Overture to continuous models

- Baxter/Rennie 2.4, “Overture to continuous models”
- Discrete techniques offer a window to the continuous world.
- We can approximate the continuous world with very small intertick time.
- A natural discrete model with constant growth rate and noise approximates a log-normal distribution under both the original measure $\mathbb{P}$ and the martingale measure $\mathbb{Q}$.
- It will be possible to ‘derive’ the **Black-Scholes** option pricing formula in a non-rigorous way.

Model with constant stock growth and noise

- The model is parametrized by the intertick time δt.
- As $\delta t \to 0$, the model should approximate continuous time.
- There are three other fixed and constant parameters: the noisiness σ, the stock growth rate μ, and the riskless interest rate r.
- The cash bond B_t has the simple form $B_t = exp(rt)$.
- The stock process follows the nodes of a recombinant tree, which moves from value s along the next up/down branch to the new value

$$\begin{cases} s\ exp\left(\mu\delta t + \sigma\sqrt{\delta t}\right) \text{ if up,} \\ s\ exp\left(\mu\delta t - \sigma\sqrt{\delta t}\right) \text{ if down.} \end{cases}$$

- The jumps are equally likely to be up as down; $p = \frac{1}{2}$ everywhere.

Model with constant stock growth and noise, continued

- For a fixed time t, if we set n to be the number of ticks until time t, then $n = t/\delta t$ and

$$S_t = S_0 exp\left(\mu t + \sigma\sqrt{t}\left(\frac{2X_n - n}{\sqrt{n}}\right)\right),$$

where X_n is the total number of n separate jumps which were up-jumps.

- The random variable X_n has the binomial distribution with mean $n/2$ and variance $n/4$
- Thus, $(2X_n - n)/\sqrt{n}$ has mean zero and variance 1.
- By the central limit theorem, this distribution converges to a normal random variable with zero mean and unit variance.
- As δt gets smaller and n gets larger, the distribution of S becomes log-normal, as log S_t is normally distributed with mean $\log S_o + \mu t$ and variance $\sigma^2 t$.

Under the martingale measure

- This is what happens under the measure $\mathbb{P}$, but what about $\mathbb{Q}$?
- Recall our formula for the martingale measure probability q :

$$q = \frac{s\ exp(r\delta t)\ - s_{down}}{s_{up} - s_{down}},$$

- We can calculate that q is approximately equal to

$$q = \frac{1}{2}\left(1 - \sqrt{\delta t}\left(\frac{\mu + \frac{1}{2}\sigma^2 - r}{\sigma}\right)\right).$$

- So, under $\mathbb{Q}$, X_n is still binomially distributed, but now has mean nq and variance $nq(1-q)$.

Under the martingale measure, continued

- Thus, $(2X_n - n)/\sqrt{n}$ has mean $-\sqrt{t}\left(\mu + \frac{1}{2}\sigma^2 - r\right)/\sigma$ and a variance that asymptotically approaches 1.
- By the **Central Limit Theorem**, this converges to a normal random variable with the same mean and variance equal to 1.
- S_t is still log-normally distributed with $\log S_t$ having mean $\log S_o + \left(r - \frac{1}{2}\sigma^2\right)t$ and variance $\sigma^2 t$. This can be written:

$$S_t = S_0 exp\left(\sigma\sqrt{t}Z + \left(r - \frac{1}{2}\sigma^2\right)t\right),$$

where Z is normal $N(0,1)$ under $\mathbb{Q}$.

- This is the marginal distribution of S_t under the martingale measure $\mathbb{Q}$.

Pricing a call option

- X is the call option maturing at date T, struck at k, with $X = (S_T - k)^+$.
- The value of the call option at time zero is worth

$$\mathbb{E}_{\mathbb{Q}}(B_T^{-1}X) = \mathbb{E}_{\mathbb{Q}}\left[\left(S_0 exp\left(\sigma\sqrt{T}Z - \frac{1}{2}\sigma^2 T\right) - k\ exp(-rT)\right)^+\right]$$

- In Chapter 3, we will see that this evaluates as

$$S_0\Phi\left(\frac{log\frac{S_0}{k} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - k\ exp(-rT)\Phi\left(\frac{log\frac{S_0}{k} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$

where Φ is the normal distribution function $\Phi(x) = \mathbb{Q}(Z \leq x)$.