Binomial Representation Theorem

- We follow Baxter and Rennie 2.3, "Binomial Representation Theorem"
- Previously we found a set of probabilities $q_j \sim (0,1)$
- This allowed us to price any derivative by a numerically trivial discounted expectation operation.
- The expectation result will carry from the discrete to the continuous case.

Process

- There are seven possible definitions illustrated by seven nodes.
- We call a set of possible stock values at each node the ***process*** S.
- One possible process is shown in **figure 2.9**.
- The random variable S_i denotes the stock process at time i.

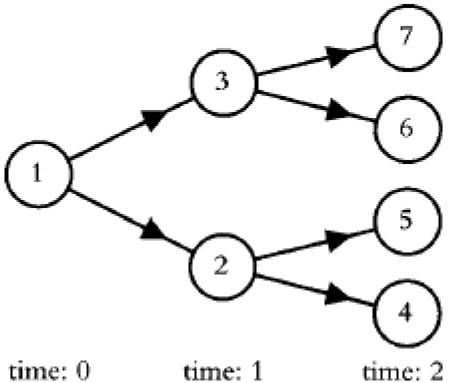

Figure 2.8 Tree with node numbers

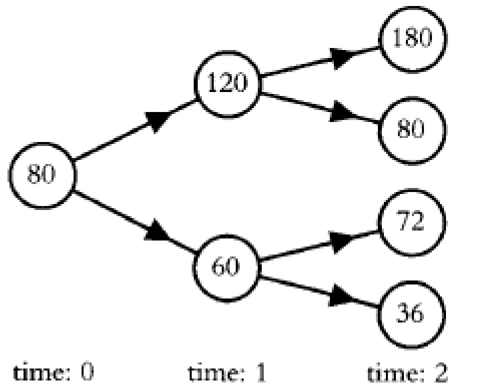

Figure 2.9 Tree with price process

Probability measure

- We will call the set of 'probabilities' (p_j) or (q_j) a *measure* $\mathbb{P}$ or $\mathbb{Q}$.
- The measure describes likelihood of up/down jump at each node.
- In the simple measure $\mathbb{P}$, all jumps are equally likely.
- Or we can choose the complex measure, $\mathbb{Q}$.
- Previously, we did not need the real-world measure $\mathbb{P}$ in order to find another measure which would allow risk-free construction.

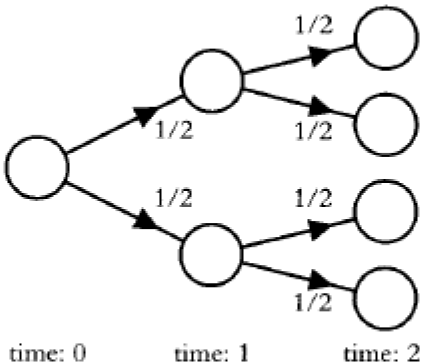

Figure 2.10a The measure $\mathbb{P}$

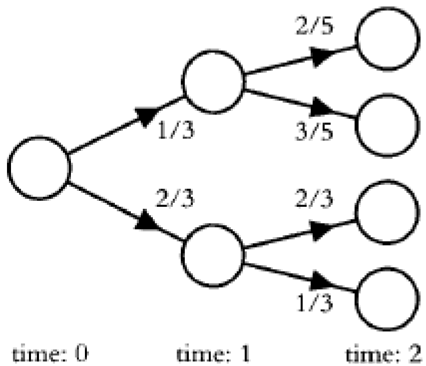

Figure 2.10b The measure $\mathbb{Q}$

Filtration

- A *filtration* $(\mathcal{F}_i)$ is the history of the stock until time *i* on the tree.
- The filtration starts at time zero with the path equal to the single node 1, $\mathcal{F}_0 = \{1\}$.
- By time 1, the filtration will be either be:
  - $\mathcal{F}_1 = \{1,2\}$, if the first jump was down.
  - $\mathcal{F}_1 = \{1,3\}$, if the first jump was up.
- The full filtration associated with each node is in **Table 2.3**.
- There is only one path at each given node.
- The filtration is a history of choices and thus fixes a node.

