consider all infinite sequences

$$u = (\ldots, u_{-1}, u_0, u_1, u_2, \ldots)$$

of eigenvalues of A; let $E := T_l X \otimes_{\mathbb{Z}_l} Q$, and let

$$E_u := \bigcap_{i \in \mathbb{Z}} \mathrm{Ker}(A_i - u_i) \subset E;$$

fix $i \in \mathbb{Z}$, and consider $\mathrm{Ker}(A_i - u) = E_{i_u}$, where u is an eigenvalue; clearly

$$\bigoplus_u E_{i_u} = \sum_u E_{iu}$$

(on E_{iu} the transformation A_i has eigenvalue u and on $\sum\limits_{v \neq u} E_{iv}$ the map A_i has no eigenvalue equal to u); thus it follows (because the A_i commute, and $\dim(E) < \infty$) that the set U of sequence u with $E_u \neq 0$ is finite and their sum is a direct sum

$$\bigoplus_{u \in U} E_u \xrightarrow{\sim} F := \Big(\sum_{u \in U} E_u\Big) \subset E.$$

Note that

$$S(E_u) = E_{S(u)},$$

where $(S(u))_i := u_{i+1}$, i.e. it is the shift to the left:

$$x \in E_u \Rightarrow S S^{-1} S^{-i} A S^i S x = S u_{i+1} x;$$

thus $S(E_u) \subset E_{S(u)}$, thus $SF \subset F$; because S is invertible $SF = F$, thus $S(E_u) = E_{S(u)}$; thus S induces a permutation

$$S\colon U \to U.$$

As U is finite, there exists an integer n so that $S^n(E_u) = E_u$ for all $u \in U$. Let $T = S^n$, let $\tau = \sigma^n$, let P be the characteristic polynomial of $r(\tau)$; note P has coefficients in $\mathbb{Z}$, and $P(1) \neq 0$ (because $r(\sigma)$ has no eigenvalues equal to a root of units, hence $r(\tau)$ does not have 1 as eigenvalue). Let λ be an eigenvalue of $A = \rho(a)$; choose $u \in U$ with $u_0 = \lambda$ (this is possible because $\mathrm{Ker}(A - \lambda) \neq 0$, etc.); as $T(E_u) = E_u$, for all $x \in E_u$,

$$(T^{-i} A T^i)\, x = \lambda x;$$

note that

$$(P(\tau))(a) = 0,$$

thus $\rho(P(\tau)(a)) = 1$,

$$\rho(P(\tau)(a))(x) = \lambda^{P(1)}\, x = 1,$$

and $P(1)$ being non-zero, this implies that λ is a root of unity. Because $\mathrm{Gal}(K^s/K)$ acts trivially on ${}_N X(K^s) = T_l X / N(T_l X)$,

$$A \equiv 1 \pmod N$$

(considered as $\mathbb{Z}_l$-linear maps), and λ being an eigenvalue of A, we conclude

$$\lambda \equiv 1 \pmod l$$

(in the ring of integers of $\mathbb{Q}_l(\lambda)$), thus λ is an l-power root of unity. Suppose $\lambda \neq 1$; in the ring of integers of $\mathbb{Q}_l(\lambda)$ the element $\pi = 1 - \lambda$ divides l, and

$$\det(A - \lambda I) = 0,$$

thus

$$\pi^{2g} + \pi^{2g-1} \cdot N \cdot a_1 + \cdots + \pi^{2g-i} \cdot N^i \cdot a_i + \cdots = 0, \qquad a_i \in \mathbb{Z}_l,$$

which implies (because $\pi^2 \mid N = l^2$):

$$\pi^{2g} \equiv 0 \pmod{\pi^{2g+1}};$$

however, from $\lambda \neq 1$ it follows that π is a uniformizing element of $\mathbb{Q}_l(\lambda)$ (e.g. cf. [35], 7.4.1), contradiction, and hence $\lambda = 1$, which concludes the proof of the first step.

Remark. In case $l > 2g+1$, one can choose $N = l$: if $\mathrm{Gal}(K^s/K)$ acts trivially on ${}_l X(K^s)$, again λ is a l-power root of unity, its degree over $\mathbb{Q}_l$

is at most $2g$ because it is an eigenvalue of $A \in \mathrm{Aut}((\mathbb{Z}_l)^{2g})$ and $[\mathbb{Q}_l(\zeta):\mathbb{Q}_l] \geqq l-1$ for any l-power root of unity not equal to one, thus it follows that $\lambda = 1$.

