KLPT^2 : Algebraic isogeny pathfinding in dimension 2

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September 12, 2025

Definition (Elliptic curve)

An elliptic curve E over a field $\mathbb{F}_q$ is the set of solution of a cubic equation, with a special *point at infinity*.

$$E = \left\{y^2 = x^3 + ax + b, \quad x, y \in \overline{\mathbb{F}}_q\right\} \cup \{\infty\}$$

with $a, b \in \mathbb{F}_q$ and $4a^3 + 27b^2 \neq 0$.

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Example

Let's take $E : y^2 = x^3 + 1$ over $\mathbb{F}_5$. It has 6 rational points :

$$E(\mathbb{F}_5) = \{(0,1), (0,4), (2,2), (2,3), (4,0), \infty\}$$

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$$\varphi : (x, y) \mapsto \left(\frac{x^2 + x - 2}{x + 1}, \frac{x^2 + 2x - 2}{x^2 + 2x + 1} y \right)$$

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The kernel of φ is $\{(4, 0), \infty\} \leftrightsquigarrow \deg(\varphi) = 2$.

Isogeny graphs

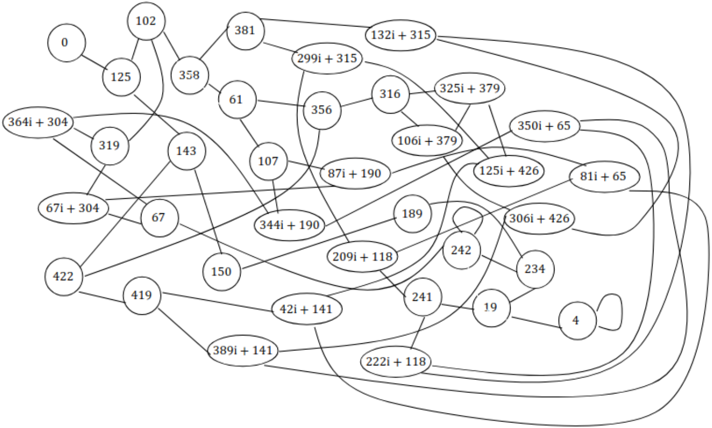

Figure: The ℓ-isogeny graph over $\mathbb{F}_{p^2} \simeq \mathbb{F}_p[i]$, for $p = 431$ and $\ell = 2$.

[Cos19] : **Costello**, *SIKE for beginners*

Definition (The quaternion algebra ramified at p and ∞)

We will make use of the quaternion algebra $B_{p,\infty}$ defined as :

$$B_{p,\infty} = \mathbb{Q} + i\mathbb{Q} + j\mathbb{Q} + k\mathbb{Q}$$

with $i^2 = -1$, $j^2 = -p$, $ij = -ji$.

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$\mathcal{O} = \mathbb{Z} + i\mathbb{Z} + j\mathbb{Z} + k\mathbb{Z}$ is an order.
$\mathcal{O}_0 = \mathbb{Z} + i\mathbb{Z} + \frac{i+j}{2}\mathbb{Z} + \frac{1+k}{2}\mathbb{Z}$ is a maximal order.

The Deuring Correspondence in one slide

Theorem (Deuring)

$$\left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{(supersingular) elliptic curves} \\ \text{over } \mathbb{F}_{p^2} \text{ and their isogenies} \end{array} \right\} \overset{\text{2-to-1}}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Maximal orders of } B_{p,\infty} \\ \text{and their connecting ideals} \end{array} \right\}$$

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The canonical example

Take $E_0 : y^2 = x^3 + x$ over $\mathbb{F}_{p^2}$, with $p = 3 \mod 4$.
Then, we have

$$\begin{array}{ccc} \mathrm{End}(E_0) & = & \mathbb{Z} + \iota\mathbb{Z} + \frac{\iota+\pi}{2}\mathbb{Z} + \frac{1+\pi\iota}{2}\mathbb{Z} \\ \wr\! \mid & & \\ \mathcal{O}_0 & = & \mathbb{Z} + i\mathbb{Z} + \frac{i+j}{2}\mathbb{Z} + \frac{1+k}{2}\mathbb{Z} \end{array}$$

Why the endomorphism ring ?

Through the Deuring correspondence, we manipulate $\mathrm{End}(E)$ with quaternions. But why do we care so much about $\mathrm{End}(E)$, to begin with ?

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$\mathrm{End}(E) \simeq \mathcal{O}$ is a non-commutative ring.

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Knowing $\mathrm{End}(E) \rightsquigarrow$ Knowing everything about E.