Table 2.3 The filtration process

node	1	2	3	4	5	6	7
filtration	$\{1\}$	$\{1,2\}$	$\{1,3\}$	$\{1,2,4\}$	$\{1,2,5\}$	$\{1,3,6\}$	$\{1,3,7\}$

Claim

- A *claim* X on the tree is a function of the nodes at a claim time-horizon *T*.
- Equivalently, X is a function of the filtration $\mathcal{F}_T$, due to uniqueness.
  - Example 1: The value of the process at time 2, S_2, is a claim.
  - Example 2: The value of a call option struck at $70 is a claim.
  - Example 3: The maximum prices the stock attains is a claim.
- A *claim* is *only* defined on the nodes at time T.
- A *process* is defined at all times up to and including T.

Table 2.4 Some claims at time 2

time 2 node	S_2	$(S_2 - 70)^+$	$\max\{S_0, S_1, S_2\}$
7	180	110	180
6	80	10	120
5	72	2	80
4	36	0	80

Conditional expectation operator

- The *conditional expectation operator* $\mathbb{E}_{\mathbb{Q}}(\cdot\,|\mathcal{F}_i)$ extends the notion of expectation to two parameters, a measure $\mathbb{Q}$ and a history $\mathcal{F}_i$.
- The measure $\mathbb{Q}$ tells us the 'probabilities' to use for each path.
- So far, however, we have only wanted paths from time zero.
- For a claim X, the quantity $\mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_i)$ is the expectation of X along some latter portion of paths with initial segment $\mathcal{F}_i$.

- Example: **Table 2.5**
- We take the $\mathbb{P}$ measure from **Figure 2.10a** and X to be the claim S_2.
- $\mathbb{E}_{\mathbb{P}}(S_2|\mathcal{F}_s) = S_2$ for every possible value of the filtration $\mathcal{F}_2$.

Table 2.5 Conditional expectation against filtration value

Expectation	Filtration value	Value
$\mathbb{E}_{\mathbb{P}}(S_2\|\mathcal{F}_0)$	$\{1\}$	$(180+80+72+36)/4 = 92$
$\mathbb{E}_{\mathbb{P}}(S_2\|\mathcal{F}_1)$	$\{1,3\}$	$\frac{1}{2}(180+80) = 130$
	$\{1,2\}$	$\frac{1}{2}(72+36) = 54$
$\mathbb{E}_{\mathbb{P}}(S_2\|\mathcal{F}_2)$	$\{1,3,7\}$	180
	$\{1,3,6\}$	80
	$\{1,2,5\}$	72
	$\{1,2,4\}$	36

Conditional expectation operator, cont'd

- We can also see $\mathbb{E}_{\mathbb{P}}(X|\mathcal{F}_i)$ as a process in i.
- We show the case where $X = S_2$ in **figure 2.11**.
- In this way, we can convert a *claim* into a *process*, given a *measure*.

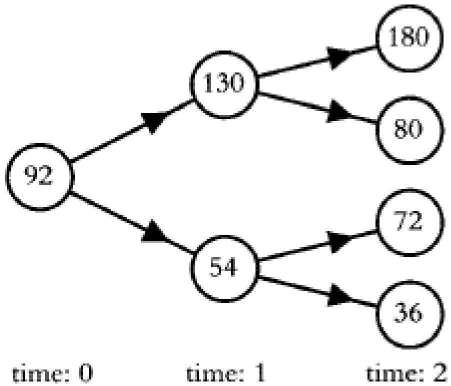

Figure 2.11 Conditional expectation process $\mathbb{E}_{\mathbb{P}}(S_2|\mathcal{F}_i)$

Previsible process

- A <u>*previsible process* $\phi = \phi_i$</u> is a process on the *same* tree whose value at any given node at time-tick i is <u>dependent only on the history up to the previous time-tick</u>, $\mathcal{F}_{i-1}$.
- The *previsible process* is defined at each node later than time zero.
- Compared to the main process S_2, the previsible process is known one node in advance.