Last Step of the Proof. Consider

$$Y := \mathrm{Tr}_{K/k} X,$$

i.e. this is an AV over k, plus a purely inseparable homomorphism

$$t\colon\; Y \otimes_k K \to X$$

(with finite kernel) which has a certain universal property (cf. [16], VIII.3, Theorem 8 on p. 213, and Corollary 2 on p. 216); we want to show t is an isogeny (i.e. t is epimorphic); let Z be an AV over K defined by the exact sequence

$$Y \otimes K \xrightarrow{\ t\ } X \longrightarrow Z \longrightarrow 0;$$

because t is purely inseparable, and $l \neq \mathrm{char}(k)$, we obtain an exact sequence

$$0 \to T_l Y \to T_l X \to T_l Z \to 0,$$

the action of $\mathrm{Gal}(K^s/K k^s)$ on $T_l Y$ is trivial and by the first step we conclude all images of

$$\rho\colon\; \mathrm{Gal}(K^s/K k^s) \to \mathrm{Aut}(T_l Z)$$

have eigenvalues all equal to one, and this image is commutative; this implies that if $Z \neq 0$, there exists

$$0 \neq v \in \bigcap \mathrm{Ker}(\rho(b) - 1),$$

the intersection taken over all $b \in \mathrm{Gal}(K^s/K k^s)$ (or all $b \in T_l J$); we define

$$Z' := \mathrm{Tr}_{K/k} Z, \quad \text{thus} \quad T_l Z' \hookrightarrow T_l Z;$$

by the Lang-Néron version of the Mordell-Weil theorem we conclude (cf. [17], p. 97, Theorem 1) that $v \in T_l Z'$, thus $Z' \neq 0$; let

$$Z'' := Z/\mathrm{Im}(Z' \otimes K \to Z);$$

by [16], II.1, Theorem 6, we conclude there exists an isogeny

$$(Y \otimes K) \oplus (Z' \otimes K) \oplus Z'' \to X,$$

thus the image of the K/k-trace of X contains the image of $(Y \oplus Z') \otimes K$ (universal property of the trace), the K/k-trace of X is $t\colon Y \otimes K \to X$, thus $Z' = 0$; thus $Z = 0$, thus t is epi, i.e. t is a purely inseparable isogeny, which concludes the proof of (2.1).

Remark (2.7). Probably the methods above can also be used to prove the statements (2.1) and (2.2) with C replaced by (the function field of) an AV over k.

Remark (2.8). In case genus $(C) = 0$, the result (2.2) was proved by Grothendieck (cf. [8], pp. 74/75, Proposition 4.4). Theorem (2.1) was inspired by the result of Grothendieck which says that an AV with sufficiently many CM is isogenous to an AV defined over an algebraic extension of the prime field (cf. [22], Theorem on p. 220, and cf. [26]). Possibly these theorems are special cases of more general results.

Remark (2.9). It is not difficult to construct examples of non-trivial families $\mathsf{X} \to C$ over a complete elliptic curve over any given base field: take an elliptic curve D, a point $d \in D(k)$ of order $(d) > 1$, and an AV Y over k with $a \in \mathrm{Aut}_k(Y)$, with order $(a) = $ order (d); let $C = D/\langle d \rangle$ and let X be the quotient of $D \times Y$ by the equivalence relation $(u, y) \sim (u+d, a(y))$.

3. Subvarieties of Moduli Spaces Defined by Kodaira Surfaces[1]

In the previous section we showed that a *fine* moduli scheme of algebraic curves in characteristic zero does not contain a complete elliptic or rational curve (use Corollary (2.3) plus Torelli's theorem which says that the Jacobi-morphism from moduli schemes of nonsingular irreducible curves to the moduli schemes of principally AV is injective); in this section we follow a construction suggested by Parshin of certain surfaces, so that we can prove:

Theorem (3.1). *In any characteristic there exists a number g such that the* coarse *moduli scheme M_g of (irreducible and nonsingular) algebraic curves contains a complete* rational *curve $E \subset M_g$.*

By a Kodaira surface, or an irregular algebraic surface, we mean a complete (nonsingular) algebraic surface M plus a smooth morphism $h \colon \mathsf{M} \to D$ onto a complete nonsingular irreducible algebraic curve D; such surfaces were constructed by Kodaira (cf. [15], also cf. Kas, [14]). The family h of curves parametrized by D defines a morphism $M \colon D \to M_g$, where g is the genus of each of the fibres of $\mathsf{M} \to D$; we construct M and D such that $D \to M_g$ factors through a rational curve.