[Wes21] : **Benjamin Wesolowski**, *The supersingular path and endomorphism ring problems are equivalent*

Connecting ideals

Example

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- I_φ is a right-ideal of $\mathrm{End}(E_2)$ (as $I_\varphi = \varphi\,\mathrm{Hom}(E_2, E_1)$).

$$\begin{tikzcd} E_1 \arrow[r, bend right, "\varphi"'] & E_2 \arrow[l, bend right, "\mathrm{Hom}(E_2,E_1)"'] \end{tikzcd}$$

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Definition (Connecting ideal)

A connecting ideal $I : \mathcal{O}_1 \to \mathcal{O}_2$ between two maximal orders is an ideal that is a left-order of $\mathcal{O}_1$ and a right-order of $\mathcal{O}_2$.

$$\mathcal{O}_1 \xrightarrow{I} \mathcal{O}_2$$

The Kohel-Lauter-Petit-Tignol paradigm

Translating the ℓ-isogeny path problem

The ℓ-isogeny path problem

Let E_1, E_2 be two elliptic curves over $\mathbb{F}_{p^2}$. Let ℓ be a small prime.

Compute an isogeny $\varphi : E_1 \to E_2$ with degree ℓ^e.

$$E_1 \xrightarrow{\varphi} E_2$$

The quaternion ℓ-isogeny path problem

Let $\mathcal{O}_1, \mathcal{O}_2$ be two maximal orders in the quaternion algebra $B_{p,\infty}$.

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[Isogeny Club – S1E4] : **Antonin Leroux**, *A new algorithm for the constructive Deuring correspondence: making SQISign faster*

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Geometric world	$\boxed{E_1 \qquad E_2}$	$\boxed{E_1 \xrightarrow{\varphi} E_2}$

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Kohel-Lauter-Petit-Tignol (2014)

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1. We start with a *bad* ideal I connecting $\mathcal{O}_1$ to $\mathcal{O}_2$.

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2. We find an element $\alpha \in I$ with smooth (reduced) norm.

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Instance of the problem | Solution of the problem

Geometric world

Quaternion world

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It requires the knowledge of $\mathrm{End}(E_1)$ and $\mathrm{End}(E_2)$!

We want an analogue in dimension 2 !

Overview of $KLPT^2$

	Instance of the problem	Solution of the problem
Geometric world	$\boxed{(A_1, \lambda_1) \qquad (A_2, \lambda_2)}$	$\boxed{(A_1, \lambda_1) \xrightarrow{\varphi} (A_2, \lambda_2)}$

- (A_1, λ_1) and (A_2, λ_2) are *principally polarized superspecial abelian surfaces*.

$\rightsquigarrow$ analogue of supersingular elliptic curves in dimension 2.

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	IKO86 $\Big\downarrow$	
Quaternion world	$\boxed{g_1 \qquad g_2}$	

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[IKO86] : **Ibukiyama-Katsura-Oort**, *Supersingular curves of genus two and class numbers*

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Quaternion world	$\boxed{g_1 \qquad g_2}$	$\xrightarrow[\text{KLPT}^2]{}$	$\boxed{g_1 \xrightarrow{\gamma} g_2}$

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Instance of the problem — Solution of the problem

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Setting the frame

For everything that follows, we fix

- A prime $p = 3 \mod 4$ of cryptographic size,
- A small prime ℓ. Typically $\ell \in \{2, 3\}$
- $E_0 : y^2 : x^3 + x$, the curve with j-invariant 1728 over $\mathbb{F}_{p^2}$,
- $\mathrm{End}(E_0) \simeq \mathcal{O}_0 = \mathbb{Z} + i\mathbb{Z} + \frac{i+j}{2}\mathbb{Z} + \frac{1+k}{2}\mathbb{Z}$,
- $B_{p,\infty} = \mathcal{O}_0 \otimes \mathbb{Q}$, the underlying quaternion algebra,

- Let x be a quaternion. Its norm is $\mathbf{n}(x)$, its trace is $\mathbf{tr}(x)$.

KLPT in dimension 2

Quaternion path problem in dimension 2

Given $g_1, g_2 \in \mathrm{Mat}(\mathcal{O}_0)$, find $\gamma \in \mathrm{M}_2(\mathcal{O}_0)$ such that :

$$\gamma^* g_2 \gamma = \ell^n g_1$$

for some small prime ℓ and with :

- $\mathrm{Mat}(\mathcal{O}_0) = \left\{ \begin{pmatrix} s & r \\ \bar{r} & t \end{pmatrix}, \quad s, t \in \mathbb{Z}_>, st - \mathbf{n}(r) = 1 \right\}$.
- $-^* : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$ is the conjugate-transpose.

KLPT in dimension 2

Quaternion path problem in dimension 2

Given $g_1, g_2 \in \mathsf{Mat}(\mathcal{O}_0)$, find $\gamma \in \mathsf{M}_2(\mathcal{O}_0)$ such that :

$$\gamma^* g_2 \gamma = \ell^n g_1$$

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Theorem (KLPT2)

This problem can be solved in polynomial time with output norm $\ell^n = O(p^{25})$.

