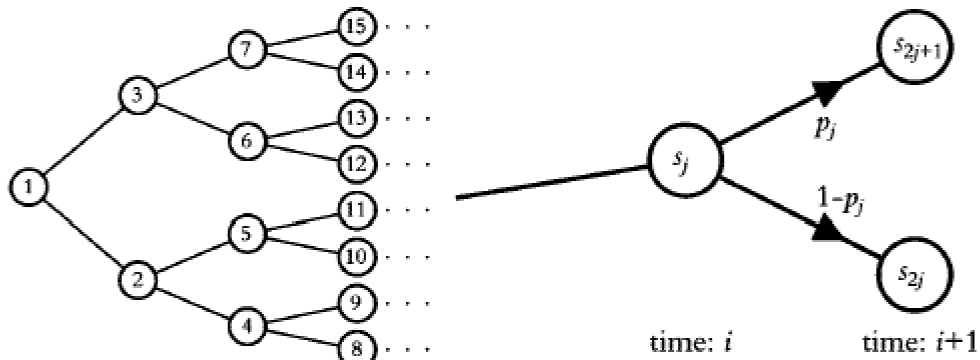
The Binomial Tree Model

- Baxter and Rennie 2.2, “The Binomial Tree Model”
- We will modify the branch model to a tree model
- We will move from 1 step to 2 steps.
- We still choose between two instruments:
  - A stock S
  - A discount bond B
- Assumptions:
  - Unlimited amounts of B and S can be bought or sold.
  - No transactions costs, default risks, or spreads.

The stock

- At tick-time *i*, the stock can have 2^i possible values.
- Given the value at tick time $(1 - i)$, there are two possibilities from s_j:
  - Up to 2_{j+1} with probability p_j
  - Down to 2_j with probability $1 - p_j$



Binary tree with numbered nodes Stock price development

Figure 2.3

The cash bond

- Before, we assumed a *constant* interest rate *r*
- More realistically, we will have a sequence of interest rates,

$$R_o, R_1 \ldots,$$

where each rate is known at the start of the appropriate period.

- The value of the cash bond, B, at time $n\delta t$ would thus be

$$B_o exp\left(\sum_{i=0}^{n-1} R_i \partial t\right)$$

- B_0 would be worth $B_0 \exp(r\delta t)$ after time unit δt.
- For simplicity, for now, we will keep the constant r assumption
- Thus, at time $n\delta t$, the cash bond is worth $B_0 \exp(rn\delta t)$

Trees are complex

- This structure may appear simplistic and arbitrary.
- Would not a better design allow for continuous fluid changes in stock and bond values?
- Our final goal is to understand “risk free construction” in continuous time.
- As δt tends to zero, the discrete model will provide a useful roadmap.

Backwards induction

- A fundamental feature of binomial models is the “boundary condition.”
- We know the value of the derivative at expiration, or terminal t.
- We *also* know the history of S to that point.
  - Later, we will call this a “filtration.”
- Each node is unique, and we can therefore associate a claim value with a node in the tree.
- A key assumption, therefore, is finiteness of the tree.
  - In the financial world, this is entirely reasonable, as most contingent claims have an expiration date.

The two-step

- Assume the interest rate over any branch is a constant r.
- There exists some $q_j s$ s.t. the value at node j, at tick-time i is

$$f(j) = e^{-r\partial t}\left(q_j f(2_j + 1) + (1 - q_j) f(2_j)\right)$$

- This is the discounted expectation under q_j of the time $i + 1$ claim values $f(2_j + 1)$ and $f(2_j)$

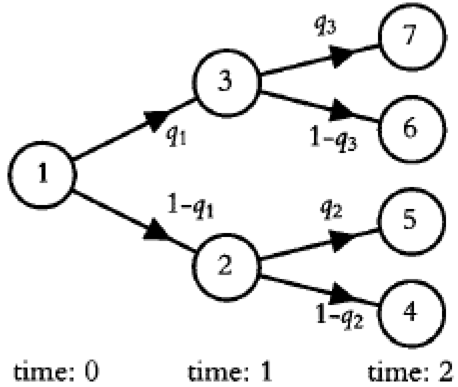


Figure 2.4 Double fork at time 0

The two-step, cont'd

- One-step forks are equivalent to branches in our one-step model.
- Thus, $f(3)$ can be derived from $f(6)$ and $f(7)$ via

$$f(3) = e^{-r\partial t}\big(q_3 f(7) + (1 - q_3) f(6)\big)$$

- Here, q_j is the modified probability operator defined as before:

$$q_j = \frac{s_j \exp(r\delta t) - s_{2j}}{s_{2j+1} - s_{2j}}$$

- So, $q_2 = \frac{s_2 \exp(r\delta t) - s_4}{s_5 - s_4}$ and $q_3 = \frac{s_3 \exp(r\delta t) - s_6}{s_7 - s_6}$