(3.2) *Parshin's Construction* (cf. [28], pp. 1168/1169, and 1163–1167). Suppose given a field k, a smooth, irreducible, complete algebraic curve B over k, two positive integers m, n both prime to char(k); then we arrive at an etale covering $D \to B$ and a Kodaira surface $\mathsf{M} \to D$; moreover if genus $(B) \geqq 2$, and $m > 1$, and $n > 1$, then no (etale) covering of D makes the fibration $\mathsf{M} \to D$ trivial (equivalently: at least two fibres of $\mathsf{M} \to D$ are non-isomorphic). The construction is performed as follows:

$$\begin{tikzcd}
\mathsf{D}, \mathsf{P} \subset C \arrow[r, hook] \arrow[d, "\pi"'] \arrow[dr, phantom, "\text{cartesian}"] & J \times B \arrow[d, "(\times n) \times id"] \\
B \times B \arrow[r, hook, "i"] & J \times B
\end{tikzcd}$$

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[1] I thank Knud Lønsted for drawing my attention to the paper by Parshin and for stimulating discussion on this topic.

consider $J := \mathrm{Jac}(B)$, let $B \to J$ be the canonical embedding of B into its Jacobian variety (e.g. cf. [30], V. 6 and V. 9), and define i so that $i(b,b) = (0,b)$ (i.e. the "variable" point b is used to define the canonical morphism i_b: $B \times \{b\} \to J \times \{b\}$); let C be the pull-back of $i(B \times B)$ by multiplication by n on the B-group scheme $J \times B$, and let

$$\mathsf{D} \cup \mathsf{P} = \pi^{-1}(b,b)$$

be the fibre over (b,b) so that P: $B \times B \to C$ is the section defined by

$$\mathsf{P} = \{0\} \times B \subset C \subset J \times B;$$

then C is irreducible (cf. [30], VI.11, Proposition 10), and π is an etale covering because $\mathrm{char}(k)$ does not divide n; note that the degree of the relative Cartier divisor $\mathsf{D} \subset C \to B \times B$ is $n^{2 \cdot \mathrm{genus}\,(B)} - 1$, so bigger than one if $n > 1$ and $\mathrm{genus}(B) > 0$; because D is an etale covering of $B \times B$, we can choose an etale covering $D \to B$

$$\begin{tikzcd}
C \arrow[d, "\pi"] & C \times_B D =: Y \arrow[l] \arrow[dd, "\alpha"'] \\
B \times B \arrow[d, "\mathrm{proj}_2"] & \\
B & D \arrow[l] \arrow[uu, bend right=60, "P_i"']
\end{tikzcd}$$

and sections P_i: $D \to Y$ so that

$$\mathsf{D} \times_B D = \cup P_i$$

(in case all points of ${}_nJ$ are rational over k, we can choose $D = B$); we write P for the section $\mathsf{P} \times_B D$: $D \to Y$, and using D, P, P_i, and the integer m we construct

$$\begin{tikzcd}
 & M \arrow[dl, dashed, "\beta"'] \arrow[dd, "h"] \\
Y \arrow[dr, dashed, "\alpha"'] & \\
 & D
\end{tikzcd}$$

as follows: first, let $\zeta \in D$ be the generic point, with fibre Y_ζ over this point; on this fibre there is a point $P(\zeta)$ and a divisor $\delta = \sum P_i(\zeta)$, thus there results a canonical morphism

$$Y_\zeta^0 = Y_\zeta \setminus \mathrm{Supp}(\delta) \to \mathrm{Jac}_\delta(Y_\zeta) = J_\delta$$

into its generalized Jacobian variety with respect to δ, sending $P(\zeta)$ onto the zero point of $0 \in J_\delta$ (and this morphism is defined over $k(\zeta)$, because

$P(\zeta)$ and all $P_i(\zeta)$ are rational over $k(\zeta)$); then we construct

$$\begin{tikzcd} M_\zeta \supset M_\zeta^0 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{cartesian}"] & J_\delta \arrow[d, "\times m"] \\ Y_\zeta \supset Y_\zeta^0 \arrow[r, hook] & J_\delta \end{tikzcd}$$

the etale covering $M_\zeta^0 \to Y_\zeta^0$ defined by multiplication by m on J_δ (again we know the covering is irreducible, and etale because char(k) does not divide m); the curve M_ζ is the unique smooth complete irreducible curve containing M_ζ^0; the minimal fibring h: $\mathsf{M} \to D$ having M_ζ as generic fibre exists, it is smooth (use the fact that $Y \to D$ is smooth, and $\mathsf{D} \cup \mathsf{P}$ etale over $B \times B$, and apply [28], p. 1164, Lemma 10), and for every point d of D, the fibre M_d can be constructed as follows:

$$\begin{tikzcd} M_d \supset M_d^0 \arrow[r] \arrow[d] & J_{\delta(d)} \arrow[d, "\times m"] \\ Y_d \supset Y_d^0 \arrow[r, "i"] & J_{\delta(d)}, \end{tikzcd} \quad i(P(d))=0,$$

with $Y_d^0 = Y_d \setminus \operatorname{Supp} \delta(d)$ (this follows because M_d is the unique smooth complete curve containing $M_d^0 \subset \mathsf{M}$); note that if $\deg(\delta)>1$, and $m>1$ then the morphism

