Homework 4

1) a) The position of W in relative world space is $(-4, -1, 3)$

b) In order to get the scale factor for each axis, we must get the norm for each

$$S_x = \left\| \begin{pmatrix} -2.08 \\ 0 \\ 1.387 \end{pmatrix} \right\| = 2.5 \quad S_y = \left\| \begin{pmatrix} -.807 \\ 2.623 \\ -1.211 \end{pmatrix} \right\| = 3$$

$$S_z = \left\| \begin{pmatrix} -1.698 \\ -1.698 \\ -2.547 \end{pmatrix} \right\| = 3.5$$

c) In order to find the unit vectors, we must nomalize each vector. We simply divide by the norm from b)

$$\text{left} = \frac{(-2.08, 0, 1.387)}{2.5} = (-.832, 0, .5548)$$

$$\text{Up} = \frac{(-.807, 2.623, -1.211)}{3} = (-.269, .8743, -.4037)$$

$$\text{fwd} = \frac{(-1.698, -1.698, -2.547)}{3.5} = (-.485, -.485, -.7277)$$

d) In order to find the determinant of this matrix, we start by breaking it up into its component matrices in TRS form:

$$\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -.832 & -.269 & -.485 & 0 \\ 0 & .8743 & -.485 & 0 \\ .5548 & -.4037 & -.7277 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2.5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The T matrix is 1 because in order to calcutate a 4x4 matrix in the form $\begin{vmatrix} A & y \\ 0^T & 1 \end{vmatrix} = |A|$. For T this is the identity matrix.

The W matrix is 1 because of the properties of rotation matrices.

The S matrix is 26.25 because S is a triangle matrix. In order to calculate the determinant of this we simply multiply the diagonal.

To get the final determinant, we simply multiply the three values together and get $|W| = 26.25$

e) In order to find the position of this object, we must apply the transformation W^{-1}

$$W^{-1} = (TRS)^{-1} = S^{-1}R^{-1}T^{-1} =$$

$$\begin{bmatrix} .4 & 0 & 0 & 0 \\ 0 & .3333 & 0 & 0 \\ 0 & 0 & .2857 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -.832 & 0 & .5548 & 0 \\ -.269 & .8743 & -.4037 & 0 \\ -.485 & -.485 & -.7277 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -.3328 & 0 & .2219 & -1.997 \\ -.0897 & .2914 & -.1346 & .3364 \\ -.1386 & -.1386 & -.2079 & -.0691 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$W^{-1}\begin{pmatrix} -1 \\ 4 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2.108 \\ 1.8609 \\ -.0691 \\ 1 \end{pmatrix}$$

f) In order to find the vector relative to W, we must apply the transformation:

$$W^{-1}\begin{pmatrix} -4 \\ 2 \\ -5 \\ 0 \end{pmatrix} = \begin{pmatrix} .2216 \\ 1.6142 \\ 1.3166 \\ 0 \end{pmatrix}$$

② a) $q = (\cos(\theta/2), \hat{r}\sin(\theta/2)) =$

$= (\cos(\pi/6),$

$\sin(\pi/6)(.6247, .4685, -.6247)) =$

$q = (.866, .3124, .2346, -.3124)$

b) $p = (0, -2, -4, 1)$

c) $qp = (w_q, v_q)(w_p, v_p) =$

$(w_q w_p - v_q \cdot v_p, w_q v_p + w_p v_q + v_q \times v_p)$

$w_q w_p = 0 \quad v_q \cdot v_p = -1.8756$

$w_q v_p = (-1.732, -3.464, .866)$

$w_p v_q = (0, 0, 0)$

$v_q \times v_p = (-1.015, .3124, -.7804)$

$qp = (1.8756, -2.747, -3.1516, -.0856)$

d)

$q^{-1} = \frac{1}{\|q\|^2}(w, -v)$

$q^{-1} = (.866, -.3124, -.2346, .3124)$

$\cancel{(-.8756, \ldots)}$

e) $(qp)q^{-1} = (w_1, v_1)(w_2, v_2) =$
$(w_1 w_2 - v_1 \cdot v_2,\ w_1 v_2 + w_2 v_1 + v_1 \times v_2$

$w_1 w_2 = 1.6243 \quad v_1 \cdot v_2 = 1.6243$

$w_1 v_2 = (-.5859, -.44, .5859)$
$w_2 v_1 = (-2.3789, -2.7293, -.0791)$
$v_1 \times v_2 = (-1.0046, .8849, -.3401)$

$v' = (qp)q^{-1} = (0, -3.929, -2.3379, .319)$

f) $v' = v\cos(\frac{\pi}{3}) + [1-\cos(\frac{\pi}{3})](v \cdot \hat{r})\hat{r} + [\hat{r} \times v]\sin(\frac{\pi}{3})$

$\cos(\frac{\pi}{3}) = .5 \quad \sin(\frac{\pi}{3}) = .866$

$v \cdot \hat{r} = -3.7481$
$(v \cdot \hat{r})\hat{r} = (-2.3411, -1.756, 2.3414)$
$.5(v \cdot \hat{r})\hat{r} = (-1.1707, -.878, 1.1707)$

$\hat{r} \times v = (-2.0303, .6247, -1.5618)$
$.866[\hat{r} \times v] = (-1.7582, .541, -1.3525)$

$v\cos(\frac{\pi}{3}) = (-1, -2, .5)$

Close Enough !!