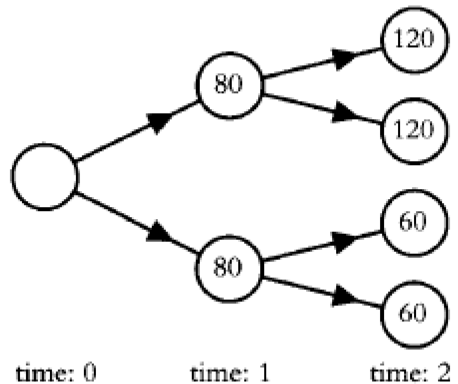

Figure 2.12 The previsible process S_{i-1}

Welborn / 86 / Binomial Rep. Thm.

Previsible process, cont'd

- The ***previsible process*** doesn't notice branches until one time-step after they have happened.
- Example 1: A random bond price process B_i would be previsible.
- Example 2: The delayed price process $\phi = S_{i-1}, i \leq 1$ (**figure 2.12**).
- Note: It is not always sensible to define the value that a previsible process has at time zero.

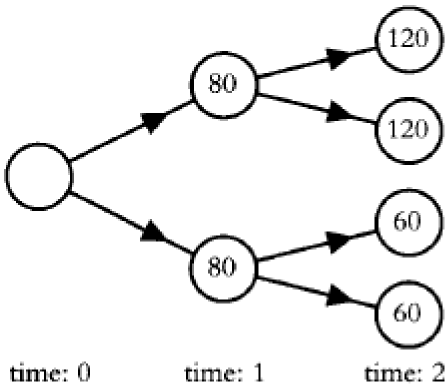

Figure 2.12 The previsible process S_{i-1}

Previsibility - Intuition

- Previsibility is closely related to the notion of predictability.
- A process whose value is “known” at any time given the “history so far” is previsible.
- Technically, a process X_t previsible, if given a filtration $\mathcal{F}_t$ over a probability space $(\Omega, \mathbb{P}, \mathcal{F})$, X_t is adapted to $\mathcal{F}_t$ or, alternatively, is $\mathcal{F}_t$ – measurable.
- Previsibility is different from the martingale property, which is a probabilistic concept relating best forecast to conditional expectation of the random variable (in finance with a clear physical interpretation of “no drift”).

Martingale

- A process S is a *martingale* with respect to a measure $\mathbb{P}$ and a filtration $(\mathcal{F}_i)$ if

$$\mathbb{E}_{\mathbb{P}}\left(S_j | \mathcal{F}_i\right) = S_i, \qquad \text{for all } i \leq j.$$

- Definition: The future expected value at time j

  …of the process S

  …under the measure $\mathbb{P}$

  …conditional on its history up until time i

  …is merely the process value at *time* i.

- Alternatively: The process S

  … has no drift under $\mathbb{P}$

  … has no bias up or down in its value under the expectation operator $\mathbb{E}_{\mathbb{P}}$.

- If the process has value 100, then the conditional expected value $\mathbb{E}_{\mathbb{P}}$ is 100 thereafter.

Martingale examples

- Example 1: The process which takes a fixed constant value.
- Example 2: The process S under the measure $\mathbb{Q}$ in figure 2.10b.
- $\mathbb{E}_{\mathbb{Q}}(S_1|\mathcal{F}_0) = \left(\frac{1}{3}\right)(120) + \left(\frac{2}{3}\right)(60) = 80$, which is the value of S_0.
- $\mathbb{E}_{\mathbb{Q}}(S_2|\mathcal{F}_1) = \left(\frac{2}{5}\right)(180) + \left(\frac{3}{5}\right)(80) = 120$, which is S_1 after 'up'.

