Malleable Commitments from Group Actions and Zero-Knowledge Proofs for Circuits based on Isogenies

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Overview

Asymmetric cryptography allows for a wide variety of schemes with interesting features :

- Threshold signatures
- Fully Homomorphic Encryption
- Zero-knowledge proofs
- Oblivious transfer
- Verifiable Delay Functions
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How ?

1. Construct a proof of knowledge for a NP-complete statement.
2. Reduce any other NP problem to this.

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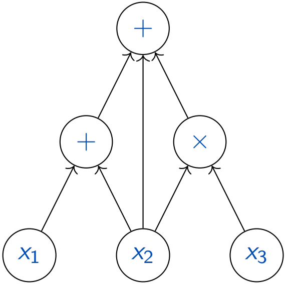

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Arithmetic Circuits

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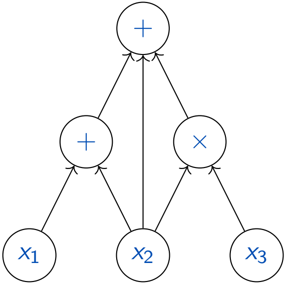

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- The SAT problem for arithmetic circuit :
  Given a polynomial f and an output value s, are there input values $x_1, \cdots, x_n$ such that $f(x_1, \cdots, x_n) = s$?

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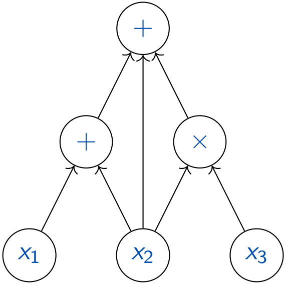

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Theorem

The satisfiability problem for arithmetic circuits is NP-complete.

Commitment Schemes

Definition (Commitment Scheme)

A *commitment scheme* is a tuple $(\mathcal{M}, \mathcal{R}, \mathcal{C}, \mathsf{Commit}, \mathsf{Verify})$ where $\mathsf{Commit} : \mathcal{M} \times \mathcal{R} \to \mathcal{C}$ and $\mathsf{Verify} : \mathcal{M} \times \mathcal{R} \times \mathcal{C} \to \{0, 1\}$ are PPTA algorithms

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Related security notions :

Hiding

An attacker cannot retrieve m or r from c.

Binding

It's hard to find $(m, r) \neq (m', r')$ that give the same commitment.

- Efficient solutions use homomophic property :

$$\forall m, m', r, r', \;\; \mathsf{Commit}(m + m', r + r') = \mathsf{Commit}(m, r) \cdot \mathsf{Commit}(m', r')$$

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Very generic $\rightsquigarrow$ We adapt depending on available structure

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Definition

A GAMC is a commitment scheme satisfying :

- $\mathcal{M}$ and $\mathcal{R}$ are groups. $\mathcal{C}$ is a set.
- We have a group action $\star : (\mathcal{M} \times \mathcal{R}) \times \mathcal{C} \to \mathcal{C}$
- $C_0 := \mathrm{Commit}(0_{\mathcal{M}}, 0_{\mathcal{R}})$
- $\mathrm{Commit}(m, r) := (m, r) \star C_0$.
- $(m', r') \star \mathrm{Commit}(m, r) = \mathrm{Commit}(m + m', r + r')$

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In our case :

- $\mathcal{M}$ and $\mathcal{R}$ are groups of isogenies (with composition).
- $\mathcal{C}$ is a set of elliptic curves (up to isomorphism).

less structure ⟵ generic malleability — GAMC — homomorphism property ⟶ more structure

How to use GAMCs

Several GAMC $\rightsquigarrow$ Proof systems for addition and multiplication gates $\rightsquigarrow$ Proof system for arithmetic circuits

Features :

- Addition : makes use of malleability $\rightsquigarrow$ small proofs.
- Multiplication : requires Merkle tree trick $\rightsquigarrow$ bigger proofs and $|\mathcal{M}|$ restricted.

Interlude : CSIDH

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$\xrightarrow{g^a}$	$\xrightarrow{\mathfrak{a}\cdot E_0}$
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- $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in $\mathcal{C}\ell(\mathcal{O})$: the *ideal class group* of $\mathbb{Z}[\pi]$.
- E_0 is a curve in SS_p : the set of supersingular curves "over $\mathbb{F}_p$".
- Natural group action of $\mathcal{C}\ell(\mathcal{O})$ over SS_p.
  This is the underlying group action of the GAMC.

An instance of a GAMC

In the CSIDH setting :

- $\mathcal{M} = \mathcal{R} := \mathcal{C}\ell(\mathcal{O})$ are groups of ideals (encoding isogenies).
- $\mathcal{C} := \mathsf{SS}_p \times \mathsf{SS}_p$.
- $C_0 := (E_0, E_1)$.

Malleability is given by

$$(\mathfrak{m}, \mathfrak{r}) \star (E, E') := (\mathfrak{r} \cdot E, \mathfrak{m}\mathfrak{r} \cdot E')$$

Enough to define a GAMC for arithmetic circuits !

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Rank 1 Constraint Systems (R1CS) and Branching programs

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Rank 1 Constraint Systems (R1CS) and **Branching programs** .

Branching programs

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Definition (Branching Program)

Given matrices $\{(A_{1,0}, A_{1,1}), \ldots, (A_{d,0}, A_{d,1})\}$ in $M_2(\mathbb{Z}_n)$, a branching program of length d takes as input :

- An initial state $M_0 \in M_2(\mathbb{Z}_n)$
- $x = (x_1, \ldots, x_d) \in \{0, 1\}^d$

and outputs :

$$A_{d,x_d} \cdot A_{d-1,x_{d-1}} \cdot \ldots \cdot A_{1,x_1} \cdot M_0$$

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Theorem (Barrington)

Every arithmetic circuit can be encoded as a branching program.

GAMC with branching programs

A GAMC where $\mathcal{M}$ and $\mathcal{R}$ are 2x2 matrices $\rightsquigarrow$ Proof systems for branching programs $\rightsquigarrow$ More efficient than arithmetic circuits

Features :

- Requires more structure than arithmetic circuits.
- $\mathcal{M}$ and $\mathcal{R}$ are rings $\rightsquigarrow$ no restriction on $|\mathcal{M}|$.
- Smaller proofs than arithmetic circuits.

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A example of GAMC for branching programs ?

GAMC with branching programs

Two ingredients :

- pSIDH, a key exchange mechanism using isogenies.
- The *malleability oracle* from [KMPW21].

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Malleability oracle f : allows the computation of a group action
$\mathrm{End}(E_0)_N \times \mathrm{SS}(E_0)_N \to \mathrm{SS}(E_0)_N$.
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But... pSIDH is no longer post-quantum secure. [CII+23]

Conclusion

Contributions :

- New framework for generic NP statements ZK proofs.
- Highly versatile.
- Proof-of-concept construction.

Performances :

- Strong security assumptions and no trusted setup.
- Proof system for an arithmetic circuit $= O(|\mathcal{M}|)$ malleability computations.
- Size of the proof $= O(\lambda|\mathcal{M}|)$ bits.

Future work :

- Proof-of-concept cannot use higher security parameters than CSIDH-512.
- Post-quantum constructions for R1CS and branching programs

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