$v' = (-3.929, -2.337, .318)$

③ a) Since the order of $R = R_x R_y R_z$ then the matrix form is:

$$\begin{bmatrix} \cos\theta_y \cos\theta_z & -\cos\theta_y \sin\theta_z & \sin\theta_y \\ \cdots & \cdots & -\sin\theta_x \cos\theta_y \\ \cdots & \cdots & \cos\theta_x \cos\theta_y \end{bmatrix}$$

$$\theta_{y1} = \sin^{-1}(.95106) = 1.2566$$

ATan2(Y/X)

$$\theta_{y2} = \begin{cases} \pi - \theta_{y1} & \text{if } \theta_{y1} \geq 0 \\ -\pi - \theta_{y1} & \text{if } \theta_{y1} < 0 \end{cases} = 1.885$$

$$\cos\theta_z = .28445 / \cos(1.2566) = .9204 = X$$
$$\sin\theta_z = -.1207 / \cos(1.2566) = -.3905 = Y$$

$$\text{ATan2}(\sin\theta_z, \cos\theta_z) = \tan^{-1}(\sin\theta_z / \cos\theta_z)$$
$$= -.4013$$

$$\sin\theta_z = -.1207 / \cos(1.885) = .3905 = Y$$
$$\cos\theta_z = .28445 / \cos(1.885) = -.9204 = X$$

$$\text{ATan2}(Y, X) = \tan^{-1}(\sin\theta_z / \cos\theta_z) + \pi$$
$$= 2.7403$$

$$\theta_{z1} = -.4013$$
$$\theta_{z2} = 2.7403$$

$\sin\theta_x = -.24015/\cos(1.2566) = -.7771 = y$
$\cos\theta_x = -.1945/\cos(1.2566) = -.6293 = x$

$ATan2(y,x) = \tan^{-1}(-.7771/-.6293) - \pi$
$= -2.2515$

$\sin\theta_x = -.24015/\cos(1.885) = .777 = y$
$\cos\theta_x = -.1945/\cos(1.885) = .6293 = x$

$ATan2(y,x) = \tan^{-1}(.777/.6293) = .89$

	axis	x	y	z
Solution 1	rad	-2.2515	1.2566	-.4013
	deg	-129.001	71.998	-22.993
Solution 2	rad	.89	1.885	2.7403
	deg	50.993	108.003	157.008

b1) $R = R_y R_x R_z =$

$$\begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix} \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_y & \sin\theta_y\sin\theta_x & \sin\theta_y\cos\theta_x \\ 0 & \cos\theta_x & -\sin\theta_x \\ -\sin\theta_y & \cos\theta_y\sin\theta_x & \cos\theta_y\cos_x \end{bmatrix} \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Only showing entries used in b2)

$$= \begin{bmatrix} \cdots & \cdots & \sin\theta_y \cos\theta_x \\ \cos\theta_x \sin\theta_z & \cos\theta_x \cos\theta_z & -\sin\theta_x \\ \cdots & \cdots & \cos\theta_y \cos\theta_x \end{bmatrix}$$

b2) $\theta_{x1} = -\sin^{-1}(-.9135) = 1.1518$

$$\theta_{x2} = \begin{cases} \pi - \theta_{x1} & \text{if } \theta_{x1} \geq 0 \\ -\pi - \theta_{x1} & \text{if } \theta_{x1} < 0 \end{cases} = 1.9898$$

$\sin\theta_y = -.2722/\cos(1.1518) = -.6691 = y$
$\cos\theta_y = .30226/\cos(1.1518) = .7429 = x$

$\text{ATan2}(y, x) = \tan^{-1}(-.6691/.7429)$
$= -.7332$

$\sin\theta_y = -.2722/\cos(1.9898) = .669 = y$
$\cos\theta_y = .30226/\cos(1.9898) = -.7429 = x$

$\text{Atan2}(y, x) = \tan^{-1}(.669/-.7429) + \pi$
$= 2.4085$

$\sin\theta_z = -.0566/\cos(1.1518) = -.1391 = y$
$\cos\theta_z = .4028/\cos(1.1518) = .9901 = x$

$\text{Atan2}(y, x) = \tan^{-1}(-.1391/.9901)$
$= -.1396$

$\sin\theta_z = -.0566/\cos(1.9898) = .1391 = y$
$\cos\theta_z = .4028/\cos(1.9898) = -.99 = x$