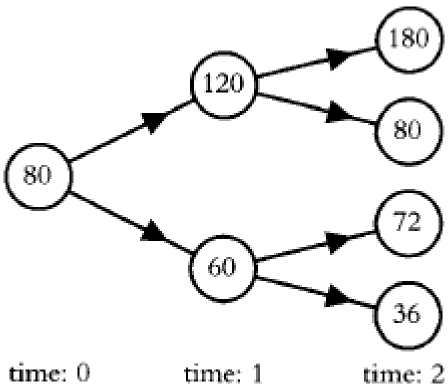

Figure 2.9 Tree with price process

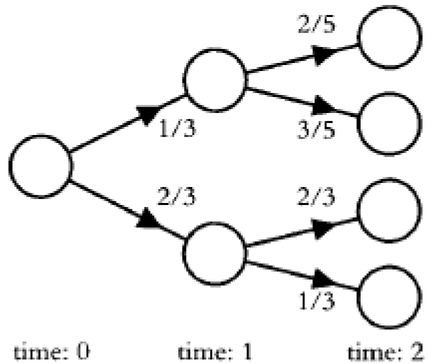

Figure 2.10b The measure $\mathbb{Q}$

Martingale examples, cont'd

- Example 3: The conditional expectation process $N_i = \mathbb{E}_{\mathbb{P}}(S_1|\mathcal{F}_0)$ is a $\mathbb{P}$-martingale.
- To check, confirm that $\mathbb{E}_{\mathbb{P}}(N_1|\mathcal{F}_0) = N_0$.
- This is just $\left(\frac{1}{2}\right)(130) + \left(\frac{1}{2}\right)(54) = 92$, which is the value of S_0.
- This is an example of a general result about $\mathbb{E}_{\mathbb{P}}(X|\mathcal{F}_i)$.

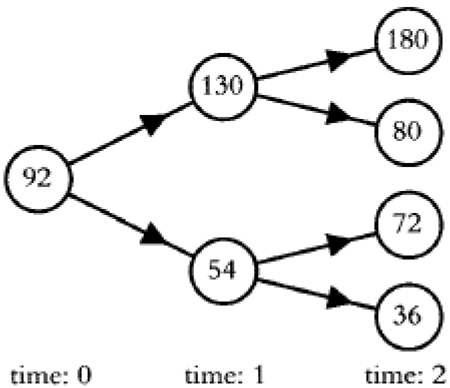

Figure 2.11 Conditional expectation process $\mathbb{E}_{\mathbb{P}}(S_2|\mathcal{F}_i)$

Conditional expectation process of a claim

- For any claim X, the process $\mathbb{E}_{\mathbb{P}}(X|\mathcal{F}_i)$ is always a $\mathbb{P}$-martingale.
- For this to be true, we need to use the fact that

$$\mathbb{E}_{\mathbb{P}}\left(\mathbb{E}_{\mathbb{P}}\left(X|\mathcal{F}_j\right)|\mathcal{F}_i\right) = \mathbb{E}_{\mathbb{P}}(X|\mathcal{F}_i), \qquad i \leq j.$$

- Translation:

  conditioning on the history up to time j,

  …and then conditioning on the history up to an earlier time i,

  …is the same as originally conditioning up to time i.

- This is called the "*tower law*".
- Given the *tower law*, we can check whether a process if a $\mathbb{P}$-martingale by comparing the process S_i to the conditional expectation process of its terminal value $\mathbb{E}_{\mathbb{P}}(S_T|\mathcal{F}_i)$.
- The process S_i is a $\mathbb{P}$-martingale only if these are identical.

Tower Law

- Another interpretation: “The conditional expectation process of a claim is always a martingale.”

