The Binomial Branch Model

- Baxter and Rennie 2.1, “The Binomial Branch Model
- We will explore the limits of arbitrage.
- First, we will use a simplified “branch” example
- Then we will expand to a more complex “tree” model.

The stock

- You own 1 share of stock, s_1, at time 0.
- One of two things can happen at time tick δt :
  - The stock moves up with probability p to s_3.
  - The stock moves down with probability $1 - p$ to s_s.
- Investors may wish to enter a forward contract for some payoff f, that is a function of values at nodes 2 and 3: $f(2)$ and $f(3)$.

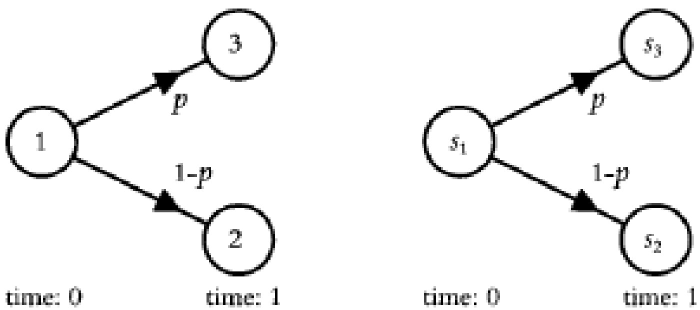

Figure 2.1 The binomial branch

The bond

- A cash bond, B, represents the time value of money.
- The bond grows at rate $\exp(r\delta t)$ per unit of time.
- Assume that we can borrow and lend at this rate.
- B_0 will be worth $B_0\exp(r\delta t)$ after time unit δt.

Risk-free construction

- Simple forward arbitrage:
  - Buy stock for s_1
  - Sell short cash bonds (to fund the stock purchase)
  - The price k of the forward must be equal to $s_1 \exp(r\delta t)$.
- Complex forward arbitrage:
  - $E(F) = (1-p)f(2) + pf(3)$
  - The stock takes one of two random values at the end.
  - We know the probabilities and the expected value of *f* at the end
  - But we do not know the actual value in advance. What do we do?
- Bond only strategy reflects discounting:
  - Cash bond grows at $\exp(r\delta t)$
  - Buying discount bonds for $B(0) = exp(-r\delta t)[(1-p)f(2) + pf(3)]$
  - … must equal, at the end, $B(1) = [(1-p)f(2) + pf(3)]$

Bond-only strategy

- Bond only strategy.
- Cash bond grows at $\exp(r\delta t)$
- Start: Buy the discount bond for $B(0) = exp(-r\delta t)[(1-p)f(2) + pf(3)]$
- End: $B(1) = [(1-p)f(2) + pf(3)]$
- This gives us the expected value of the derivative at the end of the period

Expectation for a branch

- Formally, let S be a binomial branch process
- Stock has value s_1 at time 0
- Stock may move up to s_3 with probability p
- Stock may move down to s_2 with probability $1 - p$
- Thus, expectation of S at time 1 with up move probability p :

$$E_p(S_1) = (1 - p)s_2 + ps_3$$

Stocks and bonds together

- Consider a general portfolio (φ, ψ) :
  - We have φ of the stock S worth φs_1
  - We have ψ of the bond B worth ψB_0
- Cost at T_0: $\varphi s_1 + \psi B_0$
- Value at δt:
  - After an 'up' move: $f(3) = \varphi s_3 + \psi B_0 \exp(r\delta t)$
  - After a 'down' move: $f(2) = \varphi s_2 + \psi B_0 \exp(r\delta t)$
- Thus, we have two simultaneous equations for φ and ψ.