The two-step, cont'd

- Now we can work out a value for the claim at time 1.
  - The claim is worth $f(3)$ in an "up" jump
  - The claim is worth $f(2)$ in a "down" jump
- The fork from node 1 to nodes 2 and 3 is a single branch. So:

$$f(1) = e^{-r\partial t}\left(q_1 f(3) + (1 - q_1) f(2)\right)$$

- The value at time 0 will combine the equations above:

$$\begin{aligned} &f(1) \\ &= e^{-2r\partial t}[q_1 q_3 f(7) + q_1(1 - q_3) f(6) + (1 - q_1) q_2 f(5) \\ &+ (1 - q_1)(1 - q_2) f(4)] \end{aligned}$$

Path probabilities

- The probability that the process follows a path is the product of the probabilities of each branch.
  - The probability of a “up” twice is $q_1 q_3$.
  - The probability of “up” and then “down” is $q_1(1 - q_3)$.
- The expectation of some claim on the final node is the sum of the preceding nodes weighted by the path probabilities.
- A two-step tree has four possible paths, each with two probabilities.
- The expectation of a claim is the total of the four outcomes each weighted by this path probabilities:
  - $q_1 q_3, q_1(1 - q_3), (1 - q_1)q_2, (1 - q_1)(1 - q_2)$
  - These probabilities correspond to the “probability tree” (q_1, q_2, q_3)

The inductive step

- We generalize our tree to n periods and start with the final layer.
- All nodes have claim values and are in pairs.
- Consider any of the final branches from node at time $(n - 1)$ to two nodes at time n.
- Our previous result shows that we can construct a risk-free portfolio (φ, ψ) of stock and bond that can generate the time n claim.
- Thus, each node at time $(n - 1)$ are all have arbitrage-guaranteed values for the contingent claim.
- Therefore, we can work back from the "enforced claim" at the final layer to equally strong enforced claim values at the previous layer.
- This is the *inductive* step.

The inductive result

- We repeat the inductive step and affix a value of the derivative at each layer.
- At each branch, we (re)construct (φ, ψ) portfolios for the next step.
- We reach the root of the tree with a single value at time 0.
- This is the time-zero value of the final derivative claim.
- There is a construction portfolio which, even though it changes at each tick time, will lead us to the same payoff as the claim, *regardless of the path taken*!
- We therefore need construction portfolios (φ_j, ψ_j) at each node.

A worked example

- This is a recombinant tree where different branches can recombine at the same node. (This will be computationally easier)
- The tree nodes are stock prices *s*, and the up probability is ¾, and the down probability is ¼. Assume interest rates to be 0.
- What is the value of an option to buy the stock for 100 at time 3?

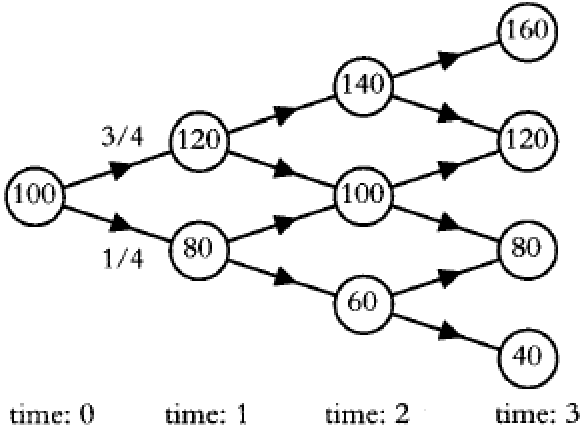


Figure 2.5 A stock price on a recombinant tree

A worked example, cont'd

- The value of the claim will be either 0, 0, 20 or 60.
- We use the equations for the "new" probability, q, and the claim, f.
- The "**risk-neutral**" probability *q is*:

$$q = \frac{s_{now} - s_{down}}{s_{up} - s_{down}}$$

- The value of a claim, f, is:

$$f_{now} = q f_{up} + (1 - q) f_{down}$$

- We can show that the q probabilities will be ½ at every node.
- Then, we solve for the option value at time 2 using the formulae.
- Finally, we can fill in the nodes at level 2, and repeat for 1, etc.