Martingale measure

- Note: the process S above is not a martingale on its own.
- This process S is a $\mathbb{P}$-martingale w.r.t the measure $\mathbb{P}$.
- The same process can be a martingale w.r.t. other measures also.
- For example, the previous process S is not a $\mathbb{P}$-martingale but a $\mathbb{Q}$-martingale. (This is because figures 2.9 and 2.11 are different).
- Thus, this process is a $\mathbb{Q}$-martingale, and $\mathbb{Q}$ is called the ***martingale measure*** for *S.*

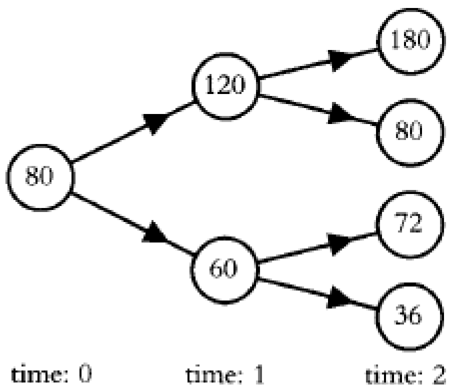

Figure 2.9 Tree with price process

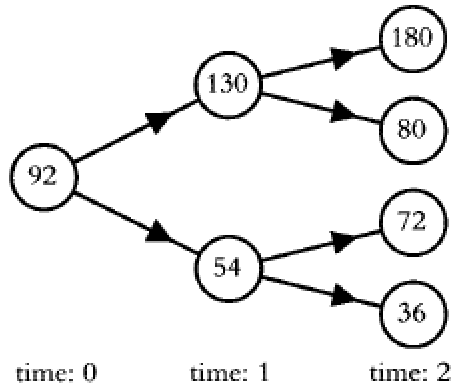

Figure 2.11 Conditional expectation process $\mathbb{E}_{\mathbb{P}}(S_2|\mathcal{F}_i)$

Welborn / 86 / Binomial Rep. Thm.

Binomial representation theorem

- Suppose the measure $\mathbb{Q}$ is such that the binomial price process S is a $\mathbb{Q}$-martingale.
- If N is any other $\mathbb{Q}$-martingale, then there exists a previsible process ϕ such that

$$N_i = N_o + \sum_{k=1}^{i} \phi_k \triangle S_k$$

where $\triangle S_k := S_i - S_{i-1}$ is the change in S from tick-time $i - 1$ to i and ϕ_k is the value of ϕ at the relevant node at tick-time i.

Binomial representation theorem - Intuition

- Basic Idea: If there are two martingale processes under the same measure, then we can find the value of one given the other in a ‘previsible’ (e.g., deterministic) way.
- As long as we know the possible states of the world in both measures, we just need to “match the widths” (how much the two processes vary, e.g. variance) and “match the offsets” (how the mean of the two processes differ).
- This is analogous to change of coordinates in geometry.
- In geometry we require the change of co-ordinates be in the same X-Y plane.
- Here, we require that both processes be martingale under the same measure $\mathbb{Q}$.

Proof of Binomial representation theorem

- Consider a single branch from a node at tick-time $i-1$ to two nodes 'up' and 'down' at tick-time i.
- The history $\mathcal{F}_i$ has two choices beyond $\mathcal{F}_{i-1}$: 'up' and 'down'.
- The increments over the branch and the processes S and N are

$$\triangle S_i = S_i - S_{i-1} \qquad \text{and} \qquad \triangle N_i = N_i - N_{i-1}$$

- The variability of the increment depends on the *geometry* of the branch (e.g., time change and price change).