$$\beta_d\colon\ M_d \to Y_d$$

ramifies exactly at all points of $\delta(d)$; this can be seen as follows: all points in $\delta(d)$ have multiplicity one, thus $J_{\delta(d)}$ is an extension over $\mathrm{Jac}(Y_d)$ with kernel $L = (\mathbb{G}_m)^{\deg(\delta)-1}$; thus

$$L \cap \operatorname{Ker}(\times m\colon\ J_{\delta(d)} \to J_{\delta(d)}) \neq 0;$$

because the unramified coverings of Y_d correspond bijectively with isogenies over $\mathrm{Jac}(Y_d)$ (cf. [30], VI.12, Corollary on p. 128) it follows β_d ramifies at each of the points $P_i(d) \in Y_d$;

$$\begin{tikzcd} & C_b \arrow[dl, "\pi"'] \arrow[d, hook] & Y_d \arrow[l, "\sim"'] \arrow[d, hook] & M_d \arrow[l] \arrow[d, hook] \\ B \times \{b\} \arrow[dr] & C \arrow[d] & Y \arrow[l] \arrow[dr, "\alpha"'] & \mathsf{M} \arrow[l, "\beta"'] \arrow[dl, "h"] \\ & B & & D \arrow[ll] \end{tikzcd}$$

let d be a point of D mapping onto a point b of B; if genus$(B)>0$ and $n>1$ (so $\deg(\delta)>1$), and $m>1$ then the composed covering $M_d \to B \times \{b\}$ ramifies at the point (b, b).

Lemma (3.3). *Let k be a field, B and Z complete smooth irreducible algebraic curves over k, and T a k-prescheme. Let*

$$\varphi\colon\ Z\times T\to B\times T$$

be a family of morphisms of Z to B parametrized by T (i.e. *φ commutes with the projections on T*); *if genus$(B)\geqq 2$, then φ is locally constant on T.*

Proof. One way of proving the lemma is the following: suppose $T=\operatorname{Spec} k[\varepsilon]$, then all $\varphi\colon Z\times T\to B\times T$ deforming a fixed $(\varphi_0\colon Z\to B)=\varphi\otimes_{k[\varepsilon]}k$ are in 1-1-correspondence with $\Gamma(B,\mathscr{D}er(\mathcal{O}_B,\varphi_{0*}\mathcal{O}_Z))$ (here $\mathscr{D}er$ stands for the sheaf of germs of k-$\mathcal{O}_B$-derivations), and one can show this set of sections to be trivial in case $\mathscr{D}er(\mathcal{O}_B,\mathcal{O}_B)$ is a negative line bundle (its degree is $2-2g$). Another way of proving is the following. Because the functor of morphisms from Z to B is representable (Grothendieck, cf. Sém. Bourbaki **13**, p. 221-20 (1960/61)), and smooth (B is a curve), it suffices to prove the lemma in case k is algebraically closed and T is connected and reduced (and this is the case which we need in applying the lemma). Let $t\in T(k)$, and consider

$$\begin{array}{ccc} Z & \xhookrightarrow{\ j\ } & \operatorname{Jac}(Z)\\ \Big\downarrow{\scriptstyle\varphi_t} & & \Big\downarrow{\scriptstyle\psi_t=\lambda+a_t}\\ B & \xhookrightarrow{\ i\ } & \operatorname{Jac}(B); \end{array}$$

because of the Albanese property of the Jacobian variety (of B) the morphism ψ_t results, making commutative the diagram; let $a_t=\psi_t(0)$; because of the rigidity lemma ([21], Corollary 6.2) the morphism $\lambda=\psi_t-a_t$ (is a homomorphism which) does not depend on t; fix $s\in T(k)$ and consider

$$b\in B,\qquad (ib)\mapsto(ib+a_s-a_t)=:\gamma_t(b)$$

because

$$\lambda jZ+a_t=iB=\lambda jZ+a_s$$

we conclude we obtain a family of automorphisms

$$\gamma\colon B\times T\to B\times T,\qquad (b,t)\mapsto(\gamma_t b,t)$$

(we identified B and $B\simeq iB$); because genus$(B)\geqq 2$ we know B has only finitely many automorphisms thus T being connected (and reduced), the family γ is constant, thus $a_s=a_t$ for all t, thus φ is constant. Q.E.D.