Definition (Connecting matrix)

Let $h_1, h_2 \in \mathrm{Mat}(A_0)$ and $u \in \mathrm{M}_2(\mathcal{O}_0)$.
We say that u is a connecting matrix between h_1 and h_2 if it satisfies

$$u^* h_2 u = \mathcal{N}(u) h_1$$

for some integer $\mathcal{N}(u)$ called its norm.
We write $u : h_1 \to h_2$.

Some useful lemmas

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Lemma (Inversion lemma)

If $u : h_1 \to h_2$ is invertible in $\mathrm{M}_2(B_{p,\infty})$,
then $\mathcal{N}(u)u^{-1} \in \mathrm{M}_2(\mathcal{O}_0)$ and $\mathcal{N}(u)u^{-1} : h_2 \to h_1$.

$$\begin{tikzcd} h_1 \arrow[r, bend left, "u"] & h_2 \arrow[l, bend left, "\mathcal{N}(u)u^{-1}"] \end{tikzcd}$$

Some useful lemmas

Lemma (Composition lemma)

Let h_1, h_2, h_3, u_1, u_2 be matrices such that

$$\begin{cases} u_1 : h_1 \to h_2 \\ u_2 : h_2 \to h_3 \end{cases}$$

Then, $u_1 u_2 : h_1 \to h_3$.

$$\begin{tikzcd} h_1 \arrow[r, "u_1"] \arrow[rr, bend right, "u_1 u_2"'] & h_2 \arrow[r, "u_2"] & h_3 \end{tikzcd}$$

How we solve KLPT2

The inputs of the algorithm

Two matrices $g_1 = \begin{pmatrix} s_1 & r_1 \\ \bar{r}_1 & t_1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} s_2 & r_2 \\ \bar{r}_2 & t_2 \end{pmatrix}$.

The strategy

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The strategy

- We note that if the inputs have a certain shape, there exists a connecting matrix τ between them.

$$g_1 \qquad \begin{pmatrix} \ell^f & r_1' \\ \bar{r}_1' & t_1' \end{pmatrix} \xrightarrow{\tau} \begin{pmatrix} \ell^f & r_2' \\ \bar{r}_2' & t_2' \end{pmatrix} \qquad g_2$$

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The strategy

- We note that if the inputs have a certain shape, there exists a connecting matrix τ between them.
- We transform our inputs so they have the aforementioned shape.
- We output the product of the three connecting matrices.

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The output of the algorithm

The composition $\gamma := u_1 \cdot \tau \cdot \mathcal{N}(u_2) u_2^{-1}$.

The norm of γ is $\mathcal{N}(u_1)\mathcal{N}(u_2)\mathcal{N}(\tau)$.

Connecting matrices between special inputs

Lemma (Step 1 : Connecting special matrices)

Let $h_1 = \begin{pmatrix} \ell^f & r_1' \\ \bar{r}_1' & t_1' \end{pmatrix}$ and $h_2 = \begin{pmatrix} \ell^f & r_2' \\ \bar{r}_2' & t_2' \end{pmatrix}$ be two “input” matrices such that $\det(h_1) = \det(h_2)$.

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Then, there exists $\tau \in \mathrm{M}_2(\mathcal{O}_0)$ connecting h_1 to h_2.

Proof.

Take $\tau = \begin{pmatrix} \ell^f & r_1 - r_2 \\ 0 & \ell^f \end{pmatrix}$.

$\square$

Computing u

Analyzing the constraints

Let us write

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad g = \begin{pmatrix} s & r \\ \bar{r} & t \end{pmatrix}, \quad h = \begin{pmatrix} s' & r' \\ \bar{r}' & t' \end{pmatrix} = u^* g u$$

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We have two constraints to satisfy :

1. The top-left entry of h must be of the form ℓ^f :
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We fix a and c such that the first constraint is satisfied,
Given a and c, we find b and d such that the second constraint is satisfied.

Fixing the top-left entry

We want to solve :

$$s \cdot \mathbf{n}(a) + t \cdot \mathbf{n}(c) + \mathbf{tr}(\bar{a}rc) = \ell^f$$

for $a, c \in \mathcal{O} = \mathbb{Z} + i\mathbb{Z} + j\mathbb{Z} + k\mathbb{Z} = \mathbb{Z}[i] + j\mathbb{Z}[i]$.

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4. Use Cornacchia's algorithm to solve $a_1^2 + a_2^2 = \frac{\ell^f - t \cdot \mathbf{n}(c)}{s}$

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We obtain $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with all the desired properties !

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Thank you for your attention !