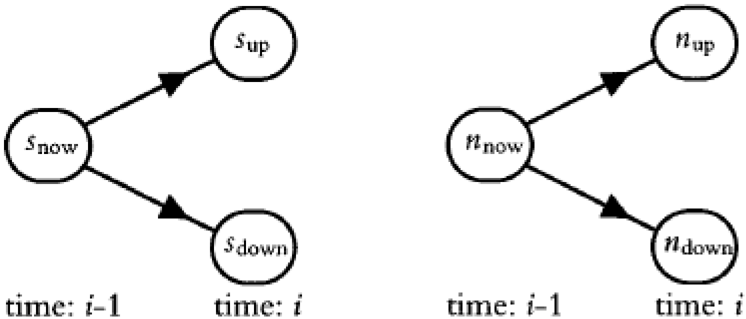

Figure 2.13 The branch geometry (process S on left; process N on right)

Proof of Binomial representation theorem

- First, consider the price scaling.
- The size of the differences between up and down jump values is

$$S\colon\ \delta s_i = s_{up} - s_{down} \quad \text{and} \quad N\colon\ \delta n_i = n_{up} - n_{down}$$

- Both processes depend on the filtration $\mathcal{F}_{i-1}$.
- We define ϕ_i to be the ratio of these branch widths:

$$\phi_i = \frac{\delta n_i}{\delta s_i}$$

Proof of Binomial representation theorem

- Second, consider the (time) shift.

$$\Delta N_i = \phi_i \Delta S_i + k, \qquad \text{for } \phi_i \text{ and } k \text{ known by } \mathcal{F}_{i-1}$$

- This says that the N-increment ΔN_i must be given by a scaled increment plus an offset k determined by $\mathcal{F}_{i-1}$.
- Both S and N are $\mathbb{Q}$-martingales:

$$\mathbb{E}_{\mathbb{Q}}(\Delta N_i | \mathcal{F}_{i-1}) = 0 \qquad \text{and} \qquad \mathbb{E}_{\mathbb{Q}}(\Delta S_i | \mathcal{F}_{i-1}) = 0,$$

which means the increments have zero expectation conditional on the history $\mathcal{F}_{i-1}$.

Proof of Binomial representation theorem

- The scaling factor ϕ_i is previsible, e.g. known by time $i - 1$, so we also have $\mathbb{E}_{\mathbb{Q}}(\phi_i \Delta S_i | \mathcal{F}_{i-1}) = 0$.

- Thus, the offset k must also be zero $(0 = 0 + k)$.

- So, for the case where S and N are both $\mathbb{Q}$-martingales, the (price) scale and (time) shift reduces to a scaling:

$$\Delta N_i = \phi_i \Delta S_i$$

Financial application

- How can we use the binomial theorem for pricing?
- Our binomial tree model is a stock that follows the binomial process S.
- If there were a measure $\mathbb{Q}$ which made S a martingale, we could use the representation theorem to represent some other martingale N_i in terms of the stock price.
- The previsible ϕ_i could act as a construction strategy.
- At each point we could buy the appropriate ϕ of the stock and follow the gains and losses of the martingale N_i.

Financial application - challenges

- There are a few challenges to applying the theorem.
- First, we have a claim X and not a martingale.
- The claim is a random variable and not a process
- Second, we have a stock and a cash bond.
- So, we need to use both ϕ_i and ψ_i to account for the stock and bond holding.

Financial application – random variables

- First, we have the tools turn a random variable (such as claim X) into a process.
- Given any measure $\mathbb{Q}$, we can take conditional expectations and form the process E:

$$E_i = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_i)$$

- Whatever measure $\mathbb{Q}$ we choose, E is automatically a $\mathbb{Q}$-martingale.
- If we find $\mathbb{Q}$, a measure under which S_i is $\mathbb{Q}$-martingale, the appropriate E is a $\mathbb{Q}$-martingale as well.

Financial application – cash bond

- The cash bond B_i represents the growth of money.
- One dollar today is B_i dollars at time i.
- The bond process B_i is <u>previsible</u> and positive. Assume $B_0 = 1$.

- More definitions:

1. The <u>*discount process*</u> is B_i^{-1} is previsible (inverse of B_i).
2. The <u>*discounted stock process*</u> $Z_i \coloneqq B_i^{-1} S_i$ is a process that can be observed on the same binomial process tree as S.
3. The <u>*discounted claim*</u> $B_T^{-1} X$ is a claim. Due to the mapping above, the discounted claim is a claim on both Z and S.