Now suppose, notation as in (3.2): genus$(B)\geqq 2$, $m>1$ and $n>1$; let $T\to D$ be a covering, then $\mathsf{M}\times_D T$ is not trivial; in fact, suppose $\mathsf{M}\times_D T\simeq Z\times T$; then apply (3.3) to

$$(\mathsf{M}\to Y\to C\to B\times B)\times_D T,$$

thus concluding that for each $d \in D$ the covering

$$M_d = (\mathsf{M} \times_D T)_t = Z \to Y_d \to C_b \to B \times \{b\}$$

with $t \mapsto d \mapsto b$ does not depend on t; however, we have seen that this covering ramifies at b in the given situation, thus while t runs through T, the point b is not constant, and $M_d \to B \times \{b\}$ cannot be a constant morphism; hence the fibring h: $\mathsf{M} \to D$ cannot be trivialized by a covering of D, and the construction and the claim in the first sentence of (3.2) are established (the arguments essentially can be found on p. 1169 of [28]).

Proof of (3.1). Choose a curve B with genus$(B) \geqq 2$, and a (finite) group $G \subset \mathrm{Aut}(B)$ with

$$q\colon\ B \to B/G \simeq \mathbb{P}^1$$

(such an example is easy to construct, e.g. take any curve B' with genus$(B') \geqq 2$; its function field $k(B')$ is a separable extension of $k(\mathbb{P}^1)$, and take for B the normalization of B' in some finite Galois extension of $k(\mathbb{P}^1)$ containing $k(B')$). Using the construction explained above, with the help of integers m and n (prime to char(k) and both at least equal to two), we arrive at a Kodaira surface

$$\begin{array}{ccccc} & & \mathsf{M} & & \\ & & \Big\downarrow{\scriptstyle h} & & \\ B & \xleftarrow{\ f\ } & D & & \\ \Big\downarrow{\scriptstyle q} & & & \searrow^{M} & \\ \mathbb{P}^1 & \dashrightarrow\!\!\!\rightarrow & E & \hookrightarrow & M_g; \end{array}$$

thus we obtain a morphism M: $D \to M_g$, with $g = \mathrm{genus}(\mathsf{M}/D)$, and we claim this factors through $\mathbb{P}^1$; because h is not locally trivial in the etale topology, the image $M(D) = E \subset M_g$ is not a point, but a curve, and because of Lüroth's theorem we conclude, $\mathbb{P}^1 \twoheadrightarrow E \subset M_g$, this to be a rational curve; the proof of the factorization follows because: if $d, e \in D(k)$, and $qfd = qfe$ then $M_d \simeq M_e$; indeed in that case there exists $\sigma \in G$ with $fd = \sigma fe$; the morphism

$$\sigma\colon\ B \times \{b\} \to B \times \{\sigma b\}$$

extends to a commutative diagram

$$\begin{array}{ccc} B \times \{b\} & \longrightarrow & \mathrm{Jac}(B) \\ \Big\downarrow{\scriptstyle \sigma} & & \Big\downarrow{\scriptstyle \sigma_*} \\ B \times \{\sigma b\} & \longrightarrow & \mathrm{Jac}(B) \end{array}$$

thus arriving at a morphism

$$\sigma_*: C_b \xrightarrow{\sim} C_{\sigma b}, \sigma_* P_b = P_{\sigma b}, \sigma_* D_b = D_{\sigma b};$$

this results in an isomorphism

$$\sigma_*: M_d^0 \xrightarrow{\sim} M_e^0,$$

and because M_d is the unique smooth curve containing M_d^0 (and the analogous statement for M_e^0 and M_e), we conclude $M_d \simeq M_e$, and Theorem (3.1) is proved.

4. Supersingular AV

In this section all fields considered will be of characteristic $p \neq 0$.

Consider an elliptic curve E over an algebraically closed field k; the following properties are equivalent:

a) E is very special (i.e. E has no points of order p);

b) $\mathrm{End}_k(E)$ has rank 4 as $\mathbb{Z}$-module (note k is algebraically closed);

c) the formal group of E is isomorphic to $G_{1,1}$ (notation of [19], p. 35);

d) α_p is a subgroup scheme of E (here α_p denotes the kernel of the Frobenius homomorphism $F: \mathbb{G}_a \to \mathbb{G}_a$).

An elliptic curve E over a field l is said to be *supersingular* if it satisfies these properties over an algebraic closure k of l; note that a supersingular curve is isomorphic over k with an elliptic curve defined over the field $\mathbb{F}_{p^2}$, the quadratic extension of the prime field; note that any two supersingular curves are isogenous over an algebraically closed field (in fact they are already isogenous over an extension of degree 12 of a common field of definition, cf. [34], pp. 537/538).

Definition (4.1). An AV X over a field l is called *supersingular* if the formal group $\hat{X}$ of X is isogenous over an algebraic closure k of l to $(G_{1,1})^g$, i.e.

$$\hat{X} \otimes k \sim \hat{E}^g \otimes k,$$

where E is a supersingular curve.