$\text{ATan2}(y,x) = \tan^{-1}(.1391/-.99) + \pi = 3.002$

	axis	X	Y	Z
Solution 1	rad	1.1518	-.7332	-.1396
	deg	65.993	-42.009	-7.998
Solution 2	rad	1.9898	2.4085	3.002
	deg	114.007	137.997	172.002

4) a) In order to find W_1, we begin by normalizing f_1 and u_1:

$$\hat{f}_1 = \frac{(-1,-2,4)}{\sqrt{22}} = (-.2132, -.4264, .8528)$$

$$\hat{u}_1 = \frac{(-2,17,8)}{\sqrt{357}} = (-.1059, .8997, .4234)$$

Next, we calculate the Left axis

$$L_1 = (-.1059, .8997, .4234) \times (-.2132, -.4264, .8528)$$
$$= (.9478, 0, .237)$$

Thus:

$$W_1 = \begin{bmatrix} L_{11} & U_{11} & f_{11} & P_{11} \\ L_{12} & U_{12} & f_{12} & P_{12} \\ L_{13} & U_{13} & f_{13} & P_{13} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} .9478 & -.1059 & -.2132 & -3 \\ 0 & .8997 & -.4264 & 2 \\ .237 & .4234 & .8528 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b)

$$W_1 = \overset{T_1}{\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \overset{R_1}{\begin{bmatrix} .9478 & -.1059 & -.2132 & 0 \\ 0 & .8997 & -.4264 & 0 \\ .237 & .4234 & .8528 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

c) We begin by normalizing f_2 and U_2

$$\hat{f}_2 = \frac{(-5,-1,3)}{\sqrt{35}} = (-.8452, -.169, .5071)$$

$$\hat{U}_2 = \frac{(-5,34,3)}{\sqrt{1184}} = (-.1453, .9881, .0872)$$

Next we calculate the left axis

$$L_2 = (-.1453, .9881, .0872) \times (-.8452, -.169, .5071)$$
$$= (.5158, 0, .8597)$$

Thus

$$W_2 = \begin{bmatrix} .5158 & -.1453 & -.8452 & -3 \\ 0 & .9881 & -.169 & 2 \\ .8597 & .0872 & .5071 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} .5158 & -.1453 & -.8452 & 0 \\ 0 & .9881 & -.169 & 0 \\ .8597 & .0872 & .5071 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

d) $W_{1\to 2} = W_2 W_1^{-1}$

$$W_1^{-1} = R_1^{-1} T_1^{-1} = \begin{bmatrix} .9478 & 0 & .237 & 0 \\ -.1059 & .8997 & .4234 & 0 \\ -.2132 & -.4264 & .8528 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} .9478 & 0 & .237 & 2.3694 \\ -.1059 & .8997 & .4234 & -2.9639 \\ -.2132 & -.4264 & .8528 & -1.4924 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$W_{1\to2} = \begin{bmatrix} .6845 & .2297 & -.6601 & -.0858 \\ -.0686 & .9611 & .2742 & -.6764 \\ .6974 & -.1378 & .6731 & -2.9783 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The reason why this works is because $W_{1\to2}$ is essentially a coordinate change of W_1 into W_2 and to calculate this we must multiply $W_2 W_1^{-1}$

e) In order to show that $W_{1\to2}$ correctly transforms p_1 into p_2, we must simply multiply $W_{1\to2}\, p_1$

$$\begin{bmatrix} .6845 & .2297 & -.6601 & -.0858 \\ -.0686 & .9611 & .2742 & -.6764 \\ .6974 & -.1378 & .6731 & -2.9783 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} .9478 & -.1059 & -.2132 & -3 \\ 0 & .8997 & -.4264 & 2 \\ .237 & .4234 & .8528 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} .5158 & -.1453 & -.8452 & -3 \\ 0 & .9881 & -.169 & 2 \\ .8597 & .0872 & .5071 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

f) In order to prove that f_1 and u_1 are transformed by $W_{1\to 2}$ into vector of the same length but in the direction of f_2 an u_2, we must multiply $W_{1\to 2} f_1$ and $W_{1\to 2} u_1$ and compare the results with their norms

$$W_{1\to 2} f_1 = \begin{pmatrix} -3.784 \\ -1.7565 \\ 2.2706 \\ 0 \end{pmatrix} = f'$$

$$W_{1\to 2} u_1 = \begin{pmatrix} -2.7451 \\ 18.6691 \\ 1.6479 \\ 0 \end{pmatrix} = u'$$ (slightly off due to rounding)

$$\|f'\| = 4.477 \approx 4.583 = \|f_1\| \checkmark$$
$$\|u'\| = 18.942 \approx 18.894 = \|u_1\| \checkmark$$