A useful trick

- We have $\mathbb{Q}$ such that Z is a $\mathbb{Q}$-martingale.
- We have a claim X.
- Thus, there is a $\mathbb{Q}$-*martingale process* E, produced from $B_T^{-1}X$ by taking conditional expectations, where $E_i = \mathbb{E}_{\mathbb{Q}}(B_T^{-1}X|\mathcal{F}_i)$.
- By the **Binomial Representation Theorem**, there is a *previsible* process ϕ such that

$$E_i = E_o + \sum_{k=1}^{i} \phi_k \triangle Z_k$$

- We need to show that the previsible process ϕ has a physical counterpart in the market (e.g., an asset).
- In other words, we will use BRT to price the claim X.

Construction strategy at time 0

- At tick-time i, buy the portfolio Π_i consisting of:
  - ϕ_{i+1} units of the stock S_i
  - $\psi_{i+1} = \left(E_i - \phi_{i+1} B_i^{-1} S_i\right)$ units of the cash bond.
- A time zero, Π_0 is worth (costs) $\phi_1 S_0 + \psi_1 B_0$.
- $\Pi_0 = \phi_1 S_0 + \psi_1 B_0$
- $\Pi_0 = B_0(\phi_1 S_0 B_0^{-1} + \psi_1)$
- $\Pi_0 = B_0(\phi_1 Z_0 + E_o - \phi_1 Z_0)$
- $\Pi_0 = B_0 E_0$
- $\Pi_0 = E_0 = \mathbb{E}_{\mathbb{Q}}(B_T^{-1} X)$
- This is what our portfolio Π_0 costs to create at time 0.
- Note that $\psi_1 = E_0 - \phi_1 B_0^{-1} S_0 = E_o - \phi Z_0$.

Construction strategy at time 1

- At time 1, the values of the stock and bond have changed.
- $\Pi_0 = \phi_1 S_1 + \psi_1 B_1$
- Substitute in $\psi_1 = E_o - \phi Z_0$,
  - $\Pi_0 = \phi_1 S_1 + B_1 E_0 - B_1 \phi Z_0 = B_1(E_0 + \phi_1 Z_1 - \phi_1 Z_0) = B_1(E_0 + \phi_1 \Delta Z_1)$
- BPT tells us: $E_1 = E_0 + \phi_1 \Delta Z_1$. Thus, at time 1
  - $\Pi_0 = B_1(Y_0 + \phi_1 \Delta Z_1) = B_1 E_1$
- Next, our strategy requires that we construct Π_1 at time 1:
  - $\Pi_1 = \phi_2 S_1 + \psi_2 B_1$
  - $\Pi_1 = B_0(\phi_1 S_1 B_0^{-1} + \psi_2)$
  - $\Pi_1 = B_0(\phi_2 Z_1 + E_1 - \phi_2 Z_1)$
  - $\Pi_1 = B_1 E_1$
- Thus, the price of Π_1 at time 1 is the same as the value of Π_0 at time 1.

Reviewing the strategy

- What have we learned?
- At time 0, $\Pi_0 = B_0 E_0$.
- At time 1, Π_0 is worth $B_1 E_1$.
- At time 1, we need to purchase a new Π_1 per our strategy.
- But the portfolio Π_1 costs precisely $B_1 E_1$, regardless of what happens to S (e.g., whatever filtration $\mathcal{F}_1$ obtains).
- Thus, we can sell Π_0 to create Π_1.
- This is what it means for the portfolio to be "self-financing."