Theorem (4.2). *Let X be a supersingular* AV *over a field l; then X is isogenous to E^g over k,*

$$X \otimes k \sim E^g \otimes k,$$

where k is an algebraic closure of l, and E is a supersingular curve.

Remark (4.3). This gives a positive answer to a question posed by Manin in the case $g=2$ (cf. [19], p. 79). In general the isogeny type of the formal group of an AV does not imply the AV can be decomposed up to isogeny in the same manner; e.g. there exists an abelian surface X whose formal group is isogenous with $G_{1,0} \oplus G_{1,1}$ such that X is a simple

AV; this can be seen as follows: consider the closed subscheme $V \subset A_2$ corresponding to abelian surfaces with p-rank equal to one (notation as in Section 1); as $\dim(A_2)=3$, from (1.6) we conclude $\dim(V)=2$; an abelian surface with p-rank equal to one has no positive dimensional families of finite subgroup schemes, thus if such an abelian surface is isogenous to a product of two elliptic curves, it can be defined over a field of transcendence degree one over the prime field; thus the generic point of V corresponds to a simple AV (this fact was already proved by Honda, cf. [11], p. 93). More generally: for any isogeny type of abelian varieties not equal to a power of $G_{1,1}$ there exist a simple AV having that isogeny type (cf. [18]).

Notation (4.4). Let X be an AV over a field l; by $a(X)$ we denote the dimension

$$a(X) := \dim_k \mathrm{HOM}(\alpha_p, X \otimes k),$$

where k is an algebraically closed field containing l (note that $\mathrm{End}_k(\alpha_p) \cong k$, a ring isomorphism, thus $\mathrm{HOM}(\alpha_p, G)$ is a right-k-module for any k-group scheme G).

Lemma (4.5). *Let X be an* AV *over a field l such that $a(X)=\dim X$; then X can be defined over a finite field.*

Proof. Take a polarization on X over l, let its degree be d, choose a large integer n prime to p so that $A_{g,d,n}$ is a fine moduli scheme, $g=\dim X$, take a level n-structure on X (make a finite extension of l if necessary), and consider the corresponding point $x \in A_{g,d,n}(l)$. Let V be the closure of x in $A_{g,d,n}$, and let $y \in V(k)$, where k is an algebraic closure of the prime field; we want to show $\dim V=0$, thus it suffices to show the tangent space to V at y to be zero. Note that $a(X)=\dim(X)$ is equivalent by saying that the p-operation in the Lie-algebra $\mathrm{Lie}(X)$ is zero. Let $\mathbf{X}$ be the abelian scheme over $V \subset A_{g,d,n}$ induced from the universal family over this fine moduli scheme; $\mathrm{Lie}(\mathbf{X})$ is a locally free sheaf of p-Lie algebras over V, and clearly the p-operation is identically zero; let

$$t\colon \operatorname{Spec}(k[\varepsilon]) \to V \subset A_{g,d,n}$$

be a tangent vector, $\varepsilon^2=0$, to V at y; then the fibre $X_t := \mathbf{X} \times_V t$ has the property $\mathrm{Lie}(X_t)$ has p-operation equal to zero; from this we are going to conclude t is a zero vector. Consider $Y=X_y$ the AV over k which is the fibre of $\mathbf{X}$ at $y \in V$; let $\mathscr{L}=\mathscr{L}(Y;\, k[\varepsilon] \to k)$ be the set of infinitesimal deformations of Y (notations of [24], p. 274 and p. 277), this set can be canonically identified with the tangent space of the local moduli space M of Y, i.e. $\mathscr{L}=M(k[\varepsilon])$, and the isomorphism class X_t can be considered as an element $t \in \mathscr{L}$. Let $G_0=\mathrm{Lie}(Y)$ be the Lie algebra of Y, and write $R=k[\varepsilon]$, $\varepsilon^2=0$; consider the set $\mathscr{K}=\mathscr{K}(G_0;\, R \to k)$ of infinitesimal deformation of G_0:

$\mathcal{K} := \{\cong \text{classes of } (G, \varphi_0) \mid G$ is a p-Lie algebra over R, and
$\varphi_0\colon G \otimes_R k \xrightarrow{\sim} G_0\}$;

this is a k-vector space in a natural way (see below, or use [29], Lemma 2.10, exactness of the infinitesimal deformation functor of G_0 is immediate); an infinitesimal deformation of Y yields the same for G_0, thus a natural map

$$\mathcal{L} \to \mathcal{K}$$

results, which is k-linear (because the map which associates with Y' over R its Lie-algebra $\mathrm{Lie}(Y') \in \mathcal{K}$ is functorial). We claim:

a) the k-linear map $\mathcal{L} \to \mathcal{K}$ is surjective, and

b) because $a(Y) = \dim Y$, this map is an isomorphism.