Stocks and bonds together

- Solving for φ and ψ :

$$\varphi = \frac{f(3) - f(2)}{s_3 - s_2},$$

$$\psi = B_0^{-1}\exp(-r\delta t)\left(f(3) - \frac{(f(3) - f(2))s_3}{s_3 - s_2}\right)$$

- Thus, if we bought the portfolio (φ,ψ) and held δt, then:
  - An "up" tick is worth $f(3)$
  - A "down" stick is worth $f(2)$.
- We have *synthesized* the derivative

The price is right

- Any derivative can be constructed by combining stocks and bonds.
- Let *V* represent buying $\varphi s_1 + \psi B_0$ of the portfolio (φ, ψ):

$$\varphi = \frac{f(3) - f(2)}{s_3 - s_2}$$

$$\mathrm{V} = s_1 \frac{f(3) - f(2)}{s_3 - s_2} + \exp(-r\partial t)\left(f(3) - \frac{(f(3) - f(2))s_3}{s_3 - s_2}\right)$$

- If a market maker sold V for $P = V - \varepsilon$, then an arbitrageur could buy the portfolio for $V - \varepsilon$, short the (φ, ψ), and earn riskless profits.
- Profits would be positive <u>regardless of stock price</u>.
- Arbitrage guarantees that V is the only rational price.

Example

- You have the following endowment at T_0 :
  - One interest-free bond priced at $1
  - One stock priced at $1
- After $T_0 + \delta t$, the stock may be worth either $2 or $0.50.
- *What is the value of a bet that pays $1 if the stock goes up?*

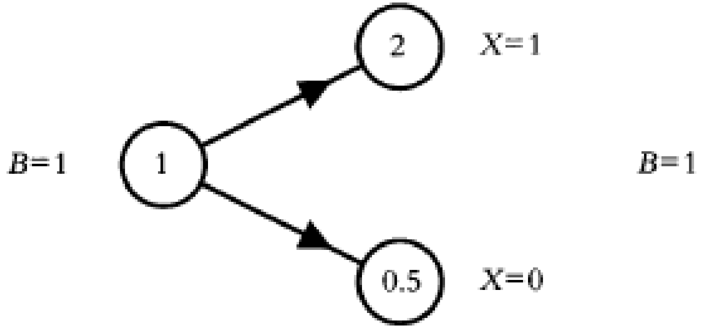

Figure 2.2 Pricing a bet

Example - Solution

- Let *B* denote the bond price, *S* the stock price, and *X* the bet payoff
- $\varphi = (1 - 0) / (2 - 0.5) = 2/3.$
- $\psi = 1 - (2/3) = 1/3$
- At T_0, the portfolio V costs $(2/3)(\$1) - (1/3)(\$1) = \$0.33.$
- At $T_0 + \delta t$:
  - Up jump: V = (2/3)($2) – (1/3)($1) = $1
  - Down jump: V = (2/3)($0.50) – (1/3)($1) = $0
- A portfolio that combines stock and bonds replicates the bet.
- Therefore, the portfolio must have the same value as the bet: $0.33
  - This reflects the "The Law of One Price."
  - This is another way of saying "No Arbitrage."

Expectation regained

- But wait, do we need *p*? **NO.** We define a new simplifying variable:

$$q = \frac{s_1 \exp(r\delta t) - s_2}{s_3 - s_2}$$

- Note that *q* must be bounded by (0,1). Why?
  - First, if $q \leq 0$, then $s_1 exp(r\delta t) \leq s_2 < s_3$. But $s_1 exp(r\delta t)$ is s_1 of the Bond.
  - Thus, we could make arbitrage profits by buying stock and financing risk free.
  - Second, if $q \geq 1$, then $s_2 < s_3 \leq s_1 exp(r\delta t)$.
  - Thus, then we could sell stock and buy cash bonds to achieve risk-free profit.
- Therefore, we can rewrite our formula for *V* of the portfolio (φ, ψ):

$$V = \exp(-r\delta t)\left((1-q)f(2) - qf(3)\right)$$

- V is the discounted *expectation* of the derivative w.r.t. *q.*
- V gives the correct strike for a forward contract: $s_1 \exp(r\delta t)$.