"Self-Financing"

- At time i, the portfolio Π_i costs $B_i E_i$ to purchase.
- At time $i+1$, the portfolio will change to be worth $B_{i+1}E_{i+1}$.
- Since $B_{i+1}E_{i+1}$ is the cost of Π_{i+1}, our strategy is "self-financing."
- At the end of our self-financing strategy, we end up with the portfolio Π_{T-1} at (maturity) time T, which is $B_T B_T^{-1} X = X$:
  - $B_T E_T = B_T E^{\mathbb{Q}}[B_T^{-1} X_T | \mathcal{F}_T] = B_T B_T^{-1} X = X$
- Thus, we have arrived at the claim X.
- The price of the claim X_t should be exactly equal to the cost of the portfolio we originally created.
- In other words, no money has entered or exited the system.

Arbitrage

- The price of the claim X is the expected value of the discounted claim, under the martingale measure $\mathbb{Q}$, for the discounted stock Z:

$$\mathbb{E}_{\mathbb{Q}}(B_T^{-1}X)$$

- This is an arbitrage price b/c any other could yield risk-free positive profits by applying the (ϕ_i, ψ_i) to duplicate the claim.
- Within the binomial two-asset model, there exists a self-financing strategy (ϕ_i, ψ_i) that duplicates the claim we want to price.
- The claim can be priced no matter what happens to the path of S or the filtration $\mathcal{F}$.
- Note: We have merely formalized the process from section 2.2.

The existence of self-financing strategies

- What do we mean by a "self-financing" (ϕ_i, ψ_i) strategy?

- Define V_i, the worth of the trading strategy at time i, to be the opening value of the portfolio Π_i at time i : $V_i = \phi_{i+1} S_i + \psi_{i+1} B_i$.
- The strategy is "self-financing" if the closing value of portfolio Π_{i-1} at time i is precisely equal to V_i.
- Mathematically, the "financing gap" must be zero:
$$D_i = V_i - \phi_i S_i - \psi_i B_i = 0$$
- Alternatively, define $\Delta V_i = V_i - V_{i-1}$. The changes of the strategy value will be
$$\Delta V_i = \phi_i \Delta S_i + \psi_i \Delta B_i + D_i$$
- The gap D_i is zero IFF ΔV_i is <u>only</u> due to changes in the stock and bond values.

Self-financing hedging strategies

- Given a binomial tree model of a market with stock S and bond B, then (ϕ_i, ψ_i) is a self-financing strategy to construct claim X if:

1. Both ϕ and ψ are previsible;
2. The change in the value V of the portfolio defined by the strategy above obeys the difference equation:

$$\Delta V_i = \phi_i \Delta S_i + \psi_i \Delta B_i$$

where $\Delta S_i := S_i - S_{i-1}$ is the change in S from tick-time i -1 to i, and $\Delta B_i := B_i - B_{i-1}$ is the corresponding change in B;

3. and $\phi_T S_T + \psi_T B_T$ is identically equal to the claim X.

Expectation of the discounted claim under the martingale measure

- The price of any derivative within the binomial tree model is the expectation of the discounted claim under the measure $\mathbb{Q}$ which makes the discounted stock a martingale.
- **Option Price Formula** (discrete time):
- The option value at tick-time i of a claim X maturing at date T is

$$B_i \mathbb{E}_{\mathbb{Q}}(B_T^{-1} X | \mathcal{F}_i)$$

- Why?
- This is because there is a self-financing strategy justified by the binomial representation theorem, which requires that amount to start off and yields the claim X without risk at T.

Uniqueness and existence of $\mathbb{Q}$

- In a discrete world, for any "sensible" stock process S, there will be a unique measure $\mathbb{Q}$ under which $B_T^{-1}S_i$, the discounted stock, is a $\mathbb{Q}$-martingale.

Conclusion

- We now have a general theorem for the discrete world.
- Any claim on a stock implies a derivative instrument tied to the underlying stock value at any time by a construction strategy capable of providing arbitrage profits if there are any "mispricings."
- The arbitrage-justified value is the expectation of the discounted claim, but expectation under just one special measure, the measure $\mathbb{Q}$ under which the discounted stock is a martingale.
- The real measure $\mathbb{P}$ which S follows is irrelevant.
- The construction strategy is self-financing and generates the claim regardless of what S does.