To prove this, consider the group scheme $N_0 = \mathrm{Ker}(F\colon Y \to Y^{(p)})$, i.e. N_0 is the group scheme of height one uniquely determined by its p-Lie algebra G_0. Consider the scheme $N := N_0 \otimes_k R$. By [4], III.3.5 we have a natural identification

$$\mathcal{K} \cong H^2_{\mathrm{symm}}(N_0, G_0)$$

(symm denotes the symmetric cocycles; they correspond to the commutative group scheme structures on N; a group scheme structure on N lifting the one on N_0 can be identified with a p-Lie algebra structure on $G_0 \otimes_k R$ extending the p-Lie structure on G_0: cf. [3], II.7.3.5), moreover

$$\mathcal{L} \cong H^2_{\mathrm{symm}}(Y, G_0)$$

(cf. [4], III.3.7), and the map $\mathcal{L} \to \mathcal{K}$ is induced by $i\colon N_0 \hookrightarrow Y$. After choice of a k-basis $G_0 \cong k^g$, one can make the following identifications, resulting in a commutative diagram:

$$\begin{array}{ccccc}
\mathcal{L} & \cong & H^2_{\mathrm{symm}}(Y, G_0) & \cong & (\mathrm{Ext}(Y, \mathbb{G}_a))^g \\
\downarrow & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\
\mathcal{K} & \cong & H^2_{\mathrm{symm}}(N_0, G_0) & \cong & (\mathrm{Ext}(N_0, \mathbb{G}_a))^g
\end{array}$$

(where Ext stands for the group of isomorphism classes of k-group scheme extensions); the exact sequence

$$0 \longrightarrow N_0 \longrightarrow Y \xrightarrow{F} Y^{(p)} \longrightarrow 0$$

results into an exact sequence

$$\ldots \to \mathrm{Ext}(Y, \mathbb{G}_a) \to \mathrm{Ext}(N_0, \mathbb{G}_a) \to E^2(Y^{(p)}, \mathbb{G}_a) = 0$$

(the last equality because of [23], Lemma 12.8), thus Claim (a) is proved. Because $a(Y)=\dim Y=g$, we have $N_0\cong(\alpha_p)^g$, thus $\dim_k \mathscr{K}=g^2$ (because $\mathrm{Ext}(\alpha_p, \mathbb{G}_a)\cong k$ as k-modules, the action of k is via $\mathrm{End}(\mathbb{G}_a, \mathbb{G}_a)$, cf. [23], Proposition 10.5); moreover $\dim_k \mathscr{L}=g^2$ (because $\dim_k \mathrm{Ext}(Y, \mathbb{G}_a)=\dim Y$, cf. [30], VII.17), thus Claim (b) is proved. The fact that $\mathrm{Lie}(X_t)$ has p-operation zero, i.e. $\mathrm{Lie}(X_t)=0\in\mathscr{K}$, implies via (b) that $t=0$. Thus $\dim V=0$, the point $x\in A_{g,d,n}$ is rational over the algebraic closure k of the prime field, and $X=\mathrm{X}_x$; thus the lemma is proved.

Remark (4.6) (Mumford). From the lemma one can deduce that abelian varieties of dimension two can be lifted to characteristic zero.

Proof of (4.2). Let X and l be as in the theorem; we assume l is algebraically closed; we can replace X within isogeny by an AV over l (again denoted by X) such that $\hat{X}\cong(G_{1,1})^g$; then $a(X)=g$, because $\alpha_p\hookrightarrow G_{1,1}$; thus by (4.5) we know X *can be defined over a finite field* K (and this AV again is denoted by X); let $f\colon X\to X$ be the geometric Frobenius of X over K, i.e. if K has $q=p^a$ elements, raising to the q-th power of the elements of $\mathcal{O}_X$ is a K-endomorphism:

$$f=(X\xrightarrow{F}X^{(p)}\to\cdots\to X^{(p^a)}\cong X).$$

Let P be the characteristic polynomial of f. Because P is monic and has integral coefficients (cf. [16], p. 187, Corollary 2; [22], p. 180, Theorem 4), the zeros of P are algebraic integers. Let v be an extension of the p-adic valuation on $\mathbb{Q}$ to $\mathbb{C}$, thus $v(p)=1$. Let $\lambda_1, \ldots, \lambda_{2g}$ be the zeros of P in $\mathbb{C}$. By the Riemann-Weil hypothesis we know

$$|\lambda_i|=\sqrt{q}, \qquad 1\leqq i\leqq 2g$$

(cf. [16], p. 139, Theorem 2; [22], p. 206, Theorem 4). Because the isogeny type of the formal group of X is $(G_{1,1})^g$ we conclude by a theorem of Manin (cf. [19], 4.1) that

$$v(\lambda_i=v(\sqrt{q}), \qquad 1\leqq i\leqq 2g.$$

Thus the elements $\lambda_i\cdot q^{-\frac{1}{2}}$ are integral, they form complete sets of conjugates and they have absolute value one; this implies they are roots of unity (e.g. cf. [1], p. 105, Theorem 2). Thus replacing K by a finite extension (the new field again denoted by K, same for X, f and P) we achieve $\lambda_i\cdot q^{-\frac{1}{2}}=1$, i.e.

$$P=(T-\sqrt{q})^{2g};$$

by a result of Tate, this implies X is isogeneous to the g-th power of a supersingular curve (cf. [33], Theorem 2.d), which ends the proof of the theorem.