A worked example, cont'd

- $F(2_{up,up})=(60+20)/2=40$
- $F(2_{up,down})=(0+20)/2=10$
- $F(2_{down,down})=(0+0)/2=0$
- $F(1_{up})=(40+10)/2=25$
- $F(1_{down})=(10+0)/2=5$

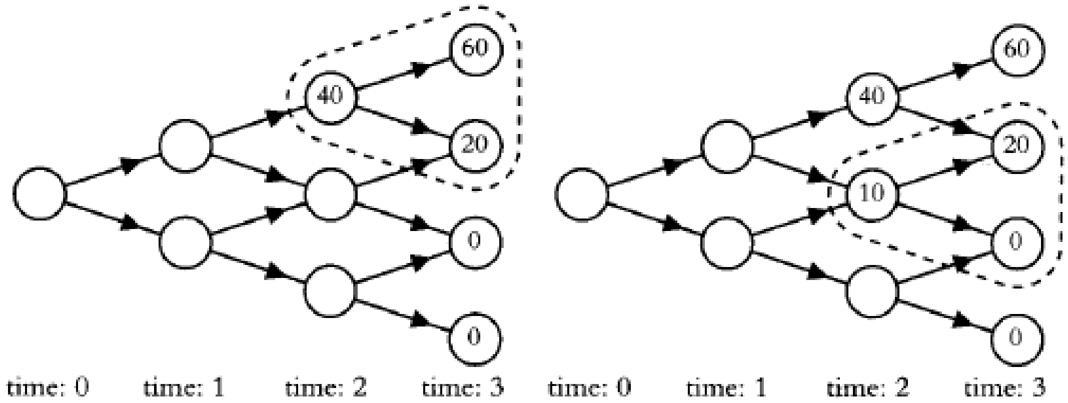


Figure 2.6 The option claims and claim-values at time 2

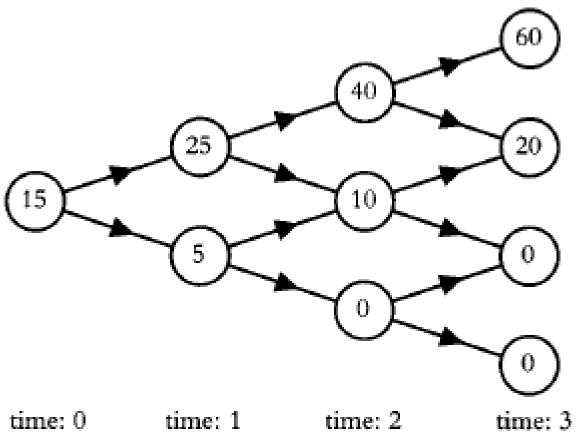


Figure 2.7 The option claim tree

- $F(0)=(25+5)/2=15$
- Thus, the price of the option at time zero is 15.

A worked example, cont'd

- We have found that the price of the option at time zero is 15.
- Next, we need φ units of stock to hedge at any current time.

$$\varphi = \frac{f_{up} - f_{down}}{s_{up} - s_{down}}$$

- *Time* 0
  - We sell the option for $\$15$.
  - We calculate $\varphi = (25 - 5) / (120 - 80) = 0.5$.
  - Buying 0.5 units of stock costs $50, so we need to borrow an additional $\$35$.
- *Time* 1_{up}: Stock goes to $120
  - $\varphi = (40 - 10) / (140 - 100) = 0.75$.
  - Buy another $(0.75 - 0.5) = 0.25$ units of stock.
  - To fund this purchase, we need to borrow $(0.25 * \$120) = \$30$.
  - Thus, our total borrowing is $\$30 + \$35 = \$65$.