Alternative proof of (4.2): Let X be defined over a finite field having $q=p^a$ elements, and f its geometric Frobenius as above; if $\hat{X} \cong (G_{1,1})^g$, we know $\mathrm{Ker}(f)$ equals the kernel of multiplication by $p^{a/2}$ (if necessary, extend K so that a is even), thus there exists a K-automorphism α of X with

$$f = p^{a/2} \cdot \alpha .$$

Clearly α is compatible with any polarization on X, and because a polarized AV has only finitely many automorphisms (cf. [20]), the order of α is finite, i.e. replacing K by a finite extension we see the geometric Frobenius of X has eigenvalues equal to $q^{\frac{1}{2}}$, and we conclude by the theorem due to Tate as above.

Corollary (4.7). *The coarse moduli scheme $A_2 = A_{2,1,1}$ of abelian varieties of dimension two with a principal polarization (over a field of characteristic $p \neq 0$) contains a complete rational curve.*

Proof. By Theorem (1.1a) A_2 contains a complete subscheme E of dimension $\frac{1}{2} \cdot 2(2-1) = 1$ (which, by the way, contradicts a suggestion by Grothendieck, cf. [8], p. 77, lines 18–20); suppose the base field k is algebraically closed, suppose E is irreducible and reduced (if necessary take one of the components of E); we now show E is a rational curve, i.e. $\mathrm{genus}(E)=0$. Let X_0 be an AV with a principal polarization λ_0, both defined over an algebraic closure of $k(E)$ such that (X_0, λ_0) corresponds to the generic point of $E \subset W_{2,1}$, cf. Sect. 1), and $\dim X_0 = 2$, thus the isogeny type of $\hat{X}_0$ is $2 \cdot G_{1,1}$; by (4.2) we conclude there exists an isogeny $\beta\colon Z \to X_0$ with

$$\hat{Z} \cong 2 \cdot G_{1,1},$$

thus $a(Z)=2$, and Z defined over k (cf. 3.5); we may assume that $Z \to X_0$ is purely inseparable (replace Z by $Z/(\mathrm{Ker}\,\beta)_{\mathrm{red}}$). let its degree be p^n, write $Z = X_n$, construct isogenies

$$Z = X_n \to X_{n-1} \to \cdots \to X_{i+1} \to X_i \to \cdots \to X_0$$

each of degree p (and thus each having α_p as kernel); lifting back λ_0 we obtain polarizations λ_i on X_i, $0 \leqq i \leqq n$; let $E = E_0$, and let $E_i \subset A_{2,p^{2i}}$ be the closure of the point given by (X_i, λ_i); let $0 < m \leqq n$ be such that $\dim(E_i) \neq 0$ for $0 \leqq i < m$ and $a(X_m)=2$ (and thus $\dim(E_m)=0$ by 4.5). We claim the isogeny correspondence between E_{i+1} and E_i, $0 \leqq i < m-1$ given by $X_{i+1} \to X_i$ is birational (because $a(X_i) = 1 = a(X_{i+1})$ for $0 \leqq i \leqq m-1$ there is only isogeny $\alpha_p \to X_{i+1} \to X_i$ possible, thus the isogeny correspondence is generically $1-1$, and because of $\alpha_p^D \to X_i^t \to X_{i+1}^t$ it is birational); next we show E_{m-1} is a rational curve: let C be the isogeny correspondence containing $X_{m-1} \to X_m$,

$$\begin{tikzcd}
& C \arrow[dl] \arrow[dr] & \\
A_{2,p^{(m-1)}} \supset E_{m-1} & & E_m \subset A_{2,p^m}
\end{tikzcd}$$

$$a(X_{m-1})=1,\ \dim(E_{m-1})=1 \qquad \dim(E_m)=0,\ a(X_m)=2,$$

note that E_m is a point, a point of C corresponds to an embedding $\alpha_p \hookrightarrow X_m$, thus it follows that C is birationally equivalent with $\mathbb{P}^1$, thus E_{m-1} is a rational curve; hence $E=E_0$ is a rational curve by what is said before, and the corollary is proved.

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