A worked example, cont'd

- *Time* $2_{up,\,up}$: Stock goes to $\$140$
  - $\varphi = (60 - 20)\,/\,(160 - 120) = 1.$
  - Buy another $(1 - 0.75) = 0.25$ units of stock.
  - To fund this purchase, we need to borrow $(0.25 * \$140) = \$35.$
  - Thus, our total borrowing is now $\$65 + \$35 = \$100.$
- *Time* $3_{up,\,up,\,down}$: Stock drops to $\$120$
  - Recall, <u>we have sold a call option struck at $100 with expiry at time 3</u>.
  - Option is "in the money" (ITM). We sell the stock for $100 and cancel our debt.
  - NOTE: Same scenario would happen if the stock went up to $160.

Table 2.1 Option and portfolio development

Time i	Last Jump	Stock Price S_i	Option Value V_i	Stock Holding ϕ_i	Bond Holding ψ_i
0	-	100	15	-	-
1	up	120	25	0.50	−35
2	up	140	40	0.75	−65
3	down	120	20	1.00	−100

A worked example, cont'd

- *Time* 1_{down}: Stock down to \$80
  - $\varphi = (10 - 0) / (100 - 60) = 0.25.$
  - Sell half our stock and reduce our debt by \$35 to \$15.
- *Time* $2_{down, up}$: Stock goes up to \$100
  - $\varphi = (20 - 0) / (120 - 80) = 0.50.$
  - Buy another $(0.50 - 0.25) = 0.25$ units of stock. Borrow \$25 for \$40 total debt.
- *Time* $3_{down, up, down}$: Stock drops to \$80
  - Our φ units of stock is worth \$40, which cancels our debt.
  - The option is "out of the money" (OTM), so we break even.

Table 2.2 Option and portfolio development along a different path

Time i	Last Jump	Stock Price S_i	Option Value V_i	Stock Holding ϕ_i	Bond Holding ψ_i
0	-	100	15	-	-
1	down	80	5	0.50	−35
2	up	100	10	0.25	−15
3	down	80	0	0.50	−40

Expectation

- The expectation under the probability *q* achieves the same result:
  - Time $3_{up,up,up}$ = (1/2)(1/2)(1/2) = (1/8)
  - Time $3_{up,up,down}$ = 3 $*$ (1/2)(1/2)(1/2) = (3/8)
  - Time $3_{down,up,down}$ = 3 $*$ (1/2)(1/2)(1/2) = (3/8)
  - Time $3_{down,down,down}$ = (1/2)(1/2)(1/2) = (1/8)
  - Claim under the expectation q: (1/8)($60) + (3/8)($20) = $15.
- Compare this to the *p* expectation:
  - Time $3_{up,up,up}$ = (3/4)(3/4)(3/4) = (27/64)
  - Time $3_{up,up,down}$ = (3/4)(3/4)(1/4) + (3/4)(1/4)(3/4) + (1/4)(3/4)(3/4) = (27/64)
  - Time $3_{down,up,down}$ = (3/4)(1/4)(1/4) + (1/4)(3/4)(1/4) + (1/4)(1/4)(3/4) = (9/64)
  - Time $3_{down,down,down}$ = (1/4)(1/4)(1/4) = (1/64)
  - Claim expectation under p: (27/64)($60) + (27/64)($20) = $33.75.

Conclusion

- Under the tree structure, there is only *one* possible value for an implied derivative instrument at every node.
  - If not, then arbitrage is possible
- Order of operations:
  - Define the claim
  - Calculate the value of the claim at time T.
  - Calculate the claim value at time 0 via arbitrage via backwards induction.
- Each branchlet carries a probability q_j
- The cost of the local hedge portfolio (φ_j, ψ_j) can be written as a discounted expectation.
- By stringing together local (re)construction portfolios, we have developed a global construction *strategy* that guarantees a value.
- The global discounted expectation gives the value of the claims.

Summary

$q = \dfrac{s_{now} - s_{down}}{s_{up} - s_{down}}$	$\varphi = \dfrac{f_{up} - f_{down}}{s_{up} - s_{down}}$
$f_{now} = qf_{up} + (1-q)f_{down}$	$\psi = B_{now}^{-1}(f_{now} - \varphi s_{now})$
$V = f(1)E_Q(B_T^{-1}X)$	
q: arbitrage probability of up-jump	r: interest rate over the period
f: claim value time-process	s: stock price process
φ: stock holding strategy	B: bond price process, $B_0 = 1$
ψ: bond holding strategy	Q: measure made up of the q's
V: claim value at time zero	X: claim payoff
δt: length of period	T: time